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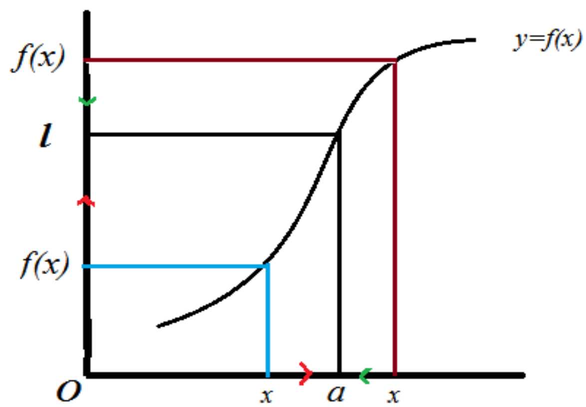
Limits, Continuity & Differentiability

Introduction: In this chapter we will study about limit that is the core tool of calculus and all other calculus concepts are based on it. A function can be undefined at a point, but we can think about what the function "approaches" as it gets closer and closer to that point (this is the "limit"). Also the function may be defined at a point, but it may approach a different limit. There are many, many times where the functional value is the same as the limit at a point. Limit is used to define continuity, derivative and integral of a function.

Limit of a function: The number " l " is called limit of a function $f(x)$ at a point $x = a$ if x approaches closer and closer to " a " from both sides and consequently $f(x)$ approaches closer and closer to " l ". Symbolically it is written as,

$$\lim_{x\to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a$$

Graphical representation of "limit of a function" at a point:



Mathematical* or $\epsilon - \delta$ *definition of limit of a function: The number " l " is called limit of a function $f(x)$ at x approaches " a " if for any given positive number ϵ $(however\ small)$, we can find another positive number $\delta(\ depending\ on\ \epsilon\ )$ such that $|f(x) - l| < \epsilon$, for all values of x satisfying $0 < |x - a| < \delta$.

Symbolically it is written as,

$$\lim_{x\to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a .$$

<u>Left Hand Limit</u>: If the values of $f(x)$ can be made as close as we like to " l " by taking values of x sufficiently close to " a " (but less than a) then we write,

$$L.H.L = \lim_{x\to a^-} f(x) = l$$

<u>Right Hand Limit</u>: If the values of $f(x)$ can be made as close as we like to " l " by taking values of x sufficiently close to " a " (but greater than a) then we write,

$$R.H.L = \lim_{x\to a^+} f(x) = l$$

<u>Existence of limit of a function $f(x)$ at $x = a$</u> :

The limit of a function $f(x)$ at $x = a$ that is $\lim_{x\to a} f(x) = l$ exists if

a) $L.H.L = \lim_{x\to a^-} f(x)\ \ exists$

b) $\boldsymbol{R.H.L} = \lim\limits_{x\to a^+} f(x)$ *also exists*

c) $\boldsymbol{L.H.L} = \boldsymbol{R.H.L} = l$.

Fundamental Properties of limit:

If $f(x)$, $g(x)$ are two functions and $\lim\limits_{x\to a} f(x)$ *and* $\lim\limits_{x\to a} g(x)$ *exists then*

a) $\lim\limits_{x\to a}\{\ f(x) \pm g(x)\ \} = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

b) $\lim\limits_{x\to a}\{\ f(x) \times g(x)\ \} = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$

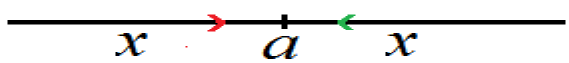
c) $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *where* $g(a) \neq 0$

d) $\lim\limits_{x\to a}\{\, constant\ \times f(x)\} = constant\ \times \lim\limits_{x\to a} f(x)$

e) $\lim\limits_{x\to a}\{\, f(x)\}^n = \{\lim\limits_{x\to a} f(x)\}^n$ *where* $n \in Z$

f) $\lim\limits_{x\to a}(constant) = Constant$

***Change of limit of a variable*:**



Left hand limit: $L.H.L = \lim\limits_{x\to a^-} f(x)$

Let $x + h = a$ and when $h \to 0$ *then* $x \to a$.

Now, $= \lim\limits_{x\to a^-} f(x) = \boldsymbol{\lim\limits_{h\to 0} f(a - h)}$, *putting value of* x .

Right hand limit: $R.H.L = \lim\limits_{x\to a^+} f(x)$

Let $x - h = a$ and when $h \to 0$ *then* $x \to a$.

Now, $= \lim\limits_{x\to a^+} f(x) = \boldsymbol{\lim\limits_{h\to 0} f(a + h)}$, *putting value of* x .

Problem-01: A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 & when \quad x < 1 \\ 2.4 & when \quad x = 1 \\ x^2 + 1 & when \quad x > 1 \end{cases}$$

Does $\lim\limits_{x\to 1} f(x)$ exist ?

***Solution*:** Given that, $f(x) = \begin{cases} x^2 & when \quad x < 1 \\ 2.4 & when \quad x = 1 \\ x^2 + 1 & when \quad x > 1 \end{cases}$

$$L.H.L = \lim_{h \to 0} f\left(1-h\right)$$
$$= \lim_{h \to 0} \left(1-h\right)^2$$
$$= \lim_{h \to 0} \left(1+2h+h^2\right)$$
$$=1$$

$$R.H.L = \lim_{h \to 0} f\left(1+h\right)$$
$$= \lim_{h \to 0} \left\{\left(1+h\right)^2+1\right\}$$
$$= \lim_{h \to 0} \left(1+2h+h^2+1\right)$$
$$=2$$

Since $L.H.L \neq R.H.L$. So $\lim_{x \to 1} f\left(x\right)$ does not exist.

<u>Problem-02</u>: A function $f\left(x\right)$ is defined as follows:

$$f\left(x\right) = \begin{cases} x^2+1 & when \quad x>0 \\ 1 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$$

Find the value of $\lim_{x \to 0} f\left(x\right)$.

***<u>Solution</u>*:** Given that, $f\left(x\right) = \begin{cases} x^2+1 & when \quad x>0 \\ 1 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$

$$L.H.L = \lim_{h \to 0} f\left(0-h\right)$$
$$= \lim_{h \to 0} \left(0-h+1\right)$$
$$=1$$

$$L.H.L = \lim_{h \to 0} f\left(0+h\right)$$
$$= \lim_{h \to 0} \left\{\left(0+h\right)^2+1\right\}$$
$$= \lim_{h \to 0} \left(h^2+1\right)$$
$$=1$$

Since $L.H.L = R.H.L$. So $\lim_{x \to 0} f\left(x\right)$ exists.

The limiting value is,

$$\lim_{x \to 0} f\left(x\right) = 1.$$

Problem-03: If $f(x)=\frac{1}{1-e^{1/x}}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist?

***Solution*:** Given that, $f(x)=\frac{1}{1-e^{1/x}}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$
$$=\lim_{h\to 0} f(-h)$$
$$=\lim_{h\to 0}\frac{1}{1-e^{-1/h}}$$
$$=\frac{1}{1-0}$$
$$=1$$

$$R.H.L=\lim_{h\to 0}(0+h)$$
$$=\lim_{h\to 0} f(h)$$
$$=\lim_{h\to 0}\frac{1}{1-e^{1/h}}$$
$$=-\frac{1}{\propto}$$
$$=0$$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same.

i.e, $L.H.L\neq R.H.L$. So $\lim_{x\to 0} f(x)$ does not exist.

Problem-04: If $f(x)=\frac{|x|}{x}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist ?

***Solution*:** Given that, $f(x)=\frac{|x|}{x}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$
$$=\lim_{h\to 0} f(-h)$$
$$=\lim_{h\to 0}\frac{|-h|}{-h}$$
$$=\lim_{h\to 0}\frac{h}{-h}$$
$$=-1$$

$$R.H.L=\lim_{h\to 0} f(0+h)$$
$$=\lim_{h\to 0} f(h)$$
$$=\lim_{h\to 0}\frac{|h|}{h}$$
$$=\lim_{h\to 0}\frac{h}{h}$$
$$=1$$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same. i.e, $L.H.L\neq R.H.L$. So $\lim_{x\to 0} f(x)$ does not exist.

<u>Problem-05</u>: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} e^{-\frac{|x|}{2}} & when \quad -1<x<0 \\ x^2 & when \quad 0\le x<2 \end{cases}$$

Discuss the existence of $\lim_{x\to 0} f(x)$.

***<u>Solution</u>*:** Given that, $f(x)=\begin{cases} e^{-\frac{|x|}{2}} & when \quad -1<x<0 \\ x^2 & when \quad 0\le x<2 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(0-h) \qquad\qquad R.H.L=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0} e^{-\frac{\{-(0-h)\}}{2}} \qquad\qquad =\lim_{h\to 0}(0+h)^2$$

$$=\lim_{h\to 0} e^{-\frac{h}{2}} \qquad\qquad =0$$

$$=1$$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same.

i.e, $L.H.L\neq R.H.L$. So $\lim_{x\to 0} f(x)$ does not exist.

<u>Problem-06</u>: A real function is defined by $f(x)=\dfrac{x}{1-x}$.

Find a). $\lim_{x\to 1} f(x)$; b) $\lim_{x\to \infty} f(x)$ and c). $\lim_{x\to -\infty} f(x)$.

***<u>Solution</u>*:** Given that, $f(x)=\dfrac{x}{1-x}$

<u>1st part a</u>: $L.H.L=\lim_{h\to 0} f(1-h)$ $\qquad\qquad$ $R.H.L=\lim_{h\to 0} f(1+h)$

$$=\lim_{h\to 0}\frac{1-h}{1-(1-h)} \qquad\qquad =\lim_{h\to 0}\frac{1+h}{1-(1+h)}$$

$$=\lim_{h\to 0}\frac{1-h}{h} \qquad\qquad =\lim_{h\to 0}\frac{1+h}{-h}$$

$$=\infty \qquad\qquad =-\infty\ .$$

Here, $L.H.L$ and $R.H.L$ both are not exist. So $\lim_{x\to 1} f(x)$ does not exist.

2nd part b:

Let $x = \dfrac{1}{y}$ then $x \to \propto \Rightarrow y \to 0$

Now, $\lim_{x\to\propto} f(x) = \lim_{y\to 0} f\left(\dfrac{1}{y}\right)$

$$= \lim_{y\to 0} \frac{\frac{1}{y}}{1-\frac{1}{y}}$$

$$= \lim_{y\to 0} \frac{1}{y-1}$$

$$= \frac{1}{0-1}$$

$$= -1.$$

3rd part c:

Let $x = \dfrac{1}{y}$ then $x \to -\propto \Rightarrow y \to 0$

Now, $\lim_{x\to -\propto} f(x) = \lim_{y\to 0} f\left(\dfrac{1}{y}\right)$

$$= \lim_{y\to 0} \frac{\frac{1}{y}}{1-\frac{1}{y}}$$

$$= \lim_{y\to 0} \frac{1}{y-1}$$

$$= \frac{1}{0-1}$$

$$= -1.$$

***<u>Homework</u>*:**

***Problem-01*:** A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x-1 & when \quad x>0 \\ 1/2 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$$

Find the value of $\lim_{x\to 0} f(x)$.

***Problem-02*:** A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} 1+2x & when \quad -1/2 \le x<0 \\ 1-2x & when \quad 0\le x<1/2 \\ 2x-1 & when \quad x>1/2 \end{cases}$$

Find the value of $\lim_{x\to 1/2} f(x)$.

Problem-03: If $f(x)=\dfrac{1}{1+e^{1/x}}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist ?

Problem-04: If $f(x)=\dfrac{1}{3+e^{1/(x-2)}}$ then find limits from the left and the right of $x=2$. Does the limit of $f(x)$ at $x=2$ exist ?

Problem-05: If $f(x)=\begin{cases} \dfrac{|x-1|}{x-1} & when \quad x\ne 1 \\ 1 & when \quad x=1 \end{cases}$ then show that $\lim_{x\to 1} f(x)$ does not exist but $\lim_{x\to 2} f(x)$ exists.

Problem-06: If $f(x)=\dfrac{1}{x}\sin\left(\dfrac{1}{x}\right)$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist?

Some important limits:

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots}{x}\right)$$

$$=\lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\cdots\right)$$

$$=1$$

2. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

Proof: Given that,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$=\lim_{x\to\infty}\left\{1+x.\frac{1}{x}+\frac{x(x-1)}{2!}.\frac{1}{x^2}+\frac{x(x-1)(x-2)}{3!}.\frac{1}{x^3}+\cdots\cdots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{\left(x^3-3x^2+2x\right)}{3!}.\frac{1}{x^3}+\cdots\cdots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{3}{x}+\frac{2}{x^2}\right)+\cdots\cdots\right\}$$

$$=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$$

$$=e$$

3. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Proof: Given that,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$=\lim_{x\to 0}\left\{1+\frac{1}{x}.x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}.x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}.x^3+\cdots\cdots\right\}$$

$$=\lim_{x\to 0}\left\{1+1+\frac{1}{2!}(1-x)+\frac{\left(\frac{1}{x^3}-\frac{3}{x^2}+\frac{2}{x}\right)}{3!}.x^3+\cdots\cdots\right\}$$

$$=\lim_{x\to 0}\left\{1+1+\frac{1}{2!}(1-x)+\frac{1}{3!}\left(1-3x+2x^2\right)+\cdots\cdots\right\}$$

$$=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$$

$$=e$$

4. $\lim\limits_{x\to 0}\dfrac{1}{x}\log(1+x)=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{1}{x}\log(1+x)$$

$$=\lim_{x\to 0}\log(1+x)^{\frac{1}{x}}$$

$$=\log\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\}$$

$$=\log e$$

$$=1$$

5. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

$$=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots\right)-1}{x}$$

$$=\lim_{x\to 0}\left(\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots}{x}\right)$$

$$=\lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\cdots\right)$$

$$=1$$

L' Hospital's Rule: If two functions $f(x)$ and $g(x)$ are continuous at $x=a$, also their derivatives $f^{'}(x), g^{'}(x)$ are continuous at this point and $f(a)=g(a)=0$ but $g^{'}(a)\neq 0$ then L' Hospital's rule states as,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^{'}(x)}{g^{'}(x)}=\frac{f^{'}(a)}{g^{'}(a)}$$

In case, $f^{'}(a)=g^{'}(a)=0$, the rule maybe extended.

Indeterminate forms: If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ then it is called an indeterminate form at $x=a$. The forms $\frac{\propto}{\propto}$, $0\times\propto$, $\propto-\propto$, 0^0, $1^{\propto}$ and $\propto^0$ are also indeterminate forms.

Evaluate the following limits:

Problem 01: Find $\lim_{x\to 0}\frac{\tan x}{x}$

Sol: Given that,

$$\lim_{x\to 0}\frac{\tan x}{x} \quad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\sec^2 x$$
$$=1$$

Problem 02: Find $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

Sol: Given that,

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x} \quad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}$$
$$=1$$

Problem 03: Find $\lim_{x\to\propto}\frac{(\ln x)^2}{x}$

Sol: Given that,

$$\lim_{x\to\propto}\frac{(\ln x)^2}{x} \quad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=\lim_{x\to\propto}2\ln x.\frac{1}{x}$$
$$=2\lim_{x\to\propto}\frac{\ln x}{x} \quad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=2\lim_{x\to\propto}\frac{1}{x}$$
$$=2.\frac{1}{\propto}$$
$$=0$$

Problem 04: Find $\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1}x}$

Sol: Given that,

$$\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1}x} \quad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x}{\cos x\sin^{-1}x+\frac{\sin x}{\sqrt{1-x^2}}} \quad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x\sqrt{1-x^2}}{\cos x\sin^{-1}x\sqrt{1-x^2}+\sin x} \quad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2\sqrt{1-x^2}+\frac{2x^2}{\sqrt{1-x^2}}}{-\sin x\sin^{-1}x\sqrt{1-x^2}+\cos x\left(1+\frac{2x}{\sqrt{1-x^2}}\right)+\cos x}$$
$$=\frac{2}{1+1}$$
$$=1$$

Problem 05: Find $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

Sol: Given that,

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}$$
$$=\frac{1+1}{1}$$
$$=2$$

Problem 05: Find $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) \qquad ;[Form\ \propto-\propto]$$
$$=\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)$$
$$=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x\sin x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right)$$
$$=\lim_{x\to 0}\left(\frac{x\sin x}{\sin x+x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right)$$
$$=\frac{0}{1+1}$$
$$=0$$

Problem 06: Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \qquad ;[Form \propto-\propto]$$
$$=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)$$
$$=\frac{0}{1+1-0}$$
$$=0$$

Problem 07: Find $\lim_{x\to 0}\sin x\ln x^2$

Sol: Given that,

$$\lim_{x\to 0}\sin x\ln x^2 \qquad ;[Form\ 0\times\propto]$$
$$=\lim_{x\to 0}\frac{2\ln x}{\cos ecx} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=2\lim_{x\to 0}\left(\frac{1/x}{-\cos ecx\cot x}\right)$$
$$=-2\lim_{x\to 0}\left(\frac{\sin^2 x}{x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=-2\lim_{x\to 0}\left(\frac{2\sin x\cos x}{\cos x-x\sin x}\right)$$
$$=-2.\frac{0}{1-0}$$
$$=0$$

Problem 08: Find $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} \qquad ;\left[Form\ \propto^{\propto}\right]$$

$$Let\ \ y=\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$$

$$\therefore\ \ln y=\frac{1}{x}\ln\left(\frac{\tan x}{x}\right)$$

$$\therefore \lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{\tan x}{x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{\ln\left(\frac{\tan x}{x}\right)}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{x}{\tan x}.\frac{x\sec^2 x-\tan x}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{x\sin 2x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{2-2\cos 2x}{\sin 2x+2x\cos 2x}\right) \quad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{4\sin 2x}{2\cos 2x+2\cos 2x-4x\sin 2x}\right)$$

$$=\frac{0}{2+2-0}$$

$$=0$$

$$\therefore \lim_{x\to 0}y=e^0$$

$$\therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}=1$$

Problem 09: Find $\lim_{x\to \pi/2}(\sin x)^{\tan x}$

Sol: Given that,

$$\lim_{x\to \pi/2}(\sin x)^{\tan x} \qquad ;\left[Form\ 1^{\propto}\right]$$

$$Let\ \ y=(\sin x)^{\tan x}$$

$$\therefore\ \ln y=\tan x\ln(\sin x)$$

$$\therefore\ \lim_{x\to \pi/2}\ln y=\lim_{x\to \pi/2}\tan x\ln(\sin x) \qquad ;\left[Form\ 0\times\propto\right]$$

$$=\lim_{x\to \pi/2}\frac{\ln(\sin x)}{\cot x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to \pi/2}\frac{\cot x}{\cos ec^2 x}$$

$$=0$$

$$\therefore\ \lim_{x\to \pi/2}y=e^0$$

$$\therefore\ \lim_{x\to \pi/2}(\sin x)^{\tan x}=1$$

Homework:

Problem 01: Find $\lim_{x\to 0}\frac{\sin x\sin^{-1}x}{x^2}$ Ans: 1

Problem 02: Find $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$ Ans: $\frac{1}{3}$

Problem 03: Find $\lim_{x\to 0}(\cos x)^{\cos ec^2 x}$ Ans: $e^{-\frac{1}{2}}$

Problem 04: Find $\lim_{x\to 0}\left(\frac{x}{x-1}-\frac{x}{\ln x}\right)$ Ans: $\frac{1}{2}$

Problem 05: Find $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ Ans: $\frac{2}{3}$

Problem 06: Find $\lim_{x\to 0}(\sin x)^x$ Ans: 1

<u>Continuity</u>: A function $f(x)$ is said to be continuous at a point $x = c$ provided the following three conditions are satisfied:

1. $\lim_{x \to c} f(x)$ exists,
2. $f(c)$ is defined,
3. $\lim_{x \to c} f(x) = f(c)$.

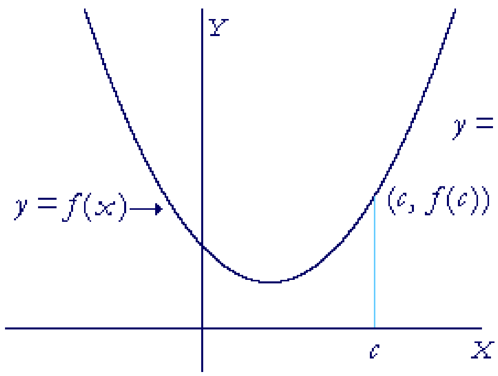


Fig. (a) Continuous function.

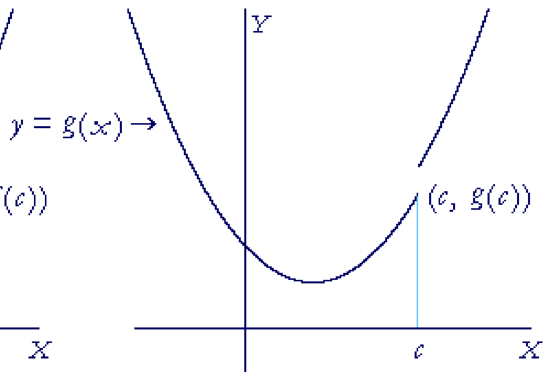


Fig. (b) Discontinuous function.

If one or more of the conditions of this definition fails to hold, then the function $f(x)$ is discontinuous at $x = c$.

<u>Problem-01</u>: A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 + 1 & when \quad x < 0 \\ x & when \quad 0 \le x \le 1 \\ 1/x & when \quad x > 1 \end{cases}$$

Discus the continuity at $x = 1$.

<u>Solution</u>: Given that, $f(x) = \begin{cases} x^2 + 1 & when \quad x < 0 \\ x & when \quad 0 \le x \le 1 \\ 1/x & when \quad x > 1 \end{cases}$

$$L.H.L = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h) = 1$$

$$R.H.L = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \left(\frac{1}{1+h} \right) = 1$$

Here, $L.H.L = R.H.L$. So $\lim_{x\to 1} f(x)$ exists and the limiting value is,

$$\lim_{x\to 1} f(x) = 1.$$

Now, the functional value at $x = 1$ is,

$$f(1) = 1$$

Since, $\lim_{x\to 1} f(x) = f(1)$, the given function is continuous at $x = 1$.

<u>Problem-02</u>: Test the continuity of the function $f(x) = |x| + |x-2|$ at the point $x = 2$.

***<u>Solution</u>*:** The given function is, $f(x) = |x| + |x-2|$

$$= \begin{cases} x + (x-2) & when \quad x \ge 2 \\ x - (x-2) & when \quad 0 \le x < 2 \\ -x - (x-2) & when \quad x < 0 \end{cases}$$

$$= \begin{cases} 2x - 2 & when \quad x \ge 2 \\ 2 & when \quad 0 \le x < 2 \\ -2x + 2 & when \quad x < 0 \end{cases}$$

$$L.H.L = \lim_{h\to 0} f(2-h)$$

$$= \lim_{h\to 0} (2)$$

$$= 2$$

$$R.H.L = \lim_{h\to 0} f(2+h)$$

$$= \lim_{h\to 0} \{2(2+h) - 2\}$$

$$= 2$$

Here, $L.H.L = R.H.L$. So $\lim_{x\to 2} f(x)$ exists and the limiting value is,

$$\lim_{x\to 2} f(x) = 2.$$

Now, the functional value at $x = 2$ is,

$$f(2) = 2\times 2 - 2$$

$$= 2$$

Since, $\lim_{x\to 2} f(x) = f(2)$, the given function is continuous at $x = 2$.

Problem-03: If $f(x)=\begin{cases} x+1 & when\ x\le 1 \\ 3-ax^2 & when\ x>1 \end{cases}$ for what value of a , $f(x)$ is continuous at $x=1$.

Solution: Given that, $f(x)=\begin{cases} x+1 & when\ x\le 1 \\ 3-ax^2 & when\ x>1 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(1-h)$$
$$=\lim_{h\to 0}(1-h+1)$$
$$=2$$

$$R.H.L=\lim_{h\to 0} f(1+h)$$
$$=\lim_{h\to 0}\left\{3-a(1+h)^2\right\}$$
$$=3-a$$

And, the functional value at $x=1$ is,

$$f(1)=1+1$$
$$=2$$

Now, the given function $f(x)$ will be continuous at $x=1$,

if $L.H.L=R.H.L=f(1)$

$or,\ 2=3-a=2$

$or,\ 3-a=2$

$or,\ a=3-2$

$or,\ a=1$ (*Ans.*)

Problem-04: If $f(x)=\begin{cases} x^2\sin\left(1/x\right) & when\ x\ne 0 \\ 0 & when\ x=0 \end{cases}$ then test the continuity at $x=0$.

Solution: Given that, $f(x)=\begin{cases} x^2\sin\left(1/x\right) & when\ x\ne 0 \\ 0 & when\ x=0 \end{cases}$

Now, $\lim_{x\to 0} f(x)$

$$=\lim_{x\to 0} x^2\sin\left(1/x\right)$$

$$=\lim_{x\to 0} x^2.\lim_{x\to 0}\sin\left(1/x\right)$$

$$= 0 \times \left(a\, number\, in\, the\, interval \left[-1,1\right] \right)$$

$$= 0$$

$\therefore \lim_{x \to 0} f(x) = 0$.

And, the functional value at $x = 0$ is,

$$f(0) = 0$$

Since, $\lim_{x \to 0} f(x) = f(0)$, the given function is continuous at $x = 0$.

<u>Problem-05</u>: If $f(x) = \begin{cases} (1+2x)^{1/x} & when\ x \neq 0 \\ e^2 & when\ x = 0 \end{cases}$ then test the continuity at $x = 0$.

<u>Solution</u>: Given that, $f(x) = \begin{cases} (1+2x)^{1/x} & when\ x \neq 0 \\ e^2 & when\ x = 0 \end{cases}$

Now, $\lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} (1+2x)^{1/x}$$

$$= \lim_{x \to 0} \left\{ 1 + \frac{1}{x}(2x) + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}(2x)^3 + \cdots\cdots \right\} \quad [By\ binomial\ theorem]$$

$$= \lim_{x \to 0} \left\{ 1 + 2 + \frac{2^2}{2!}(1-x) + \frac{2^3}{3!}(1-x)(1-2x) + \cdots\cdots \right\}$$

$$= 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots\cdots$$

$$= e^2$$

$\therefore \lim_{x \to 0} f(x) = e^2$.

And, the functional value at $x = 0$ is,

$$f(0) = e^2$$

Since, $\lim_{x \to 0} f(x) = f(0)$, the given function is continuous at $x = 0$.

Problem-06: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} -x & when \quad x\le 0 \\ x & when \quad 0<x<1 \\ 1-x & when \quad x\ge 1 \end{cases}$$

Discus the continuity at $x=1$.

Solution: Given that, $f(x)=\begin{cases} -x & when \quad x\le 0 \\ x & when \quad 0<x<1 \\ 1-x & when \quad x\ge 1 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(1-h)$$
$$=\lim_{h\to 0}(1-h)$$
$$=1$$

$$R.H.L=\lim_{h\to 0} f(1+h)$$
$$=\lim_{h\to 0}\{1-(1+h)\}$$
$$=0$$

Here, $L.H.L\ne R.H.L$. So $\lim_{x\to 1} f(x)$ does not exist.

Hence, the given function is discontinuous at $x=1$.

Homework:

***Problem-01*:** A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} 1 & when \quad -\infty<x<0 \\ 1+\sin x & when \quad 0\le x<\pi/2 \\ 2+\left(x-\pi/2\right)^2 & when \quad \pi/2< x<\infty \end{cases}$$

Test the continuity at $x=0 \ and \ \pi/2$.

***Problem-02*:** Discuss the continuity of the function $f(x)=|x|+|x-1|$ at the point $x=0$.

Problem-03: Test the continuity of the function $f(x)=|x-1|+|x-2|$ at the point $x=1$.

Problem-04: Find a non-zero value for the constant k that makes $f(x)=\begin{cases} \dfrac{\tan(kx)}{x} & if \ x<0 \\ 3x+k^2 & if \ x\ge 0 \end{cases}$ continuous at $x=0$.

Problem-05: If $f(x)=\begin{cases} x\cos\left(1/x\right) & when\ x\neq 0 \\ 0 & when\ x=0 \end{cases}$ then test the continuity at $x=0$.

Problem-06: If $f(x)=\begin{cases} (1+x)^{1/x} & when\ x\neq 0 \\ 1 & when\ x=0 \end{cases}$ then test the continuity at $x=0$.

Problem-07: If $f(x)=\begin{cases} x & when\ 0\leq x<1/2 \\ 1-x & when\ 1/2\leq x\leq 1 \end{cases}$ then test the continuity at $x=1/2$.

<u>Differentiability of a function</u>: The derivative of $y=f(x)$ with respect to x (for any particular value of) is denoted by $f^{'}(x)$ or $\frac{dy}{dx}$ and defined as,

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Provided this limit exists.

<u>Existence of Derivative</u>: A function $y=f(x)$ is called differentiable at $x=a$ if the left hand derivative and right hand derivative at this point i.e,

$$L.H.D=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

and $$R.H.D=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

are both exist and equal.

<u>Problem 01:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x^2+1 & when\ x\leq 0 \\ x & when\ 0<x<1 \\ \frac{1}{x} & when\ x\geq 1 \end{cases}$$

Discuss the differentiability at $x=0$ and $x=1$.

Solution: Given that,

$$f(x)=\begin{cases} x^2+1 & when\ x\le 0 \\ x & when\ 0<x<1 \\ \dfrac{1}{x} & when\ x\ge 1 \end{cases}$$

1st Part: For $x=0$,

$$\begin{aligned} L.H.D &= \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} \\ &= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h} \\ &= \lim_{h\to 0}\frac{\left\{(-h)^2+1\right\}-\left(0^2+1\right)}{-h} \\ &= \lim_{h\to 0}\frac{h^2+1-1}{-h} \\ &= \lim_{h\to 0}\frac{h^2}{-h} \\ &= \lim_{h\to 0}(-h) \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.D &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{h-\left\{(0)^2+1\right\}}{h} \\ &= \lim_{h\to 0}\frac{h-1}{h} \\ &= \lim_{h\to 0}\left(1-\frac{1}{h}\right) \\ &= -\infty \end{aligned}$$

Since *R.H.D* does not exist. So the function is not differentiable at $x=0$.

2nd Part: For $x=1$,

$$\begin{aligned} L.H.D &= \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} \\ &= \lim_{h\to 0}\frac{1-h-1}{-h} \\ &= \lim_{h\to 0}\frac{-h}{-h} \\ &= \lim_{h\to 0}(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} R.H.D &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h} \\ &= \lim_{h\to 0}\frac{1-1-h}{h(1+h)} \\ &= \lim_{h\to 0}\frac{-1}{1+h} \\ &= \frac{-1}{1+0} \\ &= -1 \end{aligned}$$

Since $L.H.D\ne R.H.D$ does not exist. So the function is not differentiable at $x=1$.

<u>Problem 02:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}1 & when\ x<0\\ 1+\sin x & when\ 0\le x<\dfrac{\pi}{2}\\ 2+\left(x-\pi/2\right)^2 & when\ x\ge\dfrac{\pi}{2}\end{cases}$$

Discuss the differentiability at $x=0$ and $x=\dfrac{\pi}{2}$.

<u>Solution:</u> Given that,

$$f(x)=\begin{cases}1 & when\ x<0\\ 1+\sin x & when\ 0\le x<\dfrac{\pi}{2}\\ 2+\left(x-\pi/2\right)^2 & when\ x\ge\dfrac{\pi}{2}\end{cases}$$

<u>1st Part:</u> For $x=0$,

$$\begin{aligned}L.H.D&=\lim_{h\to0}\frac{f(0-h)-f(0)}{-h}\\&=\lim_{h\to0}\frac{f(-h)-f(0)}{-h}\\&=\lim_{h\to0}\frac{1-(1+\sin 0)}{-h}\\&=\lim_{h\to0}\frac{1-1}{-h}\\&=\lim_{h\to0}\frac{0}{-h}\\&=0\end{aligned}$$

$$\begin{aligned}R.H.D&=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}\\&=\lim_{h\to0}\frac{f(h)-f(0)}{h}\\&=\lim_{h\to0}\frac{1+\sinh-(1+\sin 0)}{h}\\&=\lim_{h\to0}\frac{1+\sinh-1}{h}\\&=\lim_{h\to0}\frac{\sinh}{h}\\&=1\end{aligned}$$

Since $L.H.D\neq R.H.D$ does not exist. So the function is not differentiable at $x=0$.

$$L.H.D=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}$$
$$=\lim_{h\to 0}\frac{1+\sin\left(\frac{\pi}{2}-h\right)-\left\{2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2\right\}}{-h}$$
$$=\lim_{h\to 0}\frac{1+\cos h-2}{-h}$$
$$=\lim_{h\to 0}\frac{\cos h-1}{-h}$$
$$=\lim_{h\to 0}\frac{\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots\cdots\right)-1}{-h}$$
$$=\lim_{h\to 0}\frac{-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots\cdots}{-h}$$
$$=\lim_{h\to 0}\left(\frac{h}{2!}-\frac{h^3}{4!}+\cdots\cdots\right)$$
$$=0$$

<u>2nd Part:</u> For $x=\frac{\pi}{2}$,

$$R.H.D=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}$$
$$=\lim_{h\to 0}\frac{\left\{2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2\right\}-\left\{2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2\right\}}{h}$$
$$=\lim_{h\to 0}\frac{2+h^2-2}{h}$$
$$=\lim_{h\to 0}\frac{h^2}{h}$$
$$=\lim_{h\to 0}h$$
$$=0$$

Since $L.H.D=R.H.D$ exists. So the function is differentiable at $x=\frac{\pi}{2}$.

HOMEWORK:

<u>Problem 01:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}\ln x & when\ 0<x\le 1\\ 0 & when\ 1<x\le 2\\ 1+x^2 & when\ x>2\end{cases}$$

Discuss the differentiability at $x=1$.

<u>Problem 02:</u> Discuss the differentiability of the function $f(x)=|x|+|x-1|$ at the point $x=0$ and $x=1$.

<u>Problem 03:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}x^2 & when\ x\le 1\\ x & when\ 1<x\le 2\\ \left(\frac{1}{4}\right)x^3 & when\ x>2\end{cases}$$

Discuss the differentiability at $x=1$ and $x=2$.

<u>Problem 04:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}1+x & when\ x\le 0\\ x & when\ 0<x<1\\ 2-x & when\ 1<x\le 2\end{cases}$$

Discuss the differentiability at $x=0$ and $x=1$.

<u>Problem 05:</u> A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} 0 & when\ 0 \le x < 3 \\ 4 & when\ x=3 \\ 5 & when\ 3 < x \le 4 \end{cases}$$

Discuss the differentiability at $x=3$.