Integral Equations

Integral equation: An equation containing an unknown function under one or more integral signs is called an integral equation. The general form of an integral equation is

$$\phi(x)=F(x)+\lambda\int_{\alpha(x)}^{\beta(x)}K(x,\xi)\phi(\xi)\,d\xi \qquad (1)$$

where $\phi(x)$ is an unknown function, $F(x)$ and $K(x,\xi)$ are both known functions, $\alpha(x)$ and $\beta(x)$ are limits of integration, λ is a constant parameter.

Also $K(x,\xi)$ is known as the kernel of the integral equation (1).

There are two types integral equation such as:

(i) Linear integral equation

(ii) Non-linear integral equation.

Linear integral equation: An integral equation is called linear if only linear operations are performed in it upon the unknown function. The most general type of linear integral equation is of the form

$$\alpha(x)\phi(x)=F(x)+\lambda\int_{\Omega}K(x,\xi)\phi(\xi)\,d\xi$$

where $\phi(x)$ is the unknown function, $\alpha(x)$, $F(x)$ and $K(x,\xi)$ are known functions.

Example: 1. $\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$

2. $\phi(x)=\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$.

Non-linear integral equation: An integral equation is called non-linear if the unknown function appears under an integral sign to a power n $(n>1)$. The most general type of non-linear integral equation is of the form

$$\alpha(x)\phi(x)=F(x)+\lambda\int_{\Omega}K(x,\xi)\phi^n(\xi)\,d\xi$$

where $\phi(x)$ is the unknown function, $\alpha(x)$, $F(x)$ and $K(x,\xi)$ are known functions.

Example: 1. $\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi^2(\xi)\,d\xi$

2. $\phi(x)=\lambda\int_a^b K(x,\xi)\phi^3(\xi)\,d\xi$.

Again the linear integral equations are classified into two basic types:

1. Volterra integral equation
2. Fredholm integral equation.

Volterra integral equation: An integral equation is called Volterra integral equation if the upper limit of the integration is a variable. The most general type of Volterra integral equation is of the form

$$\alpha(x)\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi; \qquad a>-\infty$$

where $\phi(x)$ is the unknown function, $\alpha(x)$, $F(x)$ and $K(x,\xi)$ are known functions.

Case-01: If $\alpha(x)=0$, then

$$F(x)=-\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi$$

This is called the Volterra's integral equation of first kind.

Case-02: If $\alpha(x)=1$, then

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi$$

This is called the Volterra's integral equation of second kind.

Case-03: If $\alpha(x)=1$ and $F(x)=0$, then

$$\phi(x)=\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi$$

This is called the homogeneous Volterra's integral equation of second kind.

Fredholm integral equation: An integral equation is called Fredholm integral equation if the domain of the integration is fixed i.e. upper limit and lower limit are constant. The most general type of Fredholm integral equation is of the form

$$\alpha(x)\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi; \qquad a\le x\le b$$

where $\phi(x)$ is the unknown function, $\alpha(x)$, $F(x)$ and $K(x,\xi)$ are known functions.

Case-01: If $\alpha(x)=0$, then

$$F(x)=-\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$$

This is called the Fredholm integral equation of first kind.

Case-02: If $\alpha(x)=1$, then

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$$

This is called the Fredholm integral equation of second kind.

Case-03: If $\alpha(x)=1$ and $F(x)=0$, then

$$\phi(x)=\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$$

This is called the homogeneous Fredholm integral equation of second kind.

Singular integral equation: An integral equation is called Singular integral equation if one or both limits of integration become infinite or the kernel becomes infinite at one or more points within the range of integration.

Example: 1. $\phi(x)=F(x)+\lambda\int_{-\infty}^{\infty} K(x,\xi)\phi(\xi)\,d\xi$

2. $\phi(x)=\int_a^x \frac{1}{(x-\xi)^\alpha}\phi(\xi)\,d\xi; \qquad 0<\alpha<1.$

Special kinds of kernels: Some kernel are as follows:

1. **Symmetric kernel:** A kernel $K(x,\xi)$ is called symmetric (or complex symmetric or Hermitian) if

$$K(x,\xi)=\overline{K}(x,\xi)$$

where the bar denotes the complex conjugate. A real kernel $K(x,\xi)$ is symmetric if $K(x,\xi)=K(\xi,x)$.

2. **Difference kernel:** If the kernel $K(x,\xi)$ is dependent solely on the difference $x-\xi$,

$$\text{i.e. } K(x,\xi)=K(x,-\xi)$$

then $K(x-\xi)$ is called the difference kernel.

3. **Separable or Degenerate kernel:** If the kernel $K(x,\xi)$ can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of ξ only, i.e.

$$K(x,\xi)=\sum_{i=1}^{n} g_i(x)h_i(\xi)$$

then the kernel is called a separable or degenerate kernel.

4. **Iterated kernels:** (a) Consider the Volterra integral equation of second kind

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi .$$

Then, the iterated kernels $K_n(x,\xi);\quad n=1,2,3,\cdots$ are defined as

$$K_1(x,\xi)=K(x,\xi)$$

and $$K_n(x,\xi)=\int_\xi^x K(x,z)K_{n-1}(z,\xi)dz;\qquad n=2,3,\cdots$$

(b) Consider the Fredholm integral equation of second kind

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi .$$

Then, the iterated kernels $K_n(x,\xi);\quad n=1,2,3,\cdots$ are defined as

$$K_1(x,\xi)=K(x,\xi)$$

and $$K_n(x,\xi)=\int_a^b K(x,z)K_{n-1}(z,\xi)dz;\qquad n=2,3,\cdots$$

5. **Resolvent kernel or Reciprocal kernel:** Suppose the solution of the Volterra integral equation of second kind

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi$$

is of the form $$\phi(x)=F(x)+\lambda\int_a^x R(x,\xi;\lambda)F(\xi)\,d\xi$$

and the solution of the Fredholm integral equation of second kind

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi$$

is of the form $$\phi(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)\,d\xi .$$

Then $R(x,\xi;\lambda)$ is called the resolvent kernel or reciprocal kernel of the given equations.

Eigen values and Eigen functions: Let us consider the homogeneous Fredholm integral equation

$$\phi(x)=\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi. \qquad (1)$$

The values of λ for which (1) has a non-zero solution are called the eigen values of (1) or of the kernel $K(x,\xi)$ and for every eigen value of λ, the corresponding non-zero solution of (1) is called the eigen function.

Question-01: Explain the differentiation of a function under an integral sign.

Solution: Consider the function $I_n(x)$ defined by the relation

$$I_n(x)=\int_a^x (x-\eta)^{n-1} f(\eta)\,d\eta \qquad \cdots(1)$$

where η is a positive integral and a is a constant.

Here $\qquad F(x,\eta)=(x-\eta)^{n-1}f(\eta).$

Differentiating (1) w. .r. to x under the integral sign, we get

$$\frac{dI_n}{dx}=(n-1)\int_a^x (x-\eta)^{n-2}f(\eta)\,d\eta+\frac{dx}{dx}(x-x)^{n-1}f(x)-\frac{da}{dx}(x-a)^{n-1}f(a)$$

$$=(n-1)\int_a^x (x-\eta)^{n-2}f(\eta)\,d\eta+0-0$$

$$=(n-1)\int_a^x (x-\eta)^{n-2}f(\eta)\,d\eta$$

$$or, \frac{dI_n}{dx}=(n-1)I_{n-1},\ n>1. \qquad \cdots(2)$$

From (1), for $n=1$, we have

$$\frac{dI_1}{dx}=f(x) \qquad \cdots(3)$$

Differentiating (2) successively m times, we have

$$\frac{d^m I_n}{dx^m}=(n-1)(n-2)(n-3)\cdots(n-m)I_{n-m},\ n>m. \qquad \cdots(4)$$

In particular, taking $m=n-1$ in (4) we get

$$\frac{d^{n-1}I_n}{dx^{n-1}}=(n-1)!\,I_1$$

$$or, \frac{d^n I_n}{dx^n}=(n-1)!\,\frac{dI_1}{dx} \qquad \cdots(5)$$

Using (2) in (5) we get

$$\frac{d^{n-1}I_n}{dx^{n-1}}=(n-1)!\,f(x) \qquad \cdots(6)$$

Thus, we have

$$I_1=\int_a^x f(x_1)\,dx_1$$

$$I_2=\int_a^x I_1(x_2)\,dx_2=\int_a^x\int_a^{x_2} f(x_1)\,dx_1\,dx_2$$

In general, we have

$$I_n(x)=(n-1)!\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)\,dx_1\,dx_2\cdots dx_{n-1}dx_n. \qquad \cdots(7)$$

From (1) and (7), we have

$$\int_a^x\int_a^{x_n}\cdots\int_a^{x_3}\int_a^{x_2} f(x_1)\,dx_1\,dx_2\cdots dx_{n-1}dx_n=\frac{1}{(n-1)!}\int_a^x (x-\eta)^{n-1}f(\eta)\,d\eta \qquad \cdots(8)$$

This may be represented as the result of integrating the function f from a to x and then integrating $(n-1)$ times, we have

$\int_a^x f(\eta)d\eta^n = \int_a^x \frac{(x-\eta)^{n-1}}{(n-1)!} f(\eta)d\eta.$

(Complete)

Note: Leibnitz's Rule: If $\int\limits_{a(x)}^{b(x)} F(x,\xi)d\xi$ then the derivative w.r.to x is

$$\frac{d}{dx}\int\limits_{a(x)}^{b(x)} F(x,\xi)d\xi = \int\limits_{a(x)}^{b(x)} \frac{\partial F(x,\xi)}{\partial x}d\xi + \frac{db(x)}{dx}F(x,b(x)) - \frac{da(x)}{dx}F(x,a(x)).$$

Question-02: Establish the relation between differential and integral equations.

Solution: There is a fundamental relationship between integral equation and ordinary and partial differential equation with given initial values. Consider the equation

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = F(x) \qquad \cdots (1)$$

with continuous coefficients $a_i(x),\ \ i = 1,2,\cdots,n$. The initial conditions are prescribed as follows:

$$y(0) = c_0\,, y'(0) = c_1\,, \cdots, y^{n+1}(0) = c_{n-1} \qquad \cdots (2)$$

where a prime denotes differentiation with regard to x.

Consider $\frac{d^n y}{dx^n} = \varphi(x)$.

Taking into account the initial conditions (2), we get

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_0^x \varphi(u)du + c_{n-1}$$

$$or, y = \int_0^x \varphi(u)du^n + c_{n-1}\frac{x^{n-1}}{(n-1)!} + c_{n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + c_0 \qquad \cdots (3)$$

where $\int_0^x \varphi(u)du^n$ represents for a multiple integral of order n.

From (9) and (11), we have

$$\varphi(x) + a_1(x)\int_0^x \varphi(u)du + \cdots + a_n(x)\int_0^x \varphi(u)du^n = F(x) + \sum_1^n iC_i\,\chi_i(x). \qquad \cdots (4)$$

where $\chi_i(x) = a_i(x) + a_{i+1}(x)\frac{x}{1!} + \cdots + a_n(x)\frac{x^{n-i}}{(n-i)!}$ $\cdots (5)$

Putting $F(x) + \sum_1^n iC_i\,\chi_i(x) = G(x)$ in (12) we get

$$\varphi(x) + \int_0^x \left[a_1(x) + a_2(x)(x-u) + \cdots + a_n(x)\frac{(x-u)^{n-1}}{(n-1)!}\right]\varphi(u)du = G(x)$$

which represents the Volterra's integral equation of second kind.

Question-03: Show that the IVP $u'(t) = g(t,u(t)), u(t_0) = u_0$ is equivalent to the integral equation $u(t) = u_0 + \int_{t_0}^t g\,(s,u(s))ds$.

Solution: Given that $u'(t) = g(t,u(t))$

$$\Rightarrow \frac{du(t)}{dt} = g(t,u(t)) \qquad \cdots (1)$$

with $\quad u(t_0) = u_0 \qquad \cdots (2)$

From (1) we get,

$$du(t) = g(t,u(t))dt$$

Integrating both sides within t_0 to t we have

$$\int_{t_0}^{t} du(t) = \int_{t_0}^{t} g\big(s, u(s)\big)ds$$

$$\Longrightarrow [u(t)]_{t_0}^{t} = \int_{t_0}^{t} g\big(s, u(s)\big)ds$$

$$\Longrightarrow u(t) - u(t_0) = \int_{t_0}^{t} g\big(s, u(s)\big)ds$$

$$\Longrightarrow u(t) - u_0 = \int_{t_0}^{t} g\big(s, u(s)\big)ds$$

$$\Longrightarrow u(t) = u_0 + \int_{t_0}^{t} g\big(s, u(s)\big)ds$$

This is an integral equation and the given equation is equivalent to this equation. (**Showed**)

Question-04: State and prove the set of conditions that ensure the existence of a unique solution of the Volterra Integral Equation of the second kind:

$$x(t) = f(t) + \int_{t_0}^{t} \mathrm{K}(t,s)\, x(s)\, ds, \;\; t \in [t_0, t_0 + a\,].$$

<u>Solution</u>: Statement: (i) if $f(t) \neq 0$, is real and continuous on $[t_0, t_0 + a\,]$, $a > 0$

(ii) $\mathrm{K}(t,s,x)$ is real and continuous function on $\Delta = \{(t,s,x)\colon\; t_0 \le s \le\; t_0 + a,\; |x - f(t)| \le b,\; b > 0\}$

(iii) $\mathrm{K}(t,s,x)$ satisfies in Δ the Lipschitz condition $|\mathrm{K}(t,s,x) - \mathrm{K}(t,s,y)| \le L|x-y|,\; L > 0$ then there exist continuous solution $\phi(t)$ of

$$x(t) = f(t) + \int_{t_0}^{t} \mathrm{K}(t,s)\, x(s)\, ds \qquad \cdots (1)$$

defined for $t \in [t_0, t_0 + a\,]$, $\delta = min\left\{a, \frac{b}{M}\right\}$ with $M = \sup\limits_{(t,s,x)\in\Delta} |K(t,s,x)|$ $\cdots (2)$

Proof: By the method of successive approximations, we construct the sequence as follows:

$$\phi_0(t) = f(t) \qquad \cdots (3)$$

$\cdots\;\cdots\;\cdots\;\cdots$

$$\phi_n(t) = f(t) + \int_{t_0}^{t} \mathrm{K}(t,s,\phi_{n-1}(s)\,)\, ds, \quad n \ge 1 \qquad \cdots (4)$$

Since all $\phi_n(t)$ are continuous functions on $[t_0, t_0 + a\,]$ with δ given by (2).

To prove the uniform convergence of the sequence $\{\phi_n(t)\}$, we shall consider the associated series

$$\sum_{n=1}^{\infty} [\phi_n(t) - \phi_{n-1}(t)] \qquad \cdots (5)$$

Putting $n = 1$, in (4) we get

$$\phi_1(t) = f(t) + \int_{t_0}^{t} \mathrm{K}\big(t,s,\phi_0(s)\big)\, ds$$

$$\Rightarrow \phi_1(t) = \phi_0(t) + \int_{t_0}^{t} \mathrm{K}\big(t,s,\phi_0(s)\big)\, ds$$

$$\Rightarrow \phi_1(t) - \phi_0(t) = \int_{t_0}^{t} \mathrm{K}\big(t,s,\phi_0(s)\big)\, ds$$

$$\Rightarrow |\phi_1(t) - \phi_0(t)| = \left|\int_{t_0}^{t} \mathrm{K}\big(t,s,\phi_0(s)\big)\, ds\right|$$

$$\Rightarrow |\phi_1(t) - \phi_0(t)| \le \int_{t_0}^{t} \left|\mathrm{K}\big(t,s,\phi_0(s)\big)\right|\, ds$$

$$\Rightarrow |\phi_1(t)-\phi_0(t)| \le M\int_{t_0}^{t} ds$$
$$\Rightarrow |\phi_1(t)-\phi_0(t)| \le M[s]_{t_0}^{t}$$
$$\Rightarrow |\phi_1(t)-\phi_0(t)| \le M(t-t_0) \qquad \cdots (6)$$

To obtain the estimates for $|\phi_n(t)-\phi_{n-1}(t)|,\ n\ge 2$

We get from (4)

$$|\phi_n(t)-\phi_{n-1}(t)| = \left|f(t)+\int_{t_0}^{t}\mathrm{K}\big(t,s,\phi_{n-1}(s)\big)\,ds - f(t) - \int_{t_0}^{t}\mathrm{K}\big(t,s,\phi_{n-2}(s)\big)\,ds\right|$$
$$= \left|\int_{t_0}^{t}\left[\mathrm{K}\big(t,s,\phi_{n-1}(s)\big)-\mathrm{K}\big(t,s,\phi_{n-2}(s)\big)\right]ds\right|$$
$$\Rightarrow |\phi_n(t)-\phi_{n-1}(t)| \le L\int_{t_0}^{t}|\phi_{n-1}(s)-\phi_{n-2}(s)|\,ds \qquad \cdots (7)$$

Putting $n=2$ in (7) we get

$$|\phi_2(t)-\phi_1(t)| \le L\int_{t_0}^{t}|\phi_1(s)-\phi_0(s)|\,ds$$
$$\Rightarrow |\phi_2(t)-\phi_1(t)| \le L\int_{t_0}^{t}M(s-t_0)\,ds$$
$$\Rightarrow |\phi_2(t)-\phi_1(t)| \le LM\left[\frac{(s-t_0)^2}{2}\right]_{t_0}^{t}$$
$$\Rightarrow |\phi_2(t)-\phi_1(t)| \le LM\,.\frac{(t-t_0)^2}{2} \qquad \cdots (8)$$

Putting $n=3$ in (7) we get

$$|\phi_3(t)-\phi_2(t)| \le L\int_{t_0}^{t}|\phi_2(s)-\phi_1(s)|\,ds$$
$$\Rightarrow |\phi_3(t)-\phi_2(t)| \le L\int_{t_0}^{t}LM.\frac{(s-t_0)^2}{2}\,ds$$
$$\Rightarrow |\phi_3(t)-\phi_2(t)| \le ML^2\left[\frac{(s-t_0)^3}{2.3}\right]_{t_0}^{t}$$
$$\Rightarrow |\phi_3(t)-\phi_2(t)| \le ML^2\,.\frac{(t-t_0)^3}{3!} \qquad \cdots (9)$$

Thus, according to mathematical induction we get

$$|\phi_n(t)-\phi_{n-1}(t)| \le ML^{n-1}\,.\frac{(t-t_0)^n}{n!},\quad t\in[t_0,t_0+\delta] \qquad \cdots (10)$$

which holds for all positive integer $n\ge 1$.

Taking summation of (10) over n we get

$$\sum_{n=1}^{\infty}|\phi_n(t)-\phi_{n-1}(t)| \le \sum_{n=1}^{\infty}ML^{n-1}\,.\frac{(t-t_0)^n}{n!}$$

This series is convergent for all values , L and $t\in[t_0,t_0+\delta]$. As a result, the series (5) is absolutely and uniformly convergent on $[t_0,t_0+\delta]$ and so the sequence $\{\phi_n(t)\}$, since

$$\phi_n(t)-\phi_0(t) = \phi_1(t)-\phi_0(t)+[\phi_2(t)-\phi_1(t)]+\cdots+\ [\phi_n(t)-\phi_{n-1}(t)]$$

$$= \sum_{1}^{n} [\phi_n(t) - \phi_{n-1}(t)]$$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

The function $\phi(t)$ is continuous on $[t_0, t_0 + \delta]$ and satisfies (1). In (4) we can make $n \to \infty$ and obtain (1), because

$$\lim_{n\to\infty} \int_{t_0}^{t} \mathrm{K}\big(t, s, \phi_{n-1}(s)\big)\, ds = \int_{t_0}^{t} \mathrm{K}\big(t, s, \phi(s)\big)\, ds, t \in [t_0, t_0 + \delta]$$

and this relation holds, since

$\left|\mathrm{K}\big(t, s, \phi_{n-1}(s)\big) - \mathrm{K}\big(t, s, \phi(s)\big)\right| \le L|\phi_n(s) - \phi(s)|,\ t_0 \le s \le t \le\ t_0 + \delta.$

This shows that there exists a solution of (1).

Uniqueness: to prove the uniqueness, let $\psi(t)$ be another continuous solution of (1) on the same interval.

Then $\psi(t) = f(t) + \int_{t_0}^{t} \mathrm{K}\big(t, s, \psi(s)\big)\, ds \qquad \cdots (11)$

Now subtracting (4) from (11) we get

$$\psi(t) - \phi_n(t) = \int_{t_0}^{t} \left[\mathrm{K}\big(t, s, \psi(s)\big) - \mathrm{K}\big(t, s, \phi_{n-1}(s)\big)\right] ds$$

$$\Rightarrow |\psi(t) - \phi_n(t)| = \left|\int_{t_0}^{t} \left[\mathrm{K}\big(t, s, \psi(s)\big) - \mathrm{K}\big(t, s, \phi_{n-1}(s)\big)\right] ds\right|$$

$$\Rightarrow |\psi(t) - \phi_n(t)| \le L \int_{t_0}^{t} |\psi(s) - \phi_{n-1}(s)|\, ds \qquad \cdots (12)$$

From (11) we have

$$\psi(t) - f(t) = \int_{t_0}^{t} \mathrm{K}\big(t, s, \psi(s)\big)\, ds$$

$$\Rightarrow |\psi(t) - f(t)| = \left|\int_{t_0}^{t} \mathrm{K}\big(t, s, \psi(s)\big)\, ds\right|$$

$$\Rightarrow |\psi(t) - f(t)| \le \left|\mathrm{K}\big(t, s, \psi(s)\big)\right| \int_{t_0}^{t} ds$$

$$\Rightarrow |\psi(t) - f(t)| \le M[s]_{t_0}^{t}$$

$$\Rightarrow |\psi(t) - f(t)| \le M(t - t_0) \qquad \cdots (13)$$

Using this result in (12) we get

$$|\psi(t) - \phi_n(t)| \le ML^n . \frac{(t-t_0)^{n+1}}{(n+1)!}$$

This result tends to zero as $n \to \infty$

$$\therefore \lim_{n\to\infty} [\psi(t) - \phi_n(t)] = 0$$

$$\Rightarrow \psi(t) - \lim_{n\to\infty} \phi_n(t) = 0$$

$$\Rightarrow \psi(t) - \phi(t) = 0$$

$$\Rightarrow \psi(t) = \phi(t)$$

Thus the solution is unique.

Hence the proof is complete.

Question-05: State and prove a set of conditions for which Fredholm linear integral equation $x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s)\,x(s)\,ds,\ t \epsilon\, [a,b]$ has a unique solution.

<u>Solution</u>: Statement: Consider the Fredholm integral equation of second kind as

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s)\,x(s)\,ds \qquad \cdots (1)$$

where,

(i) the kernel $\mathrm{K}(t,s) \neq 0$, is real and continuous in the rectangle R, for which a ≤ x ≤ b and a ≤ ξ ≤ b.

(ii) $f(t) \neq 0$, is real and continuous in an interval I, for which a ≤ x ≤ b and λ is a constant.

Then the Fredholm integral equation in (1) has a unique solution in I. The solution is the series

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s)\,x(s)ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(t,s_1)\,x(s_1)ds_1 ds + \cdots \qquad \cdots (2)$$

which is absolutely and uniformly convergent.

<u>Proof</u>: From (1) we have

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s)\,x(s)\,ds$$

Replacing s by s_1 we get

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s_1)\,x(s_1)\,ds_1 \qquad \cdots (3)$$

Again replacing t by s in (3) we get

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(s,s_1)\,x(s_1)\,ds_1 \qquad \cdots (4)$$

Putting the value of $x(s)$ in (1) we get,

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s) \left[f(s) + \lambda \int_a^b \mathrm{K}(s,s_1)\,x(s_1)\,ds_1 \right] ds$$

$$= f(t) + \lambda \int_a^b \mathrm{K}(t,s) f(s) ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1)\,x(s_1) ds_1 ds \qquad \cdots (5)$$

Rewriting (4) by using $s_1 = s_2$, and then replacing s by s_1 we get

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(s,s_2)\,x(s_2)\,ds_2 \qquad \cdots (6)$$

and $\quad x(s_1) = f(s_1) + \lambda \int_a^b \mathrm{K}(s_1,s_2)\,x(s_2)\,ds_2 \qquad \cdots (7)$

Putting the value of $x(s_1)$ from (7) in (5) we get,

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s) f(s) ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \left[f(s_1) + \lambda \int_a^b \mathrm{K}(s_1,s_2)\,x(s_2)\,ds_2 \right] ds_1 ds$$

$$= f(t) + \lambda \int_a^b \mathrm{K}(t,s) f(s) ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) f(s_1) ds_1 ds$$

$$+ \lambda^3 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \int_a^b \mathrm{K}(s_1,s_2)\,x(s_2)\,ds_2 ds_1 ds \qquad \cdots (8)$$

Proceeding in this way up to n times we get

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s) f(s) ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) f(s_1) ds_1 ds \; + \cdots\cdots$$
$$+ \lambda^n \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \cdots \int_a^b \mathrm{K}(s_{n-2},s_{n-1})\, x(s_{n-1})\, ds_{n-1} \cdots ds_1 ds + R_{n+1}(t) \quad \cdots (9)$$

where $R_{n+1}(t) = \lambda^{n+1} \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \cdots \int_a^b \mathrm{K}(s_{n-1},s_n)\, x(s_n)\, ds_n \cdots ds_1 ds \qquad \cdots (10)$

Let us now consider the infinite series

$$x(t) = f(t) + \lambda \int_a^b \mathrm{K}(t,s) f(s) ds + \lambda^2 \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) f(s_1) ds_1 ds \; + \cdots \qquad \cdots \; (11)$$

By conditions (i) and (ii) each term of the series (11) is continuous in I. Thus, the series (11) is continuous in I, provided it converges uniformly in I.

Let $|K(t,s)| \le P$ and $|f(t)| \le Q$ contains the maximum value in R and I respectively.

Assume $U_n(t) = \lambda^n \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \cdots \int_a^b \mathrm{K}(s_{n-2},s_{n-1})\, x(s_{n-1})\, ds_{n-1} \cdots ds_1 ds$

Then $|U_n(t)| \le |\lambda^n| Q P^n (b-a)^n$

It will converge only if

$$|\lambda| P(b-a) < 1 \Rightarrow |\lambda| < \frac{1}{P(b-a)}$$

Therefore the series (11) converges absolutely and uniformly.

If (1) has a continuous solution, it must be expressed by (9). If $x(t)$ is continuous in , then $|x(t)|$ must have a maximum value, say M. Thus, from (10),

$$|R_{n+1}(t)| = \left| \lambda^{n+1} \int_a^b \mathrm{K}(t,s) \int_a^b \mathrm{K}(s,s_1) \cdots \int_a^b \mathrm{K}(s_{n-1},s_n)\, x(s_n)\, ds_n \cdots ds_1 ds \right|$$
$$\Rightarrow |R_{n+1}(t)| = |\lambda^{n+1}| M P^{n+1} (b-a)^{n+1} \Rightarrow \lim_{n\to\infty} R_{n+1}(t) = 0$$

Thus, $x(t)$ satisfying (9) is the continuous function given by the series (11). This prove our desired results.

Question-06: Solve Volterra integral equation of second kind by the method of successive substitutions.

Answer: Consider the Volterra's integral equation of second kind as

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\phi(\xi)\, d\xi \qquad (1)$$

where (i) The kernel $K(x,\xi) \ne 0$ is real and continuous in the rectangle $R: a \le x \le b,\ a \le \xi \le b$.

Consider $|K(x,\xi)| \le P$, where P is the maximum value in R.

(ii) The function $F(x) \ne 0$ is real and continuous in an interval $a \le x \le b$.

Consider $|F(x)| \le Q$, where Q is the maximum value in the interval $a \le x \le b$.

(iii) λ is non-zero numerical parameter.

Substituting the unknown function $\phi(\xi)$ under an integral sign from the relation (1), we get

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi) \left\{ F(\xi) + \lambda \int_a^\xi K(\xi,\xi_1)\phi(\xi_1)\, d\xi_1 \right\} d\xi$$
$$= F(x) + \lambda \int_a^x K(x,\xi) F(\xi)\, d\xi + \lambda^2 \int_a^x K(x,\xi) \int_a^\xi K(\xi,\xi_1)\phi(\xi_1)\, d\xi_1 d\xi$$

Performing this operation successively for $\phi(\xi)$, we have

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\left\{F(\xi_1)+\lambda\int_a^{\xi_1}K(\xi_1,\xi_2)\phi(\xi_2)d\xi_2\right\}d\xi_1 d\xi$$

$$=F(x)+\lambda\int_a^x K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi$$

$$+\lambda^3\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\int_a^{\xi_1}K(\xi_1,\xi_2)\phi(\xi_2)d\xi_2 d\xi_1 d\xi$$

In general, we have

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi+\cdots\ \cdots+$$

$$+\lambda^n\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\cdots\ \cdots\int_a^{\xi_{n-2}}K(\xi_{n-2},\xi_{n-1})F(\xi_{n-1})d\xi_{n-1}d\xi_{n-2}\cdots\cdots d\xi_1 d\xi$$

$$+\lambda^{n+1}\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\int_a^{\xi_1}K(\xi_1,\xi_2)\cdots\ \cdots\int_a^{\xi_{n-1}}K(\xi_{n-1},\xi_n)\phi(\xi_n)d\xi_n d\xi_{n-1}\cdots\cdots d\xi_1 d\xi \tag{2}$$

Now consider the infinite series

$$\phi(x)=F(x)+\lambda\int_a^x K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi+\cdots\ \cdots \tag{3}$$

Let $S_n(x)=\lambda^n\int_a^x K(x,\xi)\int_a^\xi K(\xi,\xi_1)\cdots\ \cdots\int_a^{\xi_{n-2}}K(\xi_{n-2},\xi_{n-1})F(\xi_{n-1})d\xi_{n-1}d\xi_{n-2}\cdots\cdots d\xi_1 d\xi$

Then $|S_n(x)|=|\lambda^n|\left|\int_a^x K(x,\xi)\right|\left|\int_a^\xi K(\xi,\xi_1)\right|\cdots\ \cdots\left|\int_a^{\xi_{n-2}}K(\xi_{n-2},\xi_{n-1})\right||F(\xi_{n-1})||d\xi_{n-1}|\cdots\cdots|d\xi_1||d\xi|$

Since $|K(x,\xi)|\le P$ and $|F(x)|\le Q$, then

$$|S_n(x)|\le|\lambda^n|QP^n\frac{(x-a)^n}{n!}$$

$$\le|\lambda^n|QP^n\frac{(b-a)^n}{n!}$$

It follows that the series is convergent for all values of $\lambda,P,Q,(b-a)$ and hence the series (3) is absolutely and uniformly convergent.

Again, $|S_{n+1}(x)|\le|\lambda^{n+1}|MP^{n+1}\dfrac{(b-a)^{n+1}}{(n+1)!}$

$$\therefore \lim_{n\to\infty}S_{n+1}(x)=0.$$

Therefore, we notice that the function $\phi(x)$, which satisfies the relation (2), is the continuous function given by the infinite series (3), the integral equation (1) has a unique continuous solution in the interval $a\le x\le b$.

Question-07: Solve the non-homogeneous Volterra integral equation of second kind by the method of successive approximations.

Answer: Consider the Volterra's integral equation of second kind as

$$\phi(x) = F(x) + \lambda\int_0^x K(x,\xi)\phi(\xi)\,d\xi \qquad (1)$$

where the kernel $K(x,\xi)$ is continuous function for $0 \le x \le a,\ 0 \le \xi \le a$ and $F(x)$ is continuous for $0 \le x \le a$.

Consider an infinite power series in λ as,

$$\phi(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots \cdots + \lambda^n\phi_n(x) + \cdots\cdots \qquad (2)$$

Let the series (2) is a solution of the integral equation (1), then

$$\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots + \lambda^n\phi_n(x) + \cdots = F(x) + \lambda\int_0^x K(x,\xi)\big[\phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(x) + \cdots$$
$$\cdots + \lambda^n\phi_n(x) + \cdots\cdots\big]d\xi$$

Equating the coefficients of like powers of λ, we get

$$\phi_0(x) = F(x) \qquad (3)$$
$$\phi_1(x) = \int_0^x K(x,\xi)\phi_0(\xi)\,d\xi \qquad (4)$$
$$\phi_2(x) = \int_0^x K(x,\xi)\phi_1(\xi)\,d\xi \qquad (5)$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\phi_n(x) = \int_0^x K(x,\xi)\phi_{n-1}(\xi)\,d\xi \qquad (6)$$

This yields a method for a successive approximation of the function $\phi_n(x)$, where the series (2) converges uniformly in x and λ for any $x \in [0,a]$.

Now from (3) and (4), we get

$$\phi_1(x) = \int_0^x K(x,\xi)F(\xi)\,d\xi$$

From (5), we get

$$\phi_2(x) = \int_0^x K(x,\xi)\int_0^\xi K(\xi,\xi_1)F(\xi_1)\,d\xi_1 d\xi$$

By interchanging the order of integration, we get

$$\phi_2(x) = \int_0^x F(\xi_1)\left\{\int_{\xi_1}^x K(x,\xi)K(\xi,\xi_1)d\xi\right\}d\xi_1; \qquad \because\ 0 \le \xi \le x \Rightarrow 0 \le \xi_1 \le \xi \le x$$
$$= \int_0^x K_2(x,\xi_1)F(\xi_1)d\xi_1$$

where $K_2(x,\xi_1) = \int_{\xi_1}^x K(x,\xi)K(\xi,\xi_1)d\xi$

In general, $\phi_n(x)=\int_0^x K_n(x,\xi)F(\xi)d\xi$; $\qquad n=1,2,\cdots\ \cdots \qquad$ (7)

The functions $K_n(x,\xi)$ are called iterated kernels.

So that $K_1(x,\xi)=K(x,\xi)$

Therefore, $K_2(x,\xi),\ K_3(x,\xi),\cdots\ \cdots$ are defined by

$$K_n(x,\xi)=\int_\xi^x K(x,z)K_{n-1}(z,\xi)dz\ ; \qquad n=2,3,\cdots\ \cdots \qquad (8)$$

Now from the relation (2), we get

$$\phi(x)=F(x)+\lambda\int_0^x K(x,\xi)F(\xi)d\xi+\lambda^2\int_0^x K_2(x,\xi)F(\xi)d\xi+\lambda^3\int_0^x K_3(x,\xi)F(\xi)d\xi+\cdots$$
$$+\lambda^n\int_0^x K_n(x,\xi)F(\xi)d\xi$$
$$=F(x)+\lambda\int_0^x\left[K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2K_3(x,\xi)+\cdots\cdots+\lambda^{n-1}K_n(x,\xi)\right]F(\xi)d\xi$$
$$=F(x)+\lambda\int_0^x R(x,\xi;\lambda)F(\xi)d\xi$$

where $R(x,\xi;\lambda)=K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2K_3(x,\xi)+\cdots\cdots+\lambda^{n-1}K_n(x,\xi)$

The function $R(x,\xi;\lambda)$ is called resolvent kernel or reciprocal lernel of the integral equation (1).

Thus, the solution of the integral equation (1) is given by

$$\phi(x)=F(x)+\lambda\int_0^x R(x,\xi;\lambda)F(\xi)d\xi\ .$$

Question-08: Solve Fredholm integral equation of second kind by the method of successive substitutions.

Answer: Consider the Fredholm integral equation of second kind as

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)d\xi \qquad (1)$$

where (i) The kernel $K(x,\xi)\neq 0$ is real and continuous in the rectangle $R:a\le x\le b,\ a\le\xi\le b$.

Consider $|K(x,\xi)|\le P$, where P is the maximum value in R.

(ii) The function $F(x)\neq 0$ is real and continuous in an interval $a\le x\le b$.

Consider $|F(x)|\le Q$, where Q is the maximum value in the interval $a\le x\le b$.

(iii) λ is non-zero numerical parameter.

Substituting the unknown function $\phi(\xi)$ under an integral sign from the relation (1), we get

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\left\{F(\xi)+\lambda\int_a^b K(\xi,\xi_1)\phi(\xi_1)d\xi_1\right\}d\xi$$
$$=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\phi(\xi_1)d\xi_1d\xi$$

Performing this operation successively for $\phi(\xi)$, we have

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\left\{F(\xi_1)+\lambda\int_a^b K(\xi_1,\xi_2)\phi(\xi_2)d\xi_2\right\}d\xi_1 d\xi$$

$$=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi$$

$$+\lambda^3\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\int_a^b K(\xi_1,\xi_2)\phi(\xi_2)d\xi_2 d\xi_1 d\xi$$

In general, we have

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi+\cdots\ \cdots+$$

$$+\lambda^n\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\cdots\ \cdots\int_a^b K(\xi_{n-2},\xi_{n-1})F(\xi_{n-1})d\xi_{n-1}d\xi_{n-2}\cdots\cdots d\xi_1 d\xi$$

$$+\lambda^{n+1}\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\int_a^b K(\xi_1,\xi_2)\cdots\ \cdots\int_a^b K(\xi_{n-1},\xi_n)\phi(\xi_n)d\xi_n d\xi_{n-1}\cdots\cdots d\xi_1 d\xi \qquad (2)$$

Now consider the infinite series

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)F(\xi_1)d\xi_1 d\xi+\cdots\ \cdots \qquad (3)$$

Let $S_n(x)=\lambda^n\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)\cdots\ \cdots\int_a^b K(\xi_{n-2},\xi_{n-1})F(\xi_{n-1})d\xi_{n-1}d\xi_{n-2}\cdots\cdots d\xi_1 d\xi$

Then $|S_n(x)|=|\lambda^n|\left|\int_a^b K(x,\xi)\right|\left|\int_a^b K(\xi,\xi_1)\right|\cdots\ \cdots\left|\int_a^b K(\xi_{n-2},\xi_{n-1})\right||F(\xi_{n-1})||d\xi_{n-1}|\cdots\cdots|d\xi_1||d\xi|$

Since $|K(x,\xi)|\le P$ and $|F(x)|\le Q$, then

$$|S_n(x)|\le|\lambda^n|QP^n(b-a)^n$$

It will converge only if

$$|\lambda|P(b-a)<1\Rightarrow|\lambda|<\frac{1}{P(b-a)} \qquad (4)$$

Thus, the series (2) converges absolutely and uniformly when the relation (3) holds.

Again, $|S_{n+1}(x)|\le|\lambda^{n+1}|MP^{n+1}(b-a)^{n+1}$

$$\therefore \lim_{n\to\infty}S_{n+1}(x)=0.$$

Therefore, the function $\phi(x)$, which satisfies the relation (2), is the continuous function given by the infinite series (3), the integral equation (1) has a unique continuous solution in the interval $a\le x\le b$.

Question-09: Solve the Fredholm integral equation of second kind by the method of successive approximations.

Answer: Consider the Fredholm integral equation of second kind as

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi \qquad (1)$$

where (i) The kernel $K(x,\xi)\neq 0$ is real and continuous in the rectangle $R: a\le x\le b,\ a\le \xi\le b$.

(ii) The function $F(x)\neq 0$ is real and continuous in an interval $a\le x\le b$.

(iii) λ is non-zero numerical parameter.

Consider an infinite power series in λ as,

$$\phi(x)=\phi_0(x)+\lambda\phi_1(x)+\lambda^2\phi_2(x)+\cdots\ \cdots+\lambda^n\phi_n(x)+\cdots\cdots \qquad (2)$$

Let the series (2) is a solution of the integral equation (1), then

$$\phi_0(x)+\lambda\phi_1(x)+\lambda^2\phi_2(x)+\cdots+\lambda^n\phi_n(x)+\cdots=F(x)+\lambda\int_a^b K(x,\xi)\left[\phi_0(x)+\lambda\phi_1(x)+\lambda^2\phi_2(x)+\cdots\right.$$
$$\left.\cdots+\lambda^n\phi_n(x)+\cdots\cdots\right]d\xi \qquad (3)$$

Equating the coefficients of like powers of λ, we get

$$\phi_0(x)=F(x) \qquad (4)$$

$$\phi_1(x)=\int_a^b K(x,\xi)\phi_0(\xi)\,d\xi \qquad (5)$$

$$\phi_2(x)=\int_a^b K(x,\xi)\phi_1(\xi)\,d\xi \qquad (6)$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$\phi_n(x)=\int_a^b K(x,\xi)\phi_{n-1}(\xi)\,d\xi \qquad (7)$$

This yields a method for a successive approximation of the function $\phi_n(x)$.

Now from (4) and (5), we get

$$\phi_1(x)=\int_a^b K(x,\xi)F(\xi)\,d\xi$$

From (6), we get

$$\phi_2(x)=\int_a^b K(x,\xi)\int_a^b K(\xi,\xi_1)F(\xi_1)\,d\xi_1 d\xi$$
$$=\int_a^b F(\xi_1)\left\{\int_a^b K(x,\xi)K(\xi,\xi_1)d\xi\right\}d\xi_1$$
$$=\int_a^b K_2(x,\xi_1)F(\xi_1)d\xi_1$$
$$=\int_a^b K_2(x,\xi)F(\xi)d\xi$$

where $K_2(x,\xi_1)=\int_a^b K(x,\xi)K(\xi,\xi_1)d\xi$

In general, $\phi_n(x)=\int_a^b K_n(x,\xi)F(\xi)d\xi$; $\qquad n=1,2,\cdots\ \cdots \qquad (7)$

The functions $K_n(x,\xi)$ are called iterated kernels.

So that $K_1(x,\xi)=K(x,\xi)$

Therefore, $K_2(x,\xi),\ K_3(x,\xi),\cdots\ \cdots$ are defined by

$$K_n(x,\xi)=\int_a^b K(x,z)K_{n-1}(z,\xi)dz\,; \qquad n=2,3,\cdots\ \cdots \qquad (8)$$

Now from the relation (2), we get

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)F(\xi)d\xi+\lambda^2\int_a^b K_2(x,\xi)F(\xi)d\xi+\lambda^3\int_a^b K_3(x,\xi)F(\xi)d\xi+\cdots$$
$$+\lambda^n\int_a^b K_n(x,\xi)F(\xi)d\xi$$
$$=F(x)+\lambda\int_a^b\left[K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2K_3(x,\xi)+\cdots\cdots+\lambda^{n-1}K_n(x,\xi)\right]F(\xi)d\xi$$
$$=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)d\xi$$

where $R(x,\xi;\lambda)=K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2K_3(x,\xi)+\cdots\cdots+\lambda^{n-1}K_n(x,\xi)$

The function $R(x,\xi;\lambda)$ is called resolvent kernel or reciprocal lernel of the integral equation (1).

Thus, the solution of the integral equation (1) is given by

$$\phi(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)d\xi\,.$$

Question-10: Show that all iterated kernels of symmetric kernels are also symmetric and the eigenvalues of a symmetric kernel are real.

Answer: The iterated kernel is defined as

$$\mathrm{K}_1(x,\xi)=\mathrm{K}(x,\xi)$$
$$\mathrm{K}_n(x,\xi)=\int_a^b\mathrm{K}(x,z)\,\mathrm{K}_{n-1}(z,\xi)\,dz$$
$$\therefore \mathrm{K}_n(\xi,x)=\int_a^b\cdots\int_a^b\mathrm{K}(\xi,z_1)\,\mathrm{K}(z_1,z_2)\cdots\mathrm{K}(z_{n-1},x)\,dz_1dz_2\cdots dz_{n-1}$$
$$=\int_a^b\cdots\int_a^b\mathrm{K}(\xi,z_1)\,\mathrm{K}(z_1,z_2)\cdots\mathrm{K}(z_{n-1},x)\,dz_1dz_2\cdots dz_{n-1}$$

Since the kernel is symmetric so $\mathrm{K}(\xi,z_1)=\mathrm{K}(z_1,\xi)$ and so on.

$$=\int_a^b\cdots\int_a^b\mathrm{K}(x,z_{n-1})\cdots\mathrm{K}(z_2,z_1)\,\mathrm{K}(z_1,\xi)\,dz_1dz_2\cdots dz_{n-1}$$

Now substituting $z_{n-1},\,z_{n-2},\,\cdots,\,z_2,\,z_1$ for $z_1,\,z_2,\,\cdots,\,z_{n-2},\,z_{n-1}$ we get

$$\mathrm{K}_n(\xi,x)=\int_a^b\cdots\int_a^b\mathrm{K}(x,z_1)\,\mathrm{K}(z_1,z_2)\cdots\mathrm{K}(z_{n-1},\xi)\,dz_1dz_2\cdots dz_{n-1}$$
$$\Rightarrow \mathrm{K}_n(\xi,x)=\mathrm{K}_n(x,\xi)$$

Thus, the kernel K_n is also symmetric.

Hence by induction, we conclude that every iterated kernel of a symmetric kernel is symmetric.
2nd part: We know that if λ_m and λ_n are distinct eigen values of distinct eigen functions $\phi_m(x)$ and $\phi_n(x)$ then

$$(\lambda_m - \lambda_n)\int_a^b \phi_m(x)\,\phi_n(x)dx = 0 \qquad \cdots(1)$$

Let $\lambda_m = \alpha + i\beta \quad (\beta \neq 0)$ and $\phi_m(x) = \psi_1 + i\psi_2$

Then $\overline{\lambda_m} = \alpha - i\beta \quad (\beta \neq 0)$ and $\overline{\phi_m(x)} = \psi_1 - i\psi_2$

Using these values on (1) we get

$$(\alpha + i\beta - \alpha + i\beta)\int_a^b (\psi_1 + i\psi_2)\,(\psi_1 - i\psi_2)dx = 0$$

$$\Rightarrow 2i\beta\int_a^b (\psi_1^2 + \psi_1^2)\,dx = 0$$

$$\Rightarrow \beta = 0$$

$$\Rightarrow \lambda_m = \alpha + i\beta = \alpha + i.0 = \alpha$$

Hence the eigen values of a symmetric kernel are real. (**Showed**).

Question-11: Prove that an initial value problem is equivalent to an integral equation.
Solution: Let us consider the following initial value problem

$$\frac{dy}{dx} = f(x,y) \qquad \cdots(1)$$

with $y(x_0) = y_0 \qquad \cdots(2)$

From (1) we get

$$dy = f(x,y)dx \qquad \cdots(3)$$

Integrating (3) within the limit x_0 to x we get

$$\int_{x_0}^x dy = \int_{x_0}^x f(t,y)dt$$

$$\Rightarrow [y]_{x_0}^x = \int_{x_0}^x f(t,y)dt$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^x f(t,y)dt$$

$$\Rightarrow y(x) = y(x_0) + \int_{x_0}^x f(t,y)dt$$

$$\Rightarrow y(x) = y_0 + \int_{x_0}^x f(t,y)dt$$

which is an integral equation.
Thus, the initial value problem is equivalent to an integral equation. (**Proved**)

Problem

<u>P-01:</u> Show that $\phi(x) = xe^x$, is a solution of the VIE $\phi(x) = sinx + 2\int_0^x \cos(x-\xi)\,\phi(\xi)\,d\xi$.
<u>Solution</u>: Given that, $\phi(x) = sinx + 2\int_0^x \cos(x-\xi)\,\phi(\xi)\,d\xi \qquad \cdots(1)$
Substituting the function $\phi(\xi) = \xi e^\xi$ in the right hand side of (1), we have

$$R.H.S = sinx + 2\int_0^x \cos(x-\xi)\,\xi e^{\xi}\,d\xi$$
$$= sinx + 2cosx\int_0^x \xi e^{\xi}\cos\xi\,d\xi + 2sinx\int_0^x \xi e^{\xi}\sin\xi\,d\xi$$
$$= sinx + 2cosx\left[\xi.\frac{e^{\xi}(\cos\xi+sin\xi)}{2} - \int\frac{e^{\xi}(\cos\xi+sin\xi)}{2}d\xi\right]_0^x$$
$$+2sinx\left[\xi.\frac{e^{\xi}(\sin\xi-cos\xi)}{2} - \int\frac{e^{\xi}(\sin\xi-cos\xi)}{2}d\xi\right]_0^x$$
$$= sinx + cosx\left[\xi\,e^{\xi}\cos\xi + \xi e^{\xi}sin\xi - \int e^{\xi}(\sin\xi + cos\xi)\,d\xi\right]_0^x$$
$$+sinx\left[\xi\,e^{\xi}\sin\xi - \xi e^{\xi}cos\xi + \int e^{\xi}(\cos\xi - sin\xi)\,d\xi\right]_0^x$$
$$= sinx + cosx\left[\xi\,e^{\xi}\cos\xi + \xi e^{\xi}sin\xi - e^{\xi}\sin\xi\right]_0^x$$
$$+sinx\left[\xi\,e^{\xi}\sin\xi - \xi e^{\xi}cos\xi + e^{\xi}\cos\xi\right]_0^x$$
$$= sinx + cosx(x\,e^{x}\cos x + xe^{x}sinx - e^{x}\sin x)$$
$$+sinx(x\,e^{x}\sin x - xe^{x}cosx + e^{x}\cos x - 1)$$
$$= sinx + x\,e^{x}\cos^2 x + xe^{x}sinxcosx - e^{x}\sin xcosx$$
$$+x\,e^{x}\sin^2 x - xe^{x}sinxcosx + e^{x}sinx\cos x - sinx$$
$$= x\,e^{x}(\cos^2 x + \sin^2 x)$$
$$= xe^{x}$$
$$= \phi(x)$$
$$= L.H.S$$

Hence $\phi(x) = xe^{x}$ is a solution of the given Volterra's integral equation. (**Showed**)

<u>P-02:</u> Show that $u(x) = (1+x^2)^{-3/2}$ is a solution of $u(x) = \frac{1}{1+x^2} - \int_0^x \frac{y}{1+x^2}u(y)\,dy$.

<u>Solution</u>: Given that, $u(x) = \frac{1}{1+x^2} - \int_0^x \frac{y}{1+x^2}u(y)\,dy$ $\cdots(1)$

Substituting the function $u(y) = (1+y^2)^{-3/2}$ in the right hand side of (1), we have

$$R.H.S = \frac{1}{1+x^2} - \int_0^x \frac{y}{1+x^2}u(y)\,dy$$
$$= \frac{1}{1+x^2} - \int_0^x \frac{y}{1+x^2}.\ (1+y^2)^{-3/2}\,dy$$
$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^x y\,(1+y^2)^{-3/2}\,dy \qquad \cdots(2)$$

Let $1+y^2 = t \Rightarrow ydy = \frac{1}{2}dt$

Now $\int y(1+y^2)^{-3/2}dy = \frac{1}{2}\int t^{-3/2}\,dt = \frac{1}{2}\frac{t^{-1/2}}{-1/2} = -\frac{1}{t^{1/2}}$

$$= -\frac{1}{(1+y^2)^{1/2}} \qquad \cdots(3)$$

Using (3) in (2) we get

$$R.H.S = \frac{1}{1+x^2} - \frac{1}{1+x^2}\left[-\frac{1}{(1+y^2)^{1/2}}\right]_0^x$$
$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\left[-\frac{1}{(1+x^2)^{1/2}} + 1\right]$$
$$= \frac{1}{1+x^2} + \frac{1}{(1+x^2)^{3/2}} - \frac{1}{1+x^2}$$
$$= \frac{1}{(1+x^2)^{3/2}}$$

$$= (1+x^2)^{-3/2}$$
$$= u(x)$$
$$= L.H.S$$

Hence $u(x) = (1+x^2)^{-3/2}$ is a solution of the given equation. (**Showed**)

P-03: Show that the function $\phi(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation $\int_0^x \frac{\phi(s)}{\sqrt{x-s}} ds = 1$.

Solution: Given that $\phi(x) = \frac{1}{\pi\sqrt{x}}$

$$\therefore \phi(s) = \frac{1}{\pi\sqrt{s}}$$

Now $\int_0^x \frac{\phi(s)}{\sqrt{x-s}} ds = \int_0^x \frac{1}{\sqrt{x-s}\,\pi\sqrt{s}} ds$

$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{sx-s^2}} ds$$
$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{\frac{x^2}{4} - \left(\frac{x^2}{4} - sx + s^2\right)}} ds$$
$$= \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - \left(s - \frac{x}{2}\right)^2}} ds$$
$$= \frac{1}{\pi}\left[sin^{-1}\frac{s-\frac{x}{2}}{\frac{x}{2}}\right]_0^x$$
$$= \frac{1}{\pi}\left[sin^{-1}\frac{x-\frac{x}{2}}{\frac{x}{2}} - sin^{-1}\frac{-\frac{x}{2}}{\frac{x}{2}}\right]$$
$$= \frac{1}{\pi}[sin^{-1}1 - sin^{-1}(-1)]$$
$$= \frac{1}{\pi}[sin^{-1}1 + sin^{-1}1]$$
$$= \frac{1}{\pi}.2\, sin^{-1}1$$
$$= \frac{1}{\pi}.2.\frac{\pi}{2}$$
$$= 1$$

Thus, the function $\phi(x) = \frac{1}{\pi\sqrt{x}}$ satisfies the given integral equation. Hence $\phi(x) = \frac{1}{\pi\sqrt{x}}$ is the solution of that equation. (**Showed**)

P-04: Show that $\phi(x) = 1 - x$ is a solution of the integral equation $\int_0^x e^{x-\xi}\phi(\xi)d\xi = x$.

Solution: Given that, $\int_0^x e^{x-\xi}\phi(\xi)d\xi = x$ (1)

Substituting the function $\phi(\xi) = 1 - \xi$ in the left hand side of (1), we have

$$L.H.S=\int_0^x e^{x-\xi}\phi(\xi)\,d\xi$$
$$=\int_0^x e^{x-\xi}(1-\xi)\,d\xi$$
$$=e^x\int_0^x e^{-\xi}d\xi-e^x\int_0^x e^{-\xi}\xi d\xi$$
$$=e^x\left[-e^{-\xi}\right]_0^x-e^x\left[-\xi e^{-\xi}-e^{-\xi}\right]_0^x$$
$$=e^x\left(-e^{-x}+1\right)-e^x\left(-xe^{-x}-e^{-x}+1\right)$$
$$=-1+e^x+x+1-e^x$$
$$=x$$
$$=R.H.S$$

Hence $\phi(x)=1-x$ is a solution of the given equation. (**Showed**).

<u>P-05:</u> Verify that whether $\phi(x)=\cos x$ is a solution of the equation $\phi(x)=\sin x+\int_0^\pi\left(x^2+\xi\right)\cos\xi\phi(\xi)\,d\xi$.

<u>Solution</u>: Given that, $\phi(x)=\sin x+\int_0^\pi\left(x^2+\xi\right)\cos\xi\phi(\xi)\,d\xi$ (1)

Substituting the function $\phi(\xi)=\cos\xi$ in equation (1), we have

$$\phi(x)=\sin x+\int_0^\pi\left(x^2+\xi\right)\cos\xi\phi(\xi)\,d\xi$$
$$=\sin x+\int_0^\pi\left(x^2+\xi\right)\cos^2\xi d\xi$$
$$=\sin x+x^2\int_0^\pi\cos^2\xi d\xi+\int_0^\pi\xi\cos^2\xi d\xi$$
$$=\sin x+\frac{x^2}{2}\int_0^\pi 2\cos^2\xi d\xi+\frac{1}{2}\int_0^\pi\xi\cdot 2\cos^2\xi d\xi$$
$$=\sin x+\frac{x^2}{2}\int_0^\pi(1+\cos 2\xi)\,d\xi+\frac{1}{2}\int_0^\pi\xi(1+\cos 2\xi)\,d\xi$$
$$=\sin x+\frac{x^2}{2}\int_0^\pi(1+\cos 2\xi)\,d\xi+\frac{1}{2}\int_0^\pi\xi d\xi+\frac{1}{2}\int_0^\pi\xi\cos 2\xi d\xi$$
$$=\sin x+\frac{x^2}{2}\left[\xi+\frac{\sin 2\xi}{2}\right]_0^\pi+\frac{1}{2}\left[\frac{\xi^2}{2}\right]_0^\pi+\frac{1}{2}\left[\frac{\xi\sin 2\xi}{2}+\frac{\cos 2\xi}{4}\right]_0^\pi$$
$$=\sin x+\frac{\pi x^2}{2}+\frac{\pi^2}{4}+\frac{1}{2}\cdot 0$$
$$=\sin x+\frac{\pi x^2}{2}+\frac{\pi^2}{4}$$

Hence $\phi(x)=\cos x$ is not a solution of the given integral equation. (**Verified**).

P-06: Verify that whether $\phi(x)=1-\dfrac{2\sin x}{1-\dfrac{\pi}{2}}$ is a solution of the integral equation $\phi(x)=1+\int_0^{\pi}\cos(x+\xi)\phi(\xi)\,d\xi$

Solution: Given that, $\phi(x)=1+\int_0^{\pi}\cos(x+\xi)\phi(\xi)\,d\xi$ (1)

Substituting the function $\phi(\xi)=1-\dfrac{2\sin\xi}{1-\dfrac{\pi}{2}}$ in equation (1), we have

$$\phi(x)=1+\int_0^{\pi}\cos(x+\xi)\phi(\xi)\,d\xi$$

$$=1+\int_0^{\pi}\cos(x+\xi)\left(1-\frac{2\sin\xi}{1-\frac{\pi}{2}}\right)d\xi$$

$$=1+\int_0^{\pi}\cos(x+\xi)\,d\xi-\frac{1}{1-\frac{\pi}{2}}\int_0^{\pi}2\sin\xi\cos(x+\xi)\,d\xi$$

$$=1+\left[\sin(x+\xi)\right]_0^{\pi}-\frac{1}{1-\frac{\pi}{2}}\int_0^{\pi}\left[\sin(x+2\xi)-\sin x\right]d\xi$$

$$=1-\sin x-\sin x-\frac{1}{1-\frac{\pi}{2}}\left[-\frac{\cos(x+2\xi)}{2}-\xi\sin x\right]_0^{\pi}$$

$$=1-2\sin x+\frac{\pi\sin x}{1-\frac{\pi}{2}}$$

$$=1-\frac{2\sin x}{1-\frac{\pi}{2}}$$

Hence $\phi(x)=1-\dfrac{2\sin x}{1-\dfrac{\pi}{2}}$ is a solution of the given integral equation. (**Verified**).

P-07: Show that the function $\phi(x)=e^{x}(2x-2/3)$ is a solution of the equation $\phi(x)+2\int_0^{1}e^{x-\xi}\phi(\xi)\,d\xi=2xe^{x}$.

Solution: Given that, $\phi(x)+2\int_0^{1}e^{x-\xi}\phi(\xi)\,d\xi=2xe^{x}$

$$or,\ \phi(x)=2xe^{x}-2\int_0^{1}e^{x-\xi}\phi(\xi)\,d\xi \qquad (1)$$

Substituting the function $\phi(\xi)=e^{\xi}(2\xi-2/3)$ in equation (1), we have

$$\phi(x) = 2xe^x - 2\int_0^1 e^{x-\xi}\phi(\xi)\,d\xi$$
$$= 2xe^x - 2\int_0^1 e^{x-\xi}\cdot e^{\xi}(2\xi - 2/3)\,d\xi$$
$$= 2xe^x - 2\int_0^1 e^{x}(2\xi - 2/3)\,d\xi$$
$$= 2xe^x - 2e^x\left[\xi^2 - 2\xi/3\right]_0^1$$
$$= 2xe^x - 2e^x(1 - 2/3)$$
$$= 2xe^x - 2e^x/3$$
$$= e^x(2x - 2/3)$$

Hence $\phi(x) = e^x(2x - 2/3)$ is a solution of the given integral equation. (**Showed**).

<u>P-08:</u> Show that the function $\phi(x) = \sin\dfrac{\pi x}{2}$ is a solution of the Fredholm integral equation

$$\phi(x) = \frac{x}{2} + \frac{\pi^2}{4}\int_0^1 K(x,\xi)\phi(\xi)\,d\xi$$

where $$K(x,\xi) = \begin{cases} \dfrac{x}{2}(2-\xi); & 0 \le x \le \xi \\ \dfrac{\xi}{2}(2-x); & \xi \le x \le 1 \end{cases}$$

<u>Solution</u>: Given that, $\phi(x) = \dfrac{x}{2} + \dfrac{\pi^2}{4}\displaystyle\int_0^1 K(x,\xi)\phi(\xi)\,d\xi$ (1)

where $$K(x,\xi) = \begin{cases} \dfrac{x}{2}(2-\xi); & 0 \le x \le \xi \\ \dfrac{\xi}{2}(2-x); & \xi \le x \le 1 \end{cases}$$

Replacing x by ξ, we get

$$K(x,\xi) = \begin{cases} \dfrac{\xi}{2}(2-x); & 0 \le \xi \le x \\ \dfrac{x}{2}(2-\xi); & x \le \xi \le 1 \end{cases}$$

Substituting the function $\phi(\xi) = \sin\dfrac{\pi\xi}{2}$ in equation (1), we have

$$\phi(x) = \frac{x}{2} + \frac{\pi^2}{4}\int_0^1 K(x,\xi)\phi(\xi)\,d\xi$$
$$= \frac{x}{2} + \frac{\pi^2}{4}\left[\int_0^x K(x,\xi)\phi(\xi)\,d\xi + \int_x^1 K(x,\xi)\phi(\xi)\,d\xi\right]$$

$$=\frac{x}{2}+\frac{\pi^2}{4}\left[\int_0^x \frac{\xi}{2}(2-x)\sin\frac{\pi\xi}{2}d\xi+\int_x^1 \frac{x}{2}(2-\xi)\sin\frac{\pi\xi}{2}d\xi\right]$$

$$=\frac{x}{2}+\frac{\pi^2}{4}\left[\frac{(2-x)}{2}\int_0^x \xi\sin\frac{\pi\xi}{2}d\xi+\frac{x}{2}\int_x^1 2\sin\frac{\pi\xi}{2}d\xi-\frac{x}{2}\int_x^1 \xi\sin\frac{\pi\xi}{2}d\xi\right]$$

$$=\frac{x}{2}+\frac{\pi^2(2-x)}{8}\left[-\frac{2\xi}{\pi}\cos\frac{\pi\xi}{2}+\frac{4}{\pi^2}\sin\frac{\pi\xi}{2}\right]_0^x+\frac{\pi^2 x}{4}\left[-\frac{2}{\pi}\cos\frac{\pi\xi}{2}\right]_x^1-\frac{\pi^2 x}{8}\left[-\frac{2\xi}{\pi}\cos\frac{\pi\xi}{2}+\frac{4}{\pi^2}\sin\frac{\pi\xi}{2}\right]_x^1$$

$$=\frac{x}{2}+\left(\frac{\pi^2}{4}-\frac{\pi^2 x}{8}\right)\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2}+\frac{4}{\pi^2}\sin\frac{\pi x}{2}-0\right]+\frac{\pi^2 x}{4}\left[0+\frac{2}{\pi}\cos\frac{\pi x}{2}\right]-\frac{\pi^2 x}{8}\left[\frac{4}{\pi^2}+\frac{2x}{\pi}\cos\frac{\pi x}{2}-\frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$=\frac{x}{2}-\frac{\pi x}{2}\cos\frac{\pi x}{2}+\sin\frac{\pi x}{2}+\frac{\pi x^2}{4}\cos\frac{\pi x}{2}-\frac{x}{2}\sin\frac{\pi x}{2}+\frac{\pi x}{2}\cos\frac{\pi x}{2}-\frac{x}{2}-\frac{\pi x^2}{4}\cos\frac{\pi x}{2}+\frac{x}{2}\sin\frac{\pi x}{2}$$

$$=\sin\frac{\pi x}{2}$$

Hence $\phi(x)=\sin\frac{\pi x}{2}$ is a solution of the given integral equation. (**Showed**).

P-09: Show that the function $\phi(x)=\cos 2x$ is a solution of the Fredholm integral equation

$$\phi(x)=\cos x+3\int_0^\pi K(x,\xi)\phi(\xi)d\xi$$

where $$K(x,\xi)=\begin{cases}\sin x\cos\xi; & 0\le x\le\xi\\ \cos x\sin\xi; & \xi\le x\le\pi\end{cases}$$

Solution: Given that, $\phi(x)=\cos x+3\int_0^\pi K(x,\xi)\phi(\xi)d\xi$ (1)

where $$K(x,\xi)=\begin{cases}\sin x\cos\xi; & 0\le x\le\xi\\ \cos x\sin\xi; & \xi\le x\le\pi\end{cases}$$

Replacing x by ξ, we get

$$K(x,\xi)=\begin{cases}\sin\xi\cos x; & 0\le\xi\le x\\ \cos\xi\sin x; & x\le\xi\le\pi\end{cases}$$

Substituting the function $\phi(\xi)=\cos 2\xi$ in equation (1), we have

$$\phi(x)=\cos x+3\int_0^\pi K(x,\xi)\phi(\xi)d\xi$$

$$=\cos x+3\left[\int_0^x K(x,\xi)\phi(\xi)d\xi+\int_x^\pi K(x,\xi)\phi(\xi)d\xi\right]$$

$$=\cos x+3\left[\int_0^x \sin\xi\cos x\cos 2\xi d\xi+\int_x^\pi \cos\xi\sin x\cos 2\xi d\xi\right]$$

$$=\cos x+3\left[\cos x\int_0^x \sin\xi\cos 2\xi d\xi+\sin x\int_x^\pi \cos\xi\cos 2\xi d\xi\right]$$

$$= \cos x + 3\left[\frac{\cos x}{2}\int_0^x (\sin 3\xi - \sin\xi)\,d\xi + \frac{\sin x}{2}\int_x^\pi (\cos 3\xi + \cos\xi)\,d\xi\right]$$

$$= \cos x + \frac{3\cos x}{2}\left[-\frac{\cos 3\xi}{3} + \cos\xi\right]_0^x + \frac{3\sin x}{2}\left[\frac{\sin 3\xi}{3} + \sin\xi\right]_x^\pi$$

$$= \cos x + \frac{3\cos x}{2}\left(-\frac{\cos 3x}{3} + \cos x + \frac{1}{3} - 1\right) + \frac{3\sin x}{2}\left(0 - \frac{\sin 3x}{3} - \sin x\right)$$

$$= \cos x - \frac{1}{2}\cos x\cos 3x + \frac{3}{2}\cos^2 x - \cos x - \frac{1}{2}\sin x\sin 3x - \frac{3}{2}\sin^2 x$$

$$= \frac{3}{2}\left(\cos^2 x - \sin^2 x\right) - \frac{1}{2}\left(\cos x\cos 3x + \sin x\sin 3x\right)$$

$$= \frac{3}{2}\cos 2x - \frac{1}{2}\cos 2x$$

$$= \cos 2x$$

Hence $\phi(x) = \cos 2x$ is a solution of the given integral equation. (**Showed**).

<u>P-10:</u> Transform the IVP $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$ into an integral equation.

<u>Solution:</u> Given initial value problem is

$$y'' + xy' + y = 0$$

$$or,\ \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad (1)$$

$$y(0) = 1,\ y'(0) = 0 \qquad (2)$$

Consider

$$\frac{d^2y}{dx^2} = \phi(x) \qquad (3)$$

Integrating (3) within the limit 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x \phi(u)\,du$$

$$or,\ \frac{dy}{dx} - y'(0) = \int_0^x \phi(u)\,du$$

$$or,\ \frac{dy}{dx} - 0 = \int_0^x \phi(u)\,du$$

$$or,\ \frac{dy}{dx} = \int_0^x \phi(u)\,du \qquad (4)$$

Again integrating (4) within the limit 0 to x, we get

$$\left[y(x)\right]_0^x = \int_0^x \phi(u)\,du^2$$

$$or,\ y(x)-y(0)=\int_0^x \frac{(x-u)^{2-1}}{(2-1)!}\phi(u)du$$

$$or,\ y(x)-1=\int_0^x (x-u)\phi(u)du$$

$$or,\ y(x)=1+\int_0^x (x-u)\phi(u)du \qquad (5)$$

Using the values of (3), (4) and (5) in (1) we get

$$\phi(x)+x\int_0^x \phi(u)du+1+\int_0^x (x-u)\phi(u)du=0$$

$$or,\ \phi(x)=-1-\int_0^x (2x-u)\phi(u)du$$

This is the required integral equation, which represents the Volterra's integral equation of second kind.

P-11: Transform the IVP $\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}+e^x y=x,\ \ y(0)=1,\ y'(0)=-1$ into an integral equation.

Solution: Given initial value problem is

$$\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}+e^x y=x \qquad (1)$$

$$y(0)=1,\ y'(0)=-1 \qquad (2)$$

Consider

$$\frac{d^2y}{dx^2}=\phi(x) \qquad (3)$$

Integrating (3) within the limit 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{dy}{dx}-y'(0)=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{dy}{dx}+1=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{dy}{dx}=-1+\int_0^x \phi(\xi)d\xi \qquad (4)$$

Again integrating (4) within the limit 0 to x, we get

$$[y(x)]_0^x=-\int_0^x d\xi+\int_0^x \phi(\xi)d\xi^2$$

$$or,\ y(x)-y(0)=-[\xi]_0^x+\int_0^x \frac{(x-\xi)^{2-1}}{(2-1)!}\phi(\xi)d\xi$$

$$or,\ y(x)-1=-x+\int_0^x (x-\xi)\phi(\xi)d\xi$$

$$or,\ y(x)=1-x+\int_0^x (x-\xi)\phi(\xi)d\xi \qquad (5)$$

Using the values of (3), (4) and (5) in (1) we get

$$\phi(x)-\sin x\int_0^x \phi(\xi)d\xi+1+e^x-xe^x+e^x\int_0^x (x-\xi)\phi(\xi)d\xi=x$$

$$or,\ \phi(x)=\left(x-\sin x+xe^x-e^x\right)+\int_0^x \left(\sin x-xe^x+\xi e^x\right)\phi(\xi)d\xi$$

This is the required integral equation, which represents the Volterra's integral equation of second kind.

<u>P-12:</u> Transform the IVP $\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+\left(x^2-x\right)y=xe^x+1,\quad y(0)=1=y'(0),\ y''(0)=0$ into an integral equation.

<u>Solution:</u> Given initial value problem is

$$\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+\left(x^2-x\right)y=xe^x+1 \qquad (1)$$

$$y(0)=1=y'(0),\ y''(0)=0 \qquad (2)$$

Consider

$$\frac{d^3y}{dx^3}=\phi(x) \qquad (3)$$

Integrating (3) within the limit 0 to x, we get

$$\left[\frac{d^2y}{dx^2}\right]_0^x=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{d^2y}{dx^2}-y''(0)=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{d^2y}{dx^2}-0=\int_0^x \phi(\xi)d\xi$$

$$or,\ \frac{d^2y}{dx^2}=\int_0^x \phi(\xi)d\xi \qquad (4)$$

Again integrating (4) within the limit 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x=\int_0^x \phi(\xi)d\xi^2$$

$$or,\ \frac{dy}{dx}-y'(0)=\int_0^x \phi(\xi)d\xi^2$$

$$or,\ \frac{dy}{dx}-1=\int_0^x \phi(\xi)d\xi^2$$

$$or,\ \frac{dy}{dx}=1+\int_0^x \phi(\xi)d\xi^2 \qquad (5)$$

$$\therefore\ \frac{dy}{dx}=1+\int_0^x (x-\xi)\phi(\xi)d\xi \qquad (6)$$

Again, differentiating (5) within the limit 0 to x, we get

$$\left[y(x)\right]_0^x=\int_0^x d\xi+\int_0^x \phi(\xi)d\xi^3$$

$$or,\ y(x)-y(0)=[\xi]_0^x+\int_0^x \frac{(x-\xi)^{3-1}}{(3-1)!}\phi(\xi)d\xi$$

$$or,\ y(x)-1=x+\frac{1}{2}\int_0^x (x-\xi)^2\phi(\xi)d\xi$$

$$or,\ y(x)=1+x+\frac{1}{2}\int_0^x (x-\xi)^2\phi(\xi)d\xi \qquad (7)$$

Using the values of (3), (4) and (7) in (1) we get

$$\phi(x)+x\int_0^x \phi(\xi)d\xi+\left(x^2-x\right)\left[1+x+\frac{1}{2}\int_0^x (x-\xi)^2\phi(\xi)d\xi\right]=xe^x+1$$

$$or,\ \phi(x)=\left(1+x-x^3+xe^x\right)-\int_0^x\left[x+\frac{1}{2}\left(x^2-x\right)(x-\xi)^2\right]\phi(\xi)d\xi$$

This is the required integral equation, which represents the Volterra's integral equation of second kind.

<u>P-13:</u> Transform the IVP $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0, y(0)=0, y'(0)=1$ into an integral equation and solved it.

<u>Solution:</u> Given initial value problem is

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0 \qquad \cdots(1)$$

$$y(0)=0, y'(0)=1 \qquad \cdots(2)$$

Consider

$$\frac{d^2y}{dx^2}=\varphi(x) \qquad \cdots(3)$$

Integrating (3) w.r.to 'x' within the limit 0 to , we get

$$\left[\frac{dy}{dx}\right]_0^x=\int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx}-\ y'(0)=\int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx}-\ 1=\int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx}=1+\int_0^x \varphi(u)du \qquad \cdots(4)$$

Again integrating (4) w.r.to 'x' within the limit 0 to , we get

$$[y(x)]_0^x=\int_0^x du+\int_0^x \varphi(u)du^2$$

$$or, y(x) - y(0) = [u]_0^x + \int_0^x \varphi(u)du^2$$

$$or, y(x) - 0 = x - 0 + \int_0^x (x-u)\varphi(u)du$$

$$or, y(x) = x + \int_0^x (x-u)\varphi(u)du \quad \cdots (5)$$

Using the values of (3), (4) and (5) in (1) we get

$$\varphi(x) - 5\left[1 + \int_0^x \varphi(u)du \right] + 6\left[x + \int_0^x (x-u)\varphi(u)du\right] = 0$$

$$or, \varphi(x) = 5\left[1 + \int_0^x \varphi(u)du \right] - 6\left[x + \int_0^x (x-u)\varphi(u)du\right]$$

$$or, \varphi(x) = 5 + 5\int_0^x \varphi(u)du - 6x - 6\int_0^x (x-u)\varphi(u)du$$

$$or, \varphi(x) = 5 - 6x + \int_0^x [5 - 6(x-u)]\varphi(u)du$$

$$\therefore \ \varphi(x) = f(x) + \lambda \int_0^x K(x,u)\varphi(u)du \quad \cdots (6)$$

where $(x) = 5 - 6x$, $K(x,u) = 5 - 6(x-u)$ and $\lambda = 1$.
This is the required transform integral equation, which represents the Volterra's integral equation of second kind.

<u>2nd part</u>: Now we have to find out the solution of (6) in the form of an infinite series of λ.
Let $\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x) + \cdots \quad \cdots (7)$
From (6) and (7) we have

$$\varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x) + \cdots = f(x) + \lambda\int_0^x K(x,u)[\varphi_0(u) + \lambda\varphi_1(u) + \lambda^2\varphi_2(u) + \cdots + \lambda^n\varphi_n(u) + \cdots]du \quad \cdots (8)$$

Equating the coefficients of same power of λ in both side, we get

$$\left.\begin{aligned} \varphi_0(x) &= f(x) \\ \varphi_1(x) &= \int_0^x K(x,u)\varphi_0(u)du \\ \varphi_2(x) &= \int_0^x K(x,u)\varphi_1(u)du \\ &\cdots \ \cdots \ \cdots \ \cdots \ \cdots\cdots \ \cdots \ \cdots \ \cdots \\ \varphi_n(x) &= \int_0^x K(x,u)\varphi_{n-1}(u)du \end{aligned}\right\} \quad \cdots (9)$$

Now $\varphi_0(x) = f(x) = 5 - 6x$

$$\begin{aligned} \varphi_1(x) &= \int_0^x K(x,u)\varphi_0(u)du \\ &= \int_0^x [5 - 6(x-u)](5 - 6u)du \\ &= \int_0^x (25 - 30x + 36xu - 36u^2)du \\ &= [25u - 30xu + 18xu^2 - 12u^3]_0^x \\ &= 25x - 30x^2 + 18x^3 - 12x^3 \end{aligned}$$

$$= 25x - 30x^2 + 6x^3$$

$$\varphi_2(x) = \int_0^x K(x,u)\varphi_1(u)du$$

$$= \int_0^x [5 - 6(x-u)](25u - 30u^2 + 6u^3)du$$

$$= \int_0^x (125u - 150xu + 180xu^2 - 36xu^3 - 150u^3 + 36u^4)du$$

$$= \left[\frac{125u^2}{2} - 75xu^2 + 60xu^3 - 9xu^4 - \frac{75u^4}{2} + \frac{36u^5}{5}\right]_0^x$$

$$= \frac{125x^2}{2} - 75x^3 + 60x^4 - 9x^5 - \frac{75x^4}{2} + \frac{36x^5}{5}$$

$$= \frac{125x^2}{2} - 75x^3 + \frac{45x^4}{2} - \frac{9x^5}{5}$$

Using these values in (7) we get

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots$$

$$= 5 - 6x + 1.(25x - 30x^2 + 6x^3) + 1^2.\left(\frac{125x^2}{2} - 75x^3 + \frac{45x^4}{2} - \frac{9x^5}{5}\right) + \cdots$$

$$= 5 + 19x + \frac{65}{2}x^2 + \cdots$$

This is the required solution of the integral equation obtained from the given ordinary differential equation.

P-14: Convert the IVP $y''(x) - 2y'(x) - 3y(x) = 0,\ y(0) = 1,\ y'(0) = 0$ into an integral equation and hence solve it.

Solution: Given initial value problem is

$$y''(x) - 2y'(x) - 3y(x) = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \qquad \cdots (1)$$

and $\quad y(0) = 1, y'(0) = 0 \qquad \cdots (2)$

Consider

$$\frac{d^2y}{dx^2} = \varphi(x) \qquad \cdots (3)$$

Integrating (3) w.r.to 'x' within the limit 0 to , we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} - y'(0) = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} - 0 = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} = \int_0^x \varphi(u)du \qquad \cdots (4)$$

Again integrating (4) w.r.to 'x' within the limit 0 to , we get

$$[y(x)]_0^x = \int_0^x \varphi(u)du^2$$

$$or, y(x) - y(0) = \int_0^x \varphi(u)du^2$$

$$or, y(x) - 1 = \int_0^x (x-u)\varphi(u)du \; ; \qquad \because \int_0^x y(u)du^2 = \int_0^x \frac{(x-u)^{n-1}}{(n-1)!} y(u)du$$

$$or, y(x) = 1 + \int_0^x (x-u)\varphi(u)du \qquad \cdots (5)$$

Using the values of (3), (4) and (5) in (1) we get

$$\varphi(x) - 2\int_0^x \varphi(u)du - 3\left[1 + \int_0^x (x-u)\varphi(u)du\right] = 0$$

$$or, \varphi(x) = 3 + \int_0^x [2 + 3(x-u)]\varphi(u)du$$

$$\therefore \; \varphi(x) = f(x) + \lambda \int_0^x K(x,u)\varphi(u)du \qquad \cdots (6)$$

where $(x) = 3$, $K(x,u) = 2 + 3(x-u)$ and $\lambda = 1$.
This is the required integral equation, which represents the Volterra's integral equation of second kind.

2nd part: Let $\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots\cdots$ $\cdots (7)$
From (6) and (7) we get

$$\varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots$$

$$= f(x) + \lambda \int_0^x K(x,u)[\varphi_0(u) + \lambda\varphi_1(u) + \lambda^2\varphi_2(u) + \cdots]\varphi(u)du \qquad \cdots (8)$$

Equating the coefficients of same power of λ in both side, we get

$$\varphi_0(x) = f(x)$$

$$\varphi_1(x) = \int_0^x K(x,u)\varphi_0(u)du$$

$$\varphi_2(x) = \int_0^x K(x,u)\varphi_1(u)du$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Now $\quad \varphi_0(x) = f(x) = 3$

$$\therefore \varphi_1(x) = \int_0^x K(x,u)\varphi_0(u)du$$

$$= \int_0^x [2 + 3(x-u)].3du$$

$$= 3\left[2u + 3xu - \frac{3}{2}u^2\right]_0^x$$

$$= 3\left(2x + 3x^2 - \frac{3}{2}x^2 - 0\right)$$

$$= 3\left(2x + \frac{3}{2}x^2\right)$$

$$= 6x + \frac{9}{2}x^2$$

$$\therefore \varphi_2(x) = \int_0^x K(x,u)\varphi_1(u)du$$

$$= \int_0^x [2+3(x-u)]\left(6u+\frac{9}{2}u^2\right)du$$
$$= \int_0^x \left(12u-9u^2+18ux+\frac{27}{2}xu^2-\frac{27}{2}u^3\right)du$$
$$= \left[6u^2-3u^3+9xu^2+\frac{9}{2}xu^3-\frac{27}{8}u^4\right]_0^x$$
$$= \left(6x^2-3x^3+9x^3+\frac{9}{2}x^4-\frac{27}{8}x^4-0\right)$$
$$= 6x^2+6x^3+9x^3+\frac{9}{8}x^4$$

Using these values in (7) we get

$$\varphi(x) = 3+\left(6x+\frac{9}{2}x^2\right)+\left(6x^2+6x^3+9x^3+\frac{9}{8}x^4\right)+\cdots\cdots$$
$$= 3+6x+\frac{21}{2}x^2+\cdots\cdots$$

This is the required solution.

P-15: Convert the initial value problem $x' = t^2 + x^4\ x(0) = 1$ to an integral equation.

Solution: Given initial value problem is

$$\frac{dx}{dt} = t^{\,2} + x^4 \qquad \cdots(1)$$

with $\quad x(0) = 1 \qquad \cdots(2)$

From (1), we get

$$dx = (t^{\,2} + x^4\ )dt \qquad \cdots(3)$$

Integrating (3) within the limit 0 to t we get

$$\int_0^t dx = \int_0^t (s^{\,2} + x^4\ )ds$$
$$\Rightarrow [x]_0^t = \int_0^t (s^{\,2} + x^4\ )ds$$
$$\Rightarrow x(t) - x(0) = \int_0^t (s^{\,2} + x^4\ )ds$$
$$\Rightarrow x(t) - 1 = \int_0^t (s^{\,2} + x^4\ )ds$$
$$\Rightarrow x(t) = 1 + \int_0^t (s^{\,2} + x^4\ )ds$$

This is the required integral equation.

P-16: Convert the initial value problem $y'' - 3y' + 2y = 4sinx,\ y(0) = 1,\ y'(0) = -2$ into a Volterra Integral Equation of second kind.

Solution: Given initial value problem is

$$y'' - 3y' + 2y = 4sinx$$
$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4sinx \qquad \cdots(1)$$

and $\quad y(0) = 1, y'(0) = -2 \qquad \cdots(2)$

Consider

$$\frac{d^2y}{dx^2} = \varphi(x) \qquad \cdots (3)$$

Integrating (3) w.r.to 'x' within the limit 0 to , we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} - y'(0) = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} + 2 = \int_0^x \varphi(u)du$$

$$or, \frac{dy}{dx} = -2 + \int_0^x \varphi(u)du \qquad \cdots (4)$$

Again integrating (4) w.r.to 'x' within the limit 0 to , we get

$$[y(x)]_0^x = -2\int_0^x du + \int_0^x \varphi(u)du^2$$

$$or, y(x) - y(0) = -2[u]_0^x + \int_0^x \varphi(u)du^2$$

$$or, y(x) - 1 = -2(x-0) + \int_0^x (x-u)\varphi(u)du$$

$$or, y(x) = 1 - 2x + \int_0^x (x-u)\varphi(u)du \qquad \cdots (5)$$

Using the values of (3), (4) and (5) in (1) we get

$$\varphi(x) - 3\left[-2 + \int_0^x \varphi(u)du\right] + 2\left[1 - 2x + \int_0^x (x-u)\varphi(u)du\right] = 4sinx$$

$$or, \varphi(x) = 3\left[-2 + \int_0^x \varphi(u)du\right] - 2\left[1 - 2x + \int_0^x (x-u)\varphi(u)du\right] + 4sinx$$

$$or, \varphi(x) = (-6 - 2 + 4x + 4sinx) + \int_0^x [3 - 2(x-u)]\varphi(u)du$$

$$or, \varphi(x) = -8 + 4x + 4sinx + \int_0^x [3 - 2(x-u)]\varphi(u)du$$

$$\therefore \ \varphi(x) = f(x) + \lambda\int_0^x K(x,u)\varphi(u)du \qquad \cdots (6)$$

where $(x) = -8 + 4x + 4sinx$, $K(x,u) = 3 - 2(x-u)$ and $\lambda = 1$.
This is the required transform integral equation, which represents the Volterra's integral equation of second kind.

P-17: Verify that $y(x) = \frac{1}{3}sinx - \frac{1}{6}sin2x$ is a solution of the IVP $y''(x) + 4y(x) = sinx, y(0) = y'(0) = 0$.

Solution: Given IVP is, $y''(x) + 4y(x) = sinx \qquad \cdots (1)$

and $\qquad y(0) = y'(0) = 0 \qquad \cdots (2)$

Also the given function is

$$y(x) = \frac{1}{3}sinx - \frac{1}{6}sin2x \qquad \cdots (3)$$

Differentiating (3) with respect to x we get

$$y' = \frac{1}{3}cosx - \frac{1}{3}cos2x$$

$$y'' = -\frac{1}{3}sinx + \frac{2}{3}sin2x$$

Putting these values in (1) we get

$$L.H.S = -\frac{1}{3}sinx + \frac{2}{3}sin2x + 4\left(\frac{1}{3}sinx - \frac{1}{6}sin2x\right)$$
$$= -\tfrac{1}{3}sinx + \tfrac{2}{3}sin2x + \tfrac{4}{3}sinx - \tfrac{2}{3}sin2x$$
$$= sinx$$
$$= R.H.S$$

Hence $y(x) = \frac{1}{3}sinx - \frac{1}{6}sin2x$ is a solution of the given initial value problem. (**Verified**)

P-18: Convert the IVP $y''(x) + y(x) = 0$, $y(0) = 1 = y'(0)$ into an integral equation.

Solution: Given IVP is

$$y'' + y = 0$$
$$\Rightarrow \frac{d^2y}{dx^2} + y = 0 \qquad \cdots (1)$$

and $\qquad y(0) = 1 = y'(0) \qquad \cdots (2)$

Integrating (1) w.r.to 'x' within the limit 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x + \int_0^x y(u)du = 0$$
$$or, \frac{dy}{dx} - y'(0) = -\int_0^x y(u)du$$
$$or, \frac{dy}{dx} - 1 = -\int_0^x y(u)du$$
$$or, \frac{dy}{dx} = 1 - \int_0^x \varphi(u)du \qquad \cdots (3)$$

Again integrating (3) w.r.to 'x' within the limit 0 to x, we get

$$[y(u)]_0^x = \int_0^x du - \int_0^x y(u)du^2$$
$$or, y(x) - y(0) = [u]_0^x - \int_0^x y(u)du^2$$
$$or, y(x) - 1 = (x-0) - \int_0^x (x-u)y(u)du \qquad \because \int_0^x y(u)du^2 = \int_0^x \frac{(x-u)^{n-1}}{(n-1)!}y(u)du$$
$$or, y(x) = 1 + x - \int_0^x (x-u)y(u)du \qquad \cdots (4)$$

This is the required integral equation, which represents the Volterra's integral equation of second kind.

P-19: Solve the VIE: $x(t) = e^{-t} + \int_0^t e^{-(t-s)}x(s)\,ds$.

Solution: Given that $x(t) = e^{-t} + \int_0^t e^{-(t-s)}\,x(s)\,ds \qquad \cdots (1)$

We know that the Volterra's integral equation of second kind is

$$x(t) = F(t) + \lambda\int_a^t K(t,s)x(s)\,ds \qquad \cdots (2)$$

and the solution of (2) is

$$x(t) = F(t) + \lambda \int_a^t R(t,s;\lambda)F(s)\,ds \qquad \cdots (3)$$

Comparing (2) with (1) we have

$F(t) = e^{-t},\ K(t,s) = \ e^{-(t-s)}, \lambda = 1$ and $a = 0$.

The Resolvent kernel is

$$R(t,s;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(t,s) \qquad \cdots (4)$$

and $K_n(t,s) = \int_s^t K(t,u)K_{n-1}(u,s)du\ ,\ n \geq 2$ $\cdots (5)$

$\mathrm{K}_1(t,s) = \mathrm{K}(t,s) = e^{-(t-s)}$ $\cdots (6)$

Putting $n = 2$ in (5) we get

$$\begin{aligned}
\mathrm{K}_2(t,s) &= \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_1(u,s)\,du \\
&= \int_s^t e^{-(t-u)}\, e^{-(u-s)}\,du \\
&= \int_s^t e^{-(t-s)}\ du \\
&= e^{-(t-s)} \int_s^t du \\
&= e^{-(t-s)} [u]_s^t \\
&= e^{-(t-s)}\,(t-s) \qquad \cdots (7)
\end{aligned}$$

Putting $n = 3$ in (5) we get

$$\begin{aligned}
\mathrm{K}_3(t,s) &= \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_2(u,s)\,du \\
&= \int_s^t e^{-(t-u)}\, e^{-(u-s)}\ (u-s)\,du \\
&= e^{-(t-s)} \int_s^t (u-s)du \\
&= e^{-(t-s)} \left[\frac{(u-s)^2}{2}\right]_s^t \\
&= e^{-(t-s)}\ \frac{(t-s)^2}{2} \qquad \cdots (8)
\end{aligned}$$

Putting $n = 4$ in (5) we get

$$\begin{aligned}
\mathrm{K}_4(t,s) &= \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_3(u,s)\,du \\
&= \int_s^t e^{-(t-u)}\, e^{-(u-s)}\ \frac{(u-s)^2}{2}du \\
&= e^{-(t-s)} \int_s^t \frac{(u-s)^2}{2}du \\
&= e^{-(t-s)} \left[\frac{(u-s)^3}{2.3}\right]_s^t
\end{aligned}$$

$$= e^{-(t-s)} \ \frac{(t-s)^3}{3!} \quad \cdots (9)$$

Now, we assume that

$$\mathrm{K}_n(t,s) \ = e^{-(t-s)} \ \frac{(t-s)^{n-1}}{(n-1)!} \quad \cdots (10)$$

Now from (4), we have

$$\mathrm{R}(t,s;\lambda) = \sum_{n=1}^{\infty} \mathrm{K}_n(t,s) \ ; \ \ since \ \lambda = 1$$

$$= \mathrm{K}_1(t,s) + \mathrm{K}_2(t,s) + \mathrm{K}_3(t,s) + \mathrm{K}_4(t,s) + \cdots + \ \mathrm{K}_n(t,s) + \ \cdots \ to \ \infty$$

$$= \ e^{-(t-s)} + \ e^{-(t-s)} \ (t-s) + e^{-(t-s)} \ \frac{(t-s)^2}{2} + e^{-(t-s)} \ \frac{(t-s)^3}{3!} + \cdots + e^{-(t-s)} \ \frac{(t-s)^{n-1}}{(n-1)!}$$

$$+ \ \cdots to \ \ \infty$$

$$= \ e^{-(t-s)} \left[1 + \ \ (t-s) + \ \frac{(t-s)^2}{2} + \ \frac{(t-s)^3}{3!} + \cdots + \ \frac{(t-s)^{n-1}}{(n-1)!} \ + \ \cdots \ \ to \ \infty \ \right]$$

$$= \ e^{-(t-s)} \ e^{(t-s)}$$

$$= \ e^{0}$$

$$= \ 1$$

Now from (3), the solution is

$$x(t) = e^{-t} + \int_0^t 1.e^{-s} \ ds$$

$$= e^{-t} + \int_0^t e^{-s} \, ds$$

$$= e^{-t} + [-e^{-s}]_0^t$$

$$= e^{-t} + (-e^{-t} + 1)$$

$$= e^{-t} + 1 - e^{-t}$$

$$= 1$$

This is the required solution. (**Solved**)

<u>P-20</u>: Solve: $(t) = 1 + t^2 + \int_0^t \frac{1+\mathrm{t}^2}{1+s^2} x(s)ds$.

<u>Solution</u>: Given that $x(t) = 1 + t^2 + \int_0^t \frac{1+\mathrm{t}^2}{1+s^2} x(s)ds \quad \cdots (1)$

We know that the Volterra's integral equation of second kind is

$$x(t) = F(t) + \lambda \int_a^t \mathrm{K}(t,s) \, x(s) \, ds \quad \cdots (2)$$

The solution of (2) is

$$x(t) = F(t) + \lambda \int_a^t \mathrm{R}(t,s;\lambda) \, F(s) \, ds \quad \cdots (3)$$

Comparing (2) with (1), we have

$\lambda = 1$, $F(t) = 1 + t^2$, $\mathrm{K}(t,s) = \frac{1+\mathrm{t}^2}{1+s^2}$ and a $= 0$

Also we have the Resolvent kernel

$$\mathrm{R}(t,s;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} \, \mathrm{K}_n(t,s) \quad \cdots (4)$$

and $\quad \mathrm{K}_n(t,s) \ = \int_s^t \mathrm{K}(t,u) \, \mathrm{K}_{n-1}(u,s) \, du, \quad n \geq 2 \quad \cdots (5)$

$$\mathrm{K}_1(t,s) = \mathrm{K}(t,s) = \frac{1+\mathrm{t}^2}{1+s^2} \qquad \cdots (6)$$

Putting $n = 2$ in (5) we get

$$\mathrm{K}_2(t,s) = \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_1(u,s)\,du$$
$$= \int_s^t \frac{1+\mathrm{t}^2}{1+u^2}\,\frac{1+\mathrm{u}^2}{1+s^2}\,du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\int_s^t du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}[u]_s^t$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\,(t-s) \qquad \cdots (7)$$

Putting $n = 3$ in (5) we get

$$\mathrm{K}_3(t,s) = \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_2(u,s)\,du$$
$$= \int_s^t \frac{1+\mathrm{t}^2}{1+u^2}\,\frac{1+\mathrm{u}^2}{1+s^2}\,(u-s)\,du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\int_s^t (u-s)du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\left[\frac{(u-s)^2}{2}\right]_s^t$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\,\frac{(t-s)^2}{2} \qquad \cdots (8)$$

Putting $n = 4$ in (5) we get

$$\mathrm{K}_4(t,s) = \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_3(u,s)\,du$$
$$= \int_s^t \frac{1+\mathrm{t}^2}{1+u^2}\,\frac{1+\mathrm{u}^2}{1+s^2}\,\frac{(u-s)^2}{2}\,du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\int_s^t \frac{(u-s)^2}{2}\,du$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\left[\frac{(u-s)^3}{2.3}\right]_s^t$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\,\frac{(t-s)^3}{3!} \qquad \cdots (9)$$

Now, we assume that

$$\mathrm{K}_n(t,s) = \frac{1+\mathrm{t}^2}{1+s^2}\,\frac{(t-s)^{n-1}}{(n-1)!} \qquad \cdots (10)$$

Now from (4) we have

$$\mathrm{R}(t,s;\lambda) = \sum_{n=1}^{\infty} \mathrm{K}_n(t,s)\ ;\ \ since\ \lambda = 1$$

$$= \mathrm{K}_1(t,s) + \mathrm{K}_2(t,s) + \mathrm{K}_3(t,s) + \mathrm{K}_4(t,s) + \cdots + \mathrm{K}_{n+1}(t,s) + \cdots \ to\ \infty$$
$$= \frac{1+\mathrm{t}^2}{1+s^2} + \frac{1+\mathrm{t}^2}{1+s^2}(t-s) + \frac{1+\mathrm{t}^2}{1+s^2}\frac{(t-s)^2}{2} + \frac{1+\mathrm{t}^2}{1+s^2}\frac{(t-s)^3}{3!} + \cdots + \frac{1+\mathrm{t}^2}{1+s^2}\frac{(t-s)^{n-1}}{(n-1)!} + \cdots to\ \infty$$

$$= \frac{1+\mathrm{t}^2}{1+s^2}\left[1 + (t-s) + \frac{(t-s)^2}{2} + \frac{(t-s)^3}{3!} + \cdots + \frac{(t-s)^{n-1}}{(n-1)!} + \cdots\ to\ \infty\right]$$
$$= \frac{1+\mathrm{t}^2}{1+s^2}\, e^{(t-s)}$$

Now from (3), the solution is

$$x(t) = 1 + t^2 + \int_0^t \frac{1+\mathrm{t}^2}{1+s^2}\, e^{(t-s)}(1+s^2)\, ds$$
$$= 1 + t^2 + (1+\mathrm{t}^2)\int_0^t e^{(t-s)}\, ds$$
$$= 1 + t^2 + (1+\mathrm{t}^2)\left[-e^{(t-s)}\right]_0^t$$
$$= 1 + t^2 - (1+\mathrm{t}^2)(1-e^t)$$
$$= (1+\mathrm{t}^2)e^t$$

This is the required solution. **(Solved)**

<u>P-21</u>: Solve: $x(t) = e^{-t} + 1 + \int_0^t e^{-(t-s)}\, x(s)\, ds$ and verify your result.

<u>Solution</u>: Given that $x(t) = e^{-t} + 1 + \int_0^t e^{-(t-s)}\, x(s)\, ds \qquad \cdots (1)$

We know that the Volterra's integral equation of second kind is

$$x(t) = F(t) + \lambda \int_a^t K(t,s)x(s)\, ds \qquad \cdots (2)$$

and the solution of (2) is

$$x(t) = F(t) + \lambda \int_a^t R(t,s;\lambda)F(s)\, ds \qquad \cdots (3)$$

Comparing (2) with (1) we have

$F(t) = e^{-t} + 1,\ K(t,s) = e^{-(t-s)}, \lambda = 1$ and $a = 0$.

The Resolvent kernel is

$$R(t,s;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(t,s) \qquad \cdots (4)$$

and $K_n(t,s) = \int_s^t K(t,u)K_{n-1}(u,s)du\ ,\ n \geq 2 \qquad \cdots (5)$

$\mathrm{K}_1(t,s) = \mathrm{K}(t,s) = e^{-(t-s)} \qquad \cdots (6)$

Putting $n = 2$ in (5) we get

$$\mathrm{K}_2(t,s) = \int_s^t \mathrm{K}(t,u)\,\mathrm{K}_1(u,s)\, du$$
$$= \int_s^t e^{-(t-u)}\, e^{-(u-s)}\, du$$
$$= \int_s^t e^{-(t-s)}\, du$$
$$= e^{-(t-s)} \int_s^t du$$
$$= e^{-(t-s)}[u]_s^t$$

$$= e^{-(t-s)}\,(t-s) \qquad \cdots (7)$$

Putting $n = 3$ in (5) we get

$$K_3(t,s) = \int_s^t K(t,u)\,K_2(u,s)\,du$$
$$= \int_s^t e^{-(t-u)}\,e^{-(u-s)}\,(u-s)\,du$$
$$= e^{-(t-s)} \int_s^t (u-s)du$$
$$= e^{-(t-s)} \left[\frac{(u-s)^2}{2}\right]_s^t$$
$$= e^{-(t-s)}\,\frac{(t-s)^2}{2} \qquad \cdots (8)$$

Putting $n = 4$ in (5) we get

$$K_4(t,s) = \int_s^t K(t,u)\,K_3(u,s)\,du$$
$$= \int_s^t e^{-(t-u)}\,e^{-(u-s)}\,\frac{(u-s)^2}{2}du$$
$$= e^{-(t-s)} \int_s^t \frac{(u-s)^2}{2}du$$
$$= e^{-(t-s)} \left[\frac{(u-s)^3}{2.3}\right]_s^t$$
$$= e^{-(t-s)}\,\frac{(t-s)^3}{3!} \qquad \cdots (9)$$

Now, we assume that

$$K_n(t,s) = e^{-(t-s)}\,\frac{(t-s)^{n-1}}{(n-1)!} \qquad \cdots (10)$$

Now from (4) we have

$$R(t,s;\lambda) = \sum_{n=1}^{\infty} K_n(t,s)\ ;\ \ since\ \lambda = 1$$
$$= K_1(t,s) + K_2(t,s) + K_3(t,s) + K_4(t,s) + \cdots + K_{n+1}(t,s) + \cdots\ to\ \infty$$
$$= e^{-(t-s)} + e^{-(t-s)}\,(t-s) + e^{-(t-s)}\,\frac{(t-s)^2}{2} + e^{-(t-s)}\,\frac{(t-s)^3}{3!} + \cdots + e^{-(t-s)}\,\frac{(t-s)^{n-1}}{(n-1)!}$$
$$+ \cdots to\ \infty$$
$$= e^{-(t-s)}\left[1 + (t-s) + \frac{(t-s)^2}{2} + \frac{(t-s)^3}{3!} + \cdots + \frac{(t-s)^{n-1}}{(n-1)!} + \cdots\ to\ \infty\right]$$
$$= e^{-(t-s)}\,e^{(t-s)}$$
$$= e^0$$
$$= 1$$

Now from (3), the solution is

$$x(t) = e^{-t} + 1 + \int_0^t 1.(e^{-s} + 1)\,ds$$

$$= e^{-t} + 1 + \int_0^t (e^{-s} + 1)\, ds$$
$$= e^{-t} + 1 + [-e^{-s} + s]_0^t$$
$$= e^{-t} + 1 + (-e^{-t} + t + 1 - 0)$$
$$= e^{-t} + 1 - e^{-t} + t + 1$$
$$= t + 2$$

This is the required solution. (**Solved**)

Verification: We have $x(t) = t + 2$

$\therefore x(s) = s + 2$

Putting $x(s) = s + 2$ in the right side of (1) we get

$$R.H.S = e^{-t} + 1 + \int_0^t e^{-(t-s)}\, (s + 2)\, ds$$
$$= e^{-t} + 1 + \left[(s + 2)e^{-(t-s)} - e^{-(t-s)}\right]_0^t$$
$$= e^{-t} + 1 + (t + 2)e^0 - e^0 - (0 + 2)e^{-t} + e^{-t}$$
$$= 2e^{-t} + 1 + t + 2 - 1 - 2e^{-t}$$
$$= 2 + t$$
$$= x(t)$$
$$= L.H.S$$ (**Verified**)

<u>P-22</u>: Find the resolvent kernel of the VIE $Q(x) = sinx + 2\int_0^x e^{(x-s)}\, Q(s)\, ds$.

<u>Solution</u>: Given that $Q(x) = sinx + 2\int_0^x e^{(x-s)}\, Q(s)\, ds$ $\cdots(1)$

We know that the Volterra's integral equation of second kind is

$$Q(x) = F(x) + \lambda \int_a^x K(x,s)x(s)\, ds \qquad \cdots(2)$$

and the solution of (2) is

$$Q(x) = F(x) + \lambda \int_a^x R(x,s;\lambda)F(s)\, ds \qquad \cdots(3)$$

Comparing (2) with (1) we have

$F(x) = sinx,\ K(x,s) = e^{(x-s)}, \lambda = 2$ and $a = 0$.

The resolvent kernel is

$$R(x,s;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,s) \qquad \cdots(4)$$

and $K_n(x,s) = \int_s^x K(x,u)K_{n-1}(u,s)du\ ,\ n \geq 2$ $\cdots(5)$

$\mathrm{K}_1(x,s) = \mathrm{K}(x,s) = e^{(x-s)}$ $\cdots(6)$

Putting $n = 2$ in (5) we get

$$\mathrm{K}_2(x,s) = \int_s^x \mathrm{K}(x,u)\,\mathrm{K}_1(u,s)\, du$$
$$= \int_s^x e^{(x-u)}\, e^{(u-s)}\, du$$
$$= \int_s^x e^{(x-s)}\, du$$
$$= e^{(x-s)} \int_s^x du$$
$$= e^{(x-s)}[u]_s^x$$

$$= e^{(x-s)}\,(x-s) \qquad \cdots (7)$$

Putting $n = 3$ in (5) we get

$$\begin{aligned} K_3(x,s) &= \int_s^x K(x,u)\,K_2(u,s)\,du \\ &= \int_s^x e^{(x-u)}\,e^{(u-s)}\,(u-s)\,du \\ &= e^{(x-s)} \int_s^x (u-s)du \\ &= e^{(x-s)} \left[\frac{(u-s)^2}{2}\right]_s^x \\ &= e^{(x-s)}\,\frac{(x-s)^2}{2} \end{aligned} \qquad \cdots (8)$$

Putting $n = 4$ in (5) we get

$$\begin{aligned} K_4(x,s) &= \int_s^x K(x,u)\,K_3(u,s)\,du \\ &= \int_s^x e^{(x-u)}\,e^{(u-s)}\,\frac{(u-s)^2}{2}du \\ &= e^{(x-s)} \int_s^x \frac{(u-s)^2}{2}du \\ &= e^{(x-s)} \left[\frac{(u-s)^3}{2.3}\right]_s^x \\ &= e^{(x-s)}\,\frac{(x-s)^3}{3!} \end{aligned} \qquad \cdots (9)$$

Now, we assume that

$$K_n(x,s) = e^{(x-s)}\,\frac{(x-s)^{n-1}}{(n-1)!} \qquad \cdots (10)$$

Now from (4) we have

$$\begin{aligned} R(x,s;\lambda) &= \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,s) \\ &= K_1(x,s) + \lambda K_2(x,s) + \lambda^2 K_3(x,s) + \lambda^3 K_4(x,s) + \cdots + \lambda^n K_{n+1}(x,s) + \cdots \ to\ \infty \\ &= e^{(x-s)} + 2e^{(x-s)}(x-s) + 2^2 e^{(x-s)}\frac{(x-s)^2}{2} + 2^3 e^{(x-s)}\frac{(x-s)^3}{3!} + \cdots + 2^{n-1}e^{(x-s)}\frac{(x-s)^{n-1}}{(n-1)!} \\ &\quad + \cdots to\ \infty \\ &= e^{(x-s)}\left[1 + 2(x-s) + \frac{2^2(x-s)^2}{2} + \frac{2^3(x-s)^3}{3!} + \cdots + \frac{2^{n-1}(x-s)^{n-1}}{(n-1)!} + \cdots \ to\ \infty\right] \\ &= e^{(x-s)}\,e^{2(x-s)} \\ &= e^{3(x-s)} \end{aligned}$$

This is the required resolvent kernel. **(Ans)**

<u>P-23</u>: Find the resolvent kernel and hence solve the integral equation $\phi(x) = e^{x^2} + \int_0^x e^{x^2-\xi^2}\,\phi(\xi)\,d\xi$.

<u>Solution</u>: Given that $\phi(x) = e^{x^2} + \int_0^x e^{x^2-\xi^2}\,\phi(\xi)\,d\xi \qquad \cdots (1)$

We know that the Volterra's integral equation of second kind is

$$\phi(x) = F(x) + \lambda \int_a^x K(x,\xi)\phi(\xi)\, d\xi \qquad \cdots (2)$$

and the solution of (2) is

$$\phi(x) = F(x) + \lambda \int_a^x R(x,\xi;\lambda)F(\xi)\, d\xi \qquad \cdots (3)$$

Comparing (2) with (1) we have

$F(x) = e^{x^2}$, $K(x,\xi) = e^{x^2-\xi^2}$, $\lambda = 1$ and $a = 0$.

The resolvent kernel is

$$R(x,\xi;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,\xi) \qquad \cdots (4)$$

and $K_n(x,\xi) = \int_\xi^x K(x,u)K_{n-1}(u,\xi)du$, $n \geq 2$ $\qquad \cdots (5)$

$$\mathrm{K}_1(x,\xi) = \mathrm{K}(x,\xi) = e^{x^2-\xi^2} \qquad \cdots (6)$$

Putting $n = 2$ in (5) we get

$$\begin{aligned}
\mathrm{K}_2(x,s) &= \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_1(u,\xi)\, du \\
&= \int_\xi^x e^{x^2-u^2}.e^{u^2-\xi^2}\, du \\
&= \int_\xi^x e^{x^2-\xi^2}\, du \\
&= e^{x^2-\xi^2}\int_\xi^x du \\
&= e^{x^2-\xi^2}[u]_\xi^x \\
&= e^{x^2-\xi^2}(x-\xi) \qquad \cdots (7)
\end{aligned}$$

Putting $n = 3$ in (5) we get

$$\begin{aligned}
\mathrm{K}_3(x,s) &= \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_2(u,\xi)\, du \\
&= \int_\xi^x e^{x^2-u^2}.e^{u^2-\xi^2}\,(u-\xi)\, du \\
&= e^{x^2-\xi^2}\int_\xi^x (u-\xi)du \\
&= e^{x^2-\xi^2}\left[\frac{(u-\xi)^2}{2}\right]_s^x \\
&= e^{x^2-\xi^2}\,\frac{(x-\xi)^2}{2} \qquad \cdots (8)
\end{aligned}$$

Putting $n = 4$ in (5) we get

$$\begin{aligned}
\mathrm{K}_4(x,s) &= \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_3(u,\xi)\, du \\
&= \int_\xi^x e^{x^2-u^2}.e^{u^2-\xi^2}\,\frac{(u-\xi)^2}{2} du
\end{aligned}$$

$$= e^{x^2-\xi^2}\int_{\xi}^{x}\frac{(u-\xi)^2}{2}du$$

$$= e^{x^2-\xi^2}\left[\frac{(u-\xi)^3}{2.3}\right]_{\xi}^{x}$$

$$= e^{x^2-\xi^2}\ \frac{(x-\xi)^3}{3!} \qquad \cdots (9)$$

Now, we assume that

$$\mathrm{K}_n(x,\xi) = e^{x^2-\xi^2}\ \frac{(x-\xi)^{n-1}}{(n-1)!} \qquad \cdots (10)$$

Now from (4) we have

$$\mathrm{R}(x,\xi;\lambda) = \sum_{n=1}^{\infty}\lambda^{n-1}\mathrm{K}_n(x,\xi)$$

$$= \mathrm{K}_1(x,\xi)+\lambda\mathrm{K}_2(x,\xi)+\lambda^2\mathrm{K}_3(x,\xi)+\lambda^3\mathrm{K}_4(x,\xi)+\cdots+\lambda^n\mathrm{K}_{n+1}(x,\xi)+\cdots\ to\ \infty$$

$$= e^{x^2-\xi^2}+e^{x^2-\xi^2}(x-\xi)+e^{x^2-\xi^2}\frac{(x-\xi)^2}{2}+e^{x^2-\xi^2}\frac{(x-\xi)^3}{3!}+\cdots+e^{x^2-\xi^2}\frac{(x-\xi)^{n-1}}{(n-1)!}$$
$$+\cdots to\ \infty$$

$$= e^{x^2-\xi^2}\left[1+(x-\xi)+\frac{(x-\xi)^2}{2}+\frac{(x-\xi)^3}{3!}+\cdots+\frac{(x-\xi)^{n-1}}{(n-1)!}+\cdots\ to\ \infty\right]$$

$$= e^{x^2-\xi^2}e^{(x-\xi)}$$

$$= e^{x^2-\xi^2+x-\xi}$$

This is the required resolvent kernel.

From (3) we get

$$\phi(x) = e^{x^2}+\int_0^x e^{x^2-\xi^2+x-\xi}.e^{\xi^2}\,d\xi$$

$$= e^{x^2}+\int_0^x e^{x^2+x-\xi}\,d\xi$$

$$= e^{x^2}+\left[\frac{e^{x^2+x-\xi}}{-1}\right]_0^x$$

$$= e^{x^2}-e^{x^2}+e^{x^2+x}$$

$$= e^{x^2+x}$$

This is the required solution.

P-24: Solve the VIE: $\phi(x) = x+\int_0^x (\xi-x)\phi(\xi)\,d\xi$.

Solution: Given that $\phi(x) = x+\int_0^x (\xi-x)\phi(\xi)\,d\xi \qquad \cdots (1)$

We know that the Volterra's integral equation of second kind is

$$\phi(x) = F(x)+\lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi \qquad \cdots (2)$$

and the solution of (2) is

$$\phi(x) = F(x)+\lambda\int_a^x R(x,\xi;\lambda)F(\xi)\,d\xi \qquad \cdots (3)$$

Comparing (2) with (1) we have

$F(x)=x,\ K(x,\xi)=(\xi-x),\ \lambda=1$ and $a=0$.

The Resolvent kernel is

$$R(x,\xi;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,\xi) \qquad \cdots (4)$$

and $K_n(x,\xi) = \int_\xi^x K(x,u)K_{n-1}(u,\xi)du$, $n \ge 2$ (5)

$$\mathrm{K}_1(x,\xi) = \mathrm{K}(x,\xi) = (\xi - x)$$

Putting $n = 2$ in (5) we get

$$\begin{aligned}
\mathrm{K}_2(x,\xi) &= \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_1(u,\xi)\,du \\
&= \int_\xi^x (u-x)\,(\xi-u)\,du \\
&= \int_\xi^x \left(u\xi - u^2 - x\xi + xu\right)du \\
&= \left[\frac{u^2\xi}{2} - \frac{u^3}{3} - ux\xi + \frac{xu^2}{2}\right]_\xi^x \\
&= \frac{x^2\xi}{2} - \frac{x^3}{3} - x^2\xi + \frac{x^3}{2} - \frac{\xi^3}{2} + \frac{\xi^3}{3} + x\xi^2 - \frac{x\xi^2}{2} \\
&= \frac{1}{6}\left(3x^2\xi - 2x^3 - 6x^2\xi + 3x^3 - 3\xi^3 + 2\xi^3 + 6x\xi^2 - x\xi^2\right) \\
&= \frac{1}{6}\left(-\xi^3 + 3\xi^2 x - 3x^2\xi + x^3\right) \\
&= -\frac{1}{3!}(\xi - x)^3
\end{aligned}$$

Again,

$$\begin{aligned}
\mathrm{K}_3(x,\xi) &= \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_2(u,\xi)\,du \\
&= \int_\xi^x (u-x)\left\{-\frac{1}{3!}(\xi-u)^3\right\}du \\
&= \frac{1}{5!}(\xi-x)^5
\end{aligned}$$

Similarly,

$$K_4(x,\xi) = -\frac{1}{7!}(\xi-x)^7$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The resolvent kernel is

$$\begin{aligned}
R(x,\xi;\lambda) &= K(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots\cdots \\
&= (\xi-x) - \frac{1}{3!}(\xi-x)^3 + \frac{1}{5!}(\xi-x)^5 - \frac{1}{7!}(\xi-x)^7 + \cdots\cdots \\
&= \sin(\xi - x)
\end{aligned}$$

The solution of the given integral equation is

$$\phi(x) = F(x) + \lambda\int_0^x R(x,\xi;\lambda)F(\xi)\,d\xi$$

$$= x + \int_0^x \sin(\xi - x)\xi\,d\xi$$

$$= x + \left[-\xi\cos(\xi - x) + \sin(\xi - x)\right]_0^x$$

$$= x - x\cos 0 + \sin 0 + 0 - \sin(0 - x)$$

$$= x - x + \sin x$$

$$= \sin x$$

This is the required solution. **(Solved)**

<u>P-25</u>: Solve the VIE: $\phi(x) = 1 + \int_0^x \phi(\xi)\,d\xi$

<u>Solution</u>: Given that $\phi(x) = 1 + \int_0^x \phi(\xi)\,d\xi$ (1)

We know that the Volterra's integral equation of second kind is

$$\phi(x) = F(x) + \lambda\int_a^x K(x,\xi)\phi(\xi)\,d\xi \tag{2}$$

and the solution of (2) is

$$\phi(x) = F(x) + \lambda\int_a^x R(x,\xi;\lambda)F(\xi)\,d\xi \tag{3}$$

Comparing (2) with (1) we have

$F(x) = 1,\ K(x,\xi) = 1, \lambda = 1$ and $a = 0$.

The Resolvent kernel is

$$R(x,\xi;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,\xi) \qquad \cdots (4)$$

and $K_n(x,\xi) = \int_\xi^x K(x,u)K_{n-1}(u,\xi)\,du\ ,\ n \geq 2$ (5)

$$\mathrm{K}_1(x,\xi) = \mathrm{K}(x,\xi) = 1$$

Putting $n = 2$ in (5) we get

$$\mathrm{K}_2(x,\xi) = \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_1(u,\xi)\,du$$

$$= \int_\xi^x du$$

$$= [u]_\xi^x$$

$$= (x - \xi)$$

Again,

$$\mathrm{K}_3(x,\xi) = \int_\xi^x \mathrm{K}(x,u)\,\mathrm{K}_2(u,\xi)\,du$$

$$= \int_{\xi}^{x} (u - \xi) du$$

$$= \left[\frac{u^2}{2} - u\xi \right]_{\xi}^{x}$$

$$= \frac{x^2}{2} - x\xi - \frac{\xi^2}{2} + \xi^2$$

$$= \frac{1}{2}\left(x^2 - 2x\xi + \xi^2\right)$$

$$= \frac{1}{2!}(x - \xi)^2$$

Similarly,

$$K_4(x,\xi) = \frac{1}{3!}(x-\xi)^3$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The resolvent kernel is

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots\cdots$$

$$= 1 + (x-\xi) + \frac{1}{2!}(x-\xi)^2 + \frac{1}{3!}(x-\xi)^3 + \cdots\cdots$$

$$= e^{(x-\xi)}$$

The solution of the given integral equation is

$$\phi(x) = F(x) + \lambda \int_0^x R(x,\xi;\lambda) F(\xi) d\xi$$

$$= 1 + \int_0^x e^{(x-\xi)} d\xi$$

$$= 1 + \left[-e^{(x-\xi)} \right]_0^x$$

$$= 1 - e^0 + e^x$$

$$= 1 - 1 + e^x$$

$$= e^x$$

This is the required solution. **(Solved)**

<u>P-26</u>: Solve $\phi(x) = sinx + \lambda \int_4^{10} x\phi(\xi)\, d\xi$.

<u>Solution</u>: Given that $\phi(x) = sinx + \lambda \int_4^{10} x\phi(\xi)\, d\xi$

$$= sinx + \lambda x \int_4^{10} \phi(\xi)\, d\xi$$

$$\therefore \phi(x) = sinx + \lambda x C \qquad \cdots (1)$$

where $C = \int_4^{10} \phi(\xi)\, d\xi \qquad \cdots (2)$

Using (1) in (2) we get

$$C = \int_4^{10} [sin\xi + \lambda \xi C]\, d\xi$$

$$= \int_4^{10} sin\xi\, d\xi + \lambda C \int_4^{10} \xi\, d\xi$$
$$= \int_4^{10} sin\xi\, d\xi + \lambda C \int_4^{10} \xi\, d\xi$$
$$= [-\cos\xi]_4^{10} + \lambda C \left[\frac{\xi^2}{2}\right]_4^{10}$$
$$= -\cos 10 + cos4 + \lambda C \left(\frac{100}{2} - \frac{16}{2}\right)$$
$$= -\cos 10 + cos4 + \lambda C(50 - 8)$$
$$= -\cos 10 + cos4 + 42\lambda C$$
$$\Rightarrow C(1 - 42\lambda) = cos4 - \cos 10$$
$$\Rightarrow C = \frac{2sin7.sin3}{1 - 42\lambda}$$

Putting the value of C in (1) we get

$$\phi(x) = sinx + \lambda x.\frac{2sin7.sin3}{1-42\lambda} \qquad \textbf{(Ans)}$$

P-27: Solve: $x(t) = (sint - \frac{t}{4}) + \frac{1}{4}\int_0^{1/2} \text{ts}\, x(s)\, ds$ and verify your result.

Solution: Given that $x(t) = \left(sint - \frac{t}{4}\right) + \frac{1}{4}\int_0^{1/2} \text{ts}\, x(s)\, ds$

$$= \left(sint - \frac{t}{4}\right) + \frac{t}{4}\int_0^{1/2} \text{s}\, x(s)\, ds$$
$$\therefore x(t) = \left(sint - \frac{t}{4}\right) + \frac{t}{4}C \qquad \cdots (1)$$

where $C = \int_0^{1/2} \text{s}\, x(s)\, ds \qquad \cdots (2)$

From (1) and (2), we get

$$C = \int_0^{1/2} \text{s}\, x(s)\, ds$$
$$= \int_0^{1/2} \text{s} \left[\left(sins - \frac{s}{4}\right) + \frac{s}{4}C\right] ds$$
$$\Rightarrow C\left\{1 - \int_0^{1/2} \frac{s^2}{4}\, ds\right\} = \int_0^{1/2} \text{s}\, sins\, ds - \int_0^{1/2} \frac{s^2}{4}\, ds$$
$$\Rightarrow C\left\{1 - \frac{1}{12}[s^3]_0^{1/2}\right\} = [-s\cos s + sins]_0^{1/2} - \frac{1}{12}[s^3]_0^{1/2}$$
$$\Rightarrow C\left\{1 - \frac{1}{96}\right\} = -\frac{1}{2}cos\frac{1}{2} + sin\frac{1}{2} - \frac{1}{96}$$
$$\Rightarrow \frac{95}{96}C = -\frac{1}{2}cos\frac{1}{2} + sin\frac{1}{2} - \frac{1}{96}$$
$$\therefore\ C = \frac{96}{95}\left(sin\frac{1}{2} - \frac{1}{2}cos\frac{1}{2}\right) - \frac{1}{95} \qquad \cdots (3)$$

Putting the value of C in (1) we get

$$x(t) = \left(sint - \frac{t}{4}\right) + \frac{t}{4}\left[\frac{96}{95}\left(sin\frac{1}{2} - \frac{1}{2}cos\frac{1}{2}\right) - \frac{1}{95}\right]$$
$$= \left(sint - \frac{t}{4}\right) + t\left[\frac{24}{95}\left(sin\frac{1}{2} - \frac{1}{2}cos\frac{1}{2}\right) - \frac{1}{380}\right]$$
$$= \left(sint - \frac{t}{4}\right) + \alpha t \qquad \cdots (4)$$

where $\alpha = \frac{24}{95}\left(sin\frac{1}{2} - \frac{1}{2}cos\frac{1}{2}\right) - \frac{1}{380}$ **(Ans)**

Verification:

L.H.S $= x(t) = \left(sint - \frac{t}{4}\right) + \alpha t \quad \cdots (4)$

Now $\quad x(s) = \left(sins - \frac{s}{4}\right) + \alpha s$

$$\Rightarrow sx(s) = s\, sins - \frac{s^2}{4} + \alpha s^2$$

$$= s\, sins + \left(\alpha - \frac{1}{4}\right) s^2$$

$$\Rightarrow \int_0^{1/2} sx(s)\, ds = \int_0^{1/2} s\, sins\, ds + \left(\alpha - \frac{1}{4}\right) \int_0^{1/2} s^2\, ds$$

$$= [-s\, sins + sins]_0^{1/2} + \left(\alpha - \frac{1}{4}\right) \left[\frac{s^3}{3}\right]_0^{1/2}$$

$$= sin\frac{1}{2} - \frac{1}{2} cos\frac{1}{2} + \frac{1}{24}\left(\alpha - \frac{1}{4}\right)$$

R.H.S $= \left(sint - \frac{t}{4}\right) + \frac{1}{4}\int_0^{1/2} \text{ts}\, x(s)\, ds$

$$= \left(sint - \frac{t}{4}\right) + \frac{t}{4}\left[sin\frac{1}{2} - \frac{1}{2} cos\frac{1}{2} + \frac{1}{24}\left(\alpha - \frac{1}{4}\right)\right]$$

$$= \left(sint - \frac{t}{4}\right) + \frac{t}{4}\left[\frac{95}{24}\left(\alpha + \frac{1}{380}\right) + \frac{1}{24}\left(\alpha - \frac{1}{4}\right)\right]$$

$$= \left(sint - \frac{t}{4}\right) + \frac{t}{96}\left(95\alpha + \frac{1}{4} + \alpha - \frac{1}{4}\right)$$

$$= \left(sint - \frac{t}{4}\right) + \frac{t}{96}(96\alpha)$$

$$= \left(sint - \frac{t}{4}\right) + \alpha t \quad \cdots (5)$$

Since L.H.S = R.H.S. Hence the result is verified.

<u>P-28:</u> Solve the IE $\phi(x) = x + \lambda \int_0^{\pi} (1 + sinxsint)\, \phi(t)\, dt$.

<u>Solution</u>: Given that $\phi(x) = x + \lambda \int_0^{\pi} (1 + sinxsint) \phi(t) dt \quad \cdots (1)$

$$\Rightarrow \phi(x) = x + \lambda \int_0^{\pi} \phi(t)\, dt + \lambda sinx \int_0^{\pi} sint\, \phi(t) dt$$

$$\Rightarrow \phi(x) = x + \lambda C_1 + \lambda C_2 sinx \quad \cdots (2)$$

where $\left.\begin{aligned} C_1 &= \int_0^{\pi} \phi(t)\ dt \\ C_2 &= \int_0^{\pi} sint\, \phi(t) dt \end{aligned}\right\} \quad \cdots (3)$

Substituting (2) into (3), we get

$$C_1 = \int_0^{\pi} [t + \lambda C_1 + \lambda C_2 sint]\, dt$$

$$\Rightarrow C_1\left[1 - \lambda \int_0^{\pi} dt\right] - \lambda C_2 \int_0^{\pi} sintdt - \int_0^{\pi} tdt = 0 \quad \cdots (4)$$

and $\quad C_2 = \int_0^{\pi} sint\, [t + \lambda C_1 + \lambda C_2 sint] dt$

$$\Rightarrow -\lambda C_1 \int_0^{\pi} sintdt + C_2\left[1 - \lambda \int_0^{\pi} sin^2 t\, dt\right] - \int_0^{\pi} tsint\, dt = 0 \quad \cdots (5)$$

Now $\int_0^{\pi} sin^2 t\, dt = \frac{1}{2}\int_0^{\pi} [1 - \cos 2t]\ dt = \frac{1}{2}\left[t - \frac{sin2t}{2}\right]_0^{\pi} = \frac{1}{2}\ (\pi - 0 - 0 + 0) = \frac{\pi}{2}$

$\int_0^{\pi} tsint\, dt = \ [-\, tcost + sint]_0^{\pi} = (-\, \pi cos\pi + sin\pi) - 0 = \pi$

$\int_0^{\pi} sintdt = \ [-\, cost]_0^{\pi} = (-cos\pi + cos0) = 1 + 1 = 2$

$$\int_0^\pi t\,dt = \left[\frac{t^2}{2}\right]_0^\pi = \frac{\pi^2}{2}$$
$$\int_0^\pi dt = [t]_0^\pi = \pi$$

Putting these values in (4) and (5) we get

$$C_1(1-\lambda\pi) - 2\lambda C_2 - \frac{\pi^2}{2} = 0$$
$$\Rightarrow (1-\lambda\pi)C_1 - 2\lambda C_2 - \frac{\pi^2}{2} = 0 \qquad \cdots (6)$$

and $-2\lambda C_1 + C_2\left(1-\frac{\lambda\pi}{2}\right) - \pi = 0$

$$\Rightarrow -2\lambda C_1 + \left(1-\frac{\lambda\pi}{2}\right)C_2 - \pi = 0 \qquad \cdots (7)$$

Thus, we get a non-homogeneous system of linear equations

$$\left.\begin{aligned}(1-\lambda\pi)C_1 - 2\lambda C_2 - \frac{\pi^2}{2} = 0\\ -2\lambda C_1 + \left(1-\frac{\lambda\pi}{2}\right)C_2 - \pi = 0\end{aligned}\right\} \qquad \cdots (8)$$

The determinant of this system is

$$\begin{vmatrix} 1-\lambda\pi & -2\lambda \\ -2\lambda & 1-\frac{\lambda\pi}{2}\end{vmatrix} = (1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2 \neq 0$$

Thus, the system has a unique solution.
The solution is given by

$$\frac{C_1}{2\lambda\pi + \frac{\pi^2}{2}\left(1-\frac{\lambda\pi}{2}\right)} = \frac{C_2}{\lambda\pi^2 + \pi(1-\lambda\pi)} = \frac{1}{(1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2}$$

$$\therefore C_1 = \frac{2\lambda\pi + \frac{\pi^2}{2}\left(1-\frac{\lambda\pi}{2}\right)}{(1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2} \quad and \quad C_2 = \frac{\lambda\pi^2 + \pi(1-\lambda\pi)}{(1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2}$$

Using these values in (1) we get

$$\phi(x) = x + \lambda.\frac{2\lambda\pi + \frac{\pi^2}{2}\left(1-\frac{\lambda\pi}{2}\right)}{(1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2} + \lambda.\frac{\pi}{(1-\lambda\pi)\left(1-\frac{\lambda\pi}{2}\right) - 4\lambda^2}\ sinx \qquad \textbf{(Ans)}$$

P-29: Find the eigen values and eigen functions of $\phi(x) = \lambda\int_0^\pi (cos^2x\, cos2s + cos3x cos^3s)\phi(s)ds$.

Solution: Given that $\phi(x) = \lambda\int_0^\pi (cos^2x\, cos2s + cos3x cos^3s)\phi(s)ds \qquad \cdots (1)$

$$\Rightarrow \phi(x) = \lambda cos^2x\int_0^\pi cos2s\ \phi(s)ds + \lambda cos3x\int_0^\pi cos^3s\,\phi(s)ds$$
$$\Rightarrow \phi(x) = \lambda C_1 cos^2x + \lambda C_2 cos3x \qquad \cdots (2)$$

where
$$\left.\begin{aligned}C_1 = \int_0^\pi cos2s\ \phi(s)ds\\ C_2 = \int_0^\pi cos^3s\,\phi(s)ds\end{aligned}\right\} \qquad \cdots (3)$$

and

Substituting (2) into (3) we get,

$$C_1 = \int_0^\pi cos2s\,[\lambda C_1 cos^2s + \lambda C_2 cos3s]ds$$
$$\Rightarrow C_1\left[1-\lambda\int_0^\pi cos2s\,cos^2s\,ds\right] - \lambda C_2\int_0^\pi cos2s\,cos3sds = 0 \qquad \cdots (4)$$

and $\quad C_2 = \int_0^{\pi} cos^3 s\,[\lambda C_1 cos^2 s\ + \lambda C_2 cos3s] ds$

$$\Rightarrow -\lambda C_1 \int_0^{\pi} cos^5 s\, ds + C_2 \left[1 - \lambda \int_0^{\pi} cos3s\, cos^3 s\, ds\right] = 0 \qquad \cdots (5)$$

Now $\int_0^{\pi} cos2s\, cos3sds = \frac{1}{2}\int_0^{\pi}[cos5s + \cos s]\ ds$

$$= \frac{1}{2}\left[\frac{sin5s}{5} + \sin s\right]_0^{\pi}$$

$$= \frac{1}{2}\ (0 - 0 + 0 - 0)$$

$$= 0$$

$$\int_0^{\pi} cos2s\, cos^2 sds = \frac{1}{2}\int_0^{\pi} cos2s(1 + \cos 2s)\ ds$$

$$= \frac{1}{2}\int_0^{\pi} cos2sds + \frac{1}{2}\int_0^{\pi} cos^2 2sds$$

$$= \frac{1}{2}\left[\frac{sin2s}{2}\right]_0^{\pi} + \frac{1}{4}\int_0^{\pi}(1 + cos4s)ds$$

$$= \frac{1}{2}\ (0 - 0) + \frac{1}{4}\left[s + \frac{sin4s}{4}\right]_0^{\pi}$$

$$= \frac{1}{4}(\pi + 0 - 0 - 0)$$

$$= \frac{\pi}{4}$$

$$\int_0^{\pi} cos^5 s\, ds = \int_0^{\pi} cos^4 s\ .\cos s\ ds$$

$$= \int_0^{\pi} (cos^2 s)^2 .\cos s\, ds$$

$$= \int_0^{\pi} (1 - sin^2 s)^2 .\cos s\, ds$$

Put $\sin s = t\ \ \therefore \cos s\, ds = dt$

when $s = 0$ then $t = 0$

when $s = \pi$ then $t = 0$

Using these in the above integral we get

$$\int_0^{\pi} cos^5 s\, ds = \int_0^{0} (1 - t^2)^2\, dt = 0$$

$$\int_0^{\pi} cos^3 s . cos3s\, ds = \frac{1}{4}\int_0^{\pi} (cos3s + 3\cos s). cos3s\, ds$$

$$= \frac{1}{4}\int_0^{\pi} cos^2 3sds + \frac{3}{4}\int_0^{\pi} \cos s . cos3s\, ds$$

$$= \frac{1}{8}\int_0^{\pi}(1 + cos6s)ds + \frac{3}{8}\int_0^{\pi}(cos4s + cos2s)\, ds$$

$$= \frac{1}{8}\left[s + \frac{sin6s}{6}\right]_0^{\pi} + \frac{3}{8}\left[\frac{sin4s}{4} + \frac{sin2s}{2}\right]_0^{\pi}$$

$$= \frac{1}{8}(\pi + 0 - 0 - 0) + \frac{3}{8}(0 + 0 - 0 - 0)$$

$$= \frac{\pi}{8}$$

Putting these values in (4) and (5) we get

$$C_1\left(1 - \lambda\frac{\pi}{4}\right) - \lambda C_2 . 0 = 0$$

$$\Rightarrow \left(1 - \frac{\lambda\pi}{4}\right) C_1 - 0 . C_2 = 0 \qquad \cdots (6)$$

and $\quad -\lambda C_1 . 0 + C_2\left(1 - \lambda\frac{\pi}{8}\right) = 0$

$$\Rightarrow 0. C_1 + \left(1 - \frac{\lambda\pi}{8}\right)C_2 = 0 \qquad \cdots(7)$$

Thus, we get a homogeneous system of linear equations

$$\left.\begin{aligned}\left(1-\frac{\lambda\pi}{4}\right)C_1 - 0. C_2 = 0\\ 0. C_1 + \left(1-\frac{\lambda\pi}{8}\right)C_2 = 0\end{aligned}\right\} \qquad \cdots(8)$$

The determinant of the eigen values is

$$\begin{vmatrix} 1-\frac{\lambda\pi}{4} & 0 \\ 0 & 1-\frac{\lambda\pi}{8}\end{vmatrix} = 0$$

$$\Rightarrow \left(1-\frac{\lambda\pi}{4}\right)\left(1-\frac{\lambda\pi}{8}\right) = 0$$

$$\Rightarrow 1-\frac{\lambda\pi}{4} = 0 \qquad or, 1-\frac{\lambda\pi}{8} = 0$$

$$\Rightarrow \lambda = \frac{4}{\pi} \qquad or, \lambda = \frac{8}{\pi}$$

Thus, the eigen values are $= \frac{4}{\pi}, \frac{8}{\pi}$.

When $\lambda = \frac{4}{\pi}$ then (8) reduces to

$$0. C_1 = 0\,, \quad \left(1 - \frac{4}{\pi}.\frac{\pi}{8}\right)C_2 = 0$$

$$\Rightarrow 0. C_1 = 0\,, \quad \frac{1}{2}. C_2 = 0$$

$\Rightarrow C_1$ is arbitrary and $C_2 = 0$

∴ From (2), the eigen function is

$$\phi_1(x) = \lambda C_1 cos^2x \ + 0$$

$$\Rightarrow \phi_1(x) = cos^2x \quad if \ \ \lambda C_1 = 1$$

When $\lambda = \frac{8}{\pi}$ then (8) reduces to

$$\left(1 - \frac{8}{\pi}.\frac{\pi}{4}\right)C_1 = 0\,, \quad 0. C_2 = 0$$

$$\Rightarrow -C_1 = 0\,, \quad 0. C_2 = 0$$

$\Rightarrow C_1 = 0$ and C_2 is arbitrary

∴ From (2), the eigen function is

$$\phi_2(x) = 0 + \lambda C_2 cos3x$$

$$\Rightarrow \phi_2(x) = cos3x \quad if \ \ \lambda C_2 = 1$$

Thus, the eigen values are $\lambda = \frac{4}{\pi}, \frac{8}{\pi}$ and the corresponding eigen functions are

$$\phi_1(x) = cos^2x,\ \phi_2(x) = cos3x \qquad \textbf{(Ans)}$$

P-30: Find the eigen values and eigen functions of $\phi(x) = cos3x + \lambda\int_0^\pi cos(x+y)\phi(y)dy$.

Solution: Given that $\phi(x) = cos3x + \lambda\int_0^\pi cos(x+y)\phi(y)dy \qquad \cdots(1)$

$$\Rightarrow \phi(x) = cos3x + \lambda\int_0^\pi (cosxcosy - sinxsiny)\,\phi(y)dy$$

$$\Rightarrow \phi(x) = cos3x + \lambda cosx\int_0^\pi cosy\,\phi(y)dy - \lambda sinx\int_0^\pi siny\,\phi(y)dy$$

$$\Rightarrow \phi(x) = cos3x + \lambda C_1 cosx - \lambda C_2 sinx \qquad \cdots (2)$$

where $\left.\begin{aligned} C_1 &= \int_0^\pi cosy \;\phi(y)dy \\ C_2 &= \int_0^\pi siny\,\phi(y)dy \end{aligned}\right\} \qquad \cdots (3)$

Substituting (2) into (3) we get,

$$C_1 = \int_0^\pi cosy\,[cos3y + \lambda C_1 cosy - \lambda C_2 siny]dy$$

$$\Rightarrow C_1\left[1 - \lambda\int_0^\pi cos^2y\,dy\right] + \lambda C_2\int_0^\pi siny\,cosydy - \int_0^\pi cosy\,cos3ydy = 0 \qquad \cdots (4)$$

and $\quad C_2 = \int_0^\pi siny\,[cos3y + \lambda C_1 cosy - \lambda C_2 siny]dy$

$$\Rightarrow -\lambda C_1\int_0^\pi sinycosy\,dy + C_2\left[1 + \lambda\int_0^\pi sin^2y\,dy\right] - \int_0^\pi sinycos3y\,dy = 0 \qquad \cdots (5)$$

Now $\int_0^\pi cos^2y\,dy = \frac{1}{2}\int_0^\pi [1 + \cos 2y]\;dy$

$$= \frac{1}{2}\left[y + \frac{sin2y}{2}\right]_0^\pi$$

$$= \frac{1}{2}(\pi + 0 - 0 - 0) = \frac{\pi}{2}$$

$$\int_0^\pi siny\,cosydy = \frac{1}{2}\int_0^\pi sin2y\;dy$$

$$= \frac{1}{2}\left[-\frac{cos2y}{2}\right]_0^\pi$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}(0) = 0$$

$$\int_0^\pi cosy\,cos3ydy = \frac{1}{2}\int_0^\pi (cos4y + cos2y)\;dy$$

$$= \frac{1}{2}\left[\frac{sin4y}{4} + \frac{sin2y}{2}\right]_0^\pi$$

$$= \frac{1}{2}(0 + 0 - 0 - 0)$$

$$= 0$$

$$\int_0^\pi sin^2y\,dy = \frac{1}{2}\int_0^\pi [1 - \cos 2y]\;dy$$

$$= \frac{1}{2}\left[y - \frac{sin2y}{2}\right]_0^\pi$$

$$= \frac{1}{2}(\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

$$\int_0^\pi sinycos3y\,dy = \frac{1}{2}\int_0^\pi (sin4y - sin2y)\;dy$$

$$= \frac{1}{2}\left[-\frac{cos4y}{4} + \frac{cos2y}{2}\right]_0^\pi$$

$$= \frac{1}{2}\left(-\frac{1}{4} + \frac{1}{2} + \frac{1}{4} - \frac{1}{2}\right)$$

$$= 0$$

Putting these values in (4) and (5) we get

$$C_1\left(1 - \frac{\lambda\pi}{2}\right) + \lambda C_2.0 = 0$$

$$\Rightarrow \left(1 - \frac{\lambda\pi}{2}\right)C_1 + 0.C_2 = 0 \qquad \cdots (6)$$

and $\quad -\lambda C_1.0 + C_2\left(1 + \frac{\lambda\pi}{2}\right) = 0$

$$\Rightarrow 0.C_1 + \left(1 + \frac{\lambda\pi}{2}\right)C_2 = 0 \qquad \cdots (7)$$

Thus, we get a homogeneous system of linear equations

$$\left.\begin{aligned} \left(1 - \frac{\lambda\pi}{2}\right)C_1 - 0.C_2 = 0 \\ 0.C_1 + \left(1 + \frac{\lambda\pi}{2}\right)C_2 = 0 \end{aligned}\right\} \qquad \cdots (8)$$

The determinant of the eigen values is

$$\begin{vmatrix} 1 - \frac{\lambda\pi}{2} & 0 \\ 0 & 1 + \frac{\lambda\pi}{2} \end{vmatrix} = 0$$

$$\Rightarrow \left(1 - \frac{\lambda\pi}{2}\right)\left(1 + \frac{\lambda\pi}{2}\right) = 0$$

$$\Rightarrow 1 - \frac{\lambda\pi}{2} = 0 \qquad or, 1 + \frac{\lambda\pi}{2} = 0$$

$$\Rightarrow \lambda = \frac{2}{\pi} \qquad or, \lambda = -\frac{2}{\pi}$$

Thus, the eigen values are $= \frac{2}{\pi}, -\frac{2}{\pi}$.

When $\lambda = \frac{4}{\pi}$ then (8) reduces to

$$\left(1 + \frac{2}{\pi} . \frac{\pi}{2}\right)C_1 = 0, \quad 0.C_2 = 0$$

$$\Rightarrow 2C_1 = 0, \quad 0.C_2 = 0$$

$\Rightarrow C_1 = 0$ and C_2 is arbitrary.

∴ From (2), the eigen function is

$$\phi_1(x) = cos3x + 0 - \left(-\frac{2}{\pi}\right)C_2 sinx$$

$$\Rightarrow \phi_1(x) = cos3x + sinx \quad if \ \frac{2}{\pi}C_1 = 1$$

When $\lambda = \frac{2}{\pi}$ then (8) reduces to

$$0.C_1 = 0, \quad \left(1 + \frac{2}{\pi} . \frac{\pi}{2}\right)C_2 = 0$$

$$\Rightarrow 0.C_1 = 0, \quad 2C_2 = 0$$

$\Rightarrow C_1$ is arbitrary and $C_2 = 0$.

∴ From (2), the eigen function is

$$\phi_2(x) = cos3x + \frac{2}{\pi}C_1 cosx - 0$$

$$\Rightarrow \phi_2(x) = cos3x + cosx \quad if \ \frac{2}{\pi}C_1 = 1$$

Thus, the eigen values are $\lambda = -\frac{2}{\pi}, \frac{2}{\pi}$ and the corresponding eigen functions are

$$\phi_1(x) = cos3x + sinx, \quad \phi_2(x) = cos3x + cosx$$ **(Ans)**

<u>P-31:</u> Find the eigen values and eigen functions for $x(t) = \lambda \int_0^\pi cos(s+t)x(s)ds$.

<u>Solution</u>: Given that $x(t) = \lambda \int_0^\pi cos(s+t)x(s)ds \qquad \cdots (1)$

$$\Rightarrow x(t) = \lambda \int_0^\pi (\cos s \ cost - sins \ sint)\, x(s)ds$$

$$\Rightarrow x(t) = \lambda cost \int_0^\pi \cos s \, x(s) ds - \lambda sint \int_0^\pi \sin s \, x(s) ds$$

$$\Rightarrow x(t) = \lambda C_1 cost - \lambda C_2 sint \qquad \cdots (2)$$

where $\left.\begin{aligned} C_1 &= \int_0^\pi \cos s \; x(s) ds \\ C_2 &= \int_0^\pi \sin s \, x(s) ds \end{aligned}\right\} \qquad \cdots (3)$

Substituting (2) into (3) we get,

$$C_1 = \int_0^\pi \cos s \, [\lambda C_1 \cos s - \lambda C_2 \sin s] ds$$

$$\Rightarrow C_1 \left[1 - \lambda \int_0^\pi cos^2 \, s \, ds\right] + \lambda C_2 \int_0^\pi \sin s \cos s \, ds = 0 \qquad \cdots (4)$$

and $\quad C_2 = \int_0^\pi \sin s \, [\lambda C_1 \cos s - \lambda C_2 \sin s] ds$

$$\Rightarrow -\lambda C_1 \int_0^\pi \sin s \cos s \;\, ds + C_2 \left[1 + \lambda \int_0^\pi sin^2 s \, ds\right] = 0 \qquad \cdots (5)$$

Now $\int_0^\pi cos^2 s \, ds = \frac{1}{2} \int_0^\pi [1 + \cos 2s] \;\, ds$

$$= \frac{1}{2} \left[s + \frac{sin2s}{2}\right]_0^\pi$$

$$= \frac{1}{2} \, (\pi + 0 - 0 - 0) = \frac{\pi}{2}$$

$$\int_0^\pi \sin s \cos s \, ds = \frac{1}{2} \int_0^\pi sin2s \;\, ds$$

$$= \frac{1}{2} \left[- \frac{cos2s}{2}\right]_0^\pi$$

$$= \frac{1}{2} \, (- \frac{1}{2} + \frac{1}{2} = \frac{1}{2} (0) = 0$$

$$\int_0^\pi sin^2 s \, ds = \frac{1}{2} \int_0^\pi [1 - \cos 2s] \;\, ds$$

$$= \frac{1}{2} \left[s - \frac{sin2s}{2}\right]_0^\pi$$

$$= \frac{1}{2} \, (\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

Putting these values in (4) and (5) we get

$$C_1 \left(1 - \frac{\lambda\pi}{2}\right) + \lambda C_2 . 0 = 0$$

$$\Rightarrow \left(1 - \frac{\lambda\pi}{2}\right) C_1 + 0. C_2 = 0 \qquad \cdots (6)$$

and $\quad -\lambda C_1 . 0 + C_2 \left(1 + \frac{\lambda\pi}{2}\right) = 0$

$$\Rightarrow 0. C_1 + \left(1 + \frac{\lambda\pi}{2}\right) C_2 = 0 \qquad \cdots (7)$$

Thus, we get a homogeneous system of linear equations

$$\left.\begin{aligned} \left(1 - \frac{\lambda\pi}{2}\right) C_1 - 0. C_2 &= 0 \\ 0. C_1 + \left(1 + \frac{\lambda\pi}{2}\right) C_2 &= 0 \end{aligned}\right\} \qquad \cdots (8)$$

The determinant of the eigen values is

$$\begin{vmatrix} 1-\frac{\lambda\pi}{2} & 0 \\ 0 & 1+\frac{\lambda\pi}{2} \end{vmatrix} = 0$$

$$\Rightarrow \left(1-\frac{\lambda\pi}{2}\right)\left(1+\frac{\lambda\pi}{2}\right) = 0$$

$$\Rightarrow 1-\frac{\lambda\pi}{2} = 0 \qquad or, 1+\frac{\lambda\pi}{2} = 0$$

$$\Rightarrow \lambda = \frac{2}{\pi} \qquad or, \lambda = -\frac{2}{\pi}$$

Thus, the eigen values are $= \frac{2}{\pi}, -\frac{2}{\pi}$.

When $\lambda = \frac{4}{\pi}$ then (8) reduces to

$$\left(1+\frac{2}{\pi}.\frac{\pi}{2}\right)C_1 = 0, \quad 0.C_2 = 0$$

$$\Rightarrow 2C_1 = 0, \quad 0.C_2 = 0$$

$\Rightarrow C_1 = 0$ and C_2 is arbitrary.

∴ From (2), the eigen function is

$$x_1(t) = 0 - \left(-\frac{2}{\pi}\right)C_2 sint$$

$$\Rightarrow x_1(t) = sint \quad if \ \frac{2}{\pi}C_1 = 1$$

When $\lambda = \frac{2}{\pi}$ then (8) reduces to

$$0.C_1 = 0, \quad \left(1+\frac{2}{\pi}.\frac{\pi}{2}\right)C_2 = 0$$

$$\Rightarrow 0.C_1 = 0, \quad 2C_2 = 0$$

$\Rightarrow C_1$ is arbitrary and $C_2 = 0$.

∴ From (2), the eigen function is

$$x_2(t) = \frac{2}{\pi}C_1 cost - 0$$

$$\Rightarrow x_2(t) = cost \quad if \ \frac{2}{\pi}C_1 = 1$$

Thus, the eigen values are $\lambda = -\frac{2}{\pi}, \frac{2}{\pi}$ and the corresponding eigen functions are

$$x_1(t) = sint, \quad x_2(t) = cost \qquad \textbf{(Ans)}$$

P-32: Find the eigen values and eigen functions of $x(t) = \lambda\int_{-1}^{1}(5ts^3 + 4t^2s + 3ts)x(s)ds$.

Solution: Given that $x(t) = \lambda\int_{-1}^{1}(5ts^3 + 4t^2s + 3ts)x(s)ds \qquad \cdots(1)$

$$\Rightarrow x(t) = 5\lambda t\int_{-1}^{1} s^3\, x(s)ds + 4\lambda t^2\int_{-1}^{1} s\, x(s)ds + + 3\lambda t\int_{-1}^{1} s\, x(s)ds$$

$$\Rightarrow x(t) = 5\lambda tC_1 + 4\lambda t^2C_2 + 3\lambda tC_3 \qquad \cdots(2)$$

where
$$\left.\begin{aligned} C_1 &= \int_{-1}^{1} s^3\, x(s)ds \\ C_2 &= \int_{-1}^{1} s\, x(s)ds \\ C_3 &= \int_{-1}^{1} s\, x(s)ds \end{aligned}\right\} \qquad \cdots(3)$$

Substituting (2) into (3) we get,

$$C_1 = \int_{-1}^{1} s^3\,[5\lambda s C_1 + 4\lambda s^2 C_2 + 3\lambda s C_3]ds$$
$$= 5\lambda C_1 \int_{-1}^{1} s^4\,ds + 4\lambda C_2 \int_{-1}^{1} s^5\,ds + 3\lambda C_3 \int_{-1}^{1} s^4\,ds$$
$$= 5\lambda C_1 \left[\frac{s^5}{5}\right]_{-1}^{1} + 4\lambda C_2 \left[\frac{s^6}{6}\right]_{-1}^{1} + 3\lambda C_3 \left[\frac{s^5}{5}\right]_{-1}^{1}$$
$$= 5\lambda C_1 \left(\frac{1}{5} + \frac{1}{5}\right) + 4\lambda C_2 \left(\frac{1}{6} - \frac{1}{6}\right) + 3\lambda C_3 \left(\frac{1}{5} + \frac{1}{5}\right)$$
$$= 2\lambda C_1 + \frac{6}{5}\lambda C_3$$
$$\Rightarrow 5C_1 = 10\lambda C_1 + 6\lambda C_3$$
$$\Rightarrow (10\lambda - 5)C_1 + 6\lambda C_3 = 0 \qquad \cdots (4)$$

And

$$C_2 = \int_{-1}^{1} s\,[5\lambda s C_1 + 4\lambda s^2 C_2 + 3\lambda s C_3]ds$$
$$= 5\lambda C_1 \int_{-1}^{1} s^2\,ds + 4\lambda C_2 \int_{-1}^{1} s^3\,ds + 3\lambda C_3 \int_{-1}^{1} s^2\,ds$$
$$= 5\lambda C_1 \left[\frac{s^3}{3}\right]_{-1}^{1} + 4\lambda C_2 \left[\frac{s^4}{4}\right]_{-1}^{1} + 3\lambda C_3 \left[\frac{s^3}{3}\right]_{-1}^{1}$$
$$= 5\lambda C_1 \left(\frac{1}{3} + \frac{1}{3}\right) + 4\lambda C_2 \left(\frac{1}{4} - \frac{1}{4}\right) + 3\lambda C_3 \left(\frac{1}{3} + \frac{1}{3}\right)$$
$$= \frac{10}{3}\lambda C_1 + 2\lambda C_3$$
$$\Rightarrow 3C_3 = 10\lambda C_1 + 6\lambda C_3$$
$$\Rightarrow 10\lambda C_1 - 3C_2 + 6\lambda C_3 = 0 \qquad \cdots (5)$$

Finally,

$$C_3 = \int_{-1}^{1} s\,[5\lambda s C_1 + 4\lambda s^2 C_2 + 3\lambda s C_3]ds$$
$$= 5\lambda C_1 \int_{-1}^{1} s^2\,ds + 4\lambda C_2 \int_{-1}^{1} s^3\,ds + 3\lambda C_3 \int_{-1}^{1} s^2\,ds$$
$$= 5\lambda C_1 \left[\frac{s^3}{3}\right]_{-1}^{1} + 4\lambda C_2 \left[\frac{s^4}{4}\right]_{-1}^{1} + 3\lambda C_3 \left[\frac{s^3}{3}\right]_{-1}^{1}$$
$$= 5\lambda C_1 \left(\frac{1}{3} + \frac{1}{3}\right) + 4\lambda C_2 \left(\frac{1}{4} - \frac{1}{4}\right) + 3\lambda C_3 \left(\frac{1}{3} + \frac{1}{3}\right)$$
$$= \frac{10}{3}\lambda C_1 + 2\lambda C_3$$
$$\Rightarrow 3C_2 = 10\lambda C_1 + 6\lambda C_3$$
$$\Rightarrow 10\lambda C_1 + (6\lambda - 3)C_3 = 0 \qquad \cdots (6)$$

Thus, we get a homogeneous system of linear equations

$$\left.\begin{aligned} (10\lambda - 5)C_1 + 6\lambda C_3 &= 0 \\ 10\lambda C_1 - 3C_2 + 6\lambda C_3 &= 0 \\ 10\lambda C_1 + (6\lambda - 3)C_3 &= 0 \end{aligned}\right\} \qquad \cdots (7)$$

The determinant of the eigen values is

$$\begin{vmatrix} 10\lambda - 5 & 0 & 6\lambda \\ 10\lambda & -3 & 6\lambda \\ 10\lambda & 0 & 6\lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow (10\lambda - 5)(-18\lambda + 9 - 0) - 0 + 6\lambda(0 + 30\lambda) = 0$$
$$\Rightarrow (10\lambda - 5)(-18\lambda + 9) + 180\lambda^2 = 0$$
$$\Rightarrow -180\lambda^2 + 90\lambda + 90\lambda - 45 + 180\lambda^2 = 0$$
$$\Rightarrow 180\lambda = 45$$
$$\Rightarrow \lambda = \frac{45}{180}$$
$$\Rightarrow \lambda = \frac{1}{4}$$

Thus, the eigen value is $= \frac{1}{4}$.

When $\lambda = \frac{1}{4}$ then Eq. (4) reduces to

$$-\frac{5}{2}C_1 + \frac{3}{2}C_3 = 0$$
$$\Rightarrow -5C_1 + 3C_3 = 0$$
$$\Rightarrow C_1 = \frac{3}{5}C_3 \qquad \cdots (8)$$

Eq. (5) reduces to

$$\frac{5}{2}C_1 - 3C_2 + \frac{3}{2}C_3 = 0$$
$$\Rightarrow \frac{5}{2}.\frac{3}{5}C_3 - 3C_2 + \frac{3}{2}C_3 = 0 \qquad using\ (8)$$
$$\Rightarrow \frac{3}{2}C_3 - 3C_2 + \frac{3}{2}C_3 = 0$$
$$\Rightarrow -3C_2 + 3C_3 = 0$$
$$\Rightarrow C_2 = C_3 \qquad \cdots (9)$$

And Eq. (6) reduces to

$$\frac{5}{2}C_1 - \frac{3}{2}C_3 = 0$$
$$\Rightarrow 5C_1 - 3C_3 = 0$$
$$\Rightarrow C_1 = \frac{3}{5}C_3 \qquad \cdots (10)$$

Taking $C_3 = 5$ we get $C_1 = 3$ and $C_2 = 5$

Putting the values of λ, C_1, C_2 and C_3 in Eq. (2) we get

$$x(t) = 5.\frac{1}{4}.t.3 + 4.\frac{1}{4}.t^2.5 + 3.\frac{1}{4}.t.5$$
$$= \frac{15}{4}t + 5t^2 + \frac{15}{4}t$$
$$= \frac{15}{2}t + 5t^2$$
$$= \frac{5}{2}(3t + 2t^2)$$

Thus, the eigen value is $\lambda = \frac{1}{4}$ and the corresponding eigen function is

$x(t) = \frac{5}{2}(3t + 2t^2)$ (**Ans**)

P-33: Find the Resolvent kernel and hence solve the Fredholm integral equation $\phi(x)=x+\int_0^{1/2}\phi(\xi)d\xi$

Solution: Given that $\phi(x)=x+\int_0^{1/2}\phi(\xi)d\xi$ (1)

We know that the Fredholm integral equation of second kind is

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)d\xi \tag{2}$$

and the solution of (2) is

$$\phi(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)d\xi \tag{3}$$

Comparing (2) with (1) we have

$F(x)=x$, $K(x,\xi)=1$, $\lambda=1, a=0$ and $b=1/2$.

The Resolvent kernel is

$$R(x,\xi;\lambda)=\sum_{n=1}^{\infty}\lambda^{n-1}K_n(x,\xi) \tag{4}$$

where $K_n(x,\xi)=\int_a^b K(x,u)K_{n-1}(u,\xi)du, \quad n\geq 2$ (5)

and $K_1(x,\xi)=K(x,\xi)=1$

Putting $n=2$ in (5) we get

$$K_2(x,\xi)=\int_a^b K(x,u)K_1(u,\xi)du$$
$$=\int_0^{1/2}du$$
$$=[u]_0^{1/2}$$
$$=\left(\frac{1}{2}-0\right)$$
$$=\frac{1}{2}$$

Again,

$$K_3(x,\xi)=\int_a^b K(x,u)K_2(u,\xi)du$$
$$=\frac{1}{2}\int_0^{1/2}du$$
$$=\frac{1}{2}[u]_0^{1/2}$$
$$=\frac{1}{2}\left(\frac{1}{2}-0\right)$$

$$=\left(\frac{1}{2}\right)^2$$

Similarly,

$$K_4(x,\xi)=\left(\frac{1}{2}\right)^3$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

The resolvent kernel is

$$R(x,\xi;\lambda)=K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2 K_3(x,\xi)+\cdots\cdots$$

$$=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\cdots$$

$$=\frac{1}{1-1/2}\ , \qquad \left[\because\ a+ar+ar^2+ar^3+\cdots=\frac{a}{1-r}\right]$$

$$=2$$

The solution of the given integral equation is

$$\phi(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)\,d\xi$$

$$=x+\int_0^{1/2}2\xi\,d\xi$$

$$=x+\left[\xi^2\right]_0^{1/2}$$

$$=x+\frac{1}{4}$$

This is the required solution. **(Solved)**

<u>P-34</u>: Solve the Fredholm integral equation $\phi(x)=\left(\sin x-\frac{x}{4}\right)+\frac{1}{4}\int_0^{\pi/2}\xi x\phi(\xi)\,d\xi$

<u>Solution</u>: Given that $\phi(x)=\left(\sin x-\frac{x}{4}\right)+\frac{1}{4}\int_0^{\pi/2}\xi x\phi(\xi)\,d\xi$ (1)

We know that the Fredholm integral equation of second kind is

$$\phi(x)=F(x)+\lambda\int_a^b K(x,\xi)\phi(\xi)\,d\xi \qquad (2)$$

and the solution of (2) is

$$\phi(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)\,d\xi \qquad (3)$$

Comparing (2) with (1) we have

$$F(x)=\left(\sin x-\frac{x}{4}\right),\ K(x,\xi)=\xi x,\ \lambda=\frac{1}{4},\ a=0 \text{ and } b=\pi/2.$$

The Resolvent kernel is

$$R(x,\xi;\lambda)=\sum_{n=1}^{\infty}\lambda^{n-1}K_n(x,\xi) \tag{4}$$

where $K_n(x,\xi)=\int_a^b K(x,u)K_{n-1}(u,\xi)\,du, \quad n\geq 2$ (5)

and $K_1(x,\xi)=K(x,\xi)=\xi x$

Putting $n=2$ in (5) we get

$$\begin{aligned}
K_2(x,\xi)&=\int_a^b K(x,u)K_1(u,\xi)\,du\\
&=\int_0^{\pi/2} ux\cdot\xi u du\\
&=\int_0^{\pi/2} x\xi u^2 du\\
&=x\xi\left[\frac{u^3}{3}\right]_0^{\pi/2}\\
&=\frac{x\xi}{3}\cdot\left(\frac{\pi}{2}\right)^3
\end{aligned}$$

Again,

$$\begin{aligned}
K_3(x,\xi)&=\int_a^b K(x,u)K_2(u,\xi)\,du\\
&=\int_0^{\pi/2} ux\cdot\left\{\frac{u\xi}{3}\cdot\left(\frac{\pi}{2}\right)^3\right\}du\\
&=\frac{x\xi}{3}\left(\frac{\pi}{2}\right)^3\int_0^{\pi/2} u^2 du\\
&=\frac{x\xi}{3}\left(\frac{\pi}{2}\right)^3\left[\frac{u^3}{3}\right]_0^{\pi/2}\\
&=\left(\frac{1}{3}\right)^2\left(\frac{\pi}{2}\right)^6 x\xi
\end{aligned}$$

Similarly,

$$K_4(x,\xi)=\left(\frac{1}{3}\right)^3\left(\frac{\pi}{2}\right)^9 x\xi$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$K_n(x,\xi)=\left[\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)^3\right]^{n-1} x\xi$$

The resolvent kernel is

$$R(x,\xi;\lambda)=K(x,\xi)+\lambda K_2(x,\xi)+\lambda^2K_3(x,\xi)+\cdots\cdots$$

$$= x\xi + \lambda \cdot \frac{1}{3}\left(\frac{\pi}{2}\right)^3 x\xi + \lambda^2 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{\pi}{2}\right)^6 x\xi + \lambda^3 \cdot \left(\frac{1}{3}\right)^3 \left(\frac{\pi}{2}\right)^9 x\xi + \cdots\cdots$$

$$= x\xi\left[1 + \lambda \cdot \frac{1}{3}\left(\frac{\pi}{2}\right)^3 + \lambda^2 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{\pi}{2}\right)^6 + \lambda^3 \cdot \left(\frac{1}{3}\right)^3 \left(\frac{\pi}{2}\right)^9 + \cdots\cdots\right]$$

$$= x\xi \cdot \frac{1}{1 - \lambda \cdot \frac{1}{3}\left(\frac{\pi}{2}\right)^3}$$

$$= \frac{x\xi}{1 - \frac{1}{4} \cdot \frac{\pi^3}{24}}$$

$$= \frac{96x\xi}{96 - \pi^3}$$

The solution of the given integral equation is

$$\phi(x) = F(x) + \lambda \int_a^b R(x,\xi;\lambda) F(\xi)\, d\xi$$

$$= \sin x - \frac{x}{4} + \frac{1}{4}\int_0^{\pi/2} \frac{96x\xi}{96-\pi^3} \cdot \left(\sin\xi - \frac{\xi}{4}\right) d\xi$$

$$= \sin x - \frac{x}{4} + \frac{24x}{96-\pi^3}\int_0^{\pi/2} \left(\xi\sin\xi - \frac{\xi^2}{4}\right) d\xi$$

$$= \sin x - \frac{x}{4} + \frac{24x}{96-\pi^3}\left[-\xi\cos\xi + \sin\xi - \frac{\xi^3}{12}\right]_0^{\pi/2}$$

$$= \sin x - \frac{x}{4} + \frac{24x}{96-\pi^3} \cdot \left(1 - \frac{\pi^3}{96}\right)$$

$$= \sin x - \frac{x}{4} + \frac{24x}{96-\pi^3} \cdot \frac{96-\pi^3}{96}$$

$$= \sin x - \frac{x}{4} + \frac{x}{4}$$

$$= \sin x$$

This is the required solution. **(Solved)**

<u>P-35</u>: Find the Resolvent kernel of $K(x,\xi) = x\xi$, $a = -1,\ b = 1$

<u>Solution</u>: Given that $K(x,\xi) = x\xi$, $a = -1,\ b = 1$ (1)

The Resolvent kernel is

$$R(x,\xi;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,\xi) \qquad (2)$$

where $K_n(x,\xi) = \int_a^b K(x,u) K_{n-1}(u,\xi)\, du, \quad n \ge 2$ (3)

and $\quad K_1(x,\xi) = K(x,\xi) = x\xi$

Putting $n=2$ in (3) we get

$$K_2(x,\xi) = \int_a^b K(x,u)K_1(u,\xi)\,du$$

$$= \int_{-1}^{1} xu \cdot u\xi du$$

$$= x\xi \int_{-1}^{1} u^2 du$$

$$= x\xi \left[\frac{u^3}{3}\right]_{-1}^{1}$$

$$= \frac{2}{3}x\xi$$

Again,

$$K_3(x,\xi) = \int_a^b K(x,u)K_2(u,\xi)\,du$$

$$= \int_{-1}^{1} xu \cdot \frac{2}{3}u\xi du$$

$$= \frac{2}{3}x\xi \int_{-1}^{1} u^2 du$$

$$= \frac{2}{3}x\xi \left[\frac{u^3}{3}\right]_{-1}^{1}$$

$$= \left(\frac{2}{3}\right)^2 x\xi$$

Similarly,

$$K_4(x,\xi) = \left(\frac{2}{3}\right)^3 x\xi$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$K_n(x,\xi) = \left(\frac{2}{3}\right)^{n-1} x\xi$$

The resolvent kernel is

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda K_2(x,\xi) + \lambda^2 K_3(x,\xi) + \cdots\cdots$$

$$= x\xi + \lambda \cdot \frac{2}{3}x\xi + \lambda^2 \cdot \left(\frac{2}{3}\right)^2 x\xi + \lambda^3 \cdot \left(\frac{2}{3}\right)^3 x\xi + \cdots\cdots$$

$$= x\xi \left[1 + \lambda \cdot \frac{2}{3} + \lambda^2 \cdot \left(\frac{2}{3}\right)^2 + \lambda^3 \cdot \left(\frac{2}{3}\right)^3 + \cdots\cdots\right]$$

$$= x\xi \cdot \frac{1}{1-\lambda \cdot \frac{2}{3}}$$

$$= \frac{3x\xi}{3-2\lambda}.$$

P-36: Find the Iterated kernel of $K(x,\xi) = x\xi$, $a=0,\ b=1$

Solution: Given that $K(x,\xi) = x\xi$, $a=0,\ b=1$

By the definition of Iterated kernel, we get

$$K_1(x,\xi) = K(x,\xi) = x\xi$$

$$K_2(x,\xi) = \int_a^b K(x,u)K_1(u,\xi)\,du$$

$$= \int_0^1 xu \cdot u\xi du$$

$$= x\xi\int_0^1 u^2 du$$

$$= x\xi\left[\frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{3}x\xi$$

$$K_3(x,\xi) = \int_a^b K(x,u)K_2(u,\xi)\,du$$

$$= \int_0^1 xu \cdot \frac{1}{3}u\xi du$$

$$= \frac{1}{3}x\xi\int_0^1 u^2 du$$

$$= \frac{1}{3}x\xi\left[\frac{u^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3}\right)^2 x\xi$$

Similarly, $K_4(x,\xi) = \left(\frac{1}{3}\right)^3 x\xi$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$$K_n(x,\xi) = \left(\frac{1}{3}\right)^{n-1} x\xi$$

The required Iterated kernel is

$$K_n(x,\xi) = \left(\frac{1}{3}\right)^{n-1} x\xi\ .$$

P-37: Find the Iterated kernel of $K(x,\xi)=x-\xi$, $a=0,\ b=1$

Solution: Given that $K(x,\xi)=x-\xi$, $a=0,\ b=1$

By the definition of Iterated kernel, we get

$$K_1(x,\xi)=K(x,\xi)=x-\xi$$

$$K_2(x,\xi)=\int_a^b K(x,u)K_1(u,\xi)\,du$$

$$=\int_0^1 (x-u)\cdot(u-\xi)\,du$$

$$=\int_0^1 \left(xu-x\xi-u^2+u\xi\right)du$$

$$=\left[\frac{xu^2}{2}-x\xi u-\frac{u^3}{3}+\frac{u^2\xi}{2}\right]_0^1$$

$$=\frac{x+\xi}{2}-x\xi-\frac{1}{3}$$

$$K_3(x,\xi)=\int_a^b K(x,u)K_2(u,\xi)\,du$$

$$=\int_0^1 xu\cdot\left(\frac{u+\xi}{2}-u\xi-\frac{1}{3}\right)du$$

$$=-\frac{x-\xi}{12}$$

Similarly, $K_4(x,\xi)=-\frac{1}{12}\left(\frac{x+\xi}{2}-x\xi-\frac{1}{3}\right)$

$$K_5(x,\xi)=\frac{(-1)^2}{(12)^2}(x-\xi)$$

if $n=2p-1$, then

$$K_{2p-1}(x,\xi)=\frac{(-1)^{p-1}}{(12)^{p-1}}(x-\xi) \qquad \text{for } p=1,2,3,\cdots$$

if $n=2p$, then

$$K_{2p}(x,\xi)=\frac{(-1)^{p-1}}{(12)^{p-1}}\left(\frac{x+\xi}{2}-x\xi-\frac{1}{3}\right) \qquad \text{for } p=1,2,3,\cdots$$