Equations of First Order and First Degree

Definition:

A differential equation of the type $M+N\frac{dy}{dx}=0$, where M and N are functions of x and y or constants is called a differential equation of the first order and first degree.

There are two standard forms of differential equations of first order and first degree namely

- ❖ $\frac{dy}{dx}=f(x,y)$
- ❖ $M(x,y)dx+N(x,y)dy=0$

We can classify the first order and first degree differential equation into followings eight categories according to its solution methods:

- ❖ Equations of variable separable form,
- ❖ Equations reducible to variable separable form,
- ❖ Homogeneous Equation,
- ❖ Equation reducible to Homogeneous form,
- ❖ Linear differential equation,
- ❖ Equation reducible to linear differential equation,
- ❖ Exact differential equation and
- ❖ Equation reducible to exact differential equation.

Equations of variable Separable form: An equation of the form

$$F(x)G(y)dx+f(x)g(y)dy=0$$

is called an equation with variables separable or simply a separable equation.

Mathematical Problems based on variable separable form:

Problem-01: Solve the differential equation $(1+y^2)dx+(1+x^2)dy=0$

Solution: Given differential equation is,

$$(1+y^2)dx+(1+x^2)dy=0$$

Separating the variables, we get

$$\frac{dy}{1+y^2}=-\frac{dx}{1+x^2}$$

Integrating both-sides, we find

$$\int\frac{dy}{1+y^2}=-\int\frac{dx}{1+x^2}+C$$

$$\Rightarrow \tan^{-1} y = -\tan^{-1} x + C$$
$$\Rightarrow \tan^{-1} x + \tan^{-1} y = C$$
$$\Rightarrow \tan^{-1} \frac{x+y}{1-xy} = \tan^{-1} a\,; \quad \text{where } C = \tan^{-1} a \text{ (say)}$$
$$\Rightarrow \frac{x+y}{1-xy} = a$$
$$\Rightarrow x + y = a(1-xy)$$

which is the complete or general solution of the given differential equation.

Problem-02: Solve the differential equation $\left(4+y^2\right)dx + \left(4+x^2\right)dy = 0$.

Solution: Given differential equation is,

$$\left(4+y^2\right)dx + \left(4+x^2\right)dy = 0$$

Separating the variables, we get

$$\frac{dy}{4+y^2} = -\frac{dx}{4+x^2}$$

Integrating both-sides, we find

$$\int \frac{dy}{4+y^2} = -\int \frac{dx}{4+x^2} + C$$
$$\Rightarrow \int \frac{dy}{2^2+y^2} = -\int \frac{dx}{2^2+x^2} + C$$
$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{y}{2} = -\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$
$$\Rightarrow \tan^{-1}\frac{x}{2} + \tan^{-1}\frac{y}{2} = 2C$$
$$\Rightarrow \tan^{-1}\frac{\frac{x}{2}+\frac{y}{2}}{1-\frac{x}{2}.\frac{y}{2}} = 2C$$
$$\Rightarrow \tan^{-1}\frac{\frac{x+y}{2}}{1-\frac{xy}{4}} = 2C$$
$$\Rightarrow \tan^{-1}\frac{\frac{x+y}{2}}{\frac{4-xy}{4}} = 2C$$
$$\Rightarrow \tan^{-1}\left(\frac{x+y}{2} \times \frac{4}{4-xy}\right) = 2C$$

$$\Rightarrow \tan^{-1}\frac{2(x+y)}{4-xy} = 2C$$

$$\Rightarrow \frac{2(x+y)}{4-xy} = \tan 2C = a \qquad [\ letting\ \ arbitrary\, const, \tan 2C = a\ ]$$

$$\Rightarrow \frac{2(x+y)}{4-xy} = a$$

$$\Rightarrow 2(x+y) = a(4-xy)$$

which is the complete or general solution of the given differential equation.

Problem-03: Solve the differential equation $\frac{dy}{dx} = \sqrt{1-x^2}\,\sqrt{1-y^2}$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = \sqrt{1-x^2}\,\sqrt{1-y^2}$$

Separating variables, we get

$$\frac{dy}{\sqrt{1-y^2}} = \sqrt{1-x^2}\,dx$$

Integrating both-sides we find,

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \sqrt{1-x^2}\,dx + C$$

$$\Rightarrow \sin^{-1} y = \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x + C$$

$$\Rightarrow y = \sin\left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x + C\right)$$

which is the complete or general solution of the given differential equation.

Problem-04: Solve the differential equation $\left(e^y+1\right)\cos x\,dx + e^y(1+\sin x)\,dy = 0$.

Solution: Given differential equation is,

$$\left(e^y+1\right)\cos x\,dx + e^y(1+\sin x)\,dy = 0$$

Separating the variables, we get

$$e^y(1+\sin x)\,dy = -\left(e^y+1\right)\cos x\,dx$$

$$\Rightarrow \frac{e^y dy}{1+e^y} = -\frac{\cos x\,dx}{1+\sin x}$$

Integrating both-sides, we find

$$\int \frac{e^y dy}{1+e^y} = -\int \frac{\cos x\,dx}{1+\sin x} + C$$

$$\Rightarrow \ln\left|1+e^y\right| = -\ln\left|1+\sin x\right| + C \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right|\right]$$

$$\Rightarrow \ln(1+\sin x) + \ln(1+e^y) = C$$
$$\Rightarrow \ln(1+\sin x)(1+e^y) = C$$
$$\Rightarrow (1+\sin x)(1+e^y) = e^C = a \quad \left[say\ e^C = a\right]$$
$$\Rightarrow (1+\sin x)(1+e^y) = a$$

which is the complete or general solution of the given differential equation.

Problem-05: Solve the differential equation $\frac{dy}{dx} = e^{x+y} + x^2e^{-y}$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = e^{x+y} + x^2e^{-y}$$

Separating the variables, we get

$$\frac{dy}{dx} = e^{x+y} + \frac{x^2}{e^y}$$
$$\Rightarrow e^y dy = \left(e^x + x^2\right)dx$$

Integrating both-sides, we find

$$\int e^y dy = \int \left(e^x + x^2\right)dx + C$$
$$\Rightarrow e^y = e^x + \frac{x^3}{3} + C$$
$$\Rightarrow y = \ln\left(e^x + \frac{x^3}{3} + C\right)$$

which is the complete or general solution of the given differential equation.

Exercise:

Solve the following differential equations:

1. $\frac{dy}{dx} = \sqrt{1+x^2}\,\sqrt{1-y^2}$
2. $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$
3. $e^{x-y}dx + e^{y-x}dy = 0$
4. $(1+\cos ecx)dy = dx$
5. $\frac{dy}{dx} = \sqrt{1-y^2}\,\cos 5x\, e^{3x}$
6. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

7. $\left(e^{y}+1\right)\cos x\,dx+e^{y}\sin x dy=0$
8. $3e^{x}\tan y dx+(1-e^{x})\sec^{2} y dy=0$
9. $(1-x^{2})(1-y)dx=xy(1+y)dy$
10. $y-x\frac{dy}{dx}=a\left(y^{2}+\frac{dy}{dx}\right)$

Equations reducible to variable separable form:

- Equation of the form $\frac{dy}{dx}=f(ax+by+c)$ can be reduced to variable separable form by choosing the transformation $ax+by+c=v$.

Mathematical Problems based on reducible to variable separable form.

Problem-01: Solve the differential equation $\frac{dy}{dx}=\left(4x+y+1\right)^{2}$.

Solution: Given differential equation is,

$$\frac{dy}{dx}=\left(4x+y+1\right)^{2} \dots \dots \dots (1)$$

Let, $4x+y+1=v$

$$\therefore \quad 4+\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=\frac{dv}{dx}-4$$

Now from equation (1), we get

$$\frac{dv}{dx}-4=v^{2}$$

$$\Rightarrow\frac{dv}{dx}=4+v^{2}$$

$$\Rightarrow\frac{dv}{4+v^{2}}=dx$$

Integrating both sides, we find

$$\int\frac{dv}{4+v^{2}}=\int dx+C$$

$$\Rightarrow\int\frac{dv}{2^{2}+v^{2}}=\int dx+C$$

$$\Rightarrow\frac{1}{2}\tan^{-1}\frac{v}{2}=x+C$$

$$\Rightarrow\tan^{-1}\frac{v}{2}=2x+2C$$

$$\Rightarrow\frac{v}{2}=\tan\left(2x+2C\right)$$

$$\Rightarrow v = 2\tan(2x+2C)$$

$$\therefore\ 4x+y+1 = 2\tan(2x+2C)$$

which is the complete integral or general solution of the given differential equation.

Problem-02: Solve the differential equation $\frac{dy}{dx} = \tan(x+y+6)$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = \tan(x+y+6) \cdots\cdots\cdots (1)$$

Let, $x+y+6 = v$

$$\therefore\ 1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now from equation (1), we get

$$\frac{dv}{dx} - 1 = \tan v$$

$$\Rightarrow \frac{dv}{dx} = 1+\tan v$$

$$\Rightarrow \frac{dv}{1+\tan v} = dx$$

Integrating both sides, we get

$$\int \frac{dv}{1+\tan v} = \int dx + C$$

$$\Rightarrow \int \frac{\cos v}{\sin v+\cos v} dv = \int dx + C$$

$$\Rightarrow \int \frac{\frac{1}{2}(\sin v+\cos v)+\frac{1}{2}(\cos v-\sin v)}{\sin v+\cos v} dv = \int dx + C$$

$$\Rightarrow \frac{1}{2}\int dv + \frac{1}{2}\int \frac{(\cos v-\sin v)}{\sin v+\cos v} dv = \int dx + C$$

$$\Rightarrow \frac{1}{2}v + \frac{1}{2}\ln|\sin v+\cos v| = x + C$$

$$\therefore (x+y+6) + \ln|\sin(x+y+6)+\cos(x+y+6)| = 2x+2C$$

which is the complete integral or general solution of the given differential equation.

Problem-03: Solve the differential equation $\frac{dy}{dx} = \sin(x+y)+\cos(x+y)$.

Solution: Given differential equation is,

$$\frac{dy}{dx}=\sin(x+y)+\cos(x+y)\ \ldots\ldots\ldots\ (1)$$

Let, $x+y=v$

$$\therefore\ \ 1+\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=\frac{dv}{dx}-1$$

Now from equation (1), we get

$$\frac{dv}{dx}-1=\sin v+\cos v$$

$$\Rightarrow\frac{dv}{dx}=\sin v+\cos v+1$$

$$\Rightarrow\frac{dv}{\sin v+\cos v+1}=dx$$

$$\Rightarrow\frac{dv}{2\sin\frac{v}{2}\cos\frac{v}{2}+2\cos^2\frac{v}{2}}=dx$$

$$\Rightarrow\frac{dv}{2\cos^2\frac{v}{2}\left(1+\frac{\sin\frac{v}{2}}{\cos\frac{v}{2}}\right)}=dx$$

$$\Rightarrow\frac{\sec^2\frac{v}{2}dv}{2\left(1+\tan\frac{v}{2}\right)}=dx$$

Integrating both sides, we get

$$\frac{1}{2}\int\frac{\sec^2\frac{v}{2}}{\left(1+\tan\frac{v}{2}\right)}dv=\int dx+C$$

$$\Rightarrow \ln\left|1+\tan\frac{v}{2}\right| = x + C$$

$$\Rightarrow \ln\left|1+\tan\frac{(x+y)}{2}\right| = x + C$$

which is the complete integral or general solution of the given differential equation.

Problem-04: Solve the differential equation $(x+y)^2\frac{dy}{dx} = a^2$.

Solution: Given differential equation is,

$$(x+y)^2\frac{dy}{dx} = a^2 \ldots \ldots \ldots (1)$$

Let, $x + y = v$

$$\therefore \quad 1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now from equation (1), we get

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2$$

$$\Rightarrow \frac{dv}{dx} - 1 = \frac{a^2}{v^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{a^2}{v^2} + 1$$

$$\Rightarrow \frac{dv}{dx} = \frac{a^2+v^2}{v^2}$$

$$\Rightarrow \frac{v^2}{a^2+v^2}dv = dx$$

Integrating both sides, we get

$$\int \frac{v^2}{a^2+v^2}dv = \int dx + C$$

$$\Rightarrow \int \frac{a^2+v^2-a^2}{a^2+v^2}dv = \int dx + C$$

$$\Rightarrow \int \frac{a^2+v^2}{a^2+v^2}\,dv - \int \frac{a^2}{a^2+v^2}dv = \int dx + C$$

$$\Rightarrow \int dv - \int \frac{a^2}{a^2+v^2}dv = \int dx + C$$

$$\Rightarrow v - a^2.\frac{1}{a}\tan^{-1}\frac{v}{a} = x + C$$

$$\Rightarrow x + y - a\tan^{-1}\frac{(x+y)}{a} = x + C$$

$$\therefore\ \ y = a\tan^{-1}\frac{(x+y)}{a} + C$$

which is the complete integral or general solution of the given differential equation.

Exercise:

Solve the following differential equations:

1. $\frac{dy}{dx} = (x+y)^2$
2. $(x-y)^2\frac{dy}{dx} = a^2$
3. $\frac{dy}{dx} = \sin(2x-3y+5)$
4. $\sin^{-1}\left(\frac{dy}{dx}\right) = (x+y)$

Homogeneous Differential Equation:

An equation of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

In which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree is called homogeneous equation.

It can be reduced to an equation in which variables are separable by choosing

$$y = vx.$$

Mathematical Problems based on homogeneous equation:

Problem-01: Solve the differential equation $\left(x^2+y^2\right)dx+2xydy=0$.

Solution: Given differential equation is,

$$\left(x^2+y^2\right)dx+2xydy=0 \text{ ... ... (1)}$$

Equation (1) can be written as,

$$\left(x^2+y^2\right)dx+2xydy=0$$

$$\Rightarrow 2xydy=-\left(x^2+y^2\right)dx$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left(x^2+y^2\right)}{2xy} \text{ ... ... (2)}$$

This is a homogeneous equation.

Put, $y=vx$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2) we get

$$\Rightarrow v+x\frac{dv}{dx}=-\frac{\left(x^2+v^2x^2\right)}{2vx^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=-\frac{\left(1+v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=-v-\frac{\left(1+v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-2v^2-\left(1+v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-\left(1+3v^2\right)}{2v}$$

$$\Rightarrow \frac{2v}{\left(1+3v^2\right)}dv=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{2v}{\left(1+3v^2\right)}dv=-\int\frac{dx}{x}+C$$

$$\Rightarrow \frac{1}{3}\int\frac{6v}{\left(1+3v^2\right)}dv=-\int\frac{dx}{x}+C$$

$$\Rightarrow \frac{1}{3}\ln\left|1+3v^2\right|=-\ln\left|x\right|+C$$

$$\Rightarrow \ln x+\ln\left(1+3v^2\right)^{1/3}=C$$

$$\Rightarrow \ln x\left(1+3v^2\right)^{1/3}=C$$

$$\Rightarrow x\left(1+3v^2\right)^{1/3}=e^C$$

$$\Rightarrow x\left(1+3\frac{y^{\,2}}{x^2}\right)^{1/3}=e^C=a \quad \left[e^C=a\ (say)\right]$$

$$\therefore \quad x\left(1+3\frac{y^{\,2}}{x^2}\right)^{1/3}=a$$

which is the complete integral or general solution of the given differential equation.

Problem-02: Solve the differential equation $x^2ydx-\left(x^3+y^3\right)dy=0$.

Solution: Given differential equation is,

$$x^2ydx-\left(x^3+y^3\right)dy=0 \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$x^2ydx-\left(x^3+y^3\right)dy=0$$

$$\Rightarrow \left(x^3+y^3\right)dy=x^2ydx$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2y}{\left(x^3+y^3\right)} \quad \ldots\ldots\ldots (2)$$

This is a homogeneous equation.

Let, $y=vx$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2), we get

$$v+x\frac{dv}{dx}=\frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v}{1+v^3}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-v-v^4}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$$

$$\Rightarrow \frac{1+v^3}{v^4}dv=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{1+v^3}{v^4}dv=-\int\frac{dx}{x}+c$$

$$\Rightarrow \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int\frac{dx}{x}+c$$

$$\Rightarrow -\frac{1}{3v^3}+\ln v = -\ln x + c$$

$$\Rightarrow \ln v + \ln x = \frac{1}{3v^3}+c$$

$$\Rightarrow \ln vx = \frac{1}{3v^3}+c$$

$$\therefore\ \ln y = \frac{x^3}{3y^3}+c$$

which is the complete integral or general solution of the given differential equation.

Problem-03: Solve the differential equation $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$.

Solution: Given differential equation is,

$$y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx} \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$$

$$\Rightarrow \left(xy-x^2\right)\frac{dy}{dx}=y^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2}{\left(xy-x^2\right)} \quad \ldots\ldots\ldots (2)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore\ \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2), we get

$$v+x\frac{dv}{dx}=\frac{v^2x^2}{vx^2-x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v^2}{v-1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2}{v-1}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-v^2+v}{v-1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v}{v-1}$$

$$\Rightarrow \frac{v-1}{v}dv=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{v-1}{v}dv=\int\frac{dx}{x}+c$$

$$\Rightarrow \int\left(1-\frac{1}{v}\right)dv=\int\frac{dx}{x}+c$$

$$\Rightarrow v-\ln v=\ln x+c$$

$$\Rightarrow \ln v+\ln x=v-c$$

$$\Rightarrow \ln vx=v+\ln c_1\ ;\ \left[\ln c_1=-c\quad (say)\right]$$

$$\Rightarrow \ln vx=\ln e^v+\ln c_1$$

$$\Rightarrow \ln vx=\ln c_1e^v$$

$$\Rightarrow vx=c_1e^v$$

$$\therefore\ \ y=c_1e^{y/x}$$

which is the complete integral or general solution of the given differential equation.

Problem-04: Solve the differential equation $\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$.

Solution: Given differential equation is,

$$\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0 \quad \dots\dots\dots (1)$$

Equation (1) can be written as,

$$\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y(x+y)}{x^2} \quad \dots\dots\dots (2)$$

This is a homogeneous equation.

Let, $y=vx$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2), we get

$$v+x\frac{dv}{dx}=\frac{vx(x+vx)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=v(1+v)$$

$$\Rightarrow x\frac{dv}{dx}=v+v^2-v$$

$$\Rightarrow x\frac{dv}{dx}=v^2$$

$$\Rightarrow \frac{1}{v^2}dv=\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1}{v^2}dv = \int \frac{dx}{x} + c$$

$$\Rightarrow -\frac{1}{v} = \ln x + c$$

$$\Rightarrow -\frac{x}{y} = \ln x + \ln c_1 \quad ;\left[\ln c_1 = c\ (say)\right]$$

$$\Rightarrow -\frac{x}{y} = \ln c_1 x$$

$$\Rightarrow \frac{x}{y} = -\ln c_1 x$$

$$\Rightarrow \frac{x}{y} = \ln\left(c_1 x\right)^{-1}$$

$$\Rightarrow \frac{x}{y} = \ln\left(\frac{1}{c_1 x}\right)$$

$$\therefore \quad y = \frac{x}{\ln\left(\frac{1}{c_1 x}\right)}$$

which is the complete integral or general solution of the given differential equation.

Problem-05: Solve the differential equation $x\frac{dy}{dx} - y = \sqrt{\left(x^2 + y^2\right)}$.

Solution: Given differential equation is,

$$x\frac{dy}{dx} - y = \sqrt{\left(x^2 + y^2\right)} \ \dots\dots\dots (1)$$

Equation (1) can be written as,

$$x\frac{dy}{dx} - y = \sqrt{\left(x^2 + y^2\right)}$$

$$\Rightarrow x\frac{dy}{dx} = y + \sqrt{\left(x^2 + y^2\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{\left(x^2 + y^2\right)}}{x} \quad \ldots \ldots \ldots (2)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (2), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{\left(x^2 + v^2x^2\right)}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx + x\sqrt{\left(1 + v^2\right)}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{\left(1 + v^2\right)}$$

$$\Rightarrow x\frac{dv}{dx} = v + \sqrt{\left(1 + v^2\right)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{\left(1 + v^2\right)}$$

$$\Rightarrow \frac{dv}{\sqrt{\left(1 + v^2\right)}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{\left(1 + v^2\right)}} = \int \frac{dx}{x} + c$$

$$\Rightarrow \ln\left|v + \sqrt{1 + v^2}\right| = \ln x + c$$

$$\Rightarrow \ln\left|v + \sqrt{1 + v^2}\right| = \ln x + \ln c_1 \ ; \left[\ln c_1 = c\right]$$

$$\Rightarrow \ln\left|v+\sqrt{1+v^2}\right| = \ln c_1 x$$

$$\Rightarrow v+\sqrt{1+v^2} = c_1 x$$

$$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}} = c_1 x$$

$$\Rightarrow \frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x} = c_1 x$$

$$\Rightarrow y+\sqrt{x^2+y^2} = c_1 x^2$$

$$\Rightarrow \sqrt{x^2+y^2} = c_1 x^2 - y$$

$$\Rightarrow x^2+y^2 = \left(c_1 x^2 - y\right)^2$$

which is the complete integral or general solution of the given differential equation.

Problem-06: Solve the differential equation

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}.$$

Solution: Given differential equation is,

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx} \quad \dots\dots\dots (1)$$

Equation (1) can be written as,

$$\Rightarrow \left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx} = \left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)}{x\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)} \quad \dots\dots\dots (2)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (2), we get

$$v + x\frac{dv}{dx} = \frac{vx\left(x\cos\frac{vx}{x} + vx\sin\frac{vx}{x}\right)}{x\left(vx\sin\frac{vx}{x} - x\cos\frac{vx}{x}\right)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{(v\sin v - \cos v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{(v\sin v - \cos v)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{(v\sin v - \cos v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = 2\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x} + c$$

$$\Rightarrow \ln|\sec v| - \ln v = 2\ln x + c$$

$$\Rightarrow \ln\left|\frac{\sec v}{v}\right| = 2\ln x + \ln c_1 \quad ; \left[\ln c_1 = c \ \ (say)\right]$$

$$\Rightarrow \ln\left|\frac{\sec v}{v}\right| = \ln c_1 x^2$$

$$\Rightarrow \frac{\sec v}{v} = c_1 x^2$$

$$\Rightarrow \frac{x}{y} \sec\left(\frac{y}{x}\right) = c_1 x^2$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = c_1 xy$$

which is the complete integral or general solution of the given differential equation.

Exercise:

Solve the following differential equations:

1. $\left(3x^2 + y^2\right)dy + \left(x^2 + 3y^2\right)dx = 0$
2. $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$
3. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$
4. $y\left(y^2 - 2x^2\right)dx + x\left(2y^2 - x^2\right)dy = 0$

Equation Reducible to Homogeneous Form:

An equation of the type

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \quad \ldots\ldots\ldots (1)$$

can be reduced to homogeneous form as follows:

❖ Case -01: If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ then putting $x = X + h$, $y = Y + h$ and $\dfrac{dy}{dx} = \dfrac{dY}{dX}$ in equation (1) we get

$$\frac{dY}{dX} = \frac{a_1 X + b_1 Y + \left(a_1 h + b_1 k + c_1\right)}{a_2 X + b_2 Y + \left(a_2 h + b_2 k + c_2\right)}$$

we choose the constants h and k in such a way that,

$$a_1 h + b_1 k + c_1 = 0 \text{ and } a_2 h + b_2 k + c_2 = 0$$

with this substitution the differential equation reduces to

$$\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

which is a homogeneous equation in X , Y and can be solved by putting $Y=vX$.

❖ Case-02: If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{m}$ (say), then the differential equation can be written as,

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m\left(a_1x+b_1y\right)+c_2}$$

put $a_1x+b_1y=v$ so that $\frac{dy}{dx}=\frac{1}{b_1}\left(\frac{dv}{dx}-a_1\right)$

the above equation becomes

$$\frac{1}{b_1}\left(\frac{dv}{dx}-a_1\right)=\frac{v+c}{mv+c}$$

which is in variables separable form.

Mathematical problem based on reducible to homogeneous form:

Problem-01: Solve the differential equation $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Solution: Given differential equation is,

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3} \text{ ... . (1)}$$

put $x=X+h$ and $y=Y+k$ where h, k are constants.

$$\therefore \frac{dy}{dx}=\frac{dY}{dX}$$

Then the equation (1) becomes,

$$\frac{dY}{dX}=\frac{X+2Y+\left(h+2k-3\right)}{2X+Y+\left(2h+k-3\right)} \text{ ... ... (2)}$$

Now choose

$$h+2k-3=0 \ldots \ldots \ldots (3)$$

$$2h+k-3=0 \ldots \ldots \ldots (4)$$

Solving equations (3) and (4) we get,

$$h=1 \text{ and } k=1$$

with this substitution equation (2) becomes,

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y} \ldots \ldots \ldots (5)$$

which is a homogeneous equation in X and Y.

So put, $Y=vX$

$$\therefore \frac{dY}{dX}=v+X\frac{dv}{dX}$$

From equation (5), we have

$$v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}$$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{1+2v}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v-2v-v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{2+v}{1-v^2}dv=\frac{dX}{X}$$

Integrating both sides, we get

$$\int \frac{2+v}{1-v^2}dv = \int \frac{dX}{X}$$

$$\Rightarrow \int \frac{2+v}{(1+v)(1-v)}dv = \int \frac{dX}{X}$$

$$\Rightarrow \int \left(\frac{1}{2}.\frac{1}{1+v} + \frac{3}{2}.\frac{1}{1-v} \right) dv = \int \frac{dX}{X}$$

$$\Rightarrow \frac{1}{2}\int \frac{dv}{1+v} + \frac{3}{2}\int \frac{dv}{1-v} = \int \frac{dX}{X}$$

$$\Rightarrow \frac{1}{2}\ln(1+v) - \frac{3}{2}\ln(1-v) = \ln X + \ln c$$

$$\Rightarrow \ln\sqrt{1+v} + \ln\frac{1}{\sqrt{(1-v)^3}} = \ln cX$$

$$\Rightarrow \ln\frac{\sqrt{1+v}}{\sqrt{(1-v)^3}} = \ln cX$$

$$\Rightarrow \frac{\sqrt{1+v}}{\sqrt{(1-v)^3}} = cX$$

$$\Rightarrow \frac{1+v}{(1-v)^3} = CX^2 \quad ;\ let,\ c^2 = C$$

$$\Rightarrow 1+v = CX^2(1-v)^3$$

$$\Rightarrow 1+\frac{Y}{X} = CX^2\left(1-\frac{Y}{X}\right)^3$$

$$\Rightarrow X+Y = C(X-Y)^3$$

$$\Rightarrow x-1+y-1 = C(x-1-y+1)^3$$

$$\Rightarrow x+y-2 = C(x-y)^3$$

which is the required solution.

Problem-02: Solve the differential equation $(3x-7y-3)\frac{dy}{dx}=(3y-7x+7)$

Solution: Given differential equation is,

$$(3x-7y-3)\frac{dy}{dx}=(3y-7x+7)$$

$$\Rightarrow \frac{dy}{dx}=\frac{(3y-7x+7)}{(3x-7y-3)} \quad \ldots \ldots \ldots (1)$$

put $x=X+h$ and $y=Y+k$ where h, k are constants.

$$\therefore \ \frac{dy}{dx}=\frac{dY}{dX}$$

Then the equation (1) becomes,

$$\frac{dY}{dX}=\frac{3Y-7X+(-7h+3k+7)}{3X-7Y+(3h-7k-3)} \quad \ldots \ldots \ldots (2)$$

Now choose

$$-7h+3k+7=0 \ldots \ldots \ldots (3)$$

$$3h-7k-3=0 \ldots \ldots \ldots (4)$$

Solving equations (3) and (4) we get,

$$h=1 \text{ and } k=0$$

with this substitution equation (2) becomes,

$$\frac{dY}{dX}=\frac{3Y-7X}{3X-7Y} \quad \ldots \ldots \ldots (5)$$

which is a homogeneous equation in X and Y.

So put, $Y=vX$

$$\therefore \ \frac{dY}{dX}=v+X\frac{dv}{dX}$$

From equation (5), we have

$$v+X\frac{dv}{dX}=\frac{3vX-7X}{3X-7vX}$$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{3v-7}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{3v-7}{3-7v}-v$$

$$\Rightarrow X\frac{dv}{dX}=\frac{3v-7-3v+7v^2}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-7+7v^2}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-7\left(1-v^2\right)}{3-7v}$$

$$\Rightarrow \frac{3-7v}{1-v^2}dv=-7\frac{dX}{X}$$

Integrating both sides, we get

$$\int\frac{3-7v}{1-v^2}dv=-7\int\frac{dX}{X}$$

$$\Rightarrow \int\frac{3-7v}{(1+v)(1-v)}dv=-7\int\frac{dX}{X}$$

$$\Rightarrow \int\left(\frac{5}{1+v}-\frac{2}{1-v}\right)dv=-7\int\frac{dX}{X}$$

$$\Rightarrow 5\int\frac{dv}{1+v}-2\int\frac{dv}{1-v}=-7\int\frac{dX}{X}$$

$$\Rightarrow 5\ln(1+v)+2\ln(1-v)=-7\ln X+\ln c$$

$$\Rightarrow \ln(1+v)^5+\ln(1-v)^2=-7\ln X+\ln c$$

$$\Rightarrow \ln(1+v)^5(1-v)^2=\ln cX^{-7}$$

$$\Rightarrow (1+v)^5(1-v)^2=\frac{c}{X^7}$$

$$\Rightarrow\left(1+\frac{Y}{X}\right)^5\left(1-\frac{Y}{X}\right)^2=\frac{c}{X^7}\qquad ;as\ \ Y=vX$$

$$\Rightarrow\left(X+Y\right)^5\left(X-Y\right)^2=c$$

$$\Rightarrow\left(x+y-1\right)^5\left(x-y-1\right)^2=c\qquad ;as\ \ x=X+1\ \ and\ \ y=Y+0$$

which is the required solution.

Problem-03: Solve the differential equation $\left(2x-2y+5\right)\frac{dy}{dx}=x-y+3$

Solution: Given differential equation is,

$$\left(2x-2y+5\right)\frac{dy}{dx}=x-y+3$$

$$\Rightarrow\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}$$

$$\Rightarrow\frac{dy}{dx}=\frac{x-y+3}{2\left(x-y\right)+5}\ \ \ldots\ \ldots.\ \ldots\ (1)$$

put $\quad x-y=v$

$$\therefore\ 1-\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=1-\frac{dv}{dx}$$

Then the equation (1) becomes,

$$1-\frac{dv}{dx}=\frac{v+3}{2v+5}$$

$$\Rightarrow\frac{dv}{dx}=1-\frac{v+3}{2v+5}$$

$$\Rightarrow\frac{dv}{dx}=\frac{2v+5-v-3}{2v+5}$$

$$\Rightarrow\frac{dv}{dx}=\frac{2+v}{2v+5}$$

$$\Rightarrow \frac{2v+5}{2+v}dv = dx$$

Integrating both sides, we get

$$\int \frac{2v+5}{2+v}dv = \int dx$$

$$\Rightarrow \int \frac{2(2+v)+1}{2+v}dv = \int dx$$

$$\Rightarrow \int \left(2+\frac{1}{2+v}\right)dv = \int dx$$

$$\Rightarrow 2v + \ln(2+v) = x + c$$

$$\Rightarrow 2(x-y) + \ln(x-y+2) = x + c$$

$$\Rightarrow x - 2y + \ln(x-y+2) = c$$

which is the required solution.

Problem-04: Solve the differential equation $(3x-2y+1)\frac{dy}{dx} = 6x-4y+3$

Solution: Given differential equation is,

$$(3x-2y+1)\frac{dy}{dx} = 6x-4y+3$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(3x-2y)+3}{3x-2y+1} \quad \dots \dots \dots (1)$$

put $3x-2y = v$

$$\therefore\ 3 - 2\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(3-\frac{dv}{dx}\right)$$

Then the equation (1) becomes,

$$\frac{1}{2}\left(3-\frac{dv}{dx}\right)=\frac{2v+3}{v+1}$$

$$\Rightarrow 3-\frac{dv}{dx}=\frac{4v+6}{v+1}$$

$$\Rightarrow \frac{dv}{dx}=3-\frac{4v+6}{v+1}$$

$$\Rightarrow \frac{dv}{dx}=\frac{3v+3-4v-6}{v+1}$$

$$\Rightarrow \frac{dv}{dx}=\frac{-v-3}{v+1}$$

$$\Rightarrow \frac{v+1}{v+3}\frac{dv}{dx}=-dx$$

Integrating both sides, we get

$$\int\frac{v+1}{v+3}dv=-\int dx$$

$$\Rightarrow \int\frac{(v+3)-2}{v+3}dv=-\int dx$$

$$\Rightarrow \int\left(1-\frac{2}{v+3}\right)dv=-\int dx$$

$$\Rightarrow (3x-2y)-2\ln(3x-2y+3)=-x+c$$

$$\Rightarrow 4x-2y-2\ln(3x-2y+3)=c$$

which is the required solution.

Exercise: Solve the following problems:

1. $(2x+y+3)\dfrac{dy}{dx}=x+2y+3$
2. $(2x+y+1)dx+(4x+2y-1)dy=0$

3. $\frac{dy}{dx}=\frac{y-x+3}{y-x-5}$
4. $(2x-5y+3)dx-(2x+4y-6)dy=0$.

Linear Differential Equation: A differential equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x or constants, is called the linear differential equation of the first order.

Note: To solve this equation, multiply both the sides by the following integrating factor.

$$I.F=e^{\int P(x)}dx$$

Mathematical problem based on linear differential equations:

Problem-01: Solve the differential equation $(1-x^2)\frac{dy}{dx}-xy=1$.

Solution: Given differential equation is,

$$(1-x^2)\frac{dy}{dx}-xy=1 \text{... ... ... (1)}$$

Equation (1) can be written as,

$$(1-x^2)\frac{dy}{dx}-xy=1$$

$$\Rightarrow \frac{dy}{dx}-\frac{x}{(1-x^2)}y=\frac{1}{(1-x^2)} \text{... ... ... (2)}$$

This is a linear equation of first order.

$$\text{I.F}=e^{\int \frac{-x}{1-x^2}dx}$$

$$=e^{\frac{1}{2}\ln(1-x^2)}$$

$$= e^{\ln\left(1-x^2\right)^{\frac{1}{2}}}$$

$$= \left(1-x^2\right)^{\frac{1}{2}}$$

$$= \sqrt{1-x^2}$$

Multiply both sides of equation (2) by $\sqrt{1-x^2}$, we get

$$\sqrt{1-x^2}\,\frac{dy}{dx} - \frac{x\sqrt{1-x^2}}{\left(1-x^2\right)}\,y = \frac{\sqrt{1-x^2}}{\left(1-x^2\right)}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}\,y = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}}$$

Integrating both sides, we get

$$y\sqrt{1-x^2} = \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$\Rightarrow y\sqrt{1-x^2} = \sin^{-1}\left(x\right) + c$$

which is the required solution.

Problem-02: Solve the differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$.

Solution: Given differential equation is,

$$x\frac{dy}{dx} + 2y = x^2 \log x \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}\,y = x\log x \quad \ldots\ldots\ldots (2)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{2}{x}dx}$$
$$= e^{2\ln x}$$
$$= e^{\ln x^2}$$
$$= x^2$$

Multiply both sides of equation (2) by x^2, we get

$$x^2 \frac{dy}{dx} + 2xy = x^3 \log x$$
$$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = x^3 \log x$$

Integrating both sides, we get

$$x^2 y = \int x^3 \log x + c$$
$$\Rightarrow x^2 y = \log x \int x^3 dx - \int \left\{ \frac{d}{dx}(\log x) \int x^3 dx \right\} dx + c$$
$$\Rightarrow x^2 y = \frac{x^4}{4}\log x - \int \frac{1}{x}.\frac{x^4}{4} dx + c$$
$$\Rightarrow x^2 y = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx + c$$
$$\Rightarrow x^2 y = \frac{x^4}{4}\log x - \frac{1}{4}.\frac{x^4}{4} + c$$
$$\Rightarrow x^2 y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$
$$\Rightarrow y = \frac{x^2}{4}\log x - \frac{x^2}{16} + cx^{-2}$$

which is the required solution.

Problem-03: Solve the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$.

Solution: Given differential equation is,

$$\frac{dy}{dx}+2y\tan x=\sin x \quad \ldots\ldots\ldots (1)$$

This is a linear equation of first order.

$$\text{I.F}=e^{\int 2\tan x dx}$$
$$=e^{2\ln(\sec x)}$$
$$=e^{\ln(\sec^2 x)}$$
$$=\sec^2 x$$

Multiply both sides of equation (1) by $\sec^2 x$, we get

$$\sec^2 x\frac{dy}{dx}+2y\sec^2 x\tan x=\sin x\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\left(y\sec^2 x\right)=\sec x\tan x$$

Integrating both sides, we get

$$y\sec^2 x=\int \sec x\tan x dx+c$$

$$\Rightarrow y\sec^2 x=\sec x+c$$

$$\Rightarrow y=\frac{1}{\sec x}+\frac{c}{\sec^2 x}$$

$$\Rightarrow y=\cos x+c\cos^2 x$$

which is the required solution.

Problem-04: Solve the differential equation $x\frac{dy}{dx}-2y=x^2+\sin\frac{1}{x^2}$.

Solution: Given differential equation is,

$$x\frac{dy}{dx}-2y=x^2+\sin\frac{1}{x^2} \quad \dots\dots\dots (1)$$

The equation (1) can be written as,

$$x\frac{dy}{dx}-2y=x^2+\sin\frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx}-\frac{2}{x}y=x+\frac{1}{x}\sin\frac{1}{x^2} \quad \dots\dots\dots (2)$$

This is a linear equation of first order.

$$\text{I.F}=e^{\int\frac{-2}{x}dx}$$

$$=e^{-2\ln x}$$

$$=e^{\ln x^{-2}}$$

$$=\frac{1}{x^2}$$

Multiply both sides of equation (2) by $\frac{1}{x^2}$, we get

$$\frac{1}{x^2}\frac{dy}{dx}-\frac{2}{x^3}y=\frac{1}{x}+\frac{1}{x^3}\sin\frac{1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^2}\right)=\frac{1}{x}+\frac{1}{x^3}\sin\frac{1}{x^2}$$

Integrating both sides, we get

$$\frac{y}{x^2}=\int\frac{dx}{x}+\int\frac{1}{x^3}\sin\frac{1}{x^2}dx+c$$

$$\Rightarrow \frac{y}{x^2}=\ln x+\int\frac{1}{x^3}\sin\frac{1}{x^2}dx+c$$

$$\Rightarrow \frac{y}{x^2}=\ln x-\frac{1}{2}\int\sin t\,dt+c \qquad ;\ \textit{putting}\ \frac{1}{x^2}=t$$

$$\Rightarrow \frac{y}{x^2}=\ln x+\frac{1}{2}\cos t+c$$

$$\Rightarrow \frac{y}{x^2} = \ln x + \frac{1}{2}\cos\frac{1}{x^2} + c$$

$$\Rightarrow y = x^2 \ln x + \frac{x^2}{2}\cos\frac{1}{x^2} + c\,x^2$$

which is the required solution.

Problem-05: Solve the differential equation $\frac{dy}{dx} - 2y\cos x = -2\sin 2x$.

Solution: Given differential equation is,

$$\frac{dy}{dx} - 2y\cos x = -2\sin 2x \quad \dots\dots\dots (1)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int -2\cos x dx}$$

$$= e^{-2\sin x}$$

Multiply both sides of equation (1) by $e^{-2\sin x}$, we get

$$e^{-2\sin x}\frac{dy}{dx} - 2ye^{-2\sin x}\cos x = -2e^{-2\sin x}\sin 2x$$

$$\Rightarrow \frac{d}{dx}\left(ye^{-2\sin x}\right) = -2e^{-2\sin x}\sin 2x$$

Integrating both sides, we get

$$ye^{-2\sin x} = -2\int e^{-2\sin x}\sin 2x dx + c$$

$$\Rightarrow ye^{-2\sin x} = -2\int e^{-2\sin x}.2\sin x\cos x dx + c$$

$$\Rightarrow ye^{-2\sin x} = -\int te^t dt + c \qquad ;\, putting\;\; -2\sin x = t$$

$$\Rightarrow ye^{-2\sin x} = -\left[t\int e^t dt - \int\left\{\frac{d}{dt}(t)\int e^t dt\right\}dt\right] + c$$

$$\Rightarrow ye^{-2\sin x} = -te^t + \int e^t dt + c$$

$$\Rightarrow ye^{-2\sin x} = -te^{t} + e^{t} + c$$

$$\Rightarrow ye^{-2\sin x} = -e^{t}\left(t-1\right) + c$$

$$\Rightarrow ye^{-2\sin x} = -e^{-2\sin x}\left(-2\sin x - 1\right) + c$$

$$\Rightarrow y = 2\sin x + 1 + c\,e^{2\sin x}$$

which is the required solution.

Problem-06: Solve the differential equation $\left(1+y^2\right)dx + \left(x - \tan^{-1} y\right)dy = 0$.

Solution: Given differential equation is,

$$\left(1+y^2\right)dx + \left(x - \tan^{-1} y\right)dy = 0 \text{ ... ... (1)}$$

The equation (1) can be written as,

$$\left(1+y^2\right)dx + \left(x - \tan^{-1} y\right)dy = 0$$

$$\Rightarrow \left(1+y^2\right)dx = -\left(x - \tan^{-1} y\right)dy$$

$$\Rightarrow \left(1+y^2\right)\frac{dx}{dy} = -x + \tan^{-1} y$$

$$\Rightarrow \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{\tan^{-1} y}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \text{ ... ... (2)}$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1} y}$$

Multiply both sides of equation (2) by $e^{\tan^{-1} y}$, we get

$$e^{\tan^{-1} y} \frac{dx}{dy} + \frac{x}{1+y^2} e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}$$

$$\Rightarrow \frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}$$

Integrating both sides, we get

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy + c$$

$$\Rightarrow xe^{\tan^{-1} y} = \int te^t dt + c \qquad ; \; putting \; \tan^{-1} y = t$$

$$\Rightarrow xe^{\tan^{-1} y} = e^t (t-1) + c$$

$$\Rightarrow xe^{\tan^{-1} y} = e^{\tan^{-1} y} \left(\tan^{-1} y - 1\right) + c$$

$$\Rightarrow x = \left(\tan^{-1} y - 1\right) + ce^{-\tan^{-1} y}$$

which is the required solution.

Problem-07: Solve the differential equation $\left(x+2y^3\right)\frac{dy}{dx} = y$.

Solution: Given differential equation is,

$$\left(x+2y^3\right)\frac{dy}{dx} = y \quad \ldots \ldots \ldots (1)$$

The equation (1) can be written as,

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2 \ldots \ldots \ldots (2)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{-1}{y} dy}$$

$$= e^{-\ln y}$$

$$= e^{\ln y^{-1}}$$

$$= \frac{1}{y}$$

Multiply both sides of equation (2) by $\frac{1}{y}$, we get

$$\Rightarrow \frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2y$$

$$\Rightarrow \frac{d}{dy}\left(\frac{x}{y}\right) = 2y$$

Integrating both sides, we get

$$\frac{x}{y} = 2\int y dy + c$$

$$\Rightarrow \frac{x}{y} = 2.\frac{y^2}{2} + c$$

$$\Rightarrow x = y^3 + cy$$

which is the required solution.

Exercise: Solve the following differential equations:

1. $\left(1+x^2\right)\frac{dy}{dx} + y = \tan^{-1} x$
2. $\frac{dy}{dx} + \frac{y}{x} = \sin x$
3. $x(x-1)\frac{dy}{dx} - y = x^2 (x-1)^2$
4. $\frac{dy}{dx} - y\sin x = \sin 2x$
5. $\left(x^2-1\right)\frac{dy}{dx} + 2y = (x+1)^2$

6. $y\log y dx+\left(x-\log y\right)dy=0$
7. $dx+xdy=e^{-y}\log y dy$

Equations reducible to linear form:

Bernoulli Equation: An equation of the form

$$\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)y^{n} \quad ; \quad n\neq 0,1$$

where $P\left(x\right)$ and $Q\left(x\right)$ are functions of x or constants is called a Bernoulli Equation of first order.

Theorem: Reduce the Bernoulli Equation to Linear form and then solve it.

Answer: The Bernoulli's equation is,

$$\frac{dy}{dx}+P(x)y=Q(x)y^{n} \ \ldots\ldots\ldots (1)$$

Dividing the equation (1) by y^{n} we get

$$y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x) \ \ldots\ldots\ldots (2)$$

put, $v=y^{1-n}$

$$\therefore \ \frac{dv}{dx}=\left(1-n\right)y^{-n}\frac{dy}{dx}$$

Now equation (2) transforms into,

$$\frac{1}{\left(1-n\right)}\frac{dv}{dx}+P(x)v=Q(x)$$

$$\Rightarrow \frac{dv}{dx}+\left(1-n\right)P(x)v=\left(1-n\right)Q(x) \ldots\ldots\ldots (3)$$

Let $P_{1}\left(x\right)=\left(1-n\right)P\left(x\right)$ and $Q_{1}\left(x\right)=\left(1-n\right)Q\left(x\right)$ then equation (3) becomes,

$$\frac{dv}{dx}+P_{1}(x)v=Q_{1}(x) \ldots\ldots\ldots (4)$$

which is a linear form.

2nd part: The integrating factor is,

$$I.F = e^{\int P_1(x)}dx$$

Multiply both sides of equation (4) by $e^{\int P_1(x)}dx$ we get,

$$e^{\int P_1(x)dx}\frac{dv}{dx} + e^{\int P_1(x)dx}P_1(x)v = e^{\int P_1(x)dx}Q_1(x)$$

$$\Rightarrow \frac{d}{dx}\left(ve^{\int P_1(x)dx}\right) = e^{\int P_1(x)dx}Q_1(x)$$

Integrating both sides, we get

$$\Rightarrow ve^{\int P_1(x)dx} = \int e^{\int P_1(x)dx}Q_1(x)dx + c$$

which is the required solution.

Mathematical problem based on Bernoulli's equation:

Problem-01: Solve the differential equation $\frac{dy}{dx} = x^3y^3 - xy$.

Solution: The differential equation is,

$$\frac{dy}{dx} = x^3y^3 - xy \quad \dots\dots\dots (1)$$

Equation (1) can be written as,

$$\frac{dy}{dx} = x^3y^3 - xy$$

$$\Rightarrow \frac{dy}{dx} + xy = x^3y^3 \dots\dots\dots (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by y^3 we get

$$y^{-3}\frac{dy}{dx}+xy^{-2}=x^3 \dots\dots\dots (3)$$

put $v=y^{-2}$

$$\therefore \frac{dv}{dx}=-2y^{-3}\frac{dy}{dx} \Rightarrow y^{-3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$$

Now the equation (3) becomes,

$$-\frac{1}{2}\frac{dv}{dx}+xv=x^3$$

$$\Rightarrow \frac{dv}{dx}-2xv=-2x^3 \dots\dots\dots (4)$$

This is a linear equation.

$$\text{I.F}=e^{\int -2x\,dx}$$

$$=e^{-x^2}$$

Multiply both sides of equation (4) by e^{-x^2} we get

$$e^{-x^2}\frac{dv}{dx}-2xve^{-x^2}=-2x^3e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}\left(ve^{-x^2}\right)=-2x^3e^{-x^2}$$

Integrating both sides we get

$$ve^{-x^2}=-2\int x^3e^{-x^2}dx+c$$

$$\Rightarrow ve^{-x^2}=-\int te^t dt+c \qquad ;\ putting\ -x^2=t$$

$$\Rightarrow ve^{-x^2}=-e^t\left(t-1\right)+c$$

$$\Rightarrow ve^{-x^2}=-e^{-x^2}\left(-x^2-1\right)+c$$

$$\Rightarrow v=x^2+1+ce^{x^2}$$

$$\Rightarrow y^{-2} = x^2 + 1 + ce^{x^2}$$

$$\Rightarrow \left(x^2 + 1 + ce^{x^2}\right) y^2 = 1$$

which is the solution.

Problem-02: Solve the differential equation $\frac{dy}{dx} + \frac{2}{x} y = \frac{y^3}{x^3}$.

Solution: The differential equation is,

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{y^3}{x^3} \quad \ldots\ldots\ldots (1)$$

This is a Bernoulli's equation.

Dividing the equation (1) by y^3 we get

$$y^{-3} \frac{dy}{dx} + \frac{2}{x} y^{-2} = \frac{1}{x^3} \ldots\ldots\ldots (2)$$

put $v = y^{-2}$

$$\therefore \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dv}{dx}$$

Now the equation (2) becomes,

$$-\frac{1}{2} \frac{dv}{dx} + \frac{2}{x} v = \frac{1}{x^3}$$

$$\Rightarrow \frac{dv}{dx} - \frac{4}{x} v = -\frac{2}{x^3} \ldots\ldots\ldots (3)$$

This is a linear equation.

$$\text{I.F} = e^{\int -\frac{4}{x} dx}$$

$$= e^{-4\int \frac{dx}{x}}$$

$$= e^{-4 \ln x}$$

$$= e^{\ln x^{-4}}$$

$$= \frac{1}{x^4}$$

Multiply both sides of equation (4) by $\frac{1}{x^4}$ we get

$$\frac{1}{x^4}\frac{dv}{dx} - \frac{4}{x^5}v = -\frac{2}{x^7}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^4}\right) = -\frac{2}{x^7}$$

Integrating both sides we get

$$\frac{v}{x^4} = -2\int \frac{dx}{x^7} + c$$

$$\Rightarrow \frac{v}{x^4} = \frac{1}{3}\frac{1}{x^6} + c$$

$$\Rightarrow y^{-2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

$$\Rightarrow \frac{1}{y^2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

which is the required solution.

Problem-03: Solve the differential equation $\left(x^2y^3 + xy\right)\frac{dy}{dx} = 1$.

Solution: The differential equation is,

$$\left(x^2y^3 + xy\right)\frac{dy}{dx} = 1 \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$\left(x^2y^3 + xy\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(x^2y^3 + xy\right)}$$

$$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \quad \ldots\ldots\ldots (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by x^2 we get

$$x^{-2}\frac{dx}{dy} - \frac{y}{x} = y^3 \ldots\ldots\ldots (3)$$

put $v = -\frac{1}{x}$

$$\therefore \frac{dv}{dy} = x^{-2}\frac{dx}{dy} \Rightarrow x^{-2}\frac{dx}{dy} = \frac{dv}{dy}$$

Now the equation (3) becomes,

$$\frac{dv}{dy} + yv = y^3 \ldots\ldots\ldots (4)$$

This is a linear equation.

$$\text{I.F} = e^{\int y\,dy}$$

$$= e^{\frac{y^2}{2}}$$

Multiply both sides of equation (4) by $e^{\frac{y^2}{2}}$ we get

$$e^{\frac{y^2}{2}}\frac{dv}{dy} + yve^{\frac{y^2}{2}} = y^3e^{\frac{y^2}{2}}$$

$$\Rightarrow \frac{d}{dy}\left(ve^{\frac{y^2}{2}}\right) = -y^3e^{\frac{y^2}{2}}$$

Integrating both sides we get

$$ve^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + c$$

$$\Rightarrow ve^{\frac{y^2}{2}} = 2\int te^t dt + c \qquad ;\ putting\ \frac{y^2}{2} = t$$

$$\Rightarrow ve^{\frac{y^2}{2}} = 2e^t\left(t-1\right) + c$$

$$\Rightarrow ve^{\frac{y^2}{2}} = 2e^{\frac{y^2}{2}}\left(\frac{y^2}{2} - 1\right) + c$$

$$\Rightarrow -\frac{1}{x} = y^2 - 2 + ce^{-\frac{y^2}{2}}$$

$$\Rightarrow \frac{1}{x} = 2 - y^2 - ce^{-\frac{y^2}{2}}$$

which is the required solution.

Problem-04: Solve the differential equation $2\frac{dy}{dx} - \frac{y}{x} = \frac{y^3}{x^3}$.

Solution: The differential equation is,

$$2\frac{dy}{dx} - \frac{y}{x} = \frac{y^3}{x^3} \quad \ldots\ldots\ldots (1)$$

The equation (1) can be written as,

$$\frac{dy}{dx} - \frac{y}{2x} = \frac{y^3}{2x^3} \quad \ldots\ldots\ldots (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by y^3 we get

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x}y^{-2} = \frac{1}{2x^3} \ldots\ldots\ldots (3)$$

put $v = y^{-2}$

$$\therefore \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Now the equation (2) becomes,

$$\Rightarrow \frac{dv}{dx} + \frac{v}{x} = -\frac{1}{x^3} \ldots\ldots\ldots (3)$$

This is a linear equation.

$$\text{I.F} = e^{\int -\frac{4}{x}dx}$$

$$= e^{-4\int \frac{dx}{x}}$$

$$= e^{-4\ln x}$$

$$= e^{\ln x^{-4}}$$

$$= \frac{1}{x^4}$$

Multiply both sides of equation (4) by $\frac{1}{x^4}$ we get

$$\frac{1}{x^4}\frac{dv}{dx} - \frac{4}{x^5}v = -\frac{2}{x^7}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^4}\right) = -\frac{2}{x^7}$$

Integrating both sides we get

$$\frac{v}{x^4} = -2\int \frac{dx}{x^7} + c$$

$$\Rightarrow \frac{v}{x^4} = \frac{1}{3}\frac{1}{x^6} + c$$

$$\Rightarrow y^{-2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

$$\Rightarrow \frac{1}{y^2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

which is the required solution.

Problem-05: Solve the differential equation $\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$.

Solution: The differential equation is,

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \quad \ldots\ldots\ldots (1)$$

This is a Bernoulli's equation.

put $\quad v = \tan y$

$$\therefore \frac{dv}{dx} = \sec^2 y\frac{dy}{dx}$$

Now the equation (1) becomes,

$$\frac{dv}{dx} + 2xv = x^3 \ldots\ldots\ldots (2)$$

This is a linear equation.

$$\text{I.F} = e^{\int 2xdx}$$

$$= e^{x^2}$$

Multiply both sides of equation (2) by e^{x^2} we get

$$e^{x^2}\frac{dv}{dx} + 2xe^{x^2}v = x^3e^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(ve^{x^2}\right) = x^3e^{x^2}$$

Integrating both sides we get

$$ve^{x^2} = \int x^3e^{x^2}dx + c$$

$$\Rightarrow ve^{x^2} = \frac{1}{2}\int te^t dt + c \quad ; \ putting \ x^2 = t$$

$$\Rightarrow ve^{x^2} = \frac{1}{2}\left(te^t - e^t\right) + c$$

$$\Rightarrow ve^{x^2} = \frac{e^t}{2}\left(t-1\right) + c$$

$$\Rightarrow ve^{x^2} = \frac{e^{x^2}}{2}\left(x^2-1\right) + c$$

$$\Rightarrow v = \frac{1}{2}\left(x^2-1\right) + ce^{-x^2}$$

$$\Rightarrow \tan y = \frac{1}{2}\left(x^2-1\right) + ce^{-x^2}.$$

which is the required solution.

Exercise:

1. $\dfrac{dy}{dx} + y = y^3 \sin x$
2. $\dfrac{dy}{dx} + y = y^2 e^x$
3. $\dfrac{dy}{dx} + xy = xy^2$
4. $\dfrac{dy}{dx} - y = xy^2$
5. $\dfrac{dy}{dx} + \dfrac{y}{x} = x\sqrt{y}$
6.

Exact Differential Equations: A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be an exact differential equation if it satisfies the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule:

1. Integrate M with respect to x keeping y as constant,

2. Find out those terms in N which are free from x and integrate them with respect to y,
3. Add the two expressions so obtained and equate the sum to an arbitrary constant.

Mathematical problem based on Exact differential equation:

Problem-01: Solve $\left(y^4+4x^3y+3x\right)dx+\left(x^4+4xy^3+y+1\right)dy=0$.

Solution: Given that,

$$\left(y^4+4x^3y+3x\right)dx+\left(x^4+4xy^3+y+1\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+y+1$

$$\therefore \quad \frac{\partial M}{\partial y}=4y^3+4x^3$$

$$\frac{\partial N}{\partial x}=4x^3+4y^3$$

since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating M with respect to x we get

$$xy^4+x^4y+\frac{3}{2}x^2$$

In N, terms free from x are $y+1$ whose integral with respect to y is

$$\frac{1}{2}y^2+y$$

Therefore the general solution is

$$xy^4+x^4y+\frac{3}{2}x^2+\frac{1}{2}y^2+y=c.$$

Problem-02: Solve $\left(x^2-2xy+3y^2\right)dx+\left(4y^3+6xy-x^2\right)dy=0$.

Solution: Given that,

$$\left(x^2-2xy+3y^2\right)dx+\left(4y^3+6xy-x^2\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=x^2-2xy+3y^2$ and $N=4y^3+6xy-x^2$

$$\therefore \quad \frac{\partial M}{\partial y}=-2x+6y$$

$$\frac{\partial N}{\partial x}=6y-2x$$

since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating *M* with respect to *x* we get

$$\frac{1}{3}x^3-x^2y+3xy^2$$

In *N*, terms free from *x* is $4y^3$ whose integral with respect to *y* is

$$y^4$$

Therefore the general solution is

$$\frac{1}{3}x^3-x^2y+3xy^2+y^4=c.$$

Problem-03: Solve $\left(2x^3+3y\right)dx+\left(3x+y-1\right)dy=0$.

Solution: Given that,

$$\left(2x^3+3y\right)dx+\left(3x+y-1\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=2x^3+3y$ and $N=3x+y-1$

$$\therefore \quad \frac{\partial M}{\partial y}=3$$

$$\frac{\partial N}{\partial x}=3$$

since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating *M* with respect to *x* we get

$$\frac{1}{2}x^4+3xy$$

In *N*, terms free from *x* are $y-1$ whose integral with respect to *y* is

$$\frac{1}{2}y^2-y$$

Therefore the general solution is

$$\frac{1}{2}x^4+3xy+\frac{1}{2}y^2-y=c.$$

Exercise:

1. $\left[\cos x\tan y+\cos\left(x+y\right)\right]dx+\left[\sin x\sec^2 y+\cos\left(x+y\right)\right]dy=0$
2. $\left(x-2e^y\right)dy+\left(y+x\sin x\right)dx=0$
3. $\left(x^2-4xy-2y^2\right)dx+\left(y^2-4xy-2x^2\right)dy=0$

Equations reducible to exact differential equation: A differential equation $M\left(x,y\right)dx+N\left(x,y\right)dy=0$ is not an exact differential equation if

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

But it can be reduced to an exact differential equation by multiplying a function of *x* and *y*, which is called an **integrating factor**.

Rules for finding integrating factor: Let the differential equation is,

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0 \quad \ldots\ldots\ldots (1)$$

1. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f\left(x\right)$ then the integrating factor is $\mu=e^{\int f(x)dx}$.

2. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)$ then the integrating factor is $\mu=e^{\int g(y)dy}$.

3. If M and N are both homogeneous function in x, y of degree n, then the integrating factor is $\mu=\dfrac{1}{Mx+Ny}$; *where*, $Mx+Ny\neq 0$.

4. If the equation (1) is of the form, $yf(xy)dx+xg(xy)dy=0$ then the integrating factor is $\mu=\dfrac{1}{Mx-Ny}$; *where*, $Mx-Ny\neq 0$

NOTE: 1. If $Mx+Ny=0$, then $\dfrac{M}{N}=-\dfrac{y}{x}$ and the equation reduces to $ydx-xdy=0$, which can be easily solved.

2. If $Mx-Ny=0$, then $\dfrac{M}{N}=\dfrac{y}{x}$ and the equation reduces to $ydx+xdy=0$, which can be easily solved.

Mathematical problem based on reducible exact differential equation:

Problem-01: Solve $\left(x^2+y^2+x\right)dx+xydy=0$.

Solution: Given that,

$$\left(x^2+y^2+x\right)dx+xydy=0 \quad \dots\dots\dots (1)$$

where, $M=x^2+y^2+x$ and $N=xy$

$$\therefore \quad \frac{\partial M}{\partial y}=2y$$

$$\frac{\partial N}{\partial x}=y$$

since, $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-y}{xy}=\dfrac{1}{x}$

Hence, I.F $=e^{\int \frac{1}{x}dx}$

$=e^{\ln x}$

$=x$

Multiplying by I.F, the equation (1) becomes,

$$\left(x^3+xy^2+x^2\right)dx+x^2ydy=0 \text{ ... ... (2)}$$

which is exact now.

Let, $M^{'}=x^3+xy^2+x^2$ and $N^{'}=x^2y$

Integrating $M^{'}$ with respect to x we get

$$\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3$$

In $N^{'}$, there is no term free from x.

Therefore the general solution is

$$\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3=c\,.$$

Problem-02: Solve $\left(x^2+y^2+2x\right)dx+2ydy=0$.

Solution: Given that,

$$\left(x^2+y^2+2x\right)dx+2ydy=0 \quad \text{... ... ... (1)}$$

where, $M=x^2+y^2+2x$ and $N=2y$

$$\therefore \quad \frac{\partial M}{\partial y}=2y$$

$$\frac{\partial N}{\partial x}=0$$

since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = 1$

Hence, I.F $= e^{\int dx}$

$= e^x$

Multiplying by I.F, the equation (1) becomes,

$$\left(e^x x^2 + e^x y^2 + 2xe^x\right)dx + xye^x dy = 0 \text{ ... ... (2)}$$

which is exact now.

Let, $M' = e^x x^2 + e^x y^2 + 2xe^x$ and $N' = x^2 ye^x$

Integrating M' with respect to x we get

$$e^x x^2 + e^x y^2$$

In N', there is no term free from x.

Therefore the general solution is

$$e^x x^2 + e^x y^2 = c.$$

Problem-03: Solve $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0$.

Solution: Given that,

$$\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}\left(x + xy^2\right)dy = 0 \text{ ... ... (1)}$$

where, $M = y + \frac{1}{3}y^3 + \frac{1}{2}x^2$ and $N = \frac{1}{4}\left(x + xy^2\right)$

$$\therefore \frac{\partial M}{\partial y} = 1 + y^2$$

$$\frac{\partial N}{\partial x}=\frac{1}{4}\left(1+y^2\right)$$

since, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1+y^2-\frac{1}{4}\left(1+y^2\right)}{\frac{1}{4}\left(x+xy^2\right)}$$

$$=\frac{\frac{1}{4}\left(4+4y^2-1-y^2\right)}{\frac{1}{4}x\left(1+y^2\right)}$$

$$=\frac{\left(3+3y^2\right)}{x\left(1+y^2\right)}$$

$$=\frac{3\left(1+y^2\right)}{x\left(1+y^2\right)}$$

$$=\frac{3}{x}$$

Hence, I.F$=e^{\int\frac{3}{x}dx}$

$$=e^{3\ln x}$$

$$=e^{\ln x^3}$$

$$=x^3$$

Multiplying by I.F, the equation (1) becomes,

$$\left(x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5\right)dx+\frac{1}{4}\left(x^4+x^4y^2\right)dy=0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M^{'}=x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5$ and $N^{'}=\frac{1}{4}\left(x^4+x^4y^2\right)$

Integrating M' with respect to x we get

$$\frac{1}{4}x^4y+\frac{1}{12}x^4y^3+\frac{1}{12}x^6$$

In N', there is no term free from x.

Therefore the general solution is

$$\frac{1}{4}x^4y+\frac{1}{12}x^4y^3+\frac{1}{12}x^6=c\,.$$

Problem-04: Solve $\left(x^4+y^4\right)dx-xy^3dy=0$.

Solution: Given that,

$$\left(x^4+y^4\right)dx-xy^3dy=0 \quad \dots\dots\dots (1)$$

where, $M=x^4+y^4$ and $N=-xy^3$

$$\therefore \quad \frac{\partial M}{\partial y}=4y^3$$

$$\frac{\partial N}{\partial x}=-y^3$$

since, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

But the equation (1) is a homogeneous differential equation.

Hence, I.F $=\frac{1}{Mx+Ny}$

$$=\frac{1}{x^5+xy^4-xy^4}$$

$$=\frac{1}{x^5}$$

Multiplying by I.F, the equation (1) becomes,

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}dy=0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M^{'}=\frac{1}{x}+\frac{y^4}{x^5}$ and $N^{'}=-\frac{y^3}{x^4}$

Integrating $M^{'}$ with respect to *x* we get

$$\ln x-\frac{y^4}{4x^4}$$

In $N^{'}$, there is no term free from *x*.

Therefore the general solution is

$$\ln x-\frac{y^4}{4x^4}=c.$$

Problem-05: Solve $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$.

Solution: Given that,

$$y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=y\left(xy+2x^2y^2\right)$ and $N=x\left(xy-x^2y^2\right)$

$$\therefore \quad \frac{\partial M}{\partial y}=2xy+6x^2y^2$$

$$\frac{\partial N}{\partial x}=2xy-6x^2y^2$$

since, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

But the equation (1) is of the form, $yf\left(xy\right)dx+xg\left(xy\right)dy=0$.

Hence, $\text{I.F}=\frac{1}{Mx-Ny}$

$$=\frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3}$$

$$=\frac{1}{3x^3y^3}$$

Multiplying by I.F, the equation (1) becomes,

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M^{'}=\frac{1}{3x^2y}+\frac{2}{3x}$ and $N^{'}=\frac{1}{3xy^2}-\frac{1}{3y}$

Integrating $M^{'}$ with respect to x we get

$$-\frac{1}{3xy}+\frac{2}{3}\ln x$$

In $N^{'}$, term free from x is $-\frac{1}{3y}$, whose integral with respect to y is,

$$-\frac{1}{3}\ln y$$

Therefore the general solution is

$$-\frac{1}{3xy}+\frac{2}{3}\ln x-\frac{1}{3}\ln y=\ln c$$

$$\Rightarrow -\frac{1}{xy}+\ln x^2-\ln y=3\ln c$$

$$\Rightarrow -\frac{1}{xy}+\ln\frac{x^2}{y}=\ln c^3$$

$$\Rightarrow \frac{x^2}{y}e^{-\frac{1}{xy}}=c^3$$

$$\Rightarrow \frac{x^2}{y} = Ce^{\frac{1}{xy}} \quad ; \; where,\; C = c^3 \quad .$$

Exercise:

1. $\left(x^2 + y^2 + 1\right)dx - 2xydy = 0$
2. $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$
3. $y^2dx + \left(x^2 - xy - y^2\right)dy = 0$
4. $y\left(x^2y^2 + 2\right)dx + x\left(2 - 2x^2y^2\right)dy = 0$