A Logo for Narayanpal

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Abstract

This paper determines the parameter pairs (a, b) for the function $f(t) = e^{-at}u(t) + e^{bt}u(-t)$, subject to normalization, such that these pairs correspond to the endpoints of the latus recta of an associated conic. The work derives the conic equation binding the parameters, applies eigen-decomposition, and uses affine transformations to identify the valid (a, b) values. Additionally, a numerical gradient descent method is proposed to determine symmetric truncation points $\theta = (\theta_0, \theta_1)$ such that the areas under $f(t)$ on either side of $t = 0$ are equal. This approach allows for a flexible choice of the truncation window.

Question 1

Given that

$$f(t) = e^{-at}u(t) + e^{bt}u(-t) \tag{3.1}$$

$$u(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases} \tag{3.2}$$

$$\int_{-\infty}^{\infty} f(t) = 1 \tag{3.3}$$

Find the possible values of (a, b) if these are the end points of the latus recta of the associated conic. Plot $f(t)$ for these values of (a, b).

Normalization: compute the integral

Split the integral:

$$\int_{-\infty}^{\infty} f(t) = \int_{-\infty}^{0} f(t) + \int_{0}^{\infty} f(t) \tag{4.1}$$

$$= \int_{-\infty}^{0} e^{bt} + \int_{0}^{\infty} e^{-at} \tag{4.2}$$

$$= \frac{1}{b} + \frac{1}{a} \tag{4.3}$$

Normalization $\Rightarrow$

$$\frac{1}{a} + \frac{1}{b} = 1 \tag{4.4}$$

$$ab - a - b = 0 \tag{4.5}$$

Interpret this as a conic in (a, b) (set $x = a,\ y = b$).

Conic as a quadratic form

Write the conic in matrix form

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$

By Comparsion,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{4.6}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{4.7}$$

$$f = 0 \tag{4.8}$$

Eigen-decompose $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top$ where

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\mathsf{T} \tag{4.9}$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.10}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{4.11}$$

Convert the conic into a standard conic using affine transformations.

$$\mathbf{y}^{\mathsf{T}} \left(\frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \tag{4.12}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{4.13}$$

Where

$$f_0 = \mathbf{u}^{\mathsf{T}} \mathbf{V}^{-1} \mathbf{u} - f = 1 \tag{4.14}$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.15}$$

The eigenvalues of $\mathbf{D}$ are $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. Using a reflection matrix and further transformation, we get the hyperbola in standard form:

$$\mathbf{z}^\mathsf{T} \left(\frac{\mathbf{D_0}}{f_0} \right) \mathbf{z} = 1 \tag{4.16}$$

$$j(\mathbf{z}) = \mathbf{z}^\mathsf{T} \mathbf{D_0} \mathbf{z} - f_0 = 0 \tag{4.17}$$

$$\mathbf{y} = \mathbf{P}_0 \mathbf{z} \tag{4.18}$$

Here $\mathbf{P}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{D}_0 = \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Now, solve for the endpoints of the latus recta:

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.19}$$

$$e = \sqrt{2} \tag{4.20}$$

$$c = \pm\frac{1}{\sqrt{2}} \tag{4.21}$$

$$\mathbf{F} = \pm 2\mathbf{e}_2 \tag{4.22}$$

Equation of latus recta:

$$\mathbf{n}^{\top}\mathbf{x} = \mathbf{n}^{\top}\mathbf{F} \tag{4.23}$$

$$\equiv \mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{4.24}$$

$$\tag{4.25}$$

Where,

$$\mathbf{h} = \begin{pmatrix} 0 \\ \pm 2 \end{pmatrix} \tag{4.26}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.27}$$

Let $\hat{\mathbf{z}}$ be the endpoints of the latus recta:

$$k = \pm\sqrt{2} \tag{4.28}$$

$$\therefore \hat{\mathbf{z}} = \begin{pmatrix} \pm\sqrt{2} \\ \pm 2 \end{pmatrix} \tag{4.29}$$

Transforming back to the original conic:

$$\hat{\mathbf{x}} = \mathbf{P}\left(\mathbf{P}_0 \hat{\mathbf{z}}\right) + \mathbf{c} \tag{4.30}$$

Endpoints of latus recta

Which gives:

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2+\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$
$$\hat{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{2} \\ 2+\sqrt{2} \end{pmatrix}$$
$$\hat{\mathbf{x}}_3 = \begin{pmatrix} 2-\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$
$$\hat{\mathbf{x}}_4 = \begin{pmatrix} -\sqrt{2} \\ 2-\sqrt{2} \end{pmatrix}$$

Only $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are valid as negative a or b will not yield a finite $f(t)$.

Conic Plot

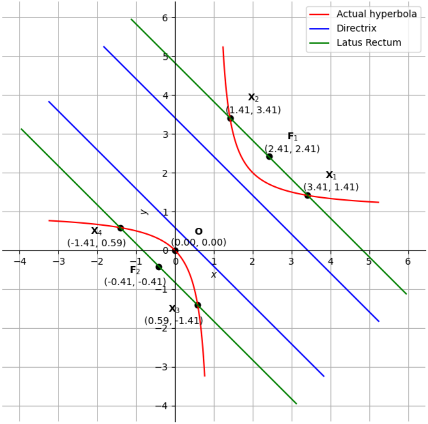

Figure: Conic Section

Function $f(t)$ for valid (a, b)

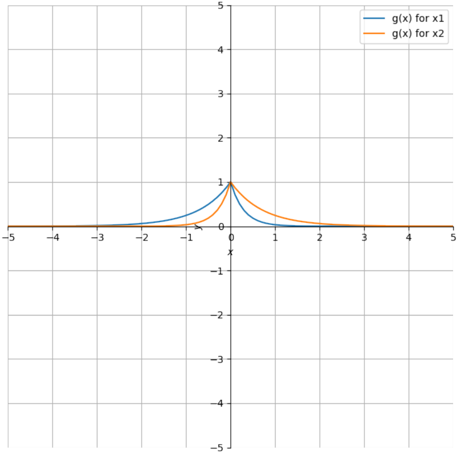

Figure: Function $f(t)$ for valid (a, b)

Numerical Area Balancing and Gradient Descent

We need to find the endpoints (θ_0, θ_1) and truncate the function f(t), such that the the area under $f(t)$ is symmetric about $t = 0$. For this, we will use gradient descent so that we get the closest solution to our guess. Define:

$$A_L(\theta_0) = \int_{\theta_0}^{0} e^{bt}\, dt = \frac{1 - e^{b\theta_0}}{b} \tag{6.1}$$

$$A_R(\theta_1) = \int_{0}^{\theta_1} e^{-at}\, dt = \frac{1 - e^{-a\theta_1}}{a} \tag{6.2}$$

We seek $\theta = (\theta_0, \theta_1)$ such that $A_L(\theta_0) = A_R(\theta_1)$. Rearranging:

$$\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a} = 0 \tag{6.3}$$

This nonlinear equation cannot be solved analytically in closed form. Thus, we define a cost function:

$$\mathcal{C}(\theta) = \left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a} \right)^2 \tag{6.4}$$

We minimize this cost using gradient descent:

$$\frac{d\mathcal{C}}{d\theta_0} = -\left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a} \right) e^{b\theta_0} \tag{6.5}$$

$$\frac{d\mathcal{C}}{d\theta_1} = -\left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a} \right) e^{-a\theta_1} \tag{6.6}$$

The update rule becomes:

$$\theta_{n+1} \leftarrow \theta_n - \eta \cdot \nabla \mathcal{C}(\theta_n) \tag{6.7}$$

where η is a learning rate.

Numerical Result

We initialize θ with a reasonable estimate and iterate until $|\mathcal{C}(\theta)| < 10^{-10}$. The resulting θ yields a numerically balanced integral under $f(t)$ on both sides of the origin.

$$\theta_{(0)} = [-2, 3]^{\mathsf{T}} \to \theta_n = [-0.37815999, 3.00038288]^{\mathsf{T}} \tag{6.8}$$

Plot for Truncated function

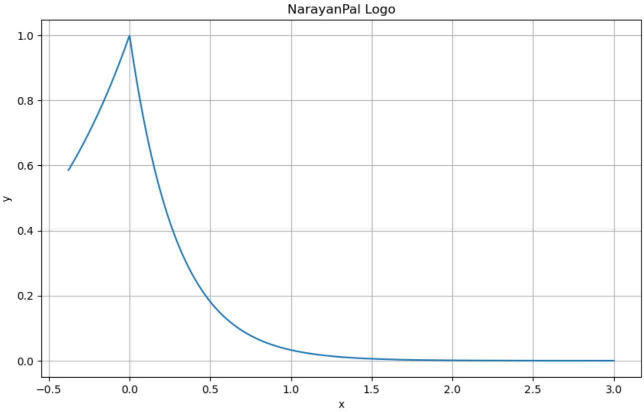

Figure: Truncated Function $f(t)$