Complex Dynamics

The study of iterated holomorphic maps

Mohith Raju
(IISc Bangalore)

Mentor: Prof Sabyasachi Mukherjee

What is complex dynamics?

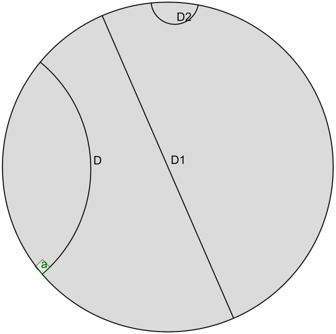
Objects

1. A holomorphic function $f: X \to X$
   e.g. $f(z) = \dfrac{p(z)}{q(z)}$ and $g(z) = a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$
2. A Riemann surface X
   e.g. $X = \mathbb{D}, \mathbb{C}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

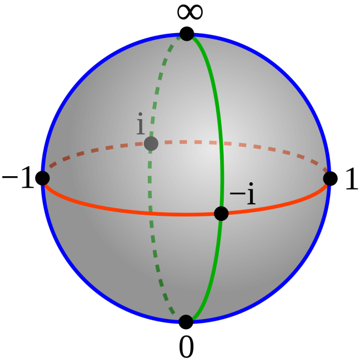
Questions

1. What happens when f is iterated, i.e., $f \circ f \circ \cdots \circ f$
2. Given a point $z \in X$, what does the forward orbit of z look like?
   Forward orbit of z is $(z, f(z), f^{\circ 2}(z), f^{\circ 3}(z), \cdots)$
   - **Periodic orbit:** $f^{\circ n}(z) = z$
   - **Convergence:** $\exists \{n_k\}$ such that $f^{\circ n_k}(z) \to \hat{z}$
   - **Divergence/Escape:** for every $K \subset\subset X$, $\exists N$ such that $f^{\circ n}(z) \notin K$ for all $n \geq N$

Examples of Riemann surfaces



The open unit disc
$\mathbb{D} = \{|z| < 1\}$



The Riemann sphere
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

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1. What happens when f is iterated, i.e., $f \circ f \circ \cdots \circ f$
2. Given a point $z \in X$, what is the forward orbit of z?
   (Periodic orbit, convergence, divergence/escape)
3. Given a small nbd N_z of z, what is the forward orbit of N_z?
   Forward orbit of z is $(N_z, f(N_z), f^{\circ 2}(N_z), f^{\circ 3}(N_z), \cdots)$

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2. Given a point $z \in X$, what is the forward orbit of z?
   (Periodic orbit, convergence, divergence/escape)
3. Given a small nbd N_z of z, what is the forward orbit of N_z?
   - Invariant: $f(N_z) = N_z$
   - Contraction: diameter of $f^{\circ n}(N_z)$ shirks to 0 as $n \to \infty$
   - Convergence: $\exists \{n_k\}$ such that $f^{\circ n_k}|_{N_z}: N_z \to X$ converges
   - Divergence/Escape: for every $K \subset\subset X$, $\exists N$ such that $f^{\circ n}(N_z) \cap K = \emptyset$ for all $n \geq N$
   - Spread out: For arbitrary small N_z, the union $\bigcup_{n=1}^{\infty} f^{\circ n}(N_z) = X \setminus \{2 \text{ or fewer exceptional points}\}$

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Questions

1. What happens when f is iterated, i.e., $f \circ f \circ \cdots \circ f$
2. Given a point $z \in X$, what is the forward orbit of z?
   (Periodic orbit, convergence, divergence/escape)
3. Given a small nbd N_z of z, what is the forward orbit of N_z?
   (invariant, contraction, convergence, divergence, spread out)

Julia and Fatou sets

Fatou set $= \{z \in X \colon N_z$ shows convergence or divergence$\}$

Julia set $= \{z \in X \colon N_z$ spreads out$\}$

Fact. $X =$ Fatou set $\bigsqcup$ Julia set

Example 1. $z \mapsto z^2$

- $f(z) = z^2$
- $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $f(0) = 0$ (fixed point)
- $f(\infty) = \infty$ (fixed point)

Question 1. What is the forward orbit of neighbourhoods of 0 and ∞?

Hint. $f^{\circ n}(z) = z^{2^n}$

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Ans. Let $N_0 := \{|z| < 1/2\}$ and $N_\infty := \{|z| > 2\}$.
Note the sets $f^{\circ n}(N_0)$ and $f^{\circ n}(N_\infty)$ shrink to a singleton as $n \to \infty$.

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Ans. Let $N_0 := \{|z| < 1 - \epsilon\}$ and $N_\infty := \{|z| > 1 + \epsilon\}$. Note the sets $f^{\circ n}(N_0)$ and $f^{\circ n}(N_\infty)$ shrink to a singleton as $n \to \infty$.

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Question 2. What happens to points on the circle $S^1 = \{|z| = 1\}$?

First step. Consider the point $z = e^{i\theta}$ where $\theta \in \mathbb{Q}^c$.

Example 1. $z \mapsto z^2$

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Question 2. What happens to points on the circle $S^1 = \{|z| = 1\}$?

First step. Consider the point $z = e^{i\theta}$ where $\theta \in \mathbb{Q}^c$.

Ans/HW. For any point $z \in S^1$ and any neighbourhood N_z of z, we have $\bigcup_{n=1}^{\infty} f^{\circ n}(N_z) = \hat{\mathbb{C}} \setminus \{0, \infty\}$.

Conclusion.

Fatou set $= \{|z| < 1\} \cup \{|z| > 1\}$

Julia set $= S^1$

Example 1. $z \mapsto z^2$

- $f(z) = z^2$
- $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $f(0) = 0$ (attracting fixed point)
- $f(\infty) = \infty$ (attracting fixed point)
- $f'(0) = 0$
- $f'(\infty) = 0$ (∞ is a double "root" of $f(z) = \infty$)

Example 2. $z \mapsto 2z$

- $f(z) = 2z$
- $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $f(0) = 0$ (fixed point)
- $f(\infty) = \infty$ (fixed point)
- $f^{\circ n}(z) = 2^n z$
- Let $N_\infty = \hat{\mathbb{C}} \setminus 0$ be a nbd of ∞
- $f^{\circ n}(N_\infty)$ shrinks to a singleton set $\{\infty\}$
- $\cup_n f^{\circ n}(B(0, \epsilon)) = \mathbb{C} = \hat{\mathbb{C}} \setminus 0$

Example 2. $z \mapsto 2z$

- $f(z) = 2z$
- $X = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $f(0) = 0$ (repelling fixed point)
- $f(\infty) = \infty$ (attracting fixed point)
- $f^{\circ n}(z) = 2^n z$
- Let $N_\infty = \hat{\mathbb{C}} \setminus 0$ be a nbd of ∞
- $f^{\circ n}(N_\infty)$ shrinks to a singleton set $\{\infty\}$
- $\cup_n f^{\circ n}(B(0,\epsilon)) = \mathbb{C} = \hat{\mathbb{C}} \setminus 0$
- Fatou set $= \hat{\mathbb{C}} \setminus 0$
- Julia set $= \{0\}$
- $f'(0) = 2$
- $f'(\infty) = 1/2$

Fixed points

- The fixed points of $f\colon X \to X$ contains a lot of information about the dynamics
- If $f(p) = p$ and $|f'(p)| < 1$, then p is an attracting fixed point and a small nbd $N_p \subseteq$ Fatou set
- If $f(p) = p$ and $|f'(p)| > 1$, then p is a repelling fixed point

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- If $f(p) = p$ and $|f'(p)| > 1$, then p is a repelling fixed point and $p \in$ Julia set

Theorem.
If $f(p) = p$ and $|f'(p)| > 1$, then any nbd N_p of p spreads out under f, that is, $\bigcup_{n=1}^{\infty} f^{\circ n}(N_p) = \hat{\mathbb{C}} \setminus \{2 \text{ or lesser points}\}$.

Remark. The proof uses "uniformization theorem" which is difficult to prove.
(Involves solving a difficult PDE $\Delta u = \rho$ on a Riemann surface.)

A question: do higher order terms matter?

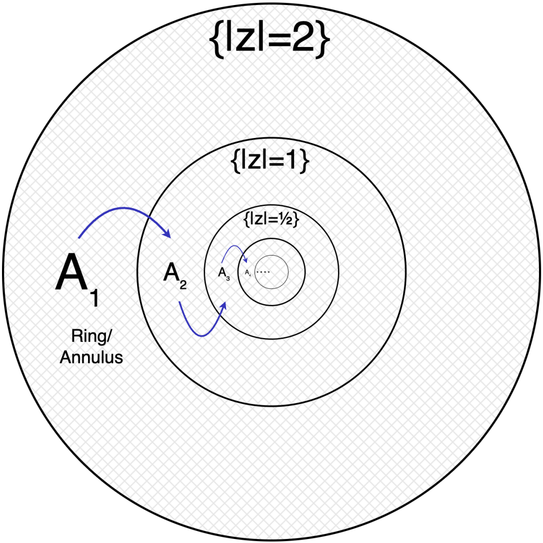
- Consider two maps f, g
- $f(z) = z/2$
- $g(z) = z/2 + a_2 z^2$
- Both f and g have an attracting fixed point at 0
- Are the dynamics of f and g same near 0?

Question 3. Does adding higher order terms affect the dynamics?

Question 4. When are two dynamics considered to be the same?

An interesting phenomenon (called X/f)

Consider $f(z) = z/2$



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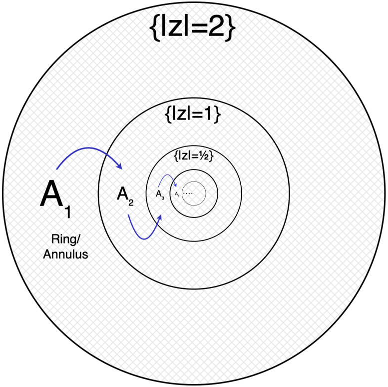

Question 5. Given $g(z) = z/2 + a_2 z^2$ can we **locally** find a system of closed curves and annuli $\{A_n\}$ such that g maps A_n homeomorphically onto A_{n+1}?

When are two dynamics the same?

One version of equivalence of dynamics.
Given $f\colon X \to X$ and $g\colon Y \to Y$, we say (f, X) and (g, Y) are *holomorphically* equivalent if $\exists$ a bijective *holomorphic* map $\varphi\colon X \to Y$ such that $g = \varphi^{-1} f \varphi$ or the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & X \arrow[d, dashed, "\varphi"'] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Want. Let $f(z) = z/2$ and $g(z) = z/2 + a_2 z^2$. Then $\exists$ nbds U, V of 0 such that (f, U) and (g, V) are *holomorphically* equivalent.

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$$\begin{tikzcd} U \arrow[r, "z/2 + a_2 z^2"] \arrow[d, dashed, "\varphi"'] & U \arrow[d, dashed, "\varphi"'] \\ V \arrow[r, "z/2"'] & V \end{tikzcd}$$

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Koenigs linearization

Theorem.

Suppose $f(z) = z/2$ and $g(z) = z/2 + a_2 z^2$. Then $\exists$ nbds U, V of 0 such that (f, U) and (g, V) are *holomorphically* equivalent.

Key idea. Work backwards and "bootstrap".

$$\begin{tikzcd}
U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[d, dashed, "\varphi"'] \\
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If φ exists, then WLOG we can assume $\varphi(0) = 0$ and $\varphi'(0) = 1$

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$$\begin{tikzcd}
z \arrow[rrrr, bend left] & & & & g^n(z) \\
U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[r, dashed] \arrow[d, dashed, "\varphi"'] & U \arrow[d, dashed, "\varphi"'] \arrow[d, bend left] \\
V \arrow[r, "z/2"'] & V \arrow[r, "z/2"'] & V \arrow[r, "z/2"'] & V \arrow[r, dashed] & V \\
\approx 2^n \cdot g^n(z) & & & & \approx g^n(z) \arrow[llll, bend left]
\end{tikzcd}$$

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$$
\begin{tikzcd}
z & & & & g^n(z) \\
U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] \arrow[rrrr, bend left, blue] & U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[r, "g(z)"] \arrow[d, dashed, "\varphi"'] & U \arrow[r, dashed] \arrow[d, dashed, "\varphi"'] & U \arrow[d, dashed, "\varphi"'] \arrow[d, bend left, blue] \\
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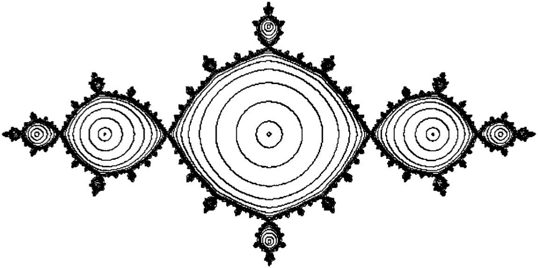
Define $\varphi(z) := \lim_n \left(2^n g^{\circ n}(z)\right)$ $\square$



Thank you

Milnor, *Dynamics in One Complex Variable*