ISA Transactions 53 (2014) 1739–1745

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Stabilization strategies of a general nonlinear car-following model with varying reaction-time delay of the drivers

Shukai Li*, Lixing Yang, Ziyou Gao, Keping Li

State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, China

ARTICLE INFO

Article history:
Received 13 February 2014
Received in revised form
14 July 2014
Accepted 31 August 2014
Available online 23 October 2014
This paper was recommended for publication by Dr. Q.-G. Wang

Keywords:
Car-following model
Reaction-time delay
State feedback control
Robust stability

ABSTRACT

In this paper, the stabilization strategies of a general nonlinear car-following model with reaction-time delay of the drivers are investigated. The reaction-time delay of the driver is time varying and bounded. By using the Lyapunov stability theory, the sufficient condition for the existence of the state feedback control strategy for the stability of the car-following model is given in the form of linear matrix inequality, under which the traffic jam can be well suppressed with respect to the varying reaction-time delay. Moreover, by considering the external disturbance for the running cars, the robust state feedback control strategy is designed, which ensures robust stability and a smaller prescribed H_∞ disturbance attenuation level for the traffic flow. Numerical examples are given to illustrate the effectiveness of the proposed methods.

1. Introduction

Traffic flow problem is an important research topic and many traffic flow models are proposed to understand the complex dynamics of traffic flow [1–5]. In particular, Bando et al. [3] proposed a car-following traffic model and derived a simple stability condition, in which the vehicle equation is governed by an optimal velocity function which depends on the headway distance. This model is called optimal velocity (OV) traffic model that describes the stop-and-go traffic characteristic and the traffic jam phenomenon of the real traffic flow. Stimulated by the above model, many new car-following traffic models were proposed to understand the complex characteristics of traffic flow. In [6], an improved traffic model was proposed to cope with particular situations like vehicles approaching standing cars. A coupled map car-following model with decentralized delayed feedback control scheme was proposed in [7] to suppress the traffic jam, which describes the dynamical behavior of a group of road vehicles traveling on a single lane without overtaking. In [8], a full velocity difference car-following model was studied, which considered both negative and positive velocity difference, and given a better description of starting process. Zhao and Gao [9] presented another simple strategy to suppress the congested state in the traffic system, in which the control signal incorporates the effect of the velocity difference between the preceding and the considering vehicles. By considering the navigation in modern traffic, a two velocity difference model for a car following theory was developed in [10], and the property of the model was investigated by using linear and nonlinear analyses. An extended car-following model which takes into account the honk effect was proposed in [11], which shows that the honk effect improves the stability of traffic flow. In [12], a modified coupled map car-following model was proposed, in which two successive vehicle headways in front of the considering vehicle were incorporated into the optimal velocity function, and a new control scheme was presented to suppress the traffic jam. With the consideration of varying road condition, a new car-following model was developed in [13], which shows that the road condition has great influences on uniform flow. In [14], a new anticipation optimal velocity model was proposed for car following theory on single lane byconsidering anticipation effect, and the linear stability condition was derived by the linear stability analysis.

Time delays play a major role in traffic behavior due to the time needed by human operators in sensing velocity and position variations of the vehicles in the traffic [4,15]. The optimal velocity model by incorporating the time-delay effect was proposed in [16], in which the time delay was constant and the properties of congestion and the delay time of car motion were analyzed. The study in [17] offered two modified optimal velocity models with time delays in order to create reasonable traffic models that better match the reality. In [18], the bifurcations and multiple traffic jams in a car-following model with reaction time delay were considered, and the stability of the uniform-flow equilibrium was studied. In [19], the finite reaction times were considered for the time-continuous microscopic traffic models, which showed that

* Corresponding author. Tel.: +86 1051683970.
E-mail address: shkli@bjtu.edu.cn (S. Li).

http://dx.doi.org/10.1016/j.isatra.2014.08.017
0019-0578/

the destabilizing effects of reaction times can be compensated by spatial and temporal anticipation. By considering the delay of the drivers response in sensing headway, an extended car-following model was proposed in [20] to describe the traffic, and the neutral stability line and the critical point were obtained by using the linear stability theory.

Although there were numerous works on the analysis of the car-following model with reaction time delays, the existing literatures mainly modeled the time-delayed actions of the drivers by a discrete delay. In practice, the reaction time-delay of the drivers should be time varying and bounded. It is more realistic to consider the time delay as time varying. The stability and stabilization problems of the time-varying delay systems were studied in [21–23]. However, little literature can be found to deal with the nonlinear optimal velocity (OV) traffic model with varying reaction-time delay. Motivated by the above discussions, in this paper, we consider the stabilization strategies of a general nonlinear car-following model with varying reaction-time delay of the drivers. By using the Lyapunov functions theory, the sufficient condition for the existence of the state feedback control strategy for the stability of the car-following model is given in the form of linear matrix inequalities, under which the traffic jam can be well suppressed with respect to the varying reaction-time delay. Additionally, the running vehicles will inevitably suffer from the uncertain external disturbances, such as irregular surfaces, bad weather, and equipment failure. To guarantee all the cars run orderly with the desired velocity with respect to the uncertain external disturbances, similar to the robust tracking control problem of nonlinear systems [24–29], the robust control method can be better applied to study the stability criteria for the car-following model with respect to the uncertain external disturbances. The H_∞ control method can not only guarantee the stability of the car-following model, but also guarantee a smaller prescribed disturbance attenuation level with respect to the external disturbances [30–33]. Then based on the H_∞ control method, we design the robust state feedback control strategy, which ensures robust stability and a smaller prescribed H_∞ disturbance attenuation level for the traffic flow. Numerical examples are given to illustrate the effectiveness of the proposed methods. From the numerical examples, we can observe that the proposed control methods can be effectively used to suppress traffic jam with respect to the varying reaction-time delay. Moreover, the proposed robust control methods effectively suppress the uncertain external disturbance to the stability of traffic flow.

The rest of this paper is organized as follows. In Section 2, a general nonlinear optimal car-following model with varying reaction-time delay is presented. In Section 3, the stabilization strategies of the nonlinear car-following model with varying reaction-time delay are considered. In Section 4, numerical examples are provided to demonstrate the effectiveness of our theoretical results. We conclude this paper in Section 5.

2. Optimal car-following model

Let us consider a general nonlinear optimal velocity traffic model with varying reaction-time delay of the drivers under an open boundary condition. The leading vehicle is supposed to be running with a constant velocity $v_0>0$, and the motion of the leading vehicle is described as follows:

$$x_0(t)=x_0(0)+v_0t, \tag{1}$$

where $x_0(t)>0$ is the position of the leading vehicle at time t, $x_0(0)>0$ is its initial position.

Assume that the leading vehicle is not influenced by other following vehicles. By considering reaction-time delay of the drivers, the motions of the following vehicles group are described by a nonlinear dynamical delayed equation as follows:

$$\begin{cases} \dot{y}_i(t)=v_{i-1}(t)-v_i(t), \\ \dot{v}_i(t)=a_i(F(y_i(t-\tau(t)))-v_i(t))+u_i(t), \quad i=1,2,\ldots,n, \end{cases} \tag{2}$$

where $y_i(t)>0$ is the headway distance between the $i-1$-th and i-th vehicles, $v_i(t)$ is the velocity of the ith vehicle, $a_i>0$ denotes the sensitivity of the driver, the different $a_i>0$ means that each driver has a different sensitivity, n is the number of following vehicles, $F(y_i(t-\tau(t)))$ is the OV function related to the behavior of individual drivers, which depends on the distance between the $(i-1)-$th and the i-th vehicles, $u_i(t)$ is the control input signal to be designed, $\tau(t)$ is the varying reaction-time delay of the drivers and satisfies that

$$0\le\tau(t)\le h, \quad \dot{\tau}(t)\le d<1, \tag{3}$$

where the bounds h and d are known constant scalars. h is the maximum reaction time delay due to the rational behavior of the driver. In addition, it is shown empirically that the variations of time delay of humans are very slow [34]. So it is reasonable to assume that $\dot{\tau}(t)\le d<1$. Additionally, for the dynamical delayed equation (2), the initial condition is given as $y_i(t)=\phi_i(t),\ t\in[-h,0],\ i=1,2,\ldots,n$, where $\phi_i(t)$ is a continuous and differentiable function.

The physical basis of the model (2) is the fact that the reaction time of the driver performing his decision is varying with the time, in contrast to the case that the reaction time is discrete, which can better match the reality to describe the behavior of individual drivers for the traffic dynamics.

Without loss of generality, we choose a general nonlinear OV function as follows:

$$F(y(t))=\frac{v_{\max}}{2}(\tanh(y(t)-y_c)+\tanh(y_c)), \tag{4}$$

where $v_{\max}$ is the maximum velocity, y_c is the safe distance. Moreover, it is clear that the nonlinear OV function (4) satisfies that

$$\|F(y_1(t))-F(y_2(t))\|\le\frac{v_{\max}}{2}\|y_1(t)-y_2(t)\|, \quad \forall y_1(t),\ y_2(t)\in R^m. \tag{5}$$

The form of condition (5) holds for the most proposed nonlinear OV functions in the existing literatures. It is very easy to prove that for the most proposed OV functions, there exists some constant $\alpha>0$, such that the following condition holds:

$$\|F(y_1(t))-F(y_2(t))\|\le\alpha\|y_1(t)-y_2(t)\|, \quad \forall y_1(t),\ y_2(t)\in R^m. \tag{6}$$

Consider that the leading vehicle runs constantly with velocity v_0. The car-following model (2) has the following steady state:

$$[y_i^*(t),v_i^*(t)]^T=[F^{-1}(v_0),v_0]^T, \quad i=1,2,\ldots,n, \tag{7}$$

which implies that, in the steady state, all the vehicles run orderly with velocity v_0 and headway distance $F^{-1}(v_0)$.

The objective of this paper is to design the state feedback control input signal $u_i(t)$ to guarantee that the following vehicles run orderly with velocity v_0 and headway distance $F^{-1}(v_0)$ by rejecting the effect of the varying reaction time delays, so as to effectively suppress the traffic jam.

To obtain the main results, the following lemmas are needed.

Lemma 2.1 (*Jensen's inequality, Gu et al. [35]*). *For any constant matrix $R>0$, scalar $b>0$, and vector function $x:[0,b]\to R^n$, one has*

$$b\int_0^b x^T(s)Rx(s)\,ds\ge\left(\int_0^b x(s)\,ds\right)^T R\left(\int_0^b x(s)\,ds\right).$$

Lemma 2.2 (*Schur Complement, Boyd et al. [36]*). *The matrix inequality*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & -S_{22} \end{bmatrix}<0,$$

where $S_{11}=S_{11}^T$, $S_{22}=S_{22}^T$ is equivalent to

$$S_{22}>0, \quad S_{11}+S_{12}S_{22}^{-1}S_{21}<0.$$

3. Stabilization strategies of the car-following model

In this section, based on the Lyapunov stability theory, we will study the stabilization strategies of the car-following model with varying reaction-time delay of the drivers.

Consider that the state feedback control input signal $u_i(t)$ for the vehicle i is based on the information of the difference between the present headway distance $y_i(t)$ and the ahead car's headway $y_{i-1}(t)$, and the difference between the ahead car's velocity $v_{i-1}(t)$ and the present velocity $v_i(t)$. Then the control input signal $u_i(t)$ for each vehicle is designed as follows:

$$u_i(t)=k_{1i}(y_i(t)-y_{i-1}(t))+k_{2i}(v_{i-1}(t)-v_i(t)), \quad i=1,2,\ldots,n, \tag{8}$$

where k_{1i},k_{2i} are the control parameters to be determined, and $y_0=F^{-1}(v_0)$. The control input signal $u_i(t)$ is designed as the control equipment of the autonomous cruise control system for the vehicles, and the ahead car's headway and velocity can be measured by the advanced radar and video techniques.

Applying the above control input signal $u_i(t)$ to the car-following model (2), one can obtain the following closed-loop car-following model:

$$\begin{cases} \dot{y}_i(t) &= v_{i-1}(t)-v_i(t), \\ \dot{v}_i(t) &= a_i(F(y_i(t-\tau(t)))-v_i(t))+k_{1i}(y_i(t)-y_{i-1}(t)) \\ & \quad +k_{2i}(v_{i-1}(t)-v_i(t)), \quad i=1,2,\ldots,n. \end{cases} \tag{9}$$

Let error vectors $\hat{y}_i(t)=y_i(t)-y_0$, $\hat{v}_i(t)=v_i(t)-v_0$. According to (7) and (9), one can obtain the error dynamic for car-following model (2) as

$$\Big\{ \dot{\hat{y}}_i(t)=\hat{v}_{i-1}(t)-\hat{v}_i(t), \dot{\hat{v}}_i(t)=a_i(\hat{F}(y_i(t-\tau(t)))-\hat{v}_i(t)) \\ +k_{1i}(\hat{y}_i(t)-\hat{y}_{i-1}(t))+k_{2i}(\hat{v}_{i-1}(t)-\hat{v}_i(t)), \quad i=1,2,\ldots,n, \tag{10}$$

where $\hat{F}(y_i(t-\tau(t)))=F(y_i(t-\tau(t)))-F(y_0)$.

Let $\hat{y}(t)=[\hat{y}_1(t),\ \hat{y}_2(t),\ldots,\hat{y}_n(t)]^T$, $\hat{v}(t)=[\hat{v}_1(t),\ \hat{v}_2(t),\ldots,\hat{v}_n(t)]^T$. Reformulate the error dynamic (10) as

$$\Big\{ \dot{\hat{y}}(t)=C_2\hat{v}(t), \dot{\hat{v}}(t)=A(\hat{F}(y(t-\tau(t)))-\hat{v}(t))+K_1C_1\hat{y}(t)+K_2C_2\hat{v}(t), \tag{11}$$

where $A=\text{diag}\{a_1,a_2,\ldots,a_n\}$, $K_1=\text{diag}\{k_{11},k_{12},\ldots,k_{1n}\}$, $K_2=\text{diag}\{k_{21},k_{22},\ldots,k_{2n}\}$,

$$C_1=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & \cdots & -1 & 1 \end{bmatrix}, \quad C_2=\begin{bmatrix} -1 & 0 & 0 & \cdots \\ 1 & -1 & 0 & \cdots \\ & \cdots & \cdots & \\ 0 & \cdots & 1 & -1 \end{bmatrix}.$$

It is obvious that the following vehicles run orderly with velocity v_0 and headway distance $F^{-1}(v_0)$ if the solution of the error dynamic (11) converges to zero.

The following theorem will provide a sufficient condition for the existence of the state feedback controller for the stability of the car-following model with varying reaction-time delay.

Theorem 3.1. *Consider the car-following model (2) with the control input signal (8). The varying reaction-time delay $\tau(t)$ satisfies condition (3). For the given control parameter matrices K_1, K_2, if there exist a positive scalar ε, positive definite matrixes P,Q_1,Q_2,R, such that the following linear matrix inequality holds:*

$$\begin{bmatrix} \Phi & 0 & \frac{1}{h}H_2^TR & PH_3 \\ 0 & \varepsilon(\frac{v_{\max}}{2})^2I-(1-d)Q_1 & 0 & 0 \\ \frac{1}{h}RH_2 & 0 & -Q_2-\frac{1}{h}R & 0 \\ H_3^TP & 0 & 0 & -\varepsilon I \end{bmatrix}<0, \tag{12}$$

where

$$\Phi=P\begin{bmatrix} 0 & C_2 \\ 0 & -A \end{bmatrix}+\begin{bmatrix} 0 & C_2 \\ 0 & -A \end{bmatrix}^TP+P\begin{bmatrix} 0 & 0 \\ K_1C_1 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ K_1C_1 & 0 \end{bmatrix}^TP \\ +P\begin{bmatrix} 0 & 0 \\ 0 & K_2C_2 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & K_2C_2 \end{bmatrix}^TP+H_2^TQ_1H_2+H_2^TQ_2H_2 \\ +hH_1^TC_2^TRC_2H_1-\frac{1}{h}H_2^TRH_2,$$

$$H_1=[0\ \ I], \quad H_2=[I\ \ 0], \quad H_3=\begin{bmatrix} 0 \\ A \end{bmatrix},$$

and I is an identity matrix with appropriate dimension, then the state feedback control gains K_1 and K_2 are obtained to guarantee the stability of the car-following model (2) with the varying reaction-time delay, under which the traffic jam can be well suppressed.

Proof. Let $\delta(t)=[\hat{y}^T(t),\hat{v}^T(t)]^T$. The error dynamic (11) is rewritten as

$$\dot{\delta}(t)=\begin{bmatrix} 0 \\ A \end{bmatrix}\hat{F}(y(t-\tau(t)))+\begin{bmatrix} 0 & C_2 \\ 0 & -A \end{bmatrix}\delta(t)+\begin{bmatrix} 0 & 0 \\ K_1C_1 & 0 \end{bmatrix}\delta(t) \\ +\begin{bmatrix} 0 & 0 \\ 0 & K_2C_2 \end{bmatrix}\delta(t). \tag{13}$$

According to the Lyapunov–Krasovskii function theory for the nonlinear time delay system, one can construct the following Lyapunov–Krasovskii function candidate for the error dynamic (13) of the car-following model (2) as

$$V(t)=V_1(t)+V_2(t)+V_3(t) \tag{14}$$

with

$$V_1(t)=\delta^T(t)P\delta(t),$$
$$V_2(t)=\int_{t-\tau(t)}^t \hat{y}^T(\alpha)Q_1\hat{y}(\alpha)\,\mathrm{d}\alpha+\int_{t-h}^t \hat{y}^T(\alpha)Q_2\hat{y}(\alpha)\,\mathrm{d}\alpha,$$
$$V_3(t)=\int_{-h}^0\int_{t+\theta}^t \dot{\hat{y}}^T(\alpha)R\dot{\hat{y}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\theta,$$

where P,Q_1,Q_2,R are positive definite weighting matrices. $V_1(t)$ is standard to the delayless nominal system while $V_2(t)$ and $V_3(t)$ correspond to the delay-dependent conditions. Moreover, the first integral term of $V_2(t)$ is related to the time-varying delay $\tau(t)$, and the second is about the maximum delay h, which will ensure that the time-varying delay satisfies the bound of the maximum delay.

Taking the derivative of $V_1(t)$ along the trajectory of the error dynamic (13) yields that

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}=2\delta^T(t)P\Big\{\begin{bmatrix} 0 \\ A \end{bmatrix}\hat{F}(y(t-\tau(t)))+\begin{bmatrix} 0 & C_2 \\ 0 & -A \end{bmatrix}\delta(t) \\ +\begin{bmatrix} 0 & 0 \\ K_1C_1 & 0 \end{bmatrix}\delta(t)+\begin{bmatrix} 0 & 0 \\ 0 & K_2C_2 \end{bmatrix}\delta(t)\Big\}. \tag{15}$$

According to the Cauchy inequality that $A^TB+B^TA\le\varepsilon^{-1}A^TA+\varepsilon B^TB$, $\forall\varepsilon>0$, and matrices A and B with appropriate dimensions, together with the condition (5), it holds that

$$2\delta^T(t)P\begin{bmatrix} 0 \\ A \end{bmatrix}\hat{F}(y(t-\tau(t)))\le\varepsilon^{-1}\delta^T(t)P\begin{bmatrix} 0 \\ A \end{bmatrix}[0\ \ A^T]P\delta(t)$$

$$+\varepsilon\left(\frac{v_{\max}}{2}\right)^2\hat{y}^T(t-\tau(t))\hat{y}(t-\tau(t)). \tag{16}$$

By the condition that $\dot{\tau}(t)\le d<1$, one has

$$\begin{aligned}\frac{dV_2(t)}{dt}&=\hat{y}^T(t)Q_1\hat{y}(t)-(1-\dot{\tau}(t))\hat{y}^T(t-\tau(t))Q_1\hat{y}(t-\tau(t))\\&\quad+\hat{y}^T(t)Q_2\hat{y}(t)-\hat{y}^T(t-h)Q_2\hat{y}(t-h)\\&\le\hat{y}^T(t)Q_1\hat{y}(t)-(1-d)\hat{y}^T(t-\tau(t))Q_1\hat{y}(t-\tau(t))\\&\quad+\hat{y}^T(t)Q_2\hat{y}(t)-\hat{y}^T(t-h)Q_2\hat{y}(t-h).\end{aligned} \tag{17}$$

In addition, by Lemma 2.1, it holds that

$$\int_{t-h}^{t}\dot{\hat{y}}^T(\alpha)R\dot{\hat{y}}(\alpha)\,d\alpha\ge\frac{1}{h}(\hat{y}(t)-\hat{y}(t-h))^TR(\hat{y}(t)-\hat{y}(t-h)). \tag{18}$$

Then direct computation gives

$$\begin{aligned}\frac{dV_3(t)}{dt}&=h\dot{\hat{y}}^T(t)R\dot{\hat{y}}(t)-\int_{t-h}^{t}\dot{\hat{y}}^T(\alpha)R\dot{\hat{y}}(\alpha)\,d\alpha\\&=h\hat{\delta}^T(t)H_1^TC_2^TRC_2H_1\delta(t)-\int_{t-h}^{t}\dot{\hat{y}}^T(\alpha)R\dot{\hat{y}}(\alpha)\,d\alpha\\&\le h\hat{\delta}^T(t)H_1^TC_2^TRC_2H_1\delta(t)\\&\quad-\frac{1}{h}(\hat{y}(t)-\hat{y}(t-h))^TR(\hat{y}(t)-\hat{y}(t-h)).\end{aligned} \tag{19}$$

Combining (15)–(19) yields

$$\frac{dV(t)}{dt}\le\begin{bmatrix}\delta(t)\\\hat{y}(t-\tau(t))\\\hat{y}(t-h)\end{bmatrix}^T\begin{bmatrix}\Phi+\varepsilon^{-1}PH_3H_3^TP & 0 & \frac{1}{h}H_2^TR\\0 & \varepsilon(\frac{v_{\max}}{2})^2I-(1-d)Q_1 & 0\\\frac{1}{h}RH_2 & 0 & -Q_2-\frac{1}{h}R\end{bmatrix}\times\begin{bmatrix}\delta(t)\\\hat{y}(t-\tau(t))\\\hat{y}(t-h)\end{bmatrix}. \tag{20}$$

Then, by Lemma 2.2, one can get that the matrix inequality (12) is equivalent to

$$\begin{bmatrix}\Phi+\varepsilon^{-1}PH_3H_3^TP & 0 & \frac{1}{h}H_2^TR\\0 & \varepsilon(\frac{v_{\max}}{2})^2I-(1-d)Q_1 & 0\\\frac{1}{h}RH_2 & 0 & -Q_2-\frac{1}{h}R\end{bmatrix}<0. \tag{21}$$

Thus according to (20), one can obtain $dV(t)/dt<0$. By the Lyapunov stability theory, the error dynamic (13) of the car-following model is stable. Therefore, the state feedback control gains K_1 and K_2 are obtained to guarantee the stability of the car-following model (2) with the varying reaction-time delay, under which the traffic jam can be well suppressed. □

Remark 3.1. Theorem 3.1 gives a sufficient condition to choose proper state feedback control gains K_1 and K_2 such that the car-following model (2) is stable at the steady state for any reaction-time delay $\tau(t)$ with condition (3). The matrix variables P, Q_1, Q_2, R reduce the conservativeness of the proposed method. Additionally, when the feedback control gains K_1 and K_2 are given, it is obvious that the condition (13) in Theorem 3.1 takes the form of linear matrix inequality, which can be easily determined by using the Matlab LMI toolbox.

In practice, the running vehicles will inevitably suffer from the uncertain external disturbances, such as irregular surfaces, bad weather, and equipment failure. Therefore, it is necessary to study the car-following model with uncertain external disturbances, and design the robust control strategy for each vehicle to inhibit the influences of uncertain external disturbances.

Let $w_i(t)$ be the external disturbance with finite energy for each vehicle i. b_i is the appropriate weighting factor. Then the car-following model with external disturbances is given as follows:

$$\begin{cases}\dot{y}_i(t) &= v_{i-1}(t)-v_i(t),\\\dot{v}_i(t) &= a_i(F(y_i(t-\tau(t)))-v_i(t))\\&\quad+u_i(t)+b_iw_i(t),\quad i=1,2,\dots,n.\end{cases} \tag{22}$$

The corresponding error dynamic is obtained as

$$\begin{aligned}\dot{\delta}(t)&=\begin{bmatrix}0\\A\end{bmatrix}\hat{F}(y(t-\tau(t)))+\begin{bmatrix}0 & C_2\\0 & -A\end{bmatrix}\delta(t)\\&\quad+\begin{bmatrix}0 & 0\\K_1C_1 & 0\end{bmatrix}\delta(t)\\&\quad+\begin{bmatrix}0 & 0\\0 & K_2C_2\end{bmatrix}\delta(t)+\begin{bmatrix}0\\B\end{bmatrix}w(t),\end{aligned} \tag{23}$$

where $w(t)=[w_1(t),\ w_2(t),\dots,w_n(t)]^T$, $B=\text{diag}\{b_1,b_2,\dots,b_n\}$, and other notations are the same to those in (11).

The robust stability of the car-following model with varying reaction-time delay of the drivers in this case consists of determining the control gains K_1 and K_2 such that the closed-loop dynamic (23) of the car-following model with $w(t)=0$ is stable, and for a certain given prescribed H_∞ disturbance attenuation level $\gamma>0$, the following condition holds:

$$\left(\int_0^{+\infty}\delta^2(t)\,dt\right)^{1/2}\le\gamma\left(\int_0^{+\infty}w^2(t)\,dt\right)^{1/2}, \tag{24}$$

for any nonzero $w(t)\in L_2[0,+\infty)$ and the varying reaction-time delay of the drivers under the zero initial condition.

Then the robust stabilization strategy for the car-following model with the varying reaction-time delay and external disturbance is given as the following theorem.

Theorem 3.2. *Consider the car-following model (22) with the control input signal (8). Let $\gamma>0$ be a given constant. For the given control parameter matrices K_1, K_2, if there exist a positive scalar ε, positive definite matrixes P, Q_1, Q_2, R, such that the following linear matrix inequality holds:*

$$\begin{bmatrix}\Phi+I & 0 & \frac{1}{h}H_2^TR & PH_3 & PH_1^TB\\0 & \varepsilon(\frac{v_{\max}}{2})^2I-(1-d)Q_1 & 0 & & \\\frac{1}{h}RH_2 & 0 & -Q_2-\frac{1}{h}R & 0 & 0\\H_3^TP & 0 & 0 & -\varepsilon I & 0\\B^TH_1P & 0 & 0 & 0 & -\gamma^2I\end{bmatrix}<0, \tag{25}$$

where the notations are the same to that in (12), then the state feedback control gains K_1 and K_2 are obtained to guarantee the robust stability of the car-following model (22) with the disturbance attenuation level γ, under which the traffic jam can be well suppressed.

Proof. At first, by Lemma 2.2, the inequality (25) implies that the condition (12) holds, so according to Theorem 3.1, the error dynamic (23) of the car-following model with $w(t)=0$ is stable. Then for any nonzero $w(t)\in L_2[0,+\infty)$, calculating the derivative of $V(t)$ with the form of (14) along the trajectory of error dynamic (23) yields that

$$\frac{dV(t)}{dt}\le\begin{bmatrix}\delta(t)\\\hat{y}(t-\tau(t))\\\hat{y}(t-h)\\w(t)\end{bmatrix}^T\begin{bmatrix}\Phi+\varepsilon^{-1}PH_3H_3^TP & 0 & \frac{1}{h}H_2^TR & PH_1^TB\\0 & \Psi & 0 & 0\\\frac{1}{h}R^TH_2 & 0 & -Q_2-\frac{1}{h}R & 0\\B^TH_1P & 0 & 0 & 0\end{bmatrix}$$

$$\times \begin{bmatrix} \delta(t) \\ \hat{y}(t-\tau(t)) \\ \hat{y}(t-h) \\ w(t) \end{bmatrix}, \tag{26}$$

where $\Psi = \varepsilon(v_{\max}/2)^2 I - (1-d)Q_1$.

By Lemma 2.2, the condition (25) is equivalent to

$$\begin{bmatrix} \Phi + \varepsilon^{-1} P H_3 H_3^T P + I & 0 & \frac{1}{h} H_2^T R & P H_1^T B \\ 0 & \Psi & 0 & 0 \\ \frac{1}{h} R^T H_2 & 0 & -Q_2 - \frac{1}{h} R & 0 \\ B^T H_1 P & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{27}$$

It obviously follows from (26) and (27) that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} + \delta^T(t)\delta(t) - \gamma^2 w^T(t) w(t) < 0. \tag{28}$$

Thus, noting that $\lim_{t\to\infty} V(t) = 0$, under the zero initial condition, integrating both sides of (28) from 0 to $+\infty$, we have

$$\left(\int_0^{+\infty} \delta^2(t)\, \mathrm{d}t \right)^{1/2} \le \gamma \left(\int_0^{+\infty} w^2(t)\, \mathrm{d}t \right)^{1/2}. \tag{29}$$

Therefore, the car-following model (22) is robustly stable with the disturbance attenuation γ, and under the control input signal (8), the traffic jam can be well suppressed. □

Remark 3.2. Theorem 3.2 gives a sufficient condition to choose proper state feedback control gains K_1 and K_2 such that the car-following model (22) is robustly stable for the external disturbance $w(t)$. For the given prescribed H_∞ disturbance attenuation level $\gamma > 0$, and feedback control gains K_1 and K_2, the condition (25) in Theorem 3.2 also takes the form of linear matrix inequality, which can be easily determined by using the Matlab LMI toolbox.

Remark 3.3. It should be pointed that for the proposed car-following model (2), the varying reaction-time delays of the drivers are set to be the same. In a similar way, the proposed methods can be extended to handle the case that the varying reaction-time delays are different for each driver.

4. Numerical simulations

In this section, numerical examples are given to illustrate the effectiveness of the proposed methods for the car-following model with varying reaction-time delay.

Let us consider a case where 20 vehicles are running on a single lane without overtaking under an open boundary. The parameters are set as the maximum velocity $v_{\max} = 2$, the safe distance $y_c = 2$, and the velocity of leading vehicle $v_0 = 0.964$. The time horizon is chosen as $T = 600$.

For analysis convenience, we assume that the sensitivity and control parameters for each vehicle are the same. The sensitivity $a_i = 0.5,\ i = 1, 2, \ldots, 20$. The varying reaction-time delay $\tau(t)$ is set as $\tau(t) = 1 + 0.4\cos(t)$, which shows that $h = 1.4,\ d = 0.4$.

Then by the condition (12) of Theorem 3.1, we choose a set of smaller state feedback control parameters as $k_{1i} = 10.1,\ k_{2i} = 10.1,\ i = 1, 2, \ldots, 20$. The initial conditions $y_i(t) = 2.2,\ t \in [-h, 0]$ and $v_i(0)$ are chosen randomly from [0.764, 1.164]. Consider the situation that the leading vehicle slows down suddenly three times, i.e., $v_0(t) = v_0/2$ for $t = 100$–102, 200–202, and 300–302. Under the state feedback control, Fig. 1 plots the space–time plot of the running traffic model, and Fig. 2 shows the temporal velocity behavior of the 1st, 10th, and 20th vehicles.

From Fig. 1, we can observe that for the car-following model with reaction-time delay of the drivers, the proposed method can be effectively used to suppress traffic jam with respect to the varying reaction-time delay. The following vehicles run orderly with velocity $v_0 = 0.964$ and headway distance $F^{-1}(v_0) = 2$. Moreover, it is observed from Fig. 2 that the amplitude of speed fluctuation decreases with the increase of vehicle number i, which is consistent with the results in the literature [12], and the traffic system exhibits better temporal behavior.

Next, we will consider the robust stability of the car-following model with varying reaction-time delay and external disturbances. Suppose that the vehicles $i = 3, 4, 5$ of the car-following model are inevitably running with the following external disturbance:

$$w_i(t) = \begin{cases} 5 + \cos(t) & \text{if } 100 \le t \le 120, \\ 5 + \cos(t) & \text{if } 260 \le t \le 280, \\ 5 + \cos(t) & \text{if } 380 \le t \le 400, \\ 0 & \text{otherwise}, \end{cases} \tag{30}$$

and the weighting factor b_i is chosen as $b_i = 0.1,\ i = 3, 4, 5$.

For a given smaller prescribed H_∞ disturbance attenuation level $\gamma = 1$, according to the condition (25) in Theorem 3.2, a set

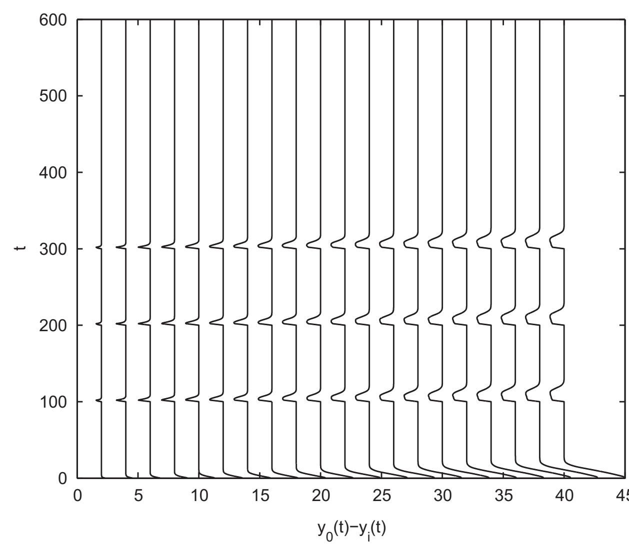

Fig. 1. The space–time plot of the running traffic model under the state feedback control.

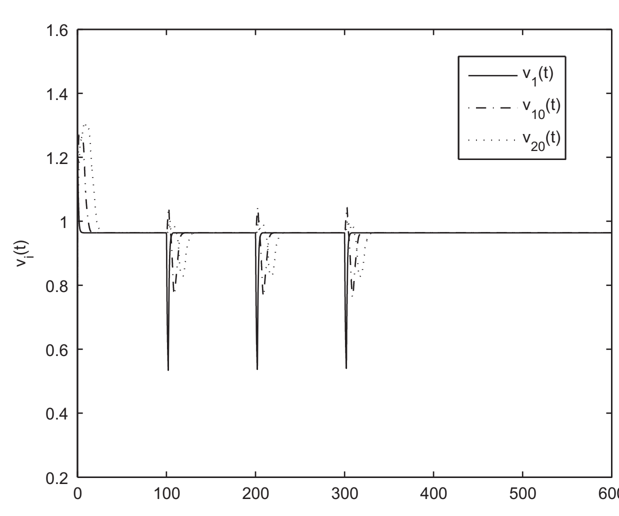

Fig. 2. The temporal velocity behavior of three vehicles.

of smaller control parameters are chosen as $k_{1i}=13.1$, $k_{2i}=13.1$. Then under the robust control, the space–time plot of the car-following model with external disturbance is shown in Fig. 3. From Fig. 3, we can see that the proposed control method effectively suppress the uncertain external disturbance to the stability of traffic flow. The following vehicles run orderly with velocity v_0=0.964 and headway distance $F^{-1}(v_0)=2$. Although the effect of the external disturbances for the 3rd, 4th, and 5th vehicles is propagated to the following cars, by adding the robust control to each car, the external disturbance to the stability of whole car-following model is effectively suppressed. Additionally, Fig. 4 plots the temporal velocity behavior of the 4th, 10th, and 20th vehicles. Under the robust control, the deviations of the velocities of the cars from the desired velocity v_0 are reduced to a reasonable range of 0.12, i.e., the speed fluctuations are effectively reduced under the robust control. When the external disturbances disappear, the running speeds of the following cars quickly converge to the desired velocity v_0. The simulations show the effectiveness of the proposed methods given in this paper. Moreover, the delay-dependent stability criteria for the multi-lane traffic flow by considering overtaking will be investigated in our future work.

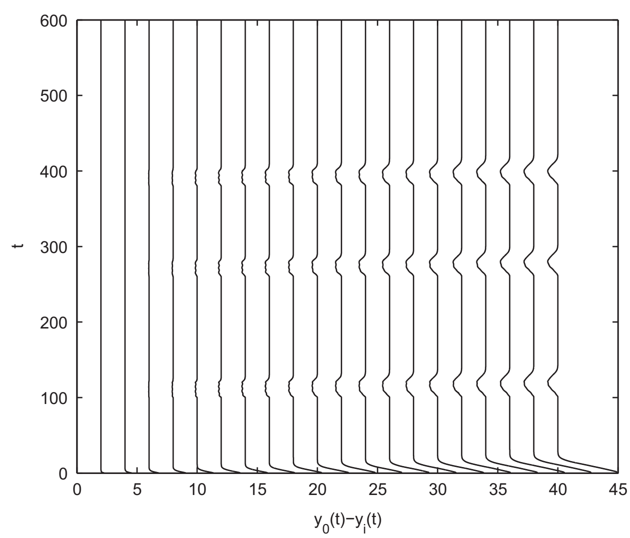

Fig. 3. The space–time plot of the running traffic model under the robust control.

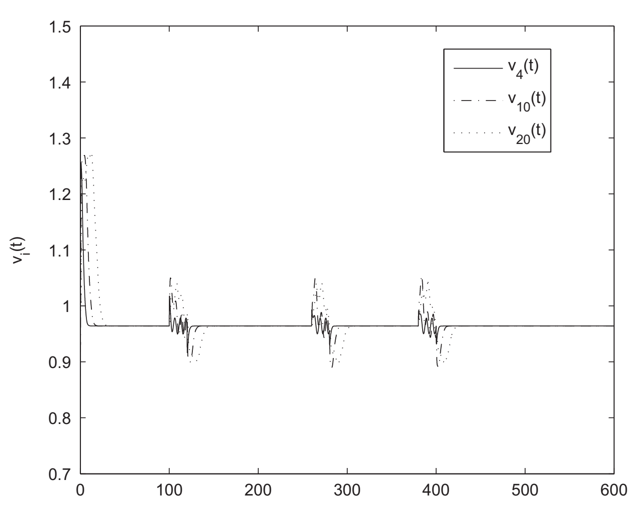

Fig. 4. The temporal velocity behavior of three vehicles.

5. Conclusion

In this paper, the stability problem of a general nonlinear car-following model is investigated by considering the varying reaction time delay of drivers. Based on the Lyapunov functions theory and the linear matrix inequalities technique, the sufficient condition for the existence of the state feedback control strategy for the stability of the car-following model is given to suppress the traffic jam with respect to the reaction-time delay. Moreover, the robust control strategy is designed to ensure robust stability and a prescribed H_∞ disturbance attenuation level for the traffic flow with external disturbances. The proposed theoretical analysis results provide an effective way to deal with the delay-dependent stability criteria of the nonlinear car-following model. Numerical examples are given to illustrate the effectiveness of the theoretical analysis results. Moreover, the delay-dependent stability criteria for the multi-lane traffic flow by considering overtaking will be investigated in our future work.

Acknowledgments

The authors thank the anonymous referees for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China (Nos. 71401007, 71422002, 71271020), the Beijing Natural Science Foundation (No. 9144031), and the Research Foundation of the State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University (No. RCS2014ZT21).

References

[1] Konishi K, Kokame H, Hirata K. Coupled map car-following model and its delayed-feedback control. Phys Rev E 1999;60(4):4000.
[2] Nagel K, Wolf DE, Wagner P, Simon P. Two-lane traffic rules for cellular automata: a systematic approach. Phys Rev E 1998;58(2):1425.
[3] Bando M, Hasebe K, Nakayama A, Shibata A, Sugiyama Y. Dynamical model of traffic congestion and numerical simulation. Phys Rev E 1995;51:1035–42.
[4] Helbing D. Traffic and related self-driven many-particle systems. Rev Mod Phys 2001;73(4):1067.
[5] Zhao X, Gao Z. The stability analysis of the full velocity and acceleration velocity model. Physica A 2007;375(2):679–86.
[6] Helbing D, Tilch B. Generalized force model of traffic dynamics. Phys Rev E 1998;58(1):133.
[7] Konishi K, Kokame H, Hirata K. Decentralized delayed-feedback control of an optimal velocity traffic model. Eur Phys J B 2000;15(4):715–22.
[8] Jiang R, Wu Q, Zhu Z. Full velocity difference model for a car-following theory. Phys Rev E 2001;64(1):017101.
[9] Zhao X, Gao Z. A new car-following model: full velocity and acceleration difference model. Eur Phys J B 2005;47(1):145–50.
[10] Ge HX, Cheng RJ, Li ZP. Two velocity difference model for a car following theory. Physica A 2008;387:5239–45.
[11] Tang TQ, Li CY, Wu YH, Huang HJ. Impact of the honk effect on the stability of traffic flow. Physica A 2011;390:3362–8.
[12] Ge HX. Modified coupled map car-following model and its delayed feedback control scheme. Chin Phys B 2011;20(9):090502.
[13] Tang TQ, Wang YP, Yang XB, Wu YH. A new car-following model accounting for varying road condition. Nonlinear Dyn 2012;70(2):1397–405.
[14] Peng GH, Cheng RJ. A new car-following model with the consideration of anticipation optimal velocity. Physica A 2013;392:3563–9.
[15] Herman R, Montroll EW, Potts RB, Rothery RW. Traffic dynamics: analysis of stability in car following. Oper Res 1959;7(1):86–106.
[16] Bando M, Hasebe K, Nakanishi K, Nakayama A. Analysis of optimal velocity model with explicit delay. Phys Rev E 1998;57:5429–35.
[17] Davis L. Modifications of the optimal velocity traffic model to include delay due to driver reaction time. Physica A 2003;319:557–67.
[18] Orosz G, Krauskopf B, Wilson RE. Bifurcations and multiple traffic jams in a car-following model with reaction-time delay. Physica D 2005;211(3):277–93.
[19] Treiber M, Kesting A, Helbing D. Delays, inaccuracies and anticipation in microscopic traffic models. Physica A 2006;360(1):71–88.
[20] Ge HX, Meng XP, Cheng RJ, Lo SM. Time-dependent Ginzburg–Landau equation in a car-following model considering the drivers physical delay. Physica A 2011;390:3348–53.
[21] Padhan DG, Majhi S. Modified Smith predictor based cascade control of unstable time delay processes. ISA Trans 2012;51(1):95–104.
[22] Liu P. New results on stability analysis for time-varying delay systems with non-linear perturbations. ISA Trans 2013;52(3):318–25.
[23] Xiang M, Xiang ZR. Robust fault detection for switched positive linear systems with time-varying delays. ISA Trans 2014;53(1):10–6.

[24] Ge SS, Wang J. Robust adaptive tracking for time-varying uncertain nonlinear systems with unknown control coefficients. IEEE Trans Autom Control 2003;48(8):1463–9.

[25] Tong SC, He XL, Zhang HG. A combined backstepping and small-gain approach to robust adaptive fuzzy output feedback control. IEEE Trans Fuzzy Syst 2009;17(5):1059–69.

[26] Li TS, Tong SC, Feng G. A novel robust adaptive-fuzzy-tracking control for a class of nonlinear multi-input/multi-output systems. IEEE Trans Fuzzy Syst 2010;18(1):150–60.

[27] Liu YJ, Wang W, Tong SC, Liu YS. Robust adaptive tracking control for nonlinear systems based on bounds of fuzzy approximation parameters. IEEE Trans Syst Man Cybern A 2010;40(1):170–84.

[28] Tong SC, Li YM. Robust adaptive fuzzy backstepping output feedback tracking control for nonlinear system with dynamic uncertainties. Sci China Technol Sci 2010;53(2):307–24.

[29] Tong SC, Li YM. Fuzzy adaptive robust backstepping stabilization for SISO nonlinear systems with unknown virtual control direction. Inf Sci 2010;180(23):4619–40.

[30] Li SK, Zhang JX, Tang WS. Robust H_∞ output feedback control for uncertain complex delayed dynamical networks. Comput Math Appl 2011;62(1):497–505.

[31] Ahmadi M, Mojallali H, Wisniewski R. Robust H_∞ control of uncertain switched systems defined on polyhedral sets with Filippov solutions. ISA Trans 2012;51(6):722–31.

[32] Jiang S, Fang H. H_∞ static output feedback control for nonlinear networked control systems with time delays and packet dropouts. ISA Trans 2013;52(2):215–22.

[33] Li SK, Yang LX, Li KP, Gao ZY. Robust sampled-data cruise control scheduling of high speed train. Transp Res Part C 2014;46:274–83.

[34] Hoogendoorn SP, Ossen S. Parameter estimation and analysis of car-following models, In: Flow, dynamics and human interaction (Transportation and Traffic Theory). The 16th international symposium on transportation and traffic theory, 2005.

[35] Gu K, Kharitonov V, Chen J. Stability of time-delay systems. Boston: Birkhauser; 2003.

[36] Boyd S, Elghaoui L, Feron E, Balakrishnan V. Linear matrix inequalities in system and control theory. Philadelphia: SIAM; 1994.