MCAC 201: Design and Analysis of Algorithms

Neelima Gupta

ngupta@cs.du.ac.in

January 5, 2025

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

Asymptotic Notations 1: Big O

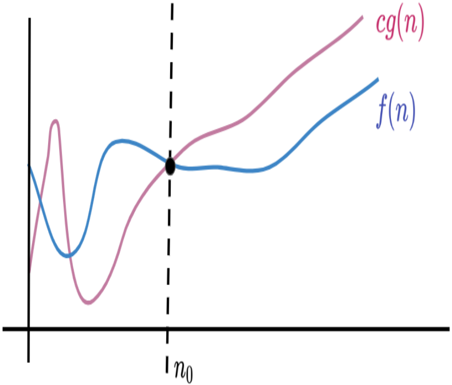
Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.



Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^{+} \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3n^2$ and $g(n) = n^2 - 2n$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3n^2$ and $g(n) = n^2 - 2n$.
  $3n^2 \leq 3n^2 + n^2 - 8n$ for all $n \geq 8$,

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.

  Examples:

  Let $f(n) = 3n^2$ and $g(n) = n^2 - 2n$.

  $3n^2 \leq 3n^2 + n^2 - 8n$ for all $n \geq 8$,

  $\quad = 4(n^2 - 2n)$

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3n^2$ and $g(n) = n^2 - 2n$.
  $3n^2 \leq 3n^2 + n^2 - 8n$ for all $n \geq 8$,
  $= 4(n^2 - 2n)$
  Thus, $c = 4$ and $n_0 = 8$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n + 3\frac{n}{\log n}, g(n) = n$
  $n + 3\frac{n}{\log n} \leq 4n$ for all $n \geq 2$
  Thus, $c = 4, n_0 = 2$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n + 3\frac{n}{\log n}, g(n) = n$
  $n + 3\frac{n}{\log n} \leq 4n$ for all $n \geq 2$
  Thus, $c = 4, n_0 = 2$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n + 3\frac{n}{\log n}, g(n) = n$
  $n + 3\frac{n}{\log n} \leq 4n$ for all $n \geq 2$
  Thus, $c = 4, n_0 = 2$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

c is a non negative finite constant. c cannot be ∞.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

c is a non negative finite constant. c cannot be ∞.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- More Examples:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^2-2n-5}{5n^2+3n+10} = \lim_{n\to\infty} \frac{7-\frac{2}{n}-\frac{5}{n^2}}{5+\frac{3}{n}+\frac{10}{n^2}} = \frac{7}{5}$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- More Examples:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^2-2n-5}{5n^2+3n+10} = \lim_{n\to\infty} \frac{7-\frac{2}{n}-\frac{5}{n^2}}{5+\frac{3}{n}+\frac{10}{n^2}} = \frac{7}{5}$.
  Computing c and n_0: $7\mathrm{n}^2 - 2n - 5 \leq \frac{7}{5}(5n^2 + 3n + 10)$ for all $n \geq 1$.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- More Examples: $\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = \lim_{n\rightarrow\infty} \frac{n+10}{n^2-100} = 0.$

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- More Examples: $\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = \lim_{n\rightarrow\infty} \frac{n+10}{n^2-100} = 0$.
  Computing c and n_0: $2(n^2 - 100) = n^2 + n^2 - 200 \geq n + 10$ for $n \geq \sqrt{210}$.
  $\geq n + 10$ for $n \geq 1$. $c = 2, n_0 = \max\{1, \sqrt{210}\}$

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- Notation: $f(n) \leq_{\infty} g(n)$ asymptotically.

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^+ \rightarrow \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

$$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- Notation: $f(n) \leq_\infty g(n)$ asymptotically.
  By asymptotically we mean within constant multiplicative factor and for large n

Asymptotic Notations 1: Big O

Consider a set $\mathcal{F}$ of non-negative, monotonically non-decreasing functions $f : \mathbf{I}^{+} \to \mathbf{R}$. Let $f, g \in \mathcal{F}$.

- $f(n) = O(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \leq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = O(g(n))$ iff

  $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 \leq c < \infty$$

  c is a non negative finite constant. c cannot be ∞.
- Notation: $f(n) \leq_{\infty} g(n)$ asymptotically.
  By asymptotically we mean within constant multiplicative factor and for large n
  We say that $f(n)$ grows at most as fast as $g(n)$ with n asymptotically.

Asymptotic Notations 2: Omega

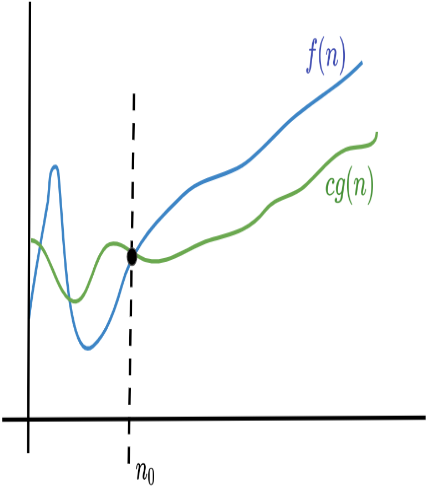
- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.



$f(n) = \Omega(g(n))$

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $4(n^2 - 2n) = 3n^2 + n^2 - 8n$ for all $n \geq 8$,

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $$\begin{aligned} 4(n^2 - 2n) &= 3n^2 + n^2 - 8n \text{ for all } n \geq 8, \\ &\geq 3n^2 \end{aligned}$$

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $4(n^2 - 2n) = 3n^2 + n^2 - 8n$ for all $n \geq 8$,
  $\geq 3n^2$
  Thus, $c = 1/4$ and $n_0 = 8$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $$\begin{aligned} 4(n^2 - 2n) &= 3n^2 + n^2 - 8n \text{ for all } n \geq 8, \\ &\geq 3n^2 \end{aligned}$$
  Thus, $c = 1/4$ and $n_0 = 8$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3\frac{n^2}{\log n} + n, g(n) = n$
  $3\frac{n^2}{\log n} + n \geq 4n$ for all $n \geq 1$
  Thus, $c = 4, n_0 = 1$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

c is a (non zero) positive constant. It can be ∞ but not 0.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3\frac{n^2}{\log n} + n, g(n) = n$
  $3\frac{n^2}{\log n} + n \geq 4n$ for all $n \geq 1$
  Thus, $c = 4, n_0 = 1$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

c is a (non zero) positive constant. It can be ∞ but not 0.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = 3\frac{n^2}{\log n} + n, g(n) = n$
  $3\frac{n^2}{\log n} + n \geq 4n$ for all $n \geq 1$
  Thus, $c = 4, n_0 = 1$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

c is a (non zero) positive constant. It can be ∞ but not 0.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

c is a (non zero) positive constant. It can be ∞ but not 0.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition:$f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

  c is a (non zero) positive constant. It can be ∞ but not 0.
- More Examples:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^2+3n+10}{7n^2-2n-5} = \lim_{n\to\infty} \frac{5+\frac{3}{n}+\frac{10}{n^2}}{7-\frac{2}{n}-\frac{5}{n^2}} = \frac{5}{7}$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

  $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

  c is a (non zero) positive constant. It can be ∞ but not 0.
- More Examples:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^2+3n+10}{7n^2-2n-5} = \lim_{n\to\infty} \frac{5+\frac{3}{n}+\frac{10}{n^2}}{7-\frac{2}{n}-\frac{5}{n^2}} = \frac{5}{7}$.
  Computing c and n_0: $(5n^2 + 3n + 10) \leq \frac{5}{7}(7n^2 - 2n - 5)$ for all $n \geq 1$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

  c is a (non zero) positive constant. It can be ∞ but not 0.
- More Examples: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2-100}{n+10} = \infty$.

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

  c is a (non zero) positive constant. It can be ∞ but not 0.
- More Examples: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2-100}{n+10} = \infty$.
  Computing c and n_0:
  $2(n^2 - 100) = (n^2 + 10) + n^2 - 210 \geq n^2 + 10$ for $n \geq \sqrt{210}$.
  $\geq n + 10$ for $n \geq 1$. $c = 1/2, n_0 = \max\{1, \sqrt{210}\}$

Asymptotic Notations 2: Omega

- $f(n) = \Omega(g(n))$ iff $\exists$ constants $c > 0, n_o > 0 : f(n) \geq cg(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c \leq \infty$$

  c is a (non zero) positive constant. It can be ∞ but not 0.

- Notation: $f(n) \geq_\infty g(n)$ asymptotically.
  We say that $f(n)$ grows at least as fast as $g(n)$ with n asymptotically.

Asymptotic Notations 3: theta

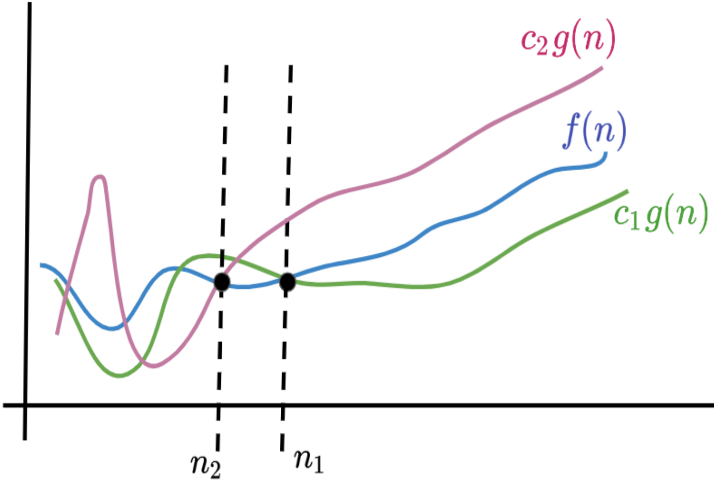
- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.



$$f(n) = \theta(g(n))$$

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $\frac{1}{4}3n^2 \leq (n^2 - 2n) \leq 3n^2$ for all $n \geq 8$,

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$:
  $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $\frac{1}{4}3n^2 \le (n^2 - 2n) \le 3n^2$ for all $n \ge 8$,
  i.e., Thus, $(1/4)g(n) \le f(n) \le g(n)$ for all $n \ge 8$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$:
  $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $\frac{1}{4}3n^2 \leq (n^2 - 2n) \leq 3n^2$ for all $n \geq 8$,
  i.e., Thus, $(1/4)g(n) \leq f(n) \leq g(n)$ for all $n \geq 8$.
  Thus, $c_1 = 1/4, c_2 = 1$ and $n_0 = 8$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$:
  $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = n^2 - 2n$ and $g(n) = 3n^2$.
  $\frac{1}{4} 3n^2 \leq (n^2 - 2n) \leq 3n^2$ for all $n \geq 8$,
  i.e., Thus, $(1/4)g(n) \leq f(n) \leq g(n)$ for all $n \geq 8$.
  Thus, $c_1 = 1/4, c_2 = 1$ and $n_0 = 8$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$:
  $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = \log_2 n, g(n) = \log_3 n$
  $\log_2 n = \log_3 2 \log_3 n$ for all n.
  Thus, $c_1 = c_2 = \log_3 2, n_0 = 1$.
- Equivalent Definition: $f(n) = \theta(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$$

  c is a (non-zero) positive finite constant. c is neither 0 nor ∞.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_o$.
  Examples:
  Let $f(n) = \log_2 n, g(n) = \log_3 n$
  $\log_2 n = \log_3 2 \log_3 n$ for all n.
  Thus, $c_1 = c_2 = \log_3 2, n_0 = 1$.
- Equivalent Definition: $f(n) = \theta(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$$

  c is a (non-zero) positive finite constant. c is neither 0 nor ∞.
- Example:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^2-2n-5}{5n^2+3n+10} = \lim_{n\to\infty} \frac{7-\frac{2}{n}-\frac{5}{n^2}}{5+\frac{3}{n}+\frac{10}{n^2}} = \frac{7}{5}$.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
  Examples:
  Let $f(n) = \log_2 n, g(n) = \log_3 n$
  $\log_2 n = \log_3 2 \log_3 n$ for all n.
  Thus, $c_1 = c_2 = \log_3 2, n_0 = 1$.
- Equivalent Definition: $f(n) = \theta(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$$

  c is a (non-zero) positive finite constant. c is neither 0 nor ∞.
- Example:
  $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^2-2n-5}{5n^2+3n+10} = \lim_{n\to\infty} \frac{7-\frac{2}{n}-\frac{5}{n^2}}{5+\frac{3}{n}+\frac{10}{n^2}} = \frac{7}{5}$.
- Notation: $f(n) =_\infty g(n)$ asymptotically.

Asymptotic Notations 3: theta

- $f(n) = \theta(g(n))$ iff $\exists$ constants $c_1, c_2 > 0, n_o > 0$: $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o$.
- Equivalent Definition: $f(n) = \theta(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$$

  c is a (non-zero) positive finite constant. c is neither 0 nor ∞.
- Example: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{7n^2-2n-5}{5n^2+3n+10} = \lim_{n\to\infty} \frac{7-\frac{2}{n}-\frac{5}{n^2}}{5+\frac{3}{n}+\frac{10}{n^2}} = \frac{7}{5}$.
- Notation: $f(n) =_\infty g(n)$ asymptotically. We say that $f(n)$ grows at the same rate as $g(n)$ with n, asymptotically.

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0$ $\exists$ $n_c > 0$: $f(n) < cg(n)$ for all $n \geq n_c$.

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0 \; \exists \; n_c > 0$: $f(n) < cg(n)$ for all $n \geq n_c$.
- Equivalent Definition: $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0$ $\exists$ $n_c > 0$: $f(n) < cg(n)$ for all $n \geq n_c$.
- Equivalent Definition: $f(n) = o(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- Example: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n+10}{n^2-100} = 0.$

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0$ $\exists$ $n_c > 0$: $f(n) < cg(n)$ for all $n \geq n_c$.
- Equivalent Definition: $f(n) = o(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- Example: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log^2 n}{\sqrt{n}} = 0.$

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0$ $\exists$ $n_c > 0$: $f(n) < cg(n)$ for all $n \geq n_c$.
- Equivalent Definition: $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

- Example: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n^{10}}{2^n} = 0.$
- Notation: $f(n) <_{\infty} g(n)$ asymptotically.
  We say that $f(n)$ grows strictly slower than $g(n)$ with n, asymptotically.

Asymptotic Notations 4: small o

- $f(n) = o(g(n))$ iff for every constant $c > 0 \ \exists \ n_c > 0 :$ $f(n) < cg(n)$ for all $n \geq n_c$.
- Equivalent Definition: $f(n) = o(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

- Example: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{2^n}{3^n} = 0.$
- Notation: $f(n) <_{\infty} g(n)$ asymptotically.
  We say that $f(n)$ grows strictly slower than $g(n)$ with n, asymptotically.

Asymptotic Notations 5: small omega

- $f(n) = \omega(g(n))$ iff for every constant $c > 0 \; \exists \; n_c > 0$: $f(n) > cg(n)$ for all $n \geq n_c$

Asymptotic Notations 5: small omega

- $f(n) = \omega(g(n))$ iff for every constant $c > 0 \ \exists \ n_c > 0 :$ $f(n) > cg(n)$ for all $n \geq n_c$
- Equivalent Definition: $f(n) = \omega(g(n))$ iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

Asymptotic Notations 5: small omega

- $f(n) = \omega(g(n))$ iff for every constant $c > 0 \; \exists \; n_c > 0$: $f(n) > cg(n)$ for all $n \geq n_c$
- Equivalent Definition: $f(n) = \omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

- Example: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2-100}{n+10} = \infty$.

Asymptotic Notations 5: small omega

- $f(n) = \omega(g(n))$ iff for every constant $c > 0 \; \exists \; n_c > 0$: $f(n) > cg(n)$ for all $n \geq n_c$
- Equivalent Definition: $f(n) = \omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

- Example: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2-100}{n+10} = \infty$.
- Notation: $f(n) >_\infty g(n)$ asymptotically.
  We say that $f(n)$ grows strictly faster than $g(n)$ with n, asymptotically.

Some easy to prove Observations

1. $f(n) \leq g(n)$ iff either $f(n) < g(n)$ or $f(n) = g(n)$ is true.

Some easy to prove Observations

1. $f(n) \leq g(n)$ iff either $f(n) < g(n)$ or $f(n) = g(n)$ is true.
2. $f(n) \leq g(n)$ iff $g(n) \geq f(n)$.

Some easy to prove Observations

1. $f(n) \leq g(n)$ iff either $f(n) < g(n)$ or $f(n) = g(n)$ is true.
2. $f(n) \leq g(n)$ iff $g(n) \geq f(n)$.
3. $f(n) = g(n)$ iff $f(n) \leq g(n)$ and $f(n) \geq g(n)$.

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4$, $g(n) = 2^n$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4$, $g(n) = 2^n$

$$\lim_{n\to\infty} \frac{n^4}{2^n} = \lim_{n\to\infty} \frac{4n^3}{2^n} = \lim_{n\to\infty} \frac{12n^2}{2^n} = \lim_{n\to\infty} \frac{24n}{2^n} = \lim_{n\to\infty} \frac{24}{2^n} = 0$$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n,\ g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4,\ g(n) = 2^n$

$$\lim_{n\to\infty} \frac{n^4}{2^n} = \lim_{n\to\infty} \frac{4n^3}{2^n} = \lim_{n\to\infty} \frac{12n^2}{2^n} = \lim_{n\to\infty} \frac{24n}{2^n} = \lim_{n\to\infty} \frac{24}{2^n} = 0$$

3. $f(n) = \log^2 n,\ g(n) = \sqrt{n}$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n,\ g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4,\ g(n) = 2^n$

$$\lim_{n\to\infty} \frac{n^4}{2^n} = \lim_{n\to\infty} \frac{4n^3}{2^n} = \lim_{n\to\infty} \frac{12n^2}{2^n} = \lim_{n\to\infty} \frac{24n}{2^n} = \lim_{n\to\infty} \frac{24}{2^n} = 0$$

3. $f(n) = \log^2 n,\ g(n) = \sqrt{n}$

$$\lim_{n\to\infty} \frac{\log^2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{2\log n}{n}.2\sqrt{n} = \lim_{n\to\infty} \frac{4\log n}{\sqrt{n}} = \lim_{n\to\infty} \frac{4}{n}.2\sqrt{n} = 0$$

Let's compare some popular functions (use *L'Hôpital's* Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4$, $g(n) = 2^n$

$$\lim_{n\to\infty} \frac{n^4}{2^n} = \lim_{n\to\infty} \frac{4n^3}{2^n} = \lim_{n\to\infty} \frac{12n^2}{2^n} = \lim_{n\to\infty} \frac{24n}{2^n} = \lim_{n\to\infty} \frac{24}{2^n} = 0$$

3. $f(n) = \log^2 n$, $g(n) = \sqrt{n}$

$$\lim_{n\to\infty} \frac{\log^2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{2\log n}{n}.2\sqrt{n} = \lim_{n\to\infty} \frac{4\log n}{\sqrt{n}} = \lim_{n\to\infty} \frac{4}{n}.2\sqrt{n} = 0$$

4. $f(n) = 2^n$, $g(n) = 3^n$

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$$

Let's compare some popular functions (use $L'Hôpital's$ Rule wherever required)

1. $f(n) = \log n$, $g(n) = n$

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{1}{n \cdot \ln 2} = 0$$

2. $f(n) = n^4$, $g(n) = 2^n$

$$\lim_{n\to\infty} \frac{n^4}{2^n} = \lim_{n\to\infty} \frac{4n^3}{2^n} = \lim_{n\to\infty} \frac{12n^2}{2^n} = \lim_{n\to\infty} \frac{24n}{2^n} = \lim_{n\to\infty} \frac{24}{2^n} = 0$$

3. $f(n) = \log^2 n$, $g(n) = \sqrt{n}$

$$\lim_{n\to\infty} \frac{\log^2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{2\log n}{n}.2\sqrt{n} = \lim_{n\to\infty} \frac{4\log n}{\sqrt{n}} = \lim_{n\to\infty} \frac{4}{n}.2\sqrt{n} = 0$$

4. $f(n) = 2^n$, $g(n) = 3^n$

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$$

Practice Questions

Use the equivalent definitions (limits) to prove the following:

1. Show that a polynomial of degree d, with positive leading coefficient is $\Theta(n^d)$.
2. For $g(n) = f(n) + o(f(n))$, show that $g(n) = \Theta(f(n)$.
3. Show that
   a. $\log n = o(n)$.
   b. $\log^M n = o(n^\epsilon)$ where M and ϵ are positive constants.
4. $a^n = o(b^n)$ for all $a < b$.
5. $\log_a n = \theta(\log_b n)$.