Geometry of Principal Bundles
Lecture 2: Connection on a Principal Bundle

Abstract

A connection on a principal G-bundle is introduced in two equivalent ways: (i) a $Lie(G)$-valued 1-form on P and (ii) a collection of local $Lie(G)$-valued 1-form on open sets of M. The key property of a connection is that it induces an affine-connection/covariant derivative on the associated vector bundles $P \times_\rho V$. Then, we note that the space of connections is an affine space modeled on $\Omega^1_M(P \times_{ad} Lie(G))$ and discuss the action of the gauge group on the space of connections.

Main Goal: An object on P that induces an affine connection on every vector bundle $P \times_\rho V$ associated with P.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Vector bundles over } M \\ \text{with structure group } G\end{matrix}\right\} \arrow[r, dashed, shift right, "\text{frame bundle}"'] & \left\{\begin{matrix}\text{Principal } G\text{-bundles} \\ \text{over } M\end{matrix}\right\} \arrow[l, dashed, shift right, "P \times_\rho V"'] \\
+ & + \\
\left\{\begin{matrix}\text{affine connections} \\ \text{on vector bundles}\end{matrix}\right\} \arrow[r, dashed, shift right] & \left\{???\right\} \arrow[l, dashed, shift right]
\end{tikzcd}$$

Connection on trivial principal G-bundle

Preliminary: the tangent bundle of a Lie group.

Recall, that for $\mathbb{R}^n$, the tangent spaces in the tangent bundle $T\mathbb{R}^n$ can be canonically identified with $\mathbb{R}^n$ itself. To see this, let $T\mathbb{R}^n = \left\{(x, v_x) \in \mathbb{R}^n \times \mathbb{R}^n\right\}$ and consider the map:

$$\begin{aligned} Q : T\mathbb{R}^n &\to \mathbb{R}^n \\ : (x, v_x) &\mapsto v_x. \end{aligned}$$

Similarly, for a Lie group G, the tangent spaces in the tangent bundle TG can be canonically identified with $Lie(G)$. To see this, start with the canonical isomorphism $Q_0 : T_eG \to Lie(G)$. Next, let $R_g : G \to G$ denote the right action $h * g \mapsto hg$. Differentiating, gives $(R_g)_* : TG \to TG$. Also, let $r_g : Lie(G) \to Lie(G)$ denote the right action $m * g \mapsto g^{-1}mg$. With these in hand, we use Q_0 to define $Q_g : T_gG \to Lie(G)$ as follows:

$$\begin{tikzcd} T_gG \arrow[r, dashed, "Q_g"] \arrow[d, "(R_{g^{-1}})^*"'] & Lie(G) \\ T_eG \arrow[r, "Q_0"'] & Lie(G) \arrow[u, "r_g"'] \end{tikzcd}$$

This gives a map $Q: TG \to Lie(G)$ which is G-equivariant wrt the right action of G on TG and $Lie(G)$.

<u>Example 1</u>. The tangent bundle of $GL_n(\mathbb{R})$.
Recall $T_e(GL_n(\mathbb{R})) = Lie(GL_n(\mathbb{R})) = M_n(\mathbb{R})$. Now given a non-identity matrix $g \in GL_n(\mathbb{R})$, for each $m \in M_n(\mathbb{R})$ we get a path passing through g as follows:

$$t \mapsto g \exp(tm).$$

It turns out that paths of the above form captures all the possible paths passing through g up to equivalence. Now differentiating the path at $t = 0$ gives the matrix gm. Thus,

$$T_gGL_n(\mathbb{R}) = \{(g, gm) : m \in M_n(\mathbb{R})\}.$$

Under this representation of $T_gGL_n(\mathbb{R})$, the map $Q: TG \to Lie(G)$ is just

$$\begin{aligned} Q &: TG \to Lie(G) \\ &: (g, gm) \mapsto m \\ &: (g, \zeta) \mapsto g^{-1}\zeta \\ &: (g, \zeta) \mapsto \left(\zeta g^{-1}\right) * g. \end{aligned}$$

Note: the above also holds for any other matrix Lie group G.

Connection on trivial bundle
Definition 0. Let $P = M \times G$ be a trivial principal G-bundle over M. A <u>*connection*</u> on $M \times G$ denoted by A is a C^∞-linear map

$$A: TM \times TG \xrightarrow[C^\infty\text{-lin}]{G\text{-equi}} TG$$

that is (i) G-equivariant and (ii) satisfies $A(x, 0, g, \zeta) = (g, \zeta)$. The second condition says that A is a projection of $TM \times TG$ onto TG.

<u>Remark</u>: Although the above is a good definition of a connection, it is usually better to have the image of A to be the vector space $Lie(G)$ instead of TG. This is achieved by simply post-composing A with Q which maps TG canonically to $Lie(G)$.

In light of the remark, we redefine a <u>*connection*</u> on $M \times G$ as a $Lie(G)$-valued 1-form, i.e., a C^∞-linear map

$$A: TM \times TG \xrightarrow[C^\infty\text{-lin}]{G\text{-equi}} Lie(G)$$

that satisfies (i) G-equivariant and (ii) $A(x, 0, g, \zeta) = Q(g, \zeta) = g^{-1}\zeta$. Next, to understand the map A better we decompose it into two parts.

Decomposition of A.

Lemma 1. *Given a connection $A : TM \times TG \to Lie(G)$ on the principal bundle $M \times G$, we have the following decomposition*

$$A = g^{-1}dg + g^{-1}ag^{-1} \tag{1}$$

where $a : TM \to Lie(G)$ is a $Lie(G)$-valued 1-form on M.

Proof. Let $(x, v_x, g, \zeta) \in TM \times TG$ and define a $Lie(G)$-valued 1-form on $M \times G$

$$\begin{aligned} a : TM &\to Lie(G) \\ a(v_x) &:= A(x, v_x, e, 0). \end{aligned}$$

Now we have the following computation:

$$\begin{aligned} A(x, v_x, g, \zeta) &= A(x, 0, g, \zeta) + A(x, v_x, g, 0) \\ &= g^{-1}\zeta + (A(x, v_x, e, 0)) * g \\ &= g^{-1}\zeta + g^{-1}(A(x, v_x, e, 0))g \\ &= g^{-1}\zeta + g^{-1}a(v_x)g \end{aligned} \tag{2}$$

where (i) we have used linearity of A, (ii) $A(x, 0, g, \zeta) = Q(g, \zeta) = g^{-1}\zeta$ is by definition, and (iii) $A(x, v_x, g, 0) = (A(x, v_x, e, 0)) * g$ is by definition of G-equivariance and $*$ denotes action of G on $Lie(G)$. Calculation (2) now implies the required decomposition. More explicitly, $g^{-1}dg$ denotes the $Lie(G)$-valued 1-form on $M \times G$ that maps (x, v_x, g, ζ) to $g^{-1}\zeta$; and $g^{-1}ag^{-1}$ is the $Lie(G)$-valued 1-form on $M \times G$ that maps (x, v_x, g, ζ) to $g^{-1}a(v_x)g$. □

Remark: Recall that a covariant derivative on a trivial vector bundle $M \times \mathbb{R}^n$ can be written as:

$$\nabla = d + \Gamma$$

where d is the standard derivative on $\mathbb{R}^n$ and $\Gamma : TM \to M_n(\mathbb{R}) = Lie(GL_n(\mathbb{R}))$ is the connection matrix or connection 1-form associated to the covariant derivative. Now note the similarity in decomposition (1) and $\nabla = d + \Gamma$. The first term is the fixed part and the second term is an extra term. Furthermore, both Γ and a appearing in the second terms are $Lie(G)$-valued 1-forms on M.

Remark: Examining the proof closely, we observe that a connection on $M \times G$ can be defined using $a : TM \to Lie(G)$, a $Lie(G)$-valued 1-form on M.

Principal bundles over the circle (Digression)

Fix the base manifold $M = S^1$ and a lie group G. Recall, that all principal G-bundles over

S^1 are isomorphic to the trivial bundle $S^1 \times G$.

Question 1: What is the set of isomorphism classes of connections on $S^1 \times G$? Are they all isomorphic to the trivial connection $g^{-1}dg$?

Answer: The set of isomorphism classes of connections on $S^1 \times G$ is in bijection with the set:

$$\mathrm{Hom}\left(\pi_1\left(S^1\right), G\right) / G \simeq \left\{\text{conjugacy classes in } G\right\}$$

We shall only provide a partial proof here; we refer the interested reader to section 13.2 of [Taubes] for a complete proof. To get a glimpse of the matter, fix a connection A and consider parallel transport around the circle S^1 starting at a point $0 \in S^1$. This defines a map from the fiber $P|_0$ over 0 to itself:

$$\begin{aligned} f_{A,0} &: P|_0 \to P|_0 \\ &: G \to G \\ &: g \mapsto f_{A,0}(g) \end{aligned}$$

Now because the connection A is G-equivariant, the parallel transport defined by A is also G-equivariant. Hence,

$$\begin{aligned} f_{A,0} &: P|_0 \to P|_0 \\ &: g \mapsto f_{A,0}(e) \cdot g \end{aligned}$$

Hence, the map $f_{A,0}$ is completely determined by $f_{A,0}(e)$. Now if $f_{A,0}(e) = e$, then $f_{A,0} \equiv \mathrm{Id}$. In this case we can show A is isomorphic the trivial connection $g^{-1}dg$. On the other hand, if $f_{A,0}(e) \neq e$, it can be shown that A is not isomorphic the trivial connection $g^{-1}dg$.

Connection on a principal bundle

Given a possibly non-trivial principal G-bundle, a connection is a map $A : TP \to Lie(G)$ that satisfies certain properties. To understand this better, we turn our attention to TP.

The structure of TP.

The projection $\pi : P \to M$ induces the following sequence of bundle homomorphisms.

$$0 \to \ker(\pi_*) \hookrightarrow TP \xrightarrow{"\pi_*"} \pi^* TM \to 0 \tag{3}$$

More about the sequence:

1. Each element of the sequence is a bundle over P.
2. $\ker(\pi_*)$ is a sub-bundle of TP and is called the *vertical subbundle* of TP because vectors in $\ker(\pi_*)$ are tangent to the fibers of P. Furthermore, the bundle $\ker(\pi_*)$ is canonically isomorphic to $P \times Lie(G)$. This isomorphism is given by

$$\Psi : P \times Lie(G) \to \ker(\pi_*)$$
$$\Psi : (p, m) \mapsto (t \mapsto p \cdot \exp(tm)).$$

Further, this map is G-equivariant: the right action of G on $P \times Lie(G)$ by $(p, m) \cdot g = \left(pg, g^{-1}mg\right)$ and the right action of G on $\ker(\pi_*)$ by $(t \mapsto p \cdot \exp(tm)) \cdot g = (t \mapsto p \cdot \exp(tm) \cdot g)$.

3. The map $TP \to \pi^*TM$ is obtained from $\pi_* : TP \to TM$ by sending $(p_\xi, \xi) \in TP$ to $(p_\xi, \pi_*(\xi)) \in P \times TM$.
4. The sequence is exact, that is, for each $p \in P$, the following sequence of vector spaces is exact.

$$0|_p \to \ker(\pi_*)|_p \hookrightarrow TP|_p \to \pi^*TM|_p \to 0|_p$$

5. The group G has a right action on each bundle in the sequence (3). Moreover, the sequence of maps is G-equivariant wrt this G-action. Note: the right action on TP is obtained by the derivative of $R_g : P \to P$ (the right action on G).

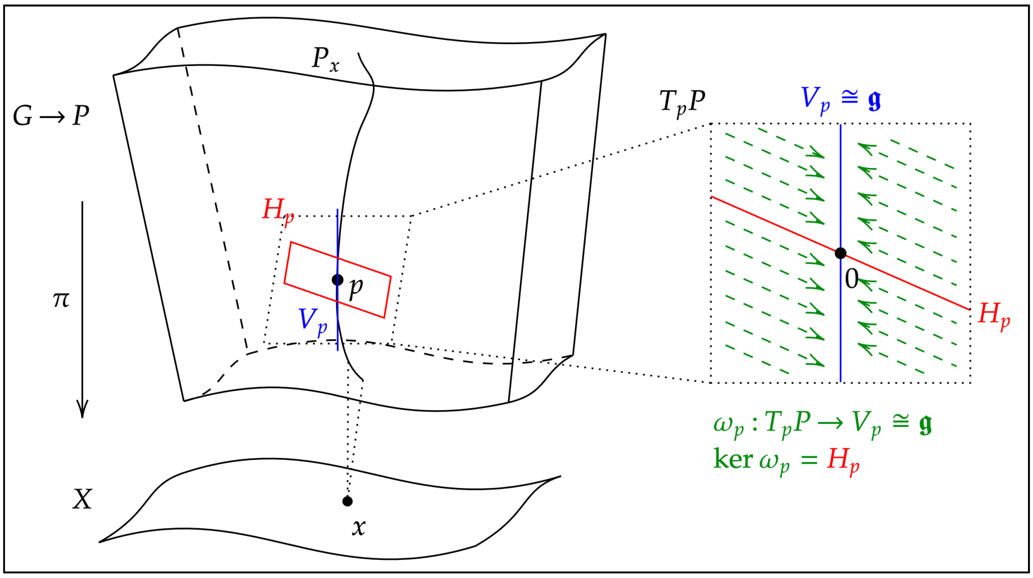

Figure 1: Diagram showing how the connection form may be thought of as a projection operator on the tangent space of the principal bundle. (Created by Tazerenix and licensed under CC BY-SA 4.0.)

Definition 1 (global)**.** A *connection* A on a principal G-bundle P is a $Lie(G)$-valued 1-form on P such that

(i) A is G-equivariant

(ii) and satisfies $A(\Psi(p, m)) = m$, that is $A(t \mapsto p \cdot \exp(tm)) = m$.

$$A : TP \xrightarrow[C^\infty\text{-lin}]{G\text{-equi}} Lie(G)$$
$$A((R_g)_*\xi) = g^{-1}A(\xi)g \quad \forall\ \xi \in TP$$

Here, $R_g : P \to P$ is the right action of G and $(R_g)_* : TP \to TP$ is the derivative of R_g.

Remark: The condition $A(\Psi(p,m)) = m$ can be described as follows: A restricts to the canonical right-invariant form on the vertical sub-bundle of TP. Put differently $A|_{\ker \pi_*} = A|_{P\times Lie(G)} : P \times Lie(G) \to Lie(G)$ is just the projection onto the right factor.

Definition 2 (local)**.** Let P be a principal G-bundle with trivializations $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$. A connection is a collection of $Lie(G)$ valued 1-forms $\{a_\alpha\}$:

$$a_\alpha : TU_\alpha \to Lie(G)$$

that transform as follows under the change of trivializations.

$$\begin{aligned} a_\beta &= \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1}, \text{ or equivalently,} \\ a_\beta &= \tau_{\alpha\beta}^{-1} a_\alpha \tau_{\alpha\beta} + \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta} \end{aligned}$$

Relation between definitions.

Lemma 2 (local picture of A)**.** Let $A : TP \to Lie(G)$ be a connection on P. Suppose $\psi_\alpha^{-1} : U_\alpha \times G \to P|_{U_\alpha}$ is a trivialization of P. Let $\left(\psi_\alpha^{-1}\right)^* A$ be the pullback $Lie(G)$-valued 1-form on $U_\alpha \times G$. Then, $\left(\psi_\alpha^{-1}\right)^* A$ can be decomposed as follows:

$$\left(\psi_\alpha^{-1}\right)^* A = g^{-1}dg + g^{-1}a_\alpha g. \tag{4}$$

where $a_\alpha : TU_\alpha \to Lie(G)$ is a $Lie(G)$-valued 1-form on U_α.

Proof. Consider $U_\alpha \times G$ as trivial principal G-bundle over U_α. Now note that $\left(\psi_\alpha^{-1}\right)^* A$ is a connection on $U_\alpha \times G$ as per Definition 0. With this, Lemma 2 follows from Lemma 1.

$\square$

Let $A : TP \to Lie(G)$ be a connection from Definition 1. Now, Lemma 2 gives a candidate for definition 2: collect all the $\{a_\alpha\}$ obtained by decomposing $\left(\psi_\alpha^{-1}\right)^* A$. To see that $\{a_\alpha\}$ is a connection as per Definition 2, we need to check that these $\{a_\alpha\}$ transform in the right way. This is provided by the Lemma 3 below. Conversely, equation (4) can be used to go the other way round too: start from the RHS and go to LHS.

Lemma 3 (transformation of $Lie(G)$-valued 1-forms under change of trivializations)**.** *Start with the simplicity assumption $U_\alpha = U_\beta$. Let $U_\alpha \times G$ and $U_\beta \times G$ be two principal G-bundles. Suppose $U_\beta \times G$ has a connection given by*

$$A = g^{-1}dg + g^{-1}a_\alpha g.$$

Let $T : U_\beta \times G \to U_\alpha \times G$ be a G-equivariant bundle isomorphism that is given by

$$T : U_\beta \times G \to U_\alpha \times G$$
$$T : (x, g) \mapsto (x, \tau_{\alpha\beta}(x) g)$$
where $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to G$.

Then, the pullback connection $T^* A$ on $U_\beta \times G$ is given by

$$\begin{aligned} T^* A &= g^{-1} dg + g^{-1}\left(\tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta} + \tau_{\alpha\beta}^{-1} a_\alpha \tau_{\alpha\beta}\right) g \\ &= g^{-1} dg + g^{-1} a_\beta g \end{aligned}$$

$$\text{where } a_\beta := \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta} + \tau_{\alpha\beta}^{-1} a_\alpha \tau_{\alpha\beta} = -d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1} + \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1}.$$

Proof. The key is to calculate $T_* : TU_\alpha \times TG \to TU_\beta \times TG$.

$$T_* : TU_\alpha \times TG \to TU_\beta \times TG$$
$$T_* : \begin{bmatrix} x \\ v_x \\ g \\ \zeta_g \end{bmatrix} \mapsto \begin{bmatrix} x \\ v_x \\ \tau_{\beta\alpha}(x) \cdot g \\ D\tau_{\beta\alpha}(v_x) \cdot g + \tau_{\beta\alpha}(x) \cdot \zeta_g \end{bmatrix}$$

Next, we compute

$$\begin{aligned} \left(T^* A\right)(x, v_x, g, \zeta_g) &= A(T_*(x, v_x, g, \zeta_g)) \\ &= A(x, v_x, \tau_{\beta\alpha} g, D\tau_{\beta\alpha}(v_x) g + \tau_{\beta\alpha}\zeta_g) \\ &= g^{-1}\tau_{\beta\alpha}^{-1}(D\tau_{\beta\alpha}(v_x) g + \tau_{\beta\alpha}\zeta_g) + g^{-1}\tau_{\beta\alpha}^{-1} a_\alpha(v_x)\tau_{\beta\alpha} g \\ &= g^{-1}\zeta_g + g^{-1}\left(\tau_{\beta\alpha}^{-1} D\tau_{\beta\alpha}(v_x) + \tau_{\beta\alpha}^{-1} a_\alpha(v_x)\tau_{\beta\alpha}\right) g. \end{aligned}$$ □

Defining an affine connection on associated vector bundles $P \times_\rho V$.

Let P be a principal G-bundle with a connection $A / \{a_\alpha\}$ and let $P \times_\rho V$ be an associated vector bundle. We want to induce an affine connection on $P \times_\rho V$. An affine connection on
a vector bundle can be described in three ways: (1) by a connection matrix, i.e., matrix valued 1-forms, (2) by a covariant derivative $\nabla : T(P \times_\rho V) \to T^* M \otimes T(P \times_\rho V)$, and (3) parallel transport. We shall see all three descriptions.

1. (Connection matrix). If $a_\alpha : TU_\alpha \to Lie(G)$ is the connection on P, define the connection matrices
$$\Gamma_\alpha := \rho_* \circ a_\alpha : TU_\alpha \to M(n, \mathbb{R})$$
where $\rho_* : Lie(G) \to M(n, \mathbb{R})$ is the derivative of $\rho : G \to GL(n, \mathbb{R})$ at the identity.

For the viewpoint of covariant derivative and parallel transport, see the appendix at the end of this note.

(Affine) space of connections on a principal G-bundle

Proposition 1 ($\mathcal{A}$ is an affine space)**.** Suppose $\pi : P \to M$ is a principal G-bundle. Let $\mathcal{A}$ denote the set of all connections on the principal bundle P. We have the following:

1. The difference of two connections is captured by a map $c : TM \xrightarrow{C^\infty} P \times_{ad} Lie(G)$
2. If $A : TP \to Lie(G)$ is a connection, and $c : TM \xrightarrow{C^\infty} P \times_{ad} Lie(G)$, then $A + "c"$ gives another connection.

Takeaway: The set of connections $\mathcal{A}$ on a principal bundle P is an affine space modeled on $C^\infty\left(M; \left(P \times_{ad} Lie(G) \otimes T^*M\right)\right) =: \Omega^1_M(P \times_{ad} Lie(G))$. Note that this is an infinite dimensional space. Alternatively, the space $C^\infty\left(M; \left(P \times_{ad} Lie(G) \otimes T^*M\right)\right)$ can be thought of as $C^\infty\left(\pi^*TM; Lie(G)\right)$.

Motivation: Here is a related fact: let V be a vector space and $W \subseteq V$ be a subspace. We say a linear map $T : V \to W$ is a projection if $T(w) = w$. Now we have:

$$\left\{ \begin{array}{c} \text{The set of projections} \\ T : V \to W \end{array} \right\} \underset{\text{bijection}}{\leftrightsquigarrow} \mathrm{Hom}(V/W, W)$$

The analogy is that connections A are projections from TP to vertical bundle $\ker \pi_*$ and they are in bijection with homorphims from π^*TM to $\ker \pi_*$.

Remark: Recall that the difference of two affine connections on a vector bundle E is a section $TM \to \mathrm{End}(E)$. Now if $E = P \times_\rho V$ and the two connections on E are induced from two connections A, A' on P, then we have

$$\nabla_{A'} - \nabla_A = \rho_*(c)$$

where $c : TM \to P \times_{ad} Lie(G)$ is the difference of two connections $A - A'$; and $\rho_* : P \times_{ad} Lie(G) \to P \times_\rho \mathrm{End}(V)$ is the map induced by $\rho_* : Lie(G) \to \mathrm{End}(V)$ which is the derivative of $\rho : G \to GL(V)$ at the identity.

Local viewpoint of observation 1: Let the two connections be $\{a_\alpha\}$ and $\{a'_\alpha\}$. Now we have $(a_\alpha - a'_\alpha) : TU_\alpha \to Lie(G)$ which transform as follows:

$$\begin{aligned}(a_\beta - a'_\beta) &= \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1} - \tau_{\beta\alpha} a'_\alpha \tau_{\beta\alpha}^{-1} + d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1} \\ &= \tau_{\beta\alpha}(a_\alpha - a'_\alpha)\tau_{\beta\alpha}^{-1} \\ &= (ad \circ \tau_{\beta\alpha})(a_\alpha - a'_\alpha).\end{aligned}$$

Thus, the collection of data $(a_\alpha - a'_\alpha)$ extends to a map $c : TM \to P \times_{ad} Lie(G)$.

Local viewpoint of observation 2: Let $c : TM \to P \times_{ad} Lie(G)$ and $\{a_\alpha\}$ be a connection on P. Write c in local trivializations as maps $c_\alpha : TU_\alpha \to Lie(G)$ that transform as:

$$\begin{aligned} c_\beta &= (ad \circ \tau_{\beta\alpha})(c_\alpha) \\ &= \tau_{\beta\alpha} c_\alpha \tau_{\beta\alpha}^{-1} \end{aligned}$$

where $\tau_{\beta\alpha}$ are the transition maps of P. Now consider the collection $\{a_\alpha + c_\alpha\}$ consisting of maps $a_\alpha + c_\alpha : TU_\alpha \to Lie(G)$. This collection defines a connection on P because it transforms in the following way:

$$\begin{aligned} a_\beta + c_\beta &= \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1} + \tau_{\beta\alpha} c_\alpha \tau_{\beta\alpha}^{-1} \\ &= \tau_{\beta\alpha}(a_\alpha + c_\alpha)\tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1}. \end{aligned}$$

Global viewpoint of observation 1: Let $A, A' : TP \to Lie(G)$ be two connections on P. The difference $A - A'$ also maps $TP \to Lie(G)$. But observe that vertical subbundle $\ker(\pi_*)$ of TP is in the kernel of $A - A'$; this is because

$$\begin{aligned} (A - A')(t \mapsto p\exp(tm)) &= m - m \\ &= 0. \end{aligned}$$

Put differently, in local trivializations, we have:

$$\begin{aligned} \left(\psi_\alpha^{-1}\right)^*(A - A') &= g^{-1}dg + g^{-1}a_\alpha g - g^{-1}dg - g^{-1}a'_\alpha g \\ &= g^{-1}(a_\alpha - a'_\alpha)g. \end{aligned}$$

Coming back to $A - A'$, we claim that $A - A' : TP \to Lie(G)$ reduces to a map $b : \pi^*TM \to Lie(G)$. More precisely we have the following fact without proof.

Lemma 4. *There is a canonical 1-1 correspondence between G-equivariant maps $TP \to Lie(G)$ that take $\ker(\pi_*)$ to 0 and G-equivariant maps $\pi^*TM \to Lie(G)$.* □

Note: Lemma 4 is the analog of this linear algebra result: the set of linear maps $T : V_1 \to V_2$ that map the subspace $W \subseteq V_1$ to 0 is in 1-1 correspondence with the set of maps $T : V_1 / W \to V_2$. In our case, we should think of π^*TM as $TP/\ker(\pi_*)$. Also, note that the above correspondence does not depend on the connection A or A'.

By Lemma 4, the difference $A - A'$ corresponds to a unique G-equivariant map $b : \pi^*TM \to Lie(G)$. Lastly, any G-equivariant map $b : \pi^*TM \to Lie(G)$ can be reinterpreted as a map $c : TM \to P \times_{ad} Lie(G)$. To see this, given a $(x, v_x) \in TM$, pick any $p \in P_x$ and consider $(p_x, x, v_x) \in \pi^*TM$. Now simply define $c(x, v_x) := [(p, b(p_x, x, v_x))]$. Thus, the $A - A'$ corresponds to a G-equivariant map $c : TM \to P \times_{ad} Lie(G)$ in a canonical way.

Further reading

1. Chapter 11, Taubes, C. H. (2011). *Differential geometry: bundles, connections, metrics and curvature* (Vol. 23). OUP Oxford.

2. Section 2.1.1. Donaldson, S. K., & Kronheimer, P. B. (1990). *The geometry of four-manifolds*. Oxford university press.

Appendix

In this appendix we give two more viewpoints of how a connection on P induces a connection on $P \times_\rho V$.

2. (Covariant derivative). Let $s : M \to P \times_\rho V$ be a section of $P \times_\rho V$. We want to define

$$\nabla s : TM \to P \times_\rho V.$$

It is easier to work with $P \times V$ instead of $P \times_\rho V$, so we lift the section s to a section $\tilde{s} : P \to P \times V$ which is of the form $\tilde{s}(p) = (p, s_P(p))$.

$$\begin{tikzcd}
P \times V \arrow[r] \arrow[d] & P \times_\rho V \arrow[d] \\
P \arrow[u, dashed, bend left, "\tilde{s} = \mathrm{Id} \times s_p"] \arrow[r, "\pi"] & M \arrow[u, dashed, bend left, "s"]
\end{tikzcd}$$

Similarly, instead of looking for $\nabla s : TM \to P \times_\rho V$, we instead look for a lift $\widetilde{\nabla s} : TP \to P \times V$ which is of the form $\widetilde{\nabla s}(p, \xi) = (p, \nabla s_P(p, \xi))$.

$$\begin{tikzcd}
P \times V \arrow[r] \arrow[d] & P \times_\rho V \arrow[d] \\
TP \arrow[u, dashed, bend left, "\widetilde{\nabla s} = \mathrm{Id} \times ds_p"] \arrow[r, "\pi_*"] & TM \arrow[u, dashed, bend left, "\nabla s"]
\end{tikzcd}$$

Now $s_P : P \to V$. Ordinary differentiation gives $ds_P : TP \to V$. This defines the required lift $\widetilde{\nabla s} : TP \to P \times V$ as $\widetilde{\nabla s}(p, \xi) = (p, ds_P(p, \xi))$. The last step is to find a ∇s whose lift is $\widetilde{\nabla s}$. A priori, there are many sections whose lift is $\widetilde{\nabla s}$, but the connection A, picks out a unique section $\nabla s : TM \to P \times_\rho V$ whose lift is $\widetilde{\nabla s}$. The section ∇s is defined as follows:

$$\begin{tikzcd}
P \times V \arrow[r, "\mathrm{Id}"] & P \times V \arrow[r] & P \times_\rho V \\
TP \arrow[u, "\widetilde{\nabla s} = \mathrm{Id} \times ds_p"] & \pi^* TM \arrow[l, "B_A"'] \arrow[u, dashed, "(G\text{-equiv})"] \arrow[r] & TM \arrow[u, dashed, "\nabla s"]
\end{tikzcd}$$

Here, B_A is a map canonical inverse of $\pi_* : TP \to \pi^* TM$ obtained from A, that is, B_A is the unique map that satisfies $\pi_* \circ B_A = \mathrm{Id}$ and $A \circ B_A \equiv 0$.

3. (Parallel transport 1). Suppose $\gamma : I \to M$ is a smooth curve and let $[(p, v)]$ be a

starting point. Then we obtain a unique lift $\sigma : I \to P \times_\rho V$ starting at $[(p, v)]$ as follows.

(a) First, lift γ to $\gamma_1 : I \to P$ starting at p. This lift should satisfy $A \circ \dot{\gamma}_1 \equiv 0$. Such a lift is unique by local calculation.

(b) Second, lift γ_1 to $\gamma_2 : I \to P \times V$. The lift $\gamma_2(t) := (\gamma_1(t), v)$.

(c) Lastly, pushforward γ_2 via the quotient map $P \times V \to P \times_\rho V$ to get σ, i.e. put $\sigma(t) = [\gamma_2(t)]$.

4. (Parallel transport 2). To see parallel transport locally, let $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ be a trivialization. Given $\gamma : I \to U_\alpha$, the lift $\gamma_1(t) = (\gamma(t), g(t))$ is such that:

$$A \circ \dot{\gamma}_1 \equiv 0$$
$$\gamma_1(0) = (\gamma(0), g_0)$$

Now $A = g^{-1}dg + g^{-1}a_\alpha g$ in local coordinates; hence:

$$g^{-1}\dot{g} + g^{-1}a_\alpha(\dot{\gamma})g = 0$$
$$\gamma_1(0) = (\gamma(0), g_0).$$

Solving the above linear ODE for $g(t)$ with $g(0) = g_0$, gives the required lift γ_1. Now, σ is just

$$\sigma(t) = (\gamma(t), \rho(g(t))v).$$