Gauge theory and special connections

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1. Introduction

Gauge theory is the study of connections (geometric objects) on principal bundles or vector bundles. A basic example is that of flat connections. These are connections that have zero curvature. In this note, we introduce two other special classes of connections – Yang-Mills connections and anti-self-dual connections – as a generalization of flat connections. These connections are special in that they are solutions to certain non-linear partial differential equations (PDE). Principal $SU(2)$-bundles over the 4-dimensional sphere S^4 are important examples where we can explicitly describe and write down these connections. Hence, we begin this note by describing these bundles. Lastly, we have listed and explained our notations in the appendix.

Remark. Some of the most important uses of gauge theory and the special connections we describe have been to construct new invariants of smooth manifolds and certain exotic geometric structures. In fact, this is what makes these connections mathematically interesting. Unfortunately, we do not discuss these results in this note. For a glimpse of these results, the interested reader may refer to [UF12, Chapter 1]. For a detailed discussion, see [DK97].

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2. Principal $SU(2)$-bundles over S^4

Given an integer $k \in \mathbb{Z}$, we describe a principal $SU(2)$-bundle $P^{(k)}$ over S^4. Note that S^4 is covered by two open sets $U_N := S^4 \setminus \{\texttt{SP}\}$ and $U_S := S^4 \setminus \{\texttt{NP}\}$. Now, a principal $SU(2)$-bundle $P^{(k)}$ over S^4 can be described by a transition map

$$\tau^{(k)} : U_N \cap U_S \to SU(2)\,.$$

Next, let $r : U_N \cap U_S \to S^3$ be a retraction of $U_N \cap U_S$ onto the equatorial S^3 of S^4 and $\eta : S^3 \to SU(2)$ be the standard identification of $SU(2)$ with S^3. Finally, the transition map $\tau^{(k)} : U_N \cap U_S \to SU(2)$ is defined as:

$$\tau^{(k)}(x) := \eta(r(x))^k\,,$$

where the exponentiation is computed using group multiplication and group inversion of $SU(2)$. (This construction is a special case of the so-called *clutching construction.*) It turns out that there are no other principal $SU(2)$-bundles other than the integer-many bundles $P^{(k)}$ obtained from $\tau^{(k)}$ as above.

Proposition 1. *Every $SU(2)$-bundle over S^4 is isomorphic to one of the $P^{(k)}$. Further, with the standard orientation and metric on S^4, the second Chern number of $P^{(k)}$ is exactly k,*[1] *i.e.,*

$$\int_{S^4} c_2(P^{(k)}) = k\,.$$

In particular, the $P^{(k)}$ are distinct.

Proof idea. Given an arbitrary principal $SU(2)$-bundle P over S^4, it can be trivialized over U_N and U_M (i.e., $P|_{U_N} \simeq U_N \times SU(2)$ and $P|_{U_S} \simeq U_S \times SU(2)$) because U_N and U_M are contractible open sets. Now, we have a single transition map $\tau : U_N \cap U_S \to SU(2)$ describing P. Next, it turns out that (i) the isomorphism class of P only depends on the homotopy class of the map $\tau : U_N \cap U_S \to SU(2)$, (ii) the homotopy class of a map $\tau : (U_N \cap U_S) \simeq S^3 \to SU(2) \simeq S^3$ is completely determined by its degree (an integer), and (iii) the degree of the map $\tau^{(k)}$ constructed above is exactly k. From these three facts it follows that P is isomorphic to one of the $P^{(k)}$.

To compute the second Chern number of $P^{(k)}$, recall that we need to pick an $SU(2)$-connection A and compute

$$\frac{1}{8\pi^2} \int_{S^4} \mathrm{Tr}(F_A \wedge F_A).$$

[1] If the orientation of S^4 is reversed, the Chern number changes sign.

From Chern-Weil theory, we know that the integral does not depend on the choice of connection, so we pick a connection $A^{(k)}$ on $P^{(k)}$ that conveniently 'vanishes' near the poles and is 'concentrated' in a neighborhood of the equitorial S^3. For a such a connection, we have:

$$\frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}(F_{A^{(k)}} \wedge F_{A^{(k)}}) = \frac{1}{8\pi^2}\int_{S^3} \mathrm{Tr}\left(a^{(k)} \wedge da^{(k)} + \frac{2}{3}a^{(k)} \wedge a^{(k)} \wedge a^{(k)}\right).$$

Moreover, it turns out that the 3-form on the RHS is the pullback of the volume form on S^3 using a degree k map, up to a scalar factor. As a result of this, the expression in the RHS evaluates to k. For more details, see [Tau11, Example 14.11]. □

3. Flat connections

Let A be a connection on a principal G-bundle P over a manifold M. The connection A is said to be *flat* if its curvature satisfies $F_A = 0$. Sometimes, a slightly different perspective is more helpful: given a manifold M and a Lie group G, we say (P, A), or just A, is a solution to $F_A = 0$ over M if it is a smooth solution to the first order system of PDEs:

$$\begin{aligned} da_\alpha + [a_\alpha \wedge a_\alpha]/2 &= 0 && \text{in } U_\alpha, \\ \tau_{\beta\alpha} a_\alpha \tau_{\beta\alpha}^{-1} - d\tau_{\beta\alpha}\tau_{\beta\alpha}^{-1} &= a_\beta && \text{in } U_\alpha \cap U_\beta, \\ \tau_{\gamma\beta} \cdot \tau_{\beta\alpha} \cdot \tau_{\alpha\gamma} &= 1 && \text{in } U_\alpha \cap U_\beta \cap U_\gamma, \text{ and} \\ \tau_{\alpha\alpha} &= 1 && \text{in } U_\alpha\,. \end{aligned} \tag{FC}$$

In this point of view, the bundle P is not fixed; rather it is a piece of the solution itself. The four main observations about flat connections are:

1. **Gauge invariance.** Under the action of the infinite-dimensional gauge group $\mathcal{G}_P$, a flat connection on a principal bundle P is mapped to another flat connection on P, i.e., if A is a flat connection and $u : P \to P$ is an automorphism (gauge transformation) of P, then $F_{u^*A} = u^*F_A = 0$.

2. **Moduli space of flat connections.** Given a manifold M and a Lie group G, the collection of flat collections modulo the gauge group, denoted by $\mathcal{F}_{M,G}$[2], is in one-to-one correspondence with

$$\mathrm{Hom}(\pi_1(M); G)/\!\sim, \tag{1}$$

where $\rho_1, \rho_2 \colon \pi_1(M) \to G$ are equivalent ($\rho_1 \sim \rho_2$), if there exists $g \in G$ such that $\rho_1(\gamma) = g \cdot \rho_2(\gamma) \cdot g^{-1}$ for all $\gamma \in \pi_1(M)$. We emphasize that $\mathcal{F}_{M,G}$ contains flat connections from *all* possible principal G-bundles on M.

[2] More precisely, $\mathcal{F}_{M,G}$ is the set of equivalence classes of pairs (P, A) where (i) P is a principal G-bundle and A is a flat connection on P; and (ii) $(P, A) \sim (P', A')$ when there exists an isomorphism $u : P \to P'$ such that $u^*A' = A$.

3. **Compactness of moduli space.** Assume M is closed and G is compact. Let $(A^\nu)_{\nu\in\mathbb{N}}$ be a sequence of flat connections on M. Then there exists a subsequence $(A^{\nu'})_{\nu'\in\mathbb{N}}$ of connections on a bundle P and a sequence of gauge transformations $u^{\nu'} \in \mathcal{G}(P)$ such that $u^{\nu'*}A^{\nu'}$ converges uniformly with all derivatives to a flat connection A on P.

4. **Special gauge/ integrability theorem.** If A is a flat connection, then there exists a local trivialization such that A is locally represented by $a_\alpha = 0$.[3] Moreover, such a local trivialization is unique up to a constant gauge transformation, i.e., the transition map between two such trivializations is constant.

Remarks. Observation (2) tells us that the moduli space $\mathcal{F}_{M,G}$ of flat connections over a given manifold M is a topological invariant of M. If two manifolds M_1 and M_2 have non-isomorphic moduli spaces of flat connections, then M_1 and M_2 are not homeomorphic. Though interesting in theory, it is perhaps not very useful as one can work with $\pi_1(M)$ directly. But, this line of thought has been very fruitful when working with 'anti-self-dual' connections over four-manifolds – the moduli space of anti-self-dual connections contain rich information about the topology and differentiable structure of the manifold.

Proof sketch of observations. Observation (1) follows from the fact that in local trivializations, u^*F_A is given by $u_\alpha^{-1} \cdot F_{a_\alpha} \cdot u_\alpha$, which is zero if $F_A = 0$. For a rough explanation of observation (2), note that the holonomy of a flat connection around a loop γ only depends on the homotopy class $[\gamma]$ in $\pi_1(M)$ because of the Ambrose-Singer theorem. As a consequence, we can capture the data of the holonomy in terms of a representation $\rho\colon \pi_1(M, x_0) \to G$. Conversely, it turns out that up to gauge equivalence, there is a unique flat connection associated to every representation $\rho\colon \pi_1(M, x_0) \to G$. The detailed proof can be found in [Tau11, Section 13.9].

Observation (3) follows from the general strong Uhlenbeck compactness theorem given in Section 4.

Observation (4) is equivalent to the first order PDE for $\tau_\alpha\colon U_\alpha \to GL(V)$ given by:

$$\begin{aligned} d\tau_\alpha &= \tau_\alpha a_\alpha &&\text{in } U_\alpha \\ \text{where}\ \ da_\alpha + [a_\alpha \wedge a_\alpha]/2 &= 0 &&\text{in } U_\alpha \text{ (given)}\,. \end{aligned}$$

By induction, the above reduces to solving first-order ODEs with prescribed initial values:

$$\begin{aligned} \frac{\partial \tau_\alpha}{\partial x_{p+1}} &= \tau_\alpha a_{p+1} \\ \text{with}\ \ \tau_\alpha(x_1, \dots, x_p, 0, x_{p+1}, \dots) &= 0\,. \end{aligned}$$

The detailed proof for the case of $G = U(n)$ can be found in [DK97, Section 2.2.1].

[3]Technical note: Since $G \subseteq GL(V)$ is a matrix Lie group, the principal bundle P is naturally associated with a vector bundle $E_P = P \times_\iota V$. As a vector bundle, E_P has more trivializations compared to P because we can change trivializations of E_P using any element of $GL(V)$, but this is not the case for P, where we can only use elements of $G \subseteq GL(V)$. In main text, by local trivialization, we mean a local trivialization of the vector bundle E_P; this may not be a local trivialization of P.

4. Yang-Mills connections

In this section, M is always an oriented Riemannian manifold and G is a compact Lie group. This gives a metric on $\Lambda^2 TM$, a volume element dvol_M, and an Hodge star operator $*\colon \Lambda^2 TM \to \Lambda^{n-2} TM$. The Lie group G is taken to be compact so that we have an ad-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$; this means:

$$\langle g m_1 g^{-1}, g m_2 g^{-1} \rangle_{\mathfrak{g}} = \langle m_1, m_2 \rangle_{\mathfrak{g}} \quad \forall\, m_1, m_2 \in \mathfrak{g} \text{ and } g \in G\,.$$

Now, a connection A on a principal G-bundle over M is said to be a **Yang-Mills connection** if it satisfies the following second order non-linear PDE, called the Yang-Mills equation:

$$d_A^* F_A = \pm * d_A * F_A = 0\,. \tag{YME}$$

A few observations and results about Yang-Mills equations are:

1. **Critical points of Yang-Mills functional.** Given a connection A, its 'energy' is the L^2 norm of the curvature F_A. This is called the Yang-Mills functional:

$$\mathcal{YM}(A) = ||F_A||_{L^2}^2 = \int_M ||F_A||^2 \,\mathrm{dvol}_M\,. \tag{YMF}$$

   If M is closed (ensures energy is finite) and A is a smooth critical point of the Yang-Mills functional, then it satisfies the Yang-Mills equation (YME). In other words, (YME) are the Euler-Lagrange equations of (YMF).

2. **Gauge invariance.** The Yang-Mills equation and functional is gauge invariant. More elaborately, if A is a solution to $d_A^* F_A = 0$ and $u \in \mathcal{G}(P)$ is a gauge transformation, then u^*A is a solution to $d_{u^*A}^* F_{u^*A} = 0$ because $d_{u^*A}^* F_{u^*A} = u^{-1}(d_A^* F_A)u$. Also, we have

$$\mathcal{YM}(u^*A) = \mathcal{YM}(A) \quad \forall\, u \in \mathcal{G}(P)\,.$$

3. **Conformal invariance in $n = 4$.** When $n = 4$, the Yang-Mills equation and functional only depend on the conformal class of the Riemannian metric on M, i.e., if two metrics g_1, g_2 are related by $g_1 = e^f g_2$, then $d_{A,g_1}^* F_A = 0$ if and only if $d_{A,g_2}^* F_A = 0$ and $\mathcal{YM}_{g_1}(A) = \mathcal{YM}_{g_2}(A)$.

4. **Non-compactness of moduli space (L^2 energy).** Even under the assumption that M is closed, G is compact, and L^2 energy is uniformly bounded, we might observe non-compactness of the moduli space of Yang-Mills connections. More precisely, for $M = S^4$, there is a sequence of Yang-Mills connections $\{A^\nu\}_{\nu \in \mathbb{N}}$ and a constant C such that (i) $\mathcal{YM}(A^\nu) = ||F_{A^\nu}||_{L^2} < C$ and (ii) there is no subsequence $\{A^{\nu'}\}_{\nu' \in \mathbb{N}}$ and no gauge transformations $u^{\nu'} \in \mathcal{G}^{2,2}(P)$ such that $u^{\nu' *} A^{\nu'}$ converges in $\mathcal{A}^{1,2}(P)$.

5. **Uhlenbeck's existence of special gauge.** For every point in M there is a small trivializing neighborhood $U \subseteq M$ with smooth boundary and constants C_U and ϵ_U such that for every Sobolev connection $a \in \mathcal{A}^{1,p}(U)$ and $p > n/2$ with small L^2 energy, i.e.,

$$\int_U ||F_a||^2 \, \mathrm{dvol}_M \leq \epsilon_U \, ,$$

there is a choice of a Sobolev gauge transformation $u \in \mathcal{G}^{2,p}(U)$ such that

(i) $d^*(u^*a) = 0$ (u^*a is in *Coulomb gauge* w.r.t. the zero connection),

(ii) $*(u^*a)|_{\partial U} = 0$ (natural boundary condition for ellipticity),

(iii) $||u^*a||_{W^{1,2}} \leq C_U ||F_a||_{L^2}$ (small energy implies small connection).

6. **Strong Uhlenbeck compactness (L^p energy with $p > n/2$).** Assume M is closed and G is compact. Let $1 < p < \infty$ be such that $p > n/2$ and in case $n = 2$ assume $p > 4/3$. Let $(A^\nu)_{\nu \in \mathbb{N}}$ be a sequence of Yang-Mills connections and suppose that $||F_{A^\nu}||_{L^p}$ is uniformly bounded. Then there exists a subsequence $(A^{\nu'})_{\nu' \in \mathbb{N}}$ and a sequence of Sobolev gauge transformations $u^{\nu'} \in \mathcal{G}^{2,p}(P)$ such that $u^{\nu' *} A^{\nu'}$ converges uniformly with all derivatives to a smooth connection $A \in \mathcal{A}(P)$.

Remarks. Flat connections satisfy the Yang-Mills equation (YME) and are, in fact, absolute minimizers of the Yang-Mills functional (YMF) because $\mathcal{YM} \geq 0$. For principal bundles that do not admit flat connections, it is natural to study connections that have the least energy. In this sense, Yang-Mills connections generalize flat connections. As an example, consider the sphere S^4 and the bundle $P^{(k)}$ introduced in Section 2. Proposition 1 says that for $k \neq 0$, the bundle $P^{(k)}$ has non-zero second Chern class. In particular, it does not admit any flat connections. Nonetheless, in Section 6, we shall see that each $P^{(k)}$ admits a Yang-Mills connection. We remark that the conformal invariance described in observation (3) plays a crucial role in the construction of these Yang-Mills connections on S^4.

Observation (4) tells us that unlike flat connections, the moduli space of Yang-Mills connections is non-compact. Next, observation (6) tells us that with a slightly stronger assumption on L^p energy, there is a compactness result. Observation (5) and (6) are remarkable analytic results of Uhlenbeck. The proof of existence of the special Coulomb gauge uses L^p-theory and estimates for the Neumann boundary value problem along with a 'continuity' argument. Next, the existence of this Coulomb gauge plays an important role in the proof of Uhlenbeck's strong compactness. Roughly, the Coulomb gauge $d^*a = 0$ along with the Yang-Mills equation $d_A^* F_A = d_A^* da + d_A^*[a \wedge a]/2 = 0$ forms an elliptic system of PDEs for which we have nice estimates.

Explanation of observations.

Lemma 1. *Suppose M is closed. The Euler-Lagrange equations of the Yang-Mills functional* (YMF) *is the Yang-Mills equation $d_A^* F_A = 0$.*

Proof. To compute the Euler-Lagrange pick an element $b \in \Omega^1(\mathfrak{g}_P)$ and consider the variation $A^t = A + t \cdot b$. Recall, that the space of connections is an affine space modelled on the vector space $\Omega^1(\mathfrak{g}_P)$; hence, we can 'add' an element of $\Omega^1(\mathfrak{g}_P)$ to obtain another connection. The corresponding variation in curvature is:

$$\begin{aligned} F_{A+tb} &= d(a_\alpha + tb_\alpha) + [(a_\alpha + tb_\alpha) \wedge (a_\alpha + tb_\alpha)]/2 \\ &= da_\alpha + [a_\alpha \wedge a_\alpha]/2 + t(db_\alpha + [a_\alpha \wedge b_\alpha]/2 + [b_\alpha \wedge a_\alpha]/2) + t^2[b_\alpha \wedge b_\alpha]/2 \\ &= F_A + t(db_\alpha + [a_\alpha \wedge b_\alpha]) + t^2[b_\alpha \wedge b_\alpha]/2 \\ &= F_A + td_A b + t^2[b \wedge b]/2. \end{aligned}$$

Next, the corresponding variation of the Yang-Mills functional (YMF) is:

$$\begin{aligned} \mathcal{YM}(A + tb) &= \int_M \langle F_{A+tb}, F_{A+tb} \rangle \,\mathrm{dvol}_M \\ &= \int_M \big(\langle F_A, F_A \rangle + 2t \langle F_A, d_A b \rangle + \mathcal{O}(t^2) \big) \,\mathrm{dvol}_M \,. \end{aligned}$$

Now, if A is a critical point, then:

$$0 = \frac{d}{dt} \mathcal{YM}(A + tb)\Big|_{t=0} = 2 \int_M \langle F_A, d_A b \rangle = 2 \int_M \langle d_A^* F_A, b \rangle \qquad \forall\, b \in \Omega^1(\mathfrak{g}_P).$$

Thus, $d_A^* F_A = 0$ is the Euler-Lagrange of the Yang-Mills functional (YMF). □

The gauge invariance of Yang-Mills equation (observation (2)) is because

$$d^*_{u^*A} F_{u^*A} = u^*(d_A^* F_A) = u^{-1}(d_A^* F_A) u \,,$$

where the first equality is by the naturality of the curvature and d_A^*; and the second equality is how the gauge group acts on elements of $\Omega^1(\mathfrak{g}_P)$, i.e., by conjugation. Next, the gauge invariance of the Yang-Mills functional comes from the ad-invariance of the inner product on $\Omega^2(\mathfrak{g}_P)$:

$$\mathcal{YM}(u^*A) = \int_M ||F_{u^*A}||^2 = \int_M ||u^{-1} F_A u||^2 = \int_M ||F_A||^2 = \mathcal{YM}(A).$$

Towards observation 3, we have the following Lemma.

Lemma 2. *If* $\dim(M) = n$*, then the Hodge star automorphism* $* : \Lambda^{n/2} T^* M \to \Lambda^{n/2} T^* M$ *is unchanged under conformal change of metric; i.e., if* $g_1 = e^f g_2$*, then* $*_{g_1} = *_{g_2}$.

Proof. If g_1, g_2 satisfy $g_1 = e^f g_2$, then the point-wise inner products $\langle \cdot, \cdot \rangle_{g_1}$ and $\langle \cdot, \cdot \rangle_{g_2}$ on $\Lambda^{n/2} T_p^* M$ are related by $\langle \cdot, \cdot \rangle_{g_1} = e^{-nf/2} \langle \cdot, \cdot \rangle_{g_2}$. Next, the volume elements are related by $\mathrm{dvol}_{g_1} = e^{nf/2} \,\mathrm{dvol}_{g_2}$ because, locally, the volume element is given by the square root of the determinant of metric. As a result, for all $\alpha, \beta \in \Lambda^{n/2} T_p^* M$ we have:

$$\alpha \wedge (*_{g_1} \beta) = \langle \alpha, \beta \rangle_{g_1} \mathrm{dvol}_{g_1} = e^{\frac{-nf+nf}{2}} \langle \alpha, \beta \rangle_{g_2} \mathrm{dvol}_{g_2} = \alpha \wedge (*_{g_2} \beta) \,.$$

This implies $*_{g_1} = *_{g_2}$. □

The conformal invariance of Yang-Mills equation (observation (3)) is a consequence of the above lemma. To see this, note that $d_A^* = (-1)^{n(k-1)+1} * d_A *$ where the rightmost $*$ acts on $2 = n/2$-forms. As a result $d_A *$ is conformally invariant and we have:

$$d^*_{A,g_1} F_A = 0 \quad \Longleftrightarrow \quad d_A(*_{g_1} F_A) = d_A(*_{g_2} F_A) = 0 \quad \Longleftrightarrow \quad d^*_{A,g_2} F_A = 0\,.$$

To prove observation (4), let A be any Yang-Mills connection over S^4 with non-zero L^2 energy. Over $U_S := S^4 \setminus \{\mathtt{NP}\}$, we can write A as a $\mathfrak{g}$-valued 1-form $a \in C^\infty(U_S; (\mathbb{R}^4)^* \otimes \mathfrak{g})$. Consider the sequence

$$a^\nu(x) := \nu^{-1} \cdot a(\nu^{-1}x), \text{ where } \nu \in \mathbb{N}\,.$$

It turns out that these $\mathfrak{g}$-valued 1-forms extend to a Yang-Mills connection A^ν over S^4. This is due to the famous removable singularity theorem of Uhlenbeck (see [Uhl82]). The curvature of a^ν over U_S is:

$$F_{a^\nu}(x) = \nu^{-2} \cdot F_a(\nu^{-1}x)\,.$$

Using the above expression, the integral of $||F_{a^\nu}||^2$ over U_S is equal to the L^2 energy of A. Despite the L^2 energy of all A^ν being equal, there is no subsequence $(u^{\nu *}A^\nu)_{\nu\in\mathbb{N}}$ which converges in $\mathcal{A}^{1,2}(P)$. If there were such a subsequence, then $(u^\nu)^{-1} F_{a^\nu} u^\nu$ would converge in $L^2(U_S; (\mathbb{R}^4)^* \otimes \mathfrak{g})$. In particular, the function $p \mapsto ||F_{a^\nu}(p)|| = ||u^{\nu-1} F_{a^\nu} u^\nu||$ would converge in $L^2(U_S; \mathbb{R})$ to the point-wise limit of $||F_{a^\nu}||$. However, the point-wise limit of $||F_{a^\nu}||$ can be shown to be 0, which contradicts the fact that the L^2 integrals of $||F_{a^\nu}||$ are all non-zero and equal to each other. In section 6, we explicitly write down a Yang-Mills connection A and demonstrate this phenomena.

A self-contained and detailed exposition of observation (5) and (6) is given in [Weh04].

5. Anti-self-dual (ASD) connections

In this section, M is always a 4-dimensional oriented Riemannian manifold and G is the compact Lie group $SU(2)$. Dimension 4 ensures that (i) the Hodge star $* : \Lambda^2 T_p M \to \Lambda^2 T_p M$ is a linear automorphism with eigen-values $+1$ and -1, and (ii) we have conformal invariance, i.e., the special ASD connections only depend on the conformal class of the metric on M. The Lie group G is taken to be $SU(2)$ for simplicity.[4]

Coming back to Yang-Mills connections, note that flat connections satisfying $F_A = 0$ form a special subset of Yang-Mills connections. Another special subset of Yang-Mills connections are the **anti-self-dual connections (ASD)**. A connection A is said to be ASD if it satisfies the first order PDE:

$$*F_A = -F_A. \tag{ASD}$$

A few observations and results about ASD connections are:

[4] With suitable modifications, the theory we discuss extends for $G = SU(n)$ (see [DK97, Section 3.3]) and for $G = Sp(n)$, the so-called *compact symplectic group* (see [Ati79, Chapter II]).

1. **ASD are absolute minimizers of Yang-Mills.** ASD connections satisfy the Yang-Mills equation because of the Bianchi identity:

$$d_A^*(F_A) = \pm * (d_A * F_A) = \mp * (d_A F_A) = \mp * 0 = 0.$$

   Moreover, they are absolute minimizers of the Yang-Mills functional (YMF).

2. **Gauge invariance.** The ASD equation is gauge invariant. More elaborately, if A is an ASD connection and $u \in \mathcal{G}(P)$ is a gauge transformation, then u^*A is also ASD, i.e., $* F_{u^*A} = *(u^{-1}F_A u) = u^{-1}(* F_A)u = -u^{-1}F_A u = F_{u^*A}$.

3. **Conformal invariance in $n = 4$.** The ASD equation and the Hodge star $* : \Lambda^2 T_pM \to \Lambda^2 T_pM$ depend only on the conformal class of the Riemannian metric on M, i.e., if two metrics g_1, g_2 are related by $g_1 = e^f g_2$, then $*_{g_1} = *_{g_2}$ and $*_{g_1} F_A = *_{g_2} F_A$ (see Lemma 2).

4. **ASD connections on S^4 (ADHM theorem).** Given $k \in \mathbb{Z}$, let $P^{(k)}$ be the principal $SU(2)$ bundle with Chern number k. Then:
   a) If $k \leq -1$, the bundle $P^{(k)}$ has no ASD connections.[5]
   b) For $k = 0$, there is only one ASD connection modulo the gauge group. This is the unique flat connection on $P^{(0)} \simeq S^4 \times SU(2)$.
   c) If $k \geq 1$, there are lots of ASD connections on $P^{(k)}$. Moreover, the moduli space of ASD connections on $P^{(k)}$ modulo the gauge group is a finite dimensional object that is described by $8k - 3$ real parameters. Additionally, these ASD connections can be explicitly described as a certain 'projection' of the trivial connection on the trivial bundle $S^4 \times \mathbb{C}^{2k+2}$.

5. Observation (4) and (6) of Yang-Mills hold true even for ASD connections, i.e., a uniform L^2 bound on the energy is not good enough to ensure compactness of a sequence of ASD connections, while a uniform L^p bound ($p > 2$) on the energy of the ASD connections does imply compactness of a sequence of ASD connections.

Explanation of observations. The proof of observation (2), (3), and (5) is similar to, if not the same as, the case of Yang-Mills. We shall postpone the discussion of the ADHM theorem to Section 6. Now, we focus on observation (1). Given any F_A, we have the orthogonal decomposition $F_A = F_A^+ + F_A^-$, where F_A^+ and F_A^- are the self-dual and anti-self-dual components, respectively, i.e., $* F_A^+ = +F_A^+$ and $* F_A^- = -F_A^-$. This is a consequence of the orthogonal splitting of $\Lambda^2 T_pM$ into the two eigen-spaces of the Hodge star: $\Lambda^+ T_pM \oplus^\perp \Lambda^- T_pM$. As a consequence, we have the point-wise identity $||F_A(p)||^2 = ||F_A^+(p)||^2 + ||F_A^-(p)||^2$. Further, we have the following Lemma.

[5]Nonetheless, for $k \leq -1$ the bundle $P^{(k)}$ has lots of *self-dual connections*, i.e., connections A that satisfy $* F_A = +F_A$ instead of $* F_A = -F_A$. To see the existence of self-dual connections, let A be an ASD connection on $P^{(-k)}$ for $-k \geq 1$ and let $r : S^4 \to S^4$ be an orientation reversing isometry of S^4. It turns out that r^*A is a self-dual connection on $r^*P^{(-k)}$ and $r^*P^{(-k)}$ is isomorphic $P^{(k)}$.

Lemma 3. *Let M be a 4-dimensional oriented Riemannian manifold and A be a $SU(n)$-connection on a $SU(n)$-bundle P over M. Then:*

(i) *Energy:* $\mathcal{YM}(A) = \int_M ||F_A||^2 \,\mathrm{dvol} = \int_M ||F_A^+||^2 \,\mathrm{dvol} + \int_M ||F_A^-||^2 \,\mathrm{dvol}$

(ii) *Chern number:* $k := \int_M c_2(P) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}(F_A \wedge F_A) = \frac{1}{8\pi^2} \Big(- \int_M ||F_A^+||^2 + \int_M ||F_A^-||^2 \Big)$

(iii) $\mathcal{YM}(A) \geq 8\pi^2 |k|$ *with equality if and only if A is anti-self-dual (when $k \geq 0$) or A is self-dual (when $k \leq 0$). In particular, the L^2 energy of an ASD connection is exactly the Chern number of the underlying bundle up to a factor of* $8\pi^2$.

Proof. The key point is that for $SU(n)$, the Lie algebra $\mathfrak{su}(n)$ consists of skew-Hermitian matrices ($m^* = -m$) and the ad-invariant inner product on $\mathfrak{su}(n)$ takes the form:

$$\langle m_1, m_2 \rangle_{\mathfrak{su}(n)} := \mathrm{Tr}(m_1^* m_2) = -\,\mathrm{Tr}(m_1 m_2)$$

Combining the above with $\langle \alpha, \beta \rangle \,\mathrm{dvol}_M = \alpha \wedge (*\,\beta)$, we get:

$$\langle F_1, F_2 \rangle \,\mathrm{dvol}_M = -\,\mathrm{Tr}(F_1 \wedge *\, F_2) \quad \forall\, F_1, F_2 \in \Omega^2(\mathfrak{g}_P)\,.$$

This gives (ii):

$$\begin{aligned} 8\pi^2 k &= \int_M \mathrm{Tr}(F_A \wedge F_A) = \int_M \mathrm{Tr}(F_A^+ \wedge F_A^+) + \int_M \mathrm{Tr}(F_A^- \wedge F_A^-) \\ &= \int_M \mathrm{Tr}(F_A^+ \wedge *\, F_A^+) - \int_M \mathrm{Tr}(F_A^- \wedge *\, F_A^-) \\ &= -\int_M ||F_A^+||^2 \,\mathrm{dvol} + \int_M ||F_A^-||^2 \,\mathrm{dvol}\ . \end{aligned}$$

Combining (i) and (ii) we have:

$$\mathcal{YM}(A) = 8\pi^2 k + 2\int_M ||F_A^+||^2 \,\mathrm{dvol} = -8\pi^2 k + 2\int_M ||F_A^-||^2 \,\mathrm{dvol}\ .$$

From the above, we get part (iii) by noting that k is a topological constant of the bundle P. □

6. Explicit ASD connections over S^4

In this section, we concretely write down an ASD connection and use conformal transformations to produce a family of distinct ASD connections.

“The basic instanton”: a rotationally invariant ASD connection on $P^{(1)}$

It turns out that $P^{(1)}$ is diffeomorphic to the standard 7-sphere S^7 (see [Tau11, Example 10.6]). This is analogous to the case of the Hopf bundle S^3 which is a principal $U(1)$-bundle over S^2. Further, notice that:

$$\begin{aligned} SU(2) &\simeq \mathrm{Unit}(\mathbb{H}) & \qquad\qquad U(1) &\simeq \mathrm{Unit}(\mathbb{C}) \\ \mathfrak{su}(2) &\simeq \mathrm{Imag}(\mathbb{H}) & \mathfrak{u}(1) &\simeq \mathrm{Imag}(\mathbb{C}) \\ S^4 &\simeq \mathbb{H}P^1 & S^2 &\simeq \mathbb{C}P^1\,, \end{aligned}$$

where $\mathbb{H}$ are the quaternions which form a normed division algebra, $\mathrm{Unit}(\mathbb{H})$ are quaternions of unit length, i.e., $\overline{x}x = 1$, the set $\mathrm{Imag}(\mathbb{H})$ contains the purely imaginary quaternions of the form $ix_2 + jx_3 + kx_4$, and $\mathbb{H}P^1$ is the set of all equivalence classes $[x, y] \in \mathbb{H}^2$ under the action of $\mathbb{H}^*$ by $\lambda : [x, y] \mapsto [\lambda x, \lambda y]$.

Next, note that the Hopf bundle has a natural connection – the Chern connection. Replacing all the complex numbers and variables in the Chern connection with quaternion numbers and variables, we get a $SU(2)$-connection on $P^{(1)}$. More elaborately, let $U_N, U_S \subseteq S^4 \simeq \mathbb{H}P^1$ be the set of points given by $[1, y], [x, 1] \in \mathbb{H}P^1$, respectively. In these trivializations, consider two $\mathfrak{su}(2)$-valued 1-forms $a_N, a_S : TU_N, TU_S \to \mathfrak{su}(2) \simeq \mathrm{Imag}(\mathbb{H})$ given by:

$$\begin{aligned} a_N([1,y]) &:= \mathrm{Imag}\left(\frac{\overline{y}dy}{1+|y|^2}\right) & \qquad a_S([x,1]) &:= \mathrm{Imag}\left(\frac{\overline{x}dx}{1+|x|^2}\right) \\ &= \frac{1}{2}\left(\frac{\overline{y}dy - d\overline{y}y}{1+|y|^2}\right) & &= \frac{1}{2}\left(\frac{\overline{x}dx - d\overline{x}x}{1+|x|^2}\right)\,, \end{aligned}$$

where dx is a $\mathbb{H}$-valued 1-form on U_S that is defined as $dx := dx_1 + idx_2 + jdx_3 + kdx_4$, where x_1, x_2, x_3, x_4 are taken to be the coordinates functions of $U_S \simeq \mathbb{R}^4$ using the identification $x_1 + ix_2 + jx_3 + kx_4 \mapsto (x_1, x_2, x_3, x_4)$. These connections transform as:

$$a_N = \tau \cdot a_S \cdot \tau^{-1} - d\tau \cdot \tau^{-1} \qquad \text{where } \tau([x,1]) := \frac{x}{|x|} \in \mathrm{Unit}(\mathbb{H}) \simeq SU(2)\,.$$

To see this, note that:

$$\begin{aligned} a_N([x,1]) &= a_N([1, \overline{x}|x|^{-2}]) & \qquad \tau \cdot a_S([x,1]) \cdot \tau^{-1} &= \mathrm{Imag}\left(\frac{-xd\overline{x}}{1+|x|^2}\right) \\ &= \mathrm{Imag}\left(\frac{xd\overline{x}}{|x|^2+|x|^4}\right) & -d\tau \cdot \tau^{-1} &= \mathrm{Imag}\left(\frac{xd\overline{x}}{|x|^2}\right)\,. \end{aligned}$$

Thus, a_N, a_S combine to give an $SU(2)$-connection A on a $SU(2)$-bundle with transition map τ. In fact, A is a $SU(2)$-connection on $P^{(1)}$ because the transition map τ is equal to $\tau^{(1)}$ introduced in Section 2. A computation shows that the curvature of A in the trivialization U_S can be written as:

$$F_{a_S}([x,1]) = \frac{d\overline{x} \wedge dx}{(1+|x|^2)^2}\,. \tag{2}$$

To see F_A is anti-self-dual, one decomposes $d\overline{x} \wedge dx$ in terms of the basis elements $q \cdot dx_i \wedge dx_j$, where $q \in \mathbb{H}$. Then, one checks that it is a linear combination of only the anti-self-dual 2-forms. For reference, the three basis elements of anti-self-dual and self-dual forms using the standard Euclidean metric of $\mathbb{R}^4$ are:

$$\begin{array}{cc} \text{anti-self-dual 2-forms:} & \text{self-dual 2-forms:} \\ dx_1dx_2 - dx_3dx_4 & dx_1dx_2 + dx_3dx_4 \\ dx_1dx_3 + dx_2dx_4 & dx_1dx_3 - dx_2dx_4 \\ dx_1dx_4 - dx_2dx_3 & dx_1dx_4 + dx_2dx_3\,. \end{array}$$

Note that because $* : \Lambda^2 T_M \to \Lambda^2 T_M$ is conformally invariant, the above continue to be anti-self-dual/self-dual on S^4 whose metric is conformal to $\mathbb{R}^4$. This completes the construction of an ASD connection A on $P^{(1)}$ over S^4. The connection A is sometimes called the *basic instanton* as it is the easiest ASD connection to write down.

More ASD connections using conformal transformations

An orientation preserving map $w : S^4 \to S^4$ is said to be a *conformal transformation* if it is diffeomorphism that preserves angles, i.e., $w^* g_{S^4} = e^f g_{S^4}$, where g_{S^4} is the standard round metric on S^4 and $f \in C^\infty(S^4, \mathbb{R})$. It turns out that the group of all such maps, denoted by $\mathrm{Conf}_+(S^4)$, is isomorphic to $SO(5,1)$ and is generated by rotations and dilations.[6] From this we can conclude that $\mathrm{Conf}_+(S^4)$ is path-connected and hence $w^*P \simeq P$ for all $w \in \mathrm{Conf}_+(S^4)$.

Given an ASD connection A on $P^{(k)}$ and $w \in \mathrm{Conf}_+(S^4)$, we consider the connection w^*A on $w^*P^{(k)} \simeq P^{(k)}$. This is also an ASD connection because its curvature is F_{w^*A} and

$$*_{g_{S^2}} F_{w^*A} = *_{w^* g_{S^2}} F_{w^*A} = w^*(*_{g_{S^2}} F_A) = w^*(-F_A) = -F_{w^*A}\,,$$

where the first equality is by conformal invariance of the Hodge star on 2-forms and the rest is by naturality principle. In conclusion, the action of conformal transformations produces a family of ASD connections starting from a single ASD connection.

For the rest of this section, we specialize to the case where A is the basic instanton constructed in the previous subsection. It turns out that A is preserved under rotations of S^4, i.e., if $w : S^4 \to S^4$ is a rotation, there is a bundle isomorphism $u : w^*P^{(1)} \to P^{(1)}$ such that $w^*A = u^*A$ (in words, the two connection w^*A and A are equivalent modulo gauge transformations). A consequence is that $||F_A(p)||^2$ is constant for all $p \in S^4$. Alternatively, this can be seen directly by computing the norm using the explicit expression in equation (2).

On the other hand, if w is a dilation, then w^*A and A are not gauge equivalent, so we get new ASD connections. Introduce two parameters $(z, \lambda) \in S^4 \times [1, \infty)$ and

[6] This is analogous to the fact that orientation preserving conformal transformations of S^2 are just the Möbius maps which are generated by rotations and dilations. Moreover, recall that the Möbius group is $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm \mathrm{Id}\}$. Similarly, for S^4, the group of orientation preserving conformal transformations is $PSL_2(\mathbb{H}) = SL_2(\mathbb{H})/\{\pm \mathrm{Id}\} \simeq SO(5,1)$.

define $w_{z,\lambda}$ to be the dilation by λ centered at $z \in S^4$. For $z = 0$, we can write down the dilation explicitly using the trivialization over $U_S \subset S^4$ as:

$$w_{0,\lambda}(x) = \lambda x \,.$$

Now, the connection $w_{0,\lambda}^* A$ and curvature $w_{0,\lambda}^* F_A$ become:

$$\begin{aligned} w_{0,\lambda}^* a_S(x) &= \frac{\lambda^2 \operatorname{Imag}(\overline{x} dx)}{1 + \lambda^2 |x|^2} & \qquad w_{0,\lambda}^* F_{a_S}(x) &= \frac{\lambda^2 \operatorname{Imag}(d\overline{x} \wedge dx)}{(1 + \lambda^2 |x|^2)^2} \\ &= \lambda a_S(\lambda x) & &= \lambda^2 F_{a_S}(\lambda x) \,. \end{aligned} \tag{3}$$

Note that $||w_{0,\lambda}^* F_A(p)||$ is no longer a constant and has a global maximum at 0 where the maximum value is a scalar multiple of λ^2:

$$||w_{0,\lambda}^* F_{a_S}(x)|| = 3^{1/2} \lambda^2 \cdot \frac{(1 + |x|^2)^2}{(1 + \lambda^2 |x|^2)^2} \,.$$ [7]

Similarly, it turns out that $||w_{z,\lambda}^* F_A(p)||$ has a global maximum at the point $z \in S^4$ with maximum value that is a scalar multiple of λ^2 for $\lambda \neq 1$. As the norm $|| \cdot ||_x$ on $\Lambda_x^2(\mathfrak{g}_P)$ is gauge invariant, $\{w_{z,\lambda}^* A\}_{z,\lambda}$ is a collection of *distinct* ASD connections on $P^{(1)}$ indexed by:

$$(z, \lambda) \in \frac{S^4 \times [1, \infty)}{S^4 \times \{1\}} \simeq \mathbb{R}^5.$$

The celebrated ADHM theorem says there are no other ASD connections on $P^{(1)}$. Before we end the section, let us observe what happens to $\{w_{0,\lambda}^* A\}_\lambda$ as $\lambda \to \infty$. First, the 'energy density' $||w_{0,\lambda}^* F_{a_S}(x)||^2$ converges in the sense of distributions to $8\pi^2$ times the Dirac delta centered at 0 (this is the so-called 'bubbling' phenomenon). Second, if we look away from the point $0 = \mathtt{SP}$, the connections $w_{0,\lambda}^* A$ 'converge' to the zero connection over $S^4 \setminus \{\mathtt{SP}\}$, which in turn extends over S^4 as an ASD connection on a different bundle $P^{(0)}$. To see this, we write down $w_{0,\lambda}^* A$ using the trivialization over $U_N = S^4 \setminus \{\mathtt{SP}\}$:

$$w_{0,\lambda}^* a_N(y) = \frac{\operatorname{Imag}(\overline{y} dy)}{\lambda^2 + |y|^2} \xrightarrow{\lambda \to \infty} 0 \,.$$

For more details about the moduli space of ASD and Yang-Mills connections, we refer the reader to [DK97, Chapter 4].

Atiyah-Drinfeld-Hitchin-Manin classification of ASD connections on S^4

In the above subsection, we only focused on ASD connections on $P^{(1)}$. With more work, we can generalize the case of the basic instanton to get an explicit family of

[7] The expression $w_{0,\lambda}^* F_{a_S}(x) = \lambda^2 F_{a_S}(\lambda x)$ in equation (3) seems to suggest $||w_{0,\lambda}^* F_{a_S}(x)|| = \lambda^2 ||F_{a_S}(\lambda x)||$ and hence $||w_{0,\lambda}^* F_{a_S}(x)||$ is a constant just like $||F_A(p)||$. However, this is flawed and the fault is in our notation. Let $|| \cdot ||_p$ denote the norm on the vector space $\Lambda_p^2(\mathfrak{g}_P)$. Now, equation (3) gives us $||w_{0,\lambda}^* F_{a_S}(x)||_x = \lambda^2 ||F_{a_S}(\lambda x)||_x$, but the latter is not equal to $\lambda^2 ||F_{a_S}(\lambda x)||_{\lambda x}$.

ASD connections on $P^{(k)}$ that is described by $8k-3$ real parameters when $k \geq 1$ (see [Ati79, Chapter 2]). The remarkable ADHM theorem shows that there are no other ASD connections other than the ones that can be explicitly constructed. In particular, the solution space of a non-trivial system of PDE over S^4 is described by a collection of matrix data!

Theorem 1 (ADHM theorem '78)**.** *Given $k \geq 1$, every ASD connection on $P^{(k)}$ arises from parameters (λ, B), where (λ, B) are certain matrix data. More precisely, B can be taken to be a symmetric $k \times k$ matrix of quaternions and λ is a row vector $(\lambda_1, \dots, \lambda_k)$ of quaternions satisfying:*

(i) *$B^*B + \lambda^*\lambda$ is a real $k \times k$ matrix (ensures that the connection arrising from B, λ is ASD),*

(ii) *The matrix*

$$\begin{bmatrix} \lambda \\ B - x \end{bmatrix}$$

has full rank for all $x \in \mathbb{H}$ (a non-degeneracy/ open condition).

Moreover, the ASD connections given by (λ, B) and (λ', B') are gauge-equivalent if and only if there is a quaternion $q \in \mathrm{Unit}(\mathbb{H})$ and $T \in O(k, \mathbb{R})$ such that $\lambda' = q\lambda T$ and $B' = T^{-1}BT$.

The detailed proof of the ADHM theorem using Twistor theory and the Ward correspondence is given in [Ati79]. For a somewhat different proof that generalizes to construct 'monopole' solutions, see [DK97, Chapter 3].

A. List of Symbols

$\langle\cdot,\cdot\rangle_{\mathfrak{g}}$	ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of a compact Lie group G
$\langle\cdot,\cdot\rangle$	inner product induced by the Riemannian metric of M (and possibly $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$)
$[\cdot\wedge\cdot]$	product on $\Omega^k(\mathfrak{g}_P)$ by wedge product on differential forms and Lie product on $\mathfrak{g}_P$ (see [Weh04, Appendix A])
$\|\|\cdot\|\|_{L^p}$	L^p-norm obtained by integrating p-th power over the manifold M
A	a connection on a bundle
a_α	an element of $\Omega^1(U_\alpha;\mathfrak{g})$ that represents the connection A in a trivialization over U_α (see [Tau11, Chapter 11])
$\mathcal{A}(P)$	space of smooth connections on the bundle P
$\mathcal{A}^{1,p}(P)$	Sobolev space of connections on the bundle P (see [Weh04, Appendix A])
$\mathcal{A}^{1,p}(U)$	Sobolev space $W^{1,p}(U;T^*U\otimes\mathfrak{g})$ of connections in a local trivialization over U
d_A	exterior covariant derivative on $\Omega^k(\mathfrak{g}_P)$
d_A^*	formal adjoint of d_A
F_A	the curvature of a connection A
G	a matrix Lie group (closed subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$)
$\mathfrak{g}$	the Lie algebra of a Lie group G
$\mathfrak{g}_P$	the vector bundle $P\times_{\mathrm{ad}}\mathfrak{g}$ with fiber $\mathfrak{g}$ associated with the bundle P
$\mathcal{G}(P)$	group of gauge transformations on P
$\mathcal{G}^{2,p}(P)$	Sobolev space of gauge transformations on P
$\mathbb{H}$	normed division algebra of quaternions with elements of the form $x_1+ix_2+jx_3+kx_4$
Imag	purely imaginary part of a quaternion (or complex number) given by $\mathrm{Imag}(x_1+ix_2+jx_3+kx_4)=ix_2+jx_3+kx_4$
M	a smooth manifold that is (mostly) closed
$\mathtt{NP}$	the point $(0,0,0,1)$ of S^4, i.e., the north pole
$\Omega^k(\mathfrak{g}_P)$	space of $\mathfrak{g}_P$ bundle-valued k-forms on M, i.e., the space of sections of $\Lambda^kT^*M\otimes\mathfrak{g}_P$
$P^{(k)}$	principal $SU(2)$-bundle over S^4 with Chern number k
$\mathtt{SP}$	the point $(0,0,0,-1)$ of S^4, i.e., the south pole
$\mathrm{Unit}(\mathbb{H})$	elements of unit length in $\mathbb{H}$, i.e., $\overline{x}x=1$
u_α	a local gauge transformation: an element of $C^\infty(U_\alpha;G)$ that represents the gauge transformation u in a trivialization over U_α
V	a finite dimensional complex or real vector space
$\mathcal{YM}(A)$	Yang-Mills functional given by $\int_M \|F_A\|^2$

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