Geometry of Principal Bundles
Lecture 3: Curvature and Flat Connections

Abstract
We start by introducing curvature and flat connections. Then, we show that a flat connection is preserved under the action of the gauge group and discuss the classification of flat connections over a given base manifold ($\mathcal{F}_{M,G}$ is in bijection with $\mathrm{Hom}(\pi_1(\mathrm{M});\mathrm{G})/\mathrm{G}$). Lastly, we take a glimpse into characteristic classes through the basic example of $U(1)$-connections.

Exercise 0. Let $G \subseteq GL(V)$ and P be the frame bundle of a vector bundle E, or equivalently, $E = P \times_{\iota} V$, where $\iota : G \hookrightarrow GL(V)$ is the inclusion map. Show that $ad(P) := P \times_{ad} Lie(G)$ is canonically isomorphic to $\mathrm{End}(E)$.

Recall
We have three definitions of a connection on a principal G-bundle P:

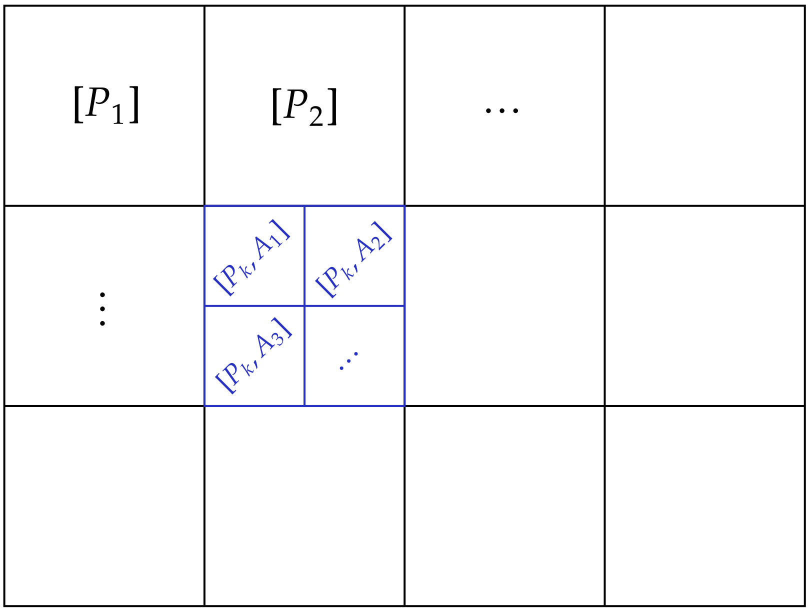
1. A connection A is a a $Lie(G)$-valued 1-form on P such that
   (a) A is G-equivariant
   (b) A satisfies $A(\Psi(p, m)) = m$, that is $A(t \mapsto p \cdot \exp(tm)) = m$.
$$A : TP \xrightarrow[C^\infty\text{-lin}]{G\text{-equi}} Lie(G)$$
2. A connection in local trivializations is a collection of $Lie(G)$-valued 1-forms
$$a_\alpha : TU_\alpha \to Lie(G)$$
   that transform as follows under the change of trivializations.
$$a_\beta = \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1}$$
3. A connection can also be defined as a horizontal sub-bundle H_A of TP that has the following properties:
   (a) Each tangent space T_pP can be decomposed as $T_pP = (H_A)_p \oplus (\ker \pi_*)_p$.
   (b) H_A is G-invariant under the right action of G on TP, i.e.,
$$H_{pg} = \mathrm{d}(R_g)_p(H_p)$$
   where $p \in P, g \in G$, and R_g is right action of G on P.

The three definitions are relation by the following
$$\left(\psi_\alpha^{-1}\right)^* A = g^{-1}dg + g^{-1}a_\alpha g$$
$$H_A = \ker A.$$

Lastly, recall two principal bundles are equivalent if they are isomorphic principal bundles.

Two connections A_1, A_2 on a principal G-bundle $P$are equivalent if there is an automorphism $\eta : P \rightarrow P$ such that $\eta^* A_1 = A_2$.

At this stage we have the following diagram:



The set of equivalence classes of principal G-bundles
and connections over a manifold M

In this diagram, each box represents an equivalence class of a principal G-bundle over M. Further, each box $[P_k]$ can be subdivided into smaller pieces, where each piece $[P_k, A_l]$ represents an equivalence class of a connection A_l on P_k.

Curvature of a principal bundle

As a preamble, recall, the curvature F_∇ of a vector bundle is a section of $\Lambda^2 T^* M \otimes \left(\mathrm{End}\, E\right) = \Omega^2_M \left(\mathrm{End}(E)\right)$ and is locally described as:

$$F_\nabla = d\Gamma + \Gamma \wedge \Gamma$$

Exercise 1. Let P be a principal G-bundle with a connection $A = \{a_\alpha\}$. Consider the following collection of local $Lie(G)$-valued 2-forms:

$$\begin{aligned} F_{A,\alpha} &: \Lambda^2 TU_\alpha \rightarrow Lie(G) \\ F_{A,\alpha} &= da_\alpha + a_\alpha \wedge a_\alpha \\ &= da_\alpha + [a_\alpha, a_\alpha]/2 \\ &= da_\alpha + [a_\alpha \wedge a_\alpha]/2. \end{aligned}$$

Show that $F_{A,\alpha}$ describes a section of $\Lambda^2 T^* M \otimes (P \times_{ad} Lie(G)) = \Omega^2_M (P \times_{ad} Lie(G))$, that is, it transforms as:

$$F_{A,\beta} = \tau_{\beta\alpha} F_{A,\alpha} \tau^{-1}_{\beta\alpha}.$$

Explanation of notation: let $a_\alpha = \sum_k f_k w_k$, where $f_k : U_\alpha \to Lie(G)$ and $w_k : TU_\alpha \to \mathbb{R}$; then,

$$\begin{aligned}
da_\alpha &= \textstyle\sum_k (df_k \wedge w_k + f_k dw_k) \\
a_\alpha \otimes a_\alpha &= \left(\textstyle\sum_k f_k w_k\right) \otimes \left(\textstyle\sum_l f_l w_l\right) \\
&= \textstyle\sum_{k,l} f_k f_l \; w_k \otimes w_l \\
a_\alpha \wedge a_\alpha &= \textstyle\sum_{k,l} f_k f_l \; w_k \otimes w_l - \sum_{k,l} f_k f_l \; w_l \otimes w_k \\
&= \textstyle\sum_{k,l} (f_k f_l - f_l f_k) \cdot (w_k \otimes w_l) \\
&= \frac{1}{2} \textstyle\sum_{k,l} (f_k f_l - f_l f_k) \cdot (w_k \wedge w_l) \\
&= \frac{1}{2} \textstyle\sum_{k,l} [f_k, f_l] \cdot (w_k \wedge w_l) \\
&=: [a_\alpha, a_\alpha]/2 \\
&=: [a_\alpha \wedge a_\alpha]/2.
\end{aligned}$$

Definition 1. Let P be a principal G-bundle with a connection $A = \{a_\alpha\}$. The *curvature* of A, denoted by F_A, is an element of $\Omega^2_M(P \times_{ad} Lie(G))$ that is defined as

$$\begin{aligned}
F_{A,\alpha} &: \Lambda^2 TU_\alpha \to Lie(G) \\
F_{A,\alpha} &= da_\alpha + a_\alpha \wedge a_\alpha \\
&= da_\alpha + [a_\alpha, a_\alpha]/2 \\
&= da_\alpha + [a_\alpha \wedge a_\alpha]/2
\end{aligned}$$

Lemma 1 (naturality of the definition)**.** Let P be a principal G-bundle with a connection A and $P \times_\rho V$ be an associated vector bundle with the associated connection $\rho_*(A)$ Now, there are two curvatures:

(i) the curvature $F_A : \Lambda^2 TM \to P \times_{ad} Lie(G)$ of the principal G-bundle with connection A

(ii) the curvature tensor $F_\nabla : \Lambda^2 TM \to \mathrm{End}(P \times_\rho V) = P \times_{\rho_* \circ ad} \mathrm{End}(V)$ of the vector bundle $P \times_\rho V$ with connection $\rho_*(A)$.

The two are related as follows:

$$\rho_* \circ F_A = F_\nabla.$$

Explanation of notation: $\rho_* \circ F_A$

Locally, if F_A and F_∇ are given by $F_{A,\alpha} : \Lambda^2 TU_\alpha \to Lie(G)$ and $F_{\nabla,\alpha} : \Lambda^2 TU_\alpha \to M_n(\mathbb{R})$ respectively, then

$$(\rho_* \circ F_{A,\alpha}) : \Lambda^2 TU_\alpha \to M_n(\mathbb{R})$$

and the claim $\rho_* \circ F_A = F_\nabla$ simply says $\rho_* \circ F_{A,\alpha} = F_{\nabla,\alpha}$ in each trivialization α.
More globally, we have $F_A : \Lambda^2 TU_\alpha \to P \times_{ad} Lie(G)$ and $F_{\nabla,\alpha} : \Lambda^2 TU_\alpha \to \mathrm{End}(E)$. To relate them we make the following observations:

(i) We have $\mathrm{End}(E) = P \times_{ad\circ\rho} \mathrm{End}(V)$ where $ad \circ \rho$ is the composition of $\rho : G \to GL(V)$ with $ad : GL(V) \to GL\big(\mathrm{End}(V)\big)$. (This is a generalization of Exercise 0 above.)

(ii) The map $\rho_* : Lie(G) \to \mathrm{End}(V)$ defines a map $\left(\mathrm{Id}_P \times \rho_*\right) : P \times Lie(G) \to P \times \mathrm{End}(V)$ as $\left(\mathrm{Id}_P \times \rho_*\right)(p,m) := (p, \rho_*(m))$. We want to show this quotients to a map $\left(\mathrm{Id}_P \times \rho_*\right)_{/\sim} : P \times_{ad} Lie(G) \to P \times_{ad\circ\rho} \mathrm{End}(V) = \mathrm{End}(E)$. For this reason, we define right G-action on $P \times Lie(G)$ and $P \times \mathrm{End}(V)$ as follows:

$$\begin{aligned} (p,m)\cdot g &= (pg, ad(g)(m)) && \forall\ (p,m) \in P \times Lie(G) \\ &= \left(pg, g^{-1}mg\right) \\ (p,m)\cdot g &= (pg, ad(\rho(g))(m)) && \forall\ (p,m) \in P \times \mathrm{End}(V) \\ &= \left(pg, \rho(g)^{-1} m \rho(g)\right) \end{aligned}$$

Now, the map $\left(\mathrm{Id}_P \times \rho_*\right) : P \times Lie(G) \to P \times \mathrm{End}(V)$ is G-equivariant in the sense that

$$\begin{aligned} \left(\mathrm{Id}_P \times \rho_*\right)\left(pg, g^{-1}mg\right) &= \left(pg, \rho_*\left(g^{-1}mg\right)\right) \\ &= \left(pg, \rho(g)^{-1}\rho_*(m)\rho(g)\right) \qquad (*) \\ &= (pg, (ad \circ \rho(g))(\rho_*(m))) \end{aligned}$$

This means $\left(\mathrm{Id}_P \times \rho_*\right)$ takes orbits to orbits. Hence, $\left(\mathrm{Id}_P \times \rho_*\right)$ gives rise to $\left(\mathrm{Id}_P \times \rho_*\right)_{/\sim} : P \times_{ad} Lie(G) \to P \times_{ad\circ\rho} \mathrm{End}(V)$, where $\left(\mathrm{Id}_P \times \rho_*\right)_{/\sim}([p,m]) := \left[\left(\mathrm{Id}_P \times \rho_*\right)(p,m)\right]$.

Lastly, the step (*) needs some explanation. To see why $\rho_*\left(g^{-1}mg\right) = \rho(g)^{-1}\rho_*(m)\rho(g)$, recall $\rho_*\left(g^{-1}mg\right)$ is just the derivative of the path $t \mapsto \rho\left(\exp\left(tg^{-1}mg\right)\right)$ at the identity. But observe

$$\begin{aligned} \rho\left(\exp\left(tg^{-1}mg\right)\right) &= \rho\left(g^{-1}\exp(tm)g\right) \\ &= \rho\left(g^{-1}\right)\rho(\exp(tm))\rho(g). \end{aligned}$$

Now taking the derivative at $t = 0$ of $t \mapsto \rho\left(g^{-1}\right)\rho(\exp(tm))\rho(g)$ gives $\rho(g)^{-1}\rho_*(m)\rho(g)$, as required.

With the above notation, the claim $\rho_* \circ F_A = F_\nabla$ in Lemma 1 simply means that $\left(\mathrm{Id}_P \times \rho_*\right)_{/\sim} \circ F_A = F_\nabla$.

Proof of Lemma 1. The two curvatures in local trivializations are given by

$$\begin{aligned}
F_{A,\alpha} &: \Lambda^2 TU_\alpha \to Lie(G) \\
F_{A,\alpha} &= da_\alpha + [a_\alpha, a_\alpha]/2 \\
F_{\nabla,\alpha} &: \Lambda^2 TU_\alpha \to M_n(\mathbb{R}) \\
F_{\nabla,\alpha} &= d(\rho_*(a_\alpha)) + [\rho_*(a_\alpha), \rho_*(a_\alpha)]/2 \\
&= d(\rho_*)d(a_\alpha) + \rho_*([a_\alpha, a_\alpha])/2 \\
&= \rho_*(da_\alpha) + \rho_*([a_\alpha, a_\alpha])/2 \\
&= \rho_* \circ F_{A,\alpha}
\end{aligned}$$

In the above we have used the fact $\rho_* : Lie(G) \to M_n(\mathbb{R})$ is a Lie algebra homomorphism, i.e., it is linear and preserves the Lie bracket. □

Flat connections

A connection A on a principal bundle P is said to be flat if any one of the following holds:

1. The curvature satisfies $F_A \equiv 0$.
2. The horizontal sub-bundle $H_A \subseteq TP$ is involutive, i.e., given two vector fields $X, Y \in \mathcal{T}^1(P)$ on P, then $[X, Y] \in H_A$.
3. Trivializations can be chosen such that $a_\alpha \equiv 0$ for all α.

Note: To show 1 and 2 are equivalent, see the discussion in section 12.8, page 150 in [Taubes]. To show 1 and 3 are equivalent, see Theorem 2.2.1, page 48 in [Donaldson-Kronheimer].

Lemma 2 (the gauge group preserves flatness)**.** Let A be a flat connection on a principal G-bundle P. If $\eta : P \to P$ is an automorphism of P, then the pullback connection $\eta^* A$ is also flat.

Note. To show $\eta^* A$ is flat, we need to calculate $\eta^* A$ or $F_{\eta^* A}$ in local coordinates. For this, we need to calculate $\eta : P \to P$ in local trivializations.

Lemma 3. In local trivializations, a gauge transformation/ automorphism $\eta : P \to P$ looks similar to a transition map. More precisely, let ψ_α be trivializations of P. Further, given a connection A, let a_α and b_α be:

$$\begin{aligned}
\left(\psi_\alpha^{-1}\right)^* A &= g^{-1}dg + g^{-1}a_\alpha g \\
\left(\psi_\alpha^{-1}\right)^* \left(\eta^* A\right) &= g^{-1}dg + g^{-1}b_\alpha g.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\psi_\alpha \circ \eta \circ \psi_\alpha^{-1} &: U_\alpha \times G \to U_\alpha \times G \\
&: (x, g) \mapsto (x, \eta_\alpha(x) g)
\end{aligned}$$

and

$$b_\alpha = \eta_\alpha^{-1} a_\alpha \eta_\alpha + \eta_\alpha^{-1} d\eta_\alpha.$$

Proof. Using G-equivariance of ψ_α and η, observe

$$\begin{aligned} \psi_\alpha \circ \eta \circ \psi_\alpha^{-1} &: U_\alpha \times G \to U_\alpha \times G \\ &: (x,g) \mapsto (x, f(x,g)) \\ &: (x,g) \mapsto (x, f(x,e)\cdot g). \\ &: (x,g) \mapsto (x, \eta_\alpha(x)\cdot g). \end{aligned}$$

Now, set $\eta_\alpha(x) := f(x,e)$. The map $\eta_\alpha : U_\alpha \to G$ captures the automorphism η in local trivializations. Next, $b_\alpha = \eta_\alpha^{-1} a_\alpha \eta_\alpha + \eta_\alpha^{-1} d\eta_\alpha$ is obtained by pretending that η_α is a transition map. □

Proof of Lemma 2. Lemma 3 states that $\eta : P \to P$ locally behaves like a transition map. Thus, we have

$$F_{\eta^* A,\alpha} = \eta_\alpha^{-1} F_{A,\alpha} \eta_\alpha.$$

In particular, if $F_{A,\alpha} \equiv 0$, then $F_{\eta^* A,\alpha} \equiv 0$. □

Note. Let P be be a principal G-bundle, $\mathcal{G}$ be its Gauge group, and $\mathcal{A}$ be the set of connections on P. Fix a connection A_0 and consider the sequence of maps

(curvature-like tensors)

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathcal{A} & \longrightarrow & \Omega^2_M(P \times_{ad} Lie(G)) \\ \eta & \longmapsto & \eta^* A_0 & \longmapsto & F_{\eta^* A_0} \end{array}$$

The above discussion gives some understanding of this sequence. We can push further and calculate the derivative of this sequence of maps (see Appendix A). The derivative of this map is important while writing down the Euler-Lagrange of Yang-Mills functional and discussing Coulomb gauge.

Two questions:

1. Does every principal G-bundle admit a flat connection?
2. What is the set of flat connections?

The answer to the first question is no (if Chern class of a bundle is non-zero, then there is no flat connection). For the second question we have the following:

Theorem 1 (classification theorem for flat connections). Let $\mathcal{F}_{M,G}$ be the set of equivalence classes of pairs (P, A), where A is a flat connection on P. Then, the set $\mathcal{F}_{M,G}$ is in bijection with the set $\mathrm{Hom}(\pi_1(M); G)/G$.

Note: the set $\mathrm{Hom}(\pi_1(M); G)/G$ is the quotient of the set $\mathrm{Hom}(\pi_1(M); G)$ with the equivalence relation: $\rho : \pi_1(M) \to G$ is equivalent to $\sigma : \pi_1(M) \to G$ if there exists an $h \in H$ such that $\rho(x) = h\, \sigma(x)\, h^{-1}$ for all $x \in \pi_1(M)$.

Holonomy:

Theorem 1 is proved by understanding the holonomy of a connection A around different loop γ in the manifold M. Now we elaborate more about holonomy around a loop. Let $\gamma : S^1 \to M$ be a smooth loop. Fix a point $p \in P_{\gamma(0)}$ and parallel transport p around the loop γ. At the end of parallel transport we obtain an element $q \in P_{\gamma(0)}$. Write $q = p \cdot h_{A,\gamma}(p)$ using the right action of G on P. This defines a map

$$h_{A,\gamma} : P_{\gamma(0)} \to G$$

that satisfies $h_{A,\gamma}(p \cdot g) = g^{-1} h_{A,\gamma}(p) g$. This map is called the *holonomy* of the connection A around the loop γ. A crucial fact is that the holonomy of a flat connection takes the same value for homotopic loops.

Proposition 1. If A is a flat connection and γ_1 and γ_2 are homotopic loops based at x_0, then $h_{A,\gamma_1} = h_{A,\gamma_2}$. □

Proof of Theorem 1.

Part 1: constructing the map

Fix a base point $x_0 \in M$. Also fix a base point $p_0 \in P_{x_0}$. Now, define

$$\begin{aligned}\Psi : \mathcal{F}_{M,G} &\to \mathrm{Hom}(\pi_1(\mathrm{M}, \mathrm{x}_0); \mathrm{G})/\mathrm{G} \\ &: (P, A) \mapsto [[\gamma] \mapsto h_{A,\gamma}(p_0))]\end{aligned}$$

Three remarks are in order:

(a) Given $[\gamma] \in \pi_1(M, x_0)$, we choose a smooth representative γ for $[\gamma]$. Note that $[\gamma] \mapsto h_{A,\gamma}(p_0)$ does not depend on the smooth representative: if $[\gamma_1] = [\gamma_2]$, then $h_{A,\gamma_1}(p_0) = h_{A,\gamma_2}(p_0)$ by Proposition 1.

(b) The image of (P, A) in $\mathrm{Hom}(\pi_1(M, x_0); G)/G$ does not depend on p_0 because if $p_1 = p_0 \cdot g$, then $h_{A,\gamma}(p_1) = g^{-1} h_{A,\gamma}(p) g$ for all $[\gamma] \in \pi_1(M, x_0)$.

(c) If (P_1, A_1) and (P_2, A_2) by an isomorphism $\eta : (P_1, A_1) \to (P_2, A_2)$, then by the naturality of parallel transport, we have

$$h_{A_2,\gamma}(p) = h_{\eta^* A_2,\gamma}\left(\eta^{-1}(p)\right) = h_{A_1,\gamma}\left(\eta^{-1}(p)\right).$$

Hence, (P_1, A_1) and (P_2, A_2) have the same image under Ψ.

Part 2: injectivity of the map (section 13.9.2, page 163 of [Taubes])

Suppose (P, A) and $(\hat{P}, \hat{A})$ have the same image under Ψ, i.e., they have the same holonomy around loops. We use parallel transport to define an isomorphism from (P, A) to $(\hat{P}, \hat{A})$. Let $x_0 \in M$ be the basepoint. Start by selecting a G-equivariant isomorphism $\phi_0 : P_{x_0} \to \hat{P}_{x_0}$. Next, we want to extend this to a fiber over y, i.e., we want $P_y \to \hat{P}_y$. To do this, let $\nu:[0,1]\to M$ be a smooth path from x_0 to y. Parallel transport defines a G-equivaraint maps $h_{A,\nu}: P_{x_0} \to P_y$ and $h_{\hat{A},\nu}: \hat{P}_{x_0} \to \hat{P}_y$. Now we define $\phi_y : P_y \to \hat{P}_y$ a follows:

$$
\begin{tikzcd}
p_y \arrow[r, mapsto, dashed, "\phi_y"'] \arrow[d, mapsto, "h^{-1}_{A,\nu}"'] & \hat{p}_y \\
p_{x_0} \arrow[r, mapsto, "\phi_0"] & \hat{p}_{x_0} \arrow[u, mapsto, "h_{\hat{A},\nu}"']
\end{tikzcd}
$$

We claim the map above doesn't depend on the choice of the loop ν. To show this consider another loop that can be written as $\nu\cdot\gamma$, where γ is a loop based at x_0. Now if we perform parallel transport around $\nu\cdot\gamma$, the diagram changes, but the end result doesn't change:

$$
\begin{tikzcd}
p_y \arrow[r, mapsto, dashed, "\phi_y"'] \arrow[d, mapsto, "h^{-1}_{A,\nu}"'] & \hat{p}_y \\
p_{x_0} \arrow[d, mapsto, "h^{-1}_{A,\gamma}"'] & \hat{p}_{x_0} \arrow[u, mapsto, "h_{\hat{A},\nu}"'] \\
p_{x_0}\cdot h^{-1}_{A,\gamma}(p_{x_0}) \arrow[r, mapsto, "\phi_0"] & \hat{p}_{x_0}\cdot h^{-1}_{A,\gamma}(p_{x_0}) \arrow[u, mapsto, "h_{\hat{A},\gamma}"'] \arrow[d, equal] \\
 & \hat{p}_{x_0}\cdot h^{-1}_{\hat{A},\gamma}(\hat{p}_{x_0})
\end{tikzcd}
$$

Thus, the map $\phi_0 : P_{x_0} \to \hat{P}_{x_0}$ extends to a G-equivariant map $\phi: P \to \hat{P}$ by parallel transport. Lastly, it remains to show that ϕ is smooth and $\phi^*\hat{A} = A$. To see this, we simply note that in local trivializations where a_α and $\hat{a}_\alpha$ are identically zero, the map ϕ is locally constant.

<u>Part 3: surjectivity of the map</u>
See section 13.9.3 and 13.9.4, page 163 and 164 of [Taubes].

Introduction to characteristic classes

Given a principal bundle P over M, a characteristic class is an invariant of the principal bundle P that measures the extent to which the bundle is "twisted" and whether it possesses global sections. As an object, a characteristic class is an element of the cohomology of M.

Basic example: first Chern class of principal $U(1)$-bundles over M.
<u>Preliminary/specialty of $U(1)$</u>:

- $Lie(U(1)) = i\mathbb{R}$ (1×1 skew Hermitian matrices).
- $Lie(U(1))$ is abelian, i.e. $[m_1, m_2] = 0$ for all $m_i \in Lie(U(1))$, or equivalently, $gmg^{-1} = m$ for all $m \in Lie(U(1))$ and $g \in U(1)$.
- $ad(P) = P \times_{ad} Lie(U(1)) = M \times Lie(U(1)) = M \times i\mathbb{R}$. This is because the transition maps $(ad \circ \tau_{\beta\alpha})(m) = \tau_{\beta\alpha} m \tau_{\beta\alpha}^{-1} = m$ are identity. Furthermore, this says that a section s of $ad(P)$ is invariant under change of trivializations, i.e., $s_\alpha = s_\beta$.
- $\Omega^2_M(ad(P)) = \Omega^2_M(M \times i\mathbb{R}) = i \cdot \Omega^2_M$.

Now, suppose P is a principal $U(1)$-bundle over M. Let A be a connection on P and $F_A \in \Omega^2_M(ad(P))$ be its curvature tensor. By the above, F_A is just an imaginary 1-form. Write $F_A = i\phi$. Next note three points:

1. ϕ is a closed 2-form. As a consequence, $[\phi]$ defines an element of the deRham cohomology $H^2_{dR}(M; \mathbb{R})$. Explanation: Given a local trivialization $F_A = F_{A,\alpha} = da_\alpha + [a_\alpha \wedge a_\alpha]/2 = da_\alpha$. Thus, ϕ is locally exact and hence closed.
2. The deRham class $[\phi]$ depends only on the principal bundle P and not the connection. Suppose $A' = A + b$ is another connection, where $b \in \Omega^1_M(ad(P)) = i \cdot \Omega^1_M$. Then, $F_{A'} = F_A + db + [b \wedge b]/2 = F_A + db$. This means that a changing a connection changes the curvature by an exact 2-form. Hence the cohomology class $[\phi]$ remains unchanged.
3. Furthermore, $[\phi]$ only depends on the isomorphism class of P, i.e., if there is an isomorphism $\eta : P_1 \to P_2$, then the associated cohomology class are the same, i.e. $[\phi_1] = [\phi_2]$. To see this, let A_2 be a connection on P_2. This gives a connection $A_1 = \eta^* A_2$ on P_1. Recall, that the local picture of an isomorphism is given by $\eta_\alpha : U_\alpha \to G$ and $F_{A_1,\alpha} = F_{\eta^* A_2,\alpha} = \eta_\alpha^{-1} F_{A_2,\alpha} \eta_\alpha$. Lastly, as $Lie(U(1))$ is abelian, $F_{A_1,\alpha} = F_{A_2,\alpha}$. Thus $\phi_1 = F_{A_1}/i$ and $\phi_2 = F_{A_2}/i$ give the same 2-form on M.

The deRham class $[\phi]$ is called the first Chern class of P (denoted as $c_1(P)$).

Note:

1. If P admits a flat connection, then the first Chern class is zero, i.e., $c_1(P) = [0]$. As a consequence, if $c_1(P) \neq [0]$, then P has no flat connections!
2. Given P_1, P_2 such that $c_1(P_1) \neq c_1(P_2)$, then P_1 and P_2 are not isomorphic.

Note: Recall the family of $U(1)$-bundles over S^2 constructed in Example 2 of Lecture 1: obtain the open cover $U_1 = S^2 \setminus \{\mathtt{NP}\}$ and $U_2 = S^2 \setminus \{\mathtt{SP}\}$ of S^2. For each $m \in \mathbb{Z}$, a principal $U(1)$-bundle $P^{(m)} \to S^2$ is defined by transition map $\tau_{12}^{(m)}$ as follows.

$$\begin{tikzcd}
U_1 \cap U_2 = \mathbb{R}^2 \setminus \{0\} \arrow[rr, dashed, "\tau_{12}^{(m)}"] \arrow[dr, "\frac{(x_1, x_2)}{||x||}"'] & & S^1 \\
& S^1 \arrow[ur, "f_m(z) = z^m"'] &
\end{tikzcd}$$

It turns out that $c_2\left(P^{(m)}\right) = (-m/4\pi) \cdot \omega_{\text{vol}}$ (see Example 14.9, page 184 of [Taubes]). Here, $\omega_{\text{vol}} \in H^2_{dR}\left(S^2; \mathbb{R}\right)$ is the volume form of S^2 with standard metric. In particular, as $H^2_{dR}\left(S^2; \mathbb{R}\right)$ is one dimensional vector space generated by ω_{vol}, we get that each $P^{(m)}$ is distinct and only $P^{(0)}$ admits a flat connection.

General Chern classes of a principal G-bundle

Let P be a principal G-bundle and $\rho : G \to U_n$ be a unitary representation. For each $k \in \{1, 2, \dots\}$ we have the **k-th Chern class of** P, denoted by $c_k(P)$. This is an element of $H^{2k}_{dR}(M; \mathbb{R})$ that can be obtained as follows: fix any connection A on P and write

$$\begin{aligned}
c_1(P) &= i\frac{\text{tr}(F_A)}{2\pi} \\
c_2(P) &= \frac{\text{tr}(F_A^2) - \text{tr}(F_A)^2}{8\pi^2} \\
c_3(P) &= i\frac{-2\text{tr}(F_A^3) + 3\text{tr}(F_A^2)\text{tr}(F_A) - \text{tr}(F_A)^3}{48\pi^3} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

We have the following properties:

1. $c_k(P)$ is a closed real-valued $2k$-form on M. Hence, it defines an element of $H^{2k}_{dR}(M; \mathbb{R})$.
2. $c_k(P)$ is an invariant of the principal bundle, i.e., it doesn't depend on the choice of connection A or the isomorphism class of P. We remark that $c_k(P)$ does depend on the representation ρ.

Further reading

1. Chapter 12-14, Taubes, C. H. (2011). *Differential geometry: bundles, connections, metrics and curvature* (Vol. 23). OUP Oxford.
2. Section 2.1.2. Donaldson, S. K., & Kronheimer, P. B. (1990). *The geometry of four-manifolds*. Oxford university press.

Appendix A: Action of gauge group on space of connections

Note: there might be multiple mistakes/ vague points in the discussion of infinite dimensional manifolds and their derivatives; please proceed with caution.

Right action of $\mathcal{G}$ on $\mathcal{A}$.

Fix a principal bundle P and recall that the gauge group $\mathcal{G}$ is the group of all automorphisms of P. We define a right action of $\mathcal{G}$ on $\mathcal{A}$ (the set of connections on P) as follows:

$$\begin{aligned} \mathcal{A} \times \mathcal{G} &\to \mathcal{A} \\ (A, \eta) &\mapsto \eta^* A \end{aligned}$$

where $\eta^* A$ is the pullback of the 1-form $A : TP \to Lie(G)$ using η. Explicitly we have

$$\begin{aligned} \eta^* A : TP &\to Lie(G) \\ \xi &\mapsto A(\eta_*(\xi)). \end{aligned}$$

For a local viewpoint, we first, look at the local picture of an automorphism η. In the local trivializations $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$, η looks like $\psi_\alpha \circ \eta \circ \psi_\alpha^{-1}$:

$$\begin{aligned} \psi_\alpha \circ \eta \circ \psi_\alpha^{-1} &: U_\alpha \times G \to U_\alpha \times G \\ &: (x, g) \mapsto (x, f(x, g)) \\ &: (x, g) \mapsto (x, f(x, e) \cdot g), \end{aligned}$$

the last one follows by G-equivariance of η and ψ_α. Now setting $\eta_\alpha = f(x, e)$, we have maps $\eta_\alpha : U_\alpha \to G$ which satisfy:

$$\begin{aligned} \psi_\alpha \circ \eta \circ \psi_\alpha^{-1} &: U_\alpha \times G \to U_\alpha \times G \\ &: (x, g) \mapsto (x, \eta_\alpha(x) g). \end{aligned}$$

Moreover, the maps η_α transform as follows:

$$\eta_\beta = \tau_{\beta\alpha} \eta_\alpha \tau_{\beta\alpha}^{-1}.$$

Now given a connection $A = \{a_\alpha\}$, we want to describe $\eta^* A = \{b_\alpha\}$ locally in terms η_α. This is given by:

$$(b_\alpha)_x = \eta_\alpha(x)^{-1} (a_\alpha)_x \eta_\alpha(x) + \eta_\alpha(x)^{-1} d\eta_\alpha(x)$$

or more concisely, it takes the familiar form:

$$b_\alpha = \eta_\alpha^{-1} a_\alpha \eta_\alpha + \eta_\alpha^{-1} d\eta_\alpha. \tag{1}$$

Thus, the local effect of the action of an automorphism (gauge change) is that of a change in trivialization.

Derivative of the action of the gauge group on $\mathcal{A}$ at A_o.

Fix a P and a connection A_0 on P. The action of gauge group on $\mathcal{A}$ gives the following map:

$$\begin{aligned} W_{A_0} &: \mathcal{G} \to \mathcal{A} \\ &: \eta \mapsto \eta^* A_0. \end{aligned}$$

Now note that $\mathcal{G}$ is an infinite dimensional lie group and $\mathcal{A}$ is an infinite dimensional manifold because it is an affine space modeled on $\Omega^1_M(P \times_{ad} Lie(G))$. With this, we can ask about the derivative of W_{A_0} at the identity, i.e.

$$D_e W_{A_0} : T_e \mathcal{G} \to T_{A_0}\mathcal{A}.$$

For our purposes:

- $T_e\mathcal{G}$ is the set of all equivalence classes of smooth curves $\gamma : (-\epsilon, \epsilon) \to \mathcal{G}$ with $\gamma(0) = e$.
- $T_{A_0}\mathcal{A}$ is the set of all equivalence classes of smooth curves $\gamma : (-\epsilon, \epsilon) \to \mathcal{A}$ with $\gamma(0) = A_0$.

Here, the equivalence relation $\alpha \simeq \beta$ if $(F \circ \alpha)'(0) = (F \circ \beta)'(0)$ for all smooth maps $F : X \to \mathbb{R}$.

Proposition 2. *Let $\Omega^k_X(P \times_{ad} Lie(G))$ denote the space of sections of $\Lambda^k T^*M \otimes (P \times_{ad} Lie(G))$ over M. Then,*

1. *$T_e\mathcal{G}$ is equal to $\Omega^0_X(P \times_{ad} Lie(G))$*
2. *$T_{A_0}\mathcal{A}$ is equal to $\Omega^1_X(P \times_{ad} Lie(G))$.*
3. *Further, the map*

$$\begin{aligned} D_e W_{A_0} &: T_e\mathcal{G} \to T_{A_0}\mathcal{A} \\ &: \Omega^0_X(P \times_{ad} Lie(G)) \to \Omega^1_X(P \times_{ad} Lie(G)) \end{aligned}$$

is given by:

$$\nabla_{ad_* \circ A_0} : \Omega^0_X(P \times_{ad} Lie(G)) \to \Omega^1_X(P \times_{ad} Lie(G)),$$

where $\nabla_{ad_* \circ A_0}$ is the covariant derivative defined by the connection A_0 on the associated vector bundle $P \times_{ad} Lie(G)$.

Proof.

Part 1: $T_e\mathcal{G} = \Omega^0_X(P \times_{ad} Lie(G))$.

Let $t \mapsto \eta_t$ be a smooth path in $\mathcal{G}$ with $\eta_0 = e$. In local trivializations, η_t is given by $\{\eta_{t,\alpha}\}$, where $\eta_{t,\alpha} : U_\alpha \to G$. Now we have a map $(-\epsilon, \epsilon) \times U_\alpha \to G$ given by $(t, x) \mapsto \eta_{t,\alpha}(x)$. Taking the partial derivative wrt t at $t = 0$, we get a map:

$$\eta'_{0,\alpha} : U_\alpha \to Lie(G).$$

We claim the collection $\{\eta'_{0,\alpha}\}$ form a section of $P \times_{ad} Lie(G)$. To see this, recall that $\eta_{t,\beta}$ and $\eta_{t,\alpha}$ are related by

$$\eta_{t,\beta} = \tau_{\beta\alpha}\eta_{t,\alpha}\tau^{-1}_{\beta\alpha}.$$

Hence, partial derivative wrt t shows that $\eta'_{0\alpha}$ transform as a section of $P \times_{ad} Lie(G)$:

$$\begin{aligned} \eta'_{0,\beta} &= \tau_{\beta\alpha}\eta'_{0,\alpha}\tau^{-1}_{\beta\alpha} \\ &= (ad \circ \tau_{\beta\alpha})\eta'_{0,\alpha}. \end{aligned}$$

Next, conversely given a section of $P \times_{ad} Lie(G)$ that is locally given by the data $\{\eta'_{0,\alpha}\}$, we can produce a smooth path η_t by defining:

$$\begin{aligned} \eta_{t,\alpha} &: U_\alpha \to G \\ &: x \mapsto \exp(t\eta'_{0,\alpha}). \end{aligned}$$

This shows $T_e\mathcal{G}$ is equal to $\Omega^0_X(P \times_{ad} Lie(G))$.

Part 1: $T_{A_0}\mathcal{A} = \Omega^1_X(P \times_{ad} Lie(G))$.

Now we provide a similar argument for $T_{A_0}\mathcal{A}$. Let $t \mapsto A_t$ be a smooth path in $\mathcal{A}$ with $A_0 = A_0$. In local trivializations, A_t is given by $\{a_{t,\alpha}\}$, where $a_{t,\alpha} : TU_\alpha \to Lie(G)$. Now we have a map $(-\epsilon, \epsilon) \times TU_\alpha \to Lie(G)$ given by $(t, v) \mapsto a_{t,\alpha}(v)$. Taking the partial derivative wrt t at $t = 0$, we get a map:

$$a'_{0,\alpha} : TU_\alpha \to Lie(G).$$

Here, we have identified $T(Lie(G))$ with $Lie(G)$ because $Lie(G)$ is a finite dimensional vector space. We claim the collection $\{a'_{0,\alpha}\}$ form a section of $T^*M \otimes (P \times_{ad} Lie(G))$. To see this, recall that $a_{t,\beta}$ and $a_{t,\alpha}$ are related by

$$a_{t,\beta} = \tau_{\beta\alpha} a_{t,\alpha} \tau^{-1}_{\beta\alpha} - d\tau_{\beta\alpha}\tau^{-1}_{\beta\alpha}.$$

Hence, partial derivative wrt t shows that $a'_{0\alpha}$ transform as a section of $T^*M \otimes (P \times_{ad} Lie(G))$:

$$\begin{aligned} a'_{0,\beta} &= \tau_{\beta\alpha} a'_{0,\alpha} \tau^{-1}_{\beta\alpha} \\ &= (ad \circ \tau_{\beta\alpha}) a'_{0,\alpha}. \end{aligned}$$

Next, conversely given a section of $T^*M \otimes (P \times_{ad} Lie(G))$ that is locally given by the data $\{a'_{0,\alpha}\}$, we can produce a smooth path $\{a_{t,\alpha}\}$ by defining:

$$\begin{aligned} a_{t,\alpha} &: TU_\alpha \to Lie(G) \\ &: v \mapsto a_{0,\alpha}(v) + t \cdot a'_{0,\alpha}(v). \end{aligned}$$

The collection $\{a_{t,\alpha}\}$ define a connection because they transform as follows:

$$\begin{aligned} a_{t,\beta} &= a_{0,\beta} + t \cdot a'_{0,\beta} \\ &= \tau_{\beta\alpha} a_{0,\alpha} \tau^{-1}_{\beta\alpha} - d\tau_{\beta\alpha}\tau^{-1}_{\beta\alpha} + t \cdot \tau_{\beta\alpha} a'_{0,\alpha} \tau^{-1}_{\beta\alpha} \\ &= \tau_{\beta\alpha}(a_{0,\alpha} + t \cdot a'_{0,\alpha}(v))\tau^{-1}_{\beta\alpha} - d\tau_{\beta\alpha}\tau^{-1}_{\beta\alpha}. \end{aligned}$$

This shows $T_{A_0}\mathcal{A}$ is equal to $\Omega^1_X(P \times_{ad} Lie(G))$.

Part 3: computing $D_eW : T_e\mathcal{G} \to T_{A_0}\mathcal{A}$.

Let $t \mapsto \eta_t$ be a smooth path in $\mathcal{G}$. Consider the smooth path in $\mathcal{A}$ given by $t \mapsto \eta_t^* A_0$; we denote this in local trivializations by $a_{t,\alpha}$. Now recall equation (1) where we describe the action of the gauge group on $\mathcal{A}$. By equation (1), we have:

$$a_{t,\alpha} = \eta^{-1}_{t,\alpha} a_{0,\alpha} \eta_{t,\alpha} + \eta^{-1}_{t,\alpha} d\eta_{t,\alpha}.$$

Taking the partial t derivative gives:

$$a'_{t,\alpha} = \frac{\partial \eta_{t,\alpha}^{-1}}{\partial t} a_{t,\alpha} \eta_{t,\alpha} + \eta_{t,\alpha}^{-1} a_{t,\alpha} \frac{\partial \eta_{t,\alpha}}{\partial t} + \frac{\partial \eta_{t,\alpha}^{-1}}{\partial t} d\eta_{t,\alpha} + \eta_{t,\alpha}^{-1} \frac{\partial d\eta_{t,\alpha}}{\partial t}$$

Now observe (i) $\partial \eta_{t,\alpha} / \partial t|_{t=0} = \eta'_{0,\alpha}$, (ii) $\partial \eta_{0,\alpha}^{-1} / \partial t|_{t=0} = -\eta'_{0,\alpha}$, (iii) $\eta_{0,\alpha} = e$, (iv) $d\eta_{0,\alpha} = de = 0$, and (v) $\partial d\eta_{t,\alpha} / \partial t|_{t=0} = d\eta'_{0,\alpha}$. Putting all this together, we get

$$\begin{aligned} a'_{0,\alpha} &= -\eta'_{0,\alpha} a_{0,\alpha} + a_{0,\alpha} \eta'_{0,\alpha} + d\eta'_{0,\alpha} \\ &= d\eta'_{0,\alpha} + [a_{0,\alpha}, \eta'_{0,\alpha}] \\ &= d\eta'_{0,\alpha} + (ad_* \circ a_{0,\alpha})(\eta'_{0,\alpha}). \end{aligned}$$

Thus, $D_e W = \nabla_{ad_* \circ A_0}$, the covariant derivative on the vector bundle $P \times_{ad} Lie(G)$. □

<u>Remark</u>: Donaldson-Kronheimer's book defines a <u>left</u> action of $\mathcal{G}$ on $\mathcal{A}$ given by

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A}$$
$$(\eta, A) \mapsto (\eta^{-1})^* A.$$

In this case, $D_e W : T_e \mathcal{G} \to T_{A_0} \mathcal{A}$ will be $-\nabla_{ad_* \circ A_0}$ instead of $\nabla_{ad_* \circ A_0}$ because of the inverse.