Complex Number

Introduction: Complex analysis (traditionally known as the theory of functions of a complex variable) is the study of complex numbers together with their derivatives, manipulation, and other properties and investigates function of complex numbers. Complex analysis is an extremely powerful tool with an unexpectedly large number of practical applications to the solution of physical problems. Contour integration, for example, provides a method of computing difficult integrals by investigating the singularities of the function in regions of the complex plane near and between the limits of integration.

Why complex number system is introduced?

Mathematics has its own language in which Alphabets are numbers, so number system is so important in mathematics**.** The prince of mathematics Gauss said that every polynomial has at least one root and that Polynomial has maximum roots exactly equal to its order or degree. Above statement is called the fundamental theorem of Algebra. When we consider the polynomial equation like as $x^2+1=0$ that implies $x^2=-1$ but which is not possible in the real field because square of any real number is non-negative so real field fails to give the solution of this polynomial equation. But fundamental theorem of Algebra tells it has maximum two roots. To permit the solution of the equation $x^2+1=0$ and similar types, the set of complex numbers is introduced. Therefore, complex number system includes real number system as a subset, so complex number system is the extended form of real number system that solved the above considered problem. By considering $i^2=-1$ the problem $x^2+1=0$ provides two complex roots that covered the fundamental theorem of Algebra stated by great mathematician Gauss .It is notable that the mathematician Euler first use the symbol i for imaginary unit and its geometrical meaning in the complex plane is the point $(0,1)$. Solution of the above arises problem is as follows:

$$x^2+1=0 \Rightarrow x^2=-1$$

$$\Rightarrow x^2=i^2 \;\; [\because i^2=-1]$$

$$\Rightarrow x=\pm\sqrt{i^2} \Rightarrow x=\pm i$$

Finally, we conclude that the roots of the aroused problem are $x=i$ and $x=-i$.

Complex variable: A complex variable z is a linear combination of two real variables x and y with the special sign i and it is defined as,

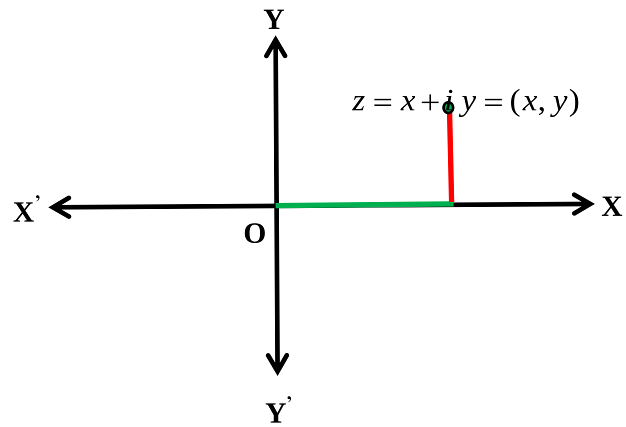
$$z=x+i\,y \qquad \text{where } x\in R, y\in R \text{ and } i=\sqrt{-1}.$$

Alternatively, the symbol z, which stands for any of a set of complex numbers, is called a complex variable. Geometrically, a complex variable represents a general point in the complex plane/Argand Plane/Argand diagram/Gaussian Plane.

Complex number: Any number of the form $x+iy$, where $x\in R, y\in R$ and $i=\sqrt{-1}$, is called a complex number and it is denoted by z.

i.e. $z=x+i\,y$

In complex number z, x is the real part of z denoted by the symbol $\mathrm{Re}(z)=x$ and y is the imaginary part of z denoted by the symbol $\mathrm{Im}(z)=y$ and also i is called imaginary unit. Geometrically, a complex number represents a unique point in the complex plane/Argand Plane/Argand diagram/Gaussian Plane. Also geometrically, $\mathrm{Re}(z)$ is the projection of $z=(x,y)$ on to the x axis, and $\mathrm{Im}(z)$ is the projection of z on to the y axis.

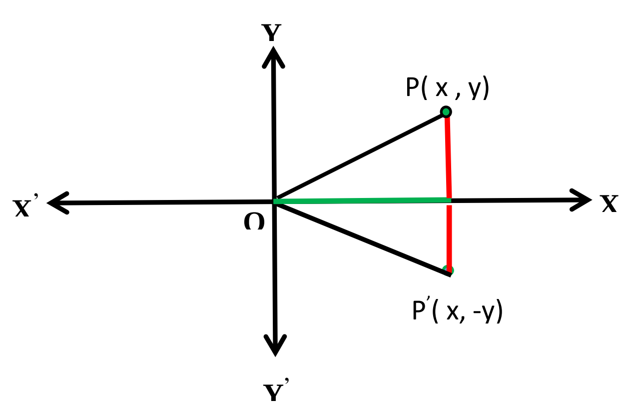


Properties of complex numbers: If z_1, z_2, z_3 belong to the set S of complex numbers, the following properties hold.

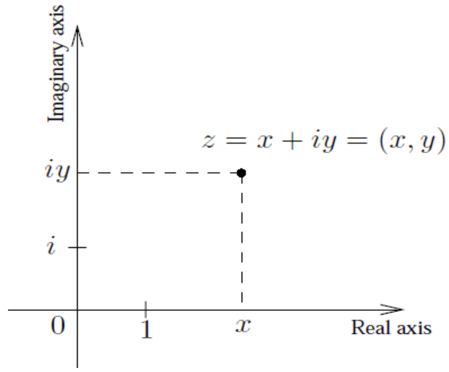
1. $z_1 + z_2$ *and* $z_1 z_2 \in S$ — Closure law
2. $z_1 + z_2 = z_2 + z_1$ — Commutative law of addition
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ — Associative law of addition
4. $z_1 z_2 = z_2 z_1$ — Commutative law of multiplication
5. $z_1 (z_2 z_3) = (z_1 z_2) z_3$ — Associative law of multiplication
6. $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ — Distributive law
7. $z_1 + 0 = 0 + z_1 = z_1$, $1 \cdot z_1 = z_1 \cdot 1 = z_1$ — 0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication.
8. For any complex number z_1 there is a unique number z in S such that $z + z_1 = 0$; z is called the inverse of z_1 with respect to addition and is denoted by $-z_1$.
9. For any $z_1 \neq 0$ there is a unique number z in S such that $z_1 z = z z_1 = 1$; z is called the inverse of z_1 with respect to multiplication and is denoted by $1/z_1$.

In general, any set which satisfy above conditions is called a field.

Conjugate of complex number: The conjugate of a complex number $z = x + i\,y$ is obtained by changing the sign of y and is denoted by the symbol $\bar{z}$ i.e. $\bar{z} = x - iy$.Geometrically, the conjugate of a complex number represents the reflection or image of the complex number z about the real axis x.



Cartesian form/Rectangular form: The form of the complex number $z = x + i\,y$, where $x \in R, y \in R$ *and* $i = \sqrt{-1}$ is called Cartesian form.



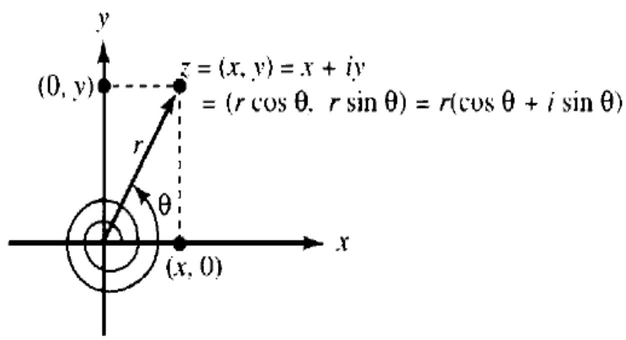
Polar Form: The Cartesian form of a complex number is $z = x + i\,y$

From the relation of Cartesian and polar system, we have

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Now,

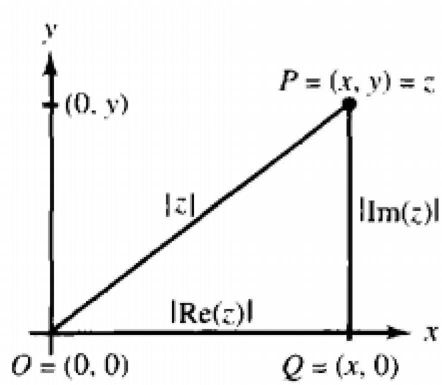
$$\begin{aligned} z &= x + i\,y \\ &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \qquad (1) \\ &= re^{i\theta} \qquad (2) \end{aligned}$$



The form of equation (1) is called the polar form and the form of equation (2) is called the exponential form.

Modulus/Absolute value: The modulus of a complex number $z = x + i\,y$ is a nonnegative real number denoted by $|z| = r \geq 0$ and is defined by the equation $|z| = \sqrt{x^2 + y^2}$.

The number |z| is the distance between the origin and the point (*x*, *y*) in Argand Plane or Gaussian plane. The only complex number with modulus zero is the number 0. The numbers | Re(z) | , | lm(z) |, and | z | are the lengths of the sides of the right triangle OPQ, which is shown in the following figure



Properties of moduli: If $z_1,\ z_2,\ z_3,\ \cdots,\ z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1||z_2| \qquad or,\ |z_1 z_2 \cdots z_m| = |z_1||z_2|\cdots|z_m|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \qquad if\ \ z_2 \neq 0$
3. $|z_1 + z_2| \leq |z_1| + |z_2| \qquad or,\ |z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$
4. $|z_1 + z_2| \geq |z_1| - |z_2| \qquad or,\ |z_1 - z_2| \geq |z_1| - |z_2|.$

Argument or Amplitude of a complex number: Consider the complex number is, $z = x + i\,y$. We know that the relations between the Cartesian coordinates (x, y) and Polar coordinates (r, θ) are,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

We can write from above relations,

$$\tan\theta = \frac{y}{x} \qquad or, \theta = \tan^{-1}\frac{y}{x}$$

This quantity is called the argument or amplitude of the complex number z. It is denoted by arg(z)or amp(z). A complex number has infinite many possible arguments. But in the range $-\pi < \theta \leq \pi$ every complex number has unique argument called principal argument of that number and is denoted by **Argz.**

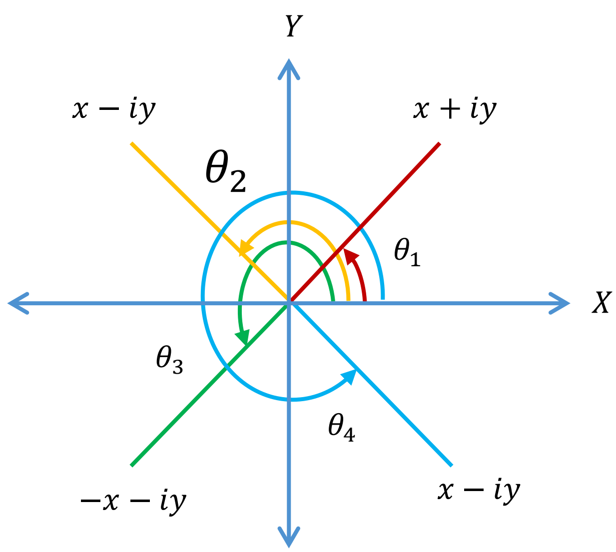
$$\theta_1 = \tan^{-1}\frac{y}{x}$$

$$\theta_2 = \pi - \tan^{-1}\left|\frac{y}{x}\right|$$

$$\theta_3 = \pi + \tan^{-1}\left|\frac{y}{x}\right|$$

$$\theta_4 = 2\pi - \tan^{-1}\left|\frac{y}{x}\right|$$

Argument = arg(z)



Theorem on argument:

- ❖ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- ❖ $\arg(\dfrac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

Euler's Formula: The relation $e^{i\theta} = \cos\theta + i\sin\theta$ is called Euler's formula·

Deduction: We know that the infinite series of trigonometric functions are

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots\cdots\cdots\cdots\cdots\cdots + \infty$$

and $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots\cdots\cdots + \infty$

We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\cdots\cdots\cdots + \infty$$

Replacing x by $i\theta$ in the above equation we get,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots\cdots\cdots\cdots + \infty$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots\cdots\cdots + \infty\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots\cdots\cdots + \infty\right)$$

$$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta$$

which is known as **Euler's Formula . (Deduced)**

De Movire's theorem: For all rational values of n the De Movire's theorem is defined as,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta .$$

Deduction: If $z_1 = x_1 + iy_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = x_2 + iy_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$, then

$$z_1z_2 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)\cdot r_2\left(\cos\theta_2 + i\sin\theta_2\right)$$
$$= r_1r_2\left\{\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right\}$$
$$\therefore\ z_1z_2z_3 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)\cdot r_2\left(\cos\theta_2 + i\sin\theta_2\right)\cdot r_3\left(\cos\theta_3 + i\sin\theta_3\right)$$
$$= r_1r_2r_3\left\{\cos\left(\theta_1+\theta_2+\theta_3\right) + i\sin\left(\theta_1+\theta_2+\theta_3\right)\right\}$$

In general,

$$z_1z_2\cdots z_n = r_1r_2\cdots r_n\left\{\cos\left(\theta_1+\theta_2+\cdots+\theta_n\right) + i\sin\left(\theta_1+\theta_2+\cdots+\theta_n\right)\right\} \qquad (1)$$

If $z_1 = z_2 = \cdots\ \cdots = z_n = z$, then (1) becomes

$$z^n = \left\{r\left(\cos\theta + i\sin\theta\right)\right\}^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

which is called De Moivre's theorem.

Roots of a complex numbers:

A number w is called an n^{th} root of a complex number z if $w^n = z$ and it can be written as

$$w = \sqrt[n]{z}$$

By assuming $z = r(\cos\theta + i\sin\theta), z \neq 0$ we get,

$$w = \sqrt[n]{r(\cos\theta + i\sin\theta)}$$
$$= \left\{r(\cos\theta + i\sin\theta)\right\}^{\frac{1}{n}}$$
$$= r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$$
$$= r^{\frac{1}{n}}(\cos\left(2k\pi+\theta\right) + i\sin\left(2k\pi+\theta\right))^{\frac{1}{n}}, \qquad k = 0,1,2,\ldots\ldots\ldots,n-1.$$
$$\therefore\ w = r^{\frac{1}{n}}\left\{\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right\}, \qquad k = 0,1,2,\ldots\ldots\ldots,n-1.$$

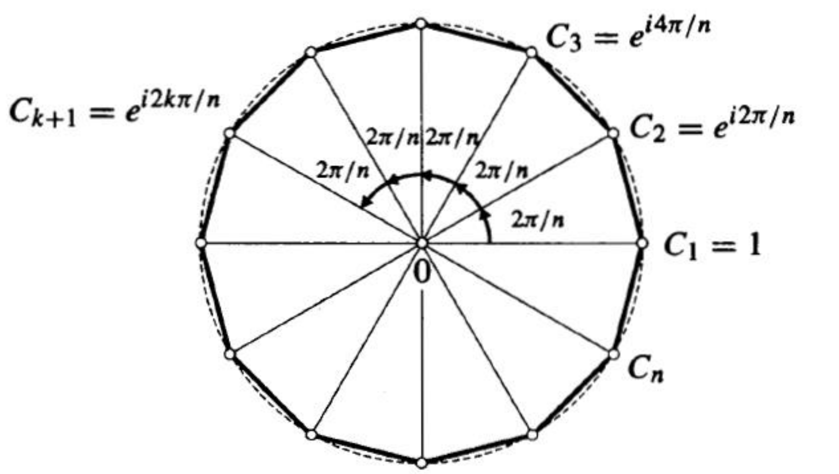
When $\theta = 0\, and\ r = 1$ then it represents n-th root of unity,

$$w = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right).$$

$$= e^{\frac{2k\pi i}{n}}, \qquad k = 0,1,2,........,n-1.$$

If we let $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{\frac{2\pi i}{n}}$, the n roots are $1, \omega, \omega^2, \omega^3, ..., \omega^{n-1}$.

Geometrically they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with C centre at origin. This circle has the equation $|z| = 1$ and is often called the unit circle.



Complex Polynomial and Equation: Any expression of the form

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

is called a complex polynomial of degree n, where $a_0 \neq 0,\ a_1, a_2, \cdots, a_n \in C\ \ and\ \ n \in N$.

If $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$ then it is called a complex equation of degree n.

If $z_1, z_2, \cdots, z_n$ are the roots of the above equation then

$$(z - z_1)(z - z_2)\cdots(z - z_n) = 0$$

It is called the factor form of the above polynomial equation.

Problems

Problem-01: Find real numbers x and y such that $3x+2iy-ix+5y=7+5i$ ..

Solution: Given that, $3x+2iy-ix+5y=7+5i$

$$or,\ 3x+5y+(2y-x)i=7+5i$$

Comparing the real parts and imaginary parts of the equation of complex numbers we get,

$$3x+5y=7$$
$$2y-x=5$$

Solving the above equations we get the values $x=-1$ and $y=2$. **(As desired)**

Problem-02: Convert the complex number $z=4e^{-\frac{i\pi}{6}}$ in its Cartesian/ Rectangular Form.

Solution: Given that, $z=4e^{-\frac{i\pi}{6}}$

$$=4\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right] \quad \text{[Using Euler's Formula]}$$

$$=4\left[\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right]$$

$$=4\left[\frac{\sqrt{3}}{2}-i.\frac{1}{2}\right]$$

$$=2\sqrt{3}-2i$$

This is the required rectangular form or Cartesian form, where $x=2\sqrt{3}$ and $y=-2$. **(As desired)**

Problem-03: Find the modulus and argument of the complex number $\left(\frac{2+i}{3-i}\right)^2$.

Solution: Given Complex number is,

$$z=\left(\frac{2+i}{3-i}\right)^2$$

$$=\frac{4+4i+i^2}{9-6i+i^2}$$

$$=\frac{4+4i-1}{9-6i-1}$$

$$=\frac{3+4i}{8-6i}$$

$$=\frac{(3+4i)(8+6i)}{(8-6i)(8+6i)}$$

$$=\frac{24+18i+32i-24}{8^2-6^2i^2}$$

$$=\frac{50i}{64+36}$$

$$=\frac{50i}{100}$$

$$= 0 + i.\frac{1}{2}$$
$$= x + iy$$

where $x = 0$ *and* $y = \frac{1}{2}$.

The modulus of the given complex number is,

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

And the argument of the given complex number is,

$$\theta = \tan^{-1}\left(\frac{1/2}{0}\right) = \tan^{-1}(\infty) = \tan^{-1}\left(\tan\frac{\pi}{2}\right) = \frac{\pi}{2}$$

This is the principal argument of the complex number.
The general argument of the complex is

$$= 2n\pi + principal\ \arg ument$$
$$= 2n\pi + \frac{\pi}{2}. \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots etc.$$

Problem-04: Find the modulus and argument of the complex number $\left(\frac{1+2i}{2+i}\right)^2$.

Solution: Given Complex number is,

$$z = \left(\frac{1+2i}{2+i}\right)^2$$
$$= \frac{1+4i-4}{4+4i-1}$$
$$= \frac{-3+4i}{3+4i}$$
$$= \frac{(-3+4i)(3-4i)}{(3+4i)(3-4i)}$$
$$= \frac{-9+12i+12i+16}{3^2-4^2i^2}$$
$$= \frac{7+24i}{9+16}$$
$$= \frac{7+24i}{25}$$
$$= \frac{7}{25} + i\frac{24}{25}$$
$$= x + iy$$

where $x = \frac{7}{25}$ *and* $y = \frac{24}{25}$.

The modulus of the given complex number is,

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2} = \sqrt{\frac{49}{625} + \frac{576}{625}} = \sqrt{\frac{625}{625}} = 1$$

And the argument of the given complex number is,

$$\theta = \tan^{-1}\left(\frac{24/25}{7/25}\right) = \tan^{-1}\left(\frac{24}{7}\right)$$

This is the principal argument of the complex number.
The general argument of the complex is

$$= 2n\pi + principal\ \arg ument$$

$$= 2n\pi + \tan^{-1}\left(\frac{24}{7}\right). \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots\ etc.$$

Problem-05: Find the modulus and argument of the complex number $\dfrac{-2}{1+i\sqrt{3}}$ **.**

Solution: Given Complex number is,

$$z = \frac{-2}{1+i\sqrt{3}}$$

$$= \frac{-2\left(1-i\sqrt{3}\right)}{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}$$

$$= \frac{-2+i2\sqrt{3}}{1^2-\left(\sqrt{3}\right)^2 i^2}$$

$$= \frac{-2+i2\sqrt{3}}{1+3}$$

$$= \frac{-2+i2\sqrt{3}}{4}$$

$$= \frac{-1+i\sqrt{3}}{2}$$

$$= -\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$= x+iy$$

where $x = -\frac{1}{2}$ *and* $y = \frac{\sqrt{3}}{2}$.

The modulus of the given complex number is,

$$r = \sqrt{x^2+y^2} = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

And the argument of the given complex number is,

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{3}/2}{-1/2}\right| = \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi - \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

This is the principal argument of the complex number.
The general argument of the complex is

$$= 2n\pi + principal\ \arg ument$$

$$= 2n\pi + \frac{2\pi}{3}. \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots \; etc.$$

Problem-06: Find the modulus and argument of the complex number $\frac{1-i}{1+i}$.

Solution: Given Complex number is,

$$z = \frac{1-i}{1+i}$$
$$= \frac{(1-i)^2}{(1+i)(1-i)}$$
$$= \frac{1-2i+i^2}{1^2-i^2}$$
$$= \frac{1-2i-1}{1+1}$$
$$= \frac{-2i}{2}$$
$$= -i$$
$$= x + iy$$

where $x = 0$ *and* $y = -1$.

The modulus of the given complex number is,

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

And the argument of the given complex number is,

$$\theta = \pi + \tan^{-1}\left|\frac{-1}{0}\right| = \pi + \tan^{-1}(\infty) = \pi + \tan^{-1}\left(\tan\frac{\pi}{2}\right) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

This is the principal argument of the complex number.

The general argument of the complex is

$$= 2n\pi + principal\arg ument$$
$$= 2n\pi + \frac{3\pi}{2}. \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots \; etc.$$

Problem-07: Find the modulus and argument of the complex number $z = 1 + \sin\alpha + i\cos\alpha$.

Solution: Given complex number is,

$$z = 1 + \sin\alpha + i\cos\alpha$$
$$or, \; x + iy = 1 + \sin\alpha + i\cos\alpha$$
$$\therefore \; x = 1 + \sin\alpha \text{ and } y = \cos\alpha$$

The modulus of the given number is,

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(1+\sin\alpha)^2 + \cos^2\alpha}$$
$$= \sqrt{1 + 2\sin\alpha + \sin^2\alpha + \cos^2\alpha}$$
$$= \sqrt{1 + 2\sin\alpha + 1}$$

$$= \sqrt{2+2\sin\alpha}$$
$$= \sqrt{2(1+\sin\alpha)}$$
$$= \sqrt{2\left(1+\cos\left(\frac{\pi}{2}-\alpha\right)\right)}$$
$$= \sqrt{2.2\cos^2\frac{\left(\frac{\pi}{2}-\alpha\right)}{2}}$$
$$= 2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

And argument of the given number is,

$$\theta = \tan^{-1}\frac{y}{x}$$
$$= \tan^{-1}\frac{\cos\alpha}{1+\sin\alpha}$$
$$= \tan^{-1}\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right)}$$
$$= \tan^{-1}\frac{2\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$$
$$= \tan^{-1}\frac{\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$$
$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$
$$= \frac{\pi}{4}-\frac{\alpha}{2}$$

This is the principal argument of the complex number.
The general argument of the complex number is,

$$= 2n\pi + principal\arg ument$$
$$= 2n\pi + \frac{\pi}{4}-\frac{\alpha}{2}. \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots etc.$$

Problem-08: Find the modulus and argument of the complex number $z = \dfrac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$.

Solution: Given complex number is,

$$z = \frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$$

$$=\frac{2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\phi}{2}+i2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$$

$$=\frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2}+i\sin\frac{\phi}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}\cdot e^{i\frac{\theta}{2}}}{\cos\frac{\phi}{2}\cdot e^{i\frac{\phi}{2}}}$$

$$=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot e^{i\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\left[\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)+i\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)\right]$$

$$=x+iy$$

where $x=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)$ and $y=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)$

The modulus of the given number is,

$$r=\sqrt{x^2+y^2}$$

$$=\sqrt{\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)\right)^2+\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)\right)^2}$$

$$=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\sqrt{\cos^2\left(\frac{\theta}{2}-\frac{\phi}{2}\right)+\sin^2\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}$$

$$=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\sqrt{1}$$

$$=\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}$$

And argument of the given number is,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left\{\frac{\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}{\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\cdot\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\theta}{2}-\frac{\phi}{2}\right)\right\}$$

$$= \frac{\theta}{2}-\frac{\phi}{2}$$

This is the principal argument of the complex number.
The general argument of the complex number is,

$$= 2n\pi + principal\arg ument$$

$$= 2n\pi + \frac{\theta}{2}-\frac{\phi}{2}. \qquad \text{where } n = 0, \pm 1, \pm 2, \cdots \ etc.$$

Problem-09: Express each of the following complex numbers in polar form and exponential form.

(a) $\mathbf{2 + 2\sqrt{3}i}$ **(b)** $\mathbf{-5 + 5i}$

Solution: (a): Given that, $z = \mathbf{2 + 2\sqrt{3}i}$

Here, $x = 2$ and $y = 2\sqrt{3}$

Modulus or absolute value is,

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$$

Amplitude or Argument is,

$$\theta = tan^{-1}(2\sqrt{3}/2) = \pi/3$$

The polar form is $z = r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

The exponential form is $z = re^{i\theta} = 4e^{\frac{\pi i}{3}}$ **Ans.**

(b): Given that, $z = \mathbf{-5 + 5i}$

Modulus or absolute value is,

$$r = \sqrt{x^2 + y^2} = \sqrt{25 + 25} = 5\sqrt{2}$$

Amplitude or Argument is,

$$\theta = \pi - tan^{-1}\left|\frac{y}{x}\right| = \pi - tan^{-1}\left|\frac{5}{-5}\right| = \pi - tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

The polar form is $z = r(\cos\theta + i\sin\theta) = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

The exponential form is $z = re^{i\theta} = 5\sqrt{2}e^{\frac{3\pi i}{4}}$ **Ans.**

Problem-10: Express each of the followings to the form $A+iB$:

(a). $\left(\dfrac{1+2i}{2+i}\right)^2$ **(b).** $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ **(c).** $\dfrac{(2e^{i\,15^\circ})^7}{(4e^{i\,45^\circ})^3}$

(d). $\left[3(\cos 40^\circ + i\sin 40^\circ)\right]\left[4(\cos 80^\circ + i\sin 80^\circ)\right]$

Solution: (a). Given that, $z=\left(\dfrac{1+2i}{2+i}\right)^2$

$$=\frac{1+2.2i+4i^2}{4+2.2i+i^2}$$

$$=\frac{1+4i-4}{4+4i-1}$$

$$=\frac{-3+4i}{3+4i}$$

$$=\frac{(-3+4i)(3-4i)}{(3+4i)(3-4i)}$$ [Multiplying numerator and denominator by $(3-4i)$]

$$=\frac{(-9+12i+12i+16)}{3^2-4^2i^2}$$

$$=\frac{7+24i}{9+16}$$

$$=\frac{7+24i}{25}$$

$$=\frac{7}{25}+i\frac{24}{25}$$

$$=A+iB$$

where $A=\dfrac{7}{25}$ *and* $B=\dfrac{24}{25}$ **(As desired).**

(b). Given that, $z=\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

$$=\left\{\frac{(1+\sqrt{3}i)(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)}\right\}^{10}$$

$$=\left\{\frac{(1+\sqrt{3}i)^2}{1^2-(\sqrt{3}i)^2}\right\}^{10}$$

$$=\left\{\frac{1+2\sqrt{3}i+(\sqrt{3}i)^2}{1+3}\right\}^{10}$$

$$=\left\{\frac{1+2\sqrt{3}i-3}{4}\right\}^{10}$$

$$=\left\{\frac{-2+2\sqrt{3}i}{4}\right\}^{10}$$

$$=\left\{\frac{\left(-1+\sqrt{3}i\right)}{2}\right\}^{10}$$

$$=\frac{1}{2^{10}}\left(-1+\sqrt{3}i\right)^{10} \qquad \cdots\cdots (1)$$

Let $z_1=-1+\sqrt{3}i$ so $x_1=-1$ and $y_1=\sqrt{3}$

$r_1=\sqrt{x_1{}^2+y_1{}^2}$ and $\theta_1=\pi-\tan^{-1}\left|\frac{y}{x}\right|$

$=\sqrt{(-1)^2+(\sqrt{3})^2}$ $\qquad =\pi-\tan^{-1}\sqrt{3}$

$=\sqrt{1+3}$ $\qquad =\pi-\tan^{-1}(\tan\pi/3)$

$=\sqrt{4}$ $\qquad =\pi-\frac{\pi}{3}$

$=2$ $\qquad =\frac{2\pi}{3}$

The exponential form of z_1 is,

$$z_1=r_1e^{i\theta_1}$$

$$=2e^{\frac{2\pi i}{3}}$$

Now, from (1) we have,

$$z=\frac{1}{2^{10}}\left(2e^{\frac{2\pi i}{3}}\right)^{10}$$

$$=\frac{1}{2^{10}}\times 2^{10}e^{\frac{20\pi i}{3}}$$

$$=e^{\frac{20\pi i}{3}}$$

$$=\cos\frac{20\pi}{3}+i\sin\frac{20\pi}{3}$$

$$=\cos\left(7\pi-\frac{\pi}{3}\right)+i\sin\left(7\pi-\frac{\pi}{3}\right)$$

$$=\left(\cos 7\pi\cos\frac{\pi}{3}+\sin 7\pi\sin\frac{\pi}{3}\right)+i\left(\sin 7\pi\cos\frac{\pi}{3}-\cos 7\pi\sin\frac{\pi}{3}\right)$$

$$=\left\{(-1)^7\cos\frac{\pi}{3}+0.\sin\frac{\pi}{3}\right\}+i\left\{0.\cos\frac{\pi}{3}-(-1)^7\sin\frac{\pi}{3}\right\}$$

$$=\left\{-\frac{1}{2}+0\right\}+i\left\{0+\frac{\sqrt{3}}{2}\right\}$$

$$=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$=A+iB$$

where $A=-\frac{1}{2}$ and $B=\frac{\sqrt{3}}{2}$ **(As desired).**

(b). Given that, $z=\frac{(2e^{i\,15^\circ})^7}{(4e^{i\,45^\circ})^3}$

$$=\frac{128e^{i\,105^\circ}}{64e^{i\,135^\circ}}$$

$$=2e^{i\,105^\circ-i\,135^\circ}$$

$$=2e^{-i30^\circ}$$

$$=2\left[\cos 30^\circ - i\sin 30^\circ\right]$$

$$=2\left[\frac{\sqrt{3}}{2}-i\frac{1}{2}\right]$$

$$=\sqrt{3}-i$$

$$=A+iB$$

where $A=\sqrt{3}$ and $B=-1$ **(As desired).**

(d). Given that, $z=\left[3(\cos 40^\circ + i\sin 40^\circ)\right]\left[4(\cos 80^\circ + i\sin 80^\circ)\right]$

$$=12(\cos 40^\circ + i\sin 40^\circ)(\cos 80^\circ + i\sin 80^\circ)$$

$$=12(\cos 40^\circ\cos 80^\circ + i\sin 80^\circ\cos 40^\circ + i\cos 80^\circ\sin 40^\circ + i^2\sin 40^\circ\sin 80^\circ)$$

$$=12\left\{(\cos 40^\circ\cos 80^\circ - \sin 40^\circ\sin 80^\circ)+i(\sin 80^\circ\cos 40^\circ + \cos 80^\circ\sin 40^\circ)\right\}$$

$$=12\left\{\cos(40^\circ+80^\circ)+i\sin(80^\circ+40^\circ)\right\}$$

$$=12\left\{\cos(120^\circ)+i\sin(120^\circ)\right\}$$

$$=12\left\{\cos(180^\circ-60^\circ)+i\sin(180^\circ-60^\circ)\right\}$$

$$=12\left\{-\cos 60^\circ + i\sin 60^\circ\right\}$$

$$=12\left\{-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right\}$$

$$=-6+i6\sqrt{3}$$

$$=A+iB$$

where $A=-6$ and $B=6\sqrt{3}$ **(As desired).**

Problem-11: Prove the identities:

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$

Solution: By De Movire's Formula we can write,

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5\cos^4\theta\left(i\sin\theta\right) + \frac{5\times4}{2!}\cos^3\theta\left(i\sin\theta\right)^2 + \frac{5\times4\times3}{3!}\cos^2\theta\left(i\sin\theta\right)^3$$
$$+\frac{5\times4\times3\times2}{4!}\cos^1\theta\left(i\sin\theta\right)^4 + \left(i\sin\theta\right)^5$$
$$= \cos^5\theta + 5i\cos^4\theta\sin\theta + \frac{20}{2}i^2\cos^3\theta\sin^2\theta + \frac{60}{6}i^3\cos^2\theta\sin^3\theta + \frac{120}{24}i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$
$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$
$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

(a). Comparing Real terms on both sides, we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(\sin^2\theta)^2$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$
$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$
$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$
$$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(b). Comparing imaginary terms on both sides, we get

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$or,\ \frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + (1-2\cos^2\theta+\cos^4\theta)$$
$$= 15\cos^4\theta - 10\cos^2\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 16\cos^4\theta - 12\cos^2\theta + 1 \qquad \textbf{(Proved)}$$

Problem-12: Prove the identities:

(a) $\sin^3\theta = \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin 3\theta$

(b) $\cos^4\theta = \dfrac{1}{8}\cos 4\theta + \dfrac{1}{2}\cos 2\theta + \dfrac{3}{8}$

Solution: (a) we have, $\sin\theta = \left(\dfrac{e^{i\theta}-e^{-i\theta}}{2i}\right)$

$$\therefore\ \sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3$$
$$= \frac{\left(e^{i\theta}-e^{-i\theta}\right)^3}{8i^3}$$
$$= -\frac{1}{8i}\left(e^{3i\theta} - 3e^{2i\theta}\cdot e^{-i\theta} + 3e^{i\theta}\cdot e^{-2i\theta} - e^{-3i\theta}\right)$$

$$= -\frac{1}{8i}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right)$$

$$= \frac{3}{4}\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) - \frac{1}{4}\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \qquad \textbf{(Proved)}$$

(b). we have, $\cos\theta = \left(\dfrac{e^{i\theta} + e^{-i\theta}}{2}\right)$

$$\therefore \cos^4\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4$$

$$= \frac{\left(e^{i\theta} + e^{-i\theta}\right)^4}{16}$$

$$= \frac{1}{16}\left\{\left(e^{i\theta}\right)^4 + 4\left(e^{i\theta}\right)^3 \cdot \left(e^{-i\theta}\right) + \frac{4\cdot 3}{2!}\left(e^{i\theta}\right)^2 \cdot \left(e^{-i\theta}\right)^2 + \frac{4\cdot 3\cdot 2}{3!}\left(e^{i\theta}\right)\cdot\left(e^{-i\theta}\right)^3 + \left(e^{-i\theta}\right)^4\right\}$$

$$= \frac{1}{16}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right)$$

$$= \frac{1}{8}\left(\frac{e^{4i\theta} + e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{3}{8}$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \qquad \textbf{(Proved)}$$

Problem-13: Find the square roots of the following complex numbers:

(a) $-15-8i$

(b) $5-12i$

(c) $8+4\sqrt{5}\,i$

Solution: (a) Let the square roots of the given complex number are

$$\sqrt{-15-8i} = p + iq$$

where p and q are real numbers.

$$\therefore \left(p+iq\right)^2 = -15-8i$$

$$or,\ p^2 + 2ipq - q^2 = -15-8i$$

$$or,\ \left(p^2 - q^2\right) + 2ipq = -15-8i$$

Separating real and imaginary parts, we get

$$p^2 - q^2 = -15 \qquad \cdots(1)$$

$$pq = -4 \qquad \cdots(2)$$

From (2), we get

$$q = -\frac{4}{p} \qquad \cdots(3)$$

From (1) and (3), we get

$$p^2 - \frac{16}{p^2} = -15$$

$$or,\ p^4 + 15p^2 - 16 = 0$$

$$or,\ p^4 - p^2 + 16p^2 - 16 = 0$$

$$or,\ \left(p^2 - 1\right)\left(p^2 + 16\right) = 0$$

$$\therefore\ p^2 - 1 = 0 \qquad or,\ \ p^2 + 16 = 0$$

$$or,\ p^2 = 1 \qquad or,\ \ p^2 = -16$$

Since p is real so $p^2 = -16$ is not acceptable.

$$\therefore\ p^2 = 1$$

$$or,\ p = \pm 1$$

when $p = 1$ then from (3), we have

$$q = -4$$

$$\therefore\ p + iq = 1 - 4i$$

when $p = -1$ then from (3), we have

$$q = 4$$

$$\therefore\ p + iq = -1 + 4i$$

$$\text{or,}\ \ p + iq = -\left(1 - 4i\right)$$

Thus, the square roots of the given complex number are

$$\sqrt{-15 - 8i} = \pm\left(1 - 4i\right) \qquad \textbf{(Ans)}$$

(b) Let the square roots of the given complex number are

$$\sqrt{5 - 12i} = p + iq$$

where p and q are real numbers.

$$\therefore\ \left(p + iq\right)^2 = 5 - 12i$$

$$or,\ p^2 + 2ipq - q^2 = 5 - 12i$$

$$or,\ \left(p^2 - q^2\right) + 2ipq = 5 - 12i$$

Separating real and imaginary parts, we get

$$p^2 - q^2 = 5 \qquad \cdots(1)$$

$$pq = -6 \qquad \cdots(2)$$

From (2), we get

$$q = -\frac{6}{p} \qquad \cdots(3)$$

From (1) and (3), we get

$$p^2 - \frac{36}{p^2} = 5$$

$$or,\ p^4 - 5p^2 - 36 = 0$$

$$or,\ p^4 - 9p^2 + 4p^2 - 36 = 0$$

$$or,\ \left(p^2 - 9\right)\left(p^2 + 4\right) = 0$$

$$\therefore\ p^2 - 9 = 0 \qquad or,\ \ p^2 + 4 = 0$$

$$or,\ p^2 = 9 \qquad or,\ p^2 = -4$$

Since p is real so $p^2 = -4$ is not acceptable.

$$\therefore\ p^2 = 9$$
$$or,\ p = \pm 3$$

when $p = 3$ then from (3), we have

$$q = -2$$
$$\therefore\ p + iq = 3 - 2i$$

when $p = -3$ then from (3), we have

$$q = 2$$
$$\therefore\ p + iq = -3 + 2i$$
$$\text{or, } p + iq = -(3 - 2i)$$

Thus, the square roots of the given complex number are

$$\sqrt{5 - 12i} = \pm(3 - 2i) \qquad \textbf{(Ans)}$$

(c) Let the square roots of the given complex number are

$$\sqrt{8 + 4\sqrt{5}i} = p + iq$$

where p and q are real numbers.

$$\therefore\ (p + iq)^2 = 8 + 4\sqrt{5}i$$
$$or,\ p^2 + 2ipq - q^2 = 8 + 4\sqrt{5}i$$
$$or,\ (p^2 - q^2) + 2ipq = 8 + 4\sqrt{5}i$$

Separating real and imaginary parts, we get

$$p^2 - q^2 = 8 \qquad \cdots(1)$$
$$pq = 2\sqrt{5} \qquad \cdots(2)$$

From (2), we get

$$q = \frac{2\sqrt{5}}{p} \qquad \cdots(3)$$

From (1) and (3), we get

$$p^2 - \frac{20}{p^2} = 8$$
$$or,\ p^4 - 8p^2 - 20 = 0$$
$$or,\ p^4 - 10p^2 + 2p^2 - 20 = 0$$
$$or,\ (p^2 - 10)(p^2 + 2) = 0$$
$$\therefore\ p^2 - 10 = 0 \qquad or,\ p^2 + 2 = 0$$
$$or,\ p^2 = 10 \qquad or,\ p^2 = -2$$

Since p is real so $p^2 = -2$ is not acceptable.

$$\therefore\ p^2 = 10$$
$$or,\ p = \pm\sqrt{10}$$

when $p = \sqrt{10}$ then from (3), we have

$$q = \sqrt{2}$$

$$\therefore\ p + iq = \sqrt{10} + \sqrt{2}i$$

when $p = -\sqrt{10}$ then from (3), we have

$$q = -\sqrt{2}$$

$$\therefore\ p + iq = -\sqrt{10} - \sqrt{2}i$$

$$\text{or, } p + iq = -\left(\sqrt{10} + \sqrt{2}i\right)$$

Thus, the square roots of the given complex number are

$$\sqrt{5-12i} = \pm\left(\sqrt{10} + \sqrt{2}i\right) \qquad \textbf{(Ans)}$$

Problem-14: Find (a) fifth roots of unity (1), (b) seventh roots of -1, (c) fourth roots of i.

Solution: (a) Let z is the fifth roots of unity.

$$\text{So, } z = (1)^{1/5}$$

$$= (\cos 0 + i \sin 0)^{\frac{1}{5}}$$

$$= \left\{\cos\left(2k\pi + 0\right) + i \sin\left(2k\pi + 0\right)\right\}^{\frac{1}{5}}$$

$$= \cos\left(\frac{2k\pi}{5}\right) + i \sin\left(\frac{2k\pi}{5}\right)$$

$$\therefore\ z = e^{2k\pi i/5}, \qquad k = 0, 1, 2, 3, 4.$$

Thus, the required roots are

$$1,\ e^{2\pi i/5},\ e^{4\pi i/5},\ e^{6\pi i/5},\ e^{8\pi i/5}. \qquad \textbf{(Ans)}$$

(b) Let z is the seventh roots of -1.

$$\text{So, } z = (-1)^{1/7}$$

$$= (\cos\pi + i \sin\pi)^{\frac{1}{7}}$$

$$= \left\{\cos\left(2k\pi + \pi\right) + i \sin\left(2k\pi + \pi\right)\right\}^{\frac{1}{7}}$$

$$= \cos\left(\frac{2k\pi + \pi}{7}\right) + i \sin\left(\frac{2k\pi + \pi}{7}\right)$$

$$\therefore\ z = e^{(2k\pi + \pi)i/7}, \qquad k = 0, 1, 2, 3, 4, 5, 6.$$

Thus, the required roots are

$$e^{\pi i/7},\ e^{3\pi i/7},\ e^{5\pi i/7},\ e^{\pi i},\ e^{9\pi i/7},\ e^{11\pi i/7},\ e^{13\pi i/7}. \qquad \textbf{(Ans)}$$

(c) Let z is the fourth roots of i.

$$\text{So, } z = (i)^{1/4}$$

$$= \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{\frac{1}{4}}$$

$$=\left\{\cos\left(2k\pi+\frac{\pi}{2}\right)+i\sin\left(2k\pi+\frac{\pi}{2}\right)\right\}^{\frac{1}{4}}$$

$$=\cos\left(\frac{4k\pi+\pi}{8}\right)+i\sin\left(\frac{4k\pi+\pi}{8}\right)$$

$$\therefore\ z=e^{(4k\pi+\pi)i/8},\qquad k=0,1,2,3.$$

Thus, the required roots are

$$e^{\pi i/8},\ e^{5\pi i/8},\ e^{9\pi i/8},\ e^{13\pi i/8}. \qquad \textbf{(Ans)}$$

Problem-15: Find each of the indicated roots of the following:

(a) $(-1+i)^{1/3}$, **(b)** $(1+i)^{1/4}$, **(c)** $\left(2\sqrt{3}-2i\right)^{1/2}$, **(d)** $(-4+4i)^{1/5}$, **(e)** $(64)^{1/6}$, **(f)** $(i)^{2/3}$, (g) $(-16i)^{1/4}$

Solution: (a) Let $z=(-1+i)^{1/3}$

The polar form of this complex number is

$$z=\left\{\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right\}^{1/3}$$

$$=2^{\frac{1}{6}}\left\{\cos\left(2k\pi+\frac{3\pi}{4}\right)+i\sin\left(2k\pi+\frac{3\pi}{4}\right)\right\}^{1/3}$$

$$=2^{\frac{1}{6}}\left\{\cos\left(\frac{8k\pi+3\pi}{4}\right)+i\sin\left(\frac{8k\pi+3\pi}{4}\right)\right\}^{1/3}$$

$$=2^{\frac{1}{6}}\left\{\cos\left(\frac{8k\pi+3\pi}{12}\right)+i\sin\left(\frac{8k\pi+3\pi}{12}\right)\right\}$$

$$\therefore\ z=2^{\frac{1}{6}}e^{(8k\pi+3\pi)i/12},\qquad k=0,1,2.$$

Thus, the required roots are

$$2^{\frac{1}{6}}e^{\pi i/4},\ 2^{\frac{1}{6}}e^{11\pi i/12},\ 2^{\frac{1}{6}}e^{19\pi i/12}. \qquad \textbf{(Ans)}$$

(b) Let $z=(1+i)^{1/4}$

The polar form of this complex number is

$$z=\left\{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right\}^{1/4}$$

$$=2^{\frac{1}{8}}\left\{\cos\left(2k\pi+\frac{\pi}{4}\right)+i\sin\left(2k\pi+\frac{\pi}{4}\right)\right\}^{1/4}$$

$$=2^{\frac{1}{8}}\left\{\cos\left(\frac{8k\pi+\pi}{4}\right)+i\sin\left(\frac{8k\pi+\pi}{4}\right)\right\}^{1/4}$$

$$=2^{\frac{1}{8}}\left\{\cos\left(\frac{8k\pi+\pi}{16}\right)+i\sin\left(\frac{8k\pi+\pi}{16}\right)\right\}$$

$$\therefore\ z=2^{\frac{1}{8}}e^{(8k\pi+\pi)i/16},\qquad k=0,1,2,3.$$

Thus, the required roots are

$$2^{\frac{1}{8}}e^{\pi i/16},\ 2^{\frac{1}{8}}e^{9\pi i/16},\ 2^{\frac{1}{8}}e^{17\pi i/16},\ 2^{\frac{1}{8}}e^{25\pi i/16}. \qquad \textbf{(Ans)}$$

(c) Let $z=\left(2\sqrt{3}-2i\right)^{1/2}$

The polar form of this complex number is

$$z=\left\{4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)\right\}^{1/2}$$
$$=2\left\{\cos\left(2k\pi+\frac{11\pi}{6}\right)+i\sin\left(2k\pi+\frac{11\pi}{6}\right)\right\}^{1/2}$$
$$=2\left\{\cos\left(\frac{12k\pi+11\pi}{6}\right)+i\sin\left(\frac{12k\pi+11\pi}{6}\right)\right\}^{1/2}$$
$$=2\left\{\cos\left(\frac{12k\pi+11\pi}{12}\right)+i\sin\left(\frac{12k\pi+11\pi}{12}\right)\right\}$$
$$\therefore\ z=2e^{(12k\pi+11\pi)i/12},\qquad k=0,1.$$

Thus, the required roots are

$$2e^{11\pi i/12},\ 2e^{23\pi i/12}. \qquad \textbf{(Ans)}$$

(d) Let $z=\left(-4+4i\right)^{1/5}$

The polar form of this complex number is

$$z=\left\{\sqrt{32}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right\}^{1/5}$$
$$=\sqrt{2}\left\{\cos\left(2k\pi+\frac{3\pi}{4}\right)+i\sin\left(2k\pi+\frac{3\pi}{4}\right)\right\}^{1/5}$$
$$=\sqrt{2}\left\{\cos\left(\frac{8k\pi+3\pi}{4}\right)+i\sin\left(\frac{8k\pi+3\pi}{4}\right)\right\}^{1/5}$$
$$=\sqrt{2}\left\{\cos\left(\frac{8k\pi+3\pi}{20}\right)+i\sin\left(\frac{8k\pi+3\pi}{20}\right)\right\}$$
$$\therefore\ z=\sqrt{2}e^{(8k\pi+3\pi)i/20},\qquad k=0,1,2,3,4.$$

Thus, the required roots are

$$\sqrt{2}e^{3\pi i/20},\ \sqrt{2}e^{11\pi i/20},\ \sqrt{2}e^{19\pi i/20},\ \sqrt{2}e^{27\pi i/20},\ \sqrt{2}e^{35\pi i/20}. \qquad \textbf{(Ans)}$$

(e) Let $z=\left(64\right)^{1/6}$

The polar form of this complex number is

$$z=\left\{64\left(\cos 0+i\sin 0\right)\right\}^{1/6}$$
$$=2\left\{\cos\left(2k\pi\right)+i\sin\left(2k\pi\right)\right\}^{1/6}$$

$$= 2\left\{\cos\left(\frac{k\pi}{3}\right) + i\sin\left(\frac{k\pi}{3}\right)\right\}$$

$$\therefore\ z = 2e^{k\pi i/3}, \qquad k = 0,1,2,3,4,5.$$

Thus, the required roots are

$$2,\ 2e^{\pi i/3},\ 2e^{2\pi i/3},\ 2e^{\pi i},\ 2e^{4\pi i/3},\ 2e^{5\pi i/3}.$$ **(Ans)**

(f) Let $z = (i)^{2/3} = (-1)^{1/3}$

The polar form of this complex number is

$$z = (\cos\pi + i\sin\pi)^{\frac{1}{3}}$$

$$= \{\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\}^{\frac{1}{3}}$$

$$= \cos\left(\frac{2k\pi + \pi}{3}\right) + i\sin\left(\frac{2k\pi + \pi}{3}\right)$$

$$\therefore\ z = e^{(2k\pi + \pi)i/3}, \qquad k = 0,1,2.$$

Thus, the required roots are

$$e^{\pi i/3},\ e^{\pi i},\ e^{5\pi i/3}.$$ **(Ans)**

(g) Let $z = (-16i)^{1/4}$

The polar form of this complex number is

$$z = \left\{16\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right\}^{\frac{1}{4}}$$

$$= 2\left\{\cos\left(2k\pi + \frac{3\pi}{2}\right) + i\sin\left(2k\pi + \frac{3\pi}{2}\right)\right\}^{\frac{1}{4}}$$

$$= 2\left\{\cos\left(\frac{4k\pi + 3\pi}{2}\right) + i\sin\left(\frac{4k\pi + 3\pi}{2}\right)\right\}^{\frac{1}{4}}$$

$$= 2\left\{\cos\left(\frac{4k\pi + 3\pi}{8}\right) + i\sin\left(\frac{4k\pi + 3\pi}{8}\right)\right\}$$

$$\therefore\ z = 2e^{(4k\pi + 3\pi)i/8}, \qquad k = 0,1,2,3.$$

Thus, the required roots are

$$2e^{3\pi i/8},\ 2e^{7\pi i/8},\ 2e^{11\pi i/8},\ 2e^{15\pi i/8}.$$ **(Ans)**

Problem-16: Solve the equations

(i) $z^2 + (2i - 3)z + 5 - i = 0.$

(ii) $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0.$

(iii) $z^2(1 - z^2) = 16.$

Solution: (i). Given that,

$$z^2 + (2i - 3)z + 5 - i = 0$$

Since it is a quadratic equation so its solutions are,

$z = \frac{-(2i-3)\pm\sqrt{(2i-3)^2-4(1)(5-i)}}{2(1)}$ [If $az^2+bz+c=0$ then z = $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$]

$= \frac{3-2i\pm\sqrt{-15-8i}}{2}$

$= \frac{3-2i\pm(1-4i)}{2}$

$= 2-3i$ and $1+i$ **Ans.**

(ii). Given that, $6z^4-25z^3+32z^2+3z-10=0$ ... ... (1)

The integer factors of 6 and 10 are respectively $\pm1, \pm2, \pm3, \pm6$ and $\pm1, \pm2, \pm5, \pm10$. The possible rational solutions are $\pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm2, \pm\frac{2}{3}, \pm5, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm10, \pm\frac{10}{3}$.

Let, $f(z) = 6z^4-25z^3+32z^2+3z-10$

By trial solution we get, $f\left(-\frac{1}{2}\right)=0$;

Therefore, $z=-\frac{1}{2}$ or $2z+1$ is a factor.

Now,

$$6z^4-25z^3+32z^2+3z-10 = 3z^3(2z+1)-14z^2(2z+1)+23z(2z+1)-10(2z+1)$$
$$= (2z+1)(3z^3-14z^2+23z-10)$$

Again let $g(z)=3z^3-14z^2+23z-10$

By trial solution, $g(2/3)=0$

Therefore, $z=\frac{2}{3}$ or $(3z-2)$ is a factor of $g(z)$.

$$3z^3-14z^2+23z-10 = z^2(3z-2)-4z(3z-2)+5(3z-2)$$
$$= (3z-2)(z^2-4z+5)$$

Equation (1) becomes,

$$(2z+1)(3z-2)(z^2-4z+5)=0$$

Hence, $z=-\frac{1}{2}$, $\frac{2}{3}$ and $(z^2-4z+5)=0$.

$$\Rightarrow z = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm\sqrt{-4}}{2} = \frac{4+2i}{2} = 2\pm i$$

Thus the solutions are $-\frac{1}{2}, \frac{2}{3}$, $2+i, 2-i$. **Ans.**

(iii). Given that, $z^2(1-z^2)=16$

$\Rightarrow z^2-z^4=16$

$\Rightarrow z^4-z^2+16=0$

$\Rightarrow z^4+8z^2+16-9z^2=0$

$\Rightarrow (z^2+4)^2-9z^2=0$

$\Rightarrow (z^2+4+3z)(z^2+4-3z)=0$

$\Rightarrow (z^2+4+3z)=0$ and $(z^2+4-3z)=0$

$\therefore (z^2+4+3z)=0 \Rightarrow z = \frac{-3\pm\sqrt{9-16}}{2} = \frac{-3\pm\sqrt{7}i}{2}$

$\therefore (z^2+4-3z)=0 \Rightarrow z = \frac{3\pm\sqrt{9-16}}{2} = \frac{3\pm\sqrt{7}i}{2}$

Therefore, the roots are: $-\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$ and $\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$ **Ans.**

Problem-17: Solve the following equations.

(a) $z^5=-32$, **(b)** $z^4+81=0$, **(c)** $z^6+1=\sqrt{3}\,i$

Solution: (a) Given that $z^5=-32$

$$or,\ z=(-32)^{\frac{1}{5}}$$
$$=\{32(\cos\pi+i\sin\pi)\}^{\frac{1}{5}}$$
$$=2\{\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{\frac{1}{5}}$$
$$=2\left\{\cos\left(\frac{2k\pi+\pi}{5}\right)+i\sin\left(\frac{2k\pi+\pi}{5}\right)\right\};\quad k=0,1,2,3,4.\qquad \textbf{(Ans)}$$

(b) Given that $z^4+81=0$

$$or,\ z=(-81)^{\frac{1}{4}}$$
$$=\{81(\cos\pi+i\sin\pi)\}^{\frac{1}{4}}$$
$$=3\{\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{\frac{1}{4}}$$
$$=3\left\{\cos\left(\frac{2k\pi+\pi}{4}\right)+i\sin\left(\frac{2k\pi+\pi}{4}\right)\right\};\quad k=0,1,2,3.\qquad \textbf{(Ans)}$$

(c) Given that $z^6+1=\sqrt{3}\,i$

$$or,\ z^6=-1+\sqrt{3}\,i$$

The polar form of this complex number is

$$z^6=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$
$$or,\ z=\left\{2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right\}^{\frac{1}{6}}$$
$$=2^{\frac{1}{6}}\left\{\cos\left(2k\pi+\frac{2\pi}{3}\right)+i\sin\left(2k\pi+\frac{2\pi}{3}\right)\right\}^{\frac{1}{6}}$$
$$=2^{\frac{1}{6}}\left\{\cos\left(\frac{6k\pi+2\pi}{3}\right)+i\sin\left(\frac{6k\pi+2\pi}{3}\right)\right\}^{\frac{1}{6}}$$
$$=2^{\frac{1}{6}}\left\{\cos\left(\frac{6k\pi+2\pi}{18}\right)+i\sin\left(\frac{6k\pi+2\pi}{18}\right)\right\};\quad k=0,1,2,3,4,5.\qquad \textbf{(Ans)}$$

Problem-18: Find all solutions of the following equations:

a. $\sinh z = i$

b. $\sinh z = 2$

c. $\cosh z = 2$

Solution: (a) Given that $\sinh z = i$ (1)

We know that, $\cosh z + \sinh z = e^z$

$\cosh z - \sinh z = e^{-z}$

From (1), we can write

$$\frac{e^z - e^{-z}}{2} = i$$

$$or,\ e^z - \frac{1}{e^z} = 2i$$

$$or,\ \left(e^z\right)^2 - 2ie^z - 1 = 0$$

$$or,\ e^z = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$or,\ e^z = \frac{2i \pm \sqrt{-4+4}}{2}$$

$$or,\ e^z = \frac{2i}{2}$$

$$or,\ e^z = i$$

$$or,\ e^z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$or,\ e^z = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$or,\ e^z = e^{(4n\pi+\pi)i/2}$$

$$\therefore\quad z = \frac{(4n\pi + \pi)i}{2};\qquad where\ n = 0, \pm1, \pm2, \cdots$$ **(Ans)**

(b) Given that $\sinh z = 2$ (1)

We know that, $\cosh z + \sinh z = e^z$

$\cosh z - \sinh z = e^{-z}$

From (1), we can write

$$\frac{e^z - e^{-z}}{2} = 2$$

$$or,\ e^z - \frac{1}{e^z} = 4$$

$$or,\ \left(e^z\right)^2 - 4e^z - 1 = 0$$

$$or,\ e^z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$or,\ e^z = \frac{4 \pm \sqrt{20}}{2}$$

$$or,\ e^z = \frac{4 \pm 2\sqrt{5}}{2}$$

$$or,\ e^z = 2 \pm \sqrt{5}$$

$$\therefore\ z = \ln\left(2 \pm \sqrt{5}\right)$$ **(Ans)**

(c) Given that $\cosh z = 2$ (1)

We know that, $\cosh z + \sinh z = e^z$

$\cosh z - \sinh z = e^{-z}$

From (1), we can write

$$\frac{e^z + e^{-z}}{2} = 2$$

$$or,\ e^z + \frac{1}{e^z} = 4$$

$$or,\ \left(e^z\right)^2 - 4e^z + 1 = 0$$

$$or,\ e^z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$or,\ e^z = \frac{4 \pm \sqrt{12}}{2}$$

$$or,\ e^z = \frac{4 \pm 2\sqrt{3}}{2}$$

$$or,\ e^z = 2 \pm \sqrt{3}$$

$$\therefore\ z = \ln\left(2 \pm \sqrt{3}\right)$$ **(Ans)**.

Problem-19: For any complex number $z_1, z_2, \cdots, z_n$ prove the followings:

- **a.** $|z|^2 = z\bar{z}$
- **b.** $\bar{\bar{z}} = z$
- **c.** $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- **d.** $|z_1 z_2| = |z_1||z_2|$
- **e.** $|z_1 + z_2| \le |z_1| + |z_2|$
- **f.** $|z_1 - z_2| \ge |z_1| - |z_2|$

Solution: (a) Let $z = x + iy$. Then $\bar{z} = x - iy$

Now $|z|^2 = |x + iy|^2$

$$= \left(\sqrt{x^2 + y^2}\right)^2$$

$$= x^2 + y^2$$

and $z\bar{z}=(x+iy)(x-iy)$

$=x^2-i^2y^2$

$=x^2+y^2$

Hence $|z|^2=z\bar{z}$ **(Proved)**

(b) Let $z=x+iy$. Then $\bar{z}=x-iy$

Now $\bar{\bar{z}}=\overline{x-iy}$

$=x+iy$

$=z$

Hence $\bar{\bar{z}}=z$ **(Proved)**

(c) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$. Then $\bar{z}_1=x_1-iy_1$ and $\bar{z}_2=x_2-iy_2$.

Now $\overline{z_1+z_2}=\overline{(x_1+iy_1)+(x_2+iy_2)}$

$=\overline{(x_1+x_2)+i(y_1+y_2)}$

$=(x_1+x_2)-i(y_1+y_2)$

$=(x_1-iy_1)+(x_2-iy_2)$

$=\overline{z_1}+\overline{z_2}$

Hence $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$ **(Proved)**

(d) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

$\therefore |z_1|=\sqrt{x_1^2+y_1^2}$ *and* $|z_2|=\sqrt{x_2^2+y_2^2}$

Now $|z_1z_2|=|(x_1+iy_1)(x_2+iy_2)|$

$=|(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)|$

$=\sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$

$=\sqrt{x_1^2x_2^2-2x_1x_2y_1y_2+y_1^2y_2^2+x_1^2y_2^2+2x_1x_2y_1y_2+x_2^2y_1^2}$

$=\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2}$

$=\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$=\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$

$=|z_1||z_2|$

Hence $|z_1z_2|=|z_1||z_2|$ **(Proved)**

(e) we know that $|z_1+z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}$

$$=(z_1+z_2)(\overline{z_1}+\overline{z_2})$$
$$=z_1\overline{z_1}+z_1\overline{z_2}+z_2\overline{z_1}+z_2\overline{z_2}$$
$$=|z_1|^2+z_1\overline{z_2}+z_2\overline{z_1}+|z_2|^2$$
$$=|z_1|^2+z_1\overline{z_2}+\overline{z_1\overline{z_2}}+|z_2|^2$$
$$=|z_1|^2+2\operatorname{Re}(z_1\overline{z_2})+|z_2|^2$$

$or,\ |z_1+z_2|^2\le|z_1|^2+2|z_1\overline{z_2}|+|z_2|^2$

$or,\ |z_1+z_2|^2\le|z_1|^2+2|z_1||\overline{z_2}|+|z_2|^2$

$or,\ |z_1+z_2|^2\le|z_1|^2+2|z_1||z_2|+|z_2|^2$

$or,\ |z_1+z_2|^2\le(|z_1|+|z_2|)^2$

$or,\ |z_1+z_2|\le|z_1|+|z_2|$

Hence $|z_1+z_2|\le|z_1|+|z_2|$ **(Proved)**

(f) we know that $|z_1-z_2|^2=(z_1-z_2)\overline{(z_1-z_2)}$

$$=(z_1-z_2)(\overline{z_1}-\overline{z_2})$$
$$=z_1\overline{z_1}-z_1\overline{z_2}-z_2\overline{z_1}+z_2\overline{z_2}$$
$$=|z_1|^2-(z_1\overline{z_2}+z_2\overline{z_1})+|z_2|^2$$
$$=|z_1|^2-(z_1\overline{z_2}+\overline{z_1\overline{z_2}})+|z_2|^2$$
$$=|z_1|^2-2\operatorname{Re}(z_1\overline{z_2})+|z_2|^2$$

$or,\ |z_1-z_2|^2\ge|z_1|^2-2|z_1\overline{z_2}|+|z_2|^2$

$or,\ |z_1-z_2|^2\ge|z_1|^2-2|z_1||\overline{z_2}|+|z_2|^2$

$or,\ |z_1-z_2|^2\ge|z_1|^2-2|z_1||z_2|+|z_2|^2$

$or,\ |z_1-z_2|^2\ge(|z_1|-|z_2|)^2$

$or,\ |z_1-z_2|\ge|z_1|-|z_2|$

Hence $|z_1-z_2|\ge|z_1|-|z_2|$ **(Proved)**

Problem-20: Describe geometrically the region of the followings:

a. $|z-4|>|z|$

b. $|z-i|=|z+i|$

c. $|z+3i|>4$

d. $|z-2+i|\le 1$

e. $\left|\dfrac{z-3}{z+3}\right|=3$

f. $\left|\dfrac{z-3}{z+3}\right|>3$

g. $\left|\dfrac{z-3}{z+3}\right|<3$

h. $\operatorname{Re}\left(\dfrac{1}{z}\right)<\dfrac{1}{2}$

i. $1<|z+i|\le 2$

j. $1<|z-2i|\le 2$

k. $-\pi<\arg z<\pi$

l. $-\pi<\arg z<\pi,\ \ z\ne 0$

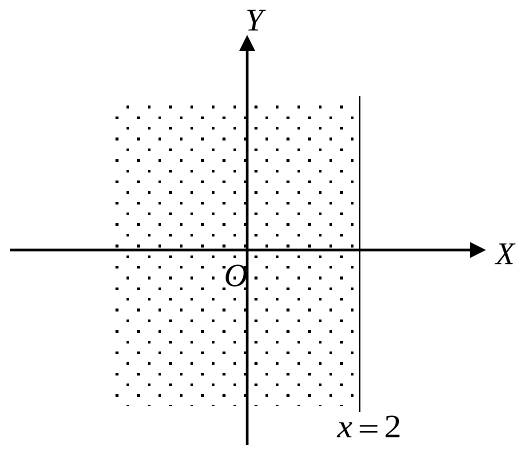
m. $0<\arg z<2\pi,\ \ |z|>0$

n. $-\pi<\arg z<\pi,\ \ |z|>2$

Solution: (a) Given that $|z-4|>|z|$

Let $z=x+iy$. The given expression reduces as,

$$|x+iy-4|>|x+iy|$$
$$or,\ |(x-4)+iy|>|x+iy|$$
$$or,\sqrt{(x-4)^2+y^2}>\sqrt{x^2+y^2}$$
$$or,x^2-8x+16+y^2>x^2+y^2$$
$$or,-8x+16>0$$
$$or,-8x>-16$$
$$or,\ \ x<2$$

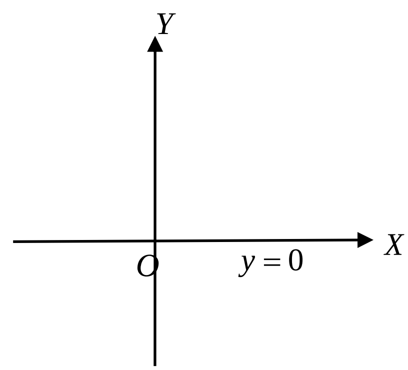


Therefore, the region is the set of all points (x,y) such that $x<2$. That is, the set of all points lie in the left hand side of the straight line $x=2$.

(b) Given that $|z-i|=|z+i|$

Let $z=x+iy$. The given equation reduces as,

$$|x+iy-i|=|x+iy+i|$$
$$or,\ |x+i(y-1)|=|x+i(y+1)|$$
$$or,\sqrt{x^2+(y-1)^2}=\sqrt{x^2+(y+1)^2}$$

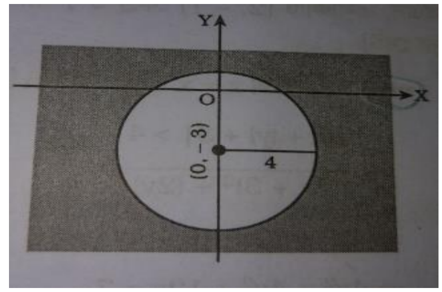


$$or, x^2+y^2-2y+1=x^2+y^2+2y+1$$
$$or, -4y=0$$
$$or,\ y=0$$

Therefore, the region is the set of all points (x,y) such that $y=0$. That is, the set of all points lie on the real axis.

(c) Given that $|z+3i|>4$

Let $z=x+iy$. The given expression reduces as,

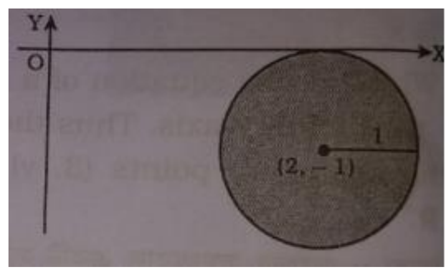


$$|x+iy+3i|>4$$
$$or, |x+i(y+3)|>4$$
$$or, \sqrt{x^2+(y+3)^2}>4$$
$$or, x^2+(y+3)^2>4^2$$

Therefore, the region is the set of all external points of the circle $x^2+(y+3)^2=4^2$, whose centre is $(0,-3)$ and radius is 4.

(d) Given that $|z-2+i|\le 1$

Let $z=x+iy$. The given expression reduces as,

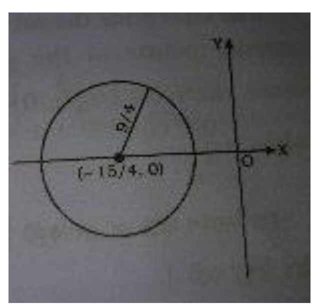


$$|x+iy-2+i|\le 1$$
$$or, |(x-2)+i(y+1)|\le 1$$
$$or, \sqrt{(x-2)^2+(y+1)^2}\le 1$$
$$or, (x-2)^2+(y+1)^2\le 1^2$$

Therefore, the region is the set of all internal points including boundary points of the circle $(x-2)^2+(y+1)^2=1^2$, whose centre is $(2,-1)$ and radius is 1.

(e) Given that $\left|\dfrac{z-3}{z+3}\right|=3$

Let $z=x+iy$. The given equation reduces as,



$$\frac{|x+iy-3|}{|x+iy+3|}=3$$
$$or, |(x-3)+iy|=3|(x+3)+iy|$$
$$or, \sqrt{(x-3)^2+y^2}=3\sqrt{(x+3)^2+y^2}$$
$$or, (x-3)^2+y^2=9(x+3)^2+9y^2$$
$$or, x^2-6x+9+y^2=9x^2+54x+81+9y^2$$
$$or, 8x^2+60x+72+8y^2=0$$
$$or, x^2+\frac{15}{2}x+9+y^2=0$$

$$or,\ x^2+2\cdot x\cdot\frac{15}{4}+\left(\frac{15}{4}\right)^2+9+y^2=\left(\frac{15}{4}\right)^2$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2=\frac{225}{16}-9$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2=\frac{225-144}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2=\frac{81}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2=\left(\frac{9}{4}\right)^2$$

Therefore, the region is the set of all boundary points of the circle $\left(x+\frac{15}{4}\right)^2+y^2=\left(\frac{9}{4}\right)^2$, whose centre is $\left(-\frac{15}{4},0\right)$ and radius is $\frac{9}{4}$.

(f) Given that $\left|\frac{z-3}{z+3}\right|>3$

Let $z=x+iy$. The given expression reduces as,

$$\frac{|x+iy-3|}{|x+iy+3|}>3$$

$$or,\ |(x-3)+iy|>3|(x+3)+iy|$$

$$or,\ \sqrt{(x-3)^2+y^2}>3\sqrt{(x+3)^2+y^2}$$

$$or,\ (x-3)^2+y^2>9(x+3)^2+9y^2$$

$$or,\ x^2-6x+9+y^2>9x^2+54x+81+9y^2$$

$$or,\ 8x^2+60x+72+8y^2<0$$

$$or,\ x^2+\frac{15}{2}x+9+y^2<0$$

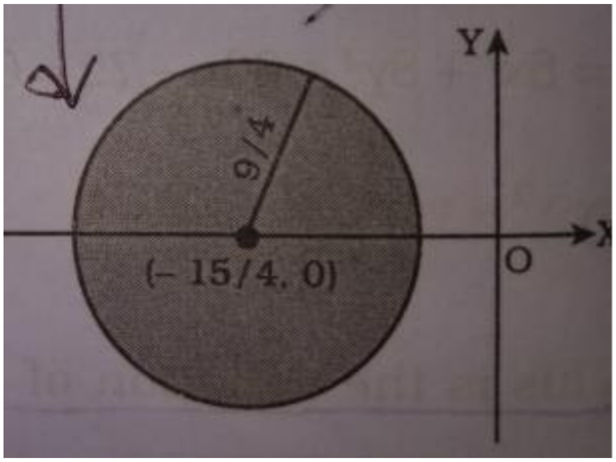
$$or,\ x^2+2\cdot x\cdot\frac{15}{4}+\left(\frac{15}{4}\right)^2+9+y^2<\left(\frac{15}{4}\right)^2$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2<\frac{225}{16}-9$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2<\frac{225-144}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2<\frac{81}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2<\left(\frac{9}{4}\right)^2$$



Therefore, the region is the set of all internal points of the circle $\left(x+\frac{15}{4}\right)^2+y^2=\left(\frac{9}{4}\right)^2$, whose centre is $\left(-\frac{15}{4},0\right)$ and radius is $\frac{9}{4}$.

(g) Given that $\left|\frac{z-3}{z+3}\right|<3$

Let $z=x+iy$. The given expression reduces as,

$$\frac{|x+iy-3|}{|x+iy+3|}<3$$

$$or,\ |(x-3)+iy|<3|(x+3)+iy|$$

$$or,\ \sqrt{(x-3)^2+y^2}<3\sqrt{(x+3)^2+y^2}$$

$$or,\ (x-3)^2+y^2<9(x+3)^2+9y^2$$

$$or,\ x^2-6x+9+y^2<9x^2+54x+81+9y^2$$

$$or,\ 8x^2+60x+72+8y^2>0$$

$$or,\ x^2+\frac{15}{2}x+9+y^2>0$$

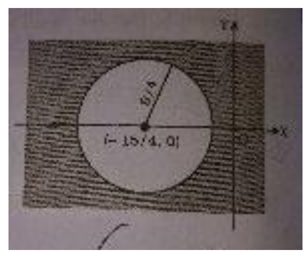
$$or,\ x^2+2\cdot x\cdot\frac{15}{4}+\left(\frac{15}{4}\right)^2+9+y^2>\left(\frac{15}{4}\right)^2$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2>\frac{225}{16}-9$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2>\frac{225-144}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2>\frac{81}{16}$$

$$or,\ \left(x+\frac{15}{4}\right)^2+y^2>\left(\frac{9}{4}\right)^2$$



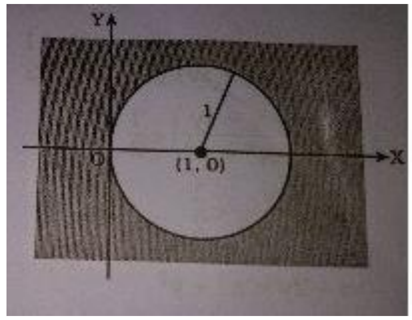
Therefore, the region is the set of all external points of the circle $\left(x+\frac{15}{4}\right)^2+y^2=\left(\frac{9}{4}\right)^2$, whose centre is $\left(-\frac{15}{4},0\right)$ and radius is $\frac{9}{4}$.

(h) Given that $\mathrm{Re}\left(\frac{1}{z}\right)<\frac{1}{2}$

Let $z=x+iy$. The given expression reduces as,

$$\mathrm{Re}\left(\frac{1}{x+iy}\right)<\frac{1}{2}$$

$$or,\ \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right)<\frac{1}{2}$$
$$or,\ \frac{x}{x^2+y^2}<\frac{1}{2}$$
$$or,\ x^2+y^2-2x>0$$
$$or,\ (x-1)^2+y^2>1$$



Therefore, the region is the set of all external points of the circle $(x-1)^2+y^2=1$, whose centre is $(1,0)$ and radius is 1.

(i) Given that $1<|z+i|\le 2$

Let $z=x+iy$. The given expression reduces as,

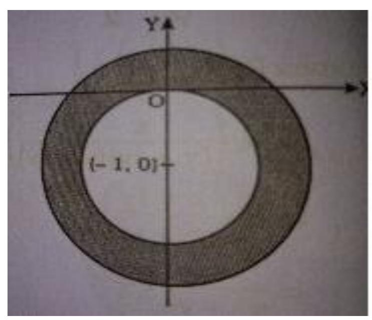
$$1<|x+iy+i|\le 2$$
$$or,\ 1<|x+i(y+1)|\le 2$$
$$or,\ 1<\sqrt{x^2+(y+1)^2}\le 2$$
$$or,\ 1^2<x^2+(y+1)^2\le 2^2$$

Therefore, the region is the set of all common points of external points of the circle $x^2+(y+1)^2=1^2$ and internal points of the circle $x^2+(y+1)^2=2^2$ including its boundary points.

(j) Given that $1<|z-2i|\le 2$

Let $z=x+iy$. The given expression reduces as,

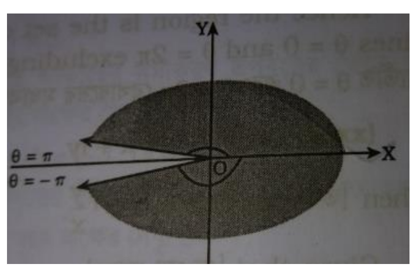
$$1<|x+iy-2i|\le 2$$
$$or,\ 1<|x+i(y-2)|\le 2$$
$$or,\ 1<\sqrt{x^2+(y-2)^2}\le 2$$
$$or,\ 1^2<x^2+(y-2)^2\le 2^2$$



Therefore, the region is the set of all common points of external points of the circle $x^2+(y-2)^2=1^2$ and internal points of the circle $x^2+(y-2)^2=2^2$ including its boundary points.

(k) Given that $-\pi<\arg z<\pi$

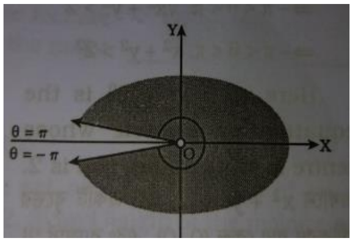
$$or,\ -\pi<\theta<\pi$$



Therefore, the region is the set of all infinite points between the lines $\theta=\pi$ and $\theta=-\pi$.

(l) Given that $-\pi < \arg z < \pi,\ \ z \neq 0$

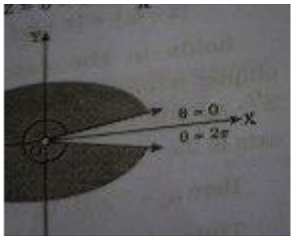
$or,\ -\pi < \theta < \pi,\ \ z \neq 0$



Therefore, the region is the set of all infinite points between the lines $\theta = \pi$ and $\theta = -\pi$ excluding the origin.

(m) Given that $0 < \arg z < 2\pi,\ |z| > 0$

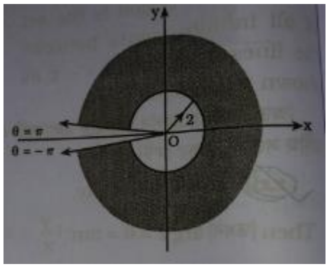
$or,\ 0 < \theta < 2\pi,\ \ x^2 + y^2 > 0$



Therefore, the region is the set of all infinite points between the lines $\theta = 0$ and $\theta = 2\pi$ excluding the origin.

(n) Given that $-\pi < \arg z < \pi,\ |z| > 2$

$or,\ -\pi < \theta < \pi,\ \ x^2 + y^2 > 2^2$



Therefore, the region is the set of all infinite common points among the lines $\theta = -\pi$, $\theta = \pi$ and external of the circle $x^2 + y^2 = 2^2$.

Exercise:

Problem-01: Express each of the following complex numbers in polar form.

(a). $2\sqrt{2} + 2\sqrt{2}i$ (b). $-2\sqrt{3} - 2i$ (c). $\frac{\sqrt{3}}{2} - \frac{3}{2}i$ (d). $-3 - 4i$

Problem-02: Express in A+iB form.

(a) $\frac{(8\ cis\ 40^o)^3}{(2\ cis\ 60^o)^4}$ (b) $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4 \left(\frac{1+i}{1-i}\right)^5$

Problem-03: Show that (a). $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b). $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Problem-04: Show that (a). $\frac{\sin 4\theta}{sin\theta} = 8\cos^3\theta - 4 = 2cos3\theta + 6cos\theta - 4$

(b). $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$

Problem-05: Solve: (a). $z^5 - 2z^4 - z^3 + 6z - 4 = 0$

(b). $z^4 + z^2 + 1 = 0.$

(c). $5z^2 + 2z + 10 = 0$

(d). $z^2 + (i-2)z + (3-i) = 0$