Equations of First Order and First Degree

Definition:

A differential equation of the type $M + N\dfrac{dy}{dx} = 0$, where M and N are functions of x and y or constants is called a differential equation of the first order and first degree.

There are two standard forms of differential equations of first order and first degree namely

- $\dfrac{dy}{dx} = f(x, y)$
- $M(x, y)dx + N(x, y)dy = 0$

We can classify the first order and first degree differential equation into followings eight categories according to its solution methods:

- Equations of variable separable form,
- Equations reducible to variable separable form,
- Homogeneous Equation,
- Equation reducible to Homogeneous form,
- Linear differential equation,
- Equation reducible to linear differential equation,
- Exact differential equation and
- Equation reducible to exact differential equation.

Equations of variable Separable form: If an equation can be written in such a way that dx and all the term containing x are on one side and dy and all the term containing y are on other side, then this an equation in which variables are separable.

$$i.e,\ F(x)dx = G(y)dy$$

This type of equation can be solved by integrating directly and adding a constant on either side.

Problem-01: Solve the differential equation $\left(1+x^2\right)\dfrac{dy}{dx} = x\left(1+y^2\right)$

Solution: Given differential equation is,

$$\left(1+x^2\right)\frac{dy}{dx} = x\left(1+y^2\right)$$

Separating the variables, we get

$$\frac{dy}{1+y^2} = \frac{x}{1+x^2}dx$$

$$or,\ \frac{dy}{1+y^2} = \frac{x}{1+x^2}dx$$

Integrating both-sides, we find

$$\int\frac{dy}{1+y^2} = \int\frac{x}{1+x^2}dx$$

$$or,\ \int\frac{dy}{1+y^2} = \frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$or,\ \tan^{-1}y = \frac{1}{2}\ln\left(1+x^2\right) + c$$

$$or,\ \tan^{-1}y = \ln\sqrt{1+x^2} + c$$

$$or,\ y = \tan\left(\ln\sqrt{1+x^2} + c\right)$$

which is the complete or general solution of the given differential equation.

Problem-02: Solve the differential equation $\left(4+y^2\right)dx+\left(4+x^2\right)dy=0$.

Solution: Given differential equation is,

$$\left(4+y^2\right)dx+\left(4+x^2\right)dy=0$$

Separating the variables, we get

$$\frac{dy}{4+y^2}=-\frac{dx}{4+x^2}$$

Integrating both-sides, we find

$$\int\frac{dy}{4+y^2}=-\int\frac{dx}{4+x^2}$$

$$\Rightarrow\int\frac{dy}{2^2+y^2}=-\int\frac{dx}{2^2+x^2}$$

$$\Rightarrow\frac{1}{2}\tan^{-1}\frac{y}{2}=-\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

$$\Rightarrow\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{y}{2}=2C$$

$$\Rightarrow\tan^{-1}\frac{\frac{x}{2}+\frac{y}{2}}{1-\frac{x}{2}.\frac{y}{2}}=2C$$

$$\Rightarrow\tan^{-1}\frac{\frac{x+y}{2}}{1-\frac{xy}{4}}=2C$$

$$\Rightarrow\tan^{-1}\frac{\frac{x+y}{2}}{\frac{4-xy}{4}}=2C$$

$$\Rightarrow\tan^{-1}\left(\frac{x+y}{2}\times\frac{4}{4-xy}\right)=2C$$

$$\Rightarrow\tan^{-1}\frac{2(x+y)}{4-xy}=2C$$

$$\Rightarrow\frac{2(x+y)}{4-xy}=\tan 2C=a \qquad [\ letting\ \ arbitrary\, const, \tan 2C=a\ ]$$

$$\Rightarrow\frac{2(x+y)}{4-xy}=a$$

$$\Rightarrow 2(x+y)=a(4-xy)$$

which is the complete or general solution of the given differential equation.

Problem-03: Solve the differential equation $\frac{dy}{dx}=\sqrt{1-x^2}\,\sqrt{1-y^2}$.

Solution: Given differential equation is,

$$\frac{dy}{dx}=\sqrt{1-x^2}\,\sqrt{1-y^2}$$

Separating variables, we get

$$\frac{dy}{\sqrt{1-y^2}}=\sqrt{1-x^2}\,dx$$

Integrating both-sides we find,

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \sqrt{1-x^2}\, dx$$

$$\Rightarrow \sin^{-1} y = \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x + C$$

$$\Rightarrow y = \sin\left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x + C\right)$$

which is the complete or general solution of the given differential equation.

Problem-04: Solve the differential equation $\left(e^y + 1\right)\cos x\, dx + e^y(1+\sin x)\, dy = 0$.

Solution: Given differential equation is,

$$\left(e^y + 1\right)\cos x\, dx + e^y(1+\sin x)\, dy = 0$$

Separating the variables, we get

$$e^y(1+\sin x)\, dy = -\left(e^y + 1\right)\cos x\, dx$$

$$\Rightarrow \frac{e^y dy}{1+e^y} = -\frac{\cos x\, dx}{1+\sin x}$$

Integrating both-sides, we find

$$\int \frac{e^y dy}{1+e^y} = -\int \frac{\cos x\, dx}{1+\sin x}$$

$$\Rightarrow \ln\left|1+e^y\right| = -\ln\left|1+\sin x\right| \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \ln\left|f(x)\right|\right]$$

$$\Rightarrow \ln(1+\sin x) + \ln(1+e^y) = \ln c$$

$$\Rightarrow \ln(1+\sin x)(1+e^y) = \ln c$$

$$\Rightarrow (1+\sin x)(1+e^y) = c$$

$$\Rightarrow (1+\sin x)(1+e^y) = c$$

which is the complete or general solution of the given differential equation.

Problem-05: Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

Separating the variables, we get

$$\frac{dy}{dx} = e^x . e^{-y} + x^2 e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = e^{-y}\left(e^x + x^2\right)$$

$$\Rightarrow e^y dy = \left(e^x + x^2\right) dx$$

Integrating both-sides, we find

$$\int e^y dy = \int \left(e^x + x^2\right) dx$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + C$$

$$\Rightarrow y = \ln\left(e^x + \frac{x^3}{3} + C\right)$$

which is the complete or general solution of the given differential equation.

Problem-06: Solve the differential equation $\left(x^2-1\right)\frac{dy}{dx} - xy = 0$.

Solution: Given differential equation is,

$$\left(x^2-1\right)\frac{dy}{dx} - xy = 0$$

Separating the variables, we get

$$\left(x^2-1\right)\frac{dy}{dx} = xy$$

$$or,\ \left(x^2-1\right)\frac{dy}{dx} = xy$$

$$or,\ \frac{dy}{y} = \frac{x}{x^2-1}dx$$

Integrating both-sides, we find

$$\int\frac{dy}{y} = \int\frac{x}{x^2-1}dx$$

$$or,\ \int\frac{dy}{y} = \frac{1}{2}\int\frac{2x}{x^2-1}dx$$

$$or,\ \ln y = \frac{1}{2}\ln\left(x^2-1\right) + \ln c$$

$$or,\ \ln y = \ln\sqrt{x^2-1} + \ln c$$

$$or,\ \ln y = \ln \mathrm{c}\sqrt{x^2-1}$$

$$or,\ y = \mathrm{c}\sqrt{x^2-1}$$

which is the complete or general solution of the given differential equation.

Problem 8: Solve $x\frac{dy}{dx} = \left(1-2x^2\right)\tan y$

Solution: Given that,

$$x\frac{dy}{dx} = \left(1-2x^2\right)\tan y$$

Separating variables we obtain,

$$\frac{dy}{\tan y} = \frac{1-2x^2}{x}dx$$

$$\cot y\,dy = \left(\frac{1}{x} - 2x\right)dx$$

Now, integrating,

$$\int\cot y\,dy = \int\left(\frac{1}{x} - 2x\right)dx$$

$$\ln siny = \ln x - 2.\frac{x^2}{2} + \ln c$$

$$\ln siny = \ln x - x^2 + \ln c$$

$$\ln siny = \ln x + \ln e^{-x^2} + \ln c$$

$$\ln \sin y = \ln\left(cx.e^{-x^2}\right)$$

$$\therefore \sin y = cx.e^{-x^2}$$ **(As desired)**

Exercise:
Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$
2. $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$
3. $x\frac{dy}{dx} = y + xy$
4. $x\sin ydx = \left(x^2+1\right)\cos ydy$
5. $\frac{dy}{dx} = \sqrt{1-y^2}\ \cos 5x\, e^{3x}$
6. $\sec^2 x\tan ydx + \sec^2 y\tan xdy = 0$
7. $\left(e^y+1\right)\cos x\,dx + e^y\sin xdy = 0$

Equations reducible to variable separable form:

- An equation of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to variable separable form by choosing the transformation $ax+by+c=v$.

Problem-01: Solve the differential equation $\frac{dy}{dx} = \left(4x+y+1\right)^2$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = \left(4x+y+1\right)^2 \text{ ... ... (1)}$$

Let, $4x+y+1=v$

$$\therefore\quad 4+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Now from equation (1), we get

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \frac{dv}{dx} = 4 + v^2$$

$$\Rightarrow \frac{dv}{4+v^2} = dx$$

Integrating both sides, we find

$$\int \frac{dv}{4+v^2} = \int dx$$

$$\Rightarrow \int \frac{dv}{2^2+v^2} = \int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{v}{2} = x + C$$

$$\Rightarrow \tan^{-1}\frac{v}{2} = 2x + 2C$$

$$\Rightarrow \frac{v}{2} = \tan(2x+2C)$$

$$\Rightarrow v = 2\tan(2x+2C)$$

$$\therefore\ 4x + y + 1 = 2\tan(2x+2C)$$

which is the complete integral or general solution of the given differential equation.

Problem-02: Solve the differential equation $\frac{dy}{dx} = 1 + \sqrt{x+y}$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = 1 + \sqrt{x+y} \ldots \ldots \ldots (1)$$

Let, $x + y = v^2$

$$\therefore\ 1 + \frac{dy}{dx} = 2v\frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 2v\frac{dv}{dx} - 1$$

Now from equation (1), we get

$$2v\frac{dv}{dx} - 1 = 1 + v$$

$$\Rightarrow 2v\frac{dv}{dx} = 2 + v$$

$$\Rightarrow \frac{2v}{2+v}dv = dx$$

Integrating both sides, we find

$$2\int \frac{v}{2+v}dv = \int dx$$

$$\Rightarrow 2\int \frac{(2+v)-2}{2+v}dv = \int dx$$

$$\Rightarrow 2\int \left(1 - \frac{2}{2+v}\right)dv = \int dx$$

$$\Rightarrow 2\int dv - 4\int \frac{dv}{2+v} = \int dx$$

$$\Rightarrow 2v - 4\ln(2+v) = x + c$$

$$\therefore\ 2\sqrt{x+y} - 4\ln\left(2+\sqrt{x+y}\right) = x + c$$

which is the complete integral or general solution of the given differential equation.

Problem-03: Solve the differential equation $\frac{dy}{dx} = \tan(x+y+6)$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = \tan(x+y+6) \cdots \cdots \cdots (1)$$

Let, $x+y+6=v$

$$\therefore\ 1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now from equation (1), we get

$$\frac{dv}{dx} - 1 = \tan v$$

$$\Rightarrow \frac{dv}{dx} = 1 + \tan v$$

$$\Rightarrow \frac{dv}{1+\tan v} = dx$$

Integrating both sides, we get

$$\int \frac{dv}{1+\tan v} = \int dx$$

$$\Rightarrow \int \frac{\cos v}{\sin v + \cos v} dv = \int dx$$

$$\Rightarrow \int \frac{\frac{1}{2}(\sin v + \cos v) + \frac{1}{2}(\cos v - \sin v)}{\sin v + \cos v} dv = \int dx$$

$$\Rightarrow \frac{1}{2}\int dv + \frac{1}{2}\int \frac{(\cos v - \sin v)}{\sin v + \cos v} dv = \int dx$$

$$\Rightarrow \frac{1}{2}v + \frac{1}{2}\ln|\sin v + \cos v| = x + C$$

$$\therefore (x+y+6) + \ln|\sin(x+y+6) + \cos(x+y+6)| = 2x + 2C$$

which is the complete integral or general solution of the given differential equation.

Problem-04: Solve the differential equation $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Solution: Given differential equation is,

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y) \ \ldots \ldots \ldots (1)$$

Let, $x+y=v$

$$\therefore\ 1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Now from equation (1), we get

$$\frac{dv}{dx}-1=\sin v+\cos v$$

$$\Rightarrow \frac{dv}{dx}=\sin v+\cos v+1$$

$$\Rightarrow \frac{dv}{\sin v+\cos v+1}=dx$$

$$\Rightarrow \frac{dv}{2\sin\frac{v}{2}\cos\frac{v}{2}+2\cos^2\frac{v}{2}}=dx$$

$$\Rightarrow \frac{dv}{2\cos^2\frac{v}{2}\left(1+\frac{\sin\frac{v}{2}}{\cos\frac{v}{2}}\right)}=dx$$

$$\Rightarrow \frac{\sec^2\frac{v}{2}dv}{2\left(1+\tan\frac{v}{2}\right)}=dx$$

Integrating both sides, we get

$$\frac{1}{2}\int\frac{\sec^2\frac{v}{2}}{\left(1+\tan\frac{v}{2}\right)}dv=\int dx$$

$$\Rightarrow \ln\left|1+\tan\frac{v}{2}\right|=x+C$$

$$\Rightarrow \ln\left|1+\tan\frac{(x+y)}{2}\right|=x+C$$

which is the complete integral or general solution of the given differential equation.

Problem-05: Solve the differential equation $\frac{dy}{dx}=\cos(x-y+5)$.

Solution: Given differential equation is,

$$\frac{dy}{dx}=\cos(x-y+5)\ldots\ldots\ldots(1)$$

Let, $x-y+5=v$

$$\therefore\quad 1-\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=1-\frac{dv}{dx}$$

Now from equation (1), we get

$$1-\frac{dv}{dx}=\cos v$$

$$\Rightarrow\frac{dv}{dx}=1-\cos v$$

$$\Rightarrow\frac{dv}{1-\cos v}=dx$$

Integrating both sides, we get

$$\int\frac{1}{1-\cos v}dv=\int dx$$

$$or,\ \int\frac{1}{2\sin^2\frac{v}{2}}dv=\int dx$$

$$or,\ \frac{1}{2}\int\cos ec^2\frac{v}{2}dv=\int dx$$

$$or,\ \frac{1}{2}\frac{\cot\frac{v}{2}}{\frac{1}{2}}=x+c$$

$$or,\ \cot\frac{v}{2}=x+c$$

$$or,\ \frac{v}{2}=\cot^{-1}\left(x+c\right)$$

$$or, v=2\cot^{-1}\left(x+c\right)$$

$$\therefore\quad x-y+5=2\cot^{-1}\left(x+c\right)$$

which is the complete integral or general solution of the given differential equation.

Problem-06: Solve the differential equation $\frac{dy}{dx}=\frac{x-2y+1}{2x-4y}$.

Solution: Given differential equation is,

$$\frac{dy}{dx}=\frac{x-2y+1}{2x-4y}$$

$$or,\ \frac{dy}{dx}=\frac{x-2y+1}{2(x-2y)}\ \dots\dots\dots (1)$$

Let, $x-2y=v$

$$\therefore\quad 1-2\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{2}\left(1-\frac{dv}{dx}\right)$$

Now from equation (1), we get

$$\frac{1}{2}\left(1-\frac{dv}{dx}\right)=\frac{v+1}{2v}$$

$$or, 1-\frac{dv}{dx}=\frac{v+1}{v}$$

$$or, \frac{dv}{dx}=1-\frac{v+1}{v}$$

$$or, \frac{dv}{dx}=-\frac{1}{v}$$

$$or, \frac{dv}{v}=-dx$$

Integrating both sides, we get

$$\int\frac{dv}{v}=-\int dx$$

$$or, \ln v=-x+\ln c$$

$$or, \ln v=\ln ce^{-x}$$

$$or, v=ce^{-x}$$

$$\therefore\quad x-2y=ce^{-x}$$

which is the complete integral or general solution of the given differential equation.

Exercise:

Solve the following differential equations:

1. $\frac{dy}{dx} = \left(2x+3y+5\right)^2$
2. $\frac{dy}{dx} = 1-\sqrt{x+y+1}$
3. $\frac{dy}{dx} = \sin(2x-3y+5)$
4. $\sin^{-1}\left(\frac{dy}{dx}\right) = \left(x+y\right)$
5. $(3x+2y+2)\frac{dy}{dx} = 3x+2y$

Homogeneous Differential Equation: An equation of the form

$$\frac{dy}{dx} = \frac{f_1\left(x,y\right)}{f_2\left(x,y\right)}$$

In which $f_1\left(x,y\right)$ and $f_2\left(x,y\right)$ are homogeneous functions of x and y of the same degree is called homogeneous equation.

It can be reduced to an equation in which variables are separable by choosing

$$y = vx.$$

Problem-01: Solve the differential equation $\left(x^2+y^2\right)dx + 2xydy = 0$.

Solution: Given differential equation is,

$$\left(x^2+y^2\right)dx + 2xydy = 0 \text{ ... ... (1)}$$

Equation (1) can be written as,

$$\left(x^2+y^2\right)dx + 2xydy = 0$$

$$\Rightarrow 2xydy = -\left(x^2+y^2\right)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(x^2+y^2\right)}{2xy} \text{ ... ... (2)}$$

This is a homogeneous equation.

Put, $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (2) we get

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{\left(x^2 + v^2x^2\right)}{2vx^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{\left(1 + v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -v - \frac{\left(1 + v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2v^2 - \left(1 + v^2\right)}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-\left(1 + 3v^2\right)}{2v}$$

$$\Rightarrow \frac{2v}{\left(1 + 3v^2\right)}dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v}{\left(1 + 3v^2\right)}dv = -\int \frac{dx}{x} + C$$

$$\Rightarrow \frac{1}{3}\int \frac{6v}{\left(1 + 3v^2\right)}dv = -\int \frac{dx}{x} + C$$

$$\Rightarrow \frac{1}{3}\ln\left|1 + 3v^2\right| = -\ln|x| + C$$

$$\Rightarrow \ln x + \ln\left(1 + 3v^2\right)^{1/3} = C$$

$$\Rightarrow \ln x\left(1 + 3v^2\right)^{1/3} = C$$

$$\Rightarrow x\left(1 + 3v^2\right)^{1/3} = e^C$$

$$\Rightarrow x\left(1 + 3\frac{y^2}{x^2}\right)^{1/3} = e^C = a \quad \left[e^C = a\ (say)\right]$$

$$\therefore \quad x\left(1 + 3\frac{y^2}{x^2}\right)^{1/3} = a$$

which is the complete integral or general solution of the given differential equation.

Problem-02: Solve the differential equation $x^2 y dx - \left(x^3 + y^3\right) dy = 0$.

Solution: Given differential equation is,

$$x^2 y dx - \left(x^3 + y^3\right) dy = 0 \quad \dots\dots\dots (1)$$

Equation (1) can be written as,

$$x^2 y dx - \left(x^3 + y^3\right) dy = 0$$

$$\Rightarrow \left(x^3 + y^3\right) dy = x^2 y dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 y}{\left(x^3 + y^3\right)} \quad \dots\dots\dots (2)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (2), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3 x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1 + v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1 + v^3} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4}{1 + v^3}$$

$$\Rightarrow \frac{1 + v^3}{v^4} dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1 + v^3}{v^4} dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \int \left(\frac{1}{v^4} + \frac{1}{v} \right) dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow -\frac{1}{3v^3} + \ln v = -\ln x + c$$

$$\Rightarrow \ln v + \ln x = \frac{1}{3v^3} + c$$

$$\Rightarrow \ln vx = \frac{1}{3v^3} + c$$

$$\therefore \ \ln y = \frac{x^3}{3y^3} + c$$

which is the complete integral or general solution of the given differential equation.

Problem-03: Solve the differential equation $xy\frac{dy}{dx} = x^2 + y^2$.

Solution: Given differential equation is,

$$xy\frac{dy}{dx} = x^2 + y^2 \quad \ldots \ldots \ldots (1)$$

Equation (1) can be written as,

$$xy\frac{dy}{dx} = x^2 + y^2$$

$$or, \ \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \ldots \ldots \ldots (2)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore \ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (2), we get

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{vx^2}$$

$$or, \ v + x\frac{dv}{dx} = \frac{1+v^2}{v}$$

$$or, \ x\frac{dv}{dx} = \frac{1+v^2}{v} - v$$

$$or,\ x\frac{dv}{dx}=\frac{1}{v}$$

$$or, vdv = xdx$$

Integrating both sides, we get

$$\int vdv=\int xdx$$

$$or,\ \frac{v^2}{2}=\frac{x^2}{2}+\frac{c}{2}$$

$$or,\ v^2=x^2+c$$

$$or,\ \left(\frac{y}{x}\right)^2=x^2+c$$

$$or,\ y^2=x^4+cx^2$$

$$\therefore\quad y=\sqrt{x^4+cx^2}$$

which is the complete integral or general solution of the given differential equation.

Problem-04: Solve the differential equation $\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$.

Solution: Given differential equation is,

$$\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0 \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y(x+y)}{x^2} \quad \ldots\ldots\ldots (2)$$

This is a homogeneous equation.

Let, $y=vx$

$$\therefore\quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2), we get

$$v+x\frac{dv}{dx}=\frac{vx(x+vx)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = v(1+v)$$

$$\Rightarrow x\frac{dv}{dx} = v + v^2 - v$$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \frac{1}{v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1}{v^2}dv = \int \frac{dx}{x} + c$$

$$\Rightarrow -\frac{1}{v} = \ln x + c$$

$$\Rightarrow -\frac{x}{y} = \ln x + \ln c_1 \quad ;\left[\ln c_1 = c\ (say)\right]$$

$$\Rightarrow -\frac{x}{y} = \ln c_1 x$$

$$\Rightarrow \frac{x}{y} = -\ln c_1 x$$

$$\Rightarrow \frac{x}{y} = \ln (c_1 x)^{-1}$$

$$\Rightarrow \frac{x}{y} = \ln\left(\frac{1}{c_1 x}\right)$$

$$\therefore \quad y = \frac{x}{\ln\left(\frac{1}{c_1 x}\right)}$$

which is the complete integral or general solution of the given differential equation.

Problem-05: Solve the differential equation $x\frac{dy}{dx} - y = \sqrt{\left(x^2 + y^2\right)}$.

Solution: Given differential equation is,

$$x\frac{dy}{dx} - y = \sqrt{\left(x^2 + y^2\right)} \; \dots \dots \dots (1)$$

Equation (1) can be written as,

$$x\frac{dy}{dx}-y=\sqrt{\left(x^2+y^2\right)}$$

$$\Rightarrow x\frac{dy}{dx}=y+\sqrt{\left(x^2+y^2\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y+\sqrt{\left(x^2+y^2\right)}}{x} \quad \ldots\ldots\ldots (2)$$

This is a homogeneous equation.

Let, $y=vx$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From equation (2), we get

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{\left(x^2+v^2x^2\right)}}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{vx+x\sqrt{\left(1+v^2\right)}}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=v+\sqrt{\left(1+v^2\right)}$$

$$\Rightarrow x\frac{dv}{dx}=v+\sqrt{\left(1+v^2\right)}-v$$

$$\Rightarrow x\frac{dv}{dx}=\sqrt{\left(1+v^2\right)}$$

$$\Rightarrow \frac{dv}{\sqrt{\left(1+v^2\right)}}=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{dv}{\sqrt{\left(1+v^2\right)}}=\int\frac{dx}{x}+c$$

$$\Rightarrow \ln\left|v+\sqrt{1+v^2}\right|=\ln x+c$$

$$\Rightarrow \ln\left|v+\sqrt{1+v^2}\right|=\ln x+\ln c_1 \ ;\ \left[\ln c_1=c\right]$$

$$\Rightarrow \ln\left|v+\sqrt{1+v^2}\right| = \ln c_1 x$$

$$\Rightarrow v+\sqrt{1+v^2} = c_1 x$$

$$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{y}{x^2}^{2}} = c_1 x$$

$$\Rightarrow \frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x} = c_1 x$$

$$\Rightarrow y+\sqrt{x^2+y^2} = c_1 x^2$$

$$\Rightarrow \sqrt{x^2+y^2} = c_1 x^2 - y$$

$$\Rightarrow x^2+y^2 = \left(c_1 x^2 - y\right)^2$$

which is the complete integral or general solution of the given differential equation.

Problem-06: Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy+3y^2}{x^2+2xy}.$$

Solution: Given differential equation is,

$$\frac{dy}{dx} = \frac{2xy+3y^2}{x^2+2xy} \quad \ldots \ldots \ldots (1)$$

This is a homogeneous equation.

Let, $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (1), we get

$$v + x\frac{dv}{dx} = \frac{2vx^2+3v^2x^2}{x^2+2vx^2}$$

$$or,\ v + x\frac{dv}{dx} = \frac{2v+3v^2}{1+2v}$$

$$or,\ x\frac{dv}{dx} = \frac{2v+3v^2}{1+2v} - v$$

$$or,\ x\frac{dv}{dx}=\frac{2v+3v^2-v-2v^2}{1+2v}$$

$$or,\ x\frac{dv}{dx}=\frac{v+v^2}{1+2v}$$

$$or,\ \frac{1+2v}{v+v^2}dv=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{1+2v}{v+v^2}dv=\int\frac{dx}{x}$$

$$or,\ \ln\left(v+v^2\right)=\ln x+\ln c$$

$$or,\ \ln\left(v+v^2\right)=\ln\ \mathrm{c}\,x$$

$$or,\ v+v^2=\mathrm{c}\,x$$

$$or,\ \frac{y}{x}+\left(\frac{y}{x}\right)^2=\mathrm{c}\,x$$

$$or,\ xy+y^2=\mathrm{c}\,x^3$$

which is the complete integral or general solution of the given differential equation.

Exercise: Solve the following differential equations:

1. $2x^2\frac{dy}{dx}=x^2+y^2$
2. $\frac{dy}{dx}=\frac{y}{x+\sqrt{xy}}$
3. $\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x}$
4. $\left(x^2+xy\right)\frac{dy}{dx}=xy-y^2$
5. $\left(x-y\right)\frac{dy}{dx}=x+y$

Equation Reducible to Homogeneous Form: An equation of the type

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \quad \dots\ \dots\ \dots\ (1)$$

can be reduced to homogeneous form as follows:

❖ Case -01: If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ then putting $x=X+h$, $y=Y+k$ and $\frac{dy}{dx}=\frac{dY}{dX}$ in equation (1) we get

$$\frac{dY}{dX}=\frac{a_1X+b_1Y+\left(a_1h+b_1k+c_1\right)}{a_2X+b_2Y+\left(a_2h+b_2k+c_2\right)}$$

we choose the constants h and k in such a way that,

$$a_1h+b_1k+c_1=0 \text{ and } a_2h+b_2k+c_2=0$$

with this substitution the differential equation reduces to

$$\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

which is a homogeneous equation in X , Y and can be solved by putting $Y=vX$.

❖ Case-02: If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{m}$ (say), then the differential equation can be written as,

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m\left(a_1x+b_1y\right)+c_2}$$

put $a_1x+b_1y=v$ so that $\frac{dy}{dx}=\frac{1}{b_1}\left(\frac{dv}{dx}-a_1\right)$

the above equation becomes

$$\frac{1}{b_1}\left(\frac{dv}{dx}-a_1\right)=\frac{v+c}{mv+c}$$

which is in variables separable form.

Problem-01: Solve the differential equation $\frac{dy}{dx}=\frac{2x-y+4}{x-2y+5}$

Solution: Given differential equation is,

$$\frac{dy}{dx}=\frac{2x-y+4}{x-2y+5} \text{ ... . (1)}$$

put $x=X+h$ and $y=Y+k$ where h , k are constants.

$$\therefore \frac{dy}{dx}=\frac{dY}{dX}$$

Then the equation (1) becomes,

$$\frac{dY}{dX}=\frac{2X-Y+\left(2h-k+4\right)}{X-2Y+\left(h-2k+5\right)} \text{ ... ... (2)}$$

Now choose

$$2h - k + 4 = 0 \dots\dots\dots (3)$$

$$h - 2k + 5 = 0 \dots\dots\dots (4)$$

Solving equations (3) and (4) we get,

$$h = -1 \text{ and } k = 2$$

with this substitution equation (2) becomes,

$$\frac{dY}{dX} = \frac{2X - Y}{X - 2Y} \dots\dots\dots (5)$$

which is a homogeneous equation in X and Y.

So put, $Y = vX$

$$\therefore \frac{dY}{dX} = v + X\frac{dv}{dX}$$

From equation (5), we have

$$v + X\frac{dv}{dX} = \frac{2X - vX}{X - 2vX}$$

$$or,\ v + X\frac{dv}{dX} = \frac{2 - v}{1 - 2v}$$

$$or,\ X\frac{dv}{dX} = \frac{2 - v}{1 - 2v} - v$$

$$or,\ X\frac{dv}{dX} = \frac{2 - v - v + 2v^2}{1 - 2v}$$

$$or,\ X\frac{dv}{dX} = \frac{2 - 2v + 2v^2}{1 - 2v}$$

$$or,\ X\frac{dv}{dX} = \frac{2 - 2v + 2v^2}{1 - 2v}$$

Integrating both sides, we get

$$\int \frac{1 - 2v}{2 - 2v + 2v^2} dv = \int \frac{dX}{X}$$

$$or,\ \int \frac{1 - 2v}{1 - v + v^2} dv = 2\int \frac{dX}{X}$$

$$or,\ -\ln\left(1-v+v^2\right)=2\ln X+\ln c$$

$$or,\ \ln\left(1-v+v^2\right)^{-1}=\ln X^2+\ln c$$

$$or,\ \left(1-v+v^2\right)^{-1}=cX^2$$

$$or,\ \left(1-\frac{Y}{X}+\frac{Y^{\ 2}}{X^2}\right)cX^2=1$$

$$or,\ \left(X^2-XY+Y^2\right)c=1$$

$$or,\ \left\{\left(x+1\right)^2-\left(x+1\right)\left(y-2\right)+\left(y-2\right)^2\right\}c=1$$

which is the required solution.

Problem-02: Solve the differential equation $\left(3x-7y-3\right)\dfrac{dy}{dx}=\left(3y-7x+7\right)$

Solution: Given differential equation is,

$$\left(3x-7y-3\right)\frac{dy}{dx}=\left(3y-7x+7\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{\left(3y-7x+7\right)}{\left(3x-7y-3\right)}\ \ldots\ \ldots.\ \ldots\ (1)$$

put $x=X+h$ and $y=Y+k$ where h, k are constants.

$$\therefore\ \frac{dy}{dx}=\frac{dY}{dX}$$

Then the equation (1) becomes,

$$\frac{dY}{dX}=\frac{3Y-7X+\left(-7h+3k+7\right)}{3X-7Y+\left(3h-7k-3\right)}\ \ldots\ \ldots\ \ldots\ (2)$$

Now choose

$$-7h+3k+7=0\ \ldots\ \ldots\ \ldots\ (3)$$

$$3h-7k-3=0\ \ldots\ \ldots\ \ldots\ (4)$$

Solving equations (3) and (4) we get,

$$h=1\text{ and }k=0$$

with this substitution equation (2) becomes,

$$\frac{dY}{dX}=\frac{3Y-7X}{3X-7Y} \text{ ... ... (5)}$$

which is a homogeneous equation in X and Y.

So put, $Y=vX$

$$\therefore \frac{dY}{dX}=v+X\frac{dv}{dX}$$

From equation (5), we have

$$v+X\frac{dv}{dX}=\frac{3vX-7X}{3X-7vX}$$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{3v-7}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{3v-7}{3-7v}-v$$

$$\Rightarrow X\frac{dv}{dX}=\frac{3v-7-3v+7v^2}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-7+7v^2}{3-7v}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{-7\left(1-v^2\right)}{3-7v}$$

$$\Rightarrow \frac{3-7v}{1-v^2}dv=-7\frac{dX}{X}$$

Integrating both sides, we get

$$\int\frac{3-7v}{1-v^2}dv=-7\int\frac{dX}{X}$$

$$\Rightarrow \int\frac{3-7v}{(1+v)(1-v)}dv=-7\int\frac{dX}{X}$$

$$\Rightarrow \int\left(\frac{5}{1+v}-\frac{2}{1-v}\right)dv=-7\int\frac{dX}{X}$$

$$\Rightarrow 5\int\frac{dv}{1+v}-2\int\frac{dv}{1-v}=-7\int\frac{dX}{X}$$

$$\Rightarrow 5\ln(1+v)+2\ln(1-v)=-7\ln X+\ln c$$

$$\Rightarrow \ln(1+v)^5 + \ln(1-v)^2 = -7\ln X + \ln c$$

$$\Rightarrow \ln(1+v)^5(1-v)^2 = \ln cX^{-7}$$

$$\Rightarrow (1+v)^5(1-v)^2 = \frac{c}{X^7}$$

$$\Rightarrow \left(1+\frac{Y}{X}\right)^5\left(1-\frac{Y}{X}\right)^2 = \frac{c}{X^7} \qquad ;as\ \ Y = vX$$

$$\Rightarrow (X+Y)^5(X-Y)^2 = c$$

$$\Rightarrow (x+y-1)^5(x-y-1)^2 = c \qquad ;as\ \ x = X+1\ \ and\ \ y = Y+0$$

which is the required solution.

Problem-03: Solve the differential equation $(2x-2y+5)\frac{dy}{dx} = x-y+3$

Solution: Given differential equation is,

$$(2x-2y+5)\frac{dy}{dx} = x-y+3$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \quad \dots \dots \dots (1)$$

put $\quad x-y = v$

$$\therefore\ 1-\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1-\frac{dv}{dx}$$

Then the equation (1) becomes,

$$1-\frac{dv}{dx} = \frac{v+3}{2v+5}$$

$$\Rightarrow \frac{dv}{dx} = 1-\frac{v+3}{2v+5}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2v+5-v-3}{2v+5}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2+v}{2v+5}$$

$$\Rightarrow \frac{2v+5}{2+v}dv = dx$$

Integrating both sides, we get

$$\int \frac{2v+5}{2+v}dv = \int dx$$

$$\Rightarrow \int \frac{2(2+v)+1}{2+v}dv = \int dx$$

$$\Rightarrow \int \left(2+\frac{1}{2+v}\right)dv = \int dx$$

$$\Rightarrow 2v+\ln(2+v) = x+c$$

$$\Rightarrow 2(x-y)+\ln(x-y+2) = x+c$$

$$\Rightarrow x-2y+\ln(x-y+2) = c$$

which is the required solution.

Problem-04: Solve the differential equation $(3x-2y+1)\frac{dy}{dx} = 6x-4y+3$

Solution: Given differential equation is,

$$(3x-2y+1)\frac{dy}{dx} = 6x-4y+3$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(3x-2y)+3}{3x-2y+1} \quad \ldots \ldots \ldots (1)$$

put $3x-2y = v$

$$\therefore \; 3-2\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(3-\frac{dv}{dx}\right)$$

Then the equation (1) becomes,

$$\frac{1}{2}\left(3-\frac{dv}{dx}\right) = \frac{2v+3}{v+1}$$

$$\Rightarrow 3-\frac{dv}{dx} = \frac{4v+6}{v+1}$$

$$\Rightarrow \frac{dv}{dx} = 3 - \frac{4v+6}{v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{3v+3-4v-6}{v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-v-3}{v+1}$$

$$\Rightarrow \frac{v+1}{v+3}\frac{dv}{dx} = -dx$$

Integrating both sides, we get

$$\int \frac{v+1}{v+3} dv = -\int dx$$

$$\Rightarrow \int \frac{(v+3)-2}{v+3} dv = -\int dx$$

$$\Rightarrow \int \left(1 - \frac{2}{v+3}\right) dv = -\int dx$$

$$\Rightarrow (3x-2y) - 2\ln(3x-2y+3) = -x + c$$

$$\Rightarrow 4x - 2y - 2\ln(3x-2y+3) = c$$

which is the required solution.

Exercise: Solve the following problems:

1. $\frac{dy}{dx} = \frac{x-2y+5}{2x-y+4}$
2. $(2x+y+1)dx + (4x+2y-1)dy = 0$
3. $\frac{dy}{dx} = \frac{y-x+1}{y+x+3}$
4. $(2x-5y+3)dx - (2x+4y-6)dy = 0$.

Linear Differential Equation: A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x or constants, is called the linear differential equation of the first order.

Note: To solve this equation, multiply both sides by the following integrating factor.

$$I.F = e^{\int P(x)} dx$$

Problem-01: Solve the differential equation $\left(1-x^2\right)\frac{dy}{dx}-xy=1$.

Solution: Given differential equation is,

$$\left(1-x^2\right)\frac{dy}{dx}-xy=1 \text{ ... ... (1)}$$

Equation (1) can be written as,

$$\left(1-x^2\right)\frac{dy}{dx}-xy=1$$

$$\Rightarrow \frac{dy}{dx}-\frac{x}{\left(1-x^2\right)}y=\frac{1}{\left(1-x^2\right)} \text{ ... ... (2)}$$

This is a linear equation of first order.

$$\text{I.F}=e^{\int\frac{-x}{1-x^2}dx}=e^{\frac{1}{2}\ln\left(1-x^2\right)}=e^{\ln\left(1-x^2\right)^{\frac{1}{2}}}=\left(1-x^2\right)^{\frac{1}{2}}=\sqrt{1-x^2}$$

Multiply both sides of equation (2) by $\sqrt{1-x^2}$, we get

$$\sqrt{1-x^2}\frac{dy}{dx}-\frac{x\sqrt{1-x^2}}{\left(1-x^2\right)}y=\frac{\sqrt{1-x^2}}{\left(1-x^2\right)}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}-\frac{x}{\sqrt{1-x^2}}y=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{1-x^2}\right)=\frac{1}{\sqrt{1-x^2}}$$

Integrating both sides, we get

$$y\sqrt{1-x^2}=\int\frac{dx}{\sqrt{1-x^2}}+c$$

$$\Rightarrow y\sqrt{1-x^2}=\sin^{-1}\left(x\right)+c$$

which is the required solution.

Problem-02: Solve the differential equation $x\frac{dy}{dx}+2y=x^2\log x$.

Solution: Given differential equation is,

$$x\frac{dy}{dx}+2y=x^2\log x \text{ ... ... (1)}$$

Equation (1) can be written as,

$$x\frac{dy}{dx}+2y=x^2\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y=x\log x \text{ ... ... (2)}$$

This is a linear equation of first order.

$$\text{I.F}=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

Multiply both sides of equation (2) by x^2, we get

$$x^2\frac{dy}{dx}+2xy=x^3\log x$$

$$\Rightarrow \frac{d}{dx}\left(x^2y\right)=x^3\log x$$

Integrating both sides, we get

$$x^2y=\int x^3\log x+c$$

$$\Rightarrow x^2y=\log x\int x^3dx-\int\left\{\frac{d}{dx}(\log x)\int x^3dx\right\}dx+c$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\int\frac{1}{x}.\frac{x^4}{4}dx+c$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\frac{1}{4}\int x^3dx+c$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\frac{1}{4}.\frac{x^4}{4}+c$$

$$\Rightarrow x^2y=\frac{x^4}{4}\log x-\frac{x^4}{16}+c$$

$$\Rightarrow y=\frac{x^2}{4}\log x-\frac{x^2}{16}+cx^{-2}$$

which is the required solution.

Problem-03: Solve the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$.

Solution: Given differential equation is,

$$\frac{dy}{dx}+2y\tan x=\sin x \text{ ... ... (1)}$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int 2\tan x dx} = e^{2\ln(\sec x)} = e^{\ln\left(\sec^2 x\right)} = \sec^2 x$$

Multiply both sides of equation (1) by $\sec^2 x$, we get

$$\sec^2 x \frac{dy}{dx} + 2y\sec^2 x \tan x = \sin x \sec^2 x$$

$$\Rightarrow \frac{d}{dx}\left(y\sec^2 x\right) = \sec x \tan x$$

Integrating both sides, we get

$$y\sec^2 x = \int \sec x \tan x dx + c$$

$$\Rightarrow y\sec^2 x = \sec x + c$$

$$\Rightarrow y = \frac{1}{\sec x} + \frac{c}{\sec^2 x}$$

$$\Rightarrow y = \cos x + c\cos^2 x$$

which is the required solution.

Problem-04: Solve the differential equation $x\frac{dy}{dx} - 2y = x^2 + \sin\frac{1}{x^2}$.

Solution: Given differential equation is,

$$x\frac{dy}{dx} - 2y = x^2 + \sin\frac{1}{x^2} \quad \dots\dots\dots (1)$$

The equation (1) can be written as,

$$x\frac{dy}{dx} - 2y = x^2 + \sin\frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{x}y = x + \frac{1}{x}\sin\frac{1}{x^2} \quad \dots\dots\dots (2)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

Multiply both sides of equation (2) by $\frac{1}{x^2}$, we get

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \frac{1}{x} + \frac{1}{x^3}\sin\frac{1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x} + \frac{1}{x^3}\sin\frac{1}{x^2}$$

Integrating both sides, we get

$$\frac{y}{x^2} = \int \frac{dx}{x} + \int \frac{1}{x^3}\sin\frac{1}{x^2}dx + c$$

$$\Rightarrow \frac{y}{x^2} = \ln x + \int \frac{1}{x^3}\sin\frac{1}{x^2}dx + c$$

$$\Rightarrow \frac{y}{x^2} = \ln x - \frac{1}{2}\int \sin t dt + c \qquad ;\ \textit{putting}\ \frac{1}{x^2} = t$$

$$\Rightarrow \frac{y}{x^2} = \ln x + \frac{1}{2}\cos t + c$$

$$\Rightarrow \frac{y}{x^2} = \ln x + \frac{1}{2}\cos\frac{1}{x^2} + c$$

$$\Rightarrow y = x^2 \ln x + \frac{x^2}{2}\cos\frac{1}{x^2} + c\,x^2$$

which is the required solution.

Problem-05: Solve the differential equation $\frac{dy}{dx} + y\cot x = \cos x$.

Solution: Given differential equation is,

$$\frac{dy}{dx} + y\cot x = \cos x \quad \dots\dots\dots (1)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$$

Multiply both sides of equation (1) by $\sin x$, we get

$$\sin x\frac{dy}{dx} + y\sin x\cot x = \sin x\cos x$$

$$\text{or, } \sin x\frac{dy}{dx} + y\cos x = \sin x\cos x$$

$$\text{or, } \frac{d}{dx}\left(y\sin x\right) = \frac{1}{2}\sin 2x$$

Integrating both sides, we get

$$y\sin x = \int \frac{1}{2}\sin 2x$$

$$or,\ y\sin x = \frac{1}{2}\times\left(-\frac{\cos 2x}{2}\right)+c$$

$$or,\ y\sin x = -\frac{1}{4}\cos 2x + c$$

which is the required solution.

Problem-06: Solve the differential equation $\left(1+y^2\right)dx+\left(x-\tan^{-1}y\right)dy=0$.

Solution: Given differential equation is,

$$\left(1+y^2\right)dx+\left(x-\tan^{-1}y\right)dy=0 \ \ldots\ldots\ldots (1)$$

The equation (1) can be written as,

$$\left(1+y^2\right)dx+\left(x-\tan^{-1}y\right)dy=0$$

$$\Rightarrow \left(1+y^2\right)dx = -\left(x-\tan^{-1}y\right)dy$$

$$\Rightarrow \left(1+y^2\right)\frac{dx}{dy} = -x+\tan^{-1}y$$

$$\Rightarrow \frac{dx}{dy} = -\frac{x}{1+y^2}+\frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy}+\frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \ \ldots\ldots\ldots (2)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Multiply both sides of equation (2) by $e^{\tan^{-1}y}$, we get

$$e^{\tan^{-1}y}\frac{dx}{dy}+\frac{x}{1+y^2}e^{\tan^{-1}y} = \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}$$

$$\Rightarrow \frac{d}{dy}\left(xe^{\tan^{-1}y}\right) = \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}$$

Integrating both sides, we get

$$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y} dy + c$$

$$\Rightarrow xe^{\tan^{-1}y} = \int te^t dt + c \qquad ;\ putting\ \tan^{-1}y = t$$

$$\Rightarrow xe^{\tan^{-1}y} = e^t\left(t-1\right) + c$$

$$\Rightarrow xe^{\tan^{-1}y} = e^{\tan^{-1}y}\left(\tan^{-1}y - 1\right) + c$$

$$\Rightarrow x = \left(\tan^{-1}y - 1\right) + ce^{-\tan^{-1}y}$$

which is the required solution.

Problem-07: Solve the differential equation $\left(x+2y^3\right)\frac{dy}{dx} = y$.

Solution: Given differential equation is,

$$\left(x+2y^3\right)\frac{dy}{dx} = y \quad \ldots\ldots\ldots (1)$$

The equation (1) can be written as,

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2 \ldots\ldots\ldots (2)$$

This is a linear equation of first order.

$$\text{I.F} = e^{\int \frac{-1}{y}dy} = e^{-\ln y} = e^{\ln y^{-1}} = \frac{1}{y}$$

Multiply both sides of equation (2) by $\frac{1}{y}$, we get

$$\Rightarrow \frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2y$$

$$\Rightarrow \frac{d}{dy}\left(\frac{x}{y}\right) = 2y$$

Integrating both sides, we get

$$\frac{x}{y} = 2\int y dy + c$$

$$\Rightarrow \frac{x}{y} = 2.\frac{y^2}{2} + c$$

$$\Rightarrow x = y^3 + cy$$

which is the required solution.

Exercise: Solve the following differential equations:

1. $\left(1+x^2\right)\frac{dy}{dx} + y = \tan^{-1} x$
2. $\frac{dy}{dx} + \frac{y}{x} = \sin x$
3. $\left(x+1\right)\frac{dy}{dx} + y = \left(x+1\right)^2$
4. $\frac{dy}{dx} - y\sin x = \sin 2x$
5. $x\frac{dy}{dx} + y = x^3$
6. $\frac{dy}{dx} + \frac{2}{x}y = e^x$

Equations reducible to linear form: Bernoulli Equation: An equation of the form

$$\frac{dy}{dx} + P\left(x\right)y = Q\left(x\right)y^n \quad ; \quad n \neq 0,1$$

where $P\left(x\right)$ and $Q\left(x\right)$ are functions of x or constants is called a Bernoulli Equation of first order.

Theorem: Reduce the Bernoulli Equation to Linear form and then solve it.

Answer: The Bernoulli's equation is,

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \dots\dots\dots (1)$$

Dividing the equation (1) by y^n we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \quad \dots\dots\dots (2)$$

put, $v = y^{1-n}$

$$\therefore \frac{dv}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

Now equation (2) transforms into,

$$\frac{1}{(1-n)} \frac{dv}{dx} + P(x)v = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x) \;\ldots\ldots\ldots (3)$$

Let $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ then equation (3) becomes,

$$\frac{dv}{dx} + P_1(x)v = Q_1(x) \;\ldots\ldots\ldots (4)$$

which is a linear form.

2nd part: The integrating factor is,

$$I.F = e^{\int P_1(x)} dx$$

Multiply both sides of equation (4) by $e^{\int P_1(x)} dx$ we get,

$$e^{\int P_1(x)dx} \frac{dv}{dx} + e^{\int P_1(x)dx} P_1(x)v = e^{\int P_1(x)dx} Q_1(x)$$

$$\Rightarrow \frac{d}{dx}\left(v e^{\int P_1(x)dx} \right) = e^{\int P_1(x)dx} Q_1(x)$$

Integrating both sides, we get

$$\Rightarrow v e^{\int P_1(x)dx} = \int e^{\int P_1(x)dx} Q_1(x) dx + c$$

which is the required solution.

Problem-01: Solve the differential equation $\frac{dy}{dx} = x^3 y^3 - xy$.

Solution: The differential equation is,

$$\frac{dy}{dx} = x^3 y^3 - xy \;\ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$\frac{dy}{dx} = x^3 y^3 - xy$$

$$\Rightarrow \frac{dy}{dx} + xy = x^3 y^3 \ldots\ldots\ldots (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by y^3 we get

$$y^{-3}\frac{dy}{dx} + xy^{-2} = x^3 \ldots\ldots\ldots (3)$$

put $v = y^{-2}$

$$\therefore \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Now the equation (3) becomes,

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$$

$$\Rightarrow \frac{dv}{dx} - 2xv = -2x^3 \ldots\ldots\ldots (4)$$

This is a linear equation.

$$\text{I.F} = e^{\int -2x dx} = e^{-x^2}$$

Multiply both sides of equation (4) by e^{-x^2} we get

$$e^{-x^2}\frac{dv}{dx} - 2xve^{-x^2} = -2x^3e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}\left(ve^{-x^2}\right) = -2x^3e^{-x^2}$$

Integrating both sides we get

$$ve^{-x^2} = -2\int x^3e^{-x^2}dx + c$$

$$\Rightarrow ve^{-x^2} = -\int te^t dt + c \qquad ; \ putting\ -x^2 = t$$

$$\Rightarrow ve^{-x^2} = -e^t\left(t-1\right) + c$$

$$\Rightarrow ve^{-x^2} = -e^{-x^2}\left(-x^2-1\right) + c$$

$$\Rightarrow v = x^2 + 1 + ce^{x^2}$$

$$\Rightarrow y^{-2} = x^2 + 1 + ce^{x^2}$$

$$\Rightarrow\left(x^2+1+ce^{x^2}\right)y^2=1$$

which is the solution.

Problem-02: Solve the differential equation $\dfrac{dy}{dx}+\dfrac{2}{x}y=\dfrac{y^3}{x^3}$.

Solution: The differential equation is,

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{y^3}{x^3} \quad \dots\dots\dots (1)$$

This is a Bernoulli's equation.

Dividing the equation (1) by y^3 we get

$$y^{-3}\frac{dy}{dx}+\frac{2}{x}y^{-2}=\frac{1}{x^3} \dots\dots\dots (2)$$

put $v=y^{-2}$

$$\therefore \frac{dv}{dx}=-2y^{-3}\frac{dy}{dx}\Rightarrow y^{-3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$$

Now the equation (2) becomes,

$$-\frac{1}{2}\frac{dv}{dx}+\frac{2}{x}v=\frac{1}{x^3}$$

$$\Rightarrow \frac{dv}{dx}-\frac{4}{x}v=-\frac{2}{x^3} \dots\dots\dots (3)$$

This is a linear equation.

$$\text{I.F}=e^{\int -\frac{4}{x}dx}=e^{-4\int\frac{dx}{x}}=e^{-4\ln x}=e^{\ln x^{-4}}=\frac{1}{x^4}$$

Multiply both sides of equation (4) by $\dfrac{1}{x^4}$ we get

$$\frac{1}{x^4}\frac{dv}{dx}-\frac{4}{x^5}v=-\frac{2}{x^7}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^4}\right)=-\frac{2}{x^7}$$

Integrating both sides we get

$$\frac{v}{x^4}=-2\int\frac{dx}{x^7}+c$$

$$\Rightarrow \frac{v}{x^4} = \frac{1}{3}\frac{1}{x^6} + c$$

$$\Rightarrow y^{-2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

$$\Rightarrow \frac{1}{y^2} = \frac{1}{3}\frac{1}{x^2} + cx^4$$

which is the required solution.

Problem-03: Solve the differential equation $2y - 3\frac{dy}{dx} = y^4 e^{3x}$.

Solution: The differential equation is,

$$2y - 3\frac{dy}{dx} = y^4 e^{3x} \quad \ldots\ldots\ldots (1)$$

Equation (1) can be written as,

$$\frac{dy}{dx} - \frac{2}{3}y = -\frac{1}{3}y^4 e^{3x} \quad \ldots\ldots\ldots (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by y^4 we get

$$y^{-4}\frac{dy}{dx} - \frac{2}{3}y^{-3} = -\frac{1}{3}e^{3x} \ldots\ldots\ldots (3)$$

put $v = -y^{-3}$

$$\therefore \frac{dv}{dx} = 3y^{-4}\frac{dy}{dx} \Rightarrow y^{-4}\frac{dy}{dx} = \frac{1}{3}\frac{dv}{dx}$$

Now the equation (3) becomes,

$$\frac{1}{3}\frac{dv}{dx} + \frac{2}{3}v = -\frac{1}{3}e^{3x}$$

$$or, \frac{dv}{dx} + 2v = -e^{3x} \quad \ldots\ldots\ldots (4)$$

This is a linear equation.

$$\text{I.F} = e^{\int 2dx} = e^{2x}$$

Multiply both sides of equation (4) by e^{2x} we get

$$e^{2x}\frac{dv}{dx} + 2ve^{2x} = -e^{2x}.e^{3x}$$

$$or,\ \frac{d}{dx}\left(ve^{2x}\right)=-e^{5x}$$

Integrating both sides we get

$$ve^{2x}=-\int e^{5x}dx$$

$$or,\ ve^{2x}=-\frac{1}{5}e^{5x}+c$$

$$or,\ v=-\frac{1}{5}e^{3x}+ce^{-2x}$$

$$or,\ -y^{-3}=-\frac{1}{5}e^{3x}+ce^{-2x}$$

$$or,\ \frac{1}{y^3}=\frac{1}{5}e^{3x}-ce^{-2x}$$

which is the required solution.

Problem-04: Solve the differential equation $x^2y-x^3\frac{dy}{dx}=y^4\cos x$.

Solution: The differential equation is,

$$x^2y-x^3\frac{dy}{dx}=y^4\cos x \quad \ldots\ldots\ldots (1)$$

The equation (1) can be written as,

$$x^3\frac{dy}{dx}-x^2y=-y^4\cos x$$

$$or,\ \frac{dy}{dx}-\frac{y}{x}=-y^4\frac{1}{x^3}\cos x \quad \ldots\ \ldots\ \ldots\ (2)$$

This is a Bernoulli's equation.

Dividing the equation (2) by y^4 we get

$$y^{-4}\frac{dy}{dx}-\frac{y^{-3}}{x}=-\frac{1}{x^3}\cos x \ldots\ldots\ldots (3)$$

put $v=-y^{-3}$

$$\therefore \frac{dv}{dx}=3y^{-4}\frac{dy}{dx}\Rightarrow y^{-4}\frac{dy}{dx}=\frac{1}{3}\frac{dv}{dx}$$

Now the equation (2) becomes,

$$\frac{1}{3}\frac{dv}{dx}+\frac{v}{x}=-\frac{1}{x^3}\cos x$$

$$or,\ \frac{dv}{dx}+\frac{3}{x}v=-\frac{3}{x^3}\cos x \quad \ldots\ldots\ldots (4)$$

This is a linear equation.

$$\text{I.F}=e^{\int\frac{3}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$$

Multiply both sides of equation (4) by x^3 we get

$$x^3\frac{dv}{dx}+3x^2v=-3\cos x$$

$$or,\ \frac{d}{dx}\left(vx^3\right)=-3\cos x$$

Integrating both sides we get

$$vx^3=-3\int\cos x dx$$

$$or,\ vx^3=-3\sin x+c$$

$$or,\ -y^{-3}x^3=-3\sin x+c$$

$$or,\ \frac{x^3}{y^3}=3\sin x-c$$

which is the required solution.

Problem-05: Solve the differential equation $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$.

Solution: The differential equation is,

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3 \quad \ldots\ldots\ldots (1)$$

This is a Bernoulli's equation.

put $v=\tan y$

$$\therefore \frac{dv}{dx}=\sec^2 y\frac{dy}{dx}$$

Now the equation (1) becomes,

$$\frac{dv}{dx}+2xv=x^3 \;\ldots\;\ldots\;\ldots\;(2)$$

This is a linear equation.

$$\text{I.F}=e^{\int 2x\,dx}=e^{x^2}$$

Multiply both sides of equation (2) by e^{x^2} we get

$$e^{x^2}\frac{dv}{dx}+2xe^{x^2}v=x^3e^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(ve^{x^2}\right)=x^3e^{x^2}$$

Integrating both sides we get

$$ve^{x^2}=\int x^3e^{x^2}dx+c$$

$$\Rightarrow ve^{x^2}=\frac{1}{2}\int te^t dt+c \quad ;\; putting\;\; x^2=t$$

$$\Rightarrow ve^{x^2}=\frac{1}{2}\left(te^t-e^t\right)+c$$

$$\Rightarrow ve^{x^2}=\frac{e^t}{2}(t-1)+c$$

$$\Rightarrow ve^{x^2}=\frac{e^{x^2}}{2}\left(x^2-1\right)+c$$

$$\Rightarrow v=\frac{1}{2}\left(x^2-1\right)+ce^{-x^2}$$

$$\Rightarrow \tan y=\frac{1}{2}\left(x^2-1\right)+ce^{-x^2}.$$

which is the required solution.

Exercise:

1. $\dfrac{dy}{dx}+y=y^3\sin x$
2. $\dfrac{dy}{dx}+y=y^2e^x$
3. $y-2x\dfrac{dy}{dx}=x\left(x+1\right)y^3$

4. $\frac{dy}{dx} - y = xy^2$
5. $\frac{dy}{dx} + \frac{y}{x} = x\sqrt{y}$
6. $\frac{dy}{dx} + \frac{y}{x} = xy^2$

Exact Differential Equations: A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be an exact differential equation if it satisfies the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule:

1. Integrate M with respect to x keeping y as constant,
2. Find out those terms in N which are free from x and integrate them with respect to y,
3. Add the two expressions so obtained and equate the sum to an arbitrary constant.

Problem-01: Solve $\left(y^4 + 4x^3y + 3x\right)dx + \left(x^4 + 4xy^3 + y + 1\right)dy = 0$.

Solution: Given that,

$$\left(y^4 + 4x^3y + 3x\right)dx + \left(x^4 + 4xy^3 + y + 1\right)dy = 0 \quad \dots\dots\dots (1)$$

where, $M = y^4 + 4x^3y + 3x$ and $N = x^4 + 4xy^3 + y + 1$

$$\therefore \quad \frac{\partial M}{\partial y} = 4y^3 + 4x^3$$

$$\frac{\partial N}{\partial x} = 4x^3 + 4y^3$$

since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating M with respect to x we get

$$xy^4 + x^4y + \frac{3}{2}x^2$$

In N, terms free from x are $y+1$ whose integral with respect to y is

$$\frac{1}{2}y^2 + y$$

Therefore the general solution is

$$xy^4 + x^4y + \frac{3}{2}x^2 + \frac{1}{2}y^2 + y = c.$$

Problem-02: Solve $\left(x^2-2xy+3y^2\right)dx+\left(4y^3+6xy-x^2\right)dy=0$.

Solution: Given that,

$$\left(x^2-2xy+3y^2\right)dx+\left(4y^3+6xy-x^2\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=x^2-2xy+3y^2$ and $N=4y^3+6xy-x^2$

$$\therefore \quad \frac{\partial M}{\partial y}=-2x+6y$$

$$\frac{\partial N}{\partial x}=6y-2x$$

since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating *M* with respect to *x* we get

$$\frac{1}{3}x^3-x^2y+3xy^2$$

In *N*, terms free from *x* is $4y^3$ whose integral with respect to *y* is

$$y^4$$

Therefore the general solution is

$$\frac{1}{3}x^3-x^2y+3xy^2+y^4=c.$$

Problem-03: Solve $\left(2x^3+3y\right)dx+\left(3x+y-1\right)dy=0$.

Solution: Given that,

$$\left(2x^3+3y\right)dx+\left(3x+y-1\right)dy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=2x^3+3y$ and $N=3x+y-1$

$$\therefore \quad \frac{\partial M}{\partial y}=3$$

$$\frac{\partial N}{\partial x}=3$$

since, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating *M* with respect to *x* we get

$$\frac{1}{2}x^4 + 3xy$$

In *N*, terms free from *x* are $y-1$ whose integral with respect to *y* is

$$\frac{1}{2}y^2 - y$$

Therefore the general solution is

$$\frac{1}{2}x^4 + 3xy + \frac{1}{2}y^2 - y = c\,.$$

Problem-04: Solve $2xydx + \left(x^2 - 1\right)dy = 0$.

Solution: Given that,

$$2xydx + \left(x^2 - 1\right)dy = 0 \quad \ldots\ldots\ldots (1)$$

where, $M = 2xy$ and $N = x^2 - 1$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ so the equation (1) is an exact differential equation.

Integrating *M* with respect to *x* we get

$$x^2 y$$

In *N*, term free from *x* is -1 whose integral with respect to *y* is

$$-y$$

Therefore the general solution is

$$x^2 y - y = c\,.$$

Exercise:

1. $\left(2xy^2 - 3\right)dx + \left(2x^2 y + 4\right)dy = 0$
2. $\left(x - 2e^y\right)dy + \left(y + x\sin x\right)dx = 0$

3. $\left(x^2-y^2\right)dx+\left(x^2-2xy\right)dy=0$
4. $\left(x^3+y^3\right)dx+3xy^2dy=0$

Equations reducible to exact differential equation: A differential equation $M\left(x,y\right)dx+N\left(x,y\right)dy=0$ is not an exact differential equation if

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

But it can be reduced to an exact differential equation by multiplying a function of x and y, which is called an **integrating factor**.

Rules for finding integrating factor: Let the differential equation is,

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0 \quad \ldots\ldots\ldots (1)$$

1. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f\left(x\right)$ then the integrating factor is $\mu=e^{\int f(x)dx}$.
2. If $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g\left(y\right)$ then the integrating factor is $\mu=e^{\int g(y)dy}$.
3. If M and N are both homogeneous function in x, y of degree n, then the integrating factor is $\mu=\dfrac{1}{Mx+Ny}$; *where*, $Mx+Ny\neq 0$.
4. If the equation (1) is of the form, $yf\left(xy\right)dx+xg\left(xy\right)dy=0$ then the integrating factor is $\mu=\dfrac{1}{Mx-Ny}$; *where*, $Mx-Ny\neq 0$

NOTE: 1. If $Mx+Ny=0$, then $\frac{M}{N}=-\frac{y}{x}$ and the equation reduces to $ydx-xdy=0$,

which can be easily solved.

2. If $Mx-Ny=0$, then $\frac{M}{N}=\frac{y}{x}$ and the equation reduces to $ydx+xdy=0$,

which can be easily solved.

Problem-01: Solve $\left(x^2+y^2+x\right)dx+xydy=0$.

Solution: Given that,

$$\left(x^2+y^2+x\right)dx+xydy=0 \quad \ldots\ldots\ldots (1)$$

where, $M=x^2+y^2+x$ and $N=xy$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = y$$

since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2y - y}{xy} = \dfrac{1}{x}$

Hence, I.F $= e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying by I.F, the equation (1) becomes,

$$\left(x^3 + xy^2 + x^2\right)dx + x^2 y dy = 0 \; \ldots \ldots \ldots (2)$$

which is exact now.

Let, $M' = x^3 + xy^2 + x^2$ and $N' = x^2 y$

Integrating M' with respect to x we get

$$\frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{3}x^3$$

In N', there is no term free from x.

Therefore the general solution is

$$\frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{3}x^3 = c\,.$$

Problem-02: Solve $\left(3x^2 y^4 + 2xy\right)dx + \left(2x^3 y^3 - x^2\right)dy = 0$.

Solution: Given that,

$$\left(3x^2 y^4 + 2xy\right)dx + \left(2x^3 y^3 - x^2\right)dy = 0 \quad \ldots \ldots \ldots (1)$$

where, $M = 3x^2 y^4 + 2xy$ and $N = 2x^3 y^3 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2 y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2 y^3 - 2x$$

since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy} = \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy} = \frac{-2\left(3x^2y^3 + 2x\right)}{y\left(3x^2y^3 + 2x\right)} = \frac{2}{y}$

Hence, I.F $= e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$

Multiplying by I.F, the equation (1) becomes,

$$\left(3x^2y^2 + 2\frac{x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M' = 3x^2y^2 + 2\frac{x}{y}$ and $N' = 2x^3y - \frac{x^2}{y^2}$

Integrating M' with respect to x we get

$$x^3y^2 + \frac{x^2}{y}$$

In N', there is no term free from x.

Therefore the general solution is

$$x^3y^2 + \frac{x^2}{y} = c.$$

Problem-03: Solve $\left(2x^2 + y\right)dx + \left(x^2y - x\right)dy = 0$.

Solution: Given that,

$$\left(2x^2 + y\right)dx + \left(x^2y - x\right)dy = 0 \quad \ldots\ldots\ldots (1)$$

where, $M = 2x^2 + y$ and $N = x^2y - x$

$$\therefore \quad \frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 2xy - 1$$

since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

However, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{1-2xy+1}{x^2y-x} = \dfrac{-2xy+2}{x^2y-x} = \dfrac{-2(xy-1)}{x(xy-1)} = -\dfrac{2}{x}$

Hence, $\text{I.F} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying by I.F, the equation (1) becomes,

$$\left(2+\frac{y}{x^2}\right)dx + \left(y-\frac{1}{x}\right)dy = 0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M' = 2+\frac{y}{x^2}$ and $N' = y-\frac{1}{x}$

Integrating M' with respect to *x* we get

$$2x-\frac{y}{x}$$

In N', the term free from *x* is *y* and integrating it with respect to y we have

$$\frac{y^2}{2}$$

Therefore the general solution is

$$2x-\frac{y}{x}+\frac{y^2}{2}=c.$$

Problem-04: Solve $\left(x^4+y^4\right)dx - xy^3dy = 0$.

Solution: Given that,

$$\left(x^4+y^4\right)dx - xy^3dy = 0 \quad \ldots\ldots\ldots (1)$$

where, $M = x^4+y^4$ and $N = -xy^3$

$$\therefore \quad \frac{\partial M}{\partial y} = 4y^3$$

$$\frac{\partial N}{\partial x} = -y^3$$

since, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

But the equation (1) is a homogeneous differential equation.

Hence, I.F $= \dfrac{1}{Mx+Ny} = \dfrac{1}{x^5+xy^4-xy^4} = \dfrac{1}{x^5}$

Multiplying by I.F, the equation (1) becomes,

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}dy = 0 \quad \ldots\ldots\ldots (2)$$

which is exact now.

Let, $M' = \dfrac{1}{x}+\dfrac{y^4}{x^5}$ and $N' = -\dfrac{y^3}{x^4}$

Integrating M' with respect to x we get

$$\ln x - \frac{y^4}{4x^4}$$

In N', there is no term free from x.

Therefore the general solution is

$$\ln x - \frac{y^4}{4x^4} = c.$$

Problem-05: Solve $y\left(xy+2x^2y^2\right)dx + x\left(xy-x^2y^2\right)dy = 0$.

Solution: Given that,

$$y\left(xy+2x^2y^2\right)dx + x\left(xy-x^2y^2\right)dy = 0 \quad \ldots\ldots\ldots (1)$$

where, $M = y\left(xy+2x^2y^2\right)$ and $N = x\left(xy-x^2y^2\right)$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy + 6x^2y^2$$

$$\frac{\partial N}{\partial x} = 2xy - 6x^2y^2$$

since, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ so the equation (1) is not an exact differential equation.

But the equation (1) is of the form, $yf(xy)dx + xg(xy)dy = 0$.

Hence, I.F $= \dfrac{1}{Mx - Ny} = \dfrac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \dfrac{1}{3x^3y^3}$

Multiplying by I.F, the equation (1) becomes,

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \quad \dots\dots\dots (2)$$

which is exact now.

Let, $M^{'} = \dfrac{1}{3x^2y} + \dfrac{2}{3x}$ and $N^{'} = \dfrac{1}{3xy^2} - \dfrac{1}{3y}$

Integrating $M^{'}$ with respect to x we get

$$-\frac{1}{3xy} + \frac{2}{3}\ln x$$

In $N^{'}$, term free from x is $-\dfrac{1}{3y}$, whose integral with respect to y is,

$$-\frac{1}{3}\ln y$$

Therefore the general solution is

$$-\frac{1}{3xy} + \frac{2}{3}\ln x - \frac{1}{3}\ln y = \ln c$$

$$\Rightarrow -\frac{1}{xy} + \ln x^2 - \ln y = 3\ln c$$

$$\Rightarrow -\frac{1}{xy} + \ln\frac{x^2}{y} = \ln c^3$$

$$\Rightarrow \frac{x^2}{y}e^{-\frac{1}{xy}} = c^3$$

$$\Rightarrow \frac{x^2}{y} = Ce^{\frac{1}{xy}} \quad ; \; where,\ C = c^3 \; .$$

Exercise:

1. $\left(x^3 - 2y^2\right)dx + 2xydy = 0$
2. $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$
3. $\left(2xy^2 + y\right)dx + \left(2y^3 - x\right)dy = 0$
4. $y\left(x^2y^2 + 2\right)dx + x\left(2 - 2x^2y^2\right)dy = 0$