Geometry of Principal Bundles
Lecture 1: Introduction to Principal Bundles

Abstract

If a vector bundle $\pi : E \to M$ with fiber $V = \mathbb{R}^n / \mathbb{C}^n$ has an additional algebraic structure, then the structure group reduces from $GL(V)$ to a subgroup $G \subseteq GL(V)$. Using this as motivation, we introduce principal G-bundles. Examples of $U(1)$-bundles over S^2 is discussed. Lastly, the relationship between vector bundles and principal bundles through frame bundles, transition maps, and Lie group representations is discussed.

Motivation: vector bundles with extra structure

Suppose $\pi : E \to M$ is a vector bundle over a smooth manifold M with fiber $E_x = \mathbb{R}^n$ over each $x \in M$. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of M and $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ be the trivializations of the vector bundle E. The transition maps of E are $\tau_{\alpha\beta}(x) := \varphi_{\alpha,x} \circ \varphi_{\beta,x}^{-1}$ and they define maps:

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

Note that E is completely determined by the collection of transition maps $\{\tau_{\alpha\beta}\}_{\alpha,\beta \in I}$.

Recall that an affine connection/covariant derivative on E is locally specified by a collection of matrix-valued 1-forms $\{\Gamma_\alpha\}$ (sometimes called connection matrix and/or Christoffel symbols) and they define maps:

$$\begin{aligned}
&\Gamma_\alpha : TU_\alpha \xrightarrow{C^\infty\text{-linear}} M(n, \mathbb{R}) = Lie(GL(n, \mathbb{R})) \\
&\Gamma_\alpha \in \Gamma\left(T^*U_\alpha \otimes M(n, \mathbb{R})\right) \\
&\Gamma_\alpha \in \Gamma\left(T^*U_\alpha \otimes Lie(GL(n, \mathbb{R}))\right).
\end{aligned}$$

Lemma 1. Suppose that each fiber $E_x = \mathbb{R}^n$ of E is given an inner product/metric that smoothly varies with $x \in M$. Let ∇ be a covariant derivative on E that is compatible with the fiber metric, i.e.,

$$\partial_i \langle X, Y \rangle = \langle \nabla_{\partial_i} X, Y \rangle + \langle X, \nabla_{\partial_i} Y \rangle.$$

Then we can choose certain special trivializations of E such that the transition maps $\tau_{\alpha\beta}$ and the connection matrix Γ_α take the following special form:

$$\begin{aligned}
\tau_{\alpha\beta} &: U_\alpha \cap U_\beta \to O(n, \mathbb{R}) \\
\Gamma_\alpha &: TU_\alpha \xrightarrow{C^\infty\text{-linear}} Lie(O(n, \mathbb{R})).
\end{aligned}$$

Proof Sk. Given a trivializations $\varphi_\alpha^{-1} : U_\alpha \times \mathbb{R}^n \to E|_{U_\alpha}$ we can pullback the fiber metric on E to the fibers of $U_\alpha \times \mathbb{R}^n$, i.e., $\{x\} \times \mathbb{R}^n$ for $x \in U_\alpha$. Now, choose trivializations φ_α

such that the metric on each $\{x\} \times \mathbb{R}^n$ is the standard Euclidean metric on $\mathbb{R}^n$; equivalently, φ_α is such that the standard basis $\{e_\mu : 1 \leq \mu \leq n\}$ of $\mathbb{R}^n$ is orthonormal. Such a trivialization can be constructed by Gram-Schmidt process.
In these trivializations, the transition map $\tau_{\alpha\beta}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ with the standard Euclidean metric. Hence, $\tau_{\alpha\beta}(x)$ sends orthonormal bases to orthonormal bases, i.e., $\tau_{\alpha\beta}(x) \in O(n, \mathbb{R})$. Next, the connection matrix in these trivializations are skew-symmetric because:

$$\begin{aligned} \Gamma^\mu_{i\nu}(x) &= \langle \nabla_i e_\mu(x), e_\nu(x) \rangle \\ &= \partial_i \langle e_\mu(x), e_\nu(x) \rangle - \langle e_\mu(x), \nabla_i e_\nu(x) \rangle \\ &= 0 - \Gamma^\nu_{i\mu}(x). \end{aligned} \qquad \square$$

General principle: If a vector bundle E has some additional algebraic structure then the trivializations can be chosen/arranged such that the transition map takes the form

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

where G is a Lie group. The group G is called the ***structure group*** of E. Furthermore, if E has an affine connection/covariant derivative that is "compatible" with the additional structure on E, then, the connection matrix in these trivializations take the form:

$$\Gamma_\alpha : TU_\alpha \xrightarrow{C^\infty\text{-linear}} Lie(G).$$

Examples:

Additional algebraic structure on fibers of E	Structure group
Orientation + metric	$SO(n)$
Complex vector space	$GL(n, \mathbb{C})$
Hermitian metric	$U(n)$

What is a Principal G-bundle?

1. If G is a Lie group, then a principal bundle and a connection on it is just the collection of the following data:

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$
$$\Gamma_\alpha : TU_\alpha \xrightarrow{C^\infty\text{-linear}} Lie(G).$$

2. Informally, a principal G-bundle captures two properties of a vector bundle E: (i) the structure group G of E and (ii) the "twist" present in the vector bundle E.

Principal G-bundles and examples

Example 1 (trivial principal bundle)**.**
Given a manifold M and Lie group G, consider the product $M \times G$ and define a right

action:

$$(x, h) \cdot g := (x, hg).$$

This is called the trivial principal G-bundle over M.

Definition 1. Fix a manifold M and a Lie group G. A *principal G-bundle* over M is a smooth manifold P with:

1. A right action of G on P:
$$P \times G \to P$$
$$(p, g) \mapsto pg$$
2. A surjective map $\pi : P \to M$ that is G-invariant, i.e., $\pi(pg) = \pi(p)$.
3. Trivializations $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ such that if $\psi(p) = (x, h)$, then $\psi(pg) = (x, hg)$. This property of ψ_α is called G-*equivariance*.

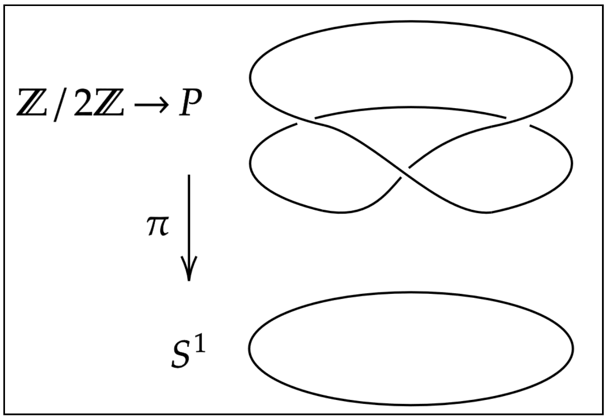

Figure 1: Non-trivial Z/2Z principal bundle over the circle.
(created by Tazerenix and licensed under CC BY-SA 4.0.)

transition maps of a principal bundle: Note the transition maps are given by:

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G.$$

Write $\psi_\alpha \circ \psi_\beta^{-1}(x, h) = (x, f(x, h))$. Now, as ψ_α, ψ_β are G-equivariant, we have

$$\begin{aligned} &\psi_\alpha \circ \psi_\beta^{-1}(x, hg) = (x, f(x, h)g), \\ \implies &f(x, h) = f(x, e)h \\ \implies &\psi_\alpha \circ \psi_\beta^{-1}(x, h) = (x, f(x, e)h). \end{aligned}$$

Thus, the transition maps reduce to $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to G$ defined by $\tau_{\alpha\beta}(x) = f(x, e)$, or equivalently,

$$\psi_\alpha \circ \psi_\beta^{-1}(x, h) = (x, \tau_{\alpha\beta}(x)h).$$

Definition 2 (co-cycle definition). Let M be a manifold with open cover $\{U_\alpha\}_{\alpha \in I}$. A *principal G-bundle* over M can be defined by a collection of transition maps $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to G$ satisfying the co-cycle condition:

$$\tau_{\alpha\beta}(x) \cdot \tau_{\beta\gamma}(x) \cdot \tau_{\gamma\alpha}(x) = e \qquad \forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Note: The two definitions of principal bundles are equivalent.

Example 2. A family of principal $U(1)$-bundles over the sphere S^2.
Let `NP` and `SP` denote the north pole and south pole of S^2, respectively. Consider the open cover of S^2 by two open sets $U_1 = S^2 \setminus \{\mathtt{NP}\}$ and $U_2 = S^2 \setminus \{\mathtt{SP}\}$. We define a principal $U(1) = S^1$-bundle over S^2 by describing the transition map

$$\tau_{12} : U_1 \cap U_2 \to S^1$$
$$\Leftrightarrow \tau_{12} : \mathbb{R}^2 \setminus \{0\} \to S^1.$$

First, for each $m \in \mathbb{Z}$, define a function $f_m : S^1 \to S^1$ by $f(z) = z^m$. Then, define

$$\begin{tikzcd} U_1 \cap U_2 = \mathbb{R}^2 \setminus \{0\} \arrow[rr, dashed, "\tau_{12}^{(m)}"] \arrow[dr, "\frac{(x_1, x_2)}{||x||}"'] & & S^1 \\ & S^1 \arrow[ur, "f_m(z) = z^m"'] & \end{tikzcd}$$

For each transition map $\tau_{12}^{(m)}$ we get a principal $U(1)$-bundle $P^{(m)}$ over S^2. For example, $P^{(0)}$ is the trivial bundle $S^2 \times U(1)$ because the transition maps are trivial.

Remark: If S^n is a Lie group, then the same construction gives a principal S^n-bundle over S^{n+1}, i.e., we define a transition map from the equator S^n of S^{n+1} to the group S^n by $f_m(g) = g^m$. Other than the circle, S^0 and S^3 are Lie groups (Recall $S^3 = SU(2)$). Following this circle of ideas, we can get "exotic spheres" that are homeomorphic but not diffeomorphic to S^7 (see example 10.6 of [Taubes]).

Gauge group of P
An *isomorphism* $\eta : P \to P'$ of two principal G-bundles P, P' over M is a bundle map that is G-equivariant. By equivariant, we mean η commutes with the right action of G, i.e., $\eta(p \cdot g) = \eta(p) \cdot g$.

$$\begin{tikzcd} P \arrow[rr, "\eta"] \arrow[dr] & & P' \\ & M \arrow[ur] & \end{tikzcd}$$

The set of principal bundle automorphisms of P is called the *gauge group* of P and is denoted by $\mathcal{G}$.

Exercise 1. Let $P = M \times G$ be the trivial bundle. Show that the gauge group $\mathcal{G}$ of P is isomorphic to $C^\infty(M; G)$. Moreover, the tangent space at identity is $C^\infty(M; Lie(G))$.

Question 1. Given a manifold M and a Lie group G, what are all the principal G-bundles over M?

Fact 1. When $M = S^2$ and $G = U(1) = S^1$, a principal $U(1)$-bundle over S^2 is completely and uniquely determined by the degree of the transition map $\tau_{12} : S^1 \to S^1$. Thus, there are only $\mathbb{Z}$ many principal $U(1)$-bundle over S^2 up to isomorphism. This can be proved using the following facts:

1. If two transition maps $\tau_{12} : U_1 \cap U_2 \to S^1$, $\tilde{\tau}_{12} : U_1 \cap U_2 \to S^1$ are homotopic, then the resulting principal bundles are isomorphic.
2. The homotopy class of $\tau_{12} : S^1 \to S^1$ is completely determined by the degree of the map τ_{12}.

Relationship between principal G-bundles and vector bundles (associated vector bundles)

General principle:

$$\left\{\begin{matrix}\text{Vector bundles over } M \\ \text{with structure group } G\end{matrix}\right\} \underset{\text{correspondence}}{\longleftrightarrow} \left\{\begin{matrix}\text{Principal } G\text{-bundles} \\ \text{over } M\end{matrix}\right\}$$

Note: This is a 1-1 correspondence for matrix Lie groups, i.e. $G \subseteq GL(V)$.

Vector bundles to principal bundles (the case of $G = O(n)$)

Let $\pi : E \to M$ denote a rank n vector bundle with a smoothly varying fiber metric. We shall construct a principal G-bundle $P_{O(E)}$ associated with E.

Description 1 (transition maps). Let

$$\left\{\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to O(n)\right\}_{\alpha\beta\in I}$$

be the transition maps of E. These transition maps also describe a principal G-bundle which we call$P_{O(E)}$.

Description 2 (frame bundles). Introduce $P_{O(E)}$ to be submanifold of $\bigoplus^n E$ such that:

$P_{O(E)} = \left\{(x, (v_1, v_2, \ldots, v_n)) \in \bigoplus^n E \,:\, (v_1, v_2, \ldots, v_n) \text{ is an orthonormal basis for } E_x\right\}.$

Further, $P_{O(E)}$ is principal bundle because we have:

1. A projection $\pi_n : P_{O(n)} \to M$ which is the restriction of the projection $\pi_n : \bigoplus^n E \to M$.
2. A right $O(n)$-action on $P_{O(n)}$ given by:

$$
\begin{aligned}
(x,(v_1,v_2,\,\dots\,,v_n))\cdot g &= (x,(v_1,v_2,\,\dots\,,v_n)\cdot g)\\
&= \left(x,\left(\sum_k g_{k,1}v_k,\sum_k g_{k,2}v_k,\,\dots\,,\sum_k g_{k,n}v_k\right)\right)
\end{aligned}
$$

3. Trivializations $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times O(n)$ defined as follows. Let $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ be a trivialization of E where the fiber metric on $\{x\}\times\mathbb{R}^n$ is the standard Euclidean metric. This gives $\varphi_{\alpha,n} : \bigoplus^n E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^{n\times n}$ whose restriction to $P|_{U_\alpha}$ is $\psi_\alpha : P|_{U_\alpha} \to U_\alpha \times O(n)$, where each element of $\mathbb{R}^{n\times n}$ is thought of an n columns vectors listed consecutively.
4. The transition maps $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)$ of $P_{O(n)}$ are defined by
$$\psi_\alpha \circ \psi_\beta^{-1} : U_\beta \times O(n) \to U_\alpha \times O(n)$$
$$\psi_\alpha \circ \psi_\beta^{-1}(x,(v_1,v_2,\,\dots\,,v_n)) = (x,\tau_{\alpha\beta}(x)\cdot(v_1,v_2,\,\dots\,,v_n)),$$
On the other hand, $\psi_\alpha \circ \psi_\beta^{-1}$ is by definition the restriction of $\varphi_{\alpha,n} \circ \varphi_{\beta,n}^{-1} : U_\beta \times \mathbb{R}^{n\times n} \to U_\alpha \times \mathbb{R}^{n\times n}$ which is given by:
$$\varphi_{\alpha,n} \circ \varphi_{\beta,n}^{-1}(x,(v_1,v_2,\,\dots\,,v_n)) = \left(x,\left(\varphi_{\alpha,x}\circ\varphi_{\beta,x}^{-1}(v_1),\,\dots\,,\varphi_{\alpha,x}\circ\varphi_{\beta,x}^{-1}(v_n)\right)\right).$$
Thus, $\tau_{\alpha\beta}(x) = \varphi_{\alpha,x}\circ\varphi_{\beta,x}^{-1}$, i.e., the transition maps of the principal bundle $O(n)$ and the vector bundle E are the same!

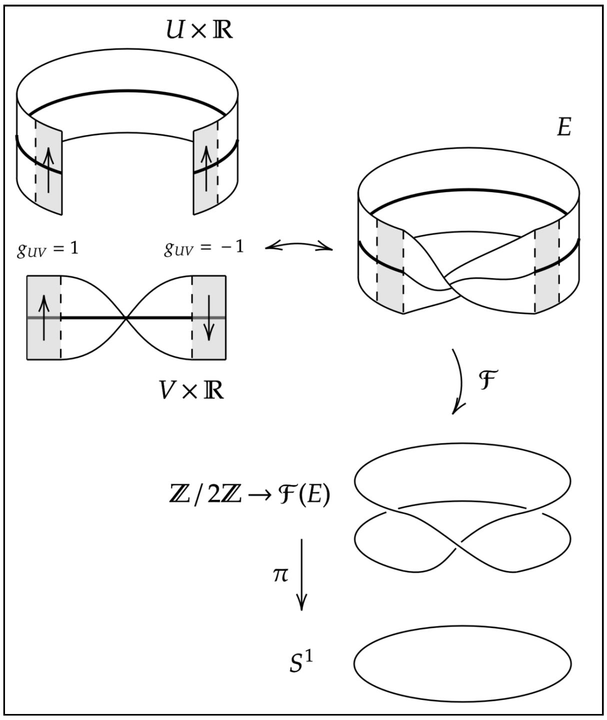

Figure 2: The orthonormal frame bundle of the mobius bundle is a non-trivial principal Z/2 bundle over the circle. (created by Tazerenix and licensed under CC BY-SA 4.0.)

Exercise 2. Let P be a principal G-bundle over M. If P has a global section $s : M \to P$, then P is isomorphic to the trivial bundle $M\times G$.

Remark. The above is analogous to the following fact: a rank n vector bundle E over M is isomorphic to the trivial bundle $M\times\mathbb{R}^n$ if and only if E has n global sections $\{s_1, \ldots, s_n\}$ such that at each point $x\in M$, the collection $\{s_1(x), \ldots, s_n(x)\}$ is a basis for the fiber $E|_x$.

Exercise 3 (principal bundles over S^1). When G is a connected Lie group, show that every principal G-bundle over $M=S^1$ is isomorphic to the trivial bundle, i.e. $S^1\times G$.

Principal bundles to vector bundles via representations
If G is a matrix subgroup, i.e., a subgroup of $GL(V)$ and $\pi: P\to M$ is a principal G-bundle over M, we can reverse the above steps to obtain a rank n vector bundle over M with structure group G. For a slightly more general construction, fix a representation $\rho: G\to GL(V)$ of G (when G is a subgroup of $GL(V)$, we can take ρ to be the inclusion map). Now, we shall construct a vector bundle $P\times_\rho V$ associated with P via the representation ρ.

<u>Description 1 (transition maps)</u>. Let

$$\left\{\tau_{\alpha\beta}:(U_\alpha\cap U_\beta)\to G\right\}_{\alpha\beta\in I}$$

be the transition maps of P. Now consider the collection of maps:

$$\left\{\rho\circ\tau_{\alpha\beta}:(U_\alpha\cap U_\beta)\to GL(V)\right\}_{\alpha\beta\in I}.$$

These transition maps describe a vector bundle over M with fiber V and structure group $\rho(G)\subseteq GL(V)$. This vector bundle is denoted by $P\times_\rho V$.

<u>Description 2 (quotient construction)</u>.
Introduce $P\times_\rho V$ as a set obtained by the quotient of $P\times V$ with the equivalence relation that relates (p,v) to all pairs of the form $\left(pg^{-1},\rho(g)v\right)$ with $g\in G$.

$$\begin{tikzcd} P\times V \arrow[r] \arrow[d] & P\times_\rho V \arrow[d] \\ P \arrow[r, "\pi"] & M \end{tikzcd}$$

Further, $P\times_\rho V$ is principal bundle because we have:

1. A projection map that sends the equivalence class $[(p,v)]$ to $\pi(p)\in M$.
2. An action of $\mathbb{R}/\mathbb{C}$ on $P\times_\rho V$ given by $\lambda\cdot(p,v)\mapsto(p,\lambda v)$ and a 0-section given by $x\mapsto[(p_x,0)]$.
3. Trivializations φ_α that are defined as follows. Let

$$\psi_\alpha: P|_{U_\alpha}\to U_\alpha\times G$$
$$\psi_\alpha: p\mapsto(\pi(p),g_p)$$

be the trivializations of P. Define the trivializations of $P\times_\rho V$ as

$$\varphi_\alpha : (P\times_\rho V)|_{U_\alpha} \to U_\alpha \times V$$
$$\varphi_\alpha : [(p,v)] \mapsto (\pi(p), \rho(g_p)v).$$

4. The transitions functions $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ is defined by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x,v) = (x, \tau_{\alpha\beta}(x)v).$$

But on the other hand, unpacking the definition of φ_α, we also obtain

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x,v) = \left(x, \rho\left(\psi_\alpha \circ \psi_\beta^{-1}(x)\right)v\right).$$

Hence, the transition maps $\tau_{\alpha\beta}$ of $P\times_\rho V$ is given by $x \mapsto \rho \circ \left(\psi_\alpha \circ \psi_\beta^{-1}(x)\right)$, i.e. ρ composed with the transition maps of P.

Pullback of principal bundles

Given $f : N \to M$ and a principal G-bundle $\pi : P \to M$, we can pullback P to get a principal G-bundle f^*P over N.

$$\begin{tikzcd}
f^*P \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\
N \arrow[r, "f"] \arrow[u, bend left, dashed, "f^*s"] & M \arrow[u, bend left, "s"]
\end{tikzcd}$$

More about pullbacks:

1. f^*P is defined as:

$$f^*P = \{(p,x) \in P\times N : \pi(p) = f(x)\},$$

with G-action given by: $(p,x)\cdot g = (pg, x)$.

2. If $\{U_\alpha\}$ is an open cover for M and $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the transition maps for P, then the transition maps for f^*P with respect to the open cover $\left\{f^{-1}(U_\alpha)\right\}$ is given by

$$\tilde{\tau}_{\alpha\beta} : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to G$$
$$\tilde{\tau}_{\alpha\beta} : x \mapsto \tau_{\alpha\beta}(f(x))$$

3. Any section $s : M \to P$ induces a section $f^*s : N \to f^*P$.
4. Pullbacks are natural in the following two ways:
   (a) Let E be a vector bundle and P_E be the principal bundle of frames in E. Then,

$$f^*P_E = P_{f^*E}.$$

   (b) Let ρ be a representation and $P\times_\rho V$ be an associated vector bundle. Then,

$$f^*(P\times_\rho V) = \left(f^*(P)\times_\rho V\right).$$

Further reading

Chapter 10, Taubes, C. H. (2011). *Differential geometry: bundles, connections, metrics and curvature* (Vol. 23). OUP Oxford.