Which functions of bounded variation are absolutely continuous?
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Introduction

Defn 1: The set of functions with ***bounded variation*** (BV) is given by

$$BV(0,1) = \left\{ f : (0,1) \to \mathbb{R} \mid V_0^1(f) < \infty \right\},$$

where

$$V_a^b(f) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

where supremum is over all $a < x_1 < \ldots < x_n < b$.

Remark: Alternatively, one can define BV functions to be difference of two non-decreasing functions.

Defn 2: $u : (0,1) \to \mathbb{R}$ is said to be ***absolutely continuous*** on $(0,1)$, i.e., an element of $AC(0,1)$ if $\exists\, v \in L^1(0,1)$ such that $u(x) = u(0) + \int_0^x v$.

Theorem 1: Let $u \in BV(0,1)$. We have $u \in AC(0,1)$ if and only if $||Du|| = ||u'||_{L^1}$ where

$$||Du|| = \sup\left\{ \left| \int_\Omega u\phi' \right| : \phi \in C_c^\infty(\Omega) \text{ such that } ||\phi||_\infty < 1 \right\}$$

Remark: $(0,1)$ can be replaced by any bounded interval. For unbounded intervals, one has to be careful about how one defines the class $AC(I)$.

We shall describe the proof of Theorem 1 in 4 parts.

Part 1: Understanding and capturing the derivative of BV functions.

Given $f \in BV(0,1)$, define $T_f(x) = V_0^x(f)$. One should think of T_f as $T_f(x) = \int_0^x |f'|$.

So T_f captures the derivative. But this is not good enough; there is no way to go back to the original function from T_f. Ideally one would want to go back to the function by integrating the derivative or something like that.

Enter, $\mathcal{M}$ = space of $\mathbb{R}$-valued borel measures on $(0,1)$ with finite total variation.

Also define $NBV(0,1) = \left\{ f \in BV(0,1) : f\left(0^+\right) = 0 \text{ and } f \text{ is left continuous} \right\}$.

Remarks

1. The left and right limits of a BV function always exist! This is because a BV function can be written as a difference of two non-decreasing functions.
2. $BV(0,1) \subseteq L^\infty(0,1)$ because $|f(x) - f\left(0^+\right)| \leq V_0^1(f) < \infty$.

Proposition 1 [Derivative of BV]:

(a) If $\mu \in \mathcal{M}$ and we set $f(x) = \mu((0,x))$, then $f \in NBV(0,1)$.

(b) For every $f \in NBV(0,1)$, there is a unique $\mu \in \mathcal{M}$ such that $f(x) - f\left(0^+\right) = \mu((0,x))$ a.e. Moreover, $T_f(x) = |\mu|((0,x))$ if $f \in NBV(0,1)$.

(c) Finally, if $f(x) - f\left(0^+\right) = \mu((0,x))$, then f is continuous precisely at those points where $\mu(\{x\}) = 0$.

Remark: We should think of the above result as

- $f(x) - f(0) = \int_0^x f'(t)dt = \int_0^x d\mu = \mu((0,x))$, i.e. $f'(t)dt = d\mu = \mu(t, t+dt)$.
- $T_f(x) = \int_0^x |f'(t)|dt = \int_0^x d|\mu| = |\mu|(0,x)$,i.e. $|f'(t)|dt = d|\mu| = |\mu|(t, t+dt)$.

Part 2: Measure theory [Reference: Rudin]

Proposition 2 [Lebesgue-Radon-Nikodym theorem]*:* Any Borel measure μ can be decomposed as $\mu = v\mathcal{L} + \mu_s$ where $\mu_s \perp \mathcal{L}$. Here addition is defined as $(v\mathcal{L} + \mu_s)(E) = \int_E v(t)\mathcal{L}(dt) + \mu_s(E)$.

Moreover we also have $||\mu|| = ||v\mathcal{L}|| + ||\mu_s||$ because $\mu_s \perp \mathcal{L}$.

Proposition 3: If $f \in BV(0,1)$, then for all $\varphi \in C_c^\infty(0,1)$ we have

$$\begin{aligned} \int_0^1 f(t)\varphi'(t)dt &= -\int_0^1 \varphi(t)d\mu(t) \\ &= -\int_0^1 \varphi(t)v(t)dt + \int_0^1 \mu_s((0,t))\varphi'(t)dt \end{aligned}$$

Proposition 4 [$\approx$ Lusin's theorem]: Given a non-negative Borel measure λ, a simple function $s \in L^1(\lambda)$, and $\epsilon > 0$, there exists a $\varphi \in C_c^\infty((0,1))$ such that $||\varphi - s||_{L^1(\lambda)} < \epsilon$ and $||\varphi||_\infty < ||s||_\infty + \epsilon$.

Proposition 5: Given the above, we have

$$\sup_{\substack{\varphi \in C_c^\infty \\ ||\varphi||_\infty < 1}} \left| \int_0^1 f(t)\varphi'(t)dt \right| = \sup_{\substack{\varphi \in C_c^\infty \\ ||\varphi||_\infty < 1}} \left| \int_0^1 \varphi(t)v(t)dt \right| + \sup_{\substack{\varphi \in C_c^\infty \\ ||\varphi||_\infty < 1}} \left| \int_0^1 \mu_s((0,t))\varphi'(t)dt \right|$$

Proposition 6: If $f(x) - f\left(0^+\right) = \mu((0,x)) = \int_0^x v(t)dt + \mu_s((0,x))$, then f is differentiable a.e. and $f' = v$ a.e.

Part 3: Return to theorem 1.

If

$$\sup_{\substack{\varphi \in C_c^\infty \\ ||\varphi||_\infty < 1}} \left| \int_0^1 f(t)\varphi'(t)dt \right| = \sup_{\substack{\varphi \in C_c^\infty \\ ||\varphi||_\infty < 1}} \left| \int_0^1 \varphi(t)v(t)dt \right|,$$

then by fundamental lemma of calculus of variations we have $\mu_s((0,t)) \equiv c$ for some constant. Hence, $\mu_s \equiv 0$ and f is absolutely continuous. □

Part 4: Proofs.

Defn 3: A Borel $\mathbb{R}$-valued measure μ on (a,b) is a **$\mathbb{R}$-valued** countable additive set function $\mu : \mathfrak{B} \to \mathbb{R}$. Then we have the total variation measure $|\mu|$ defined by

$$|\mu|(B) = \sup\left\{\sum |\mu(B_i)| : B_i \in \mathfrak{B} \text{ pairwise disjoint, } \cup B_i = B\right\}$$

Lemma: $|\mu|$ is a non-negative Borel measure. □

Define a norm $||\mu||_{\mathcal{M}} = |\mu|(a,b) =$ "total variation of μ"; and set $\mathcal{M} = \left\{\mu \text{ - a Borel measure} : ||\mu||_{\mathcal{M}} < \infty\right\}$.

Proof of proposition 1.

(a) Let $f(x) = \mu((0,x))$. Given $0 < x_0 < \cdots < x_n < 1$, observe

$$\begin{aligned} \sum |f(x_{i+1}) - f(x_i)| &= \sum |\mu([x_i, x_{i+1}))| \\ &\le \sum |\mu|([x_i, x_{i+1})) \\ &= |\mu|([x_1, x_n)) \\ &\le |\mu|((0,1)) \end{aligned}$$

(a) We know $\mu((0,x)) = f(x) - f\left(0^+\right)$. We should also have $\mu([a,b]) = f\left(b^+\right) - f(a)$. The task is it extend this to all Borel sets to make it a

Borel measure.
For simplicity assume f is non-decreasing. Associate a Borel measure μ in the following way: given any Borel set $E \subseteq (0,1)$, put

$$\mu(E) := \mathfrak{L}\left(\bigcup_{x\in E}\left[f(x), f\left(x^+\right)\right]\right)$$

Firstly, the giant union inside $\mathfrak{L}$ is a Borel set becuase

$$\bigcup_{x\in E}\left[f(x), f\left(x^+\right)\right] = f^{-1}(E) \cup \bigcup_{x\in\Delta}\left[f(x), f\left(x^+\right)\right]$$

where Δ is the countable set of discontinuities.
Secondly, μ is a countable set function, because given $E = \sqcup_n E_n$ we have

$$\begin{aligned}\mu(\sqcup_n E_n) &= \mathfrak{L}\left(\bigcup_n \bigcup_{x\in E_n}\left[f(x), f\left(x^+\right)\right]\right)\\ &= \sum_n \mathfrak{L}\left(\bigcup_{x\in E_n}\left[f(x), f\left(x^+\right)\right]\right)\end{aligned}$$

This μ satisfies the required properties.
Note: For general NBV functions we write $f = f_1 - f_2$ where f_i are non-decreasing NBV functions. (Ex: $2f = (T_f + f) - (T_f - f)$)

(c) This follows from $\mu(\{x\}) = f\left(x^+\right) - f(x)$. □

Proof of proposition 3:
Given $f \in BV((0,1))$, proposition 1 says $\exists\, \mu$ s.t. $f(x) = \mu((0,x))$. Now observe,

$$\begin{aligned}\int_0^1 f(t)\varphi'(t)dt &= \int_0^1 \mu((0,t))\varphi'(t)\,dt\\ &= \int_0^1\left(\int_0^1 \mathbb{1}\{s\le t\}d\mu(s)\right)\varphi'(t)\,dt\\ &= \int_0^1\left(\int_0^1 \mathbb{1}\{s\le t\}\varphi'(t)dt\right)d\mu(s)\\ &= \int_0^1(\varphi(1)-\varphi(s))d\mu(s)\\ &= -\int_0^1 \varphi(s)d\mu(s)\\ &= -\int_0^1 \varphi(s)d(v\mathfrak{L})(s) - \int_0^1 \varphi(s)d\mu_s(s)\\ &= -\int_0^1 \varphi(s)v(s)ds + \int_0^1 \mu_s((0,t))\varphi'(t)\,dt\end{aligned}$$ □

Proof of proposition 5:
Define three linear functionals:

$$L_1(\varphi) = \int_0^1 f(t)\varphi'(t)dt = -\int_0^1 \varphi(t)d\mu(t)$$
$$L_2(\varphi) = -\int_0^1 \varphi(t)v(t)dt$$
$$L_3(\varphi) = \int_0^1 \mu_s((0,t))\varphi'(t)dt = -\int_0^1 \varphi(s)d\mu_s(s)$$

By taking φ out of the integral, it is easy to see

- $||L_1|| \leq ||\mu||$
- $||L_2|| \leq ||v||_1 = ||v\mathfrak{L}||$
- $||L_3|| \leq ||\mu_s||$

But we would like to have an equality. We use proposition 4 to deduce this.

Write $d\mu = h \cdot d|\mu|$ where $h = \mathbb{1}_{\mu+} - \mathbb{1}_{\mu-}$. Now get a sequence $\varphi_n \in C_c^\infty((0,1))$ such that $\varphi_n \to h$ in $L^1(|\mu|)$ and $||\varphi||_\infty \to 1$. With this $L(\varphi_n) \to |\mu|((0,1))$. Hence we get $||L|| = ||\mu||$.

Once we have equality, recall $\mu = v\mathfrak{L} + \mu_s$ and $v\mathfrak{L} \perp \mu_s$. This gives $||\mu|| = ||v\mathfrak{L}|| + ||\mu_s||$ from which we get $||L_1|| = ||L_2|| + ||L_3||$. □

Proof of proposition 6:
It suffices to prove the following lemma.
Lemma: Let μ be a singular measure and $g(x) = \mu((0,x))$. Then, $g'(x) = 0$ a.e.
Proof:
(WLOG assume μ is a non-negative measure.
If there were a Lebesgue-mesure-zero compact set K such that $\mathrm{supp}(\mu) \subseteq K$, then it is easy to see that $g'(x) = 0$ for all $x \notin K$, and we would get $g'(x) = 0$ a.e.
Now such a K might not exist. It might be the case that $\overline{\mathrm{supp}(\mu)} = [0,1]$.
Fix a $\epsilon, \lambda > 0$. The regularity of μ tells us that there is a measure-zero compact set K, with $\mu(K) > ||\mu|| - \epsilon$.
Define $\mu_K(E) = \mu(E \cap K)$, and put $\mu_\epsilon = \mu - \mu_K$. Then $||\mu_\epsilon|| < \epsilon$, and for every x outside K, we can have

$$(\overline{D}\mu)(x) = (\overline{D}\mu_\epsilon)(x)$$

where

$$(\overline{D}\mu)(x) = \lim_{n\to\infty} \sup_{r<1/n} \frac{\mu(B(x,r))}{\mathfrak{L}(B(x,r)}.$$

Hence,

$$\{\overline{D}\mu > \lambda\} \subseteq K \cup \{\overline{D}\mu_\epsilon > \lambda\}$$

As $\epsilon \to 0$, we expect Lebesgue measure of $\{\overline{D}\mu_\epsilon > \lambda\}$ to go to zero. Thus we get

Lebesgue measure of $\{\overline{D}\mu > \lambda\}$ is zero for all $\lambda > 0$. $\square$

References

1. Chapter 2, Giuseppe Buttazzo, Mariano Giaquinta, and Stefan Hildebrandt (1998). *One-dimensional variational problems*. Oxford Lecture Series in Mathematics and its Applications.
2. Walter Rudin (1987). *Real and complex analysis*. McGraw-Hill Book Co., New York.