Conformal tilings of Riemann surfaces

A glimpse into how different Riemann surfaces can and can't be conformally tiled

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Abstract. Roughly speaking, a conformal tiling of a Riemann surface is a special tessellation of the surface where two adjacent tiles are antiholomorphic reflections of each other. Because of this rigidity, it is unclear whether every Riemann surface can be conformally tiled; indeed, many compact Riemann surfaces cannot be conformally tiled (Belyi's theorem). On the other hand, Bishop and Rempe show that every non-compact Riemann surface can be conformally tiled using a countable collection of equilateral triangles. The situation changes when we use bigger regular n-gons ($n \geq 7$) instead of equilateral triangles. We discuss how certain Riemann surfaces cannot be conformally tiled using a collection of regular n-gons where $n \geq 7$ (eg. non-compact surfaces with finite area). We also study a particular relationship between the combinatorics of a conformal tiling and the geometry (conformal type) of the Riemann surface it tiles.

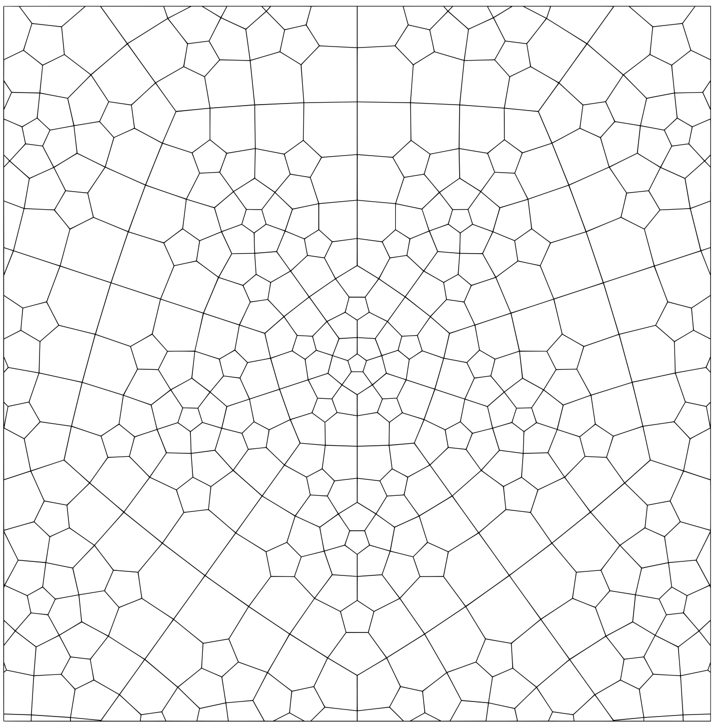

Figure 1: A conformal tiling of $\mathbb{C}$ with pentagons (Figure 1 in [BS97])

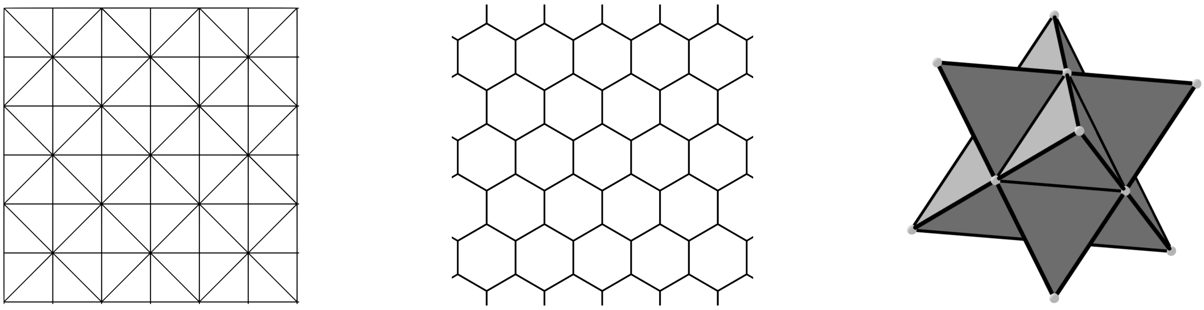

Figure 2-4: (2) A conformal tiling of $\mathbb{C}$ with triangles. (3) A conformal tiling of $\mathbb{C}$ with hexagons. (4) An equilateral surface homeomorphic to S^2 (Figure 1 in [BR21]).

1 Conformal tilings

A "tiling" of a surface is a partition of the surface into a locally finite collection of closed topological discs (tiles) which intersect only along their boundaries. In the Euclidean plane, an interesting class of tilings are those where adjacent tiles are reflections of each other along the shared edge (See Figures 2 and 3). Unlike in the Euclidean plane, the notion of lines and reflections does not make sense on a general topological surface, so we consider Riemann surfaces where we can define a notion of reflection.

A *Riemann surface* is an oriented topological surface with an additional complex structure (holomorphic charts). Given a "nice" curve on a Riemann surface, locally, there exists a unique antiholomorphic involution that fixes the curve pointwise (this is a Schwarz reflection on a Riemann surface). These are used to define a special class of tilings on a Riemann surface. (In the following definition, all tiles have three sides.)

Definition 1. A **conformal triangulation** of a Riemann surface X is a countable and locally finite collection of closed topological triangles $\mathcal{T} = \{T_\alpha\}$ that cover X, such that when two triangles T_α, T_β intersect, they either intersect at a single vertex or along a single common edge. Moreover, when $T_\alpha \cap T_\beta$ is a common edge e, there is an antiholomorphic homeomorphism $\iota : T_\alpha \to T_\beta$ that fixes e pointwise and maps the third vertex of T_α to the third vertex of T_β.

Figure 2 is an example of a conformal triangulation of $\mathbb{C}$ because a Euclidean reflection is an antiholomorphic homeomorphism. Next, to define conformal tilings having 4-sided tiles, we could simply replace 'topological triangles' with 'topological squares' in the above definition. But to make sense of a conformal tiling where both 4-sided tiles and 3-sided tiles occur, a different definition is needed. The idea is to think of a conformal tiling as a tesselation where each tile is a conformal image of a Euclidean regular polygon. Further, we require that for two adjacent tiles, the two conformal maps can be glued together to give a conformal map from a "glued polygon" to the union of adjacent tiles. Equilateral surfaces provide a neat way to think about this.

Definition 2. An **equilateral surface** E is a finite or countable collection of Euclidean unit regular polygons that are glued together by identifying each edge with exactly one edge of another polygon such that it has the following properties

- The identification map is affine-linear on the edge.
- E is connected. Further, given a vertex v, the polygons having v as a vertex can be arranged in a cyclic manner $P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_d}$ such that $(P_{\alpha_i} \cap P_{\alpha_j}) \setminus \{v\} \neq \emptyset$ if and only if $i = j + 1 \mod d$.

- Two polygons intersect along at most one edge (in particular, the degree of each vertex is at least 3).

An equilateral surface E is a topological surface. (Figure 4 shows an equilateral surface homeomorphic to the Riemann sphere). If E is orientable, then it also has a Riemann surface structure: the interior of each unit regular polygon P is a subset of the complex plane, giving charts $\varphi_\alpha : \mathring{P} \to \varphi_\alpha(\mathring{P})_{\subseteq\mathbb{C}}$. For the interior of an edge e, that is adjacent to P_α, P_β, the maps $\varphi_\alpha, \varphi_\beta$ can be glued to give a compatible chart $\varphi_{\alpha,\beta} : (P_\alpha \cup P_\beta)^\circ \to \varphi_{\alpha,\beta}((P_\alpha \cup P_\beta)^\circ)_{\subseteq\mathbb{C}}$. The maps $\varphi_\alpha, \varphi_\beta$ can be glued because the identification map between polygons is affine-linear, and E is orientable. Lastly, charts can be defined at vertices using "power maps".

Definition 3. A **conformal tiling** of a Riemann surface X is a pair $\mathcal{T} = (E, f)$, where E is an equilateral surface and $f : E \to X$ is a conformal/biholomorphic map. (If $\{P_\alpha\}$ is the set of polygons in E, then the collection of $T_\alpha := f(P_\alpha) \subseteq X$ form the tiles of the conformal tiling $\mathcal{T}$ of X.)

The honeycomb tiling in Figure 3 is a conformal tiling of $\mathbb{C}$. Its equilateral surface E is obtained by gluing Euclidian unit hexagons in the honeycomb pattern. The resulting equilateral surface is homeomorphic to the Euclidean plane via a map that takes each hexagonal tile of the plane to a unit regular hexagon in E. This map is also a biholomorphism from $\mathbb{C}$ to E, giving a conformal tiling of $\mathbb{C}$.

Figure 1 shows another conformal tiling of $\mathbb{C}$ where all the tiles are 5-sided. In [BS97], Bowers and Stephenson construct an equilateral surface E by abstractly gluing pentagons in increasing stages such that the $(n+1)^{\text{th}}$ aggregate is combinatorially equivalent to a subdivision of the n^{th} aggregate. The resulting equilateral surface is a non-compact simply connected Riemann surface. By the uniformization theorem, E is either conformal to $\mathbb{C}$ or $\mathbb{D}$. To show E is conformal to $\mathbb{C}$ and not $\mathbb{D}$, observe that the surface E has a special symmetry ("zoom-out self-similarity") due to which it admits an "expanding" automorphism $\alpha : E \to E$ satisfying $\alpha(p_0) = p_0$ and $\alpha^{-n}(x) \to p_0$ for all $x \in E$. On the other hand, an application of Schwarz lemma form complex analysis shows that the unit disc $\mathbb{D}$ does not admit such an "expanding" automorphism. This forces E to be conformal to $\mathbb{C}$ via a conformal map $f : E \to \mathbb{C}$. The pair $\mathcal{T} = (E, f)$ is the conformal tiling of $\mathbb{C}$ shown in Figure 1.[1]

The "combinatorics" of a conformal tiling $\mathcal{T}$ is captured by a special combinatorial graph G (called tiling graph) obtained by taking the 1-skeleton of the tiling. Conversely, one can construct a conformal tiling from a tiling graph.

Definition 4. A **tiling graph** $G = (V, E, F)$ is a locally finite simple graph that is embedded in a topological surface S such that for each component U of $S \setminus (V \cup E)$, there is a homeomorphism $\phi : (\overline{\mathbb{D}}, \partial\mathbb{D}) \to (\overline{U}, \partial U)$. We say $\overline{U}$ is an n-sided face of G if ∂U contains n vertices. Further, when two faces $\overline{U}_1, \overline{U}_2$ intersect, they either intersect at a single vertex or along a single common edge. (If S is homeomorphic to $\mathbb{R}^2$, we say G is a **planar tiling graph**).

[1] The uniformization theorem does not help in producing a "picture" of the pentagonal tiling as it only states the existence of a conformal map from E to $\mathbb{C}$. In [BS97], the authors generate Figure 1 by using a computer program for "circle packings".

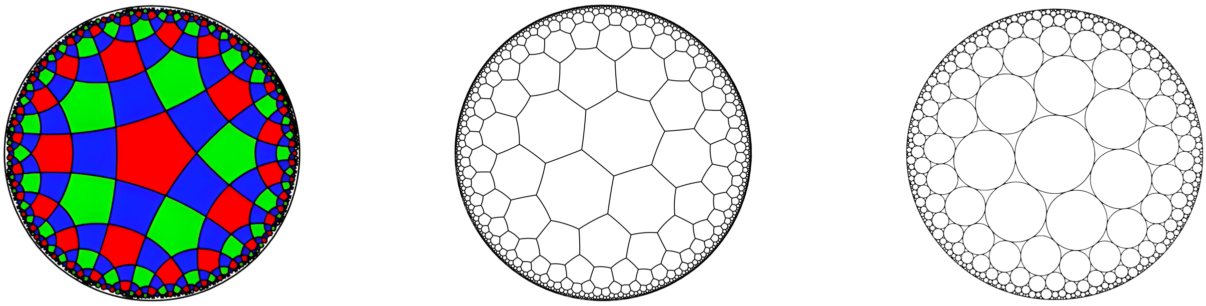

Figure 5-7: (5) A semi-regular tiling of $\mathbb{D}$ with vertex-type $[4, 5, 4, 5]$ using hyperbolic regular polygons (Figure 1 in [DG21]).[3] (6) A conformal tiling of $\mathbb{D}$ with heptagons. (7) A circle packing of $\mathbb{D}$ quasi-conformal to tiling in Figure 6.

Given any tiling graph $G = (V, E, F)$, there exists a Riemann surface X and a conformal tiling $\mathcal{T}$ of X such that the 1-skeleton of $\mathcal{T}$ is isomorphic to G. Moreover, the Riemann surface X and the conformal tiling $\mathcal{T}$ are unique up to conformal equivalence. To prove the existence, construct an equilateral surface E by gluing polygons in the pattern of faces in G. Taking $X = E$ and $\mathcal{T} = (E, \mathrm{Id})$ gives the required conformal tiling. For uniqueness, let $\mathcal{T}' = (E', f : E' \to X')$ be another conformal tiling. Then $E = E'$ because E and E' have the same combinatorics, and $f : E \to X'$ is the conformal equivalence between $\mathcal{T}$ and $\mathcal{T}'$.

Question 1. *Given a planar tiling graph G (a special infinite planar graph), by the uniformization theorem, there is a unique conformal tiling of either $\mathbb{C}$ or $\mathbb{D}$ whose 1-skeleton is G. Which planar graphs give a conformal tiling of $\mathbb{C}$, and which ones give a conformal tiling of $\mathbb{D}$? (This is called the* ***type problem*** *for conformal tilings.)*

Question 2. *Do all Riemann surfaces admit a conformal tiling? Restricting further, which Riemann surfaces can be conformally tiled using only n-sided tiles?*[2] *How does the answer change with n?*

Section 2 answers question 1 for a large class of planar tiling graphs that include semi-regular tilings. Section 3 explores question 2 and shows that non-compact hyperbolic Riemann surfaces of finite area cannot be conformally tiled using n-sided tiles for all $n \geq 7$.

2 How combinatorics decides the conformal type of the Riemann surface: elliptic (S^2), parabolic ($\mathbb{C}$), or hyperbolic ($\mathbb{D}$)

The ***vertex-type*** of a vertex v in a tiling is a cyclic tuple of integers $[k_1, k_2, \ldots, k_d]$ where d is the degree (or valence) of v, and each k_i (for $1 \leq i \leq d$) is the number of sides (the size) of the i-th polygon around v, in either clockwise or counter-clockwise order. A tiling

[2]Each tile in a conformal tiling might have a different number of sides, i.e., each polygon in the equilateral surface might have a different number of sides. Conformal tilings where every tile has the same number of sides are of special interest.

[3]This is not a conformal tiling; the conformal tiling arising from this combinatorics is only "quasi-conformal" to the hyperbolic tiling.

is called ***semi-regular*** if each vertex has the same vertex-type (eg. Figure 5). There are many examples of semi-regular conformal tilings.

Examples 1-3.

- S^2: The boundary of any Archimedean or Platonic solid is an equilateral surface. This gives a semi-regular conformal tilings of the Riemann sphere.
- $\mathbb{C}$: The 11 Archimedean tilings of the Euclidean plane with regular polygons are examples of conformal tilings of the complex plane (eg. Figure 3).
- $\mathbb{D}$: In [DG21], Datta and Gupta demonstrate a large class of semi-regular tilings of the unit disc where the tiles are regular geodesic hyperbolic polygons (eg. Figure 5). The combinatorics of these tilings can be used to produce semi-regular conformal tilings of the unit disc.

Definition 5. For a vertex v with vertex-type $[k_1, k_2, \dots, k_d]$, the **total angle** around the vertex v (denoted by $\mathcal{A}(v)$), and the **combinatorial curvature** at v (denoted by $\kappa(v)$), is defined by

$$\mathcal{A}(v) := \sum_{i=1}^{d} \left(\pi - \frac{2\pi}{k_i} \right) \qquad \kappa(v) := \frac{2\pi - \mathcal{A}(v)}{2\pi}.$$

For semi-regular tilings, $\mathcal{A}(v)$ and $\kappa(v)$ are constant across vertices. An interesting pattern emerges when $\mathcal{A}(v)$ and $\kappa(v)$ is computed for semi-regular conformal tilings in Examples 1-3.

- $\mathcal{A}(v) < 2\pi$ and $\kappa(v) > 0$ for examples of semi-regular conformal tilings of the Riemann sphere
- $\mathcal{A}(v) = 2\pi$ and $\kappa(v) = 0$ for examples of semi-regular conformal tilings of the complex plane
- $\mathcal{A}(v) > 2\pi$ and $\kappa(v) < 0$ for examples of semi-regular conformal tilings of the unit disc

We shall prove that this pattern holds for all semi-regular conformal tilings of simply connected Riemann surfaces. In fact, they hold for a larger class of tilings. A tiling is said to have ***finite local complexity (f.l.c.)*** if the vertex-type at each vertex is an element of a finite set; that is, vertices can have different vertex-types, but there are only finitely many distinct vertex-types in the tiling (cf. [BS17, §3]).

Theorem 1. *Let $\mathcal{T}$ be an f.l.c. conformal tiling of a simply connected Riemann surface X.*

- *If the total angle at each vertex of $\mathcal{T}$ is strictly lesser than 2π (i.e., $\mathcal{A}(v) < 2\pi$), then $X = \mathbb{S}^2$*
- *If the total angle at each vertex of $\mathcal{T}$ is equal to 2π (i.e., $\mathcal{A}(v) = 2\pi$), then $X = \mathbb{C}$*
- *If the total angle at each vertex of $\mathcal{T}$ is strictly greater than 2π (i.e., $\mathcal{A}(v) > 2\pi$), then $X = \mathbb{D}$*

Theorem 1 answers Question 1 in a particular case. Given an f.l.c. planar tiling graph G, where the total angle at each vertex is strictly greater than 2π, the unique conformal tiling whose 1-skeleton is G is a conformal tiling of $\mathbb{D}$ and not $\mathbb{C}$. This has an interesting consequence. If all the tiles have 7 or more sides, then the total angle at each vertex satisfies $\mathcal{A}(v) \geq 3\,(\pi - 2\pi/7) > 2\pi$, and we have the following.[4]

Corollary 1. *There is no f.l.c. conformal tiling of S^2 and $\mathbb{C}$, where each tile has 7 or more sides.*

To prove Theorem 1, we introduce a few concepts. A **circle packing** of Ω ($= \mathbb{D}$ or $\mathbb{C}$) is a collection $P = \{D_\alpha\}$ of closed non-overlapping circular discs in Ω such that the closure of each component of $S \setminus (\cup_\alpha D_\alpha)$ is compact and bounded by exactly three circular arcs. Also, the discs D_α should not accumulate in Ω (eg. Figure 7). The "combinatorics" (the pattern of tangencies) of a circle packing is captured by a planar tiling graph G, embedded in Ω, obtained by placing vertices at disc centers and joining edges between adjacent discs. Conversely, given a planar tiling graph G where every face has three sides, there exists a unique (up to conformal equivalence) circle packing P_G of either $\mathbb{C}$ or $\mathbb{D}$ with the combinatorics of G.

There is an important connection between f.l.c. conformal tilings and circle packings: suppose $\mathcal{T}$ is an f.l.c. conformal tiling of Ω ($= \mathbb{D}$ or $\mathbb{C}$) and the corresponding planar tiling graph is G. If a face of G has more than 3 sides, then we subdivide the face into a number of 3-sided faces by adding suitable extra edges (no new vertices are created in this process). The resulting planar tiling graph G' gives a circle packing $P_{G'}$ of Ω. Moreover, there is a quasi-conformal map $\varphi : \Omega \to \Omega$ that takes vertices of $\mathcal{T}$ to the disc centers of $P_{G'}$.

Proof sketch of Theorem 1. The three cases are proved separately.

- Suppose $\mathcal{A}(v) < 2\pi$ for all v (i.e., $\kappa(v) > 0$ for all v). Let E be the equilateral surface of the tiling. Due to the f.l.c. assumption, $\exists\, M, \delta > 0$ such that $\kappa(v) \geq \delta$ for all v and every polygon in E has at most M sides. In [Sto76], Stone proves a discrete version of the Bonnet-Myers theorem, which implies E is finite and compact. This, along with the uniformization theorem, gives X is compact and conformal to S^2.
- Suppose $\mathcal{A}(v) = 2\pi$ for all v. Let E be the equilateral surface of the tiling. Each Euclidean polygon $P \in E$ has a flat Riemannian metric such that the length of each edge is 1. As the total angle at each vertex is 2π, the flat metric of the polygons extends to a flat Riemannian metric on E. Also, this metric is complete as E is locally finite. Using Riemannian geometry, we get an isometry $g : E \to \mathbb{R}^2$. Lastly, g is conformal when $\mathbb{R}^2$ is viewed as $\mathbb{C}$.
- Suppose $\mathcal{A}(v) > 2\pi$ for all v (i.e., $\kappa(v) < 0$ for all v) and G is the planar tiling graph of $\mathcal{T}$.
  1. Using Euler's formula, show $\mathcal{T}$ is infinite; hence X is homeomorphic to $\mathbb{R}^2$.
  2. Obtain a perimetric inequality $|\partial S| \geq c|S|$, where $S \subseteq G$ is any finite connected subgraph. This is obtained through two inequalities (i) $-\sum_{v\in S} \kappa(v) \geq \delta|S|$ and (ii) $|\partial S| \geq -C\sum_{v\in S} \kappa(v)$. (i) follows by f.l.c. and $\kappa(v)$ assumptions; (ii) follows by careful manipulation of Euler's formula (cf. [Woe98]).
  3. In [HS95, §9], He and Schramm give a hyperbolicity condition for circle packings: if the planar tiling graph G' of a circle packing $P_{G'}$ satisfies the perimetric inequality, then $P_{G'}$ is a circle packing of $\mathbb{D}$.

[4] The 3 is because each vertex has at least 3 tiles around it (See Definition 2).

4. By the connection between f.l.c. conformal tilings and circle packings through the tiling graph G', there is a quasi-conformal map from X to the circle packing $P_{G'}$ of $\mathbb{D}$. Thus, X is conformal $\mathbb{D}$.

$\square$

3 Which Riemann surfaces can be conformally tiled using only n-sided tiles?

It is known since antiquity that $\mathbb{R}^2$ has a tiling with regular Euclidean n-gons only for $n \in \{3, 4, 6\}$. Similarly, S^2 has a tiling with regular spherical n-gons only for $3 \leq n \leq 5$. This is partly because these tilings have a rigidity property: the position of a single tile determines the position of every other tile. Conformal triangulations also have the above rigidity property; hence, it is not clear whether every Riemann surface can be conformally tiled. We now investigate which Riemann surfaces can be conformally tiled using only n-sided tiles.

$$\begin{aligned}\texttt{CT}_n := \{&X \text{ is a Riemann surface that admits an f.l.c. conformal tiling}\\ &\text{where all tiles have } n \text{ sides}\}\end{aligned}$$

The first observation is that $\texttt{CT}_n$ does not contain all compact surfaces. A compact equilateral surface has only finitely many polygons. Hence, there are only countably many compact equilateral surfaces. If $X \in \texttt{CT}_n$ and X is compact, then X is conformal to some compact equilateral surface. This shows there are only countably many compact surfaces in $\texttt{CT}_n$ (up to isomorphism). However, there are uncountably many distinct compact Riemann surfaces; hence at least one of them is left out. For $n = 3$, the Riemann surfaces which are left out are exactly known: a compact Riemann surface X is in $\texttt{CT}_3$ if and only if X can be defined as the zero loci of a set of polynomials having coefficients in algebraic numbers (see Belyi's theorem and [BR21, §1]).

For non-compact Riemann surfaces, Bishop and Rempe prove the following result ([BR21]).

Theorem 2 (Bishop and Rempe)**.** *All non-compact Riemann surfaces admit a f.l.c conformal tiling where each tile has three sides.*

The next observation is $\texttt{CT}_n \subseteq \texttt{CT}_3$ for $n \geq 3$. Informally, each n-sided tile can be subdivided into n conformal triangles by connecting the barycenter to vertices. This gives a conformal tiling where each tile is 3-sided.

For $n \leq 5$, $\texttt{CT}_3 = \texttt{CT}_4 = \texttt{CT}_5$. Roughly, if $X \in \texttt{CT}_3$, then $\exists\, f : X \to S^2$ a branched cover with at most three branch points. Since S^2 has a conformal tiling with only n-sided tiles for $n \in \{3, 4, 5\}$, this can be lifted to a conformal tiling of X. Hence $X \in \texttt{CT}_n$ for $n \in \{3, 4, 5\}$. This trick does not work for $n \geq 6$.

For $n \geq 7$, $\texttt{CT}_n \subsetneq \texttt{CT}_3$. Corollary 1 says $S^2, \mathbb{C} \notin \texttt{CT}_n$ for all $n \geq 7$. More generally, all quotients of $\mathbb{C}$ are also not in $\texttt{CT}_n$ for $n \geq 7$, because a conformal tiling of X can be lifted to its universal cover. Now only hyperbolic surfaces remain. For every $n \geq 3$, there is a tesselation of $\mathbb{D}$ using regular hyperbolic n-gons; hence $\mathbb{D} \in \texttt{CT}_n$ for all $n \geq 3$ (eg. Figure 6). Despite this, many non-compact hyperbolic surfaces are not in $\texttt{CT}_n$ for $n \geq 7$.

Theorem 3. *If X is a non-compact hyperbolic Riemann surface of finite area, then there is no f.l.c. conformal tiling of X where all tiles have 7 or more sides.*

Proof sketch. Suppose $\mathcal{T} = \{T_\alpha\}$ is an f.l.c. conformal tiling of X where $T_\alpha \subseteq X$ are the tiles of $\mathcal{T}$. For contradiction, assume each tile T_α has 7 or more sides. The hyperbolic metric of $\mathbb{D}$ induces an hyperbolic metric on X via the universal cover $\pi : \mathbb{D} \to X$. This metric can be used to measure area and we have $\text{Area}_{\mathbb{D}}(X) = \sum_\alpha \text{Area}_{\mathbb{D}}(T_\alpha)$. We shall show $\text{Area}_{\mathbb{D}}(X) = \infty$ by proving $\sum_\alpha \text{Area}_{\mathbb{D}}(T_\alpha) = \infty$. Towards this, our main claim is that $\exists\, c > 0$ such that $\text{Area}_{\mathbb{D}}(T_\alpha^{\text{ext}}) > c$ for all $T_\alpha \in \mathcal{T}$, where T_α^{ext} is the union of T_α and all of its neighboring tiles. Using this claim we can show $\exists\, M$ such that $M \cdot \sum_\alpha \text{Area}_{\mathbb{D}}(T_\alpha) \geq \sum_\alpha \text{Area}_{\mathbb{D}}(T_\alpha^{\text{ext}}) \geq \sum_\alpha c = \infty$.
The proof of the main claim proceeds in 4 steps, crucially using a corresponding result in circle packings.

1. Lift $\mathcal{T}$ to an f.l.c. conformal tiling of the disc $\tilde{\mathcal{T}} = \{\tilde{T}_\beta\}$ using the universal cover $\pi : \mathbb{D} \to X$. Let G be the planar tiling graph of $\mathcal{T}$, and G^* be the dual of G. As each tile in G has 7 or more sides, the degree of each vertex of G^* is at least 7. To get a circle packing from G^*, we add suitable extra edges and obtain $(G^*)'$ where each face is 3-sided. Let $P = \{D_\beta\}$ be the circle packing associated with $(G^*)'$.
2. Note P is a circle packing of $\mathbb{D}$ where each circle is tangent to 7 or more circles. For such circle packings, Beardon and Stephenson prove $\exists\, c_1 > 0$ such that $\text{Area}_{\mathbb{D}}(D_\beta) \geq c_1$ (see [BS91, §5]).
3. Construct a quasi-conformal map $\varphi : \mathbb{D} \to \mathbb{D}$ from the conformal tiling to the circle packing which takes the face center of $\tilde{T}_\beta$ to the center of a disc $D_\beta \in P$. Moreover, φ is such that $\varphi^{-1}(D_\beta) \subseteq \tilde{T}_\beta^{\text{ext}}$.
4. There are bounds on how quasi-conformal maps distort area of Borel sets. Using these bounds along with the fact that $\text{Area}_{\mathbb{D}}(D_\beta) \geq c_1$, we deduce $\exists\, c > 0$ such that $\text{Area}_{\mathbb{D}}(\tilde{T}_\beta^{\text{ext}}) \geq \text{Area}_{\mathbb{D}}(\varphi^{-1}(D_\beta)) > c$ for all β.

$\square$

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