An Introduction to Riemann Surfaces

Definition, examples, and holomorphic maps

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Complex Analysis

- A function $f : \mathbb{C} \to \mathbb{C}$ is said to be complex-differentiable a.k.a. *holomorphic* if
  - $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, or
  - f locally looks like $z \mapsto z^k$
- A lot of complex analysis is about understanding the logarithm
  - We **cannot** define a continuous $log : \mathbb{C} \to \mathbb{C}$
  - Is $log(1) = 0$ or $2\pi i$ or $4\pi i$?
  - Is there a reasonable way to define a continuous and differentiable $log : \mathbb{C} \to \mathbb{C}$

Riemann Surface of the logarithm

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- Recall how we solved $x^2 = -1$

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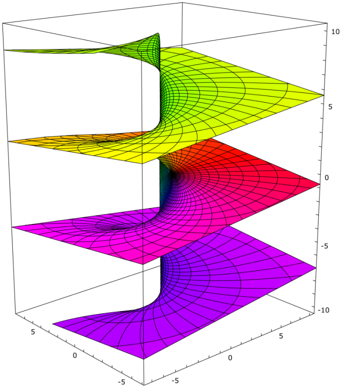
- Instead of $f : \mathbb{C} \to \mathbb{C}$, we look for $f : X \to \mathbb{C}$, where X is a bigger complex space
- This space will contain one value for each possible value of log



Riemann Surface of the logarithm

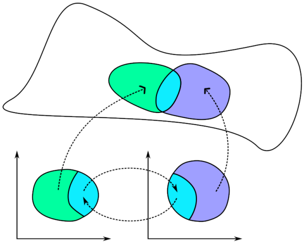
- Formally, $X = \{(z, w) \in \mathbb{C}^2 : \exp(w) = z\}$
- X locally looks like the complex plane, i.e. given $(w_0, z_0) \in X$ we have a coordinate chart:

$$\begin{aligned} \psi_{w_0} : B_{\mathbb{C}}(z_0, \epsilon) &\to U_{\subseteq X} \\ z &\mapsto (z, \log_{(z_0, w_0)}(z)) \end{aligned}$$

- And the transition maps between charts are bijective holomorphic maps:

$$\psi_{w_1}^{-1} \circ \psi_{w_0} : z \mapsto z$$

- Now we can define $\log : X \to \mathbb{C}$ to be $\log(z, w) = w$



Riemann Surfaces: Defn and examples

Defn.

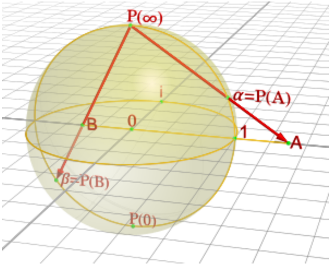
A Hasudorff second countable space X is said to be a Riemann surface if

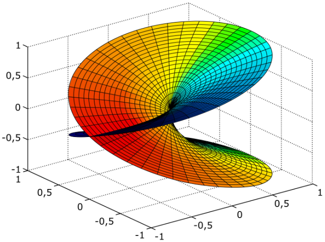
- There is a collection of coordinate charts $\{\psi_\alpha : V_\alpha \to U_\alpha\}$ where ψ_α is an homeomorphism from an open set V_α in $\mathbb{C}$ to an open set U_α in X
- And the transitions maps, $\psi_\alpha \circ \psi_\beta^{-1}$, between coordinate charts are holomorphic

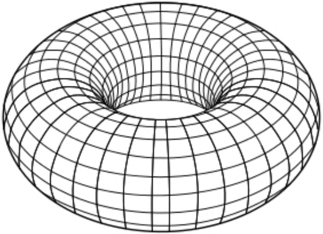
Examples

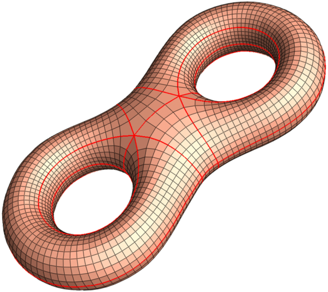
- The complex plane $\mathbb{C}$
- The unit disc $\mathbb{D}$, the upper half plane $\mathbb{H}$
- The Riemann sphere $S^2 = \mathbb{CP}^1$
- The Riemann surface of the square-root function
- A cylinder, ex: the puncture plane $\mathbb{C}^* = \mathbb{C} \backslash 0$
- A torus, ex: $\mathbb{C}/\mathbb{Z}^2$ (there are many R.S. that are homeomorphic to T^2)

Riemann Surfaces: Pictures









HW: Show every R.S. is orientable

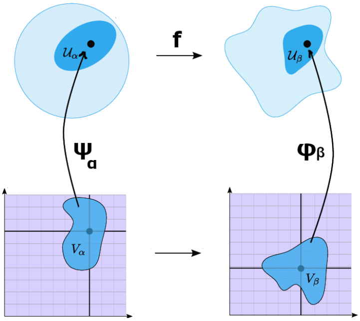
The Riemann sphere

- Consider the one-point compactification of $\mathbb{C}$, denoted by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$
- We make it into a Riemann surface using two charts:
  - $\psi_1 : \mathbb{C} \to \mathbb{C}_\infty \backslash \{\infty\}$, given by $z \mapsto z$
  - $\psi_2 : \mathbb{C} \to \mathbb{C}_\infty \backslash \{0\}$, given by $z \mapsto 1/z$
- The transition map is $\psi_2^{-1} \circ \psi_1 : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\}$, given by $z \mapsto 1/z$ which is holomorphic on its domain.
- There is a continuous bijection (homeomorphism) between $\mathbb{C}_\infty$ and the two-sphere, S^2.
- There is only one Riemann surface that is homeomorphic to a sphere. We call this *the Riemann sphere*!

Holomorphic functions between Riemann surfaces

A map between two R.S., $f : X \to Y$ is said to be holomorphic if

- f is holomorphic in *local coordinates*, i.e. $\psi_\beta^{-1} \circ f \circ \varphi_\alpha$ is holomorphic



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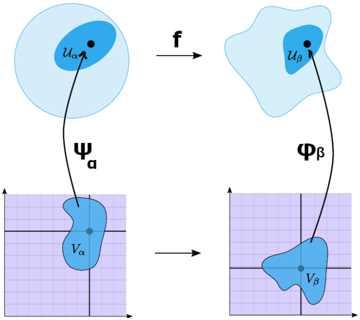
- There exists local coordinates s.t. locally f looks like $z \mapsto z^k$ for some $k \in 1, 2, 3, \dots$
  i.e. $\exists \psi_\delta, \varphi_\gamma$, such that $\phi_\delta^{-1} \circ f \circ \varphi_\delta : z \mapsto z^k$ (HW)



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Examples:

- $f : \mathbb{C} \to S^2$ given by $f : z \mapsto 1/z$
- Any meromorphic $f : \mathbb{C} \to S^2$
- Möbius map $f : S^2 \to S^2$ where $f(z) = \frac{az+b}{cz+d}$
- Most examples one comes across are $f : X \to S^2$ which are called branched covers of S^2

Fact: Given a proper holomorphic map $f : X \to S^2$, the quantity $\# f^{-1}(y)$ is finite and independent of y.

Quotient of a Riemann surface: a cylinder $\mathbb{C}\backslash\{0\}$

- Consider the map $\exp : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ given by $z \mapsto \exp(z) = \exp(x) \cdot \exp(iy)$
- Motivated by algebra class, we ask, does $\mathbb{C}/_{\text{something}} \simeq \mathbb{C}\backslash\{0\}$?

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- Observe that Γ acts on $\mathbb{C}$
- Define $x \sim y$ iff $\exists \gamma \in \Gamma$ such that $\gamma(x) = y$
- Now consider the quotient space $\mathbb{C}/\sim$
  Qn: Is $\mathbb{C}/\sim$ a Riemann surface?
- f induces a bijective holomorphic map $\tilde{f} : \mathbb{C}/\sim \to \mathbb{C}\backslash\{0\}$

Defn.
An isomorphism of Riemann surfaces is a bijective holomorphic map from one surface to the other.

HW: The inverse of such a map is also holomorphic (just as in algebra)!

Riemann surfaces of genus 1

Riemann surfaces that are homeomorphic to a torus

- Fix $w_1, w_2 \in \mathbb{C}$ and consider $\Gamma := \{z \mapsto z + nw_1 + mw_2 : n, m \in \mathbb{Z}\}$
- What does $\mathbb{C}/\Gamma$ look like?

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- Qn: Given two lattices Λ, Λ', when exactly is $\mathbb{C}/\Lambda$ equivalent to $\mathbb{C}/\Lambda'$?
- Ans: $\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda'$ iff $\Lambda = c \cdot \Lambda'$ for some $c \neq 0$
- Consider the set of all Riemann surfaces of genus 1
  $\mathcal{M} := \{[\mathbb{C}/\Lambda] : \Lambda \text{ is a lattice}\}$
- $\mathcal{M}$ is called the moduli space of Riemann surfaces of genus 1
- Fact: $\mathcal{M} \simeq \mathbb{H}/PSL(2, \mathbb{Z}) \simeq \mathbb{C}$

Pretty Picture without explanation

- Reimann surfaces of genus 1
- The Fuchsian group $PSL(2,\mathbb{Z})$
- The moduli space $\mathbb{H}/PSL(2,\mathbb{Z}) \sim_{\text{homeo}} \mathbb{C}$
- Lattices $\Lambda \subset \mathbb{C}$
- The Weierstrass $\wp_\Lambda$ function
- The pendulum equation
- Elliptic curves
- Number theory

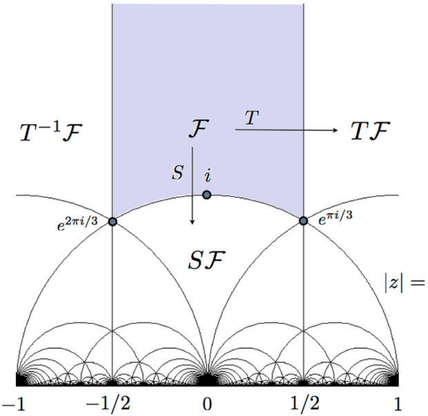

Figure: The action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$

What are all the possible Riemann Surfaces?

The Uniformization Theorem

Given a Riemann Surface X, it is isomorphic to one of the below:

1. The Riemann sphere
2. A quotient of $\mathbb{C}$, i.e.
   - $\mathbb{C}$ or,
   - Cylinder: $\mathbb{C} \backslash 0$ or,
   - A torus: $\mathbb{C}/\Lambda$
3. A quotient of $\mathbb{H}$, i.e. $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup acting freely on $\mathbb{H}$ (i.e. a Fuchsian group)

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Proof

Story for another day

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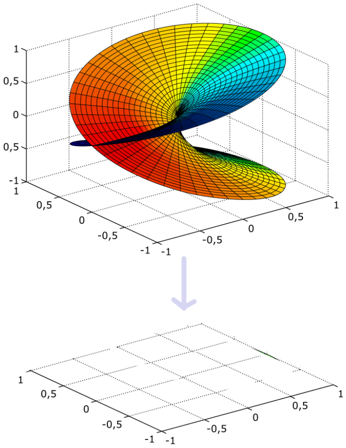
Proof uses

- Theory of universal covers and covering maps (Algebraic topology)
- Complex valued differential forms and integration over surfaces (Differential geometry)
- PDE techniques to solve the Poisson equation $\Delta f = \rho$ over a Riemann surface
- The Riesz Representation theorem: Every bounded linear functional L is $L(x) = \langle x, w_0 \rangle$ (Hilbert space theory)

The Riemann Surface of the square-root function

- Consider $X = \{(z, w) : w^2 = z\}$ (HW)
- Essentially, we throw in a point into X for each value of the square root
- Consider $f : X \to \mathbb{C}$ defined as $f((z, w)) = z$



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- Near $(0, 0)$, f looks like $\zeta \mapsto \zeta^2$ in local coordinates
- Given $\gamma : [0, 1] \to \mathbb{C}\backslash 0$ a loop based at 1, that goes around the origin in $\mathbb{C}$, we can lift it to a *path* in $X\backslash(0, 0)$
- Does the path in X, start and end at the same point? Ans: no!
- This gives an action of $\pi_1(\mathbb{C}\backslash\{0\}; 1)$ on the two points in $f^{-1}\{1\}$
- This gives to a monodromy $\rho : \pi_1(\mathbb{C}\backslash 0) \to S_2$

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- What about converse?

Riemann Existence Theorem. Let Y be a connected R.S., given a discrete set Δ, given a number k, and given $\rho : \pi_1(Y \backslash \Delta) \to S_k$, there is R.S. X and a holomorphic map $f : X \to Y$ s.t. which realises ρ as its monodromy homomorphism

References

1. S. K. Donaldson, *Riemann Surfaces*, Oxford Univ. Press, 2011.
2. C. Teleman, *Riemann Surfaces*, Lecture notes, Cambridge, Lent Term 2003.