Lecture Sheet

On

Theory of Equation

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Polynomial: A polynomial is an expression consisting of variables and coefficients that involves only the operations of addition, subtraction, multiplication and non-negative exponent of variables. A polynomial in variable *x* of the *n*-th degree is defined as,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots\cdots\cdots\cdots\cdots + a_n$$

where $a_0,\ a_1,\ \cdots,\ a_n$ are independent of *x* and $a_0 \neq 0$.

An equation consists of a polynomial is called a polynomial equation. A polynomial equation in variable *x* of the *n*-th degree is,

$$a_0 x^n + a_1 x^{n-1} + \cdots\cdots\cdots\cdots\cdots + a_n = 0.$$

Example: 1). $3x+5=0$ is a linear equation with single variable *x*.

2). $ax^2+bx+c=0$ is a quadratic equation with single variable *x*.

Note:

1. If $a_0 \neq 0$, then $f(x)$ is called a polynomial of order (or degree) *n*.
2. If $a_0 = 1$, then $f(x)$ is called monic polynomial of order *n*.
3. If $n=0$, then $f(x)$ is called a polynomial of order zero that means a constant polynomial.
4. If $n=1$, then $f(x)$ is called a polynomial of order one (1) that means a linear polynomial.
5. If $n=2$, then $f(x)$ is called a polynomial of order two means polynomial of degree 2 or Quadratic polynomial. The graph of a quadratic polynomial is a parabola.
6. If $n=3$, then $f(x)$ is called a polynomial of order three means polynomial of degree 3 or Cubic polynomial.
7. If $n=4$, then $f(x)$ is called a polynomial of order four means polynomial of degree 4 or by-quadratic polynomial.
8. If $f(x)=0$, then it is called zero polynomial with explicitly undefined degree. The graph of a zero polynomial is the *x*-axis.

Polynomials can be classified by the number of terms with nonzero coefficients such as a one-term polynomial is called a monomial; a two-term polynomial is called a binomial; and a three-term polynomial is called a trinomial. The term "quadrinomial" is occasionally used for a four-term polynomial. A polynomial in one variable is called a univariate polynomial; a polynomial in more than one variable is called a multivariate polynomial. A polynomial with two variables is called a bivariate polynomial.

Remainder Theorem: If $f(x)$ is a polynomial, then $f(h)$ is the remainder when $f(x)$ is divided by $(x-h)$.

This follows on substituting *h* for *x* in the identity,

$$f(x) = (x-h)Q + R$$

where Q and R are respectively the quotient and remainder in the division of $f(x)$ by $(x-h)$ and R is independent of *x*. If $f(h)=0$, then $(x-h)$ is a factor of $f(x)$.

Roots of Equations: Consider an equation of the type $f(x)=0$, where $f(x)$ is a polynomial. If $f(a)=0$ for $x=a$, then a is called a root of $f(x)=0$.

The general equation of the *n*-th degree is written as,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots\cdots\cdots\cdots + p_n = 0.$$

Since it is an *n*-th degree equation so it has at least one root. This is the fundamental theorem of algebra.

Theorem-01: Proved that every equation of the *n*-th degree has exactly *n* roots.

Proof: Let $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots\cdots\cdots\cdots + p_n$.

By the fundamental theorem of algebra, $f(x)=0$ has at least one root. Let α be a root of $f(x)=0$. Then by the remainder theorem, $f(x)$ is divisible by $(x-\alpha)$; we may therefore assume that,

$$f(x) = (x-\alpha)\left(x^{n-1} + q_1 x^{n-2} + q_2 x^{n-3} + \cdots\cdots\cdots\cdots + q_{n-1}\right)$$

$$or,\ f(x) = (x-\alpha)\phi(x) \qquad \cdots(1)$$

where, $\phi(x) = x^{n-1} + q_1 x^{n-2} + q_2 x^{n-3} + \cdots\cdots\cdots\cdots + q_{n-1}$.

Again let β be a root of $\phi(x)=0$; as before, $\phi(x)$ is divisible by $(x-\beta)$; and we may assume that,

$$\phi(x) = (x-\beta)\left(x^{n-2} + r_1 x^{n-3} + r_2 x^{n-4} + \cdots\cdots\cdots\cdots + r_{n-2}\right) \qquad \cdots(2)$$

From (1) and (2), we get

$$f(x) = (x-\alpha)(x-\beta)\left(x^{n-2} + r_1 x^{n-3} + r_2 x^{n-4} + \cdots\cdots\cdots\cdots + r_{n-2}\right).$$

Proceeding in this way, we can show that

$$f(x)=(x-\alpha)(x-\beta)\cdots(x-\lambda)$$

where there are *n* linear factors on the right.

Hence, $f(x)=0$ has *n* roots $\alpha,\ \beta,\ \gamma,\ \cdots,\ \lambda$ and no others. **(Proved)**

Imaginary Roots: Let the coefficients of $f(x)$ be real. Then, if $\alpha+i\beta$ is a root, so $\alpha-i\beta$ is also a root. Therefore $f(x)$ is divisible by $(x-\alpha-i\beta)(x-\alpha+i\beta)$ that is, by $(x-\alpha)^2+\beta^2$.

Thus a polynomial in *x* with real coefficients can be resolved into factors which are linear or quadratic functions of *x* with real coefficients.

Multiple Roots: If $f(x)=(x-\alpha)^r.\phi(x)$ where $\phi(x)$ is not divisible by $(x-\alpha)$, then α is called an *r*-multiple root of $f(x)=0$.

Relation between the Roots and Coefficients of an Equation: Consider a polynomial equation in *x* of the *n*-th degree,

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad ;where\, a_0\neq 0 \qquad \cdots(1)$$

Let $\alpha_1,\ \alpha_2,\cdots,\alpha_n$ be the roots of equation (1), so

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$$

$$=a_0\left[x^n-\left(\sum\alpha_1\right)x^{n-1}+\left(\sum\alpha_1\alpha_2\right)x^{n-2}-\left(\sum\alpha_1\alpha_2\alpha_3\right)x^{n-3}+\cdots+(-1)^n\alpha_1\alpha_2\alpha_3\cdots\alpha_n\right] \qquad \cdots(2)$$

Equating the coefficients of the terms having same power, we get

$$\left.\begin{array}{c} -a_0\sum\alpha_1=a_1 \\ a_0\sum\alpha_1\alpha_2=a_2 \\ -a_0\sum\alpha_1\alpha_2\alpha_3=a_3 \\ \cdots\cdots\cdots\cdots \\ (-1)^n a_0\alpha_1\alpha_2\cdots\alpha_n=a_n \end{array}\right\} \qquad \cdots(3)$$

$$or,\quad \left.\begin{aligned} \sum \alpha_1 &= -\frac{a_1}{a_0} \\ \sum \alpha_1\alpha_2 &= \frac{a_2}{a_0} \\ \sum \alpha_1\alpha_2\alpha_3 &= -\frac{a_3}{a_0} \\ &\cdots\cdots\cdots\cdots \\ \alpha_1\alpha_2\cdots\alpha_n &= (-1)^n\frac{a_n}{a_0} \end{aligned}\right\}.$$

These are the required relations between the roots and coefficients of the equation.

Example: If $\alpha_1,\ \alpha_2,\ \alpha_3$ are the roots of $2x^3+x^2-2x-1=0$, then

$$\alpha_1+\alpha_2+\alpha_3=-\frac{1}{2},\quad \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1=\frac{(-2)}{2}=-1, \text{ and } \alpha_1\alpha_2\alpha_3=-\frac{(-1)}{2}=\frac{1}{2}.$$

Transformations of Equations: Let $\alpha,\ \beta,\ \gamma,\ \cdots$ be the roots of $f(x)=0$, and suppose that we require the equation whose roots are $\phi(\alpha),\ \phi(\beta),\ \phi(\gamma),\ \cdots$ where $\phi(x)$ is a given function of x.

Let $y=\phi(x)$ and suppose that from this equation we can find x as a single-valued function of y, which we denote by $x=\phi^{-1}(y)$. Transforming the equation $f(x)=0$ by the substitution $x=\phi^{-1}(y)$, we obtain $f\left(\phi^{-1}(y)\right)=0$, which is the required equation.

Special cases: The following transformations are often required. Let $\alpha,\ \beta,\ \gamma,\ \cdots$ be the roots of $f(x)=0$, then

1. the equation whose roots are $-\alpha,\ -\beta,\ -\gamma,\ \cdots$ is $f(-x)=0$.
2. the equation whose roots are $\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma},\ \cdots$ is $f\left(\frac{1}{x}\right)=0$.
3. the equation whose roots are $k\alpha,\ k\beta,\ k\gamma,\ \cdots$ is $f\left(\frac{x}{k}\right)=0$.
4. the equation whose roots are $(\alpha-h),\ (\beta-h),\ (\gamma-h),\ \cdots$ is $f(x+h)=0$.

Problem-01: If $\alpha,\ \beta,\ \gamma$ are the roots of $x^3-x-1=0$, then find the equation whose roots are

$$\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}.$$

Solution: The given equation is,

$$x^3 - x - 1 = 0 \qquad \cdots(1)$$

The roots of equation (1) are α, β, γ.

Let $\quad y = \dfrac{1+x}{1-x}$

$$\therefore x = \frac{1-y}{1+y}.$$

From (1), we get

$$\left(\frac{1-y}{1+y}\right)^3 - \frac{1-y}{1+y} - 1 = 0$$

$$\therefore y^3 + 7y^2 - y + 1 = 0.$$

This is the required equation.

Exercise:

Problem-01: If α, β, γ are the roots of $2x^3 + 3x^2 - x - 1 = 0$, then find the equation whose roots are

1) $\alpha^2, \beta^2, \gamma^2$.
2) $\alpha + 2, \beta + 2, \gamma + 2$.
3) $\alpha - 1, \beta - 1, \gamma - 1$.
4) $\frac{1}{2}\alpha, \frac{1}{2}\beta, \frac{1}{2}\gamma$.
5) $\frac{1}{1-\alpha}, \frac{1}{1-\beta}, \frac{1}{1-\gamma}$.

Problem-02: If α, β, γ are the roots of $8x^3 - 4x^2 + 6x - 1 = 0$, then find the equation whose roots are

1) $\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \gamma + \frac{1}{2}$.

2) $2\alpha+1,\ 2\beta+1,\ 2\gamma+1$.

Theorem-02: State and prove Descarte's Rule of Signs.

Statement: The equation $f(x)=0$ cannot have more positive roots than $f(x)$ has changes of sign, or more negative roots than $f(-x)$ has changes of sign.

Proof: Let $f(x)=0$ be a polynomial equation and no term is missing in the polynomial $f(x)$. Also let the signs of the different terms of this equation are,

$$+\quad -\quad -\quad -\quad +\quad +\quad -\quad + \qquad\qquad \cdots(1)$$

Again let $\qquad g(x)=(x-a)f(x), \qquad where \quad a>0$

To prove the first part, we shall show that $g(x)$ has at least one more change of sign than $f(x)$.

The signs of the terms of the linear polynomial $(x-a)$ are "$+ \quad -$". To obtain the signs of the polynomial $g(x)$, multiplying the signs of the polynomial $f(x)$ by the sings of the linear polynomial $(x-a)$. Which are given as,

$$\begin{array}{ccccccccc}
+ & - & - & - & + & + & - & + & \\
+ & - & & & & & & & \\
\hline
+ & - & - & - & + & + & - & + & \\
 & - & + & + & + & - & - & + & - \\
\hline
+ & - & \pm & \pm & + & \pm & - & + & -
\end{array} \qquad \cdots(2)$$

where $\pm$ indicates that the sign may be $+$ or $-$, or that the corresponding term is zero.

In the diagram of corresponding signs, observe that

(i) If the r th sign of $f(x)$ is a continuation, the r th sign of $g(x)$ is ambiguous.

(ii) Unlike signs precede and follow a single ambiguity or a group of ambiguities.

(iii) A change of sign is introduced at the end of $g(x)$.

On account of (i) and (ii) , $g(x)$ has at least as many changes of sign as $f(x)$, even in the most unfavorable case in which all the ambiguities are continuations, and on account of (iii) $g(x)$ has certainly one more change of sign than $f(x)$.

That no changes of sign are lost on account of any terms which may be missing from $f(x)$ appears on considering such instances as,

$$\begin{array}{cccccc} + & + & 0 & 0 & - & \\ + & - & & & & \\ \hline + & + & 0 & 0 & - & \\ & - & - & 0 & 0 & + \\ \hline + & \pm & - & 0 & - & + \end{array} \qquad\qquad \begin{array}{cccccc} - & + & 0 & 0 & - & \\ + & - & & & & \\ \hline - & + & 0 & 0 & - & \\ & + & - & 0 & 0 & + \\ \hline - & + & - & 0 & - & + \end{array}$$

Thus, $g(x)$ has at least one more change of sign than $f(x)$.

Next let $f(x)=\{(x-\alpha)(x-\beta)\cdots\}\phi(x)$ where $\alpha, \beta, \cdots$ are the positive roots of $f(x)=0$. If $\phi(x)$ is multiplied in succession by $x-\alpha,\ x-\beta, \cdots$, each multiplication introduces at least one change of sign. Hence $f(x)$ has at least as many changes of sign as $f(x)=0$ has positive roots.

Again, the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$, with their signs changed. Hence the second part of the theorem follows from the first. **(Proved)**

Note: The equation $x^7-2x^5-3x^4-4x^3+5x^2-6x+7=0$ has 2 continuations, and 4 changes of sign, the continuations occurring at the terms $-3x^4, -4x^3$, and the changes at $-2x^5, +5x^2, -6x, +7$.

Problem-02: If 1 and 7 are two roots of $x^4-16x^3+86x^2-176x+105=0$, then solve the equation..

Solution: The given equation is,

$$x^4-16x^3+86x^2-176x+105=0 \qquad \cdots(1)$$

Let the other two roots of equation (1) are α, β. From the relation of coefficients and roots, we have

$$\alpha+\beta+1+7=16$$

$$or,\ \alpha+\beta=8 \qquad \cdots(2)$$

And
$$\alpha\beta(1)(7)=105$$

$$or,\ \alpha\beta=15 \qquad \cdots(3)$$

We know, $\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$$or,\ \alpha-\beta=\sqrt{(8)^2-4.15}$$

$$or,\ \alpha-\beta=2 \qquad \cdots(4)$$

Solving (2) and (4), we get

$$\alpha=5,\ \ \beta=3.$$

The roots or solution of the given equation are 1, 3, 5, 7.

Problem-03: Solve $4x^3-24x^2+23x+18=0$ where the roots are in arithmetic progression.

Solution: The given equation is,

$$4x^3-24x^2+23x+18=0 \qquad \cdots(1)$$

Since the roots are in arithmetic progression so the roots of equation (1) are $\alpha-\beta,\ \alpha,\ \alpha+\beta$. From the relation of coefficients and roots, we have

$$\alpha-\beta+\alpha+\alpha+\beta=\frac{24}{4}$$

$$\therefore\ \ \alpha=2$$

And $(\alpha-\beta)\alpha(\alpha+\beta)=-\frac{18}{4}$

$$or,\ 2(2-\beta)(2+\beta)=-\frac{18}{4} \qquad [\because\ \alpha=2]$$

$$or,\ 4-\beta^2=-\frac{9}{4}$$

$$or,\ \beta^2=\frac{25}{4}$$

$$\therefore\ \ \beta=\pm\frac{5}{2}$$

Now using the value of α and any one value of β, we have

$$\alpha - \beta = -\frac{1}{2}, \quad \alpha + \beta = \frac{9}{2}.$$

The roots or solution of the given equation are $-\frac{1}{2},\ 2,\ \frac{9}{2}$.

Problem-04: Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ where the roots are in arithmetic progression.

Solution: The given equation is,

$$x^4 - 2x^3 - 21x^2 + 22x + 40 = 0 \qquad \cdots(1)$$

Since the roots are in arithmetic progression so the roots of equation (1) are $\alpha - 3\beta, \alpha - \beta, \alpha + \beta, \alpha + 3\beta$. From the relation of coefficients and roots, we have

$$\alpha - 3\beta + \alpha - \beta + \alpha + \beta + \alpha + 3\beta = 2$$

$$\therefore \ \alpha = \frac{1}{2}$$

And $$(\alpha - 3\beta)(\alpha - \beta)(\alpha + \beta)(\alpha + 3\beta) = 40$$

$$or,\ (\alpha^2 - 9\beta^2)(\alpha^2 - \beta^2) = 40$$

$$or,\ \left(\frac{1}{4} - 9\beta^2\right)\left(\frac{1}{4} - \beta^2\right) = 40 \qquad \left[\because\ \alpha = \frac{1}{2}\right]$$

$$or,\ (1 - 36\beta^2)(1 - 4\beta^2) = 640$$

$$or,\ 144\beta^4 - 40\beta^2 + 1 = 640$$

$$or,\ 144\beta^4 - 40\beta^2 - 639 = 0$$

$$or,\ 144\beta^4 - 324\beta^2 + 284\beta^2 - 639 = 0$$

$$or,\ (4\beta^2 - 9)(36\beta^2 + 71) = 0$$

$$or,\ 4\beta^2 - 9 = 0 \qquad [\because\ 36\beta^2 + 71 \neq 0]$$

$$\therefore\ \beta = \pm\frac{3}{2}$$

Now using the value of α and any one value of β, we have

$$\alpha-3\beta=\frac{1}{2}-3.\frac{3}{2}=-4$$
$$\alpha-\beta=\frac{1}{2}-\frac{3}{2}=-1$$
$$\alpha+\beta=\frac{1}{2}+\frac{3}{2}=2$$
$$\alpha+3\beta=\frac{1}{2}+3.\frac{3}{2}=5$$

The roots or solution of the given equation are $-4,\ -1,\ 2,\ 5$.

Problem-05: Solve $3x^3-26x^2+52x-24=0$ where the roots are in geometric progression.

Solution: The given equation is,

$$3x^3-26x^2+52x-24=0 \qquad \cdots(1)$$

Since the roots are in geometric progression so the roots of equation (1) are $\frac{\alpha}{r},\ \alpha,\ \alpha r$. From the relation of coefficients and roots, we have

$$\frac{\alpha}{r}+\alpha+\alpha r=\frac{26}{3}$$
$$or,\ \alpha\left(\frac{1+r+r^2}{r}\right)=\frac{26}{3} \qquad \cdots(2)$$

And $$\frac{\alpha}{r}.\ \alpha.\ \alpha r=\frac{24}{4}$$

$$or,\ \alpha^3=8$$
$$\therefore\quad \alpha=2$$

Using the value of α in (2), we get

$$\frac{1+r+r^2}{r}=\frac{13}{3}$$

$$or,\ 3r^2-10r+3=0$$

$$or,\ (r-3)(3r-1)=0$$

$$\therefore \quad r = 3, \frac{1}{3}$$

Now using the value of α and any one value of r, we have

$$\frac{\alpha}{r} = \frac{2}{3}, \quad \alpha = 2, \; \alpha r = 6\,.$$

The roots or solution of the given equation are $\frac{2}{3}, \; 2, \; 6$.

Problem-06: Solve $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0$ where the roots are in geometric progression.

Solution: The given equation is,

$$27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0 \qquad \cdots(1)$$

Since the roots are in geometric progression so the roots of equation (1) are $\frac{\alpha}{r^3}, \frac{\alpha}{r}, \alpha r, \alpha r^3$.

From the relation of coefficients and roots, we have

$$\frac{\alpha}{r^3}.\frac{\alpha}{r} + \frac{\alpha}{r^3}.\alpha r + \frac{\alpha}{r^3}.\alpha r^3 + \frac{\alpha}{r}.\alpha r + \frac{\alpha}{r}.\alpha r^3 + \alpha r.\alpha r^3 = \frac{494}{27} \qquad \cdots(2)$$

and
$$\frac{\alpha}{r^3}.\frac{\alpha}{r}.\alpha r.\alpha r^3 = \frac{192}{27} \qquad \cdots(3)$$

From (3), we get

$$\alpha^4 = \frac{64}{9} \;\Rightarrow \alpha^2 = \frac{8}{3}.$$

From (2), we get

$$\alpha^2\left(\frac{1}{r^4} + \frac{1}{r^2} + 2 + r^2 + r^4\right) = \frac{494}{27}$$

$$or,\; \frac{8}{3}\left\{\left(r^2 + \frac{1}{r^2}\right)^2 + \left(r^2 + \frac{1}{r^2}\right)\right\} = \frac{494}{27}$$

$$or,\; \left(r^2 + \frac{1}{r^2}\right)^2 + \left(r^2 + \frac{1}{r^2}\right) = \frac{247}{36}$$

$$or,\ \left(r^2+\frac{1}{r^2}-\frac{13}{6}\right)\left(r^2+\frac{1}{r^2}+\frac{19}{13}\right)=0$$

$$or,\ r^2+\frac{1}{r^2}-\frac{13}{6}=0 \qquad ;\left[\because r^2+\frac{1}{r^2}+\frac{19}{13}\neq 0\right]$$

$$or,\ r^2+\frac{1}{r^2}=\frac{13}{6}$$

$$or,\ 6r^4-13r^2+6=0$$

$$or,\ \left(2r^2-3\right)\left(3r^2-2\right)=0$$

$$\therefore\ r^2=\frac{3}{2},\ \frac{2}{3}.$$

Now $$\alpha^2r^2=\frac{8}{3}.\frac{3}{2}=4$$

$$\therefore\ \alpha r=2$$

Using the value of αr, we have

$$\frac{\alpha}{r^3}=\frac{\alpha r}{r^4}=\frac{2}{\frac{9}{4}}=\frac{8}{9}$$

$$\frac{\alpha}{r}=\frac{\alpha r}{r^2}=\frac{2}{\frac{3}{2}}=\frac{4}{3}.$$

$$\alpha r^3=2.\frac{3}{2}=3$$

The roots or solution of the given equation are $\frac{8}{9},\ \frac{4}{3},\ 2,\ 3$.

Problem-07: Solve $x^4-10x^3+29x^2-22x+4=0$, where $2+\sqrt{3}$ is a root of this equation.

Solution: The given equation is,

$$x^4-10x^3+29x^2-22x+4=0 \qquad \cdots(1)$$

Since $2+\sqrt{3}$ is a root of the equation (1) so $2-\sqrt{3}$ is also root of this equation. Let the other two roots of this equation are $\alpha,\ \beta$. From the relation of coefficients and roots, we have

$$\alpha+\beta+2+\sqrt{3}+2-\sqrt{3}=10$$

$$or,\ \alpha+\beta=6 \qquad \cdots(2)$$

and
$$\alpha\beta\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=4$$

$$or,\ \alpha\beta=4 \qquad \cdots(3)$$

We know,
$$\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$or,\ \ \alpha-\beta=\sqrt{(6)^2-4.4}$$

$$or,\ \ \alpha-\beta=2\sqrt{5} \qquad \cdots(4)$$

Solving (2) and (4), we get

$$\alpha=3+\sqrt{5},\ \ \beta=3-\sqrt{5}.$$

The roots or solution of the given equation are $2\pm\sqrt{3},\ \ 3\pm\sqrt{5}$.

Problem-08: Solve $x^4-9x^3+30x^2-42x+20=0$, where $3-i$ is a root of this equation.

Solution: The given equation is,

$$x^4-9x^3+30x^2-42x+20=0 \qquad \cdots(1)$$

Since $3-i$ is a root of the equation (1) so $3+i$ is also root of this equation. Let the other two roots of this equation are $\alpha,\ \beta$. From the relation of coefficients and roots, we have

$$\alpha+\beta+3-i+3+i=9$$

$$or,\ \alpha+\beta=3 \qquad \cdots(2)$$

and
$$\alpha\beta(3-i)(3+i)=20$$

$$or,\ \alpha\beta=2 \qquad \cdots(3)$$

We know,
$$\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$or,\ \alpha-\beta=\sqrt{(3)^2-4.2}$$

$$or,\ \alpha-\beta=1 \qquad \cdots(4)$$

Solving (2) and (4), we get

$$\alpha=2,\ \beta=1.$$

The roots or solution of the given equation are $1,\ 2,\ \ 3\pm i$.

Exercise:

Problem-03: Solve $32x^3-48x^2+22x-3=0$ where the roots are in arithmetic progression.

Problem-04: Solve $54x^3-39x^2-26x+16=0$ where the roots are in geometric progression.

Problem-05: Solve $x^4-5x^3+4x^2+3x-1=0$, where $2+\sqrt{3}$ is a root of this equation.

Problem-06: Solve $8x^4+26x^3+81x^2-74x+13=0$, where $-2-3i$ is a root of this equation.