A Logo for Narayanpal

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***Abstract*—This paper determines the parameter pairs (a, b) for the function $f(t) = e^{-at}u(t) + e^{bt}u(-t)$, subject to normalization, such that these pairs correspond to the endpoints of the latus recta of an associated conic. The work derives the conic equation binding the parameters, applies eigen-decomposition, and uses affine transformations to identify the valid (a, b) values. Additionally, a numerical gradient descent method is proposed to determine symmetric truncation points $\theta = (\theta_0, \theta_1)$ such that the areas under $f(t)$ on either side of $t = 0$ are equal. This approach allows for a flexible choice of the truncation window.**

1. Question

Given that

$$f(t) = e^{-at}u(t) + e^{bt}u(-t) \tag{1}$$

$$u(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases} \tag{2}$$

$$\int_{-\infty}^{\infty} f(t) = 1 \tag{3}$$

Find the possible values of (a, b) if these are the end points of the latus recta of the associated conic. Plot $f(t)$ for these values of (a, b).

2. Solution

We expand the integral as

$$\int_{-\infty}^{\infty} f(t) = \int_{-\infty}^{0} f(t) + \int_{0}^{\infty} f(t) \tag{4}$$

$$= \int_{-\infty}^{0} e^{bt} + \int_{0}^{\infty} e^{-at} \tag{5}$$

$$= \frac{1}{b} + \frac{1}{a} \tag{6}$$

Substituting (**??**) in (**??**):

$$\frac{1}{a} + \frac{1}{b} = 1 \tag{7}$$

$$ab - a - b = 0 \tag{8}$$

This is the equation of a conic. If we take a as x and b as y and express this as a conic in standard form, we get

$$g(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathrm{T}}\mathbf{x} + f \tag{9}$$

By comparison:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{10}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{11}$$

$$f = 0 \tag{12}$$

We eigen-decompose $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}} \tag{13}$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{14}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{15}$$

Convert the conic into a standard conic using affine transformations.

$$\mathbf{y}^{\mathrm{T}}\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \tag{16}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{17}$$

Where

$$f_0 = \mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{18}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

The eigenvalues of $\mathbf{D}$ are $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. Using a reflection matrix and further transformation, we get the hyperbola in standard form:

$$\mathbf{z}^{\mathrm{T}}\left(\frac{\mathbf{D_0}}{f_0}\right)\mathbf{z} = 1 \tag{20}$$

$$j(\mathbf{z}) = \mathbf{z}^{\mathrm{T}}\mathbf{D_0}\mathbf{z} - f_0 = 0 \tag{21}$$

$$\mathbf{y} = \mathbf{P}_0\mathbf{z} \tag{22}$$

Here $\mathbf{P}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{D}_0 = \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Now, solve for the endpoints of the latus recta:

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{23}$$

$$e = \sqrt{2} \tag{24}$$

$$c = \pm\frac{1}{\sqrt{2}} \tag{25}$$

$$\mathbf{F} = \pm 2\mathbf{e}_2 \tag{26}$$

Equation of latus recta:

$$\mathbf{n}^{\mathrm{T}}\mathbf{x} = \mathbf{n}^{\mathrm{T}}\mathbf{F} \tag{27}$$

$$\equiv \mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{28}$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ \pm 2 \end{pmatrix} \tag{29}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{30}$$

Let $\hat{\mathbf{z}}$ be the endpoints of the latus recta:

$$k = \pm\sqrt{2} \tag{31}$$

$$\therefore \hat{\mathbf{z}} = \begin{pmatrix} \pm\sqrt{2} \\ \pm 2 \end{pmatrix} \tag{32}$$

Transforming back to the original conic:

$$\hat{\mathbf{x}} = \mathbf{P}\left(\mathbf{P}_0\hat{\mathbf{z}}\right) + \mathbf{c} \tag{33}$$

Which gives:

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2+\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{2} \\ 2+\sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_3 = \begin{pmatrix} 2-\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_4 = \begin{pmatrix} -\sqrt{2} \\ 2-\sqrt{2} \end{pmatrix}$$

Only $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are valid as negative a or b will not yield a finite $f(t)$.

3. Plots

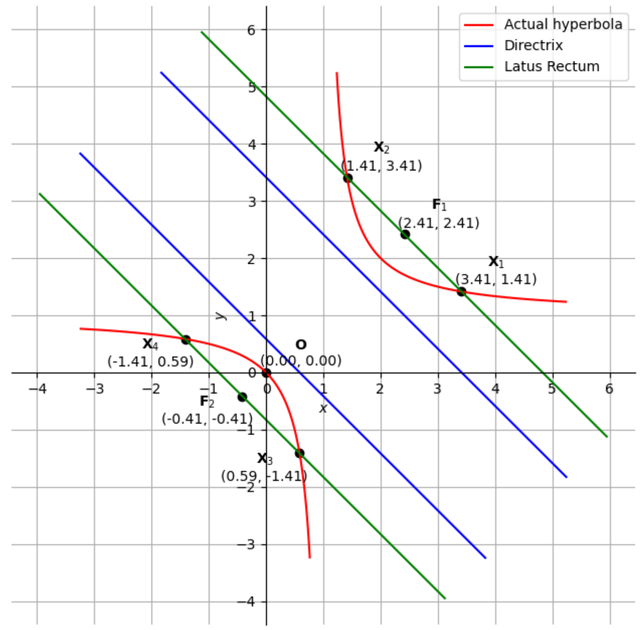

Fig. 1: Conic Section

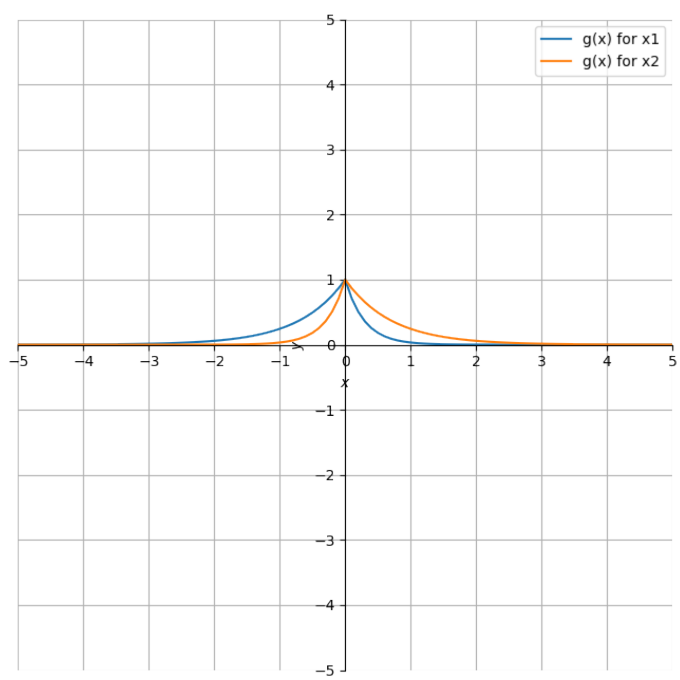

Fig. 2: Function $f(t)$ for valid (a, b)

4. Numerical Area Balancing and Gradient Descent

We need to find the endpoints (θ_0, θ_1) and truncate the function f(t), such that the the area under $f(t)$ is symmetric about $t = 0$. For this, we will use gradient descent so that we get the closest solution to our guess.

Define:

$$A_L(\theta_0) = \int_{\theta_0}^{0} e^{bt}\,dt = \frac{1 - e^{b\theta_0}}{b} \tag{34}$$

$$A_R(\theta_1) = \int_{0}^{\theta_1} e^{-at}\,dt = \frac{1 - e^{-a\theta_1}}{a} \tag{35}$$

We seek $\theta = (\theta_0, \theta_1)$ such that $A_L(\theta_0) = A_R(\theta_1)$. Rearranging:

$$\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a} = 0 \tag{36}$$

This nonlinear equation cannot be solved analytically in closed form. Thus, we define a cost function:

$$C(\theta) = \left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a}\right)^2 \tag{37}$$

We minimize this cost using gradient descent:

$$\frac{dC}{d\theta_0} = -\left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a}\right)e^{b\theta_0} \tag{38}$$

$$\frac{dC}{d\theta_1} = -\left(\frac{1 - e^{b\theta_0}}{b} - \frac{1 - e^{-a\theta_1}}{a}\right)e^{-a\theta_1} \tag{39}$$

The update rule becomes:

$$\theta_{n+1} \leftarrow \theta_n - \eta \cdot \nabla C(\theta_n) \tag{40}$$

where η is a learning rate.

We initialize θ with a reasonable estimate and iterate until $|C(\theta)| < 10^{-10}$. The resulting θ yields a numerically balanced integral under $f(t)$ on both sides of the origin.

$$\theta_{(0)} = [-2, 3]^{\mathrm{T}} \to \theta_n = [-0.37815999, 3.00038288]^{\mathrm{T}} \tag{41}$$

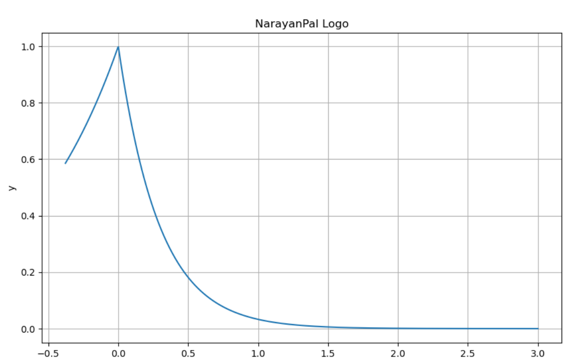

Fig. 3: Truncated Function $f(t)$