Conformal tilings

A bizarre set of tilings generated by Schwarz reflections

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Complex analysis

- A function $f : \Omega_{\mathbb{C}} \to \Omega'_{\mathbb{C}}$ is said to be *conformal* (aka biholomorphic) if both f and f^{-1} are holomorphic
- Conformal means angle-preserving
- A *Riemann surface* is topological surface with a "complex structure": it is a manifold with charts to subsets of $\mathbb{C}$ and conformal transition maps.
- The complex structure allows us to define holomorphic maps between surfaces
- Examples: $\mathbb{C}, \mathbb{D}, \{1 < |z| < 2\}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

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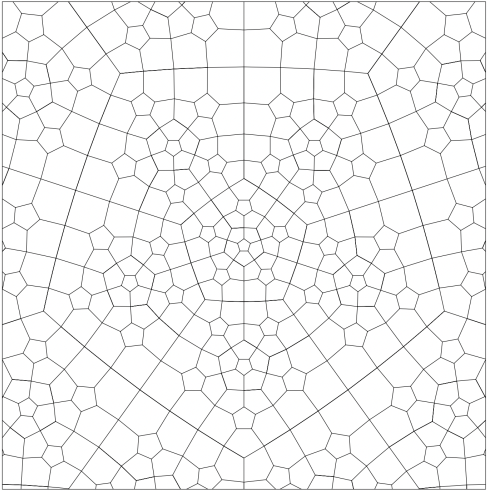
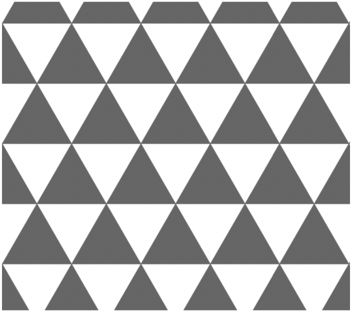
- Examples: $\mathbb{C}, \mathbb{D}, \{1 < |z| < 2\}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- Rk: Two Riemann surfaces may be topologically the same, but might not be isomorphic to each other. Ex: $\{1 < |z| < 2\}$ and $\{1 < |z| < 3\}$ are not biholomorphic to each other.
- A rich class of examples is obtained by taking the zero set $f^{-1}(0)$ of polynomial functions $f : \mathbb{C}^2 \to \mathbb{C}$.
  Ex: $X = \{(z, w) : w^2 = z^3 + az + b\}$

Examples of conformal tiling

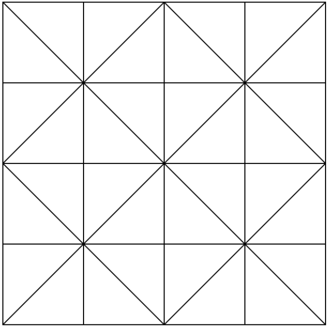


Tiling 1 (Pentagonal tiling of $\mathbb{C}$)

Examples of conformal tiling

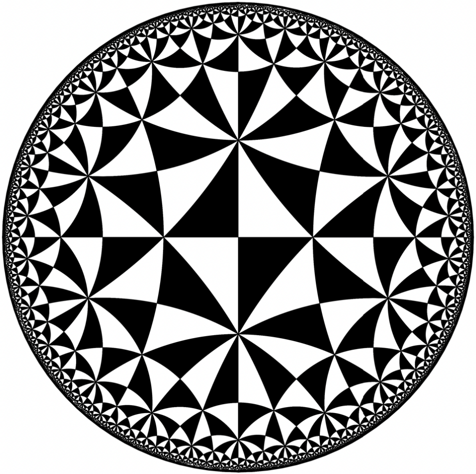


Tiling 2 (from antiquity)



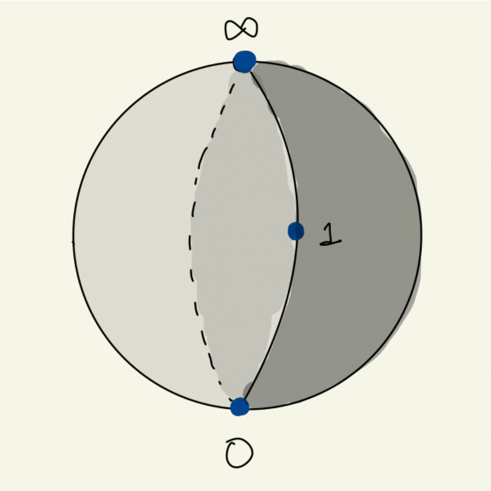
Tiling 3 (corresponding $\wp_\Lambda^2$)

Examples of conformal tiling



Tiling 4 (Hyperboling tiling of type (2,4,6))

Examples of conformal tiling



Tiling 5 (Two triangles give a sphere)

What is a conformal tiling?

- Is it an angle-preserving tiling?
  Not quite
- Obs: The internal angles of each polygon are not equal. But the angles around each vertex are equal

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- Obs: Each pair of adjacent tiles are invariant under reflection along common edge
- By reflection we mean *Schwarz reflections*, i.e. the unique anti-conformal map that fixes the edge pointwise
  Ex:
  - $z \mapsto \bar{z}$ (fixes $\mathbb{R}$)
  - $z \mapsto 1/\bar{z}$ (fixes S^1)
- Defn. $f : X \to Y$ is an anti-conformal map if it is an orientation reversing conformal map
  Formally, $\frac{\partial f}{\partial z} = 0 = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

What is a conformal tiling?

Defn. A conformal tiling of a Riemann surface X is a tiling of X such that for each pair of adjacent tiles τ_1, τ_2, there is an anti-conformal map $\phi : \tau_1 \to \tau_2$ fixing the common edge pointwise

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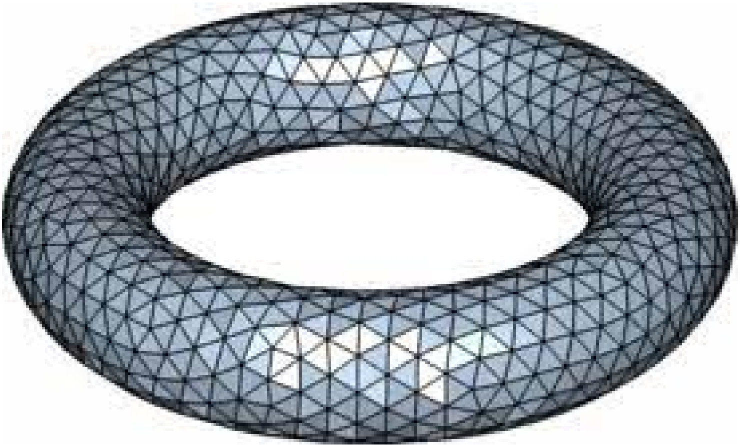
Questions:

Q1 Why is the pentagonal tiling a conformal tiling?

Q2 How can we come up conformal tilings (of say $\mathbb{C}$)?
(It is difficult to write down Schwarz reflections along a arc)

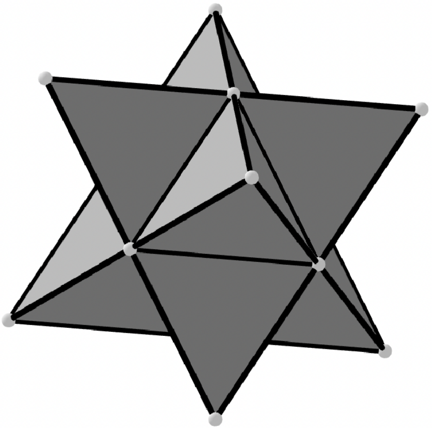
Q3 What are all the conformal tilings of $\mathbb{C}$?

Equilateral surfaces



A collection of **equilateral triangles** glued together

Equilateral surfaces



A collection of **equilateral triangles** glued together

What is an Equilateral surface

- Starting with a finite/infinite collection of unit equilateral triangles, glue these triangles together by identifying every edge with exactly one edge of another triangle
- Also ensure that each vertex is identified with only finitely many other vertices
- This gives a topological surface E
- Can we give E a complex structure, i.e. local charts to subsets of $\mathbb{C}$?

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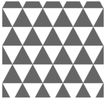
- This gives a topological surface E
- Can we give E a complex structure, i.e. local charts to subsets of $\mathbb{C}$?
- The interiors of the unit equilateral triangles can be given the complex structure of Δ
- Claim: We can add compatible charts at edges and vertices to make E into a Riemann surface
- E with the above complex structure is called an *equilateral surface*

Complex structure of E at edges and vertices

Alternative definition of conformal tiling

Defn. A conformal tiling of a Riemann surface X is a pair (E, f) such that $f : X \to E$ is conformal

Example:

$$\mathbb{C} \xrightarrow{\quad f \quad}$$ 

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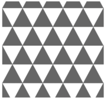
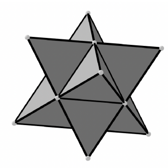


- Recall Q2: How can we come up conformal tilings?
- One way is to just put together an equilateral surface by glueing triangles in a certain pattern and then finding a Riemann surface which is conformal to it
- Eg:



Constructing conformal tilings

- The topology of E guides us to get hold of a Riemann surface X conformal to E

The Uniformization theorem

Any simply connected Riemann surface is isomorphic/conformal to either $\hat{\mathbb{C}}$ or $\mathbb{D}$ or $\mathbb{C}$.

- The uniformization theorem furnishes the required conformal map

  Drawback: The map is an abstract and not constructive :(

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The Uniformization theorem

Any simply connected Riemann surface is isomorphic/conformal to either $\hat{\mathbb{C}}$ or $\mathbb{D}$ or $\mathbb{C}$.

- The uniformization theorem furnishes the required conformal map
  Drawback: The map is an abstract and not constructive :(
- Given an equilateral surface E which is homeomorphic to the plane, the uniformization theorem tells us E is conformal to $\mathbb{D}$ or $\mathbb{C}$
- Determining between $\mathbb{C}$ or $\mathbb{D}$ is called the *type problem*

Pentagonal tiling of $\mathbb{C}$

- Recall Q3: Why is figure 1 a conformal tilings of pentagons?
- Ans: We shall construct an equilateral surface made up of pentagons and homeomorphic to the plane. Then we will show it has to conformal to $\mathbb{C}$ and not $\mathbb{D}$

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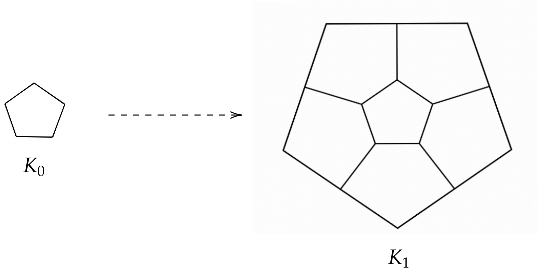
Construction of the equilateral surface

- Start with a regular pentagon and call it K_0
- Reflect K_0 along all its edges to get a flower-like structure with 5 petals
- Glue adjacent petals and form K_1
- View K_1 as a pentagon and reflect along "edges" to get a big flower-like structure with 5 big petals
- Glue adjacent petals and form K_2
- Repeat the process ad infinitum

Construction of the equilateral surface for pentagonal tiling



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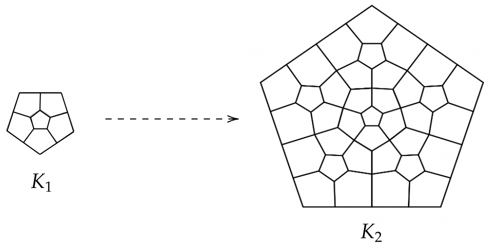

Why is this a pentagonal tiling of $\mathbb{C}$?

Why is E conformal to $\mathbb{C}$?

- Quick ans: E has a special symmetry called *"subdivisional self-similarity"* or *"zoom-in self-similarity"*
- Key observation: There is no $f \in \mathrm{Aut}(\mathbb{D})$ such that $f(0) = 0$ and f is *"shrinking"*
- We say f is "shrinking" if there is a compact set S such that $f(S) \subsetneq S$

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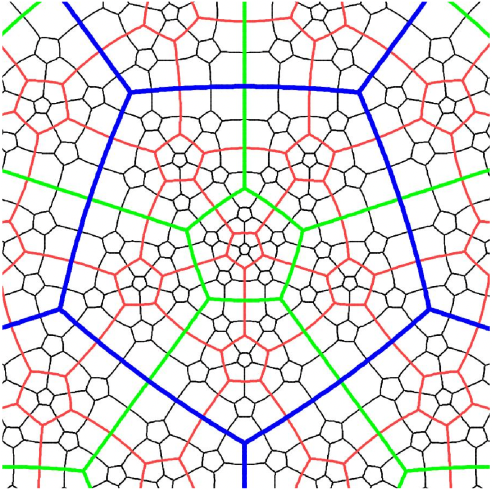
- We say f is "shrinking" if there is a compact set S such that $f(S) \subsetneq S$
- Next observe that there is a conformal map $g : E \to E$ such that g fixes a point and is "shrinking"
- This rules out the possibility that E is conformal to $\mathbb{D}$

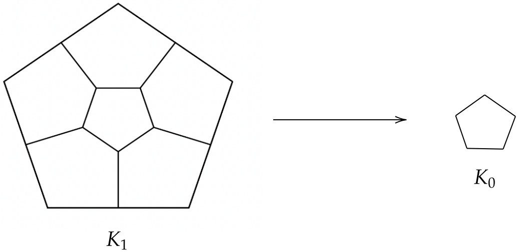
Subdivisional self-similarity of E



The pentagonal tiling with colors to highlight *"zoom-in self-similarity"*

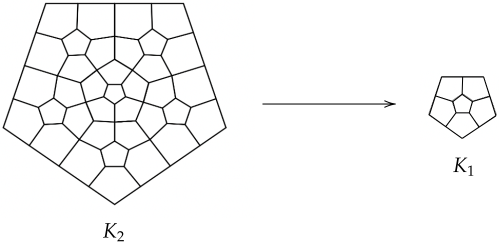
Subdivisional self-similarity of E

- Start by defining a conformal map from K_1 to K_0 fixing the center of K_0



Subdivisional self-similarity of E

- Extend the above to a conformal map from K_2 to K_1 fixing the center of K_0
- Continue this process to get a conformal from E to that fixes the center of K_0 and is "shrinking"



Summary so far

- We first defined conformal tiling using Schwarz reflections
- We gave another definition using equilateral tilings which enabled us to construct more examples of conformal tilings
- We constructed the pentagonal tiling and showed that it conformally tiles $\mathbb{C}$

Pause and ponder

- How did I get a picture of the pentagonal tiling?
  Ans: Convert to circle packing pattern + use circle packing algorithm
- Can we classify all the conformal tilings of the complex plane?
  What combinatorics give tiling of $\mathbb{C}$?

Conformal triangulations $\leftrightsquigarrow$ certain meromorphic fns

Branched cover = covering map except a discrete set (critical values)

Defn.

$f : X \to Y$ is a *branched covering* if

- f is holomorphic
- for every $p \in Y$, there is a small neighbourhood U such that
  - $f^{-1}(U) = \sqcup_\alpha V_\alpha$ where each V_α is a connected component
  - $f : V_\alpha \to U$ is $z \mapsto z^k$ in suitable coordinates

Examples:

1. $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $z \mapsto z^2$
2. $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2(1-z)^3$

Conformal triangulations $\leftrightsquigarrow$ certain meromorphic fns

Branched cover = covering map except a discrete set (critical values)

Defn.
A branched cover $f : X \to \hat{\mathbb{C}}$ is a *Belyi function* if f has at most three critical values on the sphere

Prop. Given a Riemann surface X, we have

$$\left\{ \begin{array}{c} \text{Belyi functions} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conformal} \\ \text{triangulations of } X \end{array} \right\}$$

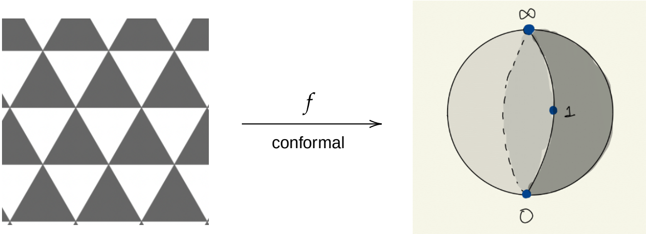
Examples:

1. $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $z \mapsto z^2$
2. $f : \mathbb{C} \to \hat{\mathbb{C}}$ given by $f(z) = \wp_\Lambda(z)^2$ where Λ is such that $\wp_\Lambda$ has $0, 1, -1, \infty$ as critical values

Conformal triangulations $\rightsquigarrow$ certain meromorphic fns

Conformal triangulation $\longrightarrow$ Belyi function

- First recall the conformal traingulation of $\hat{\mathbb{C}}$ consisting of two triangles
- Start with a particular triangle and map it to one of the triangles of $\hat{\mathbb{C}}$ using a conformal map
- Pick an adjacent triangle and extend the above map to this triangle via Schwarz reflections
- Continue extending using Schwarz reflections



Conformal triangulations $\leftrightsquigarrow$ certain meromorphic fns

Belyi function $\longrightarrow$ Conformal triangulation

- Given $f : X \to \hat{\mathbb{C}}$, we "lift/pullback" the triangulation of $\hat{\mathbb{C}}$ to get a triangulation X
- The vertices of the triangulation will be $f^{-1}\{0, 1, \infty\}$
- The edges of the triangulation will be $f^{-1}([0, 1]), f^{-1}([1, \infty]), f^{-1}([\infty, 0])$
- Claim 1: This gives a triangular tilling of X
- This triangular tiling will be a conformal tiling because we can get hold of a Schwarz reflection along each edge

Which Riemann surfaces have conformal tilings?

Lemma: There is a compact Riemann surface which does not admit a conformal tiling

Proof by pigeonhole principle

1. Fact: There are uncountably many distinct compact Riemann surfaces
2. There are only countably many compact equilateral surfaces
   - Any compact eq. surface has finitely many triangles
   - For a fixed $n \in \mathbb{N}$, there are finitely many eq. surfaces having n triangles
     (E is determined by combinatorics of how Δ are glued together)

Which Riemann surfaces have conformal tilings?

Belyi's theorem: Let X be a **compact** Riemman surface, then

X admits a conformal tiling

$$\Updownarrow$$

X admits a Belyi function

$$\Updownarrow$$

$\exists$ polynomials $p_1, \ldots, p_{n-1}$ such that

$$X \simeq \{[z_0 : \cdots : z_n] : p_1(\vec{z}) = \cdots = p_{n-1}(\vec{z}) = 0\} \subseteq \mathbb{C}P^n$$

and the coefficients of p_i are algebraic numbers

Which Riemann surfaces have conformal tilings?

Theorem [Bishop & Rempe, 21]: Every non-compact Riemann surface admits a conformal tiling

References

[BR21] Christopher J Bishop and Lasse Rempe. "Non-compact Riemann surfaces are equilaterally triangulable". In: *arXiv preprint arXiv:2103.16702* (2021).

[BS97] Philip Bowers and Kenneth Stephenson. "A "regular" pentagonal tiling of the plane". In: *Conformal Geometry and Dynamics of the American Mathematical Society* 1.5 (1997), pp. 58–86.

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