

Physics 53

Gravity

*Nature and Nature's law lay hid in night;
God said, "Let Newton be!" and all was light.
— Alexander Pope*

Kepler's laws

Explanations of the motion of the celestial bodies — sun, moon, planets and stars — are among the oldest scientific theories. The apparent rotation of these bodies around the earth every day was attributed by the ancients to their place in "celestial spheres" which revolved daily around the stationary earth, assumed to be at the center of the universe.

The wandering of the planets against the background of stars presented a challenge to this earth-centered ("geocentric") model. Their peculiar motions were explained in terms of smaller spheres rolling on larger ones. As observations became more accurate these explanations became more intricately complicated.

The sun-centered ("heliocentric") model of Copernicus (about 1540) was simpler, but it made the earth merely one of the planets orbiting the sun. This provoked outrage, since it challenged the central importance of *homo sapiens* in the universe.

In the late 1500's the Danish astronomer Tycho Brahe carried out a careful and systematic set of observations of the night sky, especially of the motions of the planets. His assistant Johannes Kepler inherited the voluminous data he had accumulated; Kepler subjected them to careful numerical analysis, and in the early years of the next century he was able to discern in the data three regularities:

Kepler's laws of planetary motion	<ol style="list-style-type: none">1. The orbits of planets are ellipses, with the sun at one focus.2. As the planet moves, the line from the sun to a planet sweeps out equal areas in equal times.3. The ratio T^2 / a^3 is the same for all planets, where T is the period and a is the semi-major axis of the orbit.
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How to explain these three laws (the "Kepler problem") was a major concern of the scientists of the 17th century, among whom was Isaac Newton.

Universal gravitation

Newton's analysis of gravity was partly motivated by the Kepler problem, but more by the idea that the moon in its orbit is continually "falling" toward the earth, much as objects near the surface of the earth — including the famous apple? — fall to the ground. He proposed that the solution to both problems lay in a universal law:

Law of universal gravitation	Every pair of point masses m_1 and m_2 attract with a force given by $F = G \frac{m_1 m_2}{r^2}$, where r is the distance between the masses and G is a universal constant.
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Because of its simplicity, elegance and its universality — note that it applies to *every* pair of point masses in the universe — this has served over the centuries as a model for what a general law of nature ought to be like.

The value of G is 6.67×10^{-11} in SI units. This is very small, so the gravitational attraction between everyday objects is too weak to be easily detectable — which is why Newton never knew its value. Gravity is much weaker than the electric force between two moderate sized charges. It is important in our everyday lives because:

- The earth is very massive (5.98×10^{24} kg). This means the attraction of the earth for ordinary objects is a substantial force.
- Atoms and molecules are electrically neutral, so ordinary objects do not usually attract or repel each other electrically unless they get very close together.

Because mass is positive, the gravitational force is *always* an attraction.

Because electric charge can be either positive or negative, electrical forces can be either attractive or repulsive, so they are subject to "shielding" which can reduce or negate their effects.

Gravity is "long range" in nature, meaning that its effects extend indefinitely far. In the universe as a whole gravity is the most important force. In very massive stars it can even overwhelm even the strong force that binds particles in the atomic nucleus.

Gravitational potential energy

Consider two masses, M and m . Let M be fixed at the origin and let the position vector of m relative to the origin be \mathbf{r} . In vector form, the gravitational force by M on m is

$$\mathbf{F}(\text{on } m) = -G \frac{Mm}{r^2} \hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector from M toward m . This shows that gravity is a central force, i.e., it is always directed toward the “force center” — in this case, M . Since all central forces are known to be conservative, there is a potential energy function for this force, which we can construct from the definition of potential energy:

$$U(r) - U(r_0) = - \int_{r_0}^r F dr = -G \left(\frac{Mm}{r} - \frac{Mm}{r_0} \right).$$

We choose the potential energy to be zero when the masses are infinitely far apart (the usual choice), so that as $r_0 \rightarrow \infty$, $U(r_0) \rightarrow 0$. This gives us a relatively simple formula:

Gravitational potential energy (point masses)

$$U = -G \frac{Mm}{r}$$

Here r is the distance between the masses. For a system of more than two point masses, each pair contributes to the total according to this formula.

Note that U is always negative, which is characteristic of an attractive force.

The gravitational field

The total gravitational force on a given particle due to a number of other particles can be obtained (by superposition) from the formula for the force between pairs of particles. In this way one could (in principle) calculate the force on a particular particle due to all the particles in the earth.

But it is more convenient to regard the particles in the earth as a set of given "sources" of gravity, and then to regard the force they exert on another particle as a collective effect. This is what we do when we say that the earth as a whole exerts a force mg on a particle of mass m near its surface. The quantity g represents the *collective* gravitational effect of all the particles in the earth.

To describe the effect of the sources collectively is an essential idea in the concept of a **field**. The sources “create” the field, and a “test” particle (the one we are interested in) exhibits the effects of that field. But what is the field itself?

A field is a physical quantity that has a specific value at each point in space.

For gravity, we make the following definition:

Gravitational field	The gravitational field \mathbf{g} at a point in space is defined as the gravitational force on a particle of unit mass placed at that point.
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That is, if the gravitation force on a particle of mass m at a particular place is \mathbf{F}_g , then the gravitational field at that place is given by $\mathbf{g} = \mathbf{F}_g / m$. Conversely, if the field \mathbf{g} is known at a point in space, and if a particle of mass m is placed there, then it will experience a gravitational force $\mathbf{F}_g = m\mathbf{g}$.

If gravity is the *only* force on the particle, then the value of the gravitational field — the force per unit mass on the particle — is the same as the particle's acceleration. (The m on both sides of $\mathbf{F} = m\mathbf{a}$ cancels, so $\mathbf{a} = \mathbf{g}$.) The “acceleration of gravity” introduced earlier for objects near the earth's surface is in fact just the gravitational field in that region.

As stated above, a field is a physical quantity that has a definite value at each point in space. The “value” of the field at a point can be a scalar or a vector, so we have scalar fields and vector fields. We see that \mathbf{g} is a vector field. The temperature at each point in a room is an example of a scalar field. Other fields are the velocity (vector) field in moving fluid, and the pressure (scalar) field in a fluid, to be discussed later in this course. In the next course much time will be spent studying electric and magnetic fields, which are vector fields, and electrostatic potential, which is a scalar field.

Why do we introduce the concept of a field? There are several reasons. Some are matters of convenience, while others are of fundamental physical importance.

- Knowledge of the field makes it relatively easy to understand what will happen to any “test particle” placed at a point in the field.
- For vector fields, one can make drawings (maps) of the field by constructing curves (“field lines”) tangent to the direction of the field. For scalar fields one can draw curves connecting points where the field's value is the same. These maps are valuable aids to the intuition.
- By introducing fields we are able to avoid the untenable assumption of “action at a distance” in which the interaction is assumed to be transmitted *instantly* from one object to the other, even though they may be light years apart. (Experiment in fact shows that no information can travel faster than the speed of light.)

- The electromagnetic interaction (to be discussed in the next course) is most easily expressed in terms of fields. The fundamental equations of electromagnetism are given in terms of these fields.
- In the electromagnetic case we know from experiment that the fields themselves possess energy and momentum; indeed, proper use of the conservation laws requires inclusion of these “mechanical” aspects of the fields as part of the system’s energy and momentum. Light itself consists entirely of energy and momentum possessed by the electromagnetic field, traveling through space as a wave in the field. (Because gravity is so weak, it is difficult to test experimentally whether the gravitational field also has these properties, although all current theories assume it does.)

The concept of a field has developed over the past two centuries into one of the most important ideas in science. Modern theories take this to an extreme, postulating that all matter (including things usually classed as “particles”) should be described in terms of fields that obey the rules of relativity and quantum theory. In this world view we and everything around us are just complicated combinations of various quantum fields.

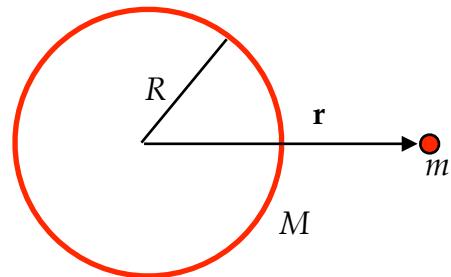
Spherical shells

In applying his gravitation law to the sun and planets, Newton assumed that the distance r between them should be taken to be the distance between their centers. Is that only an approximation?

Newton was able to show — using his newly developed integral calculus — that it is *not* an approximation, because the *external* gravitational effect of a uniform spherical mass distribution *is exactly the same* as it would be if all the mass of the sphere were concentrated at its center.

The essential step in proving this is to show that it is true for a thin spherical shell, since any spherical distribution can be considered to be a layered combination of concentric shells. Consider the gravitational field of a spherical shell, as shown. The field at the location of a test mass m can be shown (proof at the end of this section) to be:

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -G \frac{M}{r^2} \hat{\mathbf{r}} & \text{for } r > R \\ 0 & \text{for } r < R \end{cases}$$



This shows two remarkably simple properties:

- At points outside the shell the field is the same as though all the mass of the shell were concentrated at its center.

- At points inside the shell the field is zero.

Since any spherically symmetric object (such as the sun) can be treated as a superposition of concentric spherical shells — like layers of an onion — the total force exerted by such an object on an external mass is the same as if the entire mass of the sphere were at its center. This justifies Newton's treatment of the sun and planets as mass points analyzing planetary motion.

From the formulas for the field it is easy to obtain the potential energy for the system of a shell and a point test mass m :

$$U(r) = \begin{cases} -G \frac{Mm}{r} & \text{for } r > R \\ -G \frac{Mm}{R} & \text{for } r < R \end{cases}$$

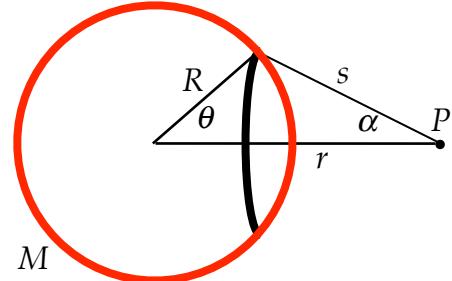
Note carefully that although the field inside the shell is zero, the potential energy is not. We have taken the potential energy to be zero at infinite distance, and that is the only place where it will be zero. The potential energy inside the shell has the same value as at the shell's surface.

These properties of shells are useful in dealing with any spherical distribution of mass.

Proof of formulas for \mathbf{g} for a spherical shell.

To find the field at the field point P shown we first find the contribution of an infinitesimal ring on the shell, all points of which are at distance s from the field point. Let the mass of the ring be dm and let the line from the center of the shell to P define the x -axis. By the symmetry the field at P has only an x -component, which is

$$dg_x = -G \frac{dm}{s^2} \cos \alpha.$$



The area of the ring is its circumference times its width, or $2\pi R \sin \theta \cdot R d\theta$. Its mass is therefore

$$dm = \frac{M}{4\pi R^2} \cdot 2\pi R^2 \sin \theta d\theta = \frac{M}{2} \sin \theta d\theta.$$

Thus we have

$$dg_x = -\frac{GM}{2} \frac{\cos \alpha}{s^2} \sin \theta d\theta.$$

The variables s , α , and θ are related by the geometry. From the law of cosines we have $R^2 = r^2 + s^2 - 2rs \cos \alpha$ and $s^2 = R^2 + r^2 - 2Rr \cos \theta$. The first of these gives

$$\cos \alpha = \frac{s^2 + r^2 - R^2}{2rs},$$

while the second gives (since r and R are constants)

$$sds = Rr \sin \theta d\theta.$$

Putting it all together we find

$$dg_x = -\frac{GM}{4Rr^2} \left(1 + \frac{r^2 - R^2}{s^2} \right) ds.$$

To find the field we integrate over s . For points outside the shell ($r > R$) we integrate from $s = r - R$ to $s = r + R$, and find

$$g_x = -\frac{GM}{r^2} \text{ for } r > R.$$

For points inside the shell we integrate from $s = R - r$ to $s = R + r$, and find

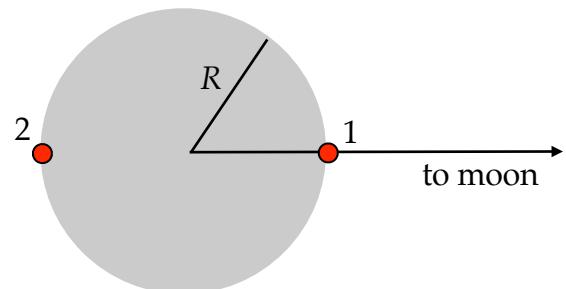
$$g_x = 0 \text{ for } r < R.$$

This proves the claimed properties.

Tidal forces

Besides solving the Kepler problem and relating the period of the moon to g at the surface of the earth, Newton also was able to explain the ocean tides as a gravitational effect caused by the moon and sun.

Shown is the earth, with the moon to the right at distance r from the center of the earth. The radius of the earth is R . Also shown are two particles of mass m located at points 1 and 2 on opposite sides of the earth. If the moon were absent, each of these masses would be attracted toward the center of the earth by the usual gravitational force mg .



We will analyze how the moon alters the forces on these two particles, as measured by observers at points 1 and 2. These observers use a reference frame fixed on the earth, which is a *non-inertial* frame because the earth is accelerating toward the moon.

We ignore (for the moment) the effects of the sun.

At the center of the earth let the gravitational field of the moon be \mathbf{g}_0 , directed toward the moon (to the right in the diagram) with magnitude

$$g_0 = \frac{GM_M}{r^2}.$$

This is also the *acceleration* of the earth's CM toward the moon, and it is therefore the acceleration of the reference frame of our two observers.

At point 1 the gravitational field of the moon is \mathbf{g}_1 with magnitude

$$g_1 = \frac{GM_M}{(r - R)^2}.$$

Similarly at point 2 the field of the moon is \mathbf{g}_2 with magnitude

$$g_2 = \frac{GM_M}{(r + R)^2}.$$

All of these vectors are directed *toward* the moon, of course.

But what the observers will measure is the *effective gravity* of the moon in their reference frame, which we get by subtracting the reference frame's acceleration \mathbf{g}_0 .

At point 1 the moon's effective gravity is $\mathbf{g}_1^{eff} = \mathbf{g}_1 - \mathbf{g}_0$. Since $g_1 > g_0$, this vector is *toward* the moon. The mass at that point experiences an *outward* force mg_1^{eff} exerted by the moon, away from the center of the earth. The total gravitational force on it (toward the center of the earth) is thus $F_1 = m(g - g_1^{eff})$.

At point 2 we have $\mathbf{g}_2^{eff} = \mathbf{g}_2 - \mathbf{g}_0$. Since $g_2 < g_0$, this vector is *away* from the moon, and therefore *away* from the center of the earth. The force of the moon on this mass is mg_2^{eff} .

The total gravitational force on it (toward the center of the earth) is $F_2 = m(g - g_2^{eff})$.

Thus the moon produces forces on *both* of the particles *away* from the earth's center, *reducing* their gravitational attraction toward the earth's center. If the particles are part of the oceans there is a small outward bulge in the water surface at both points. This gives rise to the twice daily tides. The rise or fall of the level in mid-ocean is tiny, but the effect at shorelines of the resulting shift of large volumes of water toward or away from the shore can be quite dramatic, especially if it is amplified by channel effects and wave resonance.

Now we calculate the magnitude of these effects. We find

$$g_1^{eff} = g_1 - g_0 = GM_M \left[\frac{1}{(r-R)^2} - \frac{1}{r^2} \right],$$

$$g_2^{eff} = g_0 - g_2 = GM_M \left[\frac{1}{r^2} - \frac{1}{(r+R)^2} \right].$$

These are not quite the same. But using the fact that $R \ll r$, we can make an approximation. We write

$$\frac{1}{(r \pm R)^2} = \frac{1}{r^2} \cdot \frac{1}{(1 \pm R/r)^2}$$

and use the binomial approximation on the second factor:

$$(1 \pm R/r)^{-2} \approx 1 \mp 2R/r.$$

This gives

$$g_1^{eff} \approx g_2^{eff} \approx \frac{2GM_M R}{r^3}.$$

Thus the effects on opposite sides of the earth are approximately equal and proportional to the ratio M_M / r^3 .

The sun also gives rise to tidal forces, but because of its much greater distance the ratio M / r^3 for the sun is only about half that for the moon. When earth, sun and moon are on the same line (with sun and moon either on the same or opposite sides of the earth), their effects add and give rise to especially large tides, called *spring* tides. When the lines from the sun and moon to the earth make a right angle, the effects tend to cancel and we have small *neap* tides.

An indirect effect of tidal forces is to produce dissipative torques which reduce the rotational kinetic energy of the object acted upon. This is why the moon (on which the earth's tidal forces are quite strong because the earth's mass is so large) has long since stopped rotating relative to the earth and keeps the same face toward us. Presumably the moon is also making the earth slow in its daily rotation, but very gradually.