

11.16.3.7

EE24BTECH11041 - Mohit

Question:- A fair coin is tossed four times, and a person wins Rs.1 for each head and loses Rs.1.50 for each tail. From the sample space, calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution

Let:

$$H = +1 \quad (\text{gain Rs.1 for Head}), \quad T = -1.50 \quad (\text{lose Rs.1.50 for Tail}). \quad (0.1)$$

For x , the number of heads in 4 tosses, the total net money can be calculated using the formula:

$$\text{Net Money} = x(1) + (4 - x)(-1.5) \quad (0.2)$$

$$\text{Net Money} = x - 1.5(4 - x) \quad (0.3)$$

$$\text{Net Money} = 2.5x - 6, \quad (0.4)$$

where $x = 0, 1, 2, 3, 4$.

Possible Outcomes and Net Money

- $x = 0$: All tails ($TTTT$):

$$\text{Net Money} = 2.5(0) - 6 = -6 \quad (0.5)$$

- $x = 1$: One head, three tails ($H T T T, T H T T, T T H T, T T T H$, etc.):

$$\text{Net Money} = 2.5(1) - 6 = -3.5 \quad (0.6)$$

- $x = 2$: Two heads, two tails ($H H T T, H T H T, H T T H, \dots$):

$$\text{Net Money} = 2.5(2) - 6 = -1 \quad (0.7)$$

- $x = 3$: Three heads, one tail ($H H H T, H H T H, H T H H, T H H H$):

$$\text{Net Money} = 2.5(3) - 6 = 1.5 \quad (0.8)$$

- $x = 4$: All heads ($H H H H$):

$$\text{Net Money} = 2.5(4) - 6 = 4 \quad (0.9)$$

Number of Outcomes for Each Case

The number of outcomes for each x is given by the binomial coefficient $\binom{4}{x}$:

$$x = 0 : \binom{4}{0} = 1, \quad (0.10)$$

$$x = 1 : \binom{4}{1} = 4, \quad (0.11)$$

$$x = 2 : \binom{4}{2} = 6, \quad (0.12)$$

$$x = 3 : \binom{4}{3} = 4, \quad (0.13)$$

$$x = 4 : \binom{4}{4} = 1 \quad (0.14)$$

Probabilities of Each Case

Since the coin is fair, the probability of each outcome is $\frac{1}{16}$. The probabilities for each x are:

$$x = 0 : \text{Probability} = \frac{\binom{4}{0}}{16} = \frac{1}{16}, \quad (0.15)$$

$$x = 1 : \text{Probability} = \frac{\binom{4}{1}}{16} = \frac{4}{16} = \frac{1}{4}, \quad (0.16)$$

$$x = 2 : \text{Probability} = \frac{\binom{4}{2}}{16} = \frac{6}{16} = \frac{3}{8}, \quad (0.17)$$

$$x = 3 : \text{Probability} = \frac{\binom{4}{3}}{16} = \frac{4}{16} = \frac{1}{4}, \quad (0.18)$$

$$x = 4 : \text{Probability} = \frac{\binom{4}{4}}{16} = \frac{1}{16}. \quad (0.19)$$

CODING LOGIC:-

A fair coin is tossed 4 times, and the outcomes are:

- Heads (H): Gain Rs.1.
- Tails (T): Lose Rs.1.50.

Define X_i as the random variable for the net money from the i -th toss:

$$X_i = \begin{cases} 1, & \text{if Head (H)} \quad (P(X_i = 1) = 0.5) \\ -1.5, & \text{if Tail (T)} \quad (P(X_i = -1.5) = 0.5) \end{cases} \quad (0.20)$$

The total net money after 4 tosses is:

$$S = \sum_{i=1}^4 X_i \quad (0.21)$$

Z-Transform of a Single Toss

The Z-transform of a random variable is defined as:

$$G_X(z) = \mathbb{E}[z^X] \quad (0.22)$$

where $\mathbb{E}[\cdot]$ is the expectation operator. For X_i , the Z-transform is:

$$G_{X_i}(z) = P(X_i = 1)z^1 + P(X_i = -1.5)z^{-1.5} \quad (0.23)$$

$$G_{X_i}(z) = 0.5z + 0.5z^{-1.5} \quad (0.24)$$

Z-Transform of Total Net Money

Since the tosses are independent, the Z-transform of the total net money S is the product of the Z-transforms of individual tosses:

$$G_S(z) = (G_{X_i}(z))^4 \quad (0.25)$$

$$G_S(z) = (0.5z + 0.5z^{-1.5})^4 \quad (0.26)$$

Expansion of the Z-Transform

Using the binomial theorem, expand $G_S(z)$:

$$G_S(z) = \sum_{k=0}^4 \binom{4}{k} (0.5z)^k (0.5z^{-1.5})^{4-k} \quad (0.27)$$

$$G_S(z) = \sum_{k=0}^4 \binom{4}{k} (0.5)^4 z^{k-1.5(4-k)} \quad (0.28)$$

$$G_S(z) = \sum_{k=0}^4 \binom{4}{k} (0.5)^4 z^{2.5k-6} \quad (0.29)$$

The coefficients of z^m in this expansion represent the probabilities $P(S = m)$, i.e., the PMF.

Net Money Values and PMF

The possible net money values are:

$$S \in \{-6, -3.5, -1, 1.5, 4\} \quad (0.30)$$

Substituting these values into the expansion of $G_S(z)$, the probabilities are:

$$P(S = -6) = \frac{1}{16}, \quad P(S = -3.5) = \frac{4}{16}, \quad P(S = -1) = \frac{6}{16}, \quad P(S = 1.5) = \frac{4}{16}, \quad P(S = 4) = \frac{1}{16} \quad (0.31)$$

Thus, the PMF is:

$$P(S = x) = \begin{cases} \frac{1}{16}, & x = -6 \\ \frac{4}{16}, & x = -3.5 \\ \frac{6}{16}, & x = -1 \\ \frac{4}{16}, & x = 1.5 \\ \frac{1}{16}, & x = 4 \\ 0, & \text{otherwise} \end{cases} \quad (0.32)$$

CDF of S

The Cumulative Distribution Function (CDF) is defined as:

$$F(x) = P(S \leq x) = \sum_{k \leq x} P(S = k) \quad (0.33)$$

The possible values of S and the corresponding CDF values are:

- For $x < -6$:

$$F(x) = 0 \quad (0.34)$$

- For $-6 \leq x < -3.5$:

$$F(x) = P(S = -6) = \frac{1}{16} \quad (0.35)$$

- For $-3.5 \leq x < -1$:

$$F(x) = P(S = -6) + P(S = -3.5) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16} \quad (0.36)$$

- For $-1 \leq x < 1.5$:

$$F(x) = P(S = -6) + P(S = -3.5) + P(S = -1) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16} \quad (0.37)$$

- For $1.5 \leq x < 4$:

$$F(x) = P(S = -6) + P(S = -3.5) + P(S = -1) + P(S = 1.5) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16} \quad (0.38)$$

- For $x \geq 4$:

$$F(x) = 1 \quad (0.39)$$

Thus, the CDF is:

$$F(x) = \begin{cases} 0, & x < -6 \\ \frac{1}{16}, & -6 \leq x < -3.5 \\ \frac{5}{16}, & -3.5 \leq x < -1 \\ \frac{11}{16}, & -1 \leq x < 1.5 \\ \frac{15}{16}, & 1.5 \leq x < 4 \\ 1, & x \geq 4 \end{cases} \quad (0.40)$$

