Matrix: A system of any *mn* numbers arranged in a rectangular array of *m* rows and *n* columns is called a matrix of order $m \times n$. A matrix is usually denoted by a single capital letter, namely A, B, C, … … or by the symbols $\left[a_{ij}\right]$, $\left(a_{ij}\right)$, $\left\|a_{ij}\right\|$.

The matrix of order $m \times n$ is written as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Example: A $= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3}$ **; B** $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$ **; C** $= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$**; D** $= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ **.**

Determinant: A determinant is a particular type of expression written in a special notation. The determinant of order n is a square array of n^2 quantities $a_{ij}\left(i, j = 1, 2, 3, \cdots\cdots, n\right)$ enclosed between two vertical lines and represented as follows: $\Delta = \left|a_{ij}\right| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Distinguish between matrix and determinant: The differences between a matrix and a determinant are as follows:

Matrix	Determinant
1. A matrix cannot be reduced to a single number.	**1**. A determinant can be reduced to a single number.
2. In a matrix, the number of rows may not be equal to the number of columns.	**2**. In a determinant, the number of rows must be equal to the number of columns.
3. An interchange of rows or columns gives a different matrix.	**3**. An interchange of rows or columns gives the same determinant with +*ve* or –*ve* sign.
4. Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.	**4**. Examples: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$; $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 2 & 3 & 4 \end{vmatrix}$.

Order or Dimension: The **order** or **dimension** of a matrix is given by stating the number of rows and the number of columns in the matrix.

Example: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 3 & -2 & 0 \\ -2 & 1 & 0 & 4 \end{bmatrix}$ is a matrix of order 3×4.

Rectangular Matrix: A matrix ***A*** of order $m \times n$ is called a rectangular matrix if the number of rows and the number of columns are not equal ***i.e,*** $m \neq n$.

Example: $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Square Matrix: A matrix ***A*** of order $m \times n$ is called a square matrix if the number of rows and the number of columns are equal ***i.e,*** $m = n$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

Null Matrix: A matrix is called a **Zero matrix** or **Null matrix** if each element is zero and is denoted by ***O***.

Example: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Identity Matrix: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ is when $i = j$ called the **Identity matrix or Unit matrix** and is denoted by ***I*** or ***U.***

Example: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Diagonal Matrix: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a **Diagonal matrix**. The elements a_{ij} when $i = j$ are known as diagonal elements and the line along which they lie is known as the **principal diagonal** or **leading diagonal**.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Singular matrix: A square matrix ***A*** is said to be a **singular matrix**, if the determinant of ***A*** is zero**,** ***i.e.*** $|A| = 0$.

Example: Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$**;** then $|A| = \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0.$

Non-singular matrix: A square matrix ***A*** is said to be a **non-singular matrix**, if the determinant of ***A*** is not zero**, i.e.** $|A| \neq 0$.

Example: Let $A=\begin{bmatrix}1 & 2\\ -1 & 2\end{bmatrix}$**;** then $|A|=\begin{vmatrix}1 & 2\\ -1 & 2\end{vmatrix}\neq 0.$

<u>Addition of matrices:</u> If ***A*** and ***B*** be two matrices of order $m\times n$ given by $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then the matrix $A+B$ is defined as the matrix each element of which is the sum of the corresponding elements of ***A*** and ***B*** i.e. $A+B=\left[a_{ij}+b_{ij}\right]$, where $i=1,2,3,.....,m$ and $j=1,2,3,.....,n$.

Example: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ *and* $B=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ then $A+B=\begin{bmatrix}1+2 & 2+3\\ 3+4 & 4+5\end{bmatrix}=\begin{bmatrix}3 & 5\\ 7 & 9\end{bmatrix}$.

<u>Subtraction</u>: If ***A*** and ***B*** be two matrices of order $m\times n$ given by $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then the matrix $A-B$ is defined as the matrix each element of which is obtained by subtracting the elements of ***B*** from the corresponding elements of ***A*** i.e. $A-B=\left[a_{ij}-b_{ij}\right]$, where $i=1,2,3,.....,m$ and $j=1,2,3,.....,n$.

Example: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ *and* $B=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ then $A-B=\begin{bmatrix}1-2 & 2-3\\ 3-4 & 4-5\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$.

<u>Multiplication of matrices</u>: If ***A*** and ***B*** be two matrices such that the number of columns in ***A*** is equal to the number rows in ***B*** i.e. if $A=\left[a_{ij}\right]$ and $B=\left[b_{jk}\right]$ are $m\times n$, $n\times p$ matrices then the product of the matrices ***A*** and ***B*** denoted by ***AB*** is defined as matrix

$$C=\left[c_{ik}\right]$$

$$=\sum_{j=1}^{n}a_{ij}b_{jk}$$

In the matrix product ***AB***, the matrix ***A*** is called the pre-multipliers and ***B*** is called the post- multipliers.

Example: If $A=\begin{bmatrix}1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$ then $AB=\begin{bmatrix}1+4 & 0+6\end{bmatrix}=\begin{bmatrix}5 & 6\end{bmatrix}$.

<u>Transpose of a matrix</u>: The matrix obtained from any given matrix ***A*** by interchanging its rows into columns or columns into rows is called its transpose. The transpose of ***A,*** is denoted by A^t *or* $A^{'}$**.**

Example: If $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & -2\end{bmatrix}$, then $A^t=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & -2\end{bmatrix}$.

Problem-01: If $A=\begin{pmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$ then find $A+B$ and $2A-B$.

Solution: The given matrices are,

$$A=\begin{pmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{pmatrix} \text{and } B=\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$$

Now, $A+B=\begin{pmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{pmatrix}+\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$

$$=\begin{pmatrix}1+3 & 2-1 & -3+2\\ 5+4 & 0+2 & 2+5\\ 1+2 & -1+0 & 1+3\end{pmatrix}$$

$$=\begin{pmatrix}4 & 1 & -1\\ 9 & 2 & 7\\ 3 & -1 & 4\end{pmatrix}$$

Again, $2A-B=2\begin{pmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{pmatrix}-\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$

$$=\begin{pmatrix}2 & 4 & -6\\ 10 & 0 & 4\\ 2 & -2 & 2\end{pmatrix}-\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$$

$$=\begin{pmatrix}2-3 & 4+1 & -6-2\\ 10-4 & 0-2 & 4-5\\ 2-2 & -2-0 & 2-3\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 5 & -8\\ 6 & -2 & -1\\ 0 & -2 & -1\end{pmatrix}.$$

Problem-02: If $A=\begin{pmatrix}1 & 4 & 0\\ 2 & 5 & 0\\ 3 & 6 & 0\end{pmatrix}$, $B=\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ and $C=\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 7 & 8 & 9\end{pmatrix}$ then find AB and BC.

Solution: The given matrices are,

$$A=\begin{pmatrix}1 & 4 & 0\\ 2 & 5 & 0\\ 3 & 6 & 0\end{pmatrix}, B=\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \text{ and } C=\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 7 & 8 & 9\end{pmatrix}$$

Now, $AB=\begin{pmatrix}1 & 4 & 0\\ 2 & 5 & 0\\ 3 & 6 & 0\end{pmatrix}\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$

$$=\begin{pmatrix}1.3+4.1+0.4 & 1.2+4.2+0.5 & 1.1+4.3+0.6\\ 2.3+5.1+0.4 & 2.2+5.2+0.5 & 2.1+5.3+0.6\\ 3.3+6.1+0.4 & 3.2+6.2+0.5 & 3.1+6.3+0.6\end{pmatrix}$$

$$=\begin{pmatrix}3+4+0 & 2+8+0 & 1+12+0\\ 6+5+0 & 4+10+0 & 2+15+0\\ 9+6+0 & 6+12+0 & 3+18+0\end{pmatrix}$$

$$=\begin{pmatrix}7 & 10 & 13\\ 11 & 14 & 17\\ 15 & 18 & 21\end{pmatrix}.$$

Again, $BC=\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 3\\ 7 & 8 & 9\end{pmatrix}$

$$=\begin{pmatrix}3.3+2.1+1.7 & 3.2+2.2+1.8 & 3.1+2.3+1.9\\ 1.3+2.1+3.7 & 1.2+2.2+3.8 & 1.1+2.3+3.9\\ 4.3+5.1+6.7 & 4.2+5.2+6.8 & 4.1+5.3+6.9\end{pmatrix}$$

$$=\begin{pmatrix}9+2+7 & 6+4+8 & 3+6+9\\ 3+2+21 & 2+4+24 & 1+6+27\\ 12+5+42 & 8+10+48 & 4+15+54\end{pmatrix}$$

$$=\begin{pmatrix}18 & 18 & 18\\ 26 & 30 & 34\\ 59 & 66 & 73\end{pmatrix}.$$

Problem-03: If $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 2 & 1\\ 1 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}$, then $4A+5AB-3I_3=?$

Solution: The given matrices are,

$$A=\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}-1 & 2 & 1\\ 1 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}$$

$$\therefore\ AB=\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 2 & 1\\ 1 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}-1+0+0 & 2+0+0 & 1+0+0\\ 2+3+0 & 4+0+0 & 2-3+0\\ -2-1+0 & 4+0+0 & 2+1+0\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 2 & 1\\ 5 & 4 & -1\\ -3 & 4 & 3\end{pmatrix}.$$

Now, $$4A+5AB-3I_3=4\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & -1\\ 2 & -1 & 1\end{pmatrix}+5\begin{pmatrix}-1 & 2 & 1\\ 5 & 4 & -1\\ -3 & 4 & 3\end{pmatrix}-3\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}4 & 0 & 0\\ 8 & 12 & -4\\ 8 & -4 & 4\end{pmatrix}+\begin{pmatrix}-5 & 10 & 5\\ 25 & 20 & -5\\ -15 & 20 & 15\end{pmatrix}-\begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$$

$$=\begin{pmatrix}4-5-3 & 0+10-0 & 0+5-0\\ 8+25-0 & 12+20-3 & -4-5-0\\ 8-15-0 & -4+20-0 & 4+15-3\end{pmatrix}$$

$$=\begin{pmatrix}-4 & 10 & 5\\ 23 & 29 & -9\\ -7 & 16 & 16\end{pmatrix}.$$

Homework:

1. If $A=\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}$ then find AB and BA.

2. If $A=\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$ then find $A^2-4A-5I$.

<u>Inverse of a Matrix:</u> A square matrix *A* is said to be invertible if there exists a unique matrix *B* such that *AB*=*BA*=*I*, where *I* is the identity matrix. Then *B* is called the inverse of A and it is denoted by *B*= A^{-1}.

Mathematically,

$$A^{-1}=\frac{adj.A}{|A|} \qquad ; where, |A| \neq 0$$

Note: 1. A matrix ***A*** has inverse iff it is square and non-singular. *i.e.* $|A| \neq 0$.

2. The inverse of a matrix, if it exists, is unique.

<u>Cofactors of a square matrix</u>: If ***A*** be any $n \times n$ square matrix

$$i.e, \;\; A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The determinant of ***A*** is,

$$|A|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then the cofactor of its any entry a_{ij} is defined as,

$$A_{ij}=(-1)^{i+j}\begin{pmatrix} \det er \min ent\, of\;\; submatrix\, made\, by \\ delating\;\; ith\, row\, and\;\; jth\, column\, of\;\; A \end{pmatrix}.$$

<u>Adjoint matrix</u>: Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be any $n \times n$ square matrix and A_{ij} be the cofactors of entries a_{ij} , then the matrix of cofactors from ***A*** is,

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

The transpose of this matrix is called the adjoint of ***A*** and is denoted by adj(***A***).

$$i.e,\ adj(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^t = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Problem-01: Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

Solution: The given matrix is,

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

The determinant of ***A*** is,

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\ &= 2(12-2) - 3(16-1) + 4(8-3) \\ &= 20 - 45 + 20 \\ &= -5 \end{aligned}$$

Since, $|A| \neq 0$. So the given matrix is a non-singular matrix and it has an inverse matrix.

The cofactors of each elements of $|A|$ are,

$A_{11} = 10$; $A_{12} = -15$; $A_{13} = 5$; $A_{21} = -4$; $A_{22} = 4$; $A_{23} = -1$; $A_{31} = -9$; $A_{32} = 14$; $A_{33} = -6$.

The matrix of cofactors is,

$$\begin{pmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{pmatrix}$$

$$\therefore\ adj.A = \begin{pmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{pmatrix}^t$$

$$= \begin{pmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{pmatrix}$$

Therefore,

$$A^{-1} = \frac{adj.A}{|A|}$$
$$= -\frac{1}{5}\begin{pmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{pmatrix}$$

This is required inverse matrix.

Problem-02: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ then find AB^{-1}.

Solution: The given matrices are,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} \text{and } B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

The determinant of ***B*** is,

$$|B| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$
$$= 1(18-12) - 1(9-3) + 1(4-2)$$
$$= 6 - 6 + 2$$
$$= 2$$

Since, $|B| \neq 0$. So it is a non-singular matrix and it has an inverse matrix.

The cofactors of each elements of $|B|$ are,

$B_{11}=6$; $B_{12}=-6$; $B_{13}=2$; $B_{21}=-5$; $B_{22}=8$; $B_{23}=-3$; $B_{31}=1$; $B_{32}=-2$; $B_{33}=1$.

The matrix of cofactors is,

$$\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\therefore\ adj.A = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}^t$$

$$= \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\therefore\ B^{-1} = \frac{adj.B}{|B|}$$

$$= \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Therefore,

$$AB^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -2 & 4 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & -1/2 & 3/2 \end{pmatrix}$$

This is required answer.

Exercise:

1. Find the inverse of the matrix $A=\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$.
2. Find the inverse of the matrix $A=\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$.
3. If $A=\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix}$ then find $A^{-1}B$.
4. If $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$ then show that $AA^{-1}=I$.

System of Linear Equations

Linear Equation: An equation in which the power of each unknown is one is called a linear equation. The general form of a linear equation is defined as,

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots\dots + a_nx_n = b \;\dots\;\dots\;\dots\quad (1)$$

where, $a_1, a_2, a_3, \dots\dots\dots, a_n$ and b real numbers and $x_1,\ x_2,\ x_3,\ \dots\dots\dots\ x_n$ are unknowns(or variables) which is to be determined.

If $b=0$ then the equation (1) is called a homogeneous linear equation and if $b\neq 0$ then it is called a non-homogeneous linear equation.

Example: 1. $ax+by=c$ (non-homogeneous) represents a straight line.

2. $ax+by+cz+d=0$ (homogeneous) represents a plane.

Solutions of Linear Equation: A solution of the linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots\dots + a_nx_n = b$$

in n variables is a sequence of n numbers, $\alpha_1, \alpha_2,\ \alpha_3, \dots\dots\dots, \alpha_n$ such that the equation is satisfied when we substitute $x_1=\alpha_1,\ x_2=\alpha_2, \dots\dots\dots, x_n=\alpha_n$.

System of Linear Equations: A finite set of linear equations is known as a system of linear equations. So,

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n = b_m \end{array}\right\} \ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

is a system of m linear equations in n variables $x_1,\ x_2,\ x_3, \dots \dots \dots\ x_n$.

If $b_i = 0$, then the above system is called a homogeneous system of linear equations and if $b_i \neq 0$, then it is called a non-homogeneous system of linear equations.

Classification of System of Linear Equations: Regarding the nature of solutions, systems of linear equations are classified as follows:

1. **Inconsistent:** A system of linear equations is called an inconsistent if it has no solution.
2. **Consistent:** A system of linear equations is called consistent if it has one or more solution. It is also classified as,
   a). Unique: A system of linear equations is called unique if it has only one solution.
   b).Redundant: A system of linear equations is called redundant if it has more than one solution.

Free Variables: If a system of m linear equations in n unknowns is,

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n = b_m \end{array}\right\} \dots \dots \dots (1)\ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

and its echelon form is,

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots \dots + a_{1n}x_n = b_1 \\ a_{22}x_2 + a_{13}x_3 + \dots \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m3}x_3 + \dots \dots + a_{mn}x_n = b_m \end{array}\right\} \dots \dots \dots (2)\ where,\ a_{ij}, b_i \in R,\ m \geq 2$$

The variables which do not appear at the beginning of any equation of (2) are called free variables.

Trivial and non-trivial Solution: A homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = 0 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n = 0 \end{array}\right\} \dots \dots \dots (1)$$

The above system of linear equations (1) has a solution, namely zero n-tuple $O=(0,0,0,\dots\dots,0,)$ called **zero** or **trivial solution** and any other solution, if it exists, is called **non-zero** or **non-trivial solution**.

Augmented matrix: Consider a non-homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots\dots+a_{2n}x_n=b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1+a_{m2}x_2+\dots\dots+a_{mn}x_n=b_m \end{array}\right\}\dots\dots\dots(1)\ where,\ a_{ij},b_i\in R,\ m\geq 2$$

we can write it as $AX=B$.

Where, $A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$

Here, ***A*** is called the coefficient matrix, ***X*** is called the column matrix of the variables and ***B*** is called the column matrix of the constants.

The matrix $[A\ \vdots\ B]=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$ is called augmented matrix of the system of linear equations (1). The augmented matrix is also denoted by A^* or $[A,B]$ or $[A\ B]$.

NOTE:

1. If the coefficient matrix and augmented matrix have the same rank, then the system of linear equations is said to be consistent.
2. If the coefficient matrix and augmented matrix have the different rank, then the system of linear equations is said to be inconsistent and it has no solution.
3. If the coefficient matrix and augmented matrix have the same rank and the rank is equal to the number of variables then the system of linear equations has a unique solution.
4. If the coefficient matrix and augmented matrix have the same rank and the rank is less than the number of variables then the system of linear equations has infinite number of solutions.

Gauss Elimination Method: Consider a non-homogeneous system of m linear equations in n unknowns is,

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots\dots+a_{2n}x_n=b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1+a_{m2}x_2+\dots\dots+a_{mn}x_n=b_m \end{array}\right\}\dots\dots\dots(1)\ where,\ a_{ij},b_i\in R,\ m\geq 2$$

we can write it as $AX = B$.

Where, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$

The augmented matrix is,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$$

reduce this matrix into the following form,

$$\begin{pmatrix} a_{11}' & a'_{12} & \dots & a'_{1n} & \vdots & l_1 \\ 0 & a_{22}' & \dots & a'_{2n} & \vdots & l_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & a_{mn}' & \vdots & l_m \end{pmatrix}$$

then the reduced system is,

$$\begin{aligned} &1.x_1 + a'_{1n}.x_2 + \dots\dots + a'_{1n}.x_n = l_1 \\ &0.x_1 + 1.x_2 + \dots\dots + a'_{2n}.x_n = l_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &0.x_1 + 0.x_2 + \dots\dots + 1.x_n = l_m \end{aligned}$$

Now, by back substitution, we solve for, x_n, $x_{n-1}, \dots \dots, x_1$.

This process which eliminates unknowns from succeeding equations is known as Gauss elimination.

NOTE:

1. If an equation $0x_1 + 0x_2 + \dots\dots + 0x_n = b,\ b \neq 0$ occurs, then the system is inconsistent and has no solution.
2. If an equation $0x_1 + 0x_2 + \dots\dots + 0x_n = 0$ occurs, then the equation can be deleted without affecting the solution.
3. If the number of equations is equal to the number of variables, then the system has a unique solution.
4. If the number of equations is less than the number of variables, then the system has a infinitely many solutions.

Problem-01: Show that the following system of linear equations is consistent

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

and find the solution.

Solution: The given system of linear equations is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}\right\} \dots\ \dots\ \dots\ (1)$$

the system (1) can be written as,

$$AX = B \dots\ \dots\ \dots\ (2)$$

where, $A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B] = \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 5 & -17 & \vdots & -24 \end{pmatrix} \begin{matrix} R_2' \to 2R_2 + R_1 \\ R_3' \to 2R_3 - 3R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & -76 & \vdots & -152 \end{pmatrix} R_3' \to 3R_3 - 5R_2$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ 3y + 5z &= 16 \\ -76z &= -152 \end{aligned}\right\}$$

By back substitution we get, $z = 2$, $y = 2$, $x = 2$.

Hence the given system is consistent and the solution is,

$$x = 2, y = 2, z = 2.$$

Problem-02: Solve the following system of linear equations

$$\begin{aligned} x + y - 2z + s + 3t &= 1 \\ 2x - y + 2z + 2s + 6t &= 2 \\ 3x + 2y - 4z - 3s - 9t &= 3 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x + y - 2z + s + 3t &= 1 \\ 2x - y + 2z + 2s + 6t &= 2 \\ 3x + 2y - 4z - 3s - 9t &= 3 \end{aligned}\right\} \ldots \ldots \ldots (1)$$

the system (1) can be written as,

$$AX = B \ldots \ldots \ldots (2)$$

where, $A = \begin{pmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 1 & 1 & -2 & 1 & 3 & \vdots & 1 \\ 2 & -1 & 2 & 2 & 6 & \vdots & 2 \\ 3 & 2 & -4 & -3 & -9 & \vdots & 3 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 1 & -2 & 1 & 3 & \vdots & 1 \\ 0 & -3 & 6 & 0 & 0 & \vdots & 0 \\ 0 & -1 & 2 & -6 & -18 & \vdots & 0 \end{pmatrix} \begin{matrix} R_2' \to R_2 - 2R_1 \\ R_3' \to R_3 - 3R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 1 & 1 & -2 & 1 & 3 & \vdots & 1 \\ 0 & -3 & 6 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & -18 & -54 & \vdots & 0 \end{pmatrix} R_3' \to 3R_3 - R_2$$

$$\approx \begin{pmatrix} 1 & 1 & -2 & 1 & 3 & \vdots & 1 \\ 0 & 1 & -2 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 3 & \vdots & 0 \end{pmatrix} \begin{matrix} R_2' \to -\frac{1}{3}R_2 \\ R_3' \to -\frac{1}{18}R_3 \end{matrix}$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned} x+y-2z+s+3t&=1 \\ y-2z \qquad\quad &=0 \\ s+3t&=0 \end{aligned}\right\}$$

There are 3 equations in 5 unknowns, so there are (5-3)=2 free variables which are z and t. Thus the system is consistent with an infinite number of solutions.

We put $z=a$ and $t=b$. So by back substitution we have, $s=-3b$, $y=2a$, $x=1$.

Hence the required result is, $x=1$, $y=2a$, $z=a$, $s=-3b$, $t=b$.

Problem-03: Solve the following system of linear equations

$$\begin{aligned} x+2y-2z-t&=0 \\ 2x+5y-3z-t&=1 \\ 3x+8y-4z-t&=2 \\ x+5y+z+2t&=3 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} x+2y-2z-t&=0 \\ 2x+5y-3z-t&=1 \\ 3x+8y-4z-t&=2 \\ x+5y+z+2t&=3 \end{aligned}\right\} \dots\dots\dots (1)$$

the system (1) can be written as,

$$AX=B \dots\dots\dots (2)$$

where, $A=\begin{pmatrix} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & -1 \\ 3 & 8 & -4 & -1 \\ 1 & 5 & 1 & 2 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ and $B=\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$.

The augmented matrix is,

$$[A,\ B]=\begin{pmatrix}1 & 2 & -2 & -1 & \vdots & 0\\ 2 & 5 & -3 & -1 & \vdots & 1\\ 3 & 8 & -4 & -1 & \vdots & 2\\ 1 & 5 & 1 & 2 & \vdots & 3\end{pmatrix}$$

$$\approx\begin{pmatrix}1 & 2 & -2 & -1 & \vdots & 0\\ 0 & 1 & 1 & 1 & \vdots & 1\\ 0 & 2 & 2 & 2 & \vdots & 2\\ 0 & 3 & 3 & 3 & \vdots & 3\end{pmatrix}\begin{matrix}R_2^{'}\to R_2-2R_1\\ R_3^{'}\to R_3-3R_1\\ R_4^{'}\to R_4-R_1\end{matrix}$$

$$\approx\begin{pmatrix}1 & 2 & -2 & -1 & \vdots & 0\\ 0 & 1 & 1 & 1 & \vdots & 1\\ 0 & 0 & 0 & 0 & \vdots & 0\\ 0 & 0 & 0 & 0 & \vdots & 0\end{pmatrix}\begin{matrix}R_3^{'}\to R_3-2R_1\\ R_4^{'}\to R_4-3R_2\end{matrix}$$

which is the echelon form of the augmented matrix.

The reduced system is,

$$\left.\begin{aligned}x+2y-2z-t&=0\\ y+\ \ z+t&=1\\ 0&=0\\ 0&=0\end{aligned}\right\}$$

or, $$\left.\begin{aligned}x+2y-2z-t&=0\\ y+\ \ z+t&=1\end{aligned}\right\}$$

There are 2 equations in 4 unknowns, so there are (4-2) = 2 free variables which are z and t. Thus the system is consistent with an infinite number of solutions.

We put $z=a$ and $t=b$. So by back substitution we have, $y=1-a-b$, $x=4a+3b-2$.

Hence the required result is, $x=4a+3b-2$, $y=1-a-b$, $z=a$, $t=b$.

Exercise:

1. Solve the following system of linear equations

$$\begin{aligned}x+y+z&=3\\ x+2y+3z&=4\\ x+4y+9z&=6\end{aligned}$$

2. Solve the following system of linear equations

$$\begin{aligned}x+\ \ y+\ \ z+\ \ t&=\ 4\\ x+2y+2z-\ t&=\ 4\\ x+4y+9z+2t&=16\end{aligned}$$