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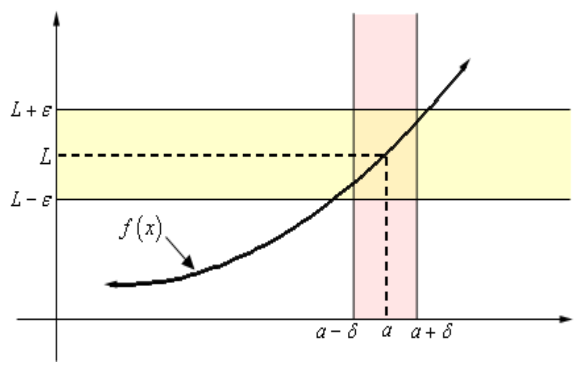
Limit & Continuity

Introduction: In this chapter we will study about limit that is the core tool of calculus and all other calculus concepts are based on it. A function can be undefined at a point, but we can think about what the function "approaches" as it gets closer and closer to that point (this is the "limit"). Also the function may be defined at a point, but it may approach a different limit. There are many, many times where the functional value is the same as the limit at a point. Limit is used to define continuity, derivative and integral of a function.

Limit of a function: The number " l " is called limit of a function $f(x)$ at a point $x = a$ if x approaches closer and closer to " a " from both sides and consequently $f(x)$ approaches closer and closer to " l ". Symbolically it is written as,

$$\lim_{x\to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a .$$

Graphical representation of "limit of a function" at a point: Let the function be $y = f(x)$. If $\lim_{x\to a} f(x) = L$, then by the definition of limit we have, if for each given $\varepsilon > 0$, there exists a positive number δ (depending on ε) such that $|f(x) - L| < \varepsilon$, whenever $0 < |x-a| < \delta$ i.e. $f(x) \in (L-\varepsilon, L+\varepsilon)$ whenever $x \in (a-\delta, a+\delta)$, i.e. $L-\varepsilon < y = f(x) < L+\varepsilon$ whenever $a-\delta < x < a+\delta$, but $x \neq a$.



The point $(x, f(x))$ of the graph of the function $f(x)$ lies between the two lines $y = L-\varepsilon$ and $y = L+\varepsilon$ provided that x lies in the interval $(a-\delta, a+\delta)$, $x \neq a$. This implies that as long as x belongs to the interval $(a-\delta, a+\delta)$, $x \neq a$, the graph of the function $f(x)$ lies within the rectangle bounded by straight lines $y = L-\varepsilon$, $y = L+\varepsilon$, $x = a-\delta$ and $x = a+\delta$, where (a, L) is the centre of the rectangle. Here $\varepsilon > 0$ can be chosen as small as we wish such that the rectangle can be made to have as small as an altitude 2ε as we wish. Thus we can explain shortly, $\lim_{x\to a} f(x) = L$ exists nearer the point (a, L).

Mathematical* or $\epsilon - \delta$ *definition of limit of a function: The number " l " is called limit of a function $f(x)$ at x approaches " a " if for any given positive number ϵ $(however\ small)$, we can find another positive number $\delta(\ depending\ on\ \epsilon\ )$ such that $|f(x) - l| < \epsilon$, for all values of x satisfying $0 < |x - a| < \delta$.

Symbolically it is written as,

$$\lim_{x \to a} f(x) = l \ \ or \ \ f(x) \to l \ as \ x \ \to a \ .$$

<u>Left Hand Limit</u>: If the values of $f(x)$ can be made as close as we like to " l " by taking values of x sufficiently close to " a " (but less than a) then we write,

$$\boldsymbol{L.H.L} = lim_{x \to a^-} f(x) = l$$

<u>Right Hand Limit</u>: If the values of $f(x)$ can be made as close as we like to " l " by taking values of x sufficiently close to " a " (but greater than a) then we write,

$$\boldsymbol{R.H.L} = lim_{x \to a^+} f(x) = l$$

<u>Existence of limit of a function $f(x)$ at $x = a$</u> :

The limit of a function $f(x)$ at $x = a$ that is $\lim_{x \to a} f(x) = l$ exists if

a) $\boldsymbol{L.H.L} = lim_{x \to a^-} f(x)\ \ exists$
b) $\boldsymbol{R.H.L} = lim_{x \to a^+} f(x)\ \ also\ exists$
c) $\boldsymbol{L.H.L} = \boldsymbol{R.H.L} = l.$

<u>Fundamental Properties of limit:</u>

If $f(x)$, $g(x)$ are two functions and $\lim_{x \to a} f(x)\ and\ \lim_{x \to a} g(x)\ \ exists\ then$

a) $\lim_{x \to a}\{\ f(x)\ \pm g(x)\ \} = \lim_{x \to a} f(x) \pm\ \lim_{x \to a} g(x)$

b) $\lim_{x \to a}\{\ f(x) \times\ g(x)\ \} = \lim_{x \to a} f(x) \times\ \lim_{x \to a} g(x)$

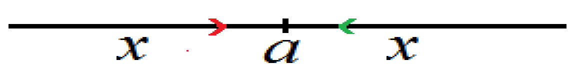
c) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\ \ where\ g(a) \neq 0$

d) $\lim_{x \to a}\{constant\ \times f(x)\} = constant\ \times\ \lim_{x \to a} f(x)$

e) $\lim_{x \to a}\{\ f(x)\}^n = \{\lim_{x \to a} f(x)\}^n\ \ where\ \ n \ \epsilon\ Z$

f) $\lim_{x \to a}(constant) = Constant$

***<u>Change of limit of a variable</u>*:**



<u>Left hand limit</u>: $L.H.L = \lim_{x \to a^-} f(x)$

Let $x + h = a$ and when $h \to 0\ then\ x \to a$.

Now, $\lim_{x \to a^-} f(x) = \boldsymbol{\lim_{h \to 0} f(a - h)}$, $putting\ value\ of\ x$.

<u>Right hand limit</u>: $R.H.L = \lim_{x \to a^+} f(x)$

Let $x - h = a$ and when $h \to 0\ then\ x \to a$.

Now, $\lim_{x \to a^+} f(x) = \boldsymbol{\lim_{h \to 0} f(a + h)}$, $putting\ value\ of\ x$.

Theorem-01: If $\lim_{x\to a} f(x)$ exists then it must be unique.

Proof: Let, $\lim_{x\to a} f(x)=l_1$ and $\lim_{x\to a} f(x)=l_2$.

Now we have to show that $l_1=l_2$.

Let $\varepsilon=|l_1-l_2|>0$, then there exists $\delta(\varepsilon)>0$, where $|f(x)-l_1|<\frac{\varepsilon}{2}$ whenever $0<|x-a|<\delta$ and $|f(x)-l_2|<\frac{\varepsilon}{2}$ whenever $0<|x-a|<\delta$.

Now
$$\begin{aligned}|l_1-l_2|&=|l_1-f(x)+f(x)-l_2|\\ &\le|l_1-f(x)|+|f(x)-l_2|\\ &\le|f(x)-l_1|+|f(x)-l_2|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &<\varepsilon\end{aligned}$$
$$\therefore |l_1-l_2|<|l_1-l_2| \qquad \because \varepsilon=|l_1-l_2|$$

This is evidently false. Hence $l_1=l_2$. **(proved)**

Theorem-02: If the two functions $f(x)$ and $g(x)$ are defined in the neighborhood of a point $x=a$ by $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$ then show that $\lim_{x\to a}\{f(x)+g(x)\}=l+m$.

Proof: Since, $\lim_{x\to a} f(x)=l$ implies that if for each given $\varepsilon>0$, there exists a positive number $\delta_1(\textit{depending on}\,\varepsilon)$ such that $|f(x)-l|<\frac{\varepsilon}{2}$, whenever $0<|x-a|<\delta_1$ $\cdots(1)$

Again as $\lim_{x\to a} g(x)=m$ implies that if for each given $\varepsilon>0$, there exists a positive number $\delta_2(\textit{depending on}\,\varepsilon)$ such that $|g(x)-m|<\frac{\varepsilon}{2}$, whenever $0<|x-a|<\delta_2$ $\cdots(2)$

Let, $\delta=\min\{\delta_1,\delta_2\}$, then from (1) and (2) we have

$|f(x)-l|<\frac{\varepsilon}{2}$ and $|g(x)-m|<\frac{\varepsilon}{2}$ whenever $0<|x-a|<\delta$.

Now
$$\begin{aligned}|f(x)+g(x)-(l+m)|&=|f(x)-l+g(x)-m|\\ &\le|f(x)-l|+|g(x)-m|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &<\varepsilon\end{aligned}$$

So by the definition we have
$$\lim_{x\to a}\{f(x)+g(x)\}=l+m.$$ **(Showed)**

Theorem-03: If the two functions $f(x)$ and $g(x)$ are defined in the neighborhood of a point $x=a$ by $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$ then show that $\lim_{x\to a}\{f(x)-g(x)\}=l-m$.

Proof: Since, $\lim_{x\to a} f(x)=l$ implies that if for each given $\varepsilon>0$, there exists a positive number $\delta_1(depending\ on\ \varepsilon)$ such that $|f(x)-l|<\frac{\varepsilon}{2}$, whenever $0<|x-a|<\delta_1$ $\cdots(1)$

Again as $\lim_{x\to a} g(x)=m$ implies that if for each given $\varepsilon>0$, there exists a positive number $\delta_2(depending\ on\ \varepsilon)$ such that $|g(x)-m|<\frac{\varepsilon}{2}$, whenever $0<|x-a|<\delta_2$ $\cdots(2)$

Let, $\delta=\min\{\delta_1,\delta_2\}$, then from (1) and (2) we have

$|f(x)-l|<\frac{\varepsilon}{2}$ and $|g(x)-m|<\frac{\varepsilon}{2}$ whenever $0<|x-a|<\delta$.

Now
$$\begin{aligned}|f(x)-g(x)-(l-m)| &= |f(x)-l+m-g(x)|\\ &\le |f(x)-l|+|g(x)-m|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &< \varepsilon\end{aligned}$$

So by the definition we have

$$\lim_{x\to a}\{f(x)-g(x)\}=l-m. \qquad \textbf{(Showed)}$$

<u>Problem-01</u>: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x^2 & when \quad x<1\\ 2.4 & when \quad x=1\\ x^2+1 & when \quad x>1\end{cases}$$

Does $\lim_{x\to 1} f(x)$ exist?

***<u>Solution</u>*:** Given that, $f(x)=\begin{cases} x^2 & when \quad x<1\\ 2.4 & when \quad x=1\\ x^2+1 & when \quad x>1\end{cases}$

$$\begin{aligned}L.H.L &= \lim_{h\to 0} f(1-h)\\ &= \lim_{h\to 0}(1-h)^2\\ &= \lim_{h\to 0}(1+2h+h^2)\\ &= 1\end{aligned}$$

$$\begin{aligned}R.H.L &= \lim_{h\to 0} f(1+h)\\ &= \lim_{h\to 0}\{(1+h)^2+1\}\\ &= \lim_{h\to 0}(1+2h+h^2+1)\\ &= 2\end{aligned}$$

Since $L.H.L \neq R.H.L$. So $\lim_{x\to 1} f(x)$ does not exist.

Problem-02: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x^2+1 & when \quad x>0 \\ 1 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$$

Find the value of $\lim_{x\to 0} f(x)$.

***Solution*:** Given that, $f(x)=\begin{cases} x^2+1 & when \quad x>0 \\ 1 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0}(0-h+1)$$

$$=1$$

$$R.H.L=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0}\left\{(0+h)^2+1\right\}$$

$$=\lim_{h\to 0}\left(h^2+1\right)$$

$$=1$$

Since $L.H.L = R.H.L$. So $\lim_{x\to 0} f(x)$ exists.

The limiting value is,

$$\lim_{x\to 0} f(x)=1.$$

Problem-03: If $f(x)=\dfrac{1}{1-e^{1/x}}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist?

***Solution*:** Given that, $f(x)=\dfrac{1}{1-e^{1/x}}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} f(-h)$$

$$=\lim_{h\to 0}\frac{1}{1-e^{-1/h}}$$

$$R.H.L=\lim_{h\to 0}(0+h)$$

$$=\lim_{h\to 0} f(h)$$

$$=\lim_{h\to 0}\frac{1}{1-e^{1/h}}$$

$=\frac{1}{1-0}$

$=1$

$=-\frac{1}{\propto}$

$=0$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same.

i.e, $L.H.L \neq R.H.L$. So $\lim_{x \to 0} f(x)$ does not exist.

Problem-04: If $f(x)=\frac{|x|}{x}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist ?

***Solution*:** Given that, $f(x)=\frac{|x|}{x}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} f(-h)$$

$$=\lim_{h\to 0}\frac{|-h|}{-h}$$

$$=\lim_{h\to 0}\frac{h}{-h}$$

$$=-1$$

$$R.H.L=\lim_{h\to 0} f(0+h)$$

$$=\lim_{h\to 0} f(h)$$

$$=\lim_{h\to 0}\frac{|h|}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=1$$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same. i.e, $L.H.L \neq R.H.L$. So $\lim_{x\to 0} f(x)$ does not exist.

Problem-05: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} e^{-\frac{|x|}{2}} & when \quad -1<x<0 \\ x^2 & when \quad 0\le x<2 \end{cases}$$

Discuss the existence of $\lim_{x\to 0} f(x)$.

***Solution*:** Given that, $f(x)=\begin{cases} e^{-\frac{|x|}{2}} & when \quad -1<x<0 \\ x^2 & when \quad 0\le x<2 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(0-h)$$

$$R.H.L=\lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} e^{-\frac{\{-(0-h)\}}{2}} \qquad\qquad = \lim_{h\to 0}(0+h)^2$$

$$= \lim_{h\to 0} e^{-\frac{h}{2}} \qquad\qquad = 0$$

$$=1$$

Here, $L.H.L$ and $R.H.L$ both are exist but they are not same.

i.e, $L.H.L \neq R.H.L$. So $\lim\limits_{x\to 0} f(x)$ does not exist.

***<u>Homework</u>*:**

***Problem-01*:** A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x-1 & when \quad x>0 \\ 1/2 & when \quad x=0 \\ x+1 & when \quad x<0 \end{cases}$$

Find the value of $\lim\limits_{x\to 0} f(x)$.

***Problem-02*:** A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} 1+2x & when \quad -1/2 \le x < 0 \\ 1-2x & when \quad 0 \le x < 1/2 \\ 2x-1 & when \quad x > 1/2 \end{cases}$$

Find the value of $\lim\limits_{x\to 1/2} f(x)$.

Problem-03: If $f(x)=\dfrac{1}{1+e^{1/x}}$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist ?

Problem-04: If $f(x)=\dfrac{1}{3+e^{1/(x-2)}}$ then find limits from the left and the right of $x=2$. Does the limit of $f(x)$ at $x=2$ exist ?

Problem-05: If $f(x)=\begin{cases} \dfrac{|x-1|}{x-1} & when \quad x\neq 1 \\ 1 & when \quad x=1 \end{cases}$ then show that $\lim\limits_{x\to 1} f(x)$ does not exist but $\lim\limits_{x\to 2} f(x)$ exists.

Problem-06: If $f(x)=\frac{1}{x}\sin\left(\frac{1}{x}\right)$ then find limits from the left and the right of $x=0$. Does the limit of $f(x)$ at $x=0$ exist?

Some important limits:

1. $\lim_{x\to 0}\frac{\sin x}{x}=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots}{x}\right)$$

$$=\lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\cdots\right)$$

$$=1$$

2. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Proof: Given that,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$=\lim_{x\to\infty}\left\{1+x.\frac{1}{x}+\frac{x(x-1)}{2!}.\frac{1}{x^2}+\frac{x(x-1)(x-2)}{3!}.\frac{1}{x^3}+\cdots\cdots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{\left(x^3-3x^2+2x\right)}{3!}.\frac{1}{x^3}+\cdots\cdots\right\}$$

$$=\lim_{x\to\infty}\left\{1+1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{3}{x}+\frac{2}{x^2}\right)+\cdots\cdots\right\}$$

$$=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$$

$$=e$$

3. $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Proof: Given that,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$=\lim_{x\to 0}\left\{1+\frac{1}{x}.x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}.x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}.x^3+\cdots\cdots\right\}$$

$$=\lim_{x\to 0}\left\{1+1+\frac{1}{2!}(1-x)+\frac{\left(\frac{1}{x^3}-\frac{3}{x^2}+\frac{2}{x}\right)}{3!}.x^3+\cdots\cdots\right\}$$

$$=\lim_{x\to 0}\left\{1+1+\frac{1}{2!}(1-x)+\frac{1}{3!}\left(1-3x+2x^2\right)+\cdots\cdots\right\}$$

$$=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$$

$$=e$$

4. $\lim_{x\to 0}\frac{1}{x}\log(1+x)=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{1}{x}\log(1+x)$$

$$=\lim_{x\to 0}\log(1+x)^{\frac{1}{x}}$$

$$=\log\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\}$$

$$=\log e$$

$$=1$$

5. $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof: Given that,

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

$$=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots\right)-1}{x}$$

$$=\lim_{x\to 0}\left(\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots}{x}\right)=\lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\cdots\right)=1$$

<u>L' Hospital's Rule</u>: If two functions $f(x)$ and $g(x)$ are continuous at $x=a$, also their derivatives $f'(x)$, $g'(x)$ are continuous at this point and $f(a)=g(a)=0$ but $g'(a)\neq 0$ then L' Hospital's rule states as,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$$

In case, $f'(a)=g'(a)=0$, the rule maybe extended.

<u>Indeterminate forms</u>: If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ then it is called an indeterminate form at $x=a$. The forms $\frac{\propto}{\propto}$, $0\times\propto$, $\propto-\propto$, 0^0, $1^{\propto}$ and $\propto^0$ are also indeterminate forms.

Evaluate the following limits:

<u>Problem 01:</u> Find $\lim_{x\to 0}\frac{\tan x}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{\tan x}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\sec^2 x$$
$$=1$$

<u>Problem 02:</u> Find $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}$$
$$=1$$

<u>Problem 03:</u> Find $\lim_{x\to\propto}\frac{(\ln x)^2}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to\propto}\frac{(\ln x)^2}{x} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=\lim_{x\to\propto}2\ln x.\frac{1}{x}$$
$$=2\lim_{x\to\propto}\frac{\ln x}{x} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=2\lim_{x\to\propto}\frac{1}{x}$$
$$=2.\frac{1}{\propto}$$
$$=0$$

<u>Problem 04:</u> Find $\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1}x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1}x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x}{\cos x\sin^{-1}x+\frac{\sin x}{\sqrt{1-x^2}}} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x\sqrt{1-x^2}}{\cos x\sin^{-1}x\sqrt{1-x^2}+\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2\sqrt{1-x^2}+\frac{2x^2}{\sqrt{1-x^2}}}{-\sin x\sin^{-1}x\sqrt{1-x^2}+\cos x\left(1+\frac{2x}{\sqrt{1-x^2}}\right)+\cos x}$$
$$=\frac{2}{1+1}$$
$$=1$$

Problem 05: Find $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$

Sol: Given that,

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}$$
$$=\frac{1+1}{1}$$
$$=2$$

Problem 05: Find $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\cot x\right)$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) \qquad ;\left[Form\ \propto-\propto\right]$$
$$=\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)$$
$$=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x\sin x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right)$$
$$=\lim_{x\to 0}\left(\frac{x\sin x}{\sin x+x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right)$$
$$=\frac{0}{1+1}=0$$

Problem 06: Find $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \qquad ;\left[Form\propto-\propto\right]$$
$$=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)$$
$$=\frac{0}{1+1-0}$$
$$=0$$

Problem 07: Find $\lim\limits_{x\to 0}\sin x\ln x^2$

Sol: Given that,

$$\lim_{x\to 0}\sin x\ln x^2 \qquad ;\left[Form\ 0\times\propto\right]$$
$$=\lim_{x\to 0}\frac{2\ln x}{\cos ecx} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=2\lim_{x\to 0}\left(\frac{1/x}{-\cos ecx\cot x}\right)$$
$$=-2\lim_{x\to 0}\left(\frac{\sin^2 x}{x\cos x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=-2\lim_{x\to 0}\left(\frac{2\sin x\cos x}{\cos x-x\sin x}\right)$$
$$=-2.\frac{0}{1-0}$$
$$=0$$

Problem 08: Find $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Sol: Given that,

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} \qquad ;\left[Form\ \infty^{\infty}\right]$$

$$Let\ \ y=\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$$

$$\therefore\ \ln y=\frac{1}{x}\ln\left(\frac{\tan x}{x}\right)$$

$$\therefore \lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{\tan x}{x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{\ln\left(\frac{\tan x}{x}\right)}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{x}{\tan x}\cdot\frac{x\sec^2 x-\tan x}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{x\sin 2x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{2-2\cos 2x}{\sin 2x+2x\cos 2x}\right) \quad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{4\sin 2x}{2\cos 2x+2\cos 2x-4x\sin 2x}\right)$$

$$=\frac{0}{2+2-0}$$

$$=0$$

$$\therefore\ \lim_{x\to 0}y=e^0$$

$$\therefore\ \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}=1$$

Problem 09: Find $\lim\limits_{x\to \pi/2}(\sin x)^{\tan x}$

Sol: Given that,

$$\lim_{x\to \pi/2}(\sin x)^{\tan x} \qquad ;\left[Form\ 1^{\infty}\right]$$

$$Let\ \ y=(\sin x)^{\tan x}$$

$$\therefore\ \ln y=\tan x\ln(\sin x)$$

$$\therefore\ \lim_{x\to \pi/2}\ln y=\lim_{x\to \pi/2}\tan x\ln(\sin x) \qquad ;\left[Form\ 0\times\infty\right]$$

$$=\lim_{x\to \pi/2}\frac{\ln(\sin x)}{\cot x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to \pi/2}\frac{\cot x}{\operatorname{cosec}^2 x}$$

$$=0$$

$$\therefore\ \lim_{x\to \pi/2}y=e^0$$

$$\therefore\ \lim_{x\to \pi/2}(\sin x)^{\tan x}=1$$

Homework:

Problem 01: Find $\lim\limits_{x\to 0}\frac{\sin x\sin^{-1}x}{x^2}$ Ans: 1

Problem 02: Find $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$ Ans: $\frac{1}{3}$

Problem 03: Find $\lim\limits_{x\to 0}(\cos x)^{\operatorname{cosec}^2 x}$ Ans: $e^{-\frac{1}{2}}$

Problem 04: Find $\lim\limits_{x\to 0}\left(\frac{x}{x-1}-\frac{x}{\ln x}\right)$ Ans: $\frac{1}{2}$

Problem 05: Find $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ Ans: $\frac{2}{3}$

Problem 06: Find $\lim\limits_{x\to 0}(\sin x)^x$ Ans: 1

Continuity: A function $f(x)$ is said to be continuous at a point $x = c$ provided the following three conditions are satisfied:

1. $\lim\limits_{x \to c} f(x)$ exists,
2. $f(c)$ is defined,
3. $\lim\limits_{x \to c} f(x) = f(c)$.

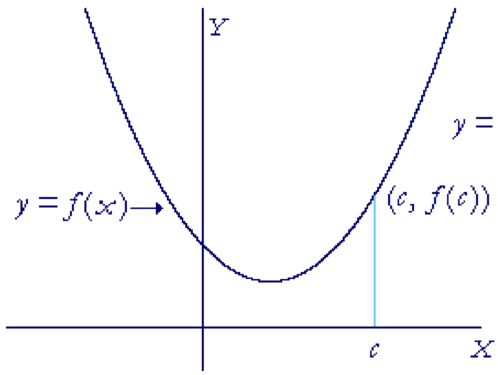


Fig. (a) Continuous function.

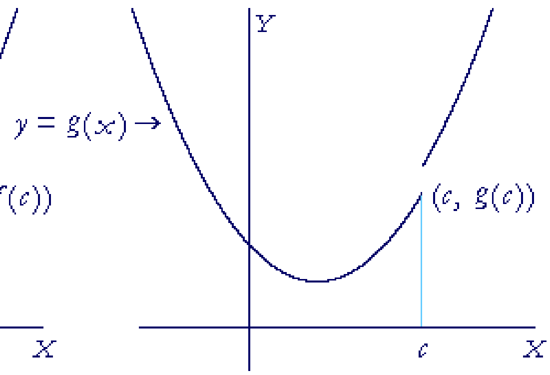


Fig. (b) Discontinuous function.

If one or more of the conditions of this definition fails to hold, then the function $f(x)$ is discontinuous at $x = c$.

Distinguish between functional value and limiting value: If $f(x)$ be a function then the functional value and the limiting value at $x = a$ are $f(a)$ and $\lim\limits_{x \to a} f(x)$ respectively. The statement $\lim\limits_{x \to a} f(x)$ stands for the value of $f(x)$ when x approaches closer and closer to a except a. In this case we do not care to know what happens when x is put equal to a. But the statement $f(a)$ stands for the value of $f(x)$ when x is exactly equal to a, obtained by substituting of a for x in the expression $f(x)$, when it exists.

Problem-01: A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 + 1 & \text{when} \quad x < 0 \\ x & \text{when} \quad 0 \le x \le 1 \\ 1/x & \text{when} \quad x > 1 \end{cases}$$

Discus the continuity at $x = 1$.

Solution: Given that, $f(x)=\begin{cases} x^2+1 & when \quad x<0 \\ x & when \quad 0\le x\le 1 \\ 1/x & when \quad x>1 \end{cases}$

$$L.H.L=\lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h) = 1$$

$$R.H.L=\lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left(1/1+h\right) = 1$$

Here, $L.H.L=R.H.L$. So $\lim_{x\to 1} f(x)$ exists and the limiting value is,

$$\lim_{x\to 1} f(x)=1.$$

Now, the functional value at $x=1$ is,

$$f(1)=1$$

Since, $\lim_{x\to 1} f(x)=f(1)$, the given function is continuous at $x=1$.

Problem-02: Test the continuity of the function $f(x)=|x|+|x-2|$ at the point $x=2$.

Solution: The given function is, $f(x)=|x|+|x-2|$

$$=\begin{cases} x+(x-2) & when \quad x\ge 2 \\ x-(x-2) & when \quad 0\le x<2 \\ -x-(x-2) & when \quad x<0 \end{cases}$$

$$=\begin{cases} 2x-2 & when \quad x\ge 2 \\ 2 & when \quad 0\le x<2 \\ -2x+2 & when \quad x<0 \end{cases}$$

$$L.H.L=\lim_{h\to 0} f(2-h) = \lim_{h\to 0}(2) = 2$$

$$R.H.L=\lim_{h\to 0} f(2+h) = \lim_{h\to 0}\{2(2+h)-2\} = 2$$

Here, $L.H.L=R.H.L$. So $\lim_{x\to 2} f(x)$ exists and the limiting value is,

$$\lim_{x \to 2} f(x) = 2.$$

Now, the functional value at $x = 2$ is,

$$f(2) = 2 \times 2 - 2$$

$$= 2$$

Since, $\lim_{x \to 2} f(x) = f(2)$, the given function is continuous at $x = 2$.

Problem-03: If $f(x) = \begin{cases} x+1 & when\ x \le 1 \\ 3 - ax^2 & when\ x > 1 \end{cases}$ for what value of a, $f(x)$ is continuous at $x = 1$.

Solution: Given that, $f(x) = \begin{cases} x+1 & when\ x \le 1 \\ 3 - ax^2 & when\ x > 1 \end{cases}$

$$L.H.L = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} (1 - h + 1)$$

$$= 2$$

$$R.H.L = \lim_{h \to 0} f(1+h)$$

$$= \lim_{h \to 0} \left\{3 - a(1+h)^2\right\}$$

$$= 3 - a$$

And, the functional value at $x = 1$ is,

$$f(1) = 1 + 1$$

$$= 2$$

Now, the given function $f(x)$ will be continuous at $x = 1$,

if $L.H.L = R.H.L = f(1)$

$or,\ 2 = 3 - a = 2$

$or,\ 3 - a = 2$

$or,\ a = 3 - 2$

$or,\ a = 1$ (*Ans.*)

Problem-04: If $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & when\ x \ne 0 \\ 0 & when\ x = 0 \end{cases}$ then test the continuity at $x = 0$.

<u>Solution</u>: Given that, $f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & when\ x\neq 0 \\ 0 & when\ x=0 \end{cases}$

$$Now,\ \lim_{x\to 0} f(x)$$

$$=\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$=\lim_{x\to 0} x^2 . \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

$$=0\times\left(a\, number\ in\ the\, \text{int}\, erval\left[-1,1\right]\right)$$

$$=0$$

$$\therefore \lim_{x\to 0} f(x)=0.$$

And, the functional value at $x=0$ is,

$$f(0)=0$$

Since, $\lim_{x\to 0} f(x)=f(0)$, the given function is continuous at $x=0$.

<u>Problem-05</u>: If $f(x)=\begin{cases} (1+2x)^{\frac{1}{x}} & when\ x\neq 0 \\ e^2 & when\ x=0 \end{cases}$ then test the continuity at $x=0$.

<u>Solution</u>: Given that, $f(x)=\begin{cases} (1+2x)^{\frac{1}{x}} & when\ x\neq 0 \\ e^2 & when\ x=0 \end{cases}$

$$Now,\ \lim_{x\to 0} f(x)$$

$$=\lim_{x\to 0} (1+2x)^{\frac{1}{x}}$$

$$=\lim_{x\to 0}\left\{1+\frac{1}{x}(2x)+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}(2x)^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}(2x)^3+\cdots\cdots\right\}\left[By\, binomial\, theorem\right]$$

$$=\lim_{x\to 0}\left\{1+2+\frac{2^2}{2!}(1-x)+\frac{2^3}{3!}(1-x)(1-2x)+\cdots\cdots\right\}$$

$$=1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\cdots$$

$$=e^2$$

$\therefore\ \lim_{x\to 0} f(x)=e^2$.

And, the functional value at $x=0$ is,

$$f(0)=e^2$$

Since, $\lim_{x\to 0} f(x)=f(0)$, the given function is continuous at $x=0$.

<u>Problem-06</u>: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases}-x & when \quad x\le 0\\ x & when \quad 0<x<1\\ 1-x & when \quad x\ge 1\end{cases}$$

Discus the continuity at $x=1$.

***<u>Solution</u>*:** Given that, $f(x)=\begin{cases}-x & when \quad x\le 0\\ x & when \quad 0<x<1\\ 1-x & when \quad x\ge 1\end{cases}$

$$L.H.L=\lim_{h\to 0} f(1-h)$$

$$=\lim_{h\to 0}(1-h)$$

$$=1$$

$$R.H.L=\lim_{h\to 0} f(1+h)$$

$$=\lim_{h\to 0}\{1-(1+h)\}$$

$$=0$$

Here, $L.H.L\neq R.H.L$. So $\lim_{x\to 1} f(x)$ does not exist.

Hence, the given function is discontinuous at $x=1$.

Problem-07: Using the (ε,δ) definition of limit to show that $\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1}=2$.

***<u>Solution</u>*:** Here, $f(x)=\frac{4x^2-1}{2x-1}$, $a=\frac{1}{2}$ and $l=2$.

We must show that for any given positive number ε, we can find a positive number $\delta(depending\ on\ \varepsilon)$ such that $|f(x)-l|<\varepsilon$, whenever $0<|x-a|<\delta$

i.e. $\left|\dfrac{4x^2-1}{2x-1}-2\right|<\varepsilon$, whenever $0<\left|x-\dfrac{1}{2}\right|<\delta$.

Now $\left|\dfrac{4x^2-1}{2x-1}-2\right|=|2x+1-2|$

$$=|2x-1|$$

$$=2\left|x-\frac{1}{2}\right|$$

$$<2\delta$$

$\therefore \left|\dfrac{4x^2-1}{2x-1}-2\right|<\varepsilon$ where $\delta=\dfrac{\varepsilon}{2}$

Hence, we have shown that $\lim\limits_{x\to\frac{1}{2}}\dfrac{4x^2-1}{2x-1}=2$.

Problem-08: Using the (ε,δ) definition of limit to show that $\lim\limits_{x\to1}\dfrac{x-1}{x^2+2x-3}=\dfrac{1}{4}$.

***Solution*:** Here, $f(x)=\dfrac{x-1}{x^2+2x-3}$, $a=1$ and $l=\dfrac{1}{4}$.

We must show that for any given positive number ε, we can find a positive number $\delta(\textit{depending on}\,\varepsilon)$ such that $|f(x)-l|<\varepsilon$, whenever $0<|x-a|<\delta$

i.e. $\left|\dfrac{x-1}{x^2+2x-3}-\dfrac{1}{4}\right|<\varepsilon$, whenever $0<|x-1|<\delta$.

Now $\left|\dfrac{x-1}{x^2+2x-3}-\dfrac{1}{4}\right|=\left|\dfrac{x-1}{(x-1)(x+3)}-\dfrac{1}{4}\right|$

$$=\left|\frac{1}{x+3}-\frac{1}{4}\right|$$

$$=\left|\frac{4-x-3}{4(x+3)}\right|=\left|\frac{1-x}{4(x+3)}\right|$$

$$=\frac{|1-x|}{4|x+3|}<\frac{|1-x|}{4}<\frac{\delta}{4}$$

$$\therefore \left| \frac{x-1}{x^2+2x-3} - \frac{1}{4} \right| < \varepsilon \quad \text{where } \delta = 4\varepsilon$$

Hence, we have shown that $\lim_{x\to 1} \frac{x-1}{x^2+2x-3} = \frac{1}{4}$.

Homework:

Problem-01: Using the (ε, δ) definition of limit to show that $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$.

Problem-02: Using the (ε, δ) definition of limit to show that $\lim_{x\to 4} (2x-2) = 6$.

***Problem-03*:** A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & \textit{when } -\infty < x < 0 \\ 1+\sin x & \textit{when } 0 \le x < \pi/2 \\ 2+\left(x-\pi/2\right)^2 & \textit{when } \pi/2 < x < \infty \end{cases}$$

Test the continuity at $x = 0$ *and* $\pi/2$.

***Problem-04*:** Discuss the continuity of the function $f(x) = |x| + |x-1|$ at the point $x = 0$.

Problem-05: Test the continuity of the function $f(x) = |x-1| + |x-2|$ at the point $x = 1$.

Problem-06: Find a non-zero value for the constant k that makes $f(x) = \begin{cases} \frac{\tan(kx)}{x} & \textit{if } x < 0 \\ 3x+k^2 & \textit{if } x \ge 0 \end{cases}$ continuous at $x = 0$.

Problem-07: If $f(x) = \begin{cases} x\cos\left(1/x\right) & \textit{when } x \ne 0 \\ 0 & \textit{when } x = 0 \end{cases}$ then test the continuity at $x = 0$.

Problem-08: If $f(x) = \begin{cases} (1+x)^{1/x} & \textit{when } x \ne 0 \\ 1 & \textit{when } x = 0 \end{cases}$ then test the continuity at $x = 0$.

Problem-09: If $f(x) = \begin{cases} x & \textit{when } 0 \le x < 1/2 \\ 1-x & \textit{when } 1/2 \le x \le 1 \end{cases}$ then test the continuity at $x = 1/2$.

Problem-09: Using the (ε, δ) definition of limit to show that $\lim_{x \to 0} \frac{x+1}{2x+1} = 1$.

<u>Solution</u>**:** Here, $f(x) = \frac{x+1}{2x+1}$, $a = 0$ and $l = 1$.

We must show that for any given positive number ε, we can find a positive number $\delta(\textit{depending on}\, \varepsilon)$ such that $|f(x) - l| < \varepsilon$, whenever $0 < |x - a| < \delta$

i.e. $\left|\frac{x+1}{2x+1} - 1\right| < \varepsilon$, whenever $0 < |x - 0| < \delta$.

$$\text{Now } \left|\frac{x+1}{2x+1} - 1\right| = \left|\frac{x+1-2x-1}{2x+1}\right|$$

$$= \left|\frac{-x}{2x+1}\right|$$

$$= \left|\frac{x}{2x+1}\right| < |x| < \delta$$

$$\therefore \left|\frac{x+1}{2x-1} - 1\right| < \varepsilon \quad \text{where } \delta = \varepsilon$$

Hence, we have shown that $\lim_{x \to 0} \frac{x+1}{2x-1} = 1$.

Problem-10: Using the (ε, δ) definition of limit to show that $\lim_{x \to 2}(3x + 4) = 10$.

<u>Solution</u>**:** Here, $f(x) = 3x + 4$, $a = 2$ and $l = 10$.

We must show that for any given positive number ε, we can find a positive number $\delta(\textit{depending on}\, \varepsilon)$ such that $|f(x) - l| < \varepsilon$, whenever $0 < |x - a| < \delta$

i.e. $|3x + 4 - 10| < \varepsilon$, whenever $0 < |x - 2| < \delta$.

$$\text{Now } |3x + 4 - 10| = |3x - 6|$$

$$= 3|x - 2|$$

$$< 3\delta$$

$$\therefore |3x + 4 - 10| < \varepsilon \quad \text{where } \delta = \frac{\varepsilon}{3}$$

Hence, we have shown that $\lim_{x \to 2}(3x+4)=10$.