Global Laplace comparison and minimum value principle$_b$

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Recall

A. Local Laplace **comparison**. Given complete (M, g), suppose $\mathrm{Rc}_g \geq (n-1)\kappa$ for some $\kappa \in \mathbb{R}$. Then, for $p \in M$,

$$\Delta \rho_p(x) \leq (n-1) \cdot \frac{\mathrm{sn}'_\kappa(\rho_p(x))}{\mathrm{sn}_\kappa(\rho_p(x))} \quad \forall\, x \in M \setminus \{p \cup \mathcal{C}_p\}$$

B. The cut locus C_p has measure zero

Goals

1. **Global** Laplace comparison. Suppose $\mathrm{Rc}_g \geq (n-1)\kappa$ for some $\kappa \in \mathbb{R}$. Then, for $p \in M$,

$$\Delta\rho_p(x) \leq_b (n-1) \cdot \frac{\mathrm{sn}'_\kappa(\rho_p(x))}{\mathrm{sn}_\kappa(\rho_p(x))} \quad \forall\, x \in M \setminus \{p\}.$$

barrier sense
defined later

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2. Regularity$_b$. Suppose a continuous f is harmonic$_b$, i.e., $\Delta f(x) \leq_b 0$ and $-\Delta f(x) \leq_b 0$ for all x. Then f is smooth.

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2. Regularity$_b$. Suppose a continuous f is harmonic$_b$, i.e., $\Delta f(x) \leq_b 0$ and $-\Delta f(x) \leq_b 0$ for all x. Then f is smooth.

3. Strong minimum principle$_b$. Suppose a continuous f is superharmonic$_b$:

$$\Delta f(x) \leq_b 0 \quad \forall\, x \in \Omega \subseteq M.$$

   If f has a local minimum at $x^* \in \Omega$, then f is constant on some nbd N_{x^*} of x^*.

Alternatively one can show

1. Laplace comparison in distributional sense. Suppose $\mathrm{Rc}_g \geq (n-1)\kappa$ for some $\kappa \in \mathbb{R}$. Then, for $p \in M$,

$$\Delta \rho_p \leq_d (n-1) \cdot \frac{\mathrm{sn}'_\kappa \circ \rho_p}{\mathrm{sn}_\kappa \circ \rho_p} \quad \text{in } M \setminus \{p\}.$$

2. Weyl's lemma. Suppose a L^1_{loc} function f is harmonic$_d$, i.e., $\Delta f \leq_d 0$ and $-\Delta f \leq_d 0$. Then f is smooth.

3. Strong minimum principle$_d$. Suppose a continuous f is superharmonic$_d$:

$$\Delta f \leq_d 0 \quad \text{in } \Omega \subseteq M.$$

If f has a local minimum at $x^* \in \Omega$, then f is constant on some nbd N_{x^*} of x^*.

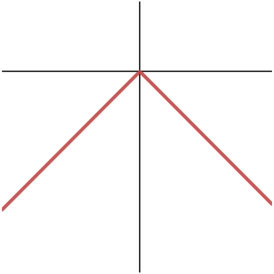
What is barrier sense
and why is it important?

Barrier function

Definition 1. Given a continuous function f, a smooth function f_ϵ is said to be a barrier function from above at the point x^* if:

- $f(x^*) = f_\epsilon(x^*)$
- $f(x) \leq f_\epsilon(x)$ for all $x \in B(x^*, \delta)$

Example 1.



Continuous function: $f(x) = -|x|$

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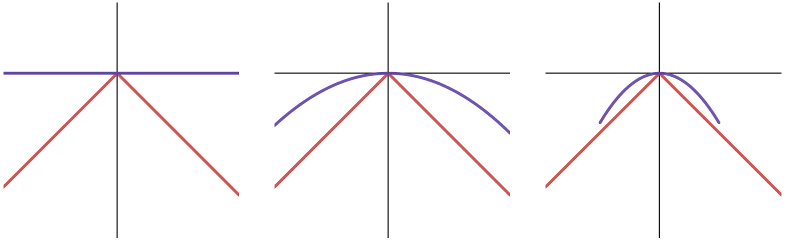


Barrier functions: $f_\epsilon(x) = -a|x|^2$

Barrier sense inequality

Definition 2. For a continuous function f we say

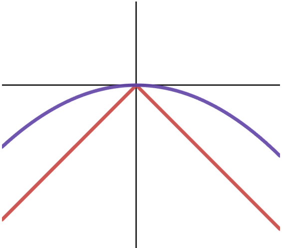
$$\Delta f(x^*) \leq_b c$$

if $\exists\, f_\epsilon$, a barrier function from above such that

$$\Delta f_\epsilon(x^*) \leq c + \epsilon\,.$$

Upshot: Barrier functions estimate Laplacian from above

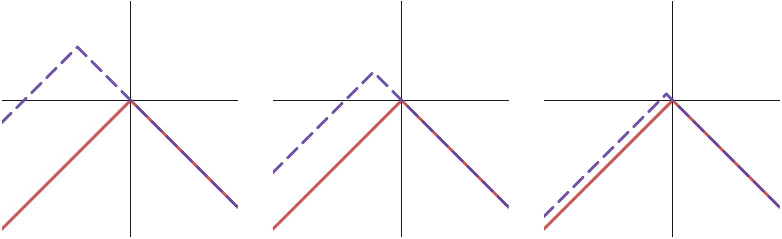
$$\Delta f(x^*) \leq_b \Delta f_\epsilon(x^*)$$



Warmup Lemma. For $f(x) = -|x|$ we have:

$$\Delta f(0) \leq_b \lim_{\epsilon \to 0} \Delta f(\epsilon) = 0.$$

Proof:



$$f_\epsilon(x) := f(x+\epsilon) + \epsilon$$

□

Barrier function f_ϵ from above:

* $f(0) = f_\epsilon(0)$
* $f(x) \leq f_\epsilon(x) \quad \forall\, x \in B(0, \delta)$

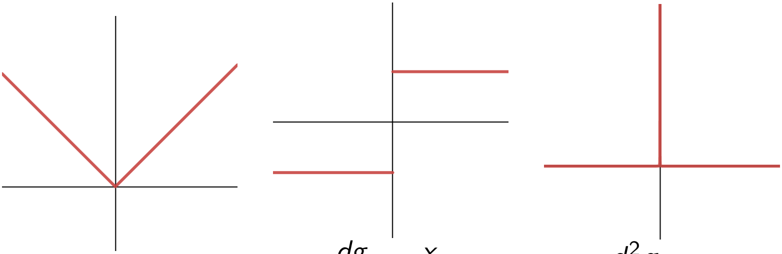
Barrier sense detects the Dirac delta

Caution. For $g(x) = +|x|$:

$$\Delta g(0) \not\leq_b \lim_{\epsilon \to 0} \Delta g(\epsilon) = 0\,.$$

Why:

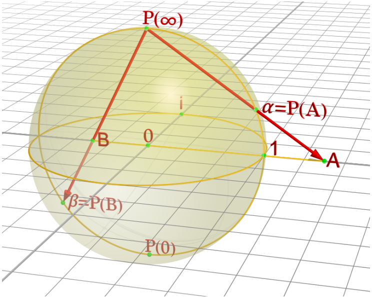
- $\Delta g(0) = +2\delta_0 \approx +\infty$ (Dirac delta distribution)
- $\Delta f(0) = -2\delta_0 \approx -\infty$



$g(x) = +|x|$ $\qquad$ $\dfrac{dg}{dx} = \dfrac{x}{|x|}$ $\qquad$ $\dfrac{d^2 g}{dx^2} = 0$

Source: Desmos calculator

Global Laplace comparison for the 2-sphere



Distance function on the 2-sphere

Proposition. Let $M = \mathbb{S}^2$ and $p = \mathtt{NP}$. Note $\mathcal{C}_p = \{\mathtt{SP}\}$. Given ρ_p we have

$$\begin{aligned}\Delta\rho_p(\mathtt{SP}) &\leq_b (n-1)\cdot\frac{\mathrm{sn}'_\kappa(\rho_p(\mathtt{SP}))}{\mathrm{sn}_\kappa(\rho_p(\mathtt{SP}))}\\ &= \lim_{t\to\pi}(n-1)\cdot\frac{\mathrm{sn}'_\kappa(t)}{\mathrm{sn}_\kappa(t)}.\end{aligned}$$

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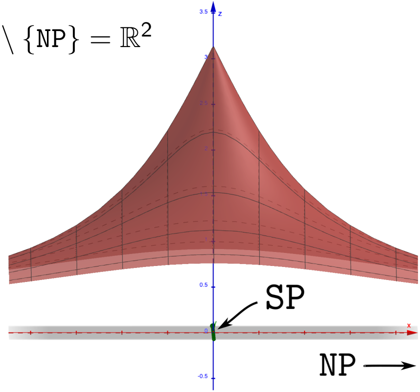
$$\Delta\rho_p(\mathtt{SP}) \leq_b (n-1)\cdot\frac{\mathrm{sn}_\kappa'(\rho_p(\mathtt{SP}))}{\mathrm{sn}_\kappa(\rho_p(\mathtt{SP}))}$$

Proof.

- Use steriographic projection: $\mathbb{S}^2 \setminus \{\mathtt{NP}\} = \mathbb{R}^2$
- Work with $\left(\mathbb{R}^2, \dfrac{4\delta_{ij}}{(1+|x|^2)^2}\right)$
- $\rho_{\mathtt{NP}}(x) = \pi - 2\arctan(|x|)$



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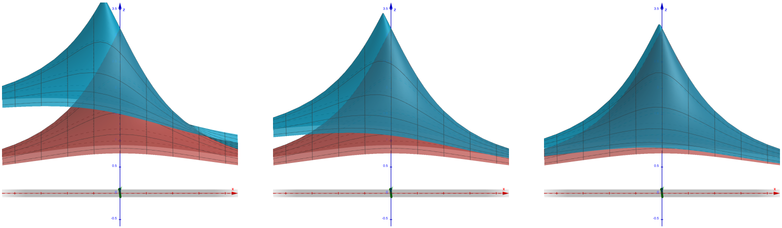

$$f_\epsilon(x) := \rho_{\mathtt{NP}+\epsilon}(x) + \epsilon$$

QED??

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Proof.

- **Calabi's idea**: $f_\epsilon(x) = \rho_{\mathtt{NP}+\epsilon}(x) + \epsilon$
- Let σ be unit speed minimal geodesic from $p = \mathtt{NP}$ to the $\mathtt{SP}$
- $\rho_{\mathtt{NP}+\epsilon}(x) = d(\text{“}\mathtt{NP} + \epsilon\text{”}, x) = d(\sigma(\epsilon), x)$

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- Properties:
  1. $\rho_p(\mathtt{SP}) = f_\epsilon(\mathtt{SP})$ – because of the $+\epsilon$
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  3. f_ϵ is smooth in a nbd of $\mathtt{SP}$
     - because σ is minimal
     - there is no pair of conjugate points on σ other than $\{\mathtt{NP}, \mathtt{SP}\}$

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- **Calabi's idea**: $f_\epsilon(x) = \rho_{p+\epsilon}(x) + \epsilon$
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  3. f_ϵ is smooth in a nbd of $\mathtt{SP}$ – because σ is minimal
- Conclusion:

$$\Delta \rho_p(\mathtt{SP}) \leq_b \Delta \rho_{p+\epsilon}(\mathtt{SP}) \leq (n-1) \cdot \frac{sn'_\kappa(\rho_{p+\epsilon}(\mathtt{SP}))}{sn'_\kappa(\rho_{p+\epsilon}(\mathtt{SP}))} \qquad \square$$

Minimum principle for barrier functions

Minimum$_b$ principle

Minimum principle$_b$. Suppose Ω is open and $x^* \in \Omega$

$$\left.\begin{array}{c} \Delta f(x) \leq_b 0 \qquad \forall\, x \in \Omega \\ f \text{ has local minima at } x^* \end{array}\right\} \implies f \equiv \text{ const. on } B_\epsilon(x^*).$$

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Proof.

Proceed in steps:

1. If f smooth and $\Delta f < 0$, then f has no local minima
2. If f cts and $\Delta f(x) \leq_b -c$ for all $x \in \Omega$, then f has no local minima
3. Prove the general case

Next:

* (1) is easy
* Prove (2) using (1)
* Prove (3) using (2)

(2) If $\Delta f(x) \leq_b -c$ then f has no local minima

$$\text{Given:} \quad \left\{ \; \Delta f(x) \leq_b -c \qquad \forall x \in \Omega \; \right\}$$

$$\text{Use barrier fns} \implies \left\{ \begin{array}{cc} f(x^*) = f_\epsilon(x^*) & \\ f(x) \leq f_\epsilon(x) & \forall x \in B_\delta(x^*) \\ \Delta f_\epsilon(x) \leq -c/2 & \forall x \in B_\delta(x^*) \end{array} \right\}$$

$$\text{By smooth case} \implies \left\{ \begin{array}{cc} f(x^*) = f_\epsilon(x^*) & \\ f(x) \leq f_\epsilon(x) & \forall x \in B_\delta(x^*) \\ f_\epsilon \text{ has no local minima} & \end{array} \right\}$$

$$\text{Using } f \leq f_\epsilon \implies \{ f \text{ has no local minima} \}$$

$\square$

(3) Minimum principle for $\Delta f(x) \leq_b 0$

General case of minimum principle:

$$\left\{ \begin{array}{c} \Delta f(x) \leq_b 0 \quad \forall x \in \Omega \\ f \text{ has local minima at } x^* \end{array} \right\} \overset{\text{want}}{\Longrightarrow} f \equiv \text{ const. on } B_\epsilon(x^*)$$

Rewrite:

$$\left\{ \begin{array}{c} \Delta f(x) \leq_b 0 \quad \forall x \in B_\delta(x^*) \\ f \text{ has local minima at } x^* \\ f \text{ is } \textbf{not} \text{ const. on } B_\epsilon(x^*) \end{array} \right\} \overset{\text{want}}{\Longrightarrow} \text{contradiction}$$

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$$\left\{ \begin{array}{c} \Delta(f+h)(x) \leq_b -c \quad \forall x \in B_\delta(x^*) \\ f+h \text{ has local minima at } x^{**} \end{array} \right\}$$

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???

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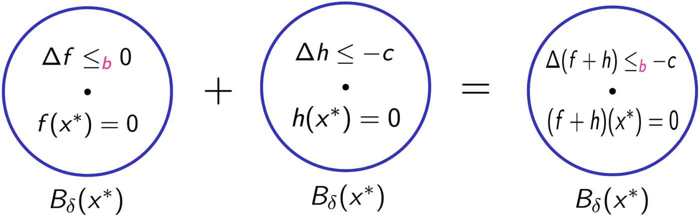
Delicate construction of h:



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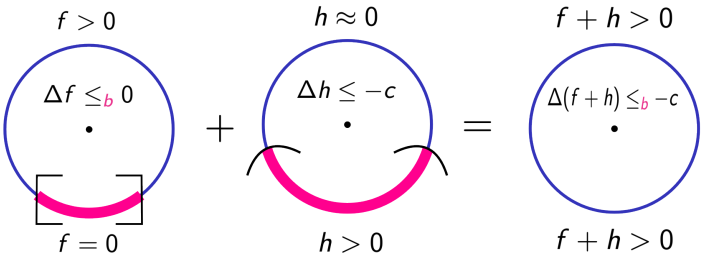
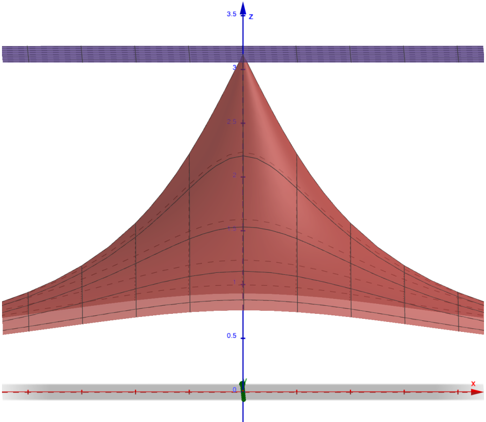
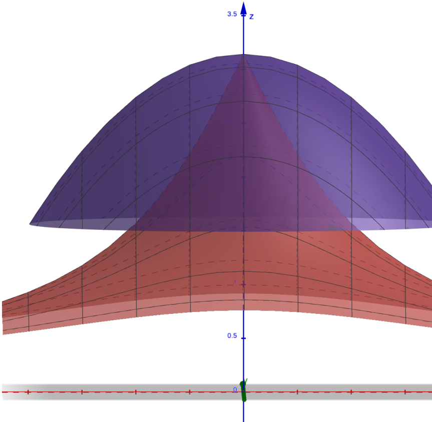


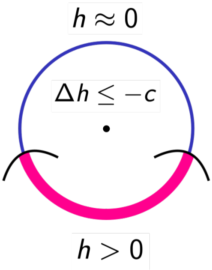
Thank you





Appendix A: Construction of h

Construction of h



Want: h such that

1. $h(x^*) = 0$
2. $h > 0$ on $V \subseteq \partial B_\delta(x^*)$
3. $\Delta h < 0$ on $\overline{B_\delta(x^*)}$

Proof.

- Put $h := 1 - e^{\alpha\varphi}$, where $\alpha \gg 1$ and $\varphi \in C^\infty(M)$
- $\Delta h = -\alpha e^{\alpha\varphi}(\alpha|\nabla\varphi|^2 + \Delta\varphi)$

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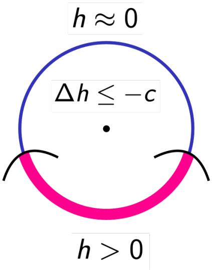

Proof.

- Put $h := 1 - e^{\alpha\varphi}$, where $\alpha \gg 1$ and $\varphi \in C^\infty(M)$
- $\Delta h = -\alpha e^{\alpha\varphi}(\alpha|\nabla\varphi|^2 + \Delta\varphi)$
- Choose φ such that
  - $\varphi(x^*) = 0$
  - $\varphi < 0$ on $V \subseteq \partial B_\delta(x^*)$
  - $\nabla\varphi \neq 0$ on $\overline{B_\delta(x^*)}$

e.g. $\varphi(x) = x_n$ $\square$

Appendix B: Regularity of harmonic$_b$ functions

Results

1. Minimum principle$_b$. Suppose Ω is open and $x^* \in \Omega$

$$\left.\begin{array}{ll} \Delta f(x) \leq_b 0 & \forall\, x \in \Omega \\ f(x^*) \leq f(x) & \forall\, x \in B_\delta(x^*) \end{array}\right\} \implies f \equiv \text{ const. on } B_\epsilon(x^*)$$

2. Uniqueness$_b$. Suppose $\Omega \subseteq M$ is relatively compact. Then,

$$\left.\begin{array}{rl} \Delta f_1(x) =_b 0 & \forall\, x \in \Omega \\ \Delta f_1(x) =_b 0 & \forall\, x \in \Omega \\ f_1(x) = f_2(x) & \forall\, x \in \partial\Omega \end{array}\right\} \implies f_1 \equiv f_2 \text{ on } \Omega$$

3. Local existence. Suppose $x^* \in M$ and $\delta \ll 1$

$$\text{Given } g \in C(\partial B_\delta(x^*)) \implies \left\{\begin{array}{c} \exists\, f \in C^\infty(B_\delta(x^*)) \cap C(\overline{B_\delta(x^*)}) \text{ s.t.} \\ \Delta f(x) = 0 \quad \forall\, x \in B_\delta(x^*) \\ f(x) = g(x) \quad \forall\, x \in \partial B_\delta(x^*) \end{array}\right.$$

4. Regularity$_b$.

$$\left.\begin{array}{c} \text{Given } f \in C(M) \\ \Delta f(x) =_b 0 \quad \forall\, x \in M \end{array}\right\} \implies f \in C^\infty(M)$$

Minimum principle$_b$ $\implies$ Regularity$_b$

1.

$$\text{Minimum principle}_b \implies \text{Uniqueness}_b$$

2.

$$\left.\begin{array}{c}\text{Uniqueness}_b \\ \text{Local existence}\end{array}\right\} \implies \text{Regularity}_b$$

$\square$

Conclusion. Regularity$_b$ of harmonic$_b$ functions follows from minimum principle$_p$ and local existence.