Lab Session 12

MA581 : Numerical Computations Lab July-November 2022 S. Bora

1. The purpose of this exercise is to demonstrate that sometimes it may be better to solve an LSP of reduced rank when making predictions from given data.

   Create the data for an overdetermined system $Az = b$ with the following commands:
   $>>$ $\mathtt{A = randi(9,7,4); A(:,4) = A(:,1) + A(:,3) + 10^{(-10)} * randi(9,7,1);}$
   $>>$ $\mathtt{x = randi(9,4,1); d = randi(9,7,1); b = A * x + 10^{(-4)} * d/(norm(d,1));}$
   The matrix A has numerical rank 4 but it is close to a rank deficient matrix. Also the residual vector from any least squares solution is expected to be small as b is almost in the column space of A.

   Consider a row vector of additional input data:
   $>>$ $\mathtt{c = A(5,:) + 10^{(-3)} * ones(1,4);}$
   that is very close to the first four entries in the fifth row of A. Therefore, given a solution $\hat{x}$ of the LSP, the predicted output $c * \hat{x}$ from these data points is expected to be close to the fifth entry of b.

   (a) Compute the relative norm of the residual $\|A * x_i - b\|_2/\|b\|_2$, and the predicted output $c * x_i$ via QR decomposition method ($\mathtt{x_1 = A\backslash b}$) and the SVD method ($\mathtt{x_2 = pinv(A) * b}$). Is the relative residual norm small in each case? Is the predicted value close to $b(5)$?

   (b) Get the column pivoted QR decomposition of A via $\mathtt{[Q,R,P] = qr(A)}$ and check $R(3,3)$ and $R(4,4)$. What do you observe?
   Set $R(4,4)$ to zero and call the resulting matrix as $\widehat{R}$. Solve the reduced rank LSP $\widehat{A}x = b$ via the QR decomposition method where $\widehat{A} = Q\widehat{R}P^T$. Set the solution to be x_3.

   (c) Check the singular values σ_3 and σ_4 of A. What do you observe? Solve the reduced rank LSP $\tilde{A}x = b$ via SVD method where $\tilde{A} = U(:,1:3) * \Sigma(1:3,1:3) * V(:,1:3)'$ is the best rank 3 approximation of A. Set the solution to be x_4.

   (d) Check the four solutions $[x_1\ x_2\ x_3\ x_4]$. What do you observe? Then check the relative residual norm $\|A * x_i - b\|_2/\|b\|_2$ and quality of the predicted output $c * x_i$ for the solutions x_3 and x_4 obtained from the reduced rank models. What do you observe? Is there any improvement relative to the solutions obtained from the original LSP?

2. The aim of the experiment in Exercise 4.2.21 of *Fundamentals of Matrix Computations* is to show that the Rank Revealing QR Decomposition is less efficient than the SVD method when detecting numerical rank deficiency. You will find it on page 273 of the second edition and pages 272 of third edition. Write a small description of your experiment.

3. This is a demonstration of image compression techniques using SVD. The following commands will first load a built-in 320×200 matrix X that represents the pixel image of a clown, computes its SVD $X = U\Sigma V^T$ and then displays the image when X is approximated by its best rank k approximation $X_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ for a chosen value of k.

```
load clown.mat; [U, S, V] = svd(X); colormap('gray');
image(U(:, 1:k)*S(1:k, 1:k)*V(:,1:k)')
```

The storage required for A_k is $k(m+n) = 520k$ words whereas the storage required for the full image is $n \times m = 6400$ words in this case. Therefore, $\frac{520k}{6400}$ gives the compression ratio for the compressed image. Also the relative error in the representation is $\frac{\sigma_{k+1}}{\sigma_1}$. Run the above commands for various choices of k and make a table that records the relative errors and compression ratios for each choice.