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ON THE FUNCTIONAL FORM OF THE SPEED–DENSITY RELATIONSHIP—II: EMPIRICAL INVESTIGATION

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Abstract—In this part we applied the theory developed in Part I. Some speed–density curves having the functional form proposed in Part I are fitted to traffic data. The goodness of fit is excellent, except in the case of the left lane detectors. A procedure for isolating stationary traffic periods is also explained. This procedure satisfactorily eliminates the dispersion of the individual measurements.

INTRODUCTION

In Part I of this work, Del Castillo and Benítez (1994), several speed–density models were proposed. All these models had in common the following functional form

$$V = V_f\left[1 - f(\lambda)\right]. \tag{1}$$

The function $f(\cdot)$ was called *generating function* and its argument, λ, *equivalent spacing*. This name was justified by its expression as a function of the spacing

$$\lambda = \frac{|C_j|}{V_f}\left(\frac{H}{H_j} - 1\right) \tag{2}$$

where H, the spacing, is the reciprocal of the traffic density. The parameters appearing in the above formulas are the free-flow speed, V_f, the kinematic wave speed at jam density, C_j and the jam spacing, H_j. These three parameters, or likewise the two first and the jam density K_j, were referred to as the fundamental parameters of traffic flow. In part I, numerous empirical evidences showed that the values of C_j range approximately from −15 km/h to −25 km/h, whereas for K_j a value of 150 veh/km may be expected. Several models of generating functions were proposed in Part I. Two of them are of special interest, the exponential curve given by

$$V = V_f\left\{1 - \exp\left[\frac{|C_j|}{V_f}\left(1 - \frac{H}{H_j}\right)\right]\right\} \tag{3}$$

and the maximum sensitivity curve

$$V = V_f\left\{1 - \exp\left[1 - \exp\left(\frac{|C_j|}{V_f}\left(\frac{H}{H_j} - 1\right)\right)\right]\right\}. \tag{4}$$

These curves stem from the exponential generating function

$$f(\lambda) = \exp\left[1 - \left(1 + \frac{\lambda}{n}\right)^n\right] \tag{5}$$

with $n = 1$ and $n = \infty$ respectively.

In this part, an application of the theory previously developed is presented. Several regression models are fitted to traffic data and the results are analyzed. Prior to this regression analysis, a procedure for isolating stationary periods of traffic flow was applied. Thanks to this procedure, the data used for the regression analysis correspond to traffic flow equilibrium conditions.

SELECTION OF EQUILIBRIUM TRAFFIC DATA

Traffic data utilized for the present study were collected from a 3-lane stretch of the freeway A2 Amsterdam–Utrecht in the Netherlands by Smulders (1992). These data were taken during the morning rush hours (6.30 to 9.30) of 5 weekdays. They consist of measurements of number of vehicles, mean speed, speed variance and occupancy over 30-sec intervals for each lane. The mean speed during the i-th period will be denoted as $\bar{V}_i$ and the number of vehicles counted N_i.

Since the speed–density curve represents traffic flow equilibrium states, the selected data should correspond to time-stationary and space-homogeneous traffic flow. First of all, we looked for potentially stationary periods lasting at least 4 or 5 min. For potentially stationary periods, we mean periods with a speed standard deviation clearly small in comparison with the mean speed (about 15% or less). Thus, data showing typical unstable behavior with stop–start and random oscillatory patterns were rejected.

In order to ensure a broad range of speed and density values, all detectors that did not record congestion periods were discarded. For congestion we mean potentially stationary periods having mean speed below 80% of the free flow speed. A second selection left out those detectors with insufficient data to cover a reasonable range of speeds (say from 30 km/h to 100 km/h) with an adequate number of potentially stationary periods: at least 15.

The first selection process left two groups of three detectors, corresponding to two consecutive detector stations on the Amsterdam-bound side. The total number of detector stations screened was 16, which implies that only 12.5% of the data turned out to be useful. If the data set had included more days, this percentage would probably have increased. The detector stations, denoted here as U for the upstream station and D for the downstream one, are separated by 1 km. The detectors have been named according to the stations (U or D) and to the lane: L for the left lane, C for the center lane and R for the right lane detectors. Then, they are: UL, UC, UR and DL, DC, DR.

The second step of the selection procedure is the assessment of space-homogeneity in the data. For this sake, the sample cross-correlation coefficient of consecutive detectors, $r(\bar{V}_i, \bar{V}_{i+k})$, was calculated for several lags k. The maximum value attained and the associated lag is shown in Table 1 for each pair of detectors and for each day. As a rule of thumb, one may regard traffic flow to be space-homogeneous if the maximum sample cross-correlation coefficient is over 0.5, provided that congestion occurs. This critical value should be increased if the distance between detectors is shorter than 1 km. On this basis, the second day records were rejected. Due to the shortage of data, the first and third days of the right lane detectors were accepted. In addition, this criterion yields a rough estimate of C_j from the time lag that maximizes the cross-correlation. If we take a maximization observation lag of 6.5 and bear in mind that each observation covers 30 sec, then

$$C_j \approx -\frac{1 \text{ km}}{6.5 \times 30 \text{ sec}} = -18.46 \text{ km/h}.$$

An adequate estimation should give values not differing too much from this one.

Table 1. Maxima of the sample cross-correlation coefficient (above) and observation lag at which it is reached (below)

Lane	Detectors	Day 1	2	3	4	5
Left	DL–UL	0.55	0.31	0.5	0.58	0.9
		6	6	6	7	7
Centre	DC–UC	0.54	0.267	0.48	0.57	0.89
		6	6	7	7	6
Right	DR–UR	0.38	0.127	0.39	0.43	0.79
		6	8	6	7	6

Now we are ready to isolate stationary flow periods from the raw data of 12 h (4 days × 3 h a day) of measurements from six detectors. Potentially stationary periods were selected by means of visual inspection of the speed data. Fig. 1 shows the selected periods from the fifth day of detector DL.

In order to determine the stationarity of the selected periods, we follow the suggestion of Breiman and Lawrence (1973) and apply a nonparametric test based on Kendall's τ test. This test measures the association between two samples of measurements and may be used to detect a trend in one sample by measuring the association with an ordered sequence $(1, 2, \ldots, n$, for example).

In a preliminary study by Del Castillo *et al.* (1993), we applied this test to the series of vehicle counts and found that constant flow periods could not correspond to constant mean speed periods. The explanation of this fact is obvious: for each flow level there are two possible speed levels. For this reason, we decided to apply the test to the series of sample mean square values of the speed, to detect any trend in the mean as well as in the variance. The statistic τ is defined as

$$\tau = \frac{2\sum_{i<k} C_{ij}}{m(m-1)}.$$

The sum is the number of pairs that occur in the sample in natural order minus the number of pairs that occur in reverse natural order

$$C_{ij} = \begin{cases} 1, & \text{if } \overline{V_i^2} < \overline{V_j^2} \\ 0, & \text{if } \overline{V_i^2} = \overline{V_j^2} \\ -1, & \text{if } \overline{V_i^2} > \overline{V_j^2} \end{cases}$$

where $\overline{V_i^2}$ is the i-th mean square speed observed, that is, the mean square value of 30 sec of vehicle speed measurements. The statistic τ may now be used in testing the hypothesis

$$H_0\text{: no trend exists in the sample } \overline{V_i^2},\ i = 1, \ldots, n$$

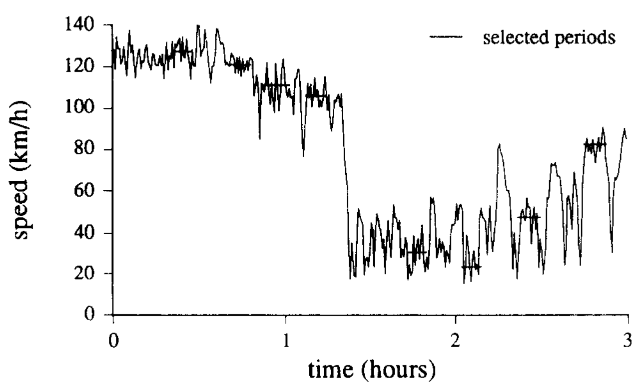

Fig. 1. Example of selected periods for testing its stationarity.

against the alternative of the existence of a trend. Under H_0 the distribution of τ is known only if the variables involved are independent. This fact represents an inconvenience at the congested regime where the vehicle speeds probably become dependent. Nevertheless, even if the vehicle speeds are strongly correlated, the 30-sec average speeds are weakly correlated due to the averaging. Furthermore, the limited sample size does not permit discerning about the autocorrelation of the speeds and we may apply the test as if they were uncorrelated.

The tests were carried out at the 5% level of significance. Critical values of τ for several levels of significance and sample sizes are given in Stoodley *et al.* (1980). Initially the test is applied to the original sample selected by visual inspection: $\overline{V_i^2}$, $i = 1, \ldots, n$. If $\tau > \tau_{\text{crit}}(n)$ this sample is rejected and a new sample $\overline{V_i^2}$, $i = 1, \ldots, n-1$ is tested. The procedure stops with the first accepted sample. Most of the potentially stationary periods were accepted at the first try which means that the visual inspection is not a crude method of selection if enough care is taken. The application of the test ensures that the speeds are free of a trend. The same procedure is then applied to the corresponding series of vehicle counts, which may further shorten the samples obtained after applying the test to the mean square speeds. In our case, this happened in few samples. The application of the test to both series, mean square speeds and vehicles counts, ensures that the data are trend-free. This, along with the earlier verification of space-homogeneity, guarantees that the selected data correspond to equilibrium traffic flow.

ESTIMATION OF MEAN QUANTITIES

The mean flow over the stationary period was estimated with the sample mean

$$\hat{N} = \frac{1}{n} \sum_{i=1}^{n} N_i = \frac{m}{n}$$

where m is the number of vehicles in the n 30-sec intervals. Moreover, it is well known that the relationship between flow, speed and spacing $Q = V_e/H$ holds for the space mean speed, which is the harmonic mean speed

$$\frac{1}{V_e} = E\left[\frac{1}{V_k}\right]$$

where $E[\cdot]$ is the expectation and V_k is the speed of vehicles. In practice, we cannot estimate the harmonic mean speed of the vehicles by using the definition of V_e, because we only have 30-sec averaged speed measurements, $\overline{V}_i$, $i = 1, \ldots, n$, and they are arithmetic means of the individual speeds, V_k, $k = 1, \ldots, m$

$$\overline{V}_i = \frac{1}{N_i} \sum_{k=1}^{N_i} V_k.$$

Instead we can estimate the mean speed and the speed variance by the usual estimates

$$\hat{V} = \frac{1}{m} \sum_{k=1}^{m} V_k = \frac{1}{m} \sum_{i=1}^{n} N_i \overline{V}_i$$

$$\hat{\sigma}^2 = \frac{1}{m-1} \sum_{k=1}^{m} (V_k - \hat{V})^2 = \frac{1}{m-1} \left[\sum_{i=1}^{n} (N_i - 1)\, \overline{V_i^2} - m\hat{V}^2\right].$$

To account for the harmonic mean, we need to introduce a correction to the estimate $\hat{V}$. Expanding the reciprocal of the individual speed around the arithmetic mean speed, V, up to the quadratic term, we get

$$\frac{1}{V_k} \approx \frac{1}{V} - \frac{V_k - V}{V^2} + \frac{(V_k - V)^2}{V^3}.$$

Therefore, an approximation for the expectation of the estimate of the harmonic mean is

$$E\left[\frac{1}{\hat{V}_e}\right] = \frac{1}{m} E\left[\sum_{k=1}^{M} \frac{1}{V_k}\right] \approx \frac{1}{V} + \frac{\sigma^2}{V^3}.$$

Replacing the mean and variance (V, σ^2) by their sample estimates ($\hat{V}$, $\hat{\sigma}^2$) we finally have the following estimate for the harmonic mean

$$\hat{V}_e = \frac{\hat{V}^3}{\hat{V}^2 + \hat{\sigma}^2}.$$

Another important question is the required size of the stationary periods for obtaining accurate estimates, that is, with a reasonably low variance. This question is actually impossible to answer because we do not know the structure of the 30-sec mean speeds and counts processes. We can have a rough idea of the size requirements if we make some assumptions on these processes. For example, assuming the vehicles counts to have a Poisson distribution with mean N, the asymptotic distribution of the mean estimate $\hat{N}$ is normal

$$\hat{N} \sim \mathcal{N}(N, N/n).$$

The required size $n_{\min}$ for the 95% probability of having an estimation error less than ϵ is then given by

$$\hat{N}_{\text{crit}} = N \pm 1.96 \sqrt{\frac{N}{n_{\min}}}\,(1 \pm \epsilon)\,N.$$

Taking $\epsilon = 0.15$ and $N = 16$ for 30-sec intervals, the above expression yields $n_{\min} = 10$, that is, 5 min of data. The estimation error increases as the traffic volume decreases. This is a serious inconvenience since, in reality, one does not find congested stationary periods longer than 4 or 5 min. Furthermore, at low traffic volumes, the counts do not follow a Poisson distribution, but are correlated, which implies that longer intervals should be required. In conclusion, the recommended minimum sample duration (5 min) is the minimum of all the minima, that is, the minimum duration when conditions for estimations are the best (Poisson vehicle counts).

As for the speed estimate $\hat{V}_e$, we can assume the individual speeds to be a first order autoregressive process with autocorrelation coefficient ϕ, then asymptotically we have

$$\hat{V}_e \sim \mathcal{N}\left[V, \frac{\sigma^2(1+\phi)}{m(1-\phi)}\right].$$

The critical region for the same estimation conditions is now

$$\hat{V}_{\text{crit}} = V_e \pm 1.96\sigma \sqrt{\frac{1+\phi}{m_{\min}(1-\phi)}} = (1 \pm \epsilon)\,V,$$

and considering the worst case as being $\phi = 0.9$ and $\sigma = V/4$ we get $m_{\min} = 200$. With 16 vehicles per interval we would need 12 intervals or 6 min of stationary traffic flow. The assumption of an AR(1) speed process is justified for the free-flow regime (low densities) by the findings of Breiman *et al.* (1977). Once again, at medium and high traffic densities the requirements would be more severe. In general, the estimation of σ with alike precision requires a number of observations one order of magnitude greater than that required for estimating V_e. Thus, we have to conform with less precise estimates of σ. The only possible solution to this drawback is to directly estimate the harmonic mean speed from the sample mean of the reciprocals of the speed.

In conclusion, the border of 5 min appears as a realistic minimum sample duration for both the counts and the speed measurements, although longer sample durations would be necessary for traffic flow at capacity and the congested regime. To permit shorter samples, say 3 or 4 min, is a question of judgement. Finally the estimate of the spacing in meters is

$$\hat{H} = \frac{1000 \times \hat{V}_e}{120 \times \hat{N}}$$

for the speed in km/h and the volume in vehicles per 30 sec.

REGRESSION ANALYSIS

In this section we will refer to the estimates of the speed and spacing of the stationary periods simply as speed and spacing. Once we have the data set speed–spacing for each detector, the problem will now be to fit a speed–spacing curve to the data. A regression analysis was carried out for this purpose. The former estimates of the speed and spacing, $\hat{V}_e$ and $\hat{H}$ become the observations for the regression model. A regression model expresses a relation between a deterministic variable and a variable subjected to error. The most common model assumes an additive independent and identically distributed normal error, ϵ. Such a model applied to the speed–spacing relationship is

$$\hat{V}_e = V_e + \epsilon = F(\hat{H}\,; \theta) + \epsilon$$

where $F(\cdot)$ is a given function and θ are the set of parameters to be estimated. We take the correspondant to any generating function as a function $F(\cdot)$. Bearing in mind the expressions (1) and (2), the resulting regression model of the speed on the spacing is

$$\hat{V} = V_f \left\{ 1 - f\left[\frac{|C_j|}{V_f} \left(\frac{\hat{H}}{H_j} - 1 \right) \right] \right\} + \epsilon, \tag{6}$$

with

$$\epsilon \sim \mathcal{N}\,(0, \sigma_\epsilon^2).$$

Once a generating function has been chosen, the problem is reduced to the estimation of the parameters $|C_j|$, V_f, and H_j. Regardless of the generating function considered, the estimation of these parameters is a nonlinear regression problem because they appear in a nonlinear manner in f. If the model (6) for a given function f is to make sense, the estimates of the parameters, $|\hat{C}_j|$, $\hat{V}_f$, and $\hat{H}_j$ should be within reasonable values. In this work, we have adopted the regression of the speed on the spacing, but in principle there is no motivation to prefer this regression to the regression of the spacing on the speed. Moreover, one may wonder why we have chosen the regression of the speed on the spacing and not on the density. This is actually irrelevant since the estimates of the parameters would be exactly the same for both regression models. We prefer to work with the speed–spacing curve because the interpretation of the parameters is much clearer from these curves than from the speed–density curves.

For the regression model (6), the maximum likelihood estimates are the least squares estimates. Since the model is nonlinear in the parameters, the minimization of the sum of square errors leads to a set of nonlinear equations. For each model these equations have been solved by an exhaustive search. This method is preferable to a gradient based one because it is possible to start from initial estimates that are close to the solution. In addition, a gradient based method could converge to a local miminum or simply not converge. A brief introduction to nonlinear regression may be found in Draper and Smith (1981).

A crucial point of the present study is the election of the generating function for the regression model (6). From the beginning, we discarded the double exponential, rational and reciprocal–exponential families proposed in Del Castillo and Benítez (1994) because the differences with the exponential curves are not significant. Then, we could have adopted the generic exponential generating function (5). Had we done so, we should have estimated the parameter n besides the physical parameters $|C_j|$, V_f, and H_j. Instead, we have chosen the exponential curve (3) and the maximum sensitivity curve (4) for the regression model. This election avoids the introduction of a new parameter. Furthermore, these two curves differ sufficiently as to enable us to clearly select the best fit by means of a regression analysis. This would have not been the case if we had permitted n to vary within a continuous range.

For the right lane detectors, DR and UR, the regression analysis was carried out assuming the exponential and the maximum sensitivity curves. The estimates of the parameters and that of the errors standard deviation, σ_ϵ are shown in Tables 2 and 3. The resulting speed–spacing curves and the observations are plotted in Figs 2 and 3, where 'max. sens.' stands for maximum sensitivity curve. The asymptotic bias and standard deviation of the estimates as a percentage of them are also given in these tables. These values are not the actual asymptotic bias and standard deviation, but estimations of them. They just provide a quantitative assesment of the accuracy of the estimates. Approximations for the bias and variance of nonlinear estimates can be derived by replacing the

Table 2. Results of the speed–spacing regression analysis for detector DR

Detector DR: exponential curve			
No. of stationary periods: 18		RMS error (km/h): 1.96	
Parameter	Estimate	% Bias	% SD
V_f	86.4 km/h	1.56	6.98
$\|C_j\|$	11.92 km/h	4.29	85.15
H_j	6.18 m	−7.46	68.73
K_j	161.75 veh/km	54.7	68.73

Detector DR: maximum sensitivity curve			
No. of stationary periods: 18		RMS error (km/h): 2.65	
Parameter	Estimate	% Bias	% SD
V_f	74.88 km/h	0.55	2.97
$\|C_j\|$	5.86 km/h	4.03	97.89
H_j	4.34 m	−6.08	86.73
K_j	230.4 veh/km	81.3	86.73

Table 3. Results of the speed–spacing regression analysis for detector UR

Detector UR: exponential curve			
No. of stationary periods: 18		RMS error (km/h): 3.1	
Parameter	Estimate	% Bias	% SD
V_f	106.85 km/h	0.41	3.9
$\|C_j\|$	21.22 km/h	0.54	28.26
H_j	8.08 m	−1.20	20.44
K_j	123.79 veh/km	4.20	20.44

Detector UR: maximum sensitivity curve			
No. of stationary periods: 18		RMS error (km/h): 3.47	
Parameter	Estimate	% Bias	% SD
V_f	94.85 km/h	0.16	2.09
$\|C_j\|$	6.14 km/h	0.88	60.12
H_j	3.64m	−2.31	54.45
K_j	274.73 veh/km	31.96	54.45

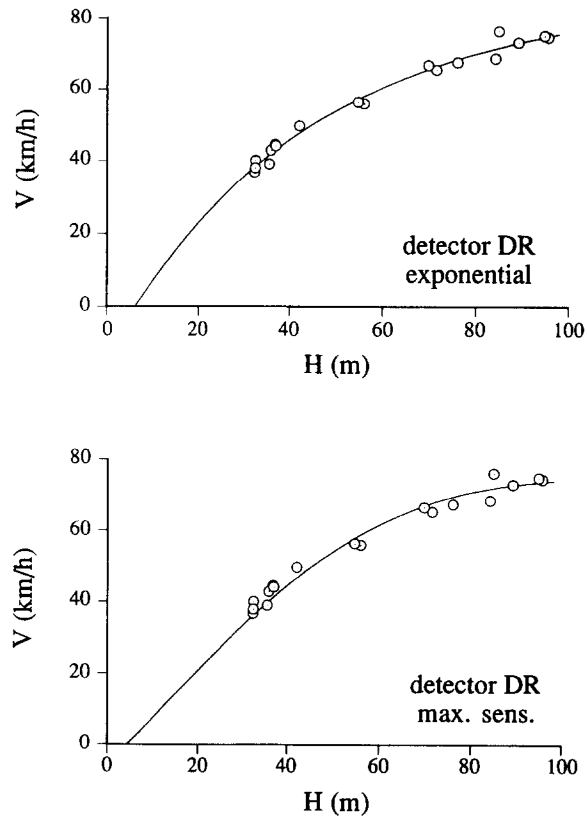

Fig. 2. Speed–spacing curves and observations for detector DR.

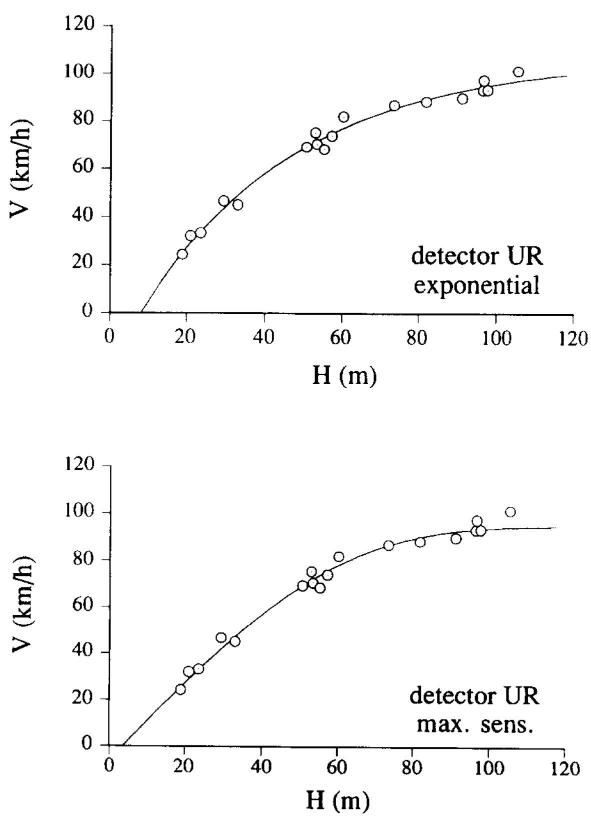

Fig. 3. Speed–spacing curves and observations for detector UR.

model function by the second-order truncation of its Taylor series in terms of the discrepancy between the estimates and the true value of the parameters. These approximations are also called asymptotic bias and variance. The explicit expressions obtained in this manner are given elsewhere, Ratkowsky (1983). The values given in the tables have been obtained by using the estimates in place of the unkown value of the parameters. The asymptotic bias and variance of K_j were obtained from those of H_j, by using the same truncation of the Taylor series. It is easy to get

$$\% \text{ bias } [\hat{K}_j] = -\% \text{ bias } [\hat{H}_j] + \% \text{ variance } [\hat{H}_j].$$

The asymptotic variance of the estimate of the jam density may be approximated by the following truncated Taylor series

$$\hat{K}_j = \frac{1}{\hat{H}_j} \approx \frac{1}{E[\hat{H}_j]} - \frac{\hat{H}_j - E[\hat{H}_j]}{E[\hat{H}_j]^2}.$$

Hence both estimates have approximately equal percentage variance

$$\frac{E[(\hat{K}_j - E[\hat{K}_j])^2]}{E[\hat{K}_j]^2} = \frac{E[(\hat{H}_j - E[\hat{H}_j])^2]}{E[\hat{H}_j]^2}.$$

The terms percentage bias and variance refer to the bias and variance expressed as a percentage of the estimate and the square of the estimate respectively.

The speed–spacing observations plots of the center and left lane detectors showed an abrupt change of slope at the transition from the congested to the uncongested regime. This feature appears more accentuated in the left lane detectors. It is obvious that the exponential curve would not fit such an abrupt slope change. Therefore, we left it out from the regression model. We took instead the maximum sensitivity curve and tried to find a generating function that could satisfactorily fit the plots. The speed–spacing curve generated by such a function should present an almost linear first piece followed by another almost linear piece with a horizontal asymptotic limit for V_f. If the generating function is to be accordant with the developed theory, it should have continuous derivative, although rapidly varying through the transition between the two regimes. In spite of our efforts, we have not found any function that satisfied all the above requirements. Further, the more complicated the functions we tried, the more parameters came up and thus, the less useful the model is. At this point, we recognized the necessity of relaxing one of the conditions of the speed–density curve: its derivability. In addition since we were to

Table 4. Results of the speed–spacing regression analysis for detector DC

Detector DC: maximum sensitivity curve			
No. of stationary periods: 21		RMS error (km/h): 2.21	
Parameter	Estimate	% Bias	% SD
V_f	113 km/h	0.04	0.98
$\lvert C_j \rvert$	17.98 km/h	0.13	11.6
H_j	6.77 m	−0.15	8.91
K_j	147.77 veh/km	0.94	8.91

Detector DC: two-linear model			
No. of stationary periods: 21			
Congested Regime, V (km/h), H (m):		$V = 2.28 \times H - 11.3$; RMS error (km/h): 1.5	
Uncongested Regime, V (km/h), H (m):		$V = 0.49 \times H + 72.32$; RMS error (km/h): 2.1	
Parameter	Estimate	% Bias	% SD
$\lvert C_j \rvert$	11.3 km/h	0	14.12
H_j	4.95 m	−0.35	11.44
K_j	202.2 veh/km	1.66	11.44

Table 5. Results of the speed–spacing regression analysis for detector UC

Detector UC: maximum sensitivity curve			
No. of stationary periods: 14		RMS error (km/h): 3.68	
Parameter	Estimate	% Bias	% SD
V_f	110.4 km/h	0.1	1.85
$\|C_j\|$	19.8 km/h	0.75	28.31
H_j	6.97 m	−0.81	22.22
K_j	143.51 veh/km	5.75	22.22

Detector UC: two-linear model			
No. of stationary periods: 14			
Congested Regime, V (km/h), H (m):		$V = 2 \times H - 2.3$ RMS error (km/h): 4.74	
Uncongested Regime, V (km/h), H (m):		$V = 0.34 \times H + 83.4$ RMS error (km/h): 2.29	
Parameter	Estimate	% Bias	% SD
$\|C_j\|$	2.3 km/h	0	44.05
H_j	1.15 m	−19.95	42.73
K_j	871.68 veh/km	38.2	42.73

Table 6. Results of the speed–spacing regression analysis for detector DL

Detector DL: two-linear model			
No. of stationary periods: 27			
Congested Regime, V (km/h), H (m):		$V = 3 \times H - 25.67$ RMS error (km/h): 2.63	
Uncongested Regime, V (km/h), H (m):		$V = 0.22 \times H + 99.41$ RMS error (km/h): 2.31	
Parameter	Estimate	% Bias	% SD
$\|C_j\|$	25.67 km/h	0	7.86
H_j	8.54 m	−0.13	5.52
K_j	117.14 veh/km	0.43	5.52

Dector DL: maximum sensitivity curve			
No. of stationary periods: 27		RMS error (km/h): 3.74	
Parameter	Estimate	% Bias	% SD
V_f	122 km/h	0.02	1.1
$\|C_j\|$	34.94 km/h	0.27	9.78
H_j	9.93 m	−0.08	6.11
K_j	100.74 veh/km	0.45	6.11

Table 7. Results of the speed–spacing regression analysis for detector UL

Detector UL: two-linear model			
No. of stationary periods: 21			
Congested Regime, V (km/h), H (m):		$V = 2.63 \times H - 18.99$ RMS error (km/h): 2.8	
Uncongested Regime, V (km/h), H (m):		$V = 0.19 \times H + 103.92$ RMS error (km/h): 2.63	
Parameter	Estimate	% Bias	% SD
$\|C_j\|$	18.99 km/h	0	16.21
H_j	7.21 m	−0.52	12.55
K_j	138.73 veh/km	2.1	12.55

Detector UL: maximum sensitivity curve			
No. of stationary periods: 21		RMS error (km/h): 5.25	
Parameter	Estimate	% Bias	% SD
V_f	126 km/h	0.02	1.34
$\|C_j\|$	38.85 km/h	0.32	15.80
H_j	10.81 m	−0.37	10.56
K_j	100.74 veh/km	1.49	10.56

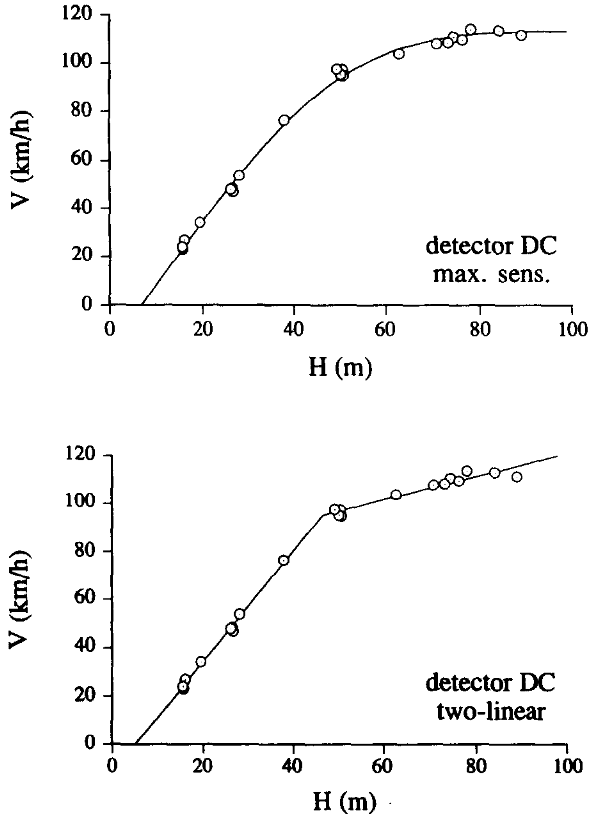

Fig. 4. Speed–spacing curves and observations for detector DC.

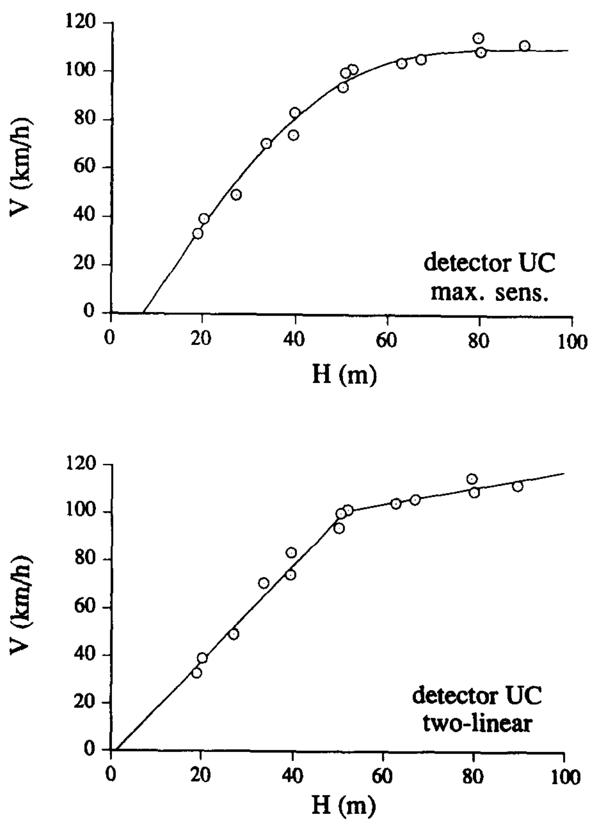

Fig. 5. Speed–spacing curves and observations for detector UC.

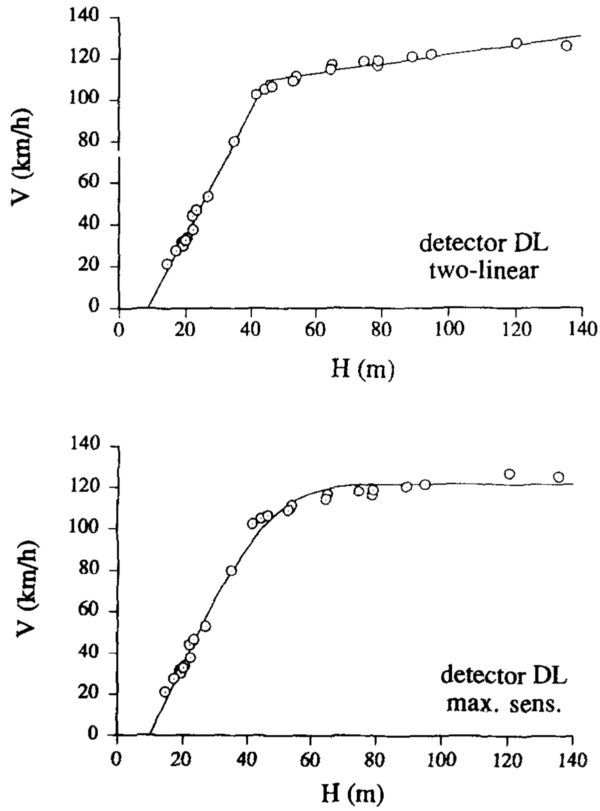

Fig. 6 Speed–spacing curves and observations for detector DL.

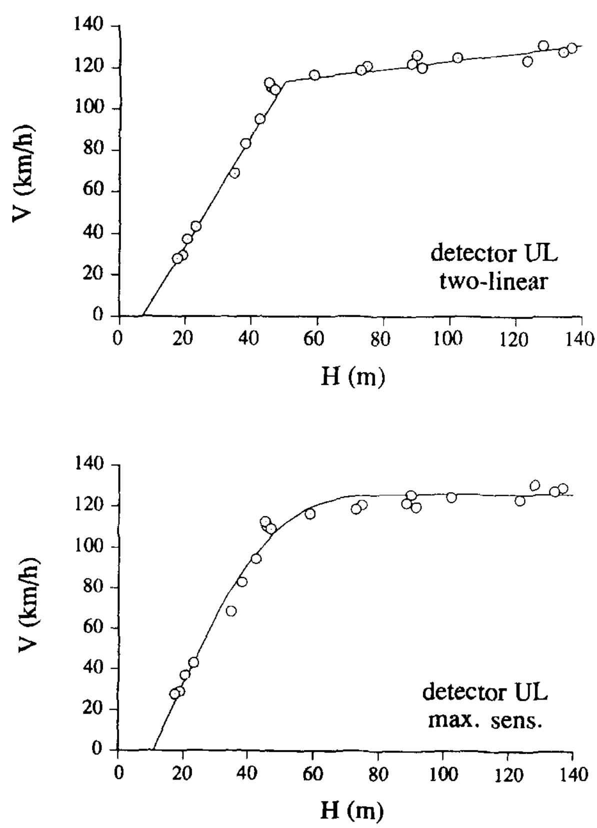

Fig. 7. Speed–spacing curves and observations for detector UL.

violate our own theory, we thought it was only worthwhile doing so if the alternative model was really simple. Then, we adopted a two-linear regression model as an alternative to the maximum sensitivity curve. This decision goes directly against our theory but, on the other hand, is a good 'test of goodness of fit' of it.

The maximum likelihood estimates of the parameters of the two-linear regime models have been obtained according to Quandt (1958). The bias of the estimates of $|C_j|$ is always zero because it is a linear parameter. The bias of the estimate of H_j has been calculated from the asymptotic variance of the estimators of the regression coefficients for the congested regime. The results for the center and left lane detectors DC, UC, DL and UL, are presented in Tables 4 to 7. The corresponding speed–spacing curves and data are depicted in Figs. 4 and 7.

Finally, it should be mentioned that the initial values of the estimates were fairly close to the final values. Namely, about a 15% of difference was obtained in the worst cases. Thus, the suitability of the exhaustive search for the minimum of the sum of square errors is manifest. The initial estimates of the parameters were simply calculated by rough fitting the corresponding regression model to the observations. This is easy to perform owing to the clear meaning of the parameters of the model.

INTERPRETATION OF THE RESULTS

In this section, we discuss how well the previous regression models fit the observations. The classical methodology for investigating whether a specified model provides a good description of the data, involves examination of the residuals, these being the differences between the observed speed and the predicted speeds. In practice, when the sample size is small and no replications of the observations exist, one is very apt to be misled if a decision on goodness of fit is based solely on the residuals. In view of this, the goodness of fit is assessed in a different manner. If the fitted regression models make sense, the estimates of the parameters $|C_j|$ and K_j should give realistic values. The wave speed values in accordance with other findings, see Del Castillo and Benítez (1994), lie within 15 and 25 km/h. Realistic values of K_j are on the order of 150 veh/km. This fact provides us with a criterion to judge the goodness of fit of the regression model. Likewise, the variance of the speed errors may serve us as a secondary criterion.

For the right lane detectors, DR and UR, the exponential curve yields a fitting definitely superior to that of the maximum sensitivity curve. The estimates of $|C_j|$ and H_j produced by this last regression model are undoubtedly unrealistic. The lack of data at very small spacings in detector DR is likely to be the cause of the high values of the percentage standard deviation of the estimates. Therefore, those estimates are not very reliable. On the contrary, the estimates found for detector UR are fairly accurate even for the small sample size fitted. Furthermore, the RMS error found with the exponential curve is smaller than that given by the maximum sensitivity curve for both detectors, although the difference is not decisive.

As for the center lane detectors, DC and UC, the fit achieved by the maximum sensitivity curve is excellent. Not only do the estimates take appropriate values, but their percentage standard deviation is also notably small, especially for detector DC. The considerably greater estimate percentage deviations from detector UC is a consequence of the small sample size: only 14 observations. The two-linear regression models are not adequate but they yield a reasonable estimation of the transition point between the congested and uncongested regimes.

Finally, for the left lane detectors, DL and UL, the maximum sensitivity regression model overestimates $|C_j|$ and gives a speed RMS error manifestly greater than the one given by the two-linear models. Hence, for these detectors, the two-linear model has the best performance. Once again the estimate standard deviations are remarkably small. This is due to the greater sample sizes, but probably also to the linearity of the regression model. In fact, the estimate of the wave speed is linear and its standard deviation is free from nonlinearity induced effects, thus closer to the actual deviation.

The best fitting models are then, the exponential for the right lane, the maximum sensitivity for the center lane and the two-linear for the left lane measurements. The results and plots of these data sets and models have been placed in the upper part of the tables and figures. An outstanding feature of these models is the extremely low bias of the estimates of V_f, $|C_j|$ and H_j. Except for detector DR, all the percentage biases are below 1%. Following Ratkowsky (1983), this percentage appears to be a good rule of thumb for indicating nonlinear behaviour of the estimates. Therefore, the parameterization of the regression model in terms of V_f, $|C_j|$ and H_j, renders a close to linear regression model. Had this not been the case, a reparameterization of the model would have been advisable. The discrepancy observed in the estimates bias for detector DR is, without doubt, motivated by the scarce number of low speed observations. However, this limitation does not considerably affect the estimate of V_f. This is logical, since all the information required for this estimate is contained in the high speed region. For this reason, the percentage bias and standard deviation of the estimate of V_f is in all the detectors, smaller than those of $|C_j|$ and H_j.

Finally, it is interesting to know the value of the capacity given by the models. The values of the capacity have been calculated for all the detectors by using the best regression model for each of them: exponential for DR and UR, maximum sensitivity for DC and UC, and two-linear model for DL and UL detectors. The values of the capacity for the left lane detectors are given by the transition point between the two linear regimes. For the right and center lane detectors, capacity is reached at the optimum equivalent spacing, λ_c. This parameter depends only on the dimensionless free-flow speed, u_f. For the exponential curve this relation is

$$\exp(\lambda_c) - 1 - \lambda_c = \frac{1}{u_f}$$

and for the maximum sensitivity curve is

$$\exp(e^{\lambda_c} - 1) - e^{\lambda_c}\,\lambda_c = \frac{e^{\lambda_c}}{u_f}.$$

The above equations are easily obtained by solving

$$\frac{\mathrm{d}q}{\mathrm{d}\rho} = 0$$

The dimensionless capacity, c, is then given by $c = q(\lambda_c)$ which becomes

$$c = \exp(-\lambda_c)$$

for the exponential curve, and

$$c = \exp(1 + \lambda_c - e^{\lambda_c})$$

for the maximum sensitivity curve. Finally, the capacity is given by $C = cK_j|C_j|$. The solution of the above equations leads to the results presented in Table 8.

In general, the results given by this analysis are in accordance with the early findings of Franklin (1965). In this memorable but overlooked work, Franklin pointed out for the first time the importance of the three fundamental parameters of traffic flow: the free flow speed, the jam density and the wave speed at jam density. The incorporation of these parameters into the functional form proposed in this work for the speed–density curve, makes the resulting speed–density curves superior to the rest appearing in the literature. The speed–density–volume relationships based on the new functional form satisfy all the required properties, see Del Castillo and Benítez (1994). It has been found that the speed

Table 8. Values of the capacity (vehicles per hour) given by the fitted models

Detector	Capacity	Detector	Capacity
DR	1216	UR	1485
DC	1970	UC	2059
DL	2429	UL	2253

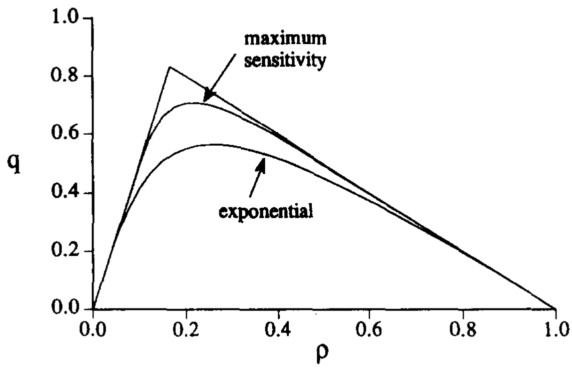

Fig. 8. Exponential and maximum sensitivity volume–density curves and the two-linear approximation.

falls off almost linearly with the spacing below a critical value. The slope of this curve is C_j, for which Franklin estimated a value of -8 mile/h. However, the value found in the present work is notably greater, about 20 km/h. Nevertheless, this value is the slope of the volume–density curve at the jam density K_j. The mean absolute value of the slope of this curve through the congested regime is slightly smaller due to the concavity of the volume–density curve. As for the jam density, Franklin found a mean value of 250 veh/mile, (156 veh/km). This value is close to the values obtained here, except for detectors DL and UR, which show a rather low jam density.

The regression analysis carried out has shown that a rough approximation for the volume–density curve is the two-linear curve

$$Q = KV_f, \qquad 0 \leqslant K \leqslant K_c;$$

$$Q = V_f K_c \frac{K_j - K}{K_j - K_c}, \qquad K_c \leqslant K \leqslant K_j.$$

In Fig. 8, this approximation, is plotted together with the exponential and maximum sensitivity curves. The y-axis represents the nondimensional volume $q = KV/(K_j |C_j|)$ and the x-axis the nondimensional density $\rho = K/K_j$. The curves have been plotted for $u_f = V_f/|C_j| = 5$. This approximation is rough only at the transition regime, whereas it is excellent outside of it. The error that it introduces is greatly compensated by its simplicity.

CONCLUSIONS

This work is aimed at demonstrating the suitability of the curves proposed in Part I for describing equilibrium traffic flow relationships. The success of this task has not been complete since it has failed for the left lane detectors. However, the fit achieved for the other lanes is excellent. This is not a conclusive result due to the limited amount of data used for this work. Nevertheless, the regression analysis shows that the functional form of Part I leads to realistic speed–spacing curves provided that the maximum speed does not exceed 115 km/h. A very nice feature of these speed–spacing models is the easiness for obtaining good initial estimates.

The regression analysis has also shown that a minimum sample size of 20 observations is needed for the estimates to have acceptable accuracy. For the same reason, it is indispensable to count with observations of spacings as short as 20 m. It would be very convenient to have a much bigger sample size than that of this paper in order to compare the goodness of fit of an additive error model with that of a multiplicative error model. In such a model, the standard deviation of the speed error would not be constant, but proportional to the mean speed.

As a by-product, the present work includes a procedure for isolating periods of stationary traffic flow and estimating the mean values of the traffic variables. This procedure eliminates, in a very satisfactory manner, the scattering of the individual measurements, as the very low variance of the speed residuals demonstrates. A further improvement to the estimation of the mean speed and spacing could be achieved by storing the vehicle speeds to calculate the sample harmonic mean speed.

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