

# Conformal tilings

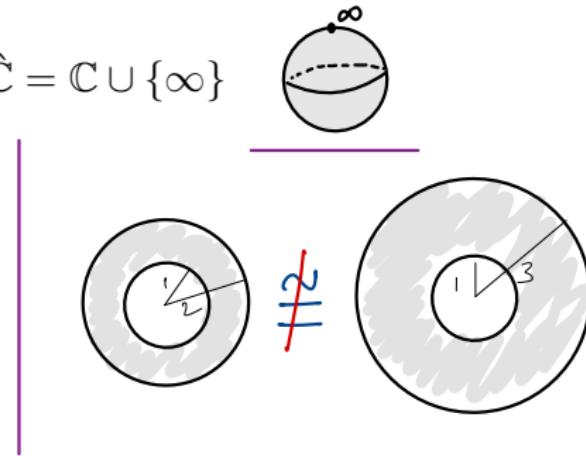
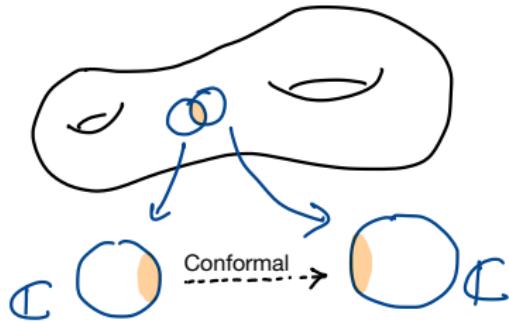
*A bizarre set of tilings generated by Schwarz reflections*

Mohith Raju N

Jan 2023

# Complex analysis

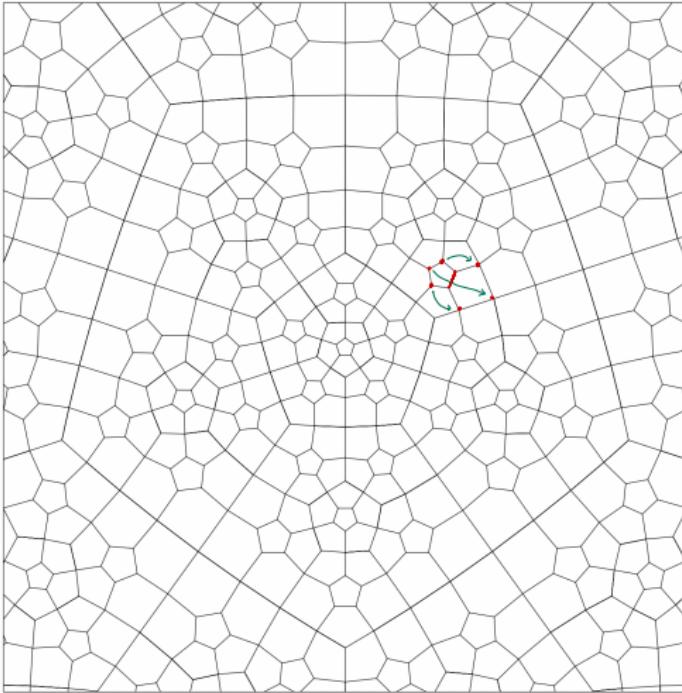
- ▶ A function  $f : \Omega_{\mathbb{C}} \rightarrow \Omega'_{\mathbb{C}}$  is said to be *conformal* (aka biholomorphic) if both  $f$  and  $f^{-1}$  are holomorphic
- ▶ Conformal means angle-preserving
- ▶ A *Riemann surface* is topological surface with a “complex structure”: it is a manifold with charts to subsets of  $\mathbb{C}$  and conformal transition maps.
- ▶ The complex structure allows us to define holomorphic maps between surfaces
- ▶ Examples:  $\mathbb{C}, \mathbb{D}, \{1 < |z| < 2\}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



# Complex analysis

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- ▶ Examples:  $\mathbb{C}, \mathbb{D}, \{1 < |z| < 2\}, \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- ▶ Rk: Two Riemann surfaces may be topologically the same, but might not be isomorphic to each other. Ex:  $\{1 < |z| < 2\}$  and  $\{1 < |z| < 3\}$  are not biholomorphic to each other.
- ▶ A rich class of examples is obtained by taking the zero set  $f^{-1}(0)$  of polynomial functions  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ .  
Ex:  $X = \{(z, w) : w^2 = z^3 + az + b\} \rightarrow$  *surely a torus!*

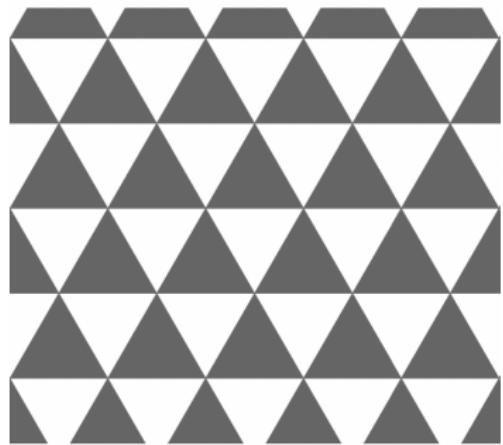
# Examples of conformal tiling



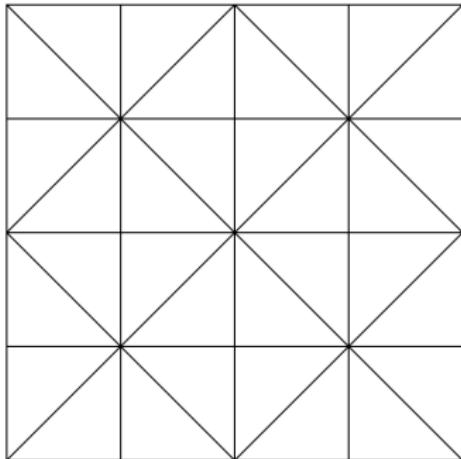
all tiles  
are pentagons!

Tiling 1 (Pentagonal tiling of  $\mathbb{C}$ )

# Examples of conformal tiling

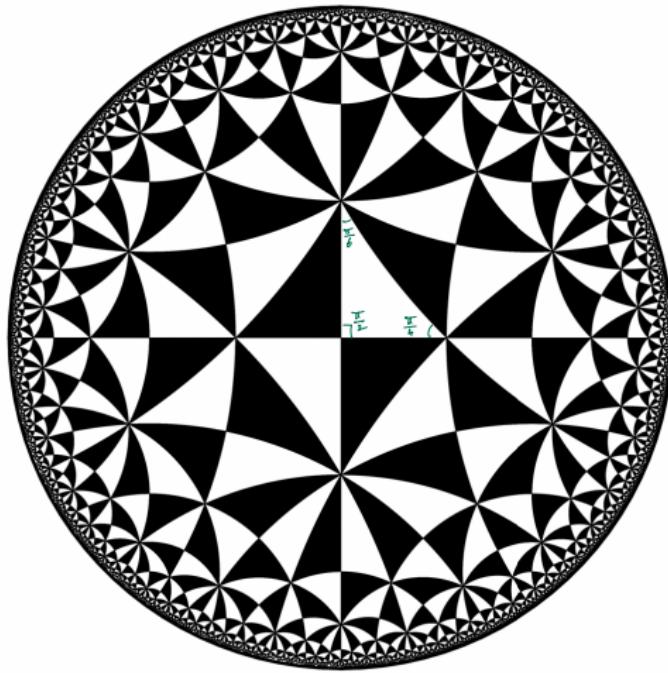


Tiling 2 (from antiquity)



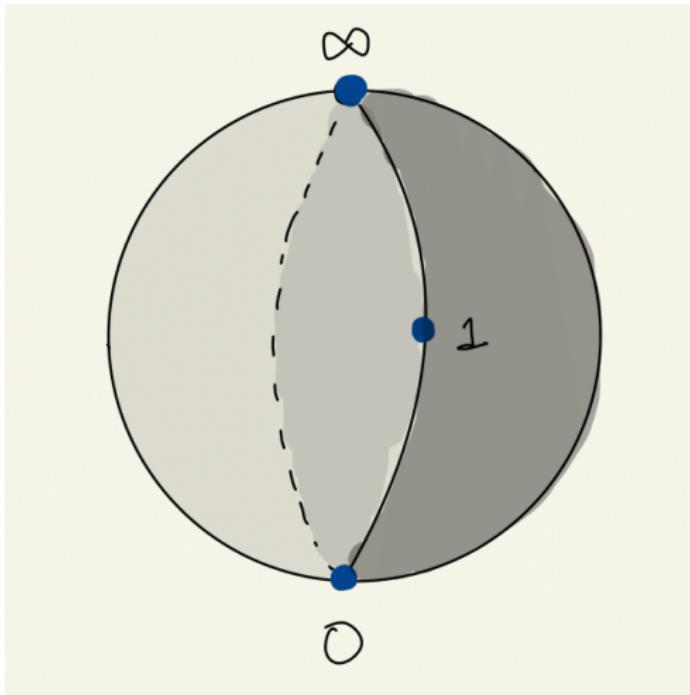
Tiling 3 (corresponding  $\wp_{\Lambda}^2$ )

# Examples of conformal tiling



Tiling 4 (Hyperboling tiling of type  $(2,4,6)$ )  
what?

## Examples of conformal tiling



Tiling 5 (Two triangles give a sphere)

# What is a conformal tiling?

- ▶ Is it an angle-preserving tiling?  
Not quite
- ▶ Obs: The internal angles of each polygon are not equal. But the angles around each vertex are equal

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- ▶ Is there some conformal symmetry?  
Yes!
- ▶ Obs: Each pair of adjacent tiles are invariant under reflection along common edge

Classical reflections

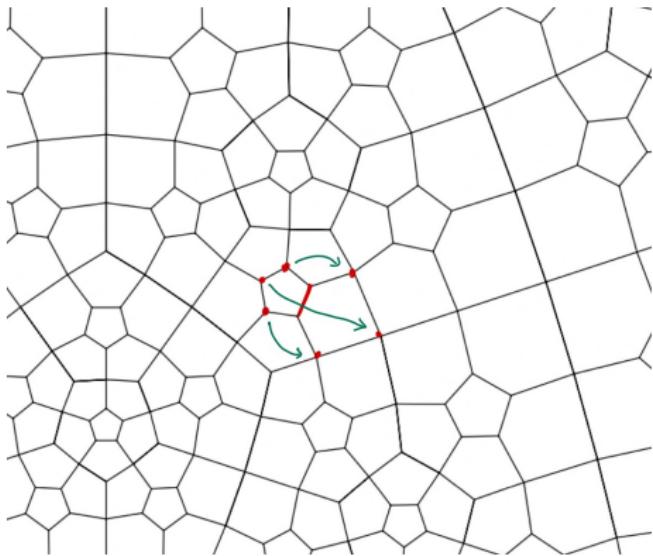
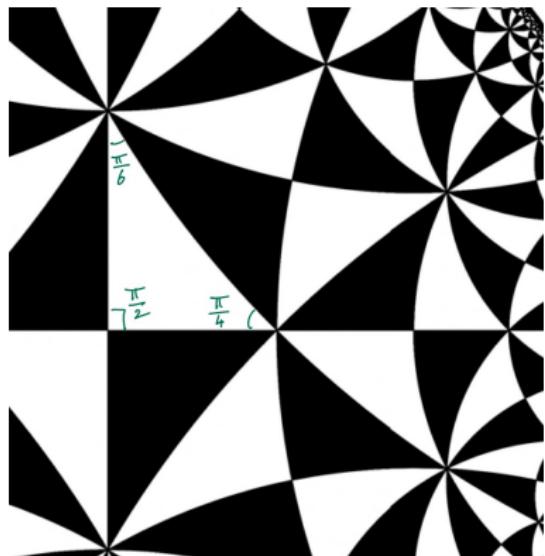
- fixes the line of symm
- orientation reversing
- It is an isometry

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- ▶ Obs: The internal angles of each polygon are not equal. But the angles around each vertex are equal
- ▶ Is there some conformal symmetry?  
Yes!
- ▶ Obs: Each pair of adjacent tiles are invariant under reflection along common edge
- ▶ By reflection we mean *Schwarz reflections*, i.e. the unique anti-conformal map that fixes the edge pointwise  
Ex:
  - ▶  $z \mapsto \bar{z}$  (fixes  $\mathbb{R}$ )
  - ▶  $z \mapsto 1/\bar{z}$  (fixes  $S^1$ )
- ▶ **Defn.**  $f : X \rightarrow Y$  is an anti-conformal map if it is an orientation reversing conformal map  
Formally,  $\frac{\partial f}{\partial z} = 0 = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$

# What is a conformal tiling?

**Defn.** A conformal tiling of a Riemann surface  $X$  is a tiling of  $X$  such that for each pair of adjacent tiles  $\tau_1, \tau_2$ , there is an anti-conformal map  $\phi : \tau_1 \rightarrow \tau_2$  fixing the common edge pointwise



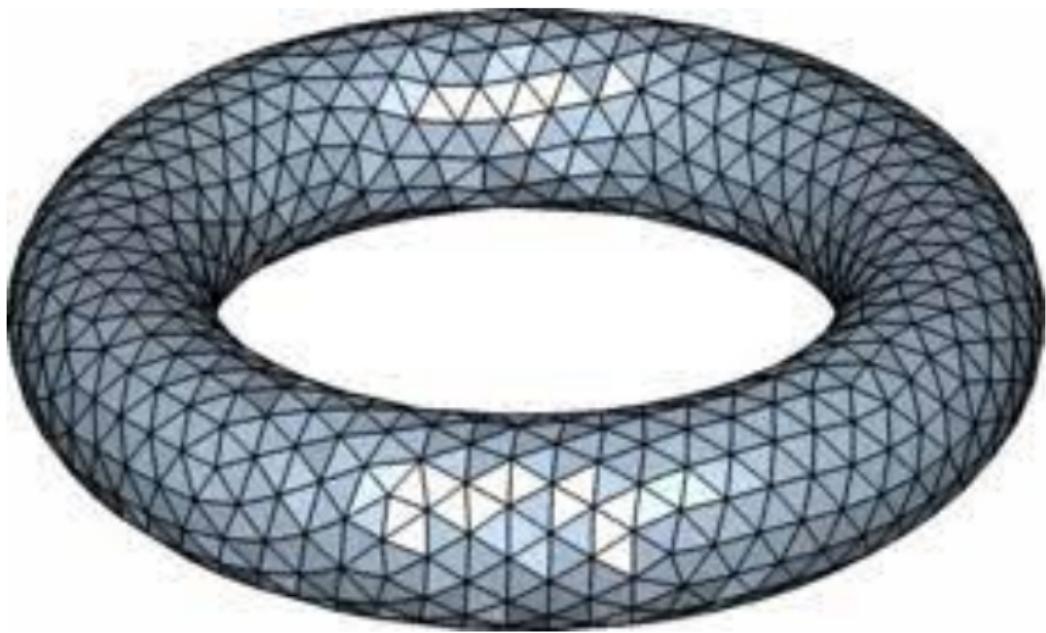
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Questions:

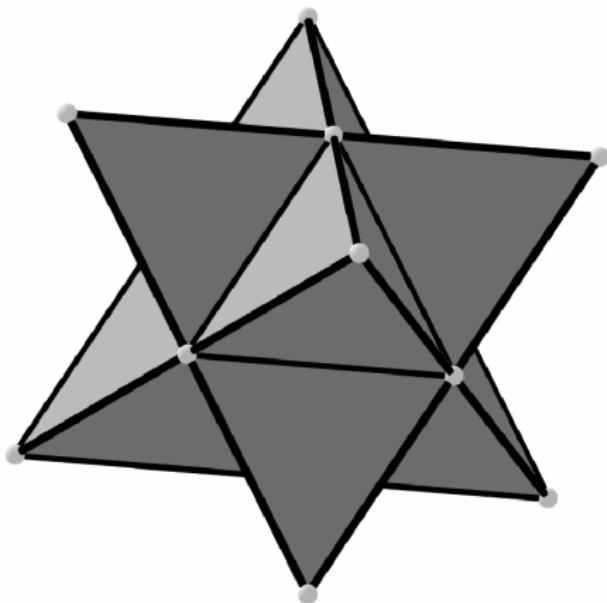
- Q1** Why is the pentagonal tiling a conformal tiling?
- Q2** How can we come up conformal tilings (of say  $\mathbb{C}$ )?  
(It is difficult to write down Schwarz reflections along a arc)
- Q3** What are all the conformal tilings of  $\mathbb{C}$ ?

## Equilateral surfaces



A collection of **equilateral triangles** glued together

# Equilateral surfaces



A collection of **equilateral triangles** glued together

# What is an Equilateral surface

- ▶ Starting with a finite/infinite collection of unit equilateral triangles, glue these triangles together by identifying every edge with exactly one edge of another triangle
- ▶ Also ensure that each vertex is identified with only finitely many other vertices
- ▶ This gives a topological surface  $E$
- ▶ Can we give  $E$  a complex structure, i.e. local charts to subsets of  $\mathbb{C}$ ?

Caution:

We are gluing the triangles "abstractly"

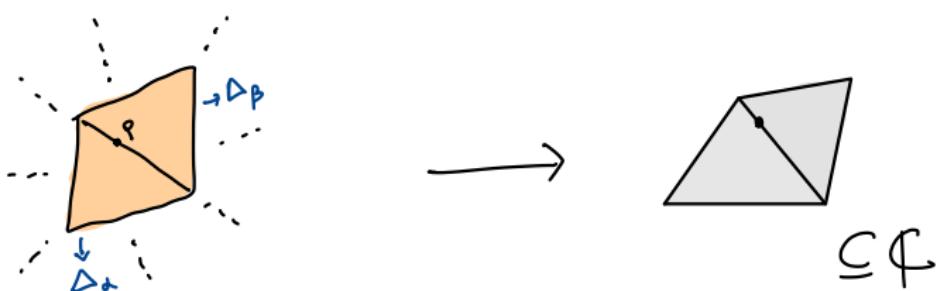
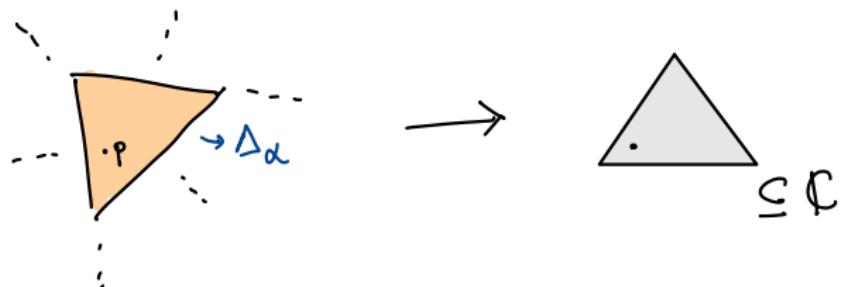
$$E = \bigsqcup_{\alpha \in I} \Delta_\alpha$$

/ *gluing data*

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- ▶ Can we give  $E$  a complex structure, i.e. local charts to subsets of  $\mathbb{C}$ ?
- ▶ The interiors of the unit equilateral triangles can be given the complex structure of  $\Delta$
- ▶ **Claim:** We can add compatible charts at edges and vertices to make  $E$  into a Riemann surface
- ▶  $E$  with the above complex structure is called an *equilateral surface*

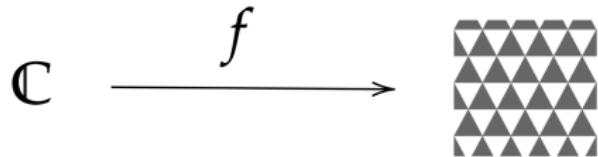
## Complex structure of $E$ at edges and vertices



# Alternative definition of conformal tiling

**Defn.** A conformal tiling of a Riemann surface  $X$  is a pair  $(E, f)$  such that  $f : X \rightarrow E$  is conformal

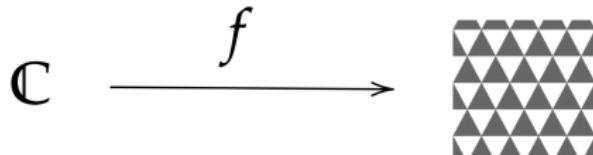
Example:



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Example:



- ▶ Recall Q2: How can we come up conformal tilings?
- ▶ One way is to just put together an equilateral surface by glueing triangles in a certain pattern and then finding a Riemann surface which is conformal to it
- ▶ Eg:



# Constructing conformal tilings

- ▶ The topology of  $E$  guides us to get hold of a Riemann surface  $X$  conformal to  $E$

## The Uniformization theorem

Any simply connected Riemann surface is isomorphic/conformal to either  $\hat{\mathbb{C}}$  or  $\mathbb{D}$  or  $\mathbb{C}$ .

- ▶ The uniformization theorem furnishes the required conformal map

Drawback: The map is an abstract and not constructive :(

# Constructing conformal tilings

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## The Uniformization theorem

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- ▶ The uniformization theorem furnishes the required conformal map  
Drawback: The map is an abstract and not constructive :(
- ▶ Given an equilateral surface  $E$  which is homeomorphic to the plane, the uniformization theorem tells us  $E$  is conformal to  $\mathbb{D}$  or  $\mathbb{C}$
- ▶ Determining between  $\mathbb{C}$  or  $\mathbb{D}$  is called the *type problem*

## Pentagonal tiling of $\mathbb{C}$

- ▶ Recall Q3: Why is figure 1 a conformal tilings of pentagons?
- ▶ Ans: We shall construct an equilateral surface made up of pentagons and homeomorphic to the plane. Then we will show it has to conformal to  $\mathbb{C}$  and not  $\mathbb{D}$

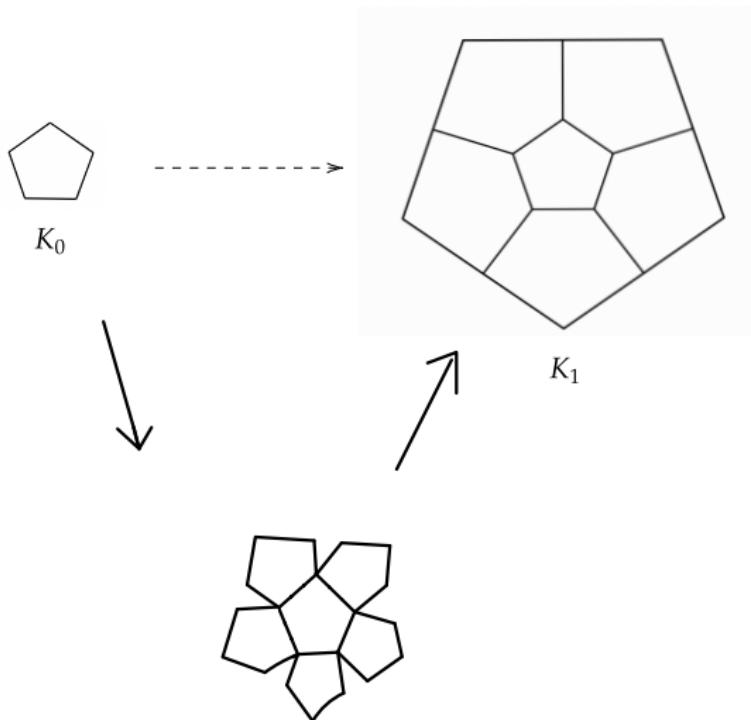
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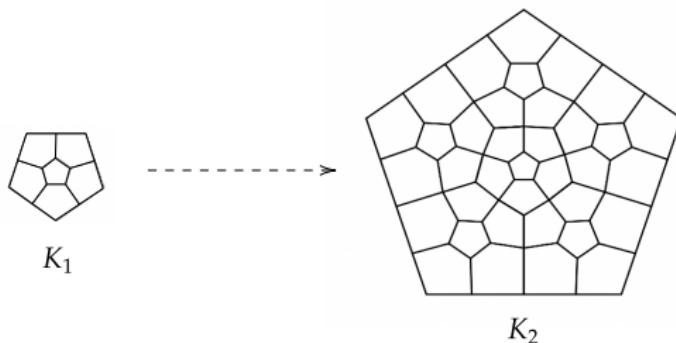
## Construction of the equilateral surface

- ▶ Start with a regular pentagon and call it  $K_0$
- ▶ Reflect  $K_0$  along all its edges to get a flower-like structure with 5 petals
- ▶ Glue adjacent petals and form  $K_1$
- ▶ View  $K_1$  as a pentagon and reflect along “edges” to get a big flower-like structure with 5 big petals
- ▶ Glue adjacent petals and form  $K_2$
- ▶ Repeat the process ad infinitum

# Construction of the equilateral surface for pentagonal tiling



# Construction of the equilateral surface for pentagonal tiling



# Why is this a pentagonal tiling of $\mathbb{C}$ ?

Why is  $E$  conformal to  $\mathbb{C}$ ?

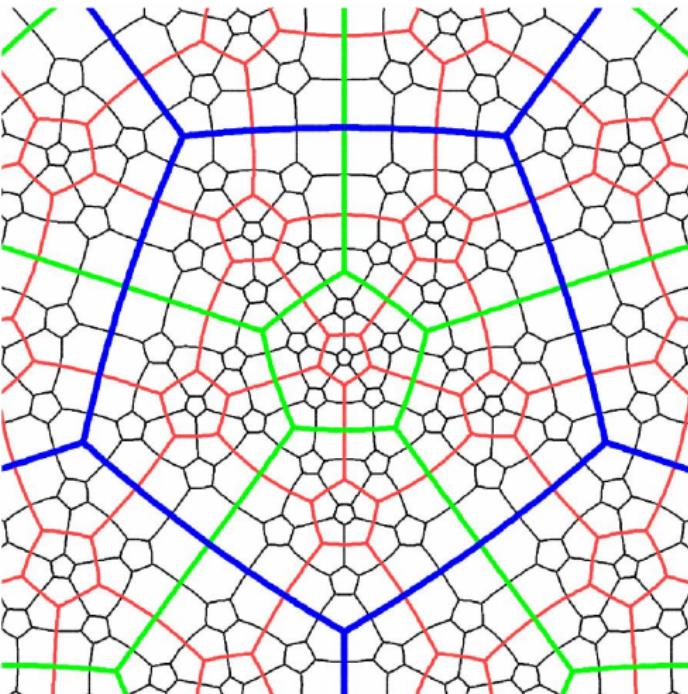
- ▶ Quick ans:  $E$  has a special symmetry called “*subdivisional self-similarity*” or “*zoom-in self-similarity*”
- ▶ Key observation: There is no  $f \in \text{Aut}(\mathbb{D})$  such that  $f(0) = 0$  and  $f$  is “*shrinking*”
- ▶ We say  $f$  is “*shrinking*” if there is a compact set  $S$  such that  $f(S) \subsetneq S$

# Why is this a pentagonal tiling of $\mathbb{C}$ ?

Why is  $E$  conformal to  $\mathbb{C}$ ?

- ▶ Quick ans:  $E$  has a special symmetry called “*subdivisional self-similarity*” or “*zoom-in self-similarity*”
- ▶ Key observation: There is no  $f \in \text{Aut}(\mathbb{D})$  such that  $f(0) = 0$  and  $f$  is “*shrinking*”
- ▶ We say  $f$  is “*shrinking*” if there is a compact set  $S$  such that  $f(S) \subsetneq S$
- ▶ Next observe that there is a conformal map  $g : E \rightarrow E$  such that  $g$  fixes a point and is “*shrinking*”
- ▶ This rules out the possibility that  $E$  is conformal to  $\mathbb{D}$

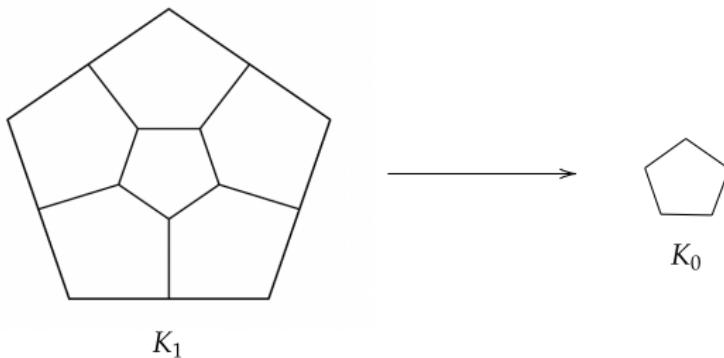
# Subdivisional self-similarity of $E$



The pentagonal tiling with colors to highlight “zoom-in self-similarity”

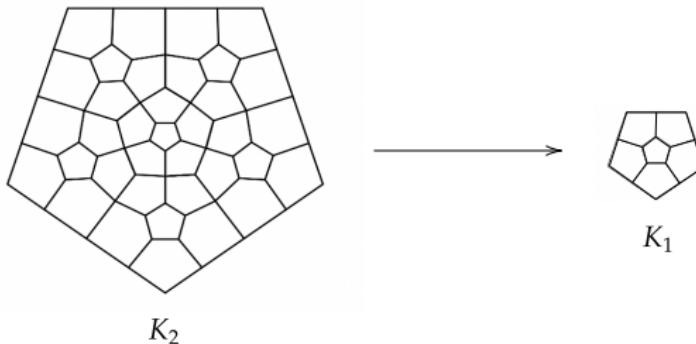
## Subdivisional self-similarity of $E$

- ▶ Start by defining a conformal map from  $K_1$  to  $K_0$  fixing the center of  $K_0$



## Subdivisional self-similarity of $E$

- ▶ Extend the above to a conformal map from  $K_2$  to  $K_1$  fixing the center of  $K_0$
- ▶ Continue this process to get a conformal from  $E$  to that fixes the center of  $K_0$  and is “shrinking”



## Summary so far

- ▶ We first defined conformal tiling using Schwarz reflections
- ▶ We gave another definition using equilateral tilings which enabled us to construct more examples of conformal tilings
- ▶ We constructed the pentagonal tiling and showed that it conformally tiles  $\mathbb{C}$

### Pause and ponder

- ▶ How did I get a picture of the pentagonal tiling?  
Ans: Convert to circle packing pattern + use circle packing algorithm
- ▶ Can we classify all the conformal tilings of the complex plane?  
What combinatorics give tiling of  $\mathbb{C}$ ?

Conformal triangulations  $\longleftrightarrow$  certain meromorphic fns

*Branched cover = covering map except a discrete set (critical values)*

Defn.

$f : X \rightarrow Y$  is a *branched covering* if

- ▶  $f$  is holomorphic
- ▶ for every  $p \in Y$ , there is a small neighbourhood  $U$  such that
  - ▶  $f^{-1}(U) = \sqcup_{\alpha} V_{\alpha}$  where each  $V_{\alpha}$  is a connected component
  - ▶  $f : V_{\alpha} \rightarrow U$  is  $z \mapsto z^k$  in suitable coordinates

Examples:

1.  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  given by  $z \mapsto z^2$
2.  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $z \mapsto z^2(1 - z)^3$

# Conformal triangulations $\longleftrightarrow$ certain meromorphic fns

*Branched cover = covering map except a discrete set (critical values)*

## Defn.

A branched cover  $f : X \rightarrow \hat{\mathbb{C}}$  is a *Belyi function* if  $f$  has at most three critical values on the sphere

**Prop.** Given a Riemann surface  $X$ , we have

$$\left\{ \begin{array}{l} \text{Belyi functions} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Conformal} \\ \text{triangulations of } X \end{array} \right\}$$

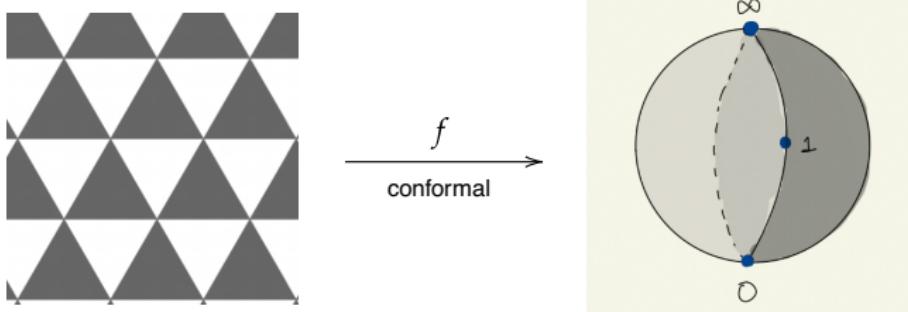
## Examples:

1.  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  given by  $z \mapsto z^2$
2.  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  given by  $f(z) = \wp_\Lambda(z)^2$  where  $\Lambda$  is such that  $\wp_\Lambda$  has  $0, 1, -1, \infty$  as critical values

# Conformal triangulations $\longleftrightarrow$ certain meromorphic fns

Conformal triangulation  $\longrightarrow$  Belyi function

- ▶ First recall the conformal triangulation of  $\hat{\mathbb{C}}$  consisting of two triangles
- ▶ Start with a particular triangle and map it to one of the triangles of  $\hat{\mathbb{C}}$  using a conformal map
- ▶ Pick an adjacent triangle and extend the above map to this triangle via Schwarz reflections
- ▶ Continue extending using Schwarz reflections



# Conformal triangulations $\longleftrightarrow$ certain meromorphic fns

Belyi function  $\longrightarrow$  Conformal triangulation

- ▶ Given  $f : X \rightarrow \hat{\mathbb{C}}$ , we “lift/pullback” the triangulation of  $\hat{\mathbb{C}}$  to get a triangulation  $X$
- ▶ The vertices of the triangulation will be  $f^{-1}\{0, 1, \infty\}$
- ▶ The edges of the triangulation will be  $f^{-1}([0, 1]), f^{-1}([1, \infty]), f^{-1}([\infty, 0])$
- ▶ Claim 1: This gives a triangular tiling of  $X$
- ▶ This triangular tiling will be a conformal tiling because we can get hold of a Schwarz reflection along each edge

# Which Riemann surfaces have conformal tilings?

**Lemma:** There is a compact Riemann surface which does not admit a conformal tiling

**Proof** by pigeonhole principle

1. Fact: There are uncountably many distinct compact Riemann surfaces
2. There are only countably many compact equilateral surfaces
  - ▶ Any compact eq. surface has finitely many triangles
  - ▶ For a fixed  $n \in \mathbb{N}$ , there are finitely many eq. surfaces having  $n$  triangles  
 $(E$  is determined by combinatorics of how  $\Delta$  are glued together)

# Which Riemann surfaces have conformal tilings?

Belyi's theorem: Let  $X$  be a **compact** Riemann surface, then

$X$  admits a conformal tiling



$X$  admits a Belyi function



$\exists$  polynomials  $p_1, \dots, p_{n-1}$  such that

$X \simeq \{[z_0 : \dots : z_n] : p_1(\bar{z}) = \dots = p_{n-1}(\bar{z}) = 0\} \subseteq \mathbb{C}P^n$

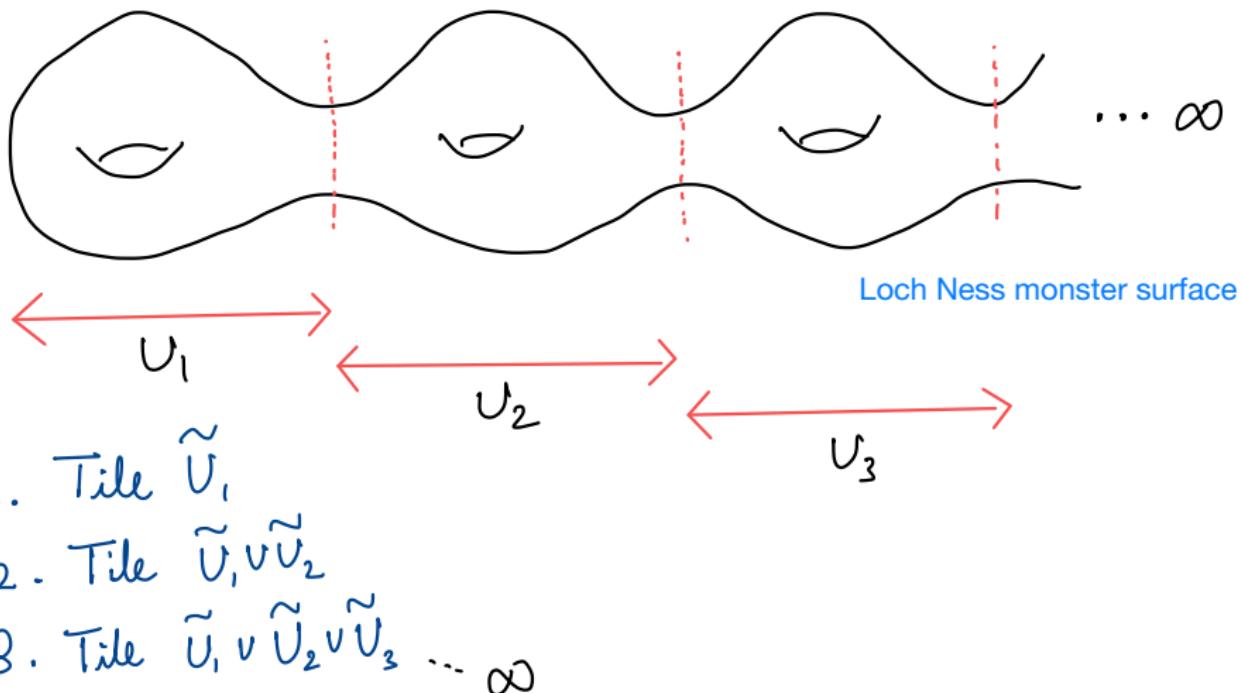
and the coefficients of  $p_i$  are **algebraic numbers**

*smooth  
algebraic  
curve  
over  $\overline{\mathbb{Q}}$*

*zero locus*

# Which Riemann surfaces have conformal tilings?

Theorem [Bishop & Rempe, 21]: Every non-compact Riemann surface admits a conformal tiling



## References

- [BR21] Christopher J Bishop and Lasse Rempe. “Non-compact Riemann surfaces are equilaterally triangulable”. In: *arXiv preprint arXiv:2103.16702* (2021).
- [BS97] Philip Bowers and Kenneth Stephenson. “A “regular” pentagonal tiling of the plane”. In: *Conformal Geometry and Dynamics of the American Mathematical Society* 1.5 (1997), pp. 58–86.
- [JW18] G.A. Jones and J. Wolfart. *Dessins d’Enfants on Riemann Surfaces*. Springer International Publishing, 2018.