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Differentiation

Introduction: The derivative is a mathematical operator, which measures the rate of change of a quantity relative to another quantity. The process of finding a derivative is called differentiation. There are many phenomena related changing quantities such as speed of a particle, inflation of currency, intensity of an earthquake and voltage of an electrical signal etc. in the world. In this chapter we will discuss about various techniques of derivative.

Outcomes: After successful completion of the chapter, the students will be able to:

1. determine the speed, velocity and acceleration of a particle with respect to time.
2. calculate the rate at which the number of bacteria , the population changes with time.
3. measure the rate at which the length of a metal rod changes with temperature.
4. find out the rate at which production cost changes with the quantity of a product .

Increment: Let $y = f(x)$ be a function of x . Let δx be an increment in the value of x and δy be the corresponding increment in the value of y so that

$$y + \delta y = f(x + \delta x)$$

$$or,\ \delta y = f(x + \delta x) - f(x)$$

$$or,\ \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Here $\frac{\delta y}{\delta x}$ is called the increment ratio.

***<u>Differentiability of a function</u>*:** The derivative of $y = f(x)$ with respect to x (for a particular value of x) is denoted by $f'(x)$ or $\frac{dy}{dx}$ and defined as,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Provided this limit exists. This is called first principle formula for derivative.

<u>Existence of Derivative</u>: A function $y = f(x)$ is called differentiable at $x = a$ if the left hand derivative and right hand derivative at this point i.e,

$$L.H.D = \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h}$$

and $$R.H.D = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

are both exist and equal.

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Problem 01: A function $f(x)$ is defined as follows:

$$f(x)=\begin{cases} x^2+1 & when\ x\le 0 \\ x & when\ 0<x<1 \\ \dfrac{1}{x} & when\ x\ge 1 \end{cases}$$

Discuss the differentiability at $x=0$ and $x=1$.

Solution: Given that,

$$f(x)=\begin{cases} x^2+1 & when\ x\le 0 \\ x & when\ 0<x<1 \\ \dfrac{1}{x} & when\ x\ge 1 \end{cases}$$

1st Part: For $x=0$,

$$\begin{aligned} L.H.D &= \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} \\ &= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h} \\ &= \lim_{h\to 0}\frac{\left\{(-h)^2+1\right\}-\left(0^2+1\right)}{-h} \\ &= \lim_{h\to 0}\frac{h^2+1-1}{-h} \\ &= \lim_{h\to 0}\frac{h^2}{-h} \\ &= \lim_{h\to 0}(-h) \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.D &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{h-\left\{(0)^2+1\right\}}{h} \\ &= \lim_{h\to 0}\frac{h-1}{h} \\ &= \lim_{h\to 0}\left(1-\frac{1}{h}\right) \\ &= -\propto \end{aligned}$$

Since *R.H.D* does not exist. So the function is not differentiable at $x=0$.

2nd Part: For $x=1$,

$$\begin{aligned} L.H.D &= \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} \\ &= \lim_{h\to 0}\frac{1-h-1}{-h} \\ &= \lim_{h\to 0}\frac{-h}{-h} \\ &= \lim_{h\to 0}(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} R.H.D &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h} \\ &= \lim_{h\to 0}\frac{1-1-h}{h(1+h)} \\ &= \lim_{h\to 0}\frac{-1}{1+h} \\ &= \frac{-1}{1+0} \\ &= -1 \end{aligned}$$

Since $L.H.D \neq R.H.D$ does not exist. So the function is not differentiable at $x = 1$.

Problem 02: A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & when\ x < 0 \\ 1 + \sin x & when\ 0 \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \pi/2\right)^2 & when\ x \ge \dfrac{\pi}{2} \end{cases}$$

Discuss the differentiability at $x = 0$ and $x = \dfrac{\pi}{2}$.

Solution: Given that,

$$f(x) = \begin{cases} 1 & when\ x < 0 \\ 1 + \sin x & when\ 0 \le x < \dfrac{\pi}{2} \\ 2 + \left(x - \pi/2\right)^2 & when\ x \ge \dfrac{\pi}{2} \end{cases}$$

1st Part: For $x = 0$,

$$\begin{aligned} L.H.D &= \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} \\ &= \lim_{h \to 0} \frac{1 - (1 + \sin 0)}{-h} \\ &= \lim_{h \to 0} \frac{1-1}{-h} \\ &= \lim_{h \to 0} \frac{0}{-h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.D &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{1 + \sinh - (1 + \sin 0)}{h} \\ &= \lim_{h \to 0} \frac{1 + \sinh - 1}{h} \\ &= \lim_{h \to 0} \frac{\sinh}{h} \\ &= 1 \end{aligned}$$

Since $L.H.D \neq R.H.D$ does not exist. So the function is not differentiable at $x = 0$.

<u>2nd Part:</u> For $x = \pi/2$,

$$L.H.D = \lim_{h\to 0}\frac{f\left(\pi/2 - h\right) - f\left(\pi/2\right)}{-h}$$

$$= \lim_{h\to 0}\frac{1+\sin\left(\pi/2 - h\right) - \left\{2+\left(\pi/2 - \pi/2\right)^2\right\}}{-h}$$

$$= \lim_{h\to 0}\frac{1+\cos h - 2}{-h}$$

$$= \lim_{h\to 0}\frac{\cos h - 1}{-h}$$

$$= \lim_{h\to 0}\frac{\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots\cdots\right)-1}{-h}$$

$$= \lim_{h\to 0}\frac{-\frac{h^2}{2!}+\frac{h^4}{4!}-\cdots\cdots}{-h}$$

$$= \lim_{h\to 0}\left(\frac{h}{2!}-\frac{h^3}{4!}+\cdots\cdots\right)$$

$$= 0$$

$$R.H.D = \lim_{h\to 0}\frac{f\left(\pi/2 + h\right) - f\left(\pi/2\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left\{2+\left(\pi/2 + h - \pi/2\right)^2\right\} - \left\{2+\left(\pi/2 - \pi/2\right)^2\right\}}{h}$$

$$= \lim_{h\to 0}\frac{2+h^2-2}{h}$$

$$= \lim_{h\to 0}\frac{h^2}{h}$$

$$= \lim_{h\to 0} h$$

$$= 0$$

Since $L.H.D = R.H.D$ exists. So the function is differentiable at $x = \pi/2$.

HOMEWORK:

<u>Problem 01:</u> A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \ln x & when\ 0 < x \le 1 \\ 0 & when\ 1 < x \le 2 \\ 1+x^2 & when\ x > 2 \end{cases}$$

Discuss the differentiability at $x = 1$.

<u>Problem 02</u>: Discuss the differentiability of the function $f(x) = |x| + |x-1|$ at the point $x = 0$ and $x = 1$.

<u>Problem 03:</u> A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2 & when\ x \le 1 \\ x & when\ 1 < x \le 2 \\ \left(1/4\right)x^3 & when\ x > 2 \end{cases}$$

Discuss the differentiability at $x = 1$ and $x = 2$.

<u>Problem 04:</u> A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1+x & when\ x \le 0 \\ x & when\ 0 < x < 1 \\ 2-x & when\ 1 < x \le 2 \end{cases}$$

Discuss the differentiability at $x = 0$ and $x = 1$.

Problem 05: A function $f\left(x\right)$ is defined as follows:

$$f\left(x\right)=\begin{cases}0 & when\ 0\le x<3\\ 4 & when\ x=3\\ 5 & when\ 3<x\le 4\end{cases}$$

Discuss the differentiability at $x=3$.

Problem-06: Find the derivative of $y=x^n$ by first principle formula.

Solution: We have $f(x)=x^n$

$$\therefore\ f(x+h)=\left(x+h\right)^n$$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(x+h\right)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{x^n+nx^{n-1}h+\frac{n\left(n-1\right)}{2!}x^{n-2}h^2+\frac{n\left(n-1\right)\left(n-2\right)}{3!}x^{n-3}h^3+\cdots\cdots+h^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{nx^{n-1}h+\frac{n\left(n-1\right)}{2!}x^{n-2}h^2+\frac{n\left(n-1\right)\left(n-2\right)}{3!}x^{n-3}h^3+\cdots\cdots+h^n}{h}$$

$$=\lim_{h\to 0}\left\{nx^{n-1}+\frac{n\left(n-1\right)}{2!}x^{n-2}h+\frac{n\left(n-1\right)\left(n-2\right)}{3!}x^{n-3}h^2+\cdots\cdots+h^{n-1}\right\}$$

$$=nx^{n-1}+\frac{n\left(n-1\right)}{2!}x^{n-2}\cdot 0+\frac{n\left(n-1\right)\left(n-2\right)}{3!}x^{n-3}\cdot 0+\cdots\cdots+0$$

$$=nx^{n-1}.$$

Problem-07: Find the derivative of $y=a^x$ by first principle formula.

Solution: We have $f(x)=a^x$

$$\therefore\ f(x+h)=a^{x+h}$$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$=\lim_{h\to 0}\frac{a^x\cdot a^h-a^x}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{e^{\ln a^h}-1}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{e^{h\ln a}-1}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{1+h\ln a+\frac{(h\ln a)^2}{2!}+\frac{(h\ln a)^3}{3!}+\cdots\cdots-1}{h}$$

$$=a^x\cdot\lim_{h\to 0}\frac{h\ln a+\frac{(h\ln a)^2}{2!}+\frac{(h\ln a)^3}{3!}+\cdots\cdots}{h}$$

$$=a^x\cdot\lim_{h\to 0}\left\{\ln a+\frac{h(\ln a)^2}{2!}+\frac{h^2(\ln a)^3}{3!}+\cdots\cdots\right\}$$

$$=a^x\cdot\{\ln a+0+0+\cdots\cdots\}$$

$$=a^x\ln a.$$

Problem-08: Find the derivative of $y=e^x$ by first principle formula.

Solution: We have $f(x)=e^x$

$$\therefore\ f(x+h)=e^{x+h}$$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$=\lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}$$

$$= e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot \lim_{h\to 0} \frac{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots \cdots - 1}{h}$$

$$= e^x \cdot \lim_{h\to 0} \frac{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots \cdots}{h}$$

$$= e^x \cdot \lim_{h\to 0} \left\{1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots \cdots\right\}$$

$$= e^x \cdot \{1 + 0 + 0 + \cdots \cdots\}$$

$$= e^x .$$

Problem-09: Find the derivative of $y = \ln x$ by first principle formula.

Solution: we have $f(x) = \ln x$

$\therefore\ f(x+h) = \ln(x+h)$

By first principle formula we can write,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots \cdots}{h}$$

$$= \lim_{h\to 0} \left\{\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \cdots \cdots\right\}$$

$$= \frac{1}{x} - 0 + 0 - \cdots \cdots$$

$$=\frac{1}{x}.$$

Problem-10: Find the derivative of $y=\cos x$ by first principle formula.

Solution: We have $f(x)=\cos x$

$$\therefore\ f(x+h)=\cos(x+h)$$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\left\{\frac{2\sin\frac{2x+h}{2}\sin\frac{-h}{2}}{h}\right\}$$

$$=-\lim_{h\to 0}\left\{\frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\right\}$$

$$=-\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\left\{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\}$$

$$=-\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h/2\to 0}\left\{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\}$$

$$=-\sin\left(x+\frac{0}{2}\right)\cdot 1$$

$$=-\sin x$$

Problem-11: Find the derivative of $y=\sin ax$ by first principle formula.

Solution: We have $f(x)=\sin a x$

$$\therefore\ f(x+h)=\sin a(x+h)$$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin a(x+h)-\sin a\,x}{h}$$

$$=\lim_{h\to 0}\left\{\frac{2\cos\frac{2ax+ah}{2}\sin\frac{ah}{2}}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{2\cos\left(ax+\frac{ah}{2}\right)\sin\frac{ah}{2}}{h}\right\}$$

$$=\lim_{h\to 0}\cos\left(ax+\frac{ah}{2}\right)\cdot a\lim_{ah/2\to 0}\left\{\frac{2\sin\frac{ah}{2}}{\frac{ah}{2}}\right\}$$

$$=\cos(ax+0)\cdot a$$

$$=a\cos ax$$

Problem-12: Find the derivative of $y=x$ by first principle formula.

Solution: We have $f(x)=x$

$\therefore\ f(x+h)=(x+h)$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h}$$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=1$$

Problem-13: Find the derivative of $y=c$ by first principle formula.

Solution: We have $f(x)=c$

$\therefore\ f(x+h)=c$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{c-c}{h}$$

$$=0$$

Problem-14: Find the derivative of $y=\tan x$ by first principle formula.

Solution: We have $f(x)=\tan x$

$\therefore\ f(x+h)=\tan(x+h)$

By first principle formula we can write,

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\{\tan(x+h)-\tan x\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{\cos(x+h)\cos x}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right\}$$

$$=\lim_{h\to 0}\frac{1}{h}\left\{\frac{\sinh}{\cos(x+h)\cos x}\right\}$$

$$=\lim_{h\to 0}\frac{\sinh}{h}\cdot\lim_{h\to 0}\left\{\frac{1}{\cos(x+h)\cos x}\right\}$$

$$=1\cdot\left\{\frac{1}{\cos(x+0)\cos x}\right\}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

Derivatives of elementary functions:

1. $\frac{d}{dx}(c) = 0$, *where c is a constant.*

2. $\frac{d}{dx}(x) = 1.$

3. $\frac{d}{dx}(x^n) = nx^{n-1}.$

4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$

5. $\frac{d}{dx}(e^x) = e^x.$

6. $\frac{d}{dx}(a^x) = a^x \ln a.$

7. $\frac{d}{dx}(\ln x) = \frac{1}{x}.$

8. $\frac{d}{dx}(\sin x) = \cos x.$

9. $\frac{d}{dx}(\cos x) = -\sin x.$

10. $\frac{d}{dx}(\tan x) = \sec^2 x.$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x.$

12. $\frac{d}{dx}(\cot x) = -\cos ec^2 x.$

13. $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x.$

14. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$

15. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}.$

16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$

17. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}.$

18. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}.$

19. $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}.$

20. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$

21. $\frac{d}{dx}(u^v) = u^v \frac{d}{dx}(v \ln u).$

22. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

where u and v are functions of x.

Find the differential coefficient ($\frac{dy}{dx}$) of the following functions:

1. $y = 5x^8$

Sol : *Given that*, $y = 5x^8$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}(5x^8)$$

$$= 5\frac{d}{dx}(x^8)$$

$$= 5 \times 8x^{8-1} = 40x^7 \quad (Ans.)$$

2. $y = 3x^7 + 2x + 1$

Sol : *Given that*, $y = 3x^7 + 2x + 1$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}(3x^7 + 2x + 1)$$

$$= 3\frac{d}{dx}(x^7) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= 21x^6 + 2 + 0 = 21x^6 + 2 \quad (Ans.)$$

3. $y = 4\sin x - \cos x$

Sol: *Given that*, $y = 4\sin x - \cos x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(4\sin x - \cos x\right)$$
$$= 4\frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$$
$$= 4\cos x - (-\sin x)$$
$$= 4\cos x + \sin x \quad (Ans.)$$

4. $y = \sec^2 x - \tan^2 x$

Sol: *Given that*, $y = \sec^2 x - \tan^2 x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sec^2 x - \tan^2 x\right)$$
$$= \frac{d}{dx}\left(\sec^2 x\right) - \frac{d}{dx}\left(\tan^2 x\right)$$
$$= 2\sec x\frac{d}{dx}(\sec x) - 2\tan x\frac{d}{dx}(\tan x)$$
$$= 2\sec x(\sec x\tan x) - 2\tan x\left(\sec^2 x\right)$$
$$= 2\sec^2 x\tan x - 2\sec^2 x\tan x = 0 \quad (Ans.)$$

5. $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

Sol: *Given that*, $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln\left(x + \sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{d}{dx}\left(\sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot \frac{d}{dx}\left(x^2 + a^2\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}} \quad (Ans.)$$

6. $y = \ln(\sec x + \tan x)$

Sol: *Given that*, $y = \ln(\sec x + \tan x)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln(\sec x + \tan x)\right\}$$
$$= \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\left(\sec x\tan x + \sec^2 x\right)}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$
$$(Ans.)$$

7. $y = e^{ax^2 + bx + c}$

Sol: *Given that*, $y = e^{ax^2 + bx + c}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{ax^2 + bx + c}\right)$$
$$= e^{ax^2 + bx + c} \cdot \frac{d}{dx}\left(ax^2 + bx + c\right)$$
$$= e^{ax^2 + bx + c}(2ax + b + 0)$$
$$= (2ax + b)e^{ax^2 + bx + c} \quad (Ans.)$$

8. $y = e^{\sqrt{\cot x}}$

Sol: *Given that*, $y = e^{\sqrt{\cot x}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{\cot x}}\right)$$
$$= e^{\sqrt{\cot x}} \cdot \frac{d}{dx}\left(\sqrt{\cot x}\right)$$
$$= e^{\sqrt{\cot x}} \cdot \frac{1}{2\sqrt{\cot x}} \cdot \frac{d}{dx}(\cot x)$$
$$= -\frac{e^{\sqrt{\cot x}}\cos ec^2 x}{2\sqrt{\cot x}} \quad (Ans.)$$

9. $y=\sqrt{x^3-2x+5}$

Sol: *Given that*, $y=\sqrt{x^3-2x+5}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x^3-2x+5}\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\frac{d}{dx}\left(x^3-2x+5\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\left(3x^2-2+0\right)$$
$$=\frac{3x^2-2}{2\sqrt{x^3-2x+5}}$$

(*Ans.*)

10. $y=\tan\ln\sin\left(e^{x^2}\right)$

Sol: *Given that*, $y=\tan\left(\ln\sin e^{x^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\tan\left(\ln\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{d}{dx}\left\{\ln\left(\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\frac{d}{dx}\left\{\sin\left(e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\cos\left(e^{x^2}\right).\frac{d}{dx}\left(e^{x^2}\right)$$
$$=\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right).e^{x^2}\frac{d}{dx}\left(x^2\right)$$
$$=2xe^{x^2}\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right)$$

(*Ans.*)

11. $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Sol: *Given that*, $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\cos^{-1}\left(e^{\cot^{-1}x}\right)\right\}$$
$$=-\frac{1}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(e^{\cot^{-1}x}\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(\cot^{-1}x\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}\left(-\frac{1}{1+x^2}\right)$$
$$=\frac{e^{\cot^{-1}x}}{\left(1+x^2\right)\sqrt{1-e^{2\cot^{-1}x}}}\quad(Ans.)$$

12. $y=e^{\sin^{-1}x}+\tan^{-1}x$

Sol: *Given that*, $y=e^{\sin^{-1}x}+\tan^{-1}x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}+\tan^{-1}x\right)$$
$$=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)+\frac{d}{dx}\left(\tan^{-1}x\right)$$
$$=e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{1}{1+x^2}$$
$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}+\frac{1}{1+x^2}$$

(*Ans.*)

13. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Sol: *Given that*, $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

put, $x=\sin\theta \quad \therefore\ \theta=\sin^{-1}x$

Now, $y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$
$$=\tan^{-1}.\tan\theta=\theta=\sin^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}\quad(Ans.)$$

14. $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Sol: *Given that*, $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

put, $x=\tan\theta \quad \therefore\ \theta=\tan^{-1}x$

Now, $y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$=\cos^{-1}.\cos 2\theta=2\theta$$
$$=2\tan^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)=\frac{2}{1+x^2}\quad(Ans.)$$

15. $y=\dfrac{\cos x}{1+\sin x}$

Sol: *Given that*, $y=\dfrac{\cos x}{1+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$
$$=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$
$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$
$$=\frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$
$$=\frac{-\sin x-1}{(1+\sin x)^2}=-\frac{1}{1+\sin x}\ (Ans.)$$

16. $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Sol: *Given that*, $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$
$$=\frac{(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)-(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)}{(\cos x+\sin x)^2}$$
$$=\frac{(\cos x+\sin x)(-\sin x-\cos x)-(\cos x-\sin x)(-\sin x+\cos x)}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$
$$=\frac{-(\cos x+\sin x)^2-(\cos x-\sin x)^2}{1+\sin 2x}$$
$$=\frac{-(1+\sin 2x)-(1-\sin 2x)}{1+\sin 2x}=-\frac{2}{1+\sin 2x}\ (Ans.)$$

17. $y=\tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

Sol: *Given that*, $y=\tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

put, $x=\tan\theta\quad\therefore\ \theta=\tan^{-1}x$

Now, $y=\tan^{-1}\left(\dfrac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$
$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)=\tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)=\tan^{-1}.\tan\theta/2=\theta/2=1/2\tan^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(1/2\tan^{-1}x\right)=\frac{1}{2(1+x^2)}\quad(Ans.)$$

18. $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Sol : *Given that*, $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)\right\}$$

$$=\frac{1}{\sqrt{1-\left(\dfrac{a+b\cos x}{b+a\cos x}\right)^2}}\cdot\frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$=\frac{(b+a\cos x)}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}\cdot\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$=\frac{1}{\sqrt{b^2-a^2+a^2\cos^2 x-b^2\cos^2 x}}\cdot\frac{-b^2\sin x-ab\sin x\cos x+a^2\sin x+ab\sin x\cos x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{1-\cos^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{\sin^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sin x}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}=\frac{-\sqrt{b^2-a^2}}{b+a\cos x}\quad (Ans.)$$

19. $y=x\sin x$

Sol : *Given that*, $y=x\sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}(x\sin x)$$

$$=x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)$$

$$=x\cos x+\sin x$$

$(Ans.)$

20. $y=e^{ax}\cos bx$

Sol : *Given that*, $y=e^{ax}\cos bx$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{ax}\cos bx\right)$$

$$=e^{ax}\frac{d}{dx}(\cos bx)+\cos bx\frac{d}{dx}\left(e^{ax}\right)$$

$$=e^{ax}(-b\sin bx)+\cos bx\left(ae^{ax}\right)$$

$$=ae^{ax}\cos bx-be^{ax}\sin bx$$

$(Ans.)$

21. $y = x^2 \cot^{-1} x$

Sol : *Given that*, $y = x^2 \cot^{-1} x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \cot^{-1} x\right)$$
$$= x^2 \frac{d}{dx}\left(\cot^{-1} x\right) + \cot^{-1} x \frac{d}{dx}\left(x^2\right)$$
$$= x^2 \left(\frac{-1}{1+x^2}\right) + \cot^{-1} x(2x)$$
$$= 2x\cot^{-1} x - \frac{x^2}{1+x^2}$$

$(Ans.)$

22. $y = x^3 \ln x$

Sol : *Given that*, $y = x^3 \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \ln x\right)$$
$$= x^3 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(x^3\right)$$
$$= x^3 . \frac{1}{x} + \ln x\left(2x^2\right)$$
$$= x^2 + 2x^2 \ln x$$

$(Ans.)$

23. $y = xe^x \sin x$

Sol : *Given that*, $y = xe^x \sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^x \sin x\right)$$
$$= xe^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(xe^x\right)$$
$$= xe^x \cos x + \sin x\left\{x\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x)\right\}$$
$$= xe^x \cos x + \sin x\left(xe^x + e^x\right)$$
$$= xe^x \cos x + xe^x \sin x + e^x \sin x$$

$(Ans.)$

24. $y = \sqrt{x}e^x \sec x$

Sol : *Given that*, $y = \sqrt{x}e^x \sec x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}e^x \sec x\right)$$
$$= \sqrt{x}e^x \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}\left(\sqrt{x}e^x\right)$$
$$= \sqrt{x}e^x \sec x \tan x + \sec x\left(\sqrt{x}e^x + \frac{e^x}{2\sqrt{x}}\right)$$

$(Ans.)$

25. $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Sol : *Given that*, $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}\right\}$$
$$= \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x\right\} + \frac{d}{dx}\left(e^{\sqrt{1+x^2}}\right)$$
$$= \left(x^2+1\right).\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x.(2x) + e^{\sqrt{1+x^2}}.\frac{1}{2\sqrt{1+x^2}}.2x$$
$$= \frac{x^2+1}{\sqrt{1-x^2}} + 2x\sin^{-1} x + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

$(Ans.)$

26. $y = e^{\sin x}\sin\left(a^x\right)$

Sol : *Given that*, $y = e^{\sin x}\sin\left(a^x\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{e^{\sin x}\sin\left(a^x\right)\right\}$$
$$= e^{\sin x}\frac{d}{dx}\left\{\sin\left(a^x\right)\right\} + \sin\left(a^x\right)\frac{d}{dx}\left(e^{\sin x}\right)$$
$$= e^{\sin x}.\cos\left(a^x\right).\frac{d}{dx}\left(a^x\right) + \sin\left(a^x\right).e^{\sin x}.\cos x$$
$$= e^{\sin x}.\cos\left(a^x\right).a^x \ln a + \sin\left(a^x\right).e^{\sin x}.\cos x$$

$(Ans.)$

Homework:-Find $\frac{dy}{dx}$ of the following functions:

1. $y=\ln\left(\sqrt{x-a}+\sqrt{x-b}\right)$ Ans: $\frac{1}{2\sqrt{(x-a)(x-b)}}$
2. $y=\ln\left(x+\sqrt{x^2\pm b^2}\right)$ Ans: $\frac{1}{\sqrt{x^2\pm b^2}}$
3. $y=\cos(\ln x)+\ln(\tan x)$ Ans: $2\cos ec2x-\frac{\sin(\ln x)}{x}$
4. $y=e^{ax}\sin^m rx$ Ans: $e^{ax}\sin^m rx(a+mr\cot rx)$
5. $y=x\sec x\ln\left(xe^x\right)$ Ans: $\sec x\left\{(x+1)+(x\tan x+1)\ln\left(xe^x\right)\right\}$
6. $y=\sin^{-1}x^2-xe^{x^2}$ Ans: $\frac{2x}{\sqrt{1-x^4}}-\left(2x^2+1\right)e^{x^2}$
7. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ Ans: $\frac{1}{\sqrt{1-x^2}}$
8. $y=\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$ Ans: $\frac{2}{\sqrt{x}(1+4x)}$
9. $y=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ Ans: $-\frac{1}{2}$
10. $y=\sin^{-1}\left(\frac{2x^{-1}}{x+x^{-1}}\right)$ Ans: $\frac{2}{\sqrt{x}(1+4x)}$

Logarithmic differentiation: If we have functions that are composed of products, quotients and powers, to differentiate such functions it would be convenient first to take logarithm of the function and then differentiate. Such a technique is called the logarithmic differentiation.

1. $y=(\sin x)^{\ln x}$

Sol : *Given that,* $y=(\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$
$$=(\sin x)^{\ln x}\frac{d}{dx}\{\ln x\ln(\sin x)\}$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{d}{dx}\{\ln(\sin x)\}+\ln(\sin x).\frac{d}{dx}(\ln x)\right]$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{1}{\sin x}.\cos x+\ln(\sin x).\frac{1}{x}\right]$$
$$=(\sin x)^{\ln x}\left[\cot x\ln x+\frac{\ln(\sin x)}{x}\right]$$

$(Ans.)$

2. $y=x^{1+x+x^2}$

Sol : *Given that,* $y=x^{1+x+x^2}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{1+x+x^2}\right)$$
$$=x^{1+x+x^2}\frac{d}{dx}\left\{\left(1+x+x^2\right)\ln x\right\}$$
$$=x^{1+x+x^2}\left[\ln x.\frac{d}{dx}\left(1+x+x^2\right)+\left(1+x+x^2\right).\frac{d}{dx}(\ln x)\right]$$
$$=x^{1+x+x^2}\left[\ln x.(0+1+2x)+\left(1+x+x^2\right).\frac{1}{x}\right]$$
$$=x^{1+x+x^2}\left[(2x+1)\ln x+\frac{\left(1+x+x^2\right)}{x}\right]$$

$(Ans.)$

3. $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Sol : *Given that*, $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\tan^{-1}x\right)^{\sin x+\cos x}\right\}$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\frac{d}{dx}\left\{\left(\sin x+\cos x\right).\ln\left(\tan^{-1}x\right)\right\}$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\left(\sin x+\cos x\right)\frac{d}{dx}\left\{\ln\left(\tan^{-1}x\right)\right\}+\ln\left(\tan^{-1}x\right).\frac{d}{dx}\left(\sin x+\cos x\right)\right]$$

$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\frac{\left(\sin x+\cos x\right)}{\tan^{-1}x}.\frac{1}{\left(1+x^2\right)}+\ln\left(\tan^{-1}x\right).\left(\cos x-\sin x\right)\right]$$

$(Ans.)$

4. $y=x^x+\left(\sin x\right)^{\ln x}$

Sol : *Given that*, $y=x^x+\left(\sin x\right)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{x^x+\left(\sin x\right)^{\ln x}\right\}$$

$$=\frac{d}{dx}\left(x^x\right)+\frac{d}{dx}\left\{\left(\sin x\right)^{\ln x}\right\}$$

$$=x^x\frac{d}{dx}\left(x\ln x\right)+\left(\sin x\right)^{\ln x}\frac{d}{dx}\left\{\ln x\ln\left(\sin x\right)\right\}$$

$$=x^x\left(x.\frac{1}{x}+\ln x\right)+\left(\sin x\right)^{\ln x}\left\{\ln x.\frac{1}{\sin x}.\cos x+\frac{\ln\left(\sin x\right)}{x}\right\}$$

$$=x^x\left(1+\ln x\right)+\left(\sin x\right)^{\ln x}\left\{\ln x.\cot x+\frac{\ln\left(\sin x\right)}{x}\right\}\quad Ans.$$

5. $y=\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}$

Sol : *Given that*, $y=\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}\right\}$$

$$=\frac{d}{dx}\left\{\left(\sin x\right)^{\cos x}\right\}+\frac{d}{dx}\left\{\left(\cos x\right)^{\sin x}\right\}$$

$$=\left(\sin x\right)^{\cos x}\frac{d}{dx}\left\{\cos x\ln\left(\sin x\right)\right\}+\left(\cos x\right)^{\sin x}\frac{d}{dx}\left\{\sin x\ln\left(\cos x\right)\right\}$$

$$=\left(\sin x\right)^{\cos x}\left[\cos x.\frac{1}{\sin x}.\cos x-\sin x\ln\left(\sin x\right)\right]+\left(\cos x\right)^{\sin x}\left[\sin x.\frac{1}{\cos x}.\left(-\sin x\right)+\cos x\ln\left(\cos x\right)\right]$$

$$=\left(\sin x\right)^{\cos x}\left[\cos x.\cot x-\sin x\ln\left(\sin x\right)\right]+\left(\cos x\right)^{\sin x}\left[\cos x\ln\left(\cos x\right)-\sin x.\tan x\right]\quad Ans.$$

6. $y = x^{\cos^{-1}x} - \sin x \ln x$

Sol: *Given that*, $y = x^{\cos^{-1}x} - \sin x \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\cos^{-1}x} - \sin x \ln x\right)$$
$$= \frac{d}{dx}\left(x^{\cos^{-1}x}\right) - \frac{d}{dx}(\sin x \ln x)$$
$$= x^{\cos^{-1}x}\frac{d}{dx}\left(\cos^{-1}x \ln x\right) - \left(\frac{\sin x}{x} + \cos x \ln x\right)$$
$$= x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] - \left(\frac{\sin x}{x} + \cos x \ln x\right) \text{ Ans.}$$

7. $y = \left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}$

Sol: *Given that*, $y = \left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}\right\}$$
$$= \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x}\right\} + \frac{d}{dx}\left\{(2-\sin x)^{\ln x}\right\}$$
$$= \left(1+x^2\right)^{\tan x}\frac{d}{dx}\left\{\tan x \ln\left(1+x^2\right)\right\} + (2-\sin x)^{\ln x}\frac{d}{dx}\left\{\ln x \ln(2-\sin x)\right\}$$
$$= \left(1+x^2\right)^{\tan x}\left[\frac{2x\tan x}{1+x^2} + \sec^2 x \ln\left(1+x^2\right)\right] + (2-\sin x)^{\ln x}\left[\frac{\ln(2-\sin x)}{x} - \frac{\cos x \ln x}{(2-\sin x)}\right] \text{ Ans.}$$

Homework: Find $\frac{dy}{dx}$ of the following functions:

1. $y = x^{\sin^{-1}x}$ — Ans: $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right]$
2. $y = (\sin x)^{\cos^{-1}x}$ — Ans: $(\sin x)^{\cos^{-1}x}\left[\cot x \cos^{-1}x - \frac{\ln \sin x}{\sqrt{1-x^2}}\right]$
3. $y = x^{x^x}$ — Ans: $x^{x^x}x^x\left[1 + \ln x + \frac{1}{x}\right]$
4. $y = x^{\cos^{-1}x} + (\sin x)^{\ln x}$ — Ans: $x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] + (\sin x)^{\ln x}\left[\ln x \cot x + \frac{\ln \sin x}{x}\right]$
5. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ — Ans: $(tanx)^{\cot x} \operatorname{cos}ec^2 x(1 - \ln \tan x) + (\cot x)^{\tan x} \operatorname{s}ec^2 x(\ln \cot x - 1)$
6. $y = x^{\ln x} + x^{\sin^{-1}x}$ — Ans: $\frac{2x^{\ln x}\ln x}{x} + x^{\cos^{-1}x}\left(\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right)$

Parametric Equation: If in the equation of a curve $y=f(x)$, x and y are expressed in terms of a third variable known as parameter i.e, $x=\varphi(t)$, $y=\psi(t)$ then the equations are called a parametric equation.

1. $x=a(t+\sin t)$, $y=a(1-\cos t)$

$sol:$ *Given that,*

$$x=a(t+\sin t)\cdots\cdots(1)$$

$$and\quad y=a(1-\cos t)\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to t we get,*

$$\frac{dx}{dt}=a(1+\cos t)$$

$$and\quad \frac{dy}{dt}=a\sin t$$

$$Now,\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1+\cos t)}=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\tan\frac{t}{2}\quad(Ans.)$$

2. $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$

$sol:$ *Given that,*

$$x=a(\cos t+t\sin t)\cdots\cdots(1)$$

$$and\quad y=a(\sin t-t\cos t)\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to t we get,*

$$\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)=at\cos t$$

$$and\quad \frac{dy}{dt}=a(\cos t+t\sin t-\cos t)=at\sin t$$

$$Now,\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{at\sin t}{at\cos t}=\tan t\quad(Ans.)$$

3. $x=a\left(\cos t+\ln\tan\frac{t}{2}\right)$, $y=a\sin t$

$sol:$ *Given that,*

$$x=a\left(\cos t+\ln\tan\frac{t}{2}\right)\cdots\cdots(1)$$

$$and\quad y=a\sin t\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to t we get,*

$$\frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$=a\left(-\sin t+\frac{1}{2}\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}\right)$$

$$=a\left(-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$=a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$and\quad \frac{dy}{dt}=a\cos t$$

$$Now,\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}=\tan t\quad(Ans.)$$

4. $x=t-\sqrt{1-t^2}$, $y=e^{\sin^{-1}t}$

$sol:$ *Given that,*

$$x=t-\sqrt{1-t^2}\cdots\cdots(1)$$

$$and\quad y=e^{\sin^{-1}t}\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to t we get,*

$$\frac{dx}{dt}=1-\frac{1}{2\sqrt{1-t^2}}.(-2t)=1+\frac{t}{\sqrt{1-t^2}}=\frac{t+\sqrt{1-t^2}}{\sqrt{1-t^2}}$$

$$and\quad \frac{dy}{dt}=e^{\sin^{-1}t}.\frac{1}{\sqrt{1-t^2}}=\frac{e^{\sin^{-1}t}}{\sqrt{1-t^2}}$$

$$Now,\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{e^{\sin^{-1}t}}{\sqrt{1-t^2}}.\frac{\sqrt{1-t^2}}{t+\sqrt{1-t^2}}=\frac{e^{\sin^{-1}t}}{t+\sqrt{1-t^2}}\quad(Ans.)$$

5. *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

sol : *Let*, $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\ ;\left[\begin{array}{l} putting\ \ x=\tan\theta \\ \therefore\ \ \theta=\tan^{-1}x \end{array}\right]$$

$$=\tan^{-1}.\tan 2\theta$$

$$=2\theta$$

$$=2\tan^{-1}x\ \cdots\cdots\ (1)$$

and $z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\ ;\left[\begin{array}{l} putting\ \ x=\tan\theta \\ \therefore\ \ \theta=\tan^{-1}x \end{array}\right]$$

$$=\sin^{-1}.\sin 2\theta$$

$$=2\theta$$

$$=2\tan^{-1}x\ \cdots\cdots\ (2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=\frac{2}{1+x^2}\qquad and\ \ \frac{dz}{dx}=\frac{2}{1+x^2}$$

Now, $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$=1\quad (Ans.)$$

6. *Differentiate* $(\sin x)^x$ *with respect to* $x^{\sin x}$.

sol : *Let*, $y=(\sin x)^x\cdots\cdots(1)$

and $z=x^{\sin x}\cdots\cdots(2)$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=(\sin x)^x\frac{d}{dx}(x\ln\sin x)$$

$$=(\sin x)^x\left(x.\frac{\cos x}{\sin x}+\ln\sin x\right)$$

$$=(\sin x)^x(x\cot x+\ln\sin x)$$

and $\frac{dz}{dx}=x^{\sin x}\frac{d}{dx}(\sin x\ln x)$

$$=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)$$

Now, $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{(\sin x)^x(x\cot x+\ln\sin x)}{x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)}\quad (Ans.)$$

7. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\sin^{-1}x$.

sol : *Let*, $y=x^{\sin^{-1}x}\ \cdots\cdots(1)$

and $z=\sin^{-1}x\cdots\cdots(2)$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=x^{\sin^{-1}x}\frac{d}{dx}\left(\sin^{-1}x\ln x\right)\ ;\left[\because\ \frac{d}{dx}(u^v)=u^v\frac{d}{dx}(v\ln u)\right]$$

$$=x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)$$

and $\frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$

Now, $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$

$$=x^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}.\sin^{-1}x}{x}+\ln x\right)\quad (Ans.)$$

8. *Differentiate* $\tan^{-1}\left(\dfrac{\sqrt{1-x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$.

sol : *Let*, $y=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}-1}{x}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}-1}{\sin\theta}\right)\;;\left[\begin{array}{l}\text{putting}\quad x=\sin\theta\\ \qquad\qquad\therefore\;\theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sqrt{\cos^2\theta}-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left\{-\frac{(1-\cos\theta)}{\sin\theta}\right\}$$

$$=\tan^{-1}\left\{-\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\tan\frac{\theta}{2}\right\}$$

$$=\tan^{-1}\left\{\tan\left(\pi-\frac{\theta}{2}\right)\right\}$$

$$=\pi-\frac{\theta}{2}$$

$$=\pi-\frac{1}{2}\sin^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}x\cdots\cdots(2)$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}\quad\text{and}\quad\frac{dz}{dx}=\frac{1}{1+x^2}$$

Now, $\dfrac{dy}{dz}=\dfrac{dy/dx}{dz/dx}$

$$=\frac{-\frac{1}{2\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$=-\frac{1+x^2}{2\sqrt{1-x^2}}\quad(Ans.)$$

9. *Differentiate* $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ *with respect to* $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

sol : *Let*, $y=\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

$$=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)\;;\left[\begin{array}{l}\text{putting}\quad x=\cos\theta\\ \qquad\qquad\therefore\;\theta=\cos^{-1}x\end{array}\right]$$

$$=\sec^{-1}\left(\frac{1}{\cos2\theta}\right)$$

$$=\sec^{-1}(\sec2\theta)$$

$$=2\theta$$

$$=2\cos^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)\;;\left[\begin{array}{l}\text{putting}\quad x=\sin\theta\\ \qquad\qquad\therefore\;\theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$=\tan^{-1}.\tan\theta$$

$$=\theta$$

$$=\sin^{-1}x\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}\quad\text{and}\quad\frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$$

Now, $\dfrac{dy}{dz}=\dfrac{dy/dx}{dz/dx}$

$$=\frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$=-2\quad(Ans.)$$

Homework:-

1. *Differentiate* $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$. Ans: $\dfrac{1}{2}$

2. *Differentiate* $e^{\sin^{-1}x}$ *with respect to* $\cos 3x$. Ans: $-\dfrac{e^{\sin^{-1}x}}{3\sqrt{1-x^2}.\sin 3x}$

3. *Differentiate* $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *with respect to* $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$. Ans: 1

4. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\ln x$. Ans: $x^{\sin^{-1}x}\left(\sin^{-1}x+\dfrac{x\ln x}{\sqrt{1-x^2}}\right)$

Theorem-01: Prove that a differentiable function is always continuous but the converse is not always true.

Proof: Let the function $f(x)$ be differentiable at $x=a$ i.e. $f'(a)$ exists, so that by the definition of differentiability we have,

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists and finite quantity.}$$

$$\text{i.e. } f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Now $\lim\limits_{x\to a}\left[f(x)-f(a)\right]=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\times(x-a)$

$or,\ \lim\limits_{x\to a}f(x)-\lim\limits_{x\to a}f(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\times\lim\limits_{x\to a}(x-a)$

$or,\ \lim\limits_{x\to a}f(x)-f(a)=f'(a)\times 0$

$or,\ \lim\limits_{x\to a}f(x)-f(a)=0$

$\therefore \lim\limits_{x\to a}f(x)=f(a)$

So by the definition of continuity we can say that the function $f(x)$ is continuous at the point $x=a$.

Again conversely, if the function $f(x)$ is continuous at a point, then it may not be differentiable at that point. As for example, we will show that the function $f(x)=|x|$ is continuous at the point $x=0$ but it is not differentiable at this point.

Continuity test: We have $f(x)=|x|$

$$i.e.\ f(x)=\begin{cases} x & if\ \ x\geq 0 \\ -x & if\ \ x<0 \end{cases}$$

$$L.H.L=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} -(0-h)=\lim_{h\to 0}\ h=0$$

$$R.H.L=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}(0+h)=\lim_{h\to 0}\ h=0$$

Also the functional value at $x=0$ is $f(0)=0$.

Since $L.H.L=R.H.L=f(0)$ so $f(x)$ is continuous at $x=0$.

Differentiability test: We have $f(x)=|x|$

$$i.e.\ f(x)=\begin{cases} x & if\ \ x\geq 0 \\ -x & if\ \ x<0 \end{cases}$$

$$L.H.D=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{-(0-h)-0}{-h}=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}\ (-1)=-1$$

$$R.H.D=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h)-0}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}\ (1)=1$$

Since $L.H.D=R.H.D$ so $f(x)$ is not differentiable at $x=0$.

Hence, a differentiable function is always continuous but the converse is not always true. **(Proved)**

Problem-01: If $f(x)=\begin{cases} x\sin\frac{1}{x} & if\ \ x\neq 0 \\ 0 & if\ \ x=0 \end{cases}$, then show that $f(x)$ is continuous at $x=0$ but not differentiable .

Solution: The given function is $f(x)=\begin{cases} x\sin\frac{1}{x} & if\ \ x\neq 0 \\ 0 & if\ \ x=0 \end{cases}$

Continuity test:

$$L.H.L=\lim_{h\to 0}(0-h)\sin\left(\frac{1}{0-h}\right)=\lim_{h\to 0}(h)\times\lim_{h\to 0}\sin\left(\frac{1}{h}\right)=0\times\left[a\ number\ in\ the\ interval\ [-1,1]\right]=0$$

$$R.H.L=\lim_{h\to 0}(0+h)\sin\left(\frac{1}{0+h}\right)=\lim_{h\to 0}(h)\times\lim_{h\to 0}\sin\left(\frac{1}{h}\right)=0\times\left[a\ number\ in\ the\ interval\ [-1,1]\right]=0$$

Also the functional value at $x=0$ is $f(0)=0$.

Since $L.H.L = R.H.L = f(0)$ so $f(x)$ is continuous at $x = 0$.

Differentiability test:

$$L.H.D = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{(0-h)\sin\left(\frac{1}{0-h}\right)-0}{-h} = -\lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

Which does not exist but oscillates between -1 and 1 for all values of h except $h = 0$.

$$L.H.D = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(0+h)\sin\left(\frac{1}{0+h}\right)-0}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

Which does not exist but oscillates between -1 and 1 for all values of h except $h = 0$.

So the function does not differentiable at the point $x = 0$ i.e. $f'(0)$ does not exist.

Thus the function $f(x)$ is continuous at $x = 0$ but not differentiable. **(Showed)**