Bonnet-Myers theorem for polygonal surfaces

Mohith Raju Nagaraju

STGS, IISc, July 2023

Bonnet-Myers theorem (Riemannian geometry)

Theorem 1. Let M be a complete, connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/r^2$. Then

- M has diameter less than or equal to πr
- M is compact
- M has finite fundamental group.

Want to prove: A "*discrete*" version Bonnet-Myers

Discrete = piecewise linear manifolds

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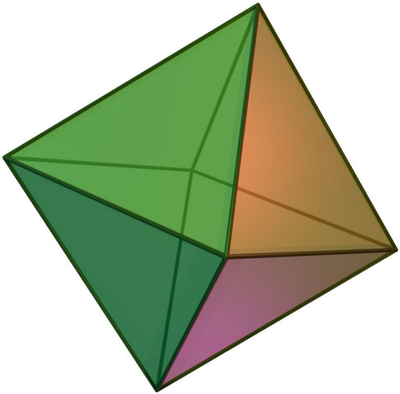
Want to prove: A "*discrete*" version Bonnet-Myers

Discrete = ~~piecewise linear manifolds~~
regular polygonal surfaces (dim 2)

Examples of regular polygonal surfaces

Example 1. The boundary of an octahedron - made up of eight **equilateral triangles glued** along the edges



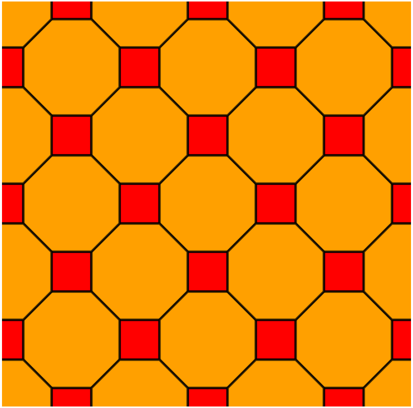
Examples of regular polygonal surfaces

Example 2. Truncated icosahedron - made up of **regular pentagons** and **regular hexagons glued** along the edges

Source: `https://polyhedra.tessera.li/`
by @tesseralis

Examples of regular polygonal surfaces

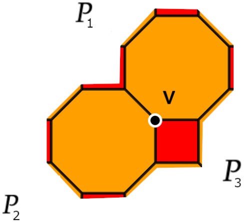
Example 3. An Archimedean tiling of $\mathbb{R}^2$ - made up of **regular octagons** and **squares glued** along the edges



What is a regular polygonal surfaces?

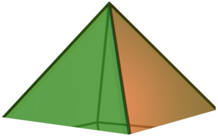
Definition 1. A regular polygonal surface E is a finite or countable collection of **Euclidean unit regular polygons** that are **glued** together such that

1. The identification map between edges is an isometry
2. E is connected
3. Each edge is identified with exactly one other edge
4. Given a vertex v, the polygons having v as a vertex can be arranged in a cyclic manner $P_1, P_2, \ldots, P_d$ such that $P_i \cap P_j$ contains an edge if and only if $i = j + 1 \mod d$

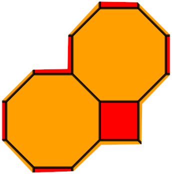


Curvature and geometry of regular polygonal surfaces

Curvature at vertices



- Angle-sum $< 2\pi$
- "Positive curvature" at vertex v



- Angle-sum $= 2\pi$
- Zero curvature at vertex v

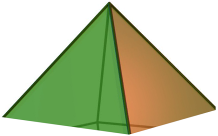
- Angle-sum $> 2\pi$
- "Negative curvature" at vertex v

- Let $P_1, P_2, \ldots, P_d$ be the cyclic arrangement at v
- Angle-sum at vertex v is

$$\mathcal{A}(v) = \sum_{i=1}^{d} \left(\pi - \frac{2\pi}{|P_i|} \right)$$

Source: "Octahedron" by Cyp and Fropuff. "uniform tiling of Euclidean tiling, faces colored by sides" by Tomruen. "A deltahedron with 1000 Faces" by TED-43.

Curvature at vertices



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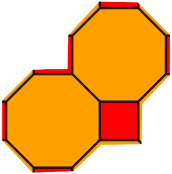

- Angle-sum $= 2\pi$
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Important.
$\kappa(v) =$ curvature at $v = 2\pi - \mathcal{A}(v)$

Source: "Octahedron" by Cyp and Fropuff. "uniform tiling of Euclidean tiling, faces colored by sides" by Tomruen. "A deltahedron with 1000 Faces" by TED-43.

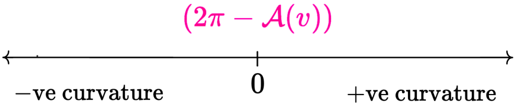
Angle-sum and curvature

Observation 1. Let v be a vertex of E. If ϵ is small, then

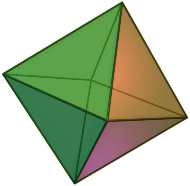
$$\begin{aligned} \operatorname{len}(\partial B(v,r)) &= \mathcal{A}(v) \cdot r \\ &= (2\pi - (2\pi - \mathcal{A}(v))) \cdot r \\ &= (2\pi - \kappa(v)) \cdot r \end{aligned}$$

Observation 2. If Δ is any geodesic triangle that contains a single vertex v in its interior, then

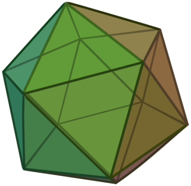
$$\begin{aligned} \text{Sum of interior angles of } \Delta &= \pi + (2\pi - \mathcal{A}(v)) \\ &= \pi + \kappa(v) \end{aligned}$$



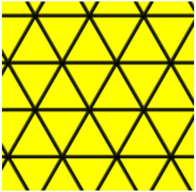
Examples of angle-sum



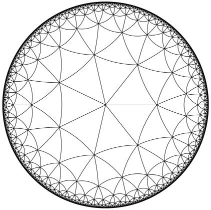
4 triangles at each vertex
$\mathcal{A}(v) = \frac{4}{3}\pi$



5 triangles at each vertex
$\mathcal{A}(v) = \frac{5}{3}\pi$



6 triangles at each vertex
$\mathcal{A}(v) = \frac{6}{3}\pi = 2\pi$



7 triangles at each vertex
$\mathcal{A}(v) = \frac{7}{3}\pi$

Statement of Main Theorem

Discrete version of Bonnet-Myers

Main Theorem. Let E be a **f.l.c.** regular polygonal surface where the angle-sum at each vertex is strictly lesser than 2π. Then E is compact.

f.l.c. $=$ There exists a big constant N such that

(a) Each polygon P in E has at most N sides

(b) The degree at each vertex in E is at most N.

Discrete version of Bonnet-Myers

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(a) Each polygon P in E has at most N sides
(b) The degree at each vertex in E is at most N.

Proof idea.

- Observe that E is a smooth manifold away from the vertex set V, that is, $E \setminus V$ is a flat Riemannian manifold
- "**Distribute/spread**" the $+$ve curvature at the vertex into the interior, that is, reduce curvature at V and add smooth $+$ve curvature in $E \setminus V$
- By **Riemannian geometry** on $E_r \setminus V$, get diameter of $E_r \setminus V$ is $< \pi r$; hence, E_r and E are compact.

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 215, 1976

GEODESICS IN PIECEWISE LINEAR MANIFOLDS

BY

DAVID A. STONE(1)

ABSTRACT. A simplicial complex M is *metrized* by assigning to each simplex $a \in \mathrm{M}$ a linear simplex a^* in some Euclidean space R^k so that face relations correspond to isometries. An equivalence class of metrized complexes under the relation generated by subdivisions and isometries is called a *metric complex*; it consists primarily of a polyhedron M with an intrinsic metric ρ_{M}. This paper studies geodesics in metric complexes. Let $P \in \mathrm{M}$; then the tangent

⋮

.... For 2-dimensional manifolds–topological and p.l.–the theory of curvature is well established (see Aleksandroff and Zalgaller [1] or W. Rinow [10]), and Theorems 1, 2 and 3, though perhaps new, are simply exercises. The present

Illinois J. Math.
20 (1976), no. 1, 12–21.

A COMBINATORIAL ANALOGUE OF A THEOREM OF MYERS

BY

DAVID A. STONE

PROPOSITION 1. *Let K be a connected cell complex which is a 2-manifold without boundary. Assume there is a number $R > 0$ such that $R^*(v) \geq R$ for every vertex v of K. Then K has diameter $\leq 1 + 2/R$.*

$$\begin{aligned} L &= |\alpha| + \sum_{j=1}^{s} (|\partial_{\beta_1} c_j| - |\partial_\alpha c_j|)/|\partial c_j| + (1/|\partial c_1| + 1/|\partial c_s|) - 2 \\ &\boxed{>} |\alpha| + (|\beta_1| - |\alpha|)/3 - 2, \end{aligned}$$

since $|\partial c| \geq 3$ for any 2-cell. Substituting into (3) gives

Illinois J. Math.
20 (1976), no. 3, 551–554.

CORRECTION TO MY PAPER "A COMBINATORIAL ANALOGUE OF A THEOREM OF MYERS"

BY

DAVID A. STONE

This is a correction of my paper [3]. The proof of Proposition 1 is mistaken (the assertion "$L > |\alpha| + (|\beta_1| - |\alpha|)/3$" does not follow from "$|\partial c| \geq 3$ for any 2-cell").

⋮

$|K'|$ is complete, Theorem 3 of [2] implies that $|K'|$ is compact. Hence K' has only finitely many vertices; and it follows that K is finite.

State of the art

- In 2001, Yusuke Higuchi asked whether f.l.c. can be relaxed?
  Combinatorial curvature for planar graphs, *J. Graph Theory*
- In 2007, Matt Devos and Bojan Mohar showed that f.l.c. is not required
  An analogue of the Descartes-Euler formula for infinite graphs and Higuchi's conjecture, *Trans. Am. Math. Soc.*

State of the art

- In 2001, Yusuke Higuchi asked whether f.l.c. can be relaxed?
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- One can ask for a diameter bound like in the classical Bonnet-Myers
- In 2022, Luca Ghidelli proved that either E is a prism/anti-prism, or E has at most 208 vertices!
- In particular, if $\mathcal{A}(v) \leq 2\pi - \epsilon_0$, i.e. $\kappa(v) \geq \epsilon_0$ for all v, then

$$\mathrm{diam}(E) \leq \frac{c}{\epsilon_0}$$

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On the largest planar graphs with everywhere positive combinatorial curvature

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A R T I C L E I N F O

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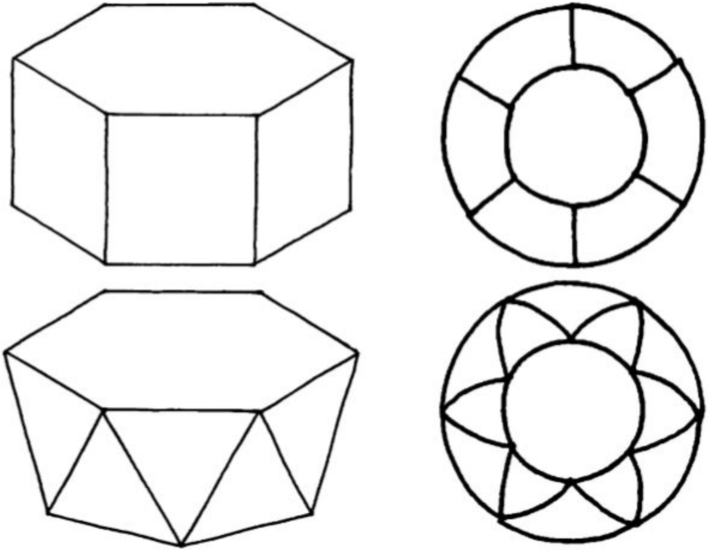
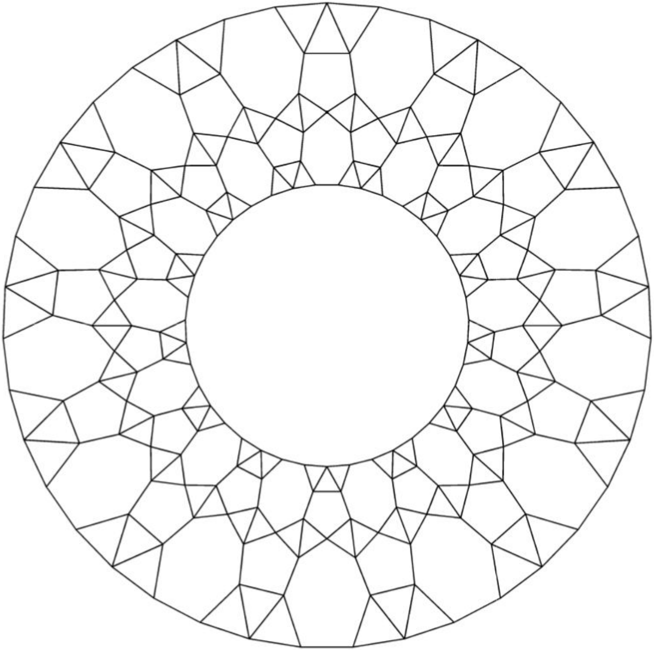
Keywords:
Planar graph
Combinatorial curvature
Positive curvature
Discharging
Linear optimization
Local-global

A B S T R A C T

A planar PCC graph is a simple connected planar graph with everywhere positive combinatorial curvature which is not a prism or an antiprism and with all vertices of degree at least 3. We prove that every planar PCC graph has at most 208 vertices, thus answering completely a question raised by DeVos and Mohar. The proof is based on a refined discharging technique and on an accurate low-scale combinatorical description of such graphs. We also prove that all faces in a planar PCC graph have at most 41 sides, and this result is sharp as well.





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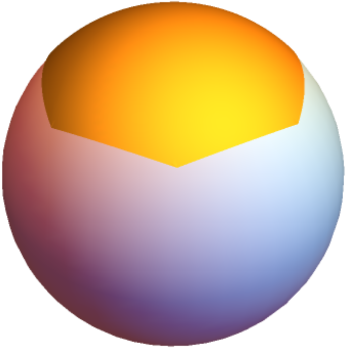
$$\mathrm{diam}(E) \leq \frac{c}{\epsilon_0}$$

- In 1995, Burago and Zalgaller proved a Nash isometric embedding theorem for polygonal surfaces

Regular r-spherical surfaces
(piecewise spherical surfaces)

Regular r-spherical polygons

- Let $r > 0$ and $n \in \mathbb{N}$ such that $2\pi r/n > 1$
- Roughly, a unit regular r-spherical n-gon is a spherical polygon in S_r^2 with n sides where each edge has unit length



Source: WOLFRAM Demonstrations Project

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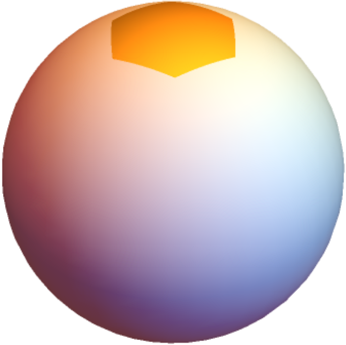

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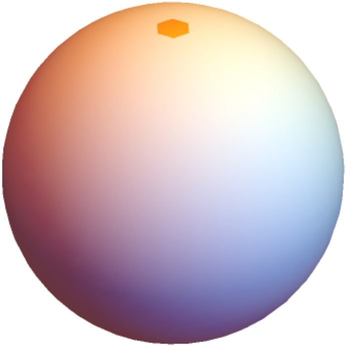


Source: WOLFRAM Demonstrations Project

Regular r-spherical surfaces

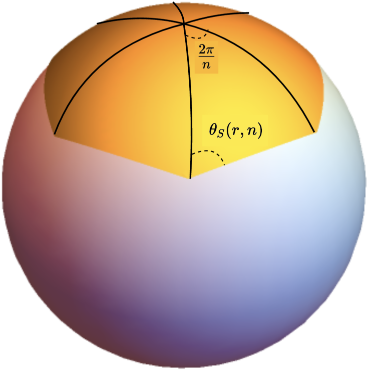
- Let $r > 0$ and $n \in \mathbb{N}$ such that $2\pi r/n > 1$
- Roughly, a unit regular r-spherical n-gon is a spherical polygon in S_r^2 with n sides where each edge has unit length
- Roughly, a regular r-spherical surface $E_r \approx$ regular polygonal surface E, but instead of Euclidean regular polygons we have spherical regular polygons.

Remarks.

1. $E \xrightarrow[\text{curvature}]{\text{Distribute}} E_r$
2. E_r might also have "curvature singularity" at the vertices
3. But $E_r \setminus V$ is (smooth) positively curved with curvature $+1/r^2$

Regular r-spherical surfaces

Definition 2. Suppose $r \in (0, \infty)$ and $n \geq 3$ such that $2\pi r/n > 1$. A unit regular r-spherical n-gon is defined to be the union of n isosceles spherical triangles having an angle $A = 2\pi/n$ and an opposite side of length $a = 1$.



Lemma. Spherical trig gives

$$\sin^2(\theta_S(r, n)) = \frac{1 + \cos(2\pi/n)}{1 + \cos(1/r)}$$

Note. $2\theta_S(\infty, n) = \pi - \frac{2\pi}{n}$
(Euclidean angles)

Regular r-spherical surfaces

Definition 3. A regular r-spherical surface E_r is a finite or countable collection of **unit regular r-spherical n-gons** that are **glued** together such that

1. The identification map between edges is a **spherical** isometry
2. E_r is connected
3. Each edge is identified with exactly one other edge
4. Given a vertex v, the **spherical** polygons having v as a vertex can be arranged in a cyclic manner $P_1, P_2, \ldots, P_d$ such that $P_i \cap P_j$ contains an edge if and only if $i = j + 1 \mod d$

Definition 4.

- Let $P_1, P_2, \ldots, P_d$ be the cyclic arrangement at v
- The r-spherical angle-sum at vertex v is

$$\mathcal{A}_r(v) = \sum_{i=1}^{d} 2\theta_S(r, |P_i|)$$

Bonnet-Myers for regular r-spherical surfaces

Friday's Theorem 3. Let $0 < r < \infty$. Suppose E_r is a regular r-spherical surface where $\mathcal{A}_r(v) < 2\pi$ at each vertex. Then

- The diameter of E_r is lesser than or equal to πr
- E_r is compact

Main Proposition (Friday) (Length-minimizing paths miss vertices.) Let $0 < r \leq \infty$. Suppose E_r is a regular r-spherical surface where $\mathcal{A}_r(v) < 2\pi$ at each vertex. Let γ be a length-minimizing path from p to q. Then, the image of γ in E_r does not intersect any vertex of E_r, except possibly the endpoints p and q.

Plan

Next time.

1. Prove the main proposition about length-minimizing paths miss vertices
2. Prove the Bonnet-Myers theorem for regular r-spherical surfaces

Today.

1. Deduce Bonnet-Myers for E using Bonnet-Myers for E_r

Proving Bonnet-Myers for E using Bonnet-Myers for E_r

The three Bonnet-Myers

Bonnet-Myers for **f.l.c.** regular polygonal surfaces. If $\mathcal{A}(v) < 2\pi$ at each vertex of E, then E is compact.

$$\Uparrow$$

Bonnet-Myers for regular r-spherical surfaces. If spherical angle-sum $\mathcal{A}_r(v) < 2\pi$ at each vertex of E_r, then $\mathrm{diam}(E_r) \leq \pi r$ and E_r is compact.

$$\Uparrow$$

Bonnet-Myers for Riemannian manifolds. If the sectional curvatures are uniformly bounded below by $+1/r^2$ and M is complete*, then $\mathrm{diam}(M) \leq \pi r$.

Proposition 2. Suppose E is a **f.l.c.** regular polygonal surface. Then for each $r > N/2\pi$, there is a r-spherical surface E_r homeomorphic to E. Furthermore, for each vertex v, we have

$$\lim_{r\to\infty} \mathcal{A}_r(v) = \mathcal{A}(v).$$

f.l.c. = There exists a big constant M such that

(a) Each polygon P in E has at most M sides

(b) The degree at each vertex in E is at most M.

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$$\lim_{r\to\infty} \mathcal{A}_r(v) = \mathcal{A}(v).$$

Proof.

- Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be the collection of polygons in E
- For each P_α, label the sides of P_α from 1 to $|P_\alpha|$
- Let $\{P_{r,\alpha}\}_{\alpha\in\Lambda}$ be a collection of r-spherical polygons such that $|P_{r,\alpha}| = |P_\alpha|$
- For each $P_{r,\alpha}$, label the sides of $P_{r,\alpha}$ from 1 to $|P_{r,\alpha}|$

$E \xrightarrow[\text{curvature}]{\text{Distribute}} E_r$

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- Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be the collection of polygons in E
- For each P_α, label the sides of P_α from 1 to $|P_\alpha|$
- Let $\{P_{r,\alpha}\}_{\alpha\in\Lambda}$ be a collection of r-spherical polygons such that $|P_{r,\alpha}| = |P_\alpha|$
- For each $P_{r,\alpha}$, label the sides of $P_{r,\alpha}$ from 1 to $|P_{r,\alpha}|$
- Now glue $\{P_{r,\alpha}\}_\alpha$ along the edges in the same pattern as in E
- If the i-th side of P_α is glued to the j-th side of P_β, then ...
- After such glueings, get $E_r = \bigcup_\alpha P_{r,\alpha}/\sim$ $\square$?

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$$\lim_{r\to\infty} \mathcal{A}_r(v) = \mathcal{A}(v).$$

Proof contd.

- Construct a homeomorphism from E to E_r by defining $f_\alpha \colon P_\alpha \xrightarrow{\sim} P_{r,\alpha}$
- Choose a homeo f_α such that f_α restricted to each edge is an isometry of the edges
- Claim. The maps f_α glue to give $f \colon \bigcup P_\alpha/\sim \xrightarrow{\sim} \bigcup_\alpha P_{r,\alpha}/\sim$ that is, $f \colon E \xrightarrow{\sim} E_r$

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- Construct a homeomorphism from E to E_r by defining $f_\alpha \colon P_\alpha \xrightarrow{\sim} P_{r,\alpha}$
- Choose a homeo f_α such that f_α restricted to each edge is an isometry of the edges
- Claim. The maps f_α glue to give $f \colon E \xrightarrow{\sim} E_r$
- Recall $\mathcal{A}_r(v) = \sum_{i=1}^{d} 2\theta_S(r, |P_{r,\alpha_i}|)$
- Now $\displaystyle\lim_{r\to\infty} \mathcal{A}_r(v) = \sum_{i=1}^{d} \lim_{r\to\infty} 2\theta_S(r, |P_{r,\alpha_i}|) = \sum_{i=1}^{d} \pi - \frac{2\pi}{|P_{r,\alpha_i}|}$

$\square$

Wrap up

Proposition 2 $\implies$ Bonnet-Myers for regular polygonal surfaces

- Let E be a **f.l.c.** regular polygonal surface where $\mathcal{A}(v) < 2\pi$ for all v
- For each $r \in (M/2\pi, \infty)$ there is a E_r homeomorphic to E
  For a fixed v_0, $\lim_{r\to\infty} \mathcal{A}_r(v_0) = \mathcal{A}(v_0) < 2\pi$
- Claim. $\exists\, r_0 \in (M/2\pi, \infty)$ such that $\mathcal{A}_{r_0}(v) < 2\pi$ for each v

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- For each $r \in (M/2\pi, \infty)$ there is a E_r homeomorphic to E
  For a fixed v_0, $\lim_{r\to\infty} \mathcal{A}_r(v_0) = \mathcal{A}(v_0) < 2\pi$
- Claim. $\exists\, r_0 \in (M/2\pi, \infty)$ such that $\mathcal{A}_{r_0}(v) < 2\pi$ for each v
- Note that $\mathcal{A}_r(v)$ only depends on the vertex-type of a vertex v
- The vertex-type of a vertex v is defined to be the tuple $[|P_1|, |P_2|, \ldots, |P_d|]$ where $P_1, \ldots, P_d$ is the cyclic arrangement of polygons around v
- Note. f.l.c $\implies$ there are only finitely many distinct vertex-types
- This is because
  $\{\text{vertex-types in E}\} \subseteq \{[n_1, n_2, \ldots, n_d] \colon n_i \le M \text{ and } d \le M\}$

Wrap up

Proposition 2 $\implies$ Bonnet-Myers for regular polygonal surfaces

- Let E be a **f.l.c.** regular polygonal surface where $\mathcal{A}(v) < 2\pi$ for all v
- For each $r \in (M/2\pi, \infty)$ there is a E_r homeomorphic to E
- Moreover, for a fixed v_0, $\lim_{r\to\infty} \mathcal{A}_r(v_0) = \mathcal{A}(v_0) < 2\pi$
- Claim. $\exists\, r_0 \in (M/2\pi, \infty)$ such that $\mathcal{A}_{r_0}(v) < 2\pi$ for each v
- For such a r_0, the surface E_{r_0} satisfies $\mathcal{A}_{r_0}(v) < 2\pi$ for each v
- $\implies E_{r_0}$ is compact
- $\implies E$ is compact $\blacksquare$

Bonnet-Myers Theorem
(Riemannian geometry)

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Theorem 1. Let M be a complete, connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/r^2$.
Then

- M has diameter less than or equal to πr
- M is compact
- M has finite fundamental group.

Bonnet-Myers theorem (Riemannian geometry)

Theorem 1'. Let M be a ~~complete,~~ connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/r^2$.
Further, suppose any two points $p, q \in M$ can be connected via a length minimizing geodesic.
Then

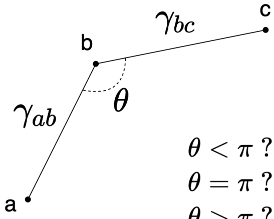
- M has diameter less than or equal to πr
- ~~M is compact~~
- ~~M has finite fundamental group.~~

Thank you!

(To be continued...)

Question to ponder

Question. Suppose M is a 2-dim Riemannian manifold and a, b, c are three points in M. Let γ_{ab} be the length-minimizing path from a to b and γ_{bc} from b to c. If $\gamma_{ab} * \gamma_{bc}$ is a length-minimizing path from a to c, then what can we say about the angle between γ_{ab} and γ_{bc} at the point b?



$\theta < \pi$?
$\theta = \pi$?
$\theta > \pi$?