An introduction to Chern-Weil theory: invariants from curvature

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Recall:

- a principal $GL(n, \mathbb{C})$-bundle is the same as a complex vector bundle
- a principal $U(n)$-bundle is the same as a complex vector bundle with Hermitian metric
- a principal $SU(n)$-bundle is the same as a complex vector bundle with Hermitian metric where $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to SU(n)$ and $a_\alpha : TU_\alpha \to Lie(SU(n))$
- a principal $O(n)$-bundle is the same as a real vector bundle with a smooth varying inner product on fiber

Exercise 0: Show that every principal G-bundle $P \to M$ admits a connection.

We introduce invariants of vector bundles (and principal bundles).

- These invariants live in the de Rham cohomology $H^*_{dR}(M; \mathbb{R})$.
- These invariants are <u>defined</u> using the curvature tensor $F_A \in \Omega^2\left(\mathrm{ad}\, P\right) \approx \Omega^2(Lie(G))$. But the end result is independent of the connection A on the vector bundle. Roughly, this is because the space of connections is an affine space which is contractible. Hence, the invariant does not depend of the choice of connection.
- We shall use these to show that the the family of $SU(2)$-bundles $\{P^m\}_{m \in \mathbb{Z}}$ over S^4 are distinct.

<u>The first Chern class of an Hermitian vector bundle</u>

In this section, we introduce an invariant for Hermitian vector bundles, i.e., complex vector bundles with an Hermitian metric, i.e., a $U(n)$-bundle. Given $E \to M$, a connection A, and curvature F_A:

Curvature tensor of E		Invariant of E
F_A		$c_1(E)$
Matrix valued 2-form	$\xrightarrow{\det, \mathrm{Tr}\ ?}$	Real valued 2-form

Definition 1. The *<u>1st Chern class</u>* of a $U(n)$-bundle $E \to M$ is obtained by picking a $U(n)$-connection A and considering the following 2-form in $H^2_{dR}(M; \mathbb{R})$:

$$c_1(E) := \left[\frac{\sqrt{-1}}{2\pi} \mathrm{Tr}(F_A) \right].$$

Remarks/ properties of 1st Chern class:

1. **(Ad-invariance of trace)** To compute $\mathrm{Tr}(F_A)$, suppose in local trivializations, $F_A = \sum_\mu f_\mu \omega_\mu$, where $f_\mu : U_\alpha \to Lie(G)$ and $\omega_\mu \in \Omega^2(U_\alpha)$. Then,
$$\mathrm{Tr}(F_A) = \textstyle\sum_\mu \mathrm{Tr}(f_\mu)\, \omega_\mu.$$
Note that F_A transforms $\eta F_A \eta^{-1}$ under change of trivializations where $\eta : U_\alpha \to G$; hence, $\mathrm{Tr}\left(\eta F_A \eta^{-1}\right) = \mathrm{Tr}(F_A)$ and the definition is independent of trivializations.
2. **(Why $U(n)$ and why $\sqrt{-1}$)** A priori, $\mathrm{Tr}(F_A)$ is a $\mathbb{C}$-valued 2-form. But, as A is a $U(n)$ connection, in local trivializations we have $F_A = \sum_\mu f_\mu \omega_\mu$ where f_μ are matrices in $Lie(U(n))$, i.e., the matrices f_μ are skew-Hermitian and $\mathrm{Tr}(f_\mu) \in \sqrt{-1}\,\mathbb{R}$. Thus, $\sqrt{-1}\,\mathrm{Tr}(F_A)$ is a $\mathbb{R}$-valued 2-form.
3. **(Chern class is an element of de Rham cohomology)** To show $\mathrm{Tr}(F_A)$ is a closed form, notice that
$$d\,\mathrm{Tr}(F_A) = \mathrm{Tr}(d_A F_A).$$
The easiest way to see this is to compute both sides in local trivialization. Lastly, the RHS is zero because $d_A F_A \equiv 0$ by Bianchi's identity. Thus, $c_1(P)$ is an element of $H^2_{dR}(M; \mathbb{R})$.
4. **(Doesn't depend on connection)** If we started with a different connection, then, $\mathrm{Tr}(F_A) - \mathrm{Tr}(F_{A'})$ would be an exact form, i.e., $\mathrm{Tr}(F_A) - \mathrm{Tr}(F_{A'}) = d\omega$ for $\omega \in \Omega^1(M)$. The idea is to consider the homotopy $A_t = A + t(A' - A)$ (see last paragraph of page 173 of [Taubes] for proof). Hence, the de Rham element $c_2(P)$ does not depend on the connection A.
5. **(Naturality of Chern class)** If $\eta : P_1 \to P_2$ is an isomorphism of principal bundles, then we can extend this to an isomorphism $\eta : (P_1, A_1) \to (P_2, A_2)$. Now, in local trivializations $U_\alpha \times G$, we have
$$\eta : (x, g) \mapsto (x, \eta_\alpha(g))$$
$$F_{A_2} = \eta_\alpha F_{A_1} \eta_\alpha^{-1}$$
As a result, $\mathrm{Tr}(F_{A_1}) = \mathrm{Tr}(F_{A_2})$ and hence $c_2(P_1) = c_2(P_2)$.

Observation 1. If $E \to M$ is a $SU(n)$-bundle, then $c_1(E) = 0$.
Proof: Let A be a $SU(n)$-connection on the bundle $E \to M$. This means that $F_A \in \Omega^2(Lie(SU(n)))$, i.e., it is a matrix-valued 2-form, where the matrices are in $Lie(SU(n))$. In particular, $\mathrm{Tr}(F_A) \equiv 0$. Hence, $c_1(E) = 0$. □

Question 1. Is the converse true? Given a $U(n)$-bundle $E \to M$ such that $c_1(E) = 0$, is $E \to M$ also a $SU(n)$ bundle? In fancy language, does the structure group of E reduce to $SU(n) \subseteq U(n)$? that is, can the determinant bundle of E be trivialized or can the trivializations of E be chosen such that $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \to SU(n)$?

Answer: The converse may not be true if we use the 1st Chern class defined above. But, it turns out that if we use 1st Chern class "with integer coefficients", then the converse is true (for a discussion, see this StackExchange post).[1] Thus, the 1st Chern class "with integer coefficients" can be thought of as the "obstruction" to reducing the structure group from $U(n)$ to $SU(n)$.

Remark: For rank 1 complex vector bundles (line bundles), the 1st Chern class is a complete invariant. Moreover the bijection between isomorphism classes of rank 1 complex vector bundles and 1st Chern class is actually a group homomorphism in the sense that:

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Generalizing the 1st Chern class: ad-invariant functions

Definition 2. A smooth function $h: M_n(\mathbb{C}) \to \mathbb{C}$ is said to be *ad-invariant* if

$$h\left(gmg^{-1}\right) = h(m) \quad \forall\, m \in M_n(\mathbb{C}), g \in GL_n(\mathbb{C}).$$

Alternatively, $h(m)$ is a symmetric function of the eigen-values of m, i.e., $h(m) = h(\lambda_1, \ldots, \lambda_n)$, where λ_i are the eigen-values of m.[2]

Examples:

- $\mathrm{Tr}(m) = \sum_{\mu=1}^{n} \lambda_\mu$
- $\det(m) = \prod_{\mu=1}^{n} \lambda_\mu$

Goal: given a ad-invariant function, we want to define a characteristic class

$$c_h(E) := \left[\sqrt{-1}^{\,d} h(F_A)\right]$$

like the way we defined the 1st Chern class.

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[1] We thank Prof Vamsi Pingali for pointing out that the Converse might not hold if we use 1st Chern class as defined in this note.

[2] Given a smooth symmetric function $\hat{h}(\lambda_1, \ldots, \lambda_n)$, setting $h(m) := \hat{h}\left(\text{eigen-values}(m)\right)$ gives an ad-invariant smooth function. Conversely, given a smooth ad-invariant function h, consider the smooth symmertic function $\hat{h}(\lambda_1, \ldots, \lambda_n) := h\left(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\right)$. We claim that $h(m) = \hat{h}\left(\text{eigen-values}(m)\right)$ for all $m \in M_n(\mathbb{C})$. First observe this is true for diagonalizable matrices. Then, by density of diagonalizable matrices in $M_n(\mathbb{C})$ and continuity of h and $\hat{h}$, the claim follows.

Attempt 1: Consider $h = \det$. If $F_A = \sum_\mu f_\mu \omega_\mu$, put

$$\det(F_A) = \sum_\mu \det(f_\mu)\omega_\mu?$$

Flaw: $\det(tf_\mu) = t^n \det(f_\mu)$, i.e., $\sum_\mu f_\mu \omega_\mu = \sum_\mu (tf_\mu)\left(t^{-1}\omega_\mu\right)$, but $\sum_\mu \det(f_\mu)\omega_\mu \neq \sum_\mu \det(tf_\mu)t^{-1}\omega_\mu$.

Attempt 2: Recall that $\det$ is a polynomial in the entries of matrices (Leibniz formula):

$$\det(\{m_{ij}\}) = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot m_{1\sigma_1} \cdots m_{n\sigma_n}.$$

Next write

$$\begin{aligned} F_A &= \{\Omega_{ij}\}_{1 \le i,j \le n} \\ \det(F_A) &= \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \cdot (\Omega_{1\sigma_1} \wedge \Omega_{2\sigma_2} \wedge \cdots \wedge \Omega_{n\sigma_n}). \end{aligned}$$

Now note that $\det(F_A)$ is scalar valued n-form and defines the cohomology class:

$$\left[\sqrt{-1}^{\,n} \det(F_A)\right].$$

Take away: if $h(m)$ is a homogeneous polynomial of degree d in the entries of m, i.e., if

$$h(\{m_{ij}\}) = \sum_{I,J} b_{IJ} \cdot m_{i_1j_1} m_{i_2j_2} \cdots m_{i_kj_k},$$

then we can do:

$$\begin{aligned} F_A &= \{\Omega_{ij}\}_{1 \le i,j \le n} \\ h(F_A) &= \sum_{I,J} b_{IJ} \cdot (\Omega_{i_1j_1} \wedge \Omega_{i_2j_2} \wedge \cdots \wedge \Omega_{i_kj_k}). \end{aligned}$$

Schematic:

$h(m)$

- $m = \{m_{ij}\}$ where $m_{ij} \in \mathbb{C}$
- h is a homogeneous polynomial in m_{ij}
- $h(m)$ is the sum of products of m_{ij}

$h(F_A)$

- $F_A = \{\Omega_{ij}\}$ where $\Omega_{ij} \in \Omega^2(\mathbb{C})$
- h is a homogeneous polynomial in Ω_{ij}
- $h(F_A)$ is the sum of wedge products of Ω_{ij}

Claim 1: If the matrices $\{\Omega_{ij}\}_{1 \le i,j \le n}$ and $\{\Omega'_{ij}\}_{1 \le i,j \le n}$ are conjugate, i.e.,

$$\{\Omega'_{ij}\} = \eta \cdot \{\Omega_{ij}\} \cdot \eta^{-1},$$

then $h(\{\Omega'_{ij}\}) = h(\{\Omega_{ij}\})$.

For proof of the above claim, see Appendix A.

Notation: Let Q^d be the vector space of ad-invariant functions that are homogeneous polynomials of degree d in the matrix entries. Let $Q^* = \bigoplus_{d=0}^{\infty} Q^d$ and note that $(Q^*, \cdot)$ is a commutative ring, where multiplication is given $(h_1 \cdot h_2)(m) = h_1(m) \cdot h_2(m)$.

Theorem 1 (Chern-Weil). *Let $E \to M$ be a $U(n)$-vector bundle with a connection A. Define the Chern-Weil map:*

$$\mathcal{CW} : (Q^*, \cdot) \to \left(H^{2*}_{dR}(M; \mathbb{R}), \wedge\right)$$
$$h \mapsto \left[\sqrt{-1}^{\,d} h(F_A)\right].$$

The theorem states:

- *For $h \in Q^d$, we have $\sqrt{-1}^{\,d} h(F_A)$ is a closed $2d$-form*
- *The cohomology class $\left[\sqrt{-1}^{\,d} h(F_A)\right]$ is independent of the connection A*
- *The Chern-Weil map is a ring homomorphism.*

Proof idea: every $h \in Q^*$ can be written as a sum of product of $\mathrm{Tr}(m^k)$. ☐

Remark: The cohomology class $\left[\sqrt{-1}^{\,d} h(F_A)\right]$ is called a *characteristic class* of E.

Chern class as an elementary symmetric polynomial in the eigen-values of curvature 2-form

It is natural to ask what is the generating set of the ring $(Q^*, \cdot)$. Recall, $h \in Q^*$ is an ad-invariant homogeneous polynomial of degree d, alternatively, $h(m)$ is a degree d symmetric polynomial in the eigen-values of m.[3] Thus, to obtain a generating set of the ring $(Q^*, \cdot)$, we turn to the ring of symmetric polynomials in $\lambda_1, \ldots, \lambda_n$. There are many generating sets for the ring of symmetric polynomials; two of them are:

1. Elementary symmetric polynomials:

[3] Given ad-invariant function h that is an homogeneous polynomial of degree d, the corresponding symmetric function $\hat{h}(\lambda_1, \ldots, \lambda_n) := h\left(\text{diag}(\lambda_1, \ldots, \lambda_n)\right)$ is also a homogeneous polynomial of degree d. Conversely, given a symmetric function $\hat{h}$ that is a homogeneous polynomial of degree d, then the ad-invariant function $h(m) := \hat{h}\left(\text{eigen-values}(m)\right)$ is also a homogeneous polynomial of degree d. The homogenity of h is clear, but a little bit more effort is needed to show h is actually a polynomial (see this post).

$$e_1(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 \le n} \lambda_{i_1}$$

$$e_2(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 < i_2 \le n} \lambda_{i_1} \lambda_{i_2}$$

$$e_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

2. Power sum symmetric polynomial:

$$p_1(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i^1$$

$$p_2(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i^2$$

$$p_k(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i^k.$$

Remark: Notice that

$$p_k(m) = \sum_{i=1}^{n} \lambda_i^k = \mathrm{Tr}\left(m^k\right).$$

This is most easily seen for diagonalizable matrices. Next, by density of diagonalizable matrices, the above is true for all matrices.

Observation 2 (ad-invariant function theorem). Any $h \in Q^*$ can be written as a sum of product of $p_k(m) = \mathrm{Tr}\left(m^k\right)$, i.e.,

$$h(m) = \sum \prod \text{(const.)} \cdot \mathrm{Tr}\left(m^d\right).$$

For example, we have:

$$\begin{aligned} e_1(m) &:= \mathrm{Tr}(m) \\ e_2(m) &:= \frac{\mathrm{Tr}(m)^2 - \mathrm{Tr}\left(m^2\right)}{2} \\ e_3(m) &:= \frac{\mathrm{Tr}(m)^3 - 3\,\mathrm{Tr}\left(m^2\right)\mathrm{Tr}(m) + 2\,\mathrm{Tr}\left(m^3\right)}{3} \\ &\vdots \end{aligned}$$

This follows as a consequence of the fact that $p_k(m) = \mathrm{Tr}\left(m^d\right)$ form a generating set for Q^*.

Definition 2. The *k-th Chern class* of a $U(n)$-bundle $E \longrightarrow M$ is obtained by picking a

$U(n)$-connection A and considering the following $2k$-form in $H^{2k}_{dR}(M;\mathbb{R})$:

$$c_k(E) := \left[\frac{\sqrt{-1}^k}{(2\pi)^k} e_k(F_A)\right].$$

For example, we have:

$$\begin{aligned}
c_1(E) &:= \left[\frac{\sqrt{-1}}{2\pi}\operatorname{Tr}(F_A)\right]\\
c_2(E) &:= \left[\frac{\sqrt{-1}^2}{(2\pi)^2}\cdot\frac{\operatorname{Tr}(F_A)^2-\operatorname{Tr}\left(F_A^2\right)}{2}\right]\\
c_3(E) &:= \left[\frac{\sqrt{-1}^3}{(2\pi)^3}\cdot\frac{\operatorname{Tr}(F_A)^3-3\operatorname{Tr}\left(F_A^2\right)\operatorname{Tr}(F_A)+2\operatorname{Tr}\left(F_A^3\right)}{3}\right]\\
&\ \vdots
\end{aligned}$$

Also, we define the *total Chern class* as:

$$\begin{aligned}
c(E) &:= 1+c_1(E)+c_2(E)+\ \cdots\ +c_n(E)\in H^*_{dR}(M;\mathbb{R})\\
&= \det\left(1+\frac{\sqrt{-1}}{2\pi}F_A\right).
\end{aligned}$$

The last equality follows from:

$$\det\left(1+t\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\right)=\prod_{i=1}^{n}(1+t\lambda_i)=1+te_1+t^2e_2+\ \cdots\ +t^ne_n.$$

Some properties of Chern class:

1. Whitney sum formula:

$$c_k(E\oplus F)=\sum_{i+j=k}c_i(E)\smile c_j(F).$$

2. $c_k(E)$ are, in fact, elements of $H^{2k}(M;\mathbb{Z})$, i.e., singular cohomology with integer coefficients. A related fact is that for all compact oriented submanifolds $\Sigma\subset M$ of dimension $2k$, we have:

$$\int_\Sigma c_k(E)\in\mathbb{Z}.$$

Vanishing of odd Chern classes of a real vector bundle and the Pontryagin class

Given a real rank n vector bundle $E \to M$ we can define the characteristic classes:

$$c_{e_k}(E) := \left[\frac{1}{(2\pi)^k} e_k(F_A)\right] \in H^{2k}_{dR}(M;\mathbb{R})$$
$$c_e(E) := \det\left(1 + \frac{1}{2\pi} F_A\right) \in H^*_{dR}(M;\mathbb{R}).$$

Observation 3. The classes $c_{e_k}(E) = 0$ for all odd k.

Proof. The structure group of E can be reduced from $GL_n(\mathbb{R})$ to $O(n)$, that is, we can choose a smoothly varying inner product on the fibers of E (p.o.u argument). Further, we can choose a compatible $O(n)$ connection A on E, that is, A is locally given by $Lie(O(n))$-valued 1-forms. Hence, F_A is locally given by $Lie(O(n))$-valued 2-forms. Now, as $Lie(O(n))$ matrices are just skew-symmetric matrices, we have

$$-F_A = F_A^T.$$

This gives:

$$\begin{aligned} e_k(F_A) &= e_k\left(F_A^T\right) && \text{(see below)} \\ &= e_k(-F_A) \\ &= (-1)^k e_k(F_A) \\ &= \begin{cases} 0 & \text{if } k \text{ is odd} \\ e_k(F_A) & \text{if } k \text{ is even} \end{cases}. \end{aligned}$$

Roughly speaking, $e_k(F_A) = e_k\left(F_A^T\right)$ is because m and m^T are always conjugate over $\mathbb{C}$ (see this post for a proof of $m = Zm^TZ^{-1}$). More precisely, note:

$$\begin{aligned} e_k(F_A) &= \sum \prod \text{(const.)} \cdot \mathrm{Tr}\left(F_A^d\right) \\ &= \sum \prod \text{(const.)} \cdot \mathrm{Tr}\left(\left(F_A^T\right)^d\right) \\ &= e_k\left(F_A^T\right). \end{aligned}$$

Now, to see the second equality let $F_A = \sum f_\mu \omega_\mu$ and $F_A^T = \sum f_\nu^T \omega_\nu$ and note:

$$
\begin{aligned}
\mathrm{Tr}\left(\left(F_A^T\right)^d\right) &= \mathrm{Tr}\left(\sum_{1\le \nu_1,\dots,\nu_d\le n}\left(f_{\nu_1}^T \cdots f_{\nu_d}^T\right)\cdot \omega_{\nu_1}\wedge \cdots \wedge \omega_{\nu_d}\right)\\
&= \sum_{1\le \nu_1,\dots,\nu_d\le n} \mathrm{Tr}\left(f_{\nu_1}^T \cdots f_{\nu_d}^T\right)\cdot \omega_{\nu_1}\wedge \cdots \wedge \omega_{\nu_d}\\
&= \sum_{1\le \nu_1,\dots,\nu_d\le n} \mathrm{Tr}\left((f_{\nu_d} \cdots f_{\nu_1})^T\right)\cdot \omega_{\nu_1}\wedge \cdots \wedge \omega_{\nu_d}\\
&= \sum_{1\le \nu_1,\dots,\nu_d\le n} \mathrm{Tr}(f_{\nu_d} \cdots f_{\nu_1})\cdot \omega_{\nu_1}\wedge \cdots \wedge \omega_{\nu_d}\\
&= \sum_{1\le \nu_1,\dots,\nu_d\le n} \mathrm{Tr}(f_{\nu_d} \cdots f_{\nu_1})\cdot \omega_{\nu_d}\wedge \cdots \wedge \omega_{\nu_1}\\
&= \sum_{1\le \mu_1,\dots,\mu_d\le n} \mathrm{Tr}(f_{\mu_1} \cdots f_{\mu_d})\cdot \omega_{\mu_1}\wedge \cdots \wedge \omega_{\mu_d}\\
&= \mathrm{Tr}\left(F_A^d\right),
\end{aligned}
$$

where we have relabeled ν to μ using $\mu_i = \nu_{d-i}$. This completes the proof of observation 3. □

Definition 2. The *k-th Pontryagin class* of a rank n real bundle $E \to M$ is obtained by picking a connection A and considering the following $4k$-form in $H^{4k}_{dR}(M;\mathbb{R})$:

$$
p_k(E) := \left[\frac{1}{(2\pi)^{2k}} e_{2k}(F_A)\right].
$$

Alternatively, we have:

$$
p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}).
$$

Computation of 2nd Chern class for $SU(2)$-bundles over S^4

Main fact: *Let $E \to M$ be an $SU(2)$ bundle over a 4-dimensional manifold.*

- *If M is non-compact, then E is trivializable, i.e., $E = M \times \mathbb{C}^2$ and $c_2(E) = 0$.*
- *If M is compact, then E is completely characterized by $c_2(E) \in H^4(M;\mathbb{Z}) \cong \mathbb{Z}$, i.e., given an integer $m \in \mathbb{Z}$, there is a unique (up to isomorphism) E such that $c_2(E) = m$. Further, for such a E we have*

$$
\int_M c_2(E) = m \in \mathbb{Z}.
$$

For a discussion/proof of the above fact, see this post.

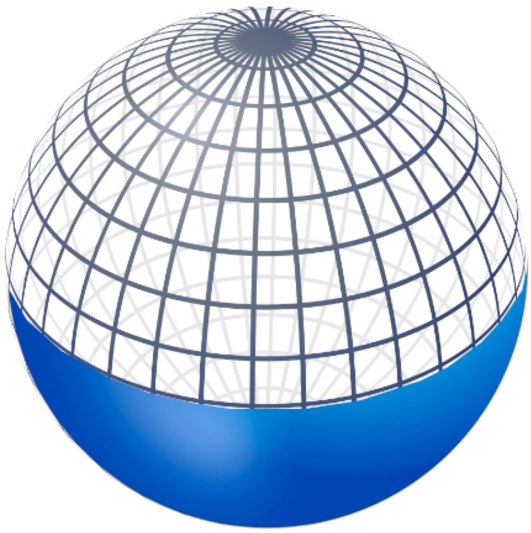
Now we specialize to $SU(2)$ bundle over S^4. Recall, that we introduced an integer family $\{P^{(m)}\}_{m\in\mathbb{Z}}$ of $SU(2)$-bundles over S^4. Our goal is to show:

$$\int_M c_2\left(P^{(m)}\right) = -m.$$

The proof given below is taken from Example 14.11 on page 186 of [Taubes].

<u>(Recall) The definition of $P^{(m)}$ over S^4</u>:

For S^4, we have the open cover by two sets U_1 and U_2:



$U_1 = \{\text{south hemisphere}\}$ $\qquad$ $U_2 = \{\text{north hemisphere} + \text{little more}\}$

Next we orient S^4 using the stereographic projection from $U_1 \subset S^4$ to $B(0;1) \subseteq \mathbb{R}^4$, where $B(0;1) \subseteq \mathbb{R}^4$ is given the standard orientation[4]. To define $P^{(m)}$, it suffice to define the transition map

$$\tau_{21}^{(m)} : U_1 \cap U_2 \to SU(2).$$

This is done by:

$$\begin{tikzcd}
U_1 \cap U_2 = \{r < |x| < 1\} \arrow[rr, dashed, "\tau_{21}^{(m)}"] \arrow[dr, swap, "\frac{(x_1, x_2, x_3, x_4)}{||x||}"] & & SU(2) \\
& S^3 = SU(2) \arrow[ur, swap, "g \mapsto g^m"] &
\end{tikzcd}$$

[4] The orientation is needed later to integrate over S^4.

Now to calculate $c_2\left(P^{(m)}\right)$ we need to pick a connection.

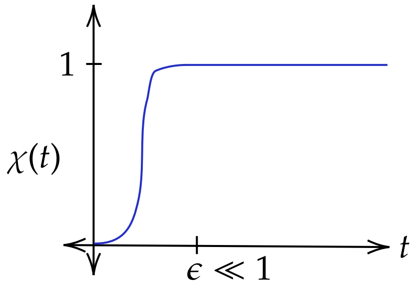
<u>A choice of $SU(2)$ connection on $P^{(m)}$</u>:
Recall, a connection is defined using $a_1 : TU_1 \to Lie(SU(2))$ and $a_2 : TU_2 \to Lie(SU(2))$ that satisfy

$$a_1 = \tau_{21}^{-1} a_2 \tau_{21} + \tau_{21}^{-1} d\tau_{21} \quad \forall\, x \in U_1 \cap U_2.$$

Now define,

$$\begin{aligned} a_2(x) &:= 0 \quad \forall\, x \in U_2 \\ a_1(x) &:= \begin{cases} \tau_{21}^{-1} d\tau_{21} & x \in U_1 \cap U_2 = \{r < |x| < 1\} \\ \chi(|x|) \cdot \tau_{21}^{-1} d\tau_{21} & x \in U_1 \cap U_2^c = \{|x| \leqslant r\}. \end{cases} \end{aligned}$$

Here, χ is the smooth function:



Now the goal is to show:

$$\int_{S^4} c_2\left(P^{(m)}\right) = \frac{1}{8\pi^2} \int_{U_1 = \{|x| \le 1\}} \mathrm{Tr}(F_{a_1} \wedge F_{a_1}) = -m.$$

To this end, we have three steps:

<u>Step 1</u>:

$$\mathrm{Tr}(F_{a_1} \wedge F_{a_1}) = d\left(\mathrm{Tr}\left(a_1 \wedge da_1 + \tfrac{2}{3} a_1 \wedge a_1 \wedge a_1\right)\right)$$

Note: $\mathrm{Tr}(F_{a_1} \wedge F_{a_1})$ is a closed 2-form over the contractible space $U_1 = \{|x| < 1\}$. Hence it is an exact form. The above gives an explicit expression for the exact form. Also, the 2-form $\mathrm{Tr}\left(a_1 \wedge da_1 + \frac{2}{3} a_1 \wedge a_1 \wedge a_1\right)$ is called a Chern-Simon form.

<u>Step 2</u>:

$$\int_{U_1=\{|x|\le 1\}} \mathrm{Tr}(F_{a_1}\wedge F_{a_1}) = \int_{U_1=\{|x|\le 1\}} d\left(\mathrm{Tr}\left(a_1\wedge da_1+\tfrac{2}{3}a_1\wedge a_1\wedge a_1\right)\right)$$
$$= \int_{S^3=\{|x|=1\}} \mathrm{Tr}\left(a_1\wedge da_1+\tfrac{2}{3}a_1\wedge a_1\wedge a_1\right)$$

<u>Step 3</u>:

$$\begin{aligned}
\int_{S^3=\{|x|=1\}} \mathrm{Tr}\left(a_1\wedge da_1+\tfrac{2}{3}a_1\wedge a_1\wedge a_1\right) &= \int_{S^3=SU(2)} \frac{-1}{3}\,\mathrm{Tr}\left(g^{-m}dg^m\wedge g^{-m}dg^m\wedge g^{-m}dg^m\right)\\
&= m\int_{S^3=SU(2)} \frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg^1\wedge g^{-1}dg^1\wedge g^{-1}dg^1\right)\\
&= m\cdot(-4)\int_{S^3=SU(2)} dVol_{S^3}\\
&= m\cdot(-4)\cdot 2\pi^2\\
&= -m\cdot 8\pi^2.
\end{aligned}$$

Thus, it follows that $\int_{S^4} c_2\left(P^{(m)}\right) = -m$.

<u>Proof of Step 1</u>: We expand the two sides:

$$\begin{aligned}
\mathrm{Tr}(F_{a_1}\wedge F_{a_1}) &= \mathrm{Tr}((da_1+a_1\wedge a_1)\wedge(da_1+a_1\wedge a_1))\\
&= \mathrm{Tr}(da_1\wedge da_1)+\mathrm{Tr}(da_1\wedge a_1\wedge a_1)+\mathrm{Tr}(a_1\wedge a_1\wedge da_1)+\cancel{\mathrm{Tr}(a_1\wedge a_1\wedge a_1\wedge a_1)}\\
&= \mathrm{Tr}(da_1\wedge da_1)+2\,\mathrm{Tr}(da_1\wedge a_1\wedge a_1)+0
\end{aligned}$$

$$\begin{aligned}
&d\left(\mathrm{Tr}\left(a_1\wedge da_1+\tfrac{2}{3}a_1\wedge a_1\wedge a_1\right)\right)\\
&\quad= \mathrm{Tr}\left(d\left(a_1\wedge da_1+\tfrac{2}{3}a_1\wedge a_1\wedge a_1\right)\right)\\
&\quad= \mathrm{Tr}\left(da_1\wedge da_1+\tfrac{2}{3}da_1\wedge a_1\wedge a_1-\tfrac{2}{3}a_1\wedge da_1\wedge a_1+\tfrac{2}{3}a_1\wedge a_1\wedge da_1\right)\\
&\quad= \mathrm{Tr}(da_1\wedge da_1)+\frac{2}{3}\mathrm{Tr}(da_1\wedge a_1\wedge a_1)-\frac{2}{3}\mathrm{Tr}(a_1\wedge da_1\wedge a_1)+\frac{2}{3}\mathrm{Tr}(a_1\wedge a_1\wedge da_1)\\
&\quad= \mathrm{Tr}(da_1\wedge da_1)+\frac{2}{3}\mathrm{Tr}(da_1\wedge a_1\wedge a_1)+\frac{2}{3}\mathrm{Tr}(da_1\wedge da_1\wedge a_1)+\frac{2}{3}\mathrm{Tr}(da_1\wedge a_1\wedge a_1)\\
&\quad= \mathrm{Tr}(da_1\wedge da_1)+2\,\mathrm{Tr}(da_1\wedge a_1\wedge a_1).
\end{aligned}$$

This gives the required identity. Along the way we have used the following facts:

- $\mathrm{Tr}(a_1\wedge a_1\wedge da_1) = \mathrm{Tr}(da_1\wedge a_1\wedge a_1)$
- $\mathrm{Tr}(a_1\wedge da_1\wedge a_1) = -\mathrm{Tr}(da_1\wedge a_1\wedge a_1)$
- $\mathrm{Tr}(a_1\wedge a_1\wedge a_1\wedge a_1) = 0$

All of these follow because of the identity:

$$\mathrm{Tr}(\alpha \wedge \beta) = (-1)^{kl}\,\mathrm{Tr}(\beta \wedge \alpha)$$

which holds for all $\alpha \in \Omega^k(Lie(G))$ and $\beta \in \Omega^l(Lie(G))$. The proof of the above identity follows by (i) writing $\alpha = \sum f_\mu \omega_\mu$ and $\beta = \sum h_\sigma \eta_\sigma$ where $f_\mu, h_\sigma \in Lie(G)$ and $\omega_\mu, \eta_\sigma \in \Omega^*(U_1)$; and (ii) noting the computation:

$$\begin{aligned}
\mathrm{Tr}(\alpha \wedge \beta) &= \mathrm{Tr}\left(\sum f_\mu \omega_\mu \wedge \sum h_\sigma \eta_\sigma\right) \\
&= \mathrm{Tr}\left(\sum\nolimits_{\mu,\sigma} f_\mu h_\sigma \cdot \omega_\mu \wedge \eta_\sigma\right) \\
&= \sum\nolimits_{\mu,\sigma} \mathrm{Tr}(f_\mu h_\sigma) \cdot \omega_\mu \wedge \eta_\sigma \\
&= (-1)^{kl} \sum\nolimits_{\mu,\sigma} \mathrm{Tr}(h_\sigma f_\mu) \cdot \eta_\sigma \wedge \omega_\mu \\
&= (-1)^{kl}\,\mathrm{Tr}(\beta \wedge \alpha).
\end{aligned}$$

For an alternative proof of Step 1 see section 14.9 on page 190 of [Taubes].

Proof of Step 3:

The first equality in Step 3 is

$$\mathrm{Tr}\left(a_1 \wedge da_1 + \tfrac{2}{3} a_1 \wedge a_1 \wedge a_1\right) = \frac{-1}{3}\,\mathrm{Tr}\left(g^{-m}dg^m \wedge g^{-m}dg^m \wedge g^{-m}dg^m\right).$$

To see this, recall that $a_1 = \tau_{21}^{-1} d\tau_{21}$ on the equator $\{|x| = 1\}$ and that $\tau_{21}^{(m)} = g^m$. Using this, we unpack the terms on LHS to get:

- $a_1 = g^{-m}dg^m$
- $da_1 = d\left(g^{-m}\right) \wedge dg^m = g^{-m}g^m d\left(g^{-m}\right) \wedge dg^m = -g^{-m}dg^m g^{-m} \wedge dg^m = -g^{-m}dg^m \wedge g^{-m}dg^m$
- $a_1 \wedge da_1 = -g^{-m}dg^m \wedge g^{-m}dg^m \wedge g^{-m}dg^m$
- $a_1 \wedge a_1 \wedge a_1 = g^{-m}dg^m \wedge g^{-m}dg^m \wedge g^{-m}dg^m$.

The second equality in Step 3 is

$$\int_{S^3 = SU(2)} \frac{-1}{3}\,\mathrm{Tr}\left(g^{-m}dg^m \wedge g^{-m}dg^m \wedge g^{-m}dg^m\right) = m \int_{S^3 = SU(2)} \frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg^1\right).$$

To see this, note that $\mathrm{Tr}\left(g^{-m}dg^m \wedge g^{-m}dg^m \wedge g^{-m}dg^m\right)$ is the pullback of the form $\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg^1\right)$ under the map $g \mapsto g^m$ from $SU(2)$ to itself. Next, the interesting fact is that the map $g \mapsto g^m$ has degree m (see section 14.8 on page 189 of [Taubes]). Because the degree is m we have an m-fold cover and hence the integral on LHS is m times integral of RHS.

The third equality in Step 3 is

$$\frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg^1\right) = (-4) \cdot dVol_{S^3}$$

To see this, note that the 3-form

$$\frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg^{1}\right)$$

is invariant under both left and right multiplication of the group. This is because:

$$\begin{aligned}\frac{-1}{3}\,\mathrm{Tr}\left((gh)^{-1}d(gh)\wedge(gh)^{-1}d(gh)\wedge(gh)^{-1}d(gh)\right) &= \frac{-1}{3}\,\mathrm{Tr}\left(h^{-1}g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\cdot h\right)\\ &= \frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right),\end{aligned}$$

where we used the cyclic invariance of Tr in the last step. In conclusion, the above 3-form is a scalar multiple of the volume form on $SU(2)$ obtained from a bi-invariant metric that identifies $SU(2)$ with S^3. To compute this scalar, it suffice to understand the behaviour at the idenity point $e\in SU(2)$. We feed in the three orthonormal vectors of $T_eSU(2)$:

$$m_1=\begin{pmatrix} & i\\ i & \end{pmatrix},\ m_2=\begin{pmatrix} & 1\\ -1 & \end{pmatrix},\ m_3=\begin{pmatrix} i & \\ & -i\end{pmatrix}$$

to get:

$$\begin{aligned}\frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)(m_1,m_2,m_3) &= \frac{-1}{3}\,\mathrm{Tr}\left(\sum_{\sigma\in S_3}\mathrm{sgn}\,\sigma\cdot m_{\sigma_1}m_{\sigma_2}m_{\sigma_3}\right)\\ &= \frac{-1}{3}\cdot 12\\ &= -4\end{aligned}$$

Thus, we have

$$\frac{-1}{3}\,\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)(m_1,m_2,m_3) = -4\cdot dVol_{S^3}.$$

Appendix A

[to be added]

Further reading

1. Chapter 14, Taubes, C. H. (2011). *Differential geometry: bundles, connections, metrics and curvature* (Vol. 23). OUP Oxford.
2. Lecture note, Zuoqin Wang (2018) *Chern-Weil theory.* [Link]