12.9.4.15

EE24BTECH11041 - Mohit

1) Find the equation of a curve passing through the point $(0, 0)$ and whose differential equation is $y^{'} = e^x \sin x$
Theoritical Solution:-

The given differential equation is:

$$y^{'} = e^x \sin x \tag{1.1}$$

Integrating both sides:

$$y = \int e^x \sin x dx \tag{1.2}$$

Using integration by parts:

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx \tag{1.3}$$

Let I = $\int e^x \sin x dx$. Then:

$$I = e^x \sin x - \int e^x \cos x dx \tag{1.4}$$

Now, solve $\int e^x \cos x dx$ using integration by parts again:

$$\int e^x \cos x dx = e^x \cos x - \int e^x(-\sin x)dx = e^x \cos x + \int e^x \sin x dx \tag{1.5}$$

Substituting back, we get:

$$I = e^x \sin x - (e^x \cos x + I) \tag{1.6}$$

$$I + I = e^x(\sin x - \cos x) \tag{1.7}$$

$$2I = e^x(\sin x - \cos x) \tag{1.8}$$

$$I = \frac{e^x(\sin x - \cos x)}{2} \tag{1.9}$$

Thus, the solution to the differential equation is:

$$y = \frac{e^x(\sin x - \cos x)}{2} + C \tag{1.10}$$

Using the initial condition $y(0) = 0$:

$$y(0) = \frac{e^0(\sin 0 - \cos 0)}{2} + C \tag{1.11}$$

$$0 = \frac{(0-1)}{2} + C \tag{1.12}$$

$$C = \frac{1}{2} \tag{1.13}$$

The final solution is:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2} \tag{1.14}$$

Laplace-Method

$$y' = e^x \sin x \tag{1.15}$$

Taking Laplace on both sides

$$\mathcal{L}\left(y'\right) = \mathcal{L}\left(e^x \sin x\right) \tag{1.16}$$

Let,

$$\mathcal{L}\left(y'\right) = sY(s) \text{ and } \mathcal{L}\left(e^x \sin x\right) = G(s) \tag{1.17}$$

Subsituting in 1.16,

$$sY(s) = G(s) \tag{1.18}$$

$$\frac{Y(s)}{G(s)} = \frac{1}{s} \tag{1.19}$$

Applying bi-linear Transformation

$$s = \frac{2}{h}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) \tag{1.20}$$

$$\tag{1.21}$$

Subsituting in 1.19,

$$\frac{Y(s)}{G(s)} = \frac{h}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) \tag{1.22}$$

$$\left(1-z^{-1}\right)Y(s) = \frac{h}{2}\left(1+z^{-1}\right)G(s) \tag{1.23}$$

$$Y(s) - z^{-1}Y(s) = \frac{h}{2}\left(G(s) - z^{-1}Gs\right) \tag{1.24}$$

Then putting the value of $Y(s)$ and $G(s)$
Final expression :-

$$y_n - y_{n-1} = \frac{h}{2}\left(e_n^x \sin x_n + e_n^{x-1} \sin x_{n-1}\right) \tag{1.25}$$

Trapezoid Rule :-

The trapezoidal rule is as follows.

$$A = \int_a^b f(x)\,dx \approx h\left(\frac{1}{2}f(a) + f(x_1) + f(x_2)\cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \quad (1.26)$$

$$h = \frac{b-a}{n} \quad (1.27)$$

$$A = j_n, \text{ where, } j_{i+1} = j_i + h\frac{f(x_{i+1}) + f(x_i)}{2} \quad (1.28)$$

$$\rightarrow j_{i+1} = j_i + h\left(\sqrt{x_{i+1}} + \sqrt{x_i}\right) \quad (1.29)$$

$$x_{i+1} = x_i + h \quad (1.30)$$

$$(1.31)$$