Differentiation

Introduction: The derivative is a mathematical operator, which measures the rate of change of a quantity relative to another quantity. The process of finding a derivative is called differentiation. There are many phenomena related changing quantities such as speed of a particle, inflation of currency, intensity of an earthquake and voltage of an electrical signal etc. in the world. In this chapter we will discuss about various techniques of derivative.

Differentiability of a function: The derivative of $y = f\left(x\right)$ with respect to x (for any particular value of x) is denoted by $f^{'}\left(x\right)$ or $\frac{dy}{dx}$ and defined as,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$$

$$= \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}$$

Provided this limit exists.

Existence of Derivative: A function $y = f\left(x\right)$ is said to have a derivative at $x = a$ if the left hand derivative and right hand derivative at this point i.e,

$$L.H.D = \lim_{h\to 0}\frac{f\left(a-h\right)-f\left(a\right)}{-h}$$

and

$$R.H.D = \lim_{h\to 0}\frac{f\left(a+h\right)-f\left(a\right)}{h}$$

are both exist and equal.

Problem-01: From the definition find the differential coefficient of $\sin x$.

Solution: we have $y = f\left(x\right) = \sin x$

$$\therefore\ f\left(x+h\right) = \sin\left(x+h\right)$$

By the definition of differentiation we have

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sin\left(x+h\right)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \lim_{\frac{h}{2} \to 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$

$$= \cos(x+0) \times 1$$

$$= \cos x$$

$$\therefore \frac{dy}{dx}(\sin x) = \cos x \,.$$

Problem-02: From the definition find the differential coefficient of e^x.

Solution: we have $y = f(x) = e^x$

$$\therefore f(x+h) = e^{x+h}$$

By the definition of differentiation we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots - 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)}{h}$$

$$= \lim_{h \to 0} e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right)$$

$$= e^x\left(1 + \frac{0}{2!} + \frac{0^2}{3!} + \cdots\right)$$

$$= e^x$$

$$\therefore \frac{dy}{dx}\left(e^x\right) = e^x.$$

Derivatives of elementary functions:

1. $\frac{d}{dx}(c) = 0,$ *where c is a constant.*

2. $\frac{d}{dx}(x) = 1.$

3. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$

4. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}.$

5. $\frac{d}{dx}\left(e^x\right) = e^x.$

6. $\frac{d}{dx}\left(a^x\right) = a^x \ln a.$

7. $\frac{d}{dx}(\ln x) = \frac{1}{x}.$

8. $\frac{d}{dx}(\sin x) = \cos x.$

9. $\frac{d}{dx}(\cos x) = -\sin x.$

10. $\frac{d}{dx}(\tan x) = \sec^2 x.$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x.$

12. $\frac{d}{dx}(\cot x) = -\cos ec^2 x.$

13. $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x.$

14. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}.$

15. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}.$

16. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$

17. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}.$

18. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}.$

19. $\frac{d}{dx}\left(\cos ec^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}.$

20. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$

21. $\frac{d}{dx}\left(u^v\right) = u^v \frac{d}{dx}(v \ln u).$

where u and v are functions of x.

22. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

- ***Find the differential coefficient ($\frac{dy}{dx}$) of the following functions with respect to x.***

1. $y=5x^8$

Sol: *Given that*, $y=5x^8$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(5x^8\right)$$
$$=5\frac{d}{dx}\left(x^8\right)$$
$$=5\times 8x^{8-1}$$
$$=40x^7 \quad (Ans.)$$

2. $y=3x^7+2x+1$

Sol: *Given that*, $y=3x^7+2x+1$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(3x^7+2x+1\right)$$
$$=3\frac{d}{dx}\left(x^7\right)+2\frac{d}{dx}(x)+\frac{d}{dx}(1)$$
$$=21x^6+2+0$$
$$=21x^6+2 \quad (Ans.)$$

3. $y=4\sin x-\cos x$

Sol: *Given that*, $y=4\sin x-\cos x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(4\sin x-\cos x\right)$$
$$=4\frac{d}{dx}(\sin x)-\frac{d}{dx}(\cos x)$$
$$=4\cos x-(-\sin x)$$
$$=4\cos x+\sin x \quad (Ans.)$$

4. $y=\sec^2 x-\tan^2 x$

Sol: *Given that*, $y=\sec^2 x-\tan^2 x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sec^2 x-\tan^2 x\right)$$
$$=\frac{d}{dx}\left(\sec^2 x\right)-\frac{d}{dx}\left(\tan^2 x\right)$$
$$=2\sec x\frac{d}{dx}(\sec x)-2\tan x\frac{d}{dx}(\tan x)$$
$$=2\sec x(\sec x\tan x)-2\tan x\left(\sec^2 x\right)$$
$$=2\sec^2 x\tan x-2\sec^2 x\tan x$$
$$=0 \quad (Ans.)$$

5. $y=\ln\left(x+\sqrt{x^2+a^2}\right)$

Sol: *Given that*, $y=\ln\left(x+\sqrt{x^2+a^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\ln\left(x+\sqrt{x^2+a^2}\right)\right\}$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left\{1+\frac{d}{dx}\left(\sqrt{x^2+a^2}\right)\right\}$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left\{1+\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}\left(x^2+a^2\right)\right\}$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left\{1+\frac{2x}{2\sqrt{x^2+a^2}}\right\}$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left\{1+\frac{x}{\sqrt{x^2+a^2}}\right\}$$
$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$$
$$=\frac{1}{\sqrt{x^2+a^2}} \quad (Ans.)$$

6. $y=\ln(\sec x+\tan x)$

Sol: *Given that*, $y=\ln(\sec x+\tan x)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\ln(\sec x+\tan x)\right\}$$
$$=\frac{1}{\sec x+\tan x}\cdot\frac{d}{dx}(\sec x+\tan x)$$
$$=\frac{\left(\sec x\tan x+\sec^2 x\right)}{\sec x+\tan x}$$
$$=\frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}$$
$$=\sec x$$
$$(Ans.)$$

7. $y=e^{ax^2+bx+c}$

Sol: *Given that,* $y=e^{ax^2+bx+c}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{ax^2+bx+c}\right)$$
$$=e^{ax^2+bx+c}.\frac{d}{dx}\left(ax^2+bx+c\right)$$
$$=e^{ax^2+bx+c}\left(2ax+b+0\right)$$
$$=\left(2ax+b\right)e^{ax^2+bx+c}$$

$(Ans.)$

8. $y=e^{\sqrt{\cot x}}$

Sol: *Given that,* $y=e^{\sqrt{\cot x}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sqrt{\cot x}}\right)$$
$$=e^{\sqrt{\cot x}}.\frac{d}{dx}\left(\sqrt{\cot x}\right)$$
$$=e^{\sqrt{\cot x}}.\frac{1}{2\sqrt{\cot x}}.\frac{d}{dx}(\cot x)$$
$$=\frac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.\left(-\cos ec^2x\right)$$
$$=-\frac{e^{\sqrt{\cot x}}\cos ec^2x}{2\sqrt{\cot x}}$$

$(Ans.)$

9. $y=\sqrt{x^3-2x+5}$

Sol: *Given that,* $y=\sqrt{x^3-2x+5}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x^3-2x+5}\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\frac{d}{dx}\left(x^3-2x+5\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\left(3x^2-2+0\right)$$
$$=\frac{3x^2-2}{2\sqrt{x^3-2x+5}}$$

$(Ans.)$

10. $y=\tan\ln\sin\left(e^{x^2}\right)$

Sol: *Given that,* $y=\tan\left(\ln\sin e^{x^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\tan\left(\ln\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{d}{dx}\left\{\ln\left(\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\frac{d}{dx}\left\{\sin\left(e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\cos\left(e^{x^2}\right).\frac{d}{dx}\left(e^{x^2}\right)$$
$$=\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right).e^{x^2}\frac{d}{dx}\left(x^2\right)$$
$$=2xe^{x^2}\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right)$$

$(Ans.)$

11. $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Sol: *Given that,* $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\cos^{-1}\left(e^{\cot^{-1}x}\right)\right\}$$
$$=-\frac{1}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(e^{\cot^{-1}x}\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(\cot^{-1}x\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}\left(-\frac{1}{1+x^2}\right)$$
$$=\frac{e^{\cot^{-1}x}}{\left(1+x^2\right)\sqrt{1-e^{2\cot^{-1}x}}}$$

$(Ans.)$

12. $y=e^{\sin^{-1}x}+\tan^{-1}x$

Sol: *Given that,* $y=e^{\sin^{-1}x}+\tan^{-1}x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}+\tan^{-1}x\right)$$
$$=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)+\frac{d}{dx}\left(\tan^{-1}x\right)$$
$$=e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{1}{1+x^2}$$
$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}+\frac{1}{1+x^2}$$

$(Ans.)$

13. $y=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

Sol : *Given that*, $y=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$put,\ x=\sin\theta \quad \therefore\ \theta=\sin^{-1}x$

$$Now,\ y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$
$$=\tan^{-1}.\tan\theta$$
$$=\theta$$
$$=\sin^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)$$
$$=\frac{1}{\sqrt{1-x^2}}$$

$(Ans.)$

14. $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Sol : *Given that*, $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$put,\ x=\tan\theta \quad \therefore\ \theta=\tan^{-1}x$

$$Now,\ y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$
$$=\cos^{-1}.\cos 2\theta$$
$$=2\theta$$
$$=2\tan^{-1}x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)$$
$$=\frac{2}{1+x^2}$$

$(Ans.)$

15. $y=\dfrac{\cos x}{1+\sin x}$

Sol : *Given that*, $y=\dfrac{\cos x}{1+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$
$$=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$
$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$
$$=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$$
$$=\frac{-\sin x-1}{(1+\sin x)^2}$$
$$=-\frac{1}{1+\sin x}$$

$(Ans.)$

16. $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Sol : *Given that*, $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$
$$=\frac{(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)-(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)}{(\cos x+\sin x)^2}$$
$$=\frac{(\cos x+\sin x)(-\sin x-\cos x)-(\cos x-\sin x)(-\sin x+\cos x)}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$
$$=\frac{-(\cos x+\sin x)^2-(\cos x-\sin x)^2}{1+\sin 2x}$$
$$=\frac{-(1+\sin 2x)-(1-\sin 2x)}{1+\sin 2x}$$
$$=-\frac{2}{1+\sin 2x}$$

$(Ans.)$

17. $y = \tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

Sol : *Given that*, $y = \tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

$put,\ x = \tan\theta \quad \therefore\ \theta = \tan^{-1} x$

$$Now,\ y = \tan^{-1}\left(\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2 \theta/2}{2\sin\theta/2 \cos\theta/2}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$= \tan^{-1}.\tan\theta/2$$

$$= \theta/2$$

$$= 1/2 \tan^{-1} x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(1/2 \tan^{-1} x\right)$$

$$= \frac{1}{2(1+x^2)}$$

$(Ans.)$

18. $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Sol : *Given that,* $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)\right\}$$

$$=\frac{1}{\sqrt{1-\left(\dfrac{a+b\cos x}{b+a\cos x}\right)^2}}\cdot\frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$=\frac{(b+a\cos x)}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}\cdot\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$=\frac{1}{\sqrt{b^2-a^2+a^2\cos^2 x-b^2\cos^2 x}}\cdot\frac{-b^2\sin x-ab\sin x\cos x+a^2\sin x+ab\sin x\cos x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{1-\cos^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{\sin^2 x}}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sin x}\cdot\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{-\sqrt{b^2-a^2}}{b+a\cos x}$$

$(Ans.)$

19. $y=x\sin x$

Sol : *Given that,* $y=x\sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}(x\sin x)$$

$$=x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)$$

$$=x\cos x+\sin x$$

$(Ans.)$

20. $y=e^{ax}\cos bx$

Sol : *Given that,* $y=e^{ax}\cos bx$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{ax}\cos bx\right)$$

$$=e^{ax}\frac{d}{dx}(\cos bx)+\cos bx\frac{d}{dx}\left(e^{ax}\right)$$

$$=e^{ax}(-b\sin bx)+\cos bx\left(ae^{ax}\right)$$

$$=ae^{ax}\cos bx-be^{ax}\sin bx$$

$(Ans.)$

21. $y = x^2 \cot^{-1} x$

Sol : *Given that*, $y = x^2 \cot^{-1} x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \cot^{-1} x\right)$$
$$= x^2 \frac{d}{dx}\left(\cot^{-1} x\right) + \cot^{-1} x \frac{d}{dx}\left(x^2\right)$$
$$= x^2 \left(\frac{-1}{1+x^2}\right) + \cot^{-1} x(2x)$$
$$= 2x \cot^{-1} x - \frac{x^2}{1+x^2}$$

(*Ans.*)

22. $y = x^3 \ln x$

Sol : *Given that*, $y = x^3 \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \ln x\right)$$
$$= x^3 \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(x^3\right)$$
$$= x^3 . \frac{1}{x} + \ln x\left(2x^2\right)$$
$$= x^2 + 2x^2 \ln x$$

(*Ans.*)

23. $y = xe^x \sin x$

Sol : *Given that*, $y = xe^x \sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^x \sin x\right)$$
$$= xe^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}\left(xe^x\right)$$
$$= xe^x \cos x + \sin x \left\{ x\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(x) \right\}$$
$$= xe^x \cos x + \sin x\left(xe^x + e^x\right)$$
$$= xe^x \cos x + xe^x \sin x + e^x \sin x$$

(*Ans.*)

24. $y = \sqrt{x}e^x \sec x$

Sol : *Given that*, $y = \sqrt{x}e^x \sec x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}e^x \sec x\right)$$
$$= \sqrt{x}e^x \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}\left(\sqrt{x}e^x\right)$$
$$= \sqrt{x}e^x \sec x \tan x + \sec x\left(\sqrt{x}e^x + \frac{e^x}{2\sqrt{x}}\right)$$

(*Ans.*)

25. $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Sol : *Given that*, $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}\right\}$$
$$= \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x\right\} + \frac{d}{dx}\left(e^{\sqrt{1+x^2}}\right)$$
$$= \left(x^2+1\right).\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x.(2x) + e^{\sqrt{1+x^2}} . \frac{1}{2\sqrt{1+x^2}} .2x$$
$$= \frac{x^2+1}{\sqrt{1-x^2}} + 2x\sin^{-1} x + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

(*Ans.*)

26. $y = e^{\sin x} \sin\left(a^x\right)$

Sol : *Given that*, $y = e^{\sin x} \sin\left(a^x\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{e^{\sin x} \sin\left(a^x\right)\right\}$$
$$= e^{\sin x} \frac{d}{dx}\left\{\sin\left(a^x\right)\right\} + \sin\left(a^x\right)\frac{d}{dx}\left(e^{\sin x}\right)$$
$$= e^{\sin x}.\cos\left(a^x\right).\frac{d}{dx}\left(a^x\right) + \sin\left(a^x\right).e^{\sin x}.\cos x$$
$$= e^{\sin x}.\cos\left(a^x\right).a^x \ln a + \sin\left(a^x\right).e^{\sin x}.\cos x$$

(*Ans.*)

Homework:-Find $\frac{dy}{dx}$ of the following functions:

1. $y=\ln\left(\sqrt{x-a}+\sqrt{x-b}\right)$ Ans: $\frac{1}{2\sqrt{(x-a)(x-b)}}$
2. $y=\ln\left(x+\sqrt{x^2\pm b^2}\right)$ Ans: $\frac{1}{\sqrt{x^2\pm b^2}}$
3. $y=\cos(\ln x)+\ln(\tan x)$ Ans: $2\cos ec2x-\frac{\sin(\ln x)}{x}$
4. $y=e^{ax}\sin^m rx$ Ans: $e^{ax}\sin^m rx(a+mr\cot rx)$
5. $y=x\sec x\ln\left(xe^x\right)$ Ans: $\sec x\left\{(x+1)+(x\tan x+1)\ln\left(xe^x\right)\right\}$
6. $y=\sin^{-1}x^2-xe^{x^2}$ Ans: $\frac{2x}{\sqrt{1-x^4}}-\left(2x^2+1\right)e^{x^2}$
7. $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ Ans: $\frac{1}{\sqrt{1-x^2}}$
8. $y=\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$ Ans: $\frac{2}{\sqrt{x}(1+4x)}$
9. $y=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ Ans: $-\frac{1}{2}$
10. $y=\sin^{-1}\left(\frac{2x^{-1}}{x+x^{-1}}\right)$ Ans: $\frac{2}{\sqrt{x}(1+4x)}$

Logarithmic differentiation: If we have functions that are composed of products, quotients and powers, to differentiate such functions it would be convenient first to take logarithm of the function and then differentiate. Such a technique is called the logarithmic differentiation.

1. $y=(\sin x)^{\ln x}$

Sol: *Given that*, $y=(\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$
$$=(\sin x)^{\ln x}\frac{d}{dx}\left\{\ln x\ln(\sin x)\right\}$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{d}{dx}\left\{\ln(\sin x)\right\}+\ln(\sin x).\frac{d}{dx}(\ln x)\right]$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{1}{\sin x}.\cos x+\ln(\sin x).\frac{1}{x}\right]$$
$$=(\sin x)^{\ln x}\left[\cot x\ln x+\frac{\ln(\sin x)}{x}\right]$$

$(Ans.)$

2. $y=x^{1+x+x^2}$

Sol: *Given that*, $y=x^{1+x+x^2}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{1+x+x^2}\right)$$
$$=x^{1+x+x^2}\frac{d}{dx}\left\{\left(1+x+x^2\right)\ln x\right\}$$
$$=x^{1+x+x^2}\left[\ln x.\frac{d}{dx}\left(1+x+x^2\right)+\left(1+x+x^2\right).\frac{d}{dx}(\ln x)\right]$$
$$=x^{1+x+x^2}\left[\ln x.(0+1+2x)+\left(1+x+x^2\right).\frac{1}{x}\right]$$
$$=x^{1+x+x^2}\left[(2x+1)\ln x+\frac{\left(1+x+x^2\right)}{x}\right]$$

$(Ans.)$

3. $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Sol : *Given that*, $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\tan^{-1}x\right)^{\sin x+\cos x}\right\}$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\frac{d}{dx}\left\{\left(\sin x+\cos x\right).\ln\left(\tan^{-1}x\right)\right\}$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\left(\sin x+\cos x\right)\frac{d}{dx}\left\{\ln\left(\tan^{-1}x\right)\right\}+\ln\left(\tan^{-1}x\right).\frac{d}{dx}\left(\sin x+\cos x\right)\right]$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\frac{\left(\sin x+\cos x\right)}{\tan^{-1}x}.\frac{1}{\left(1+x^2\right)}+\ln\left(\tan^{-1}x\right).\left(\cos x-\sin x\right)\right]$$

$(Ans.)$

4. $y=x^x+\left(\sin x\right)^{\ln x}$

Sol : *Given that*, $y=x^x+\left(\sin x\right)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{x^x+\left(\sin x\right)^{\ln x}\right\}$$
$$=\frac{d}{dx}\left(x^x\right)+\frac{d}{dx}\left\{\left(\sin x\right)^{\ln x}\right\}$$
$$=x^x\frac{d}{dx}\left(x\ln x\right)+\left(\sin x\right)^{\ln x}\frac{d}{dx}\left\{\ln x\ln\left(\sin x\right)\right\}$$
$$=x^x\left(x.\frac{1}{x}+\ln x\right)+\left(\sin x\right)^{\ln x}\left\{\ln x.\frac{1}{\sin x}.\cos x+\frac{\ln\left(\sin x\right)}{x}\right\}$$
$$=x^x\left(1+\ln x\right)+\left(\sin x\right)^{\ln x}\left\{\ln x.\cot x+\frac{\ln\left(\sin x\right)}{x}\right\}\quad Ans.$$

5. $y=\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}$

Sol : *Given that*, $y=\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\sin x\right)^{\cos x}+\left(\cos x\right)^{\sin x}\right\}$$
$$=\frac{d}{dx}\left\{\left(\sin x\right)^{\cos x}\right\}+\frac{d}{dx}\left\{\left(\cos x\right)^{\sin x}\right\}$$
$$=\left(\sin x\right)^{\cos x}\frac{d}{dx}\left\{\cos x\ln\left(\sin x\right)\right\}+\left(\cos x\right)^{\sin x}\frac{d}{dx}\left\{\sin x\ln\left(\cos x\right)\right\}$$
$$=\left(\sin x\right)^{\cos x}\left[\cos x.\frac{1}{\sin x}.\cos x-\sin x\ln\left(\sin x\right)\right]+\left(\cos x\right)^{\sin x}\left[\sin x.\frac{1}{\cos x}.\left(-\sin x\right)+\cos x\ln\left(\cos x\right)\right]$$
$$=\left(\sin x\right)^{\cos x}\left[\cos x.\cot x-\sin x\ln\left(\sin x\right)\right]+\left(\cos x\right)^{\sin x}\left[\cos x\ln\left(\cos x\right)-\sin x.\tan x\right]\quad Ans.$$

6. $y = x^{\cos^{-1}x} - \sin x \ln x$

Sol : *Given that*, $y = x^{\cos^{-1}x} - \sin x \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\cos^{-1}x} - \sin x \ln x\right)$$

$$= \frac{d}{dx}\left(x^{\cos^{-1}x}\right) - \frac{d}{dx}(\sin x \ln x)$$

$$= x^{\cos^{-1}x}\frac{d}{dx}\left(\cos^{-1}x \ln x\right) - \left(\frac{\sin x}{x} + \cos x \ln x\right)$$

$$= x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] - \left(\frac{\sin x}{x} + \cos x \ln x\right) \quad Ans.$$

7. $y = \left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}$

Sol : *Given that*, $y = \left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x} + (2-\sin x)^{\ln x}\right\}$$

$$= \frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x}\right\} + \frac{d}{dx}\left\{(2-\sin x)^{\ln x}\right\}$$

$$= \left(1+x^2\right)^{\tan x}\frac{d}{dx}\left\{\tan x \ln\left(1+x^2\right)\right\} + (2-\sin x)^{\ln x}\frac{d}{dx}\left\{\ln x \ln(2-\sin x)\right\}$$

$$= \left(1+x^2\right)^{\tan x}\left[\frac{2x\tan x}{1+x^2} + \sec^2 x \ln\left(1+x^2\right)\right] + (2-\sin x)^{\ln x}\left[\frac{\ln(2-\sin x)}{x} - \frac{\cos x \ln x}{(2-\sin x)}\right] \quad Ans.$$

Homework:-Find $\dfrac{dy}{dx}$ of the following functions:

1. $y = x^{\sin^{-1}x}$ Ans: $x^{\sin^{-1}x}\left[\dfrac{\sin^{-1}x}{x} + \dfrac{\ln x}{\sqrt{1-x^2}}\right]$
2. $y = (\sin x)^{\cos^{-1}x}$ Ans: $(\sin x)^{\cos^{-1}x}\left[\cot x \cos^{-1}x - \dfrac{\ln \sin x}{\sqrt{1-x^2}}\right]$
3. $y = x^{x^x}$ Ans: $x^{x^x}x^x\left[1 + \ln x + \dfrac{1}{x}\right]$
4. $y = x^{\cos^{-1}x} + (\sin x)^{\ln x}$ Ans: $x^{\cos^{-1}x}\left[\dfrac{\cos^{-1}x}{x} - \dfrac{\ln x}{\sqrt{1-x^2}}\right] + (\sin x)^{\ln x}\left[\ln x \cot x + \dfrac{\ln \sin x}{x}\right]$
5. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$ Ans: $(tanx)^{\cot x}\cos ec^2 x(1 - \ln\tan x) + (\cot x)^{\tan x} sec^2 x(\ln\cot x - 1)$
6. $y = x^{\ln x} + x^{\sin^{-1}x}$ Ans: $\dfrac{2x^{\ln x}\ln x}{x} + x^{\cos^{-1}x}\left(\dfrac{\cos^{-1}x}{x} - \dfrac{\ln x}{\sqrt{1-x^2}}\right)$

Parametric Equation: If in the equation of a curve $y = f(x)$, x and y are expressed in terms of a third variable known as parameter i.e, $x=\varphi(t)$, $y=\psi(t)$ then the equations are called a parametric equation.

1. $x = a(t+\sin t)$, $y = a(1-\cos t)$

sol: *Given that*,

$$x = a(t+\sin t) \cdots\cdots (1)$$

$$\text{and} \quad y = a(1-\cos t) \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get*,

$$\frac{dx}{dt} = a(1+\cos t)$$

$$\text{and} \quad \frac{dy}{dt} = a\sin t$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2} \quad (Ans.)$$

2. $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

sol: *Given that*,

$$x = a(\cos t + t\sin t) \cdots\cdots (1)$$

$$\text{and} \quad y = a(\sin t - t\cos t) \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get*,

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$$

$$\text{and} \quad \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t \quad (Ans.)$$

3. $x = a\left(\cos t + \ln\tan\frac{t}{2}\right)$, $y = a\sin t$

sol: *Given that*,

$$x = a\left(\cos t + \ln\tan\frac{t}{2}\right) \cdots\cdots (1)$$

$$\text{and} \quad y = a\sin t \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get*,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2}\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\text{and} \quad \frac{dy}{dt} = a\cos t$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \tan t \quad (Ans.)$$

4. $x = t - \sqrt{1-t^2}$, $y = e^{\sin^{-1}t}$

sol: *Given that*,

$$x = t - \sqrt{1-t^2} \cdots\cdots (1)$$

$$\text{and} \quad y = e^{\sin^{-1}t} \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to* t *we get*,

$$\frac{dx}{dt} = 1 - \frac{1}{2\sqrt{1-t^2}}.(-2t) = 1 + \frac{t}{\sqrt{1-t^2}} = \frac{t+\sqrt{1-t^2}}{\sqrt{1-t^2}}$$

$$\text{and} \quad \frac{dy}{dt} = e^{\sin^{-1}t}.\frac{1}{\sqrt{1-t^2}} = \frac{e^{\sin^{-1}t}}{\sqrt{1-t^2}}$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{\sin^{-1}t}}{\sqrt{1-t^2}}.\frac{\sqrt{1-t^2}}{t+\sqrt{1-t^2}} = \frac{e^{\sin^{-1}t}}{t+\sqrt{1-t^2}} \quad (Ans.)$$

5. *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

$$sol: Let,\ y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\ ;\left[\begin{array}{l} putting\ \ x=\tan\theta \\ \therefore\ \ \theta=\tan^{-1}x\end{array}\right]$$
$$=\tan^{-1}.\tan 2\theta$$
$$=2\theta$$
$$=2\tan^{-1}x \cdots\cdots (1)$$
$$and\quad z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\ ;\left[\begin{array}{l} putting\ \ x=\tan\theta \\ \therefore\ \ \theta=\tan^{-1}x\end{array}\right]$$
$$=\sin^{-1}.\sin 2\theta$$
$$=2\theta$$
$$=2\tan^{-1}x\ \cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to x we get,*

$$\frac{dy}{dx}=\frac{2}{1+x^2}\qquad and\ \ \frac{dz}{dx}=\frac{2}{1+x^2}$$
$$Now,\ \frac{dy}{dz}=\frac{dy/dx}{dz/dx}$$
$$=\frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$
$$=1\quad (Ans.)$$

6. *Differentiate* $(\sin x)^x$ *with respect to* $x^{\sin x}$.

$$sol: Let,\ y=(\sin x)^x \cdots\cdots(1)$$
$$and\ \ z=x^{\sin x}\cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to x we get,*

$$\frac{dy}{dx}=(\sin x)^x\frac{d}{dx}(x\ln\sin x)$$
$$=(\sin x)^x\left(x.\frac{\cos x}{\sin x}+\ln\sin x\right)$$
$$=(\sin x)^x(x\cot x+\ln\sin x)$$
$$and\ \frac{dz}{dx}=x^{\sin x}\frac{d}{dx}(\sin x\ln x)$$
$$=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)$$
$$Now,\ \frac{dy}{dz}=\frac{dy/dx}{dz/dx}$$
$$=\frac{(\sin x)^x(x\cot x+\ln\sin x)}{x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)}\quad (Ans.)$$

7. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\sin^{-1}x$.

$$sol: Let,\ y=x^{\sin^{-1}x}\ \cdots\cdots(1)$$
$$and\ \ z=\sin^{-1}x\cdots\cdots (2)$$

Differentiating (1) *and* (2) *with respect to x we get,*

$$\frac{dy}{dx}=x^{\sin^{-1}x}\frac{d}{dx}(\sin^{-1}x\ln x)\ \ ;\left[\because \frac{d}{dx}(u^v)=u^v\frac{d}{dx}(v\ln u)\right]$$
$$=x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)$$
$$and\ \frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$$
$$Now,\ \frac{dy}{dz}=\frac{dy/dx}{dz/dx}$$
$$=\frac{x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$
$$=x^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}.\sin^{-1}x}{x}+\ln x\right)\quad (Ans.)$$

8. *Differentiate* $\tan^{-1}\left(\dfrac{\sqrt{1-x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$.

sol : *Let*, $y=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}-1}{x}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}-1}{\sin\theta}\right)\ ;\left[\begin{array}{l}putting\quad x=\sin\theta\\ \qquad\quad \therefore\ \theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sqrt{\cos^2\theta}-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left\{-\frac{(1-\cos\theta)}{\sin\theta}\right\}$$

$$=\tan^{-1}\left\{-\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\tan\frac{\theta}{2}\right\}$$

$$=\tan^{-1}\left\{\tan\left(\pi-\frac{\theta}{2}\right)\right\}$$

$$=\pi-\frac{\theta}{2}$$

$$=\pi-\frac{1}{2}\sin^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}x\cdots\cdots(2)$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}\qquad and\quad \frac{dz}{dx}=\frac{1}{1+x^2}$$

Now, $\dfrac{dy}{dz}=\dfrac{dy/dx}{dz/dx}$

$$=\frac{-\frac{1}{2\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$=-\frac{1+x^2}{2\sqrt{1-x^2}}\quad (Ans.)$$

9. *Differentiate* $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ *with respect to* $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

sol : *Let*, $y=\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

$$=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)\ ;\left[\begin{array}{l}putting\quad x=\cos\theta\\ \qquad\quad \therefore\ \theta=\cos^{-1}x\end{array}\right]$$

$$=\sec^{-1}\left(\frac{1}{\cos2\theta}\right)$$

$$=\sec^{-1}(\sec2\theta)$$

$$=2\theta$$

$$=2\cos^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right);\left[\begin{array}{l}putting\quad x=\sin\theta\\ \qquad\quad \therefore\ \theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$=\tan^{-1}.\tan\theta$$

$$=\theta$$

$$=\sin^{-1}x\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}\qquad and\quad \frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$$

Now, $\dfrac{dy}{dz}=\dfrac{dy/dx}{dz/dx}$

$$=\frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$=-2\quad (Ans.)$$

Homework:-

1. $Differentiate\ \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right) with\ respect\ to\ \tan^{-1}x.$ Ans: $\dfrac{1}{2}$
2. $Differentiate\ e^{\sin^{-1}x}\ with\ respect\ to\ \cos 3x.$ Ans: $-\dfrac{e^{\sin^{-1}x}}{3\sqrt{1-x^2}.\sin 3x}$
3. $Differentiate\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) with\ respect\ to\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right).$ Ans: 1
4. $Differentiate\ x^{\sin^{-1}x}\ with\ respect\ to\ \ln x.$ Ans: $x^{\sin^{-1}x}\left(\sin^{-1}x+\dfrac{x\ln x}{\sqrt{1-x^2}}\right)$

Successive derivative: If $y=f(x)$ be a function of *x* then the first order derivative of *y* with respect to *x* is denoted by $\dfrac{dy}{dx},\ f'(x),\ y_1\ ,\ y^{(1)},\ f^{(1)}(x),\ f_x'(x)\ etc.$

Again the derivative of first ordered derivative of *y* with respect to *x* is called second order derivative and is denoted by $\dfrac{d^2y}{dx^2},\ f''(x),\ y_2\ ,\ y^{(2)},\ f^{(2)}(x),\ f_x''(x)\ etc.$

Similarly, the nth derivative of *y* with respect to *x* is denoted by
$\dfrac{d^ny}{dx^n},\ f^n(x),\ y_n\ ,\ y^{(n)},\ f^{(n)}(x),\ f_x^n(x)\ etc.$

❖ Find the nth derivative of the following functions:

1. $y=x^n$

$sol: Given\ that,\ y=x^n$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=nx^{n-1}$$
$$\therefore\ y_2=n(n-1)x^{n-2}$$
$$\therefore\ y_3=n(n-1)(n-2)x^{n-3}$$

$Similarly,$

$$y_r=n(n-1)(n-2)\cdots\cdots\{n-(r-1)\}x^{n-r}\quad ;\ where,\ r<n$$
$$\therefore\ y_n=n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}x^{n-n}$$
$$=n(n-1)(n-2)\cdots\cdots 3.2.1$$
$$=n!\quad Ans.$$

2. $y=e^{ax}$

$sol: Given\ that,\ y=e^{ax}$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=ae^{ax}$$
$$\therefore\ y_2=a^2e^{ax}$$
$$\therefore\ y_3=a^3e^{ax}$$

$Similarly,$

$$y_r=a^re^{ax}\quad ;\ where,\ r<n$$
$$\therefore\ y_n=a^ne^{ax}\quad Ans.$$

3. $y=(ax+b)^m$

sol : *Given that*, $y=(ax+b)^m$

Differentiating with respect to x we get,

$$y_1 = am(ax+b)^{m-1}$$
$$\therefore\ y_2 = a^2 m(m-1)(ax+b)^{m-2}$$
$$\therefore\ y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

Similarly,

$$y_r = a^r m(m-1)(m-2)\cdots\cdots\{m-(r-1)\}(ax+b)^{m-r}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = a^n m(m-1)(m-2)\cdots\cdots\{m-(n-1)\}(ax+b)^{m-n}$$
$$=\frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}\quad Ans.$$

4. $y=\sin(ax+b)$

sol : *Given that*, $y=\sin(ax+b)$

Differentiating with respect to x we get,

$$y_1 = a\cos(ax+b)$$
$$= a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2 = a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\sin\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3 = a^3\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\sin\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\sin\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r = a^r\sin\left\{\frac{r\pi}{2}+(ax+b)\right\}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = a^n\sin\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

5. $y=\cos(ax+b)$

sol : *Given that*, $y=\cos(ax+b)$

Differentiating with respect to x we get,

$$y_1 = -a\sin(ax+b)$$
$$= a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2 = -a^2\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\cos\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3 = -a^3\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\cos\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\cos\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r = a^r\cos\left\{\frac{r\pi}{2}+(ax+b)\right\}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = a^n\cos\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

6. $y = e^{ax} \sin(bx+c)$

sol : Given that, $y = e^{ax}\sin(bx+c)$

Differentiating with respect to x we get,

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$
$$= e^{ax}\{a\sin(bx+c) + b\cos(bx+c)\}$$

put $a = r\cos\varphi$ *and* $b = r\sin\phi$

$$\therefore\ r = \sqrt{a^2+b^2}\ \text{ and }\ \varphi = \tan^{-1}\left(\frac{b}{a}\right)$$

Now, $y_1 = e^{ax}\{r\cos\varphi\sin(bx+c) + r\sin\varphi\cos(bx+c)\}$

$$= re^{ax}\sin(bx+c+\varphi)$$
$$\therefore\ y_2 = re^{ax}\{a\sin(bx+c+\varphi) + b\cos(bx+c+\varphi)\}$$
$$= re^{ax}\{r\cos\varphi\sin(bx+c+\varphi) + r\sin\varphi\cos(bx+c+\varphi)\}$$
$$= r^2e^{ax}\sin(bx+c+2\varphi)$$
$$\therefore\ y_3 = r^3e^{ax}\sin(bx+c+3\varphi)$$

Similarly,

$$y_n = r^ne^{ax}\sin(bx+c+n\varphi)$$
$$= \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)\quad \textit{Ans.}$$

7. $y = \ln(ax+b)$

sol : Given that, $y = \ln(ax+b)$

Differentiating with respect to x we get,

$$y_1 = \frac{a}{(ax+b)}$$
$$\therefore\ y_2 = -\frac{1.a^2}{(ax+b)^2}$$
$$\therefore\ y_3 = \frac{1.2\,a^3}{(ax+b)^3}$$
$$\therefore\ y_4 = -\frac{1.2.3a^4}{(ax+b)^4}$$

Similarly,

$$\therefore\ y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}\quad \textit{Ans.}$$

8. *If* $y = \sin nx + \cos nx$ *then show that* $y_r = n^r\left[1+(-1)^r\sin 2nx\right]^{\frac{1}{2}}$.

sol : Given that, $y = \sin nx + \cos nx$

Differentiating with respect to x we get,

$$y_1 = n\cos nx - n\sin nx$$
$$= n\sin\left(\frac{\pi}{2}+nx\right) + n\cos\left(\frac{\pi}{2}+nx\right)$$
$$\therefore\ y_2 = n^2\cos\left(\frac{\pi}{2}+nx\right) - n^2\sin\left(\frac{\pi}{2}+nx\right)$$
$$= n^2\sin\left(\frac{2\pi}{2}+nx\right) + n^2\cos\left(\frac{2\pi}{2}+nx\right)$$
$$\therefore\ y_3 = n^3\cos\left(\frac{2\pi}{2}+nx\right) - n^3\sin\left(\frac{2\pi}{2}+nx\right)$$
$$= n^3\sin\left(\frac{3\pi}{2}+nx\right) + n^3\cos\left(\frac{3\pi}{2}+nx\right)$$

Similarly,

$$y_r = n^r\sin\left(\frac{r\pi}{2}+nx\right) + n^3\cos\left(\frac{r\pi}{2}+nx\right)$$
$$= n^r\left[\left\{\sin\left(\frac{r\pi}{2}+nx\right) + \cos\left(\frac{r\pi}{2}+nx\right)\right\}^2\right]^{\frac{1}{2}}$$
$$= n^r\left[\sin^2\left(\frac{r\pi}{2}+nx\right) + \cos^2\left(\frac{r\pi}{2}+nx\right) + 2\sin\left(\frac{r\pi}{2}+nx\right)\cos\left(\frac{r\pi}{2}+nx\right)\right]^{\frac{1}{2}}$$
$$= n^r\left[1+\sin 2\left(\frac{r\pi}{2}+nx\right)\right]^{\frac{1}{2}}$$
$$= n^r\left[1+\sin(r\pi+2nx)\right]^{\frac{1}{2}}$$
$$= n^r\left[1+(-1)^r\sin 2nx\right]^{\frac{1}{2}}\quad \textit{showed.}$$

9. $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$

$$sol: Given\,that, y = \frac{x^2+x-1}{x^3+x^2-6x}$$
$$= \frac{x^2+x-1}{x\left(x^2+x-6\right)}$$
$$= \frac{x^2+x-1}{x(x-2)(x+3)}$$
$$= \frac{1}{6}.\frac{1}{x}+\frac{1}{2}.\frac{1}{(x-2)}+\frac{1}{3}.\frac{1}{(x+3)}$$

Differentiating with respect to x we get,

$$y_1 = -\frac{1}{6}.\frac{1}{x^2}-\frac{1}{2}.\frac{1}{(x-2)^2}-\frac{1}{3}.\frac{1}{(x+3)^2}$$
$$\therefore\ y_2 = \frac{1.2}{6}.\frac{1}{x^3}+\frac{1.2}{2}.\frac{1}{(x-2)^3}+\frac{1.2}{3}.\frac{1}{(x+3)^3}$$
$$\therefore\ y_3 = -\frac{1.2.3}{6}.\frac{1}{x^4}-\frac{1.2.3}{2}.\frac{1}{(x-2)^4}-\frac{1.2.3}{3}.\frac{1}{(x+3)^4}$$

Similarly,

$$\therefore\ y_n = (-1)^n\, n!\left[\frac{1}{6}.\frac{1}{x^{n+1}}+\frac{1}{2}.\frac{1}{(x-2)^{n+1}}+\frac{1}{3}.\frac{1}{(x+3)^{n+1}}\right] Ans.$$

Function of Several variables: A function that contains more than one independent variables is called several variables function. For example $u = f(\mathrm{x,y,z}) = \mathrm{x}^2 + y^2 + z^2$ is a function of three variables x, y and z.

Partial Differentiation: The differentiation of a function $u = f\left(x,y\right)$, with respect to x **,** treating y as constant, is called the partial derivative of u with respect to x, and it is denoted as,

$$\frac{\partial u}{\partial x},\ u_x,\ \frac{\partial f}{\partial x},\ f_x.$$

Analytically, $\dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f\left(x+\Delta x, y\right) - f\left(x,y\right)}{\Delta x}$

when this limit exists.

Similarly, the differentiation of a function $u = f\left(x,y\right)$, with respect to y **,** treating x as constant, is called the partial derivative of u with respect to y, and it is denoted as, $\dfrac{\partial u}{\partial y},\ u_y,\ \dfrac{\partial f}{\partial y},\ f_y.$

Analytically, $\dfrac{\partial u}{\partial y}=\lim\limits_{\Delta y\to 0}\dfrac{f\left(x,y+\Delta y\right)-f\left(x,y\right)}{\Delta y}$

provided this limit exists.

Successive Partial Derivatives: Consider a function $u=f\left(x,y\right)$, which has the partial derivatives $\dfrac{\partial u}{\partial x}$ *and* $\dfrac{\partial u}{\partial y}$ with respect to the independent variables x and y respectively. Also each of them may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of u, and these are denoted as,

$$\frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial x\partial y}\,and\ \frac{\partial^2 u}{\partial y\partial x}\ .$$

Similarly, the third order partial derivatives of u are denoted as,

$$\frac{\partial^3 u}{\partial x^3},\ \frac{\partial^3 u}{\partial y^3},\frac{\partial^3 u}{\partial x^2\partial y},\frac{\partial^3 u}{\partial x\partial y^2},\frac{\partial^3 u}{\partial y\partial x^2}\ and\ \frac{\partial^3 u}{\partial y^2\partial x}\ .$$

and so on for higher order derivatives.

Symmetric Function: A function $u=f\left(x,y\right)$ is called a symmetric function if it satisfies the condition $f\left(x,y\right)=f\left(y,x\right)$.

Example: $u=x^2+y^2$ is a symmetric function.

Problem-01: *If* $u=x^3+3x^2y+3xy^2+y^3$ *then find* $\dfrac{\partial^2 u}{\partial x^2},\ \dfrac{\partial^2 u}{\partial y^2},\dfrac{\partial^2 u}{\partial x\partial y}$ *and* $\dfrac{\partial^2 u}{\partial y\partial x}$.

Sol : *Given that*, $u=x^3+3x^2y+3xy^2+y^3\cdots\cdots(1)$

Differentiating (1) partially with respect to x we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^3+3x^2y+3xy^2+y^3\right)$$

$$=3x^2+6xy+3y^2+0$$

$$\therefore\ \frac{\partial u}{\partial x}=3x^2+6xy+3y^2\cdots\cdots(2)$$

Now differentiating (2) partially with respect to x we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(3x^2+6xy+3y^2\right)$$

$$=6x+6y+0$$

$=6x+6y$ **(Ans.)**

Again Differentiating (1) partially with respect to *y* we get,

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^3+3x^2y+3xy^2+y^3\right)$$

$$=0+3x^2+6xy+3y^2$$

$$\therefore\ \frac{\partial u}{\partial y}=3x^2+6xy+3y^2\cdots\cdots(3)$$

Now differentiating (3) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(3x^2+6xy+3y^2\right)$$

$$=0+6x+6y$$

$=6x+6y$ **(Ans.)**

Again Differentiating (3) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(3x^2+6xy+3y^2\right)$$

$$=6x+6y+0$$

$=6x+6y$ **(Ans.)**

Again Differentiating (2) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(3x^2+6xy+3y^2\right)$$

$$=0+6x+6y$$

$=6x+6y$ **(Ans.)**

Problem-02: *If* $u=x^2+y^2\ln x+2e^{-x}y$ *then find* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial y^2}$.

Sol : *Given that,* $u=x^2+y^2\ln x+2e^{-x}y\ \cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2\ln x+2e^{-x}y\right)$$

$$=2x+\frac{y^2}{x}-2e^{-x}y\ \cdots\cdots(2)$$

Now differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(2x+\frac{y^2}{x}-2e^{-x}y\right)$$

$$=2-\frac{y^2}{x^2}+2e^{-x}y \text{ (Ans.)}$$

Again Differentiating (1) partially with respect to y we get,

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^2+y^2\ln x+2e^{-x}y\right)$$

$$=0+2y\ln x+2e^{-x}$$

$$=2y\ln x+2e^{-x}\cdots\cdots(3)$$

Now differentiating (3) partially with respect to y we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(2y\ln x+2e^{-x}\right)$$

$$=2\ln x+0$$

$$=2\ln x \text{ (Ans.)}$$

Problem-03: *If* $u=e^x\left(x\cos y-y\sin y\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial y^2}$. Also show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

Sol : *Giventhat*, $u=e^x\left(x\cos y-y\sin y\right)$

$$=xe^x\cos y-ye^x\sin y\cdots\cdots(1)$$

Differentiating (1) partially with respect to x we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(xe^x\cos y-ye^x\sin y\right)$$

$$=\cos y\frac{\partial}{\partial x}\left(xe^x\right)-y\sin y\frac{\partial}{\partial x}\left(e^x\right)$$

$$=\cos y\left(xe^x+e^x\right)-ye^x\sin y$$

$$=xe^x\cos y+e^x\cos y-ye^x\sin y\cdots\cdots(2)$$

Now differentiating (2) partially with respect to x we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(xe^x\cos y+e^x\cos y-ye^x\sin y\right)$$

$$=\left(xe^x+e^x\right)\cos y+e^x\cos y-ye^x\sin y$$

$$=xe^x\cos y+2e^x\cos y-ye^x\sin y\cdots\cdots(3) \text{ (Ans.)}$$

Again Differentiating (1) partially with respect to y we get,

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(xe^x\cos y-ye^x\sin y\right)$$

$$=xe^x\frac{\partial}{\partial y}(\cos y)-e^x\frac{\partial}{\partial y}(y\sin y)$$

$$=xe^x(-\sin y)-e^x(y\cos y+\sin y)$$

$$=-xe^x\sin y-ye^x\cos y-e^x\sin y\cdots\cdots(4)$$

Now differentiating (4) partially with respect to y we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(-xe^x\sin y-ye^x\cos y-e^x\sin y\right)$$

$$=-xe^x\cos y-e^x(-y\sin y+\cos y)-e^x\cos y$$

$$=-xe^x\cos y+e^xy\sin y-e^x\cos y-e^x\cos y$$

$$=-xe^x\cos y+e^xy\sin y-2e^x\cos y\cdots\cdots(5)$$ **(Ans.)**

Finally, adding (3) and (5) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(xe^x\cos y+2e^x\cos y-ye^x\sin y\right)+\left(-xe^x\cos y+e^xy\sin y-2e^x\cos y\right)$$

$$=xe^x\cos y+2e^x\cos y-ye^x\sin y-xe^x\cos y+e^xy\sin y-2e^x\cos y$$

$$=0$$ **(Showed).**

Problem-04: *If* $u=\tan^{-1}\left(\frac{y}{x}\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Sol : *Giventhat*, $u=\tan^{-1}\left(\frac{y}{x}\right)\cdots\cdots(1)$

Differentiating (1) partially with respect to x we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$=\frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(-\frac{y}{x^2}\right)$$

$$=-\frac{x^2}{x^2+y^2}.\frac{y}{x^2}$$

$$= -\frac{y}{x^2+y^2} \cdots\cdots (2)$$

Now differentiating (2) partially with respect to x we get,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\left\{-\frac{y}{\left(x^2+y^2\right)^2}.(2x+0)\right\}$$

$$= \frac{2xy}{\left(x^2+y^2\right)^2} \textbf{ (Ans.)}$$

Again Differentiating (1) partially with respect to y we get,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}.\ \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2}.\ \frac{1}{x}$$

$$= \frac{x}{x^2+y^2} \cdots\cdots (3)$$

Now differentiating (3) partially with respect to y we get,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= -\frac{x}{\left(x^2+y^2\right)^2}.\ (0+2y)$$

$$= -\frac{2xy}{\left(x^2+y^2\right)^2} \textbf{ (Ans.)}$$

Again Differentiating (2) partially with respect to y we get,

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\left\{ \frac{(x^2+y^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} \right\}$$

$$= -\left\{ \frac{(x^2+y^2) - y(0+2y)}{(x^2+y^2)^2} \right\}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \textbf{ (Ans.)}$$

Problem-05: *If* $u = x^2+y^2+z^2$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$

Sol : *Given that*, $u = x^2+y^2+z^2 \cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= 2x+0+0$$

$$= 2x$$

$$\therefore x\frac{\partial u}{\partial x} = 2x^2 \cdots\cdots(2)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to *y* and *z* we get,

$$y\frac{\partial u}{\partial y} = 2y^2 \cdots\cdots(3)$$

and $z\frac{\partial u}{\partial z} = 2z^2 \cdots\cdots(4)$

Finally adding (2), (3) and (4) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2x^2+2y^2+2z^2$$

$$= 2(x^2+y^2+z^2)$$

$$= 2u \textbf{ (Showed.)}$$

Problem-06: *If* $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u.$

Sol : *Giventhat*, $u=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\}$$

$$=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\left(2x+0+0\right)$$

$$=-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$\therefore\ x\frac{\partial u}{\partial x}=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(2)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to *y* and *z* we get,

$$y\frac{\partial u}{\partial y}=-y^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(3)$$

and $z\dfrac{\partial u}{\partial z}=-z^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(4)$

Finally adding (2), (3) and (4) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}-y^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}-z^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=-\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$=-u$ **(Showed.)**

Problem-07: *If* $u=\left(x^2+y^2+z^2\right)^{\frac{1}{2}}$ *then show that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2}{u}$.

Sol : *Giventhat*, $u=\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right\}$$

$$=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}.\left(2x+0+0\right)$$

$$=x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\cdots\cdots(2)$$

Again Differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\}$$

$$=x.\left\{-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\left(2x+0+0\right)\right\}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$=-\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

$$=\frac{-x^2+x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$=\frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\cdots\cdots(3)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to *y* and *z* we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\cdots\cdots(4)$$

and $\displaystyle\frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\cdots\cdots(5)$

Finally adding (3), (4) and (5) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$=\frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$=\frac{2}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

$=\frac{2}{u}$ **(Showed.)**

Exercise:

Problem-01: *If* $u=e^{xy}\sin x\cos x$ *then find* $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-02: *If* $u=x\cos y+y\cos x$ *then find* $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-03: *If* $u=\ln\left(x^2y+xy^2\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-04: *If* $u=\ln\left(x^3+y^3+z^3-3xyz\right)$ *then show that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$.

Problem-05: *If* $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ *then show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Problem-06: *If* $u=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$ *then show that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

Problem-07: *If* $u=z\tan^{-1}\left(\frac{y}{x}\right)$ *then show that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

Problem-08: *If* $u=\ln\sqrt{\left(x^2+y^2+z^2\right)}$ *then show that* $\left(x^2+y^2+z^2\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$.

Integration

Introduction: Integration is a mathematical technique which is used to find something whose rate of change is known. In 17th century Newton and Leibnitz discovered the idea of integration. It has a wide range application in engineering, medicine, architecture, economics, etc. The objectives of this chapter are to discuss integration and provide standard integration techniques.

Learning Outcomes: By the end of this course, students will be able to........

(a). find displacement from velocity and velocity from acceleration.

(b). calculate areas under curves, volumes of solids, arc lengths.

(c). evaluate center of mass, moment of inertia.

(d). determine work done by a force, electric charge etc.

Integration: The process of finding an anti-derivative or integral of a function is called integration. It is the inverse process of differentiation. If $f(x)$ be a function of x related with another function $F(x)$ in such a way that

$$\frac{d}{dx}\left[F(x)\right] = f(x)$$

then

$$\int f(x)dx = F(x) + c$$

which is called an indefinite integral of $f(x)$ with respect to x.

where $f(x), F(x)$ and c are called integrand, integral and constant of integration respectively.

And

$$\int_a^b f(x)dx = F(b) - F(a)$$

which is called the definite integral of $f(x)$ from a to b, and 'a' is called the lower limit and 'b' the upper limit of the definite integral.

Fundamental Properties:

1. $\int\left[f_1(x) \pm f_2(x) \pm \ldots\ldots\ldots \text{ to } n \text{ terms}\right]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \ldots\ldots \text{ to } n \text{ terms}$.

2. $\int cf(x)dx = c\int f(x)dx$

where c is a constant.

Integration Formulas:

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$ *where* $(n \neq -1)$.

2. $\int \frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}$ *where* $(n \neq 1)$.

3. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$.

4. $\int dx = x$.

5. $\int \frac{dx}{x} = \ln x$.

6. $\int e^x dx = e^x$.

7. $\int e^{mx} dx = \frac{e^{mx}}{m}$.

8. $\int a^x dx = \frac{a^x}{\ln a}$ *where* $a > 0$.

9. $\int \sin mx dx = -\frac{\cos mx}{m}$.

10. $\int \sin x dx = -\cos x$.

11. $\int \cos mx dx = \frac{\sin mx}{m}$.

12. $\int \cos x dx = \sin x$.

13. $\int \sec^2 x dx = \tan x$.

14. $\int \sec x \tan x dx = \sec x$.

15. $\int co\sec^2 x dx = -\cot x$.

16. $\int co\sec x \cot x dx = -co\sec x$.

17. $\int \tan x dx = \ln|\sec x|$.

18. $\int \cot x dx = \ln|\sin x|$.

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ *where* $a \neq 0$.

20. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$.

21. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$.

22. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)$.

23. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right)$.

24. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$.

25. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

26. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$

27. $\int co\sec x dx = \ln|co\sec x - \cot x|$.

28. $\int \sec x dx = \ln|\sec x + \tan x|$.

29. $\int uv dx = u\int v dx - \int\left(\frac{du}{dx}.\int v dx\right)dx$.

30. $\int \sqrt{x^2+a^2}\,dx = \dfrac{x\sqrt{x^2+a^2}}{2} + \dfrac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|.$

31. $\int \sqrt{x^2-a^2}\,dx = \dfrac{x\sqrt{x^2-a^2}}{2} - \dfrac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|.$

32. $\int \sqrt{a^2-x^2}\,dx = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right).$

33. $\int \dfrac{f'(x)}{f(x)}dx = \ln f(x).$ 34. $\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x).$

35. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

36. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

Illustrative Examples:

Problem-01: $\int \sin^2 x\,dx$

Exercise-01: $\int \cos^2 x\,dx$.

Ans: $\dfrac{1}{2}\left(x+\dfrac{\sin 2x}{2}\right)+c.$

Sol^n : *Let* $I = \int \sin^2 x\,dx$

$= \dfrac{1}{2}\int 2\sin^2 x\,dx$

$= \dfrac{1}{2}\int (1-\cos 2x)\,dx$

$= \dfrac{1}{2}\left(x-\dfrac{\sin 2x}{2}\right)+c$

where c is an integrating constant.

Problem-02: $\int \tan^2 x\,dx$

Exercise-02: $\int \cot^2 x\,dx$

Ans: $-\cot x - x + c.$

Sol^n : *Let* $I = \int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$=(\tan x - x) + c.$

where c is an integrating constant.

Problem-03: $\int \frac{a\sin^2 x + b\cos^2 x}{\sin^2 x \cos^2 x}dx$

Exercise-03: $\int \frac{\sin^3 x + \cos^3 x}{\cos x + \sin x}dx$

Ans: $x + \frac{1}{4}\cos 2x + c$.

$$Sol^n : Let\ I = \int \frac{a\sin^2 x + b\cos^2 x}{\sin^2 x \cos^2 x}dx$$

$$= \int \left(\frac{a}{\cos^2 x} + \frac{b}{\sin^2 x} \right) dx$$

$$= \int \left(a\sec^2 x + b\operatorname{cosec}^2 x \right) dx$$

$$= a\tan x - b\cot x + c$$

where c is an integrating constant.

Problem-04: $\int \frac{\cos x - \cos 2x}{1 - \cos x}dx$

Exercise-04: $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}}dx$

Ans: $x + c$.

$$Sol^n : Let\ I = \int \frac{\cos x - \cos 2x}{1 - \cos x}dx$$

$$= \int \frac{\cos x - \left(\cos^2 x - \sin^2 x \right)}{1 - \cos x}dx$$

$$= \int \frac{\cos x - \cos^2 x + \sin^2 x}{1 - \cos x}dx$$

$$= \int \frac{\cos x(1 - \cos x) + \left(1 - \cos^2 x\right)}{1 - \cos x}dx$$

$$= \int \left\{ \frac{\cos x(1 - \cos x)}{1 - \cos x} + \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} \right\} dx$$

$$= \int (\cos x + 1 + \cos x)dx$$

$$=\int (1+2\cos x)dx$$

$$=x+2\sin x+c$$

where c is an integrating constant.

Problem-05: $\int \sqrt{1-\sin 2x}dx$

Exercise-05: $\int \sqrt{1+\sin x}dx$.

Ans: $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$

Sol^n : *Let* $I=\int \sqrt{1-\sin 2x}dx$

.

$$=\int \sqrt{\cos^2 x+\sin^2 x-2\sin x\cos x}dx$$

$$=\int \sqrt{(\cos x-\sin x)^2}\,dx$$

$$=\int (\cos x-\sin x)dx$$

$$=\sin x+\cos x+c$$

where c is an integrating constant.

Problem-06: $\int \sqrt{1+\cos x}dx$

Exercise-06: $\int \sqrt{1-\cos 2x}dx$

Ans: $-\sqrt{2}\cos x+c$.

.

Sol^n : *Let* $I=\int \sqrt{1+\cos x}dx$

$$=\int \sqrt{2\cos^2\frac{x}{2}}dx$$

$$=\sqrt{2}\int \cos\frac{x}{2}\,dx$$

$$=\sqrt{2}\,\frac{\sin\frac{x}{2}}{1/2}+c$$

$$=2\sqrt{2}\sin\frac{x}{2}+c$$

where c is an integrating constant.

Problem-07: $\int \frac{dx}{1+\sin x}$

Exercise-07: $\int \frac{dx}{1+\cos x}$.

Ans: $-\cot x+\operatorname{cosec} x+c$.

$$Sol^n : Let\ I=\int \frac{dx}{1+\sin x}$$

$$=\int \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$=\int \frac{1-\sin x}{1-\sin^2 x}dx$$

$$=\int \frac{1-\sin x}{\cos^2 x}dx$$

$$=\int \frac{1}{\cos^2 x}dx-\int \frac{\sin x}{\cos^2 x}dx$$

$$=\int \sec^2 xdx-\int \sec x\tan xdx$$

$$=\tan x-\sec x+c$$

where c is an integrating constant.

Problem-08: $\int \cos^4 xdx$

Exercise-08: 1. $\int \sin^4 xdx$.

Ans: $\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x+c$.

$$Sol^n : Let\ I=\int \cos^4 xdx$$

$$=\frac{1}{4}\int \left(2\cos^2 x\right)^2 dx$$

$$=\frac{1}{4}\int (1+\cos 2x)^2 dx$$

$$=\frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$=\frac{1}{4}\int \left\{1+2\cos 2x+\frac{1}{2}\left(2\cos^2 2x\right)\right\}dx$$

$$=\frac{1}{4}\int\left\{1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right\}dx$$

$$=\frac{1}{4}\left[x+\sin 2x+\frac{x}{2}+\frac{\sin 4x}{8}\right]+c$$

$$=\frac{1}{4}\left[\frac{3x}{2}+\sin 2x+\frac{\sin 4x}{8}\right]+c$$

$$=\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+c$$

where c is an integrating constant.

Problem-09: $\int\frac{\sin x}{\sqrt{1+\cos x}}dx$

Exercise-09: $\int\frac{\sin 2x}{\sqrt{1-\cos 2x}}dx$

Ans: $\sqrt{2}\sin x+c.$

Sol^n : $Let\ I=\int\frac{\sin x}{\sqrt{1+\cos x}}dx$

$$=\int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}}}dx$$

$$=\int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2}\cos\frac{x}{2}}dx$$

$$=\sqrt{2}\int\sin\frac{x}{2}dx$$

$$=-\sqrt{2}\,\frac{\cos\frac{x}{2}}{\frac{1}{2}}+c$$

$$=-2\sqrt{2}\cos\frac{x}{2}+c$$

where c is an integrating constant.

Problem-10: $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

Exercise-10: $\int \frac{\sin^8 x-\cos^8 x}{1-2sin^2 x\cos^2 x}dx$

$Sol^n : Let\ I=\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

Ans: $-\frac{1}{2}\sin 2x+c$.

$$=\int \frac{\left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3}{\sin^2 x\cos^2 x}dx$$

$$=\int \frac{\left(\sin^2 x+\cos^2 x\right)^3-3\sin^2 x\cos^2 x\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x\cos^2 x}dx$$

$$=\int \frac{1-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\left(\frac{1}{\sin^2 x\cos^2 x}-3\right)dx$$

$$=\int\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-3\right)dx$$

$$=\int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}-3\right)dx$$

$$=\int\left(\sec^2 x+\cos ec^2 x-3\right)dx$$

$$=\tan x-\cot x-3x+c.$$

where c is an integrating constant.

Problem-11: $\int \frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}}dx$

$Sol^n : Let\ I=\int \frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}}dx$

$$=\int \frac{\left(\cos^2 x\right)^2-\left(\sin^2 x\right)^2}{\sqrt{2\cos^2 2x}}dx$$

$$=\frac{1}{\sqrt{2}}\int\frac{\left(\cos^2 x+\sin^2 x\right)\left(\cos^2 x-\sin^2 x\right)}{\cos 2x}dx$$

$$=\frac{1}{\sqrt{2}}\int\frac{\cos 2x}{\cos 2x}dx$$

$$=\frac{1}{\sqrt{2}}\int dx$$

$$=\frac{x}{\sqrt{2}}+c$$

where c is an integrating constant.

Method of substitution

Sometimes, the integration of given integral $\int f(x)dx$ is relatively difficult. In this case, we can replace x by $\varphi(z)$ and dx by $\varphi'(z)dz$ for integrating easily. This process is known as method of substitution.

Problem-01: $\int(a+bx)^n\,dx$

sol^n : Let $I=\int(a+bx)^n dx$

put $z=a+bx$ $\therefore dz=bdx$

$$\Rightarrow\frac{1}{b}dz=dx$$

Now $I=\int z^n\frac{1}{b}dz$

$$=\frac{1}{b}\int z^n dz$$

$$=\frac{1}{b}\frac{z^{n+1}}{n+1}+c$$

$$=\frac{(a+bx)^{n+1}}{b(n+1)}+c$$

Exercise-01: $\int\frac{2\sin x}{5+3\cos x}dx$

Ans: $-\frac{2}{3}\ln(5+3\cos x)+c$.

where c is an integrating constant.

Problem-02: $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$

$sol^n : Let\ I = \int \frac{\sin^{-1} x dx}{\sqrt{1-x^2}}$

put $z = \sin^{-1} x \quad \therefore dz = \frac{dx}{\sqrt{1-x^2}}$

Now $I = \int z dz$

$= \frac{z^2}{2} + c$

$= \frac{\left(\sin^{-1} x\right)^2}{2} + c$

where c is an integrating constant.

Exercise-02: $\int \frac{e^{m\tan^{-1} x}}{1+x^2}dx$

Ans: $\frac{e^{m\tan^{-1} x}}{m} + c$.

Problem-03: $\int \frac{(1+x)e^x}{\cos^2\left(xe^x\right)}dx$

$sol^n :\ Let\ I = \int \frac{(1+x)e^x}{\cos^2\left(xe^x\right)}dx$

put $xe^x = z \therefore (1+x)e^x dx = dz$

Now $I = \int \frac{dz}{\cos^2 z}$

$= \int \sec^2 z dz$

$= \tan z + c$

$= \tan\left(xe^x\right) + c$

where c is an integrating constant.

Exercise-03: $\int \frac{(x+1)(x+\ln x)^2}{x}dx$

Ans: $\frac{1}{3}(x+\ln x)^3 + c$.

Integration by Parts

The formula for the integration of a product of two functions is referred to as integration by parts. *i.e,*

$$\int (uv)dx = u\int vdx - \int\left\{\frac{du}{dx}\int vdx\right\}dx\ .$$

While applying the above rule for integration by parts to the product of two functions, care should be taken to choose properly the first function, i.e., the function not to be integrated.

Problem-01: $\int xe^x dx$

$sol^n: Let\ I = \int xe^x dx$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx$$

$$= xe^x - \int 1.e^x dx$$

$$= xe^x - e^x + c$$

where c is an integration constant.

Exercise-01: $\int x^2 \cos x dx$

Ans: $x^2 \sin x + 2x\cos x - 2\sin x + c$

Problem-02: $\int \tan^{-1} x dx$

$sol^n: Let\ I = \int \tan^{-1} x dx$

$$= \tan^{-1} x\int dx - \int\left\{\frac{d}{dx}(\tan^{-1} x)\int dx\right\}dx$$

$$= x\tan^{-1} x - \int \frac{1}{1+x^2}.xdx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + c$$

Exercise-02: $\int \cos^{-1} x dx$

Ans: $x\cos^{-1} x - \sqrt{1-x^2} + c$

where c is an integration constant

Problem-03: $\int e^{ax}\cos bx dx$

Exercise-03: $\int e^{ax}\sin(bx+d)dx$

Ans: $\dfrac{e^{ax}\left[a\sin(bx+d)-b\cos(bx+d)\right]}{a^2+b^2}+c$

Sol^n : $Let\ I=\int e^{ax}\cos bx dx$

$$=e^{ax}\int\cos bx dx-\int\left\{\frac{d}{dx}\left(e^{ax}\right)\int\cos bx dx\right\}dx$$

$$=\frac{e^{ax}\sin bx}{b}-\int\left\{ae^{ax}\frac{\sin bx}{b}\right\}dx$$

$$=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx dx$$

$$=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\left[e^{ax}\int\sin bx dx-\int\left\{\frac{d}{dx}\left(e^{ax}\right)\int\sin bx dx\right\}dx\right]$$

$$=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\left[\frac{-e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx dx\right]$$

$$=\frac{e^{ax}\sin bx}{b}+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}I$$

$$\therefore\ I+\frac{a^2}{b^2}I=\frac{e^{ax}\sin bx}{b}+\frac{a}{b^2}e^{ax}\cos bx$$

$$\Rightarrow\frac{I\left(a^2+b^2\right)}{b^2}=\frac{be^{ax}\sin bx+ae^{ax}\cos bx}{b^2}$$

$$\Rightarrow I=\frac{e^{ax}\left(a\cos bx+b\sin bx\right)}{a^2+b^2}$$

$$\therefore\ I=\frac{e^{ax}\left(a\cos bx+b\sin bx\right)}{a^2+b^2}+c$$

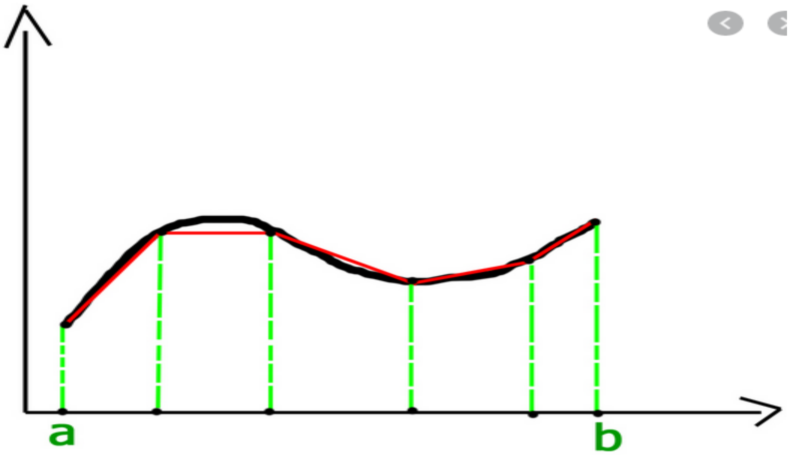
where c is an integrating constant.

Fundamental Theorem of Integral Calculus: If $f(x)$ be a bounded and continuous function defined in the interval $[a, b]$ where, $b>a$ and there exists a function $\varphi(x)$ such that $\varphi'(x) = f(x)$, then

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$$

This is called the fundamental theorem of integral calculus.

Integration as the limit of a sum: Let, $f(x)$ be a continuous, bounded and single-valued function defined in the interval $[a, b]$ where a, b are finite quantities and $b > a$.



If the interval $[a, b]$ be divided into n equal sub-intervals, each of length $h(h \to 0)$, by the points $a+h,\ a+2h, \cdots\cdots a+(n-1)h$ so that $nh = b-a$, then the area enclosed by $f(x)$ is defined as

$$S = \lim_{h\to 0}\left[hf(a) + hf(a+h) + hf(a+2h) + \cdots\cdots + hf\{a+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) \qquad where,\ nh = b-a$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(a+\frac{r}{n}\right) \qquad \text{where } h = \frac{1}{n} \text{ if } h \to 0 \text{ then } n \to \infty.$$

Which is also defined as the definite integral of $f(x)$ with respect to x between the limits a and b, and is denoted by the symbol,

$$\int_a^b f(x)dx$$

where, *a* is called the lower limit and *b* is called the upper limit.

Therefore, $\int_a^b f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(a+\frac{r}{n}\right)$ where $nh = b-a$.

NOTE:

1. $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx$;OR, $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx$;OR, $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right)=\int_0^1 f(x)dx$
2. $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n-1} f\left(\frac{r}{n}\right)=\int_0^2 f(x)dx$ OR, $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right)=\int_0^2 f(x)dx$
3. $\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n-1} f\left(\frac{r}{n}\right)=\int_0^3 f(x)dx$ OR, $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n} f\left(\frac{r}{n}\right)=\int_0^3 f(x)dx$

Problem-01: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{n+n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)}$$

$$=\int_0^1\frac{dx}{1+x}$$

$$=\left[\ln(1+x)\right]_0^1 = \ln(1+1)-\ln(1+0) = \ln 2$$

Problem-02: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

$$=\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{n^2}{(n+n)^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}$$

$$=\left[-\frac{1}{2}\frac{1}{(1+x)^2}\right]_0^1$$

$$=\left[-\frac{1}{2}\frac{1}{(1+1)^2}+\frac{1}{2}\frac{1}{(1+0)^2}\right]$$

$$=-\frac{1}{8}+\frac{1}{2}\;=\frac{3}{8}$$

Problem-03: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\sqrt{1-x^2}\,dx$$

$$=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=\left[\frac{1.\sqrt{1-1^2}}{2}+\frac{1}{2}\sin^{-1}.1-\frac{0.\sqrt{1-0^2}}{2}-\frac{1}{2}\sin^{-1}.0\right]$$

$$=\frac{1}{2}\sin^{-1}.\sin\frac{\pi}{2}\ =\frac{\pi}{4}$$

Problem-04: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

Solution: Given that, $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n.1-1^2}}+\frac{1}{\sqrt{2n.2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{2n.n-n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2nr-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\left\{\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2\right\}}}$$

$$=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(1-x)^2}}$$

$$=-\left[\sin^{-1}(1-x)\right]_0^1$$

$$=-\left[\sin^{-1}(1-1)-\sin^{-1}(1-0)\right]$$

$$=-\sin^{-1}.0+\sin^{-1}.1$$

$$=-\sin^{-1}.\sin 0+\sin^{-1}.\sin\frac{\pi}{2}$$

$$=0+\frac{\pi}{2}\ =\frac{\pi}{2}$$

Problem-05: Evaluate $\int_a^b xdx$ from the definition of the integral as the limit of a sum.

Solution: We have $I = \int_a^b xdx$

Here $f(x) = x$

$$\therefore f(a) = a,\ f(a+h) = a+h,\ f(a+2h) = a+2h,\ \cdots,\ f\{a+(n-1)h\} = a+(n-1)h$$

Since $\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots + f\{a+(n-1)h\}\right]$

where $nh = b-a$

$$\therefore I = \lim_{h\to 0} h\left[a + (a+h) + (a+2h) + \cdots + \{a+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[na + h\{1+2+\cdots+(n-1)\}\right]$$

$$= \lim_{h\to 0} h\left[na + h.\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[nha + \frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[(b-a)a + \frac{(b-a)(b-a-h)}{2}\right]$$

$$= (b-a)a + \frac{(b-a)(b-a)}{2} = \frac{b^2-a^2}{2}.$$

Problem-06: Evaluate $\int_a^b \sin xdx$ from the definition of the integral as the limit of a sum.

Solution: We have $I = \int_a^b \sin xdx$

Here $f(x) = \sin x$

$$\therefore f(a) = \sin a,\ f(a+h) = \sin(a+h),\ \cdots,\ f\{a+(n-1)h\} = \sin\{a+(n-1)\}h$$

Since $\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots + f\{a+(n-1)h\}\right]$

where $nh = b-a$

$$\therefore I = \lim_{h\to 0} h\left[\sin a + \sin(a+h) + \sin(a+2h) + \cdots + \sin\{a+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(a+\frac{n-1}{2}h\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$= 2\lim_{\frac{h}{2}\to 0}\frac{\frac{h}{2}}{\sin\left(\frac{h}{2}\right)}.\lim_{h\to 0}\sin\left(a+\frac{nh-h}{2}\right)\sin\left(\frac{nh}{2}\right)$$

$$= 2.1.\lim_{h\to 0}\sin\left(a+\frac{b-a-h}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= 2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= \cos a - \cos b\,.$$

Assignment:

Problem-01: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

Problem-02: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Problem-03: Evaluate $\int_a^b x^2dx$ from the definition of the integral as the limit of a sum.

Problem-04: Evaluate $\int_a^b \cos xdx$ from the definition of the integral as the limit of a sum.

Problem-05: Evaluate $\int_a^b e^xdx$ from the definition of the integral as the limit of a sum.

Problem-06: Evaluate $\int_0^{\frac{\pi}{2}} \sin xdx$ from the definition of the integral as the limit of a sum.

Some Definite integrations

Problem-01: Evaluate $\int_0^{\pi/2} \cos^2 x dx$

Solution: Let, $I = \int_0^{\pi/2} \cos^2 x dx$

$$= \frac{1}{2}\int_0^{\pi/2} 2\cos^2 x dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin 2.\frac{\pi}{2}}{2}\right) - \left(0 + \frac{\sin 2.0}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}$$

Problem-02: Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{dx}{1+\cos x}$

$$= \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2}\left[\frac{\tan\frac{x}{2}}{1/2}\right]_0^{\pi/2}$$

$$= \left[\tan\frac{x}{2}\right]_0^{\pi/2}$$

$$= \tan\frac{\pi}{4} - \tan\frac{0}{2} = 1$$

Problem-03: Evaluate $\int_0^{\ln 2} \frac{e^x}{1+e^x}dx$

Solution: Let, $I = \int_0^{\ln 2} \frac{e^x}{1+e^x}dx$

$$= \left[\ln\left(1+e^x\right)\right]_0^{\ln 2}$$

$$= \ln\left(1+e^{\ln 2}\right) - \ln\left(1+e^0\right)$$

$$= \ln(1+2) - \ln(1+1)$$

$$= \ln 3 - \ln 2 = \ln\frac{3}{2}$$

Problem-04: Evaluate $\int_0^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

Solution: Let, $I = \int_0^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

$$= \frac{1}{4}\int_0^{\pi/3} \frac{4\cos x dx}{3+4\sin x}$$

$$= \frac{1}{4}\left[\ln(3+4\sin x)\right]_0^{\pi/3}$$

$$= \frac{1}{4}\left[\ln\left(3+4\sin\frac{\pi}{3}\right) - \ln(3+4\sin 0)\right]$$

$$= \frac{1}{4}\left[\ln\left(3+4.\frac{\sqrt{3}}{2}\right) - \ln 3\right]$$

$$=\frac{1}{4}\left[\ln\left(3+2\sqrt{3}\right)-\ln 3\right]$$

$$=\frac{1}{4}\ln\left(\frac{3+2\sqrt{3}}{3}\right)$$

Problem-05: Evaluate $\int_0^{\pi/2}\cos^7 x dx$

Solution: Let, $I=\int_0^{\pi/2}\cos^7 x dx$

$$=\int_0^{\pi/2}\cos^6 x\cos x dx$$

$$=\int_0^{\pi/2}\left(\cos^2 x\right)^3\cos x dx$$

$$=\int_0^{\pi/2}\left(1-\sin^2 x\right)^3\cos x dx$$

put, $\sin x=t \therefore \cos x dx=dt$

when $x=0$ then $t=0$

when $x=\frac{\pi}{2}$ then $t=1$

Now, $I=\int_0^1\left(1-t^2\right)^3 dt$

$$=\int_0^1\left(1-3t^2+3t^4-t^6\right)dt$$

$$=\left[t-t^3+3\frac{t^5}{5}-\frac{t^7}{7}\right]_0^1$$

$$=1-1+\frac{3}{5}-\frac{1}{7}$$

$$=\frac{16}{35}$$

Problem-06: Evaluate $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$= \frac{1}{b^2}\int_0^{\pi/2} \frac{dx}{\cos^2 x\left\{\left(\frac{a}{b}\right)^2 + \tan^2 x\right\}}$$

$$= \frac{1}{b^2}\int_0^{\pi/2} \frac{\sec^2 x dx}{\left(\frac{a}{b}\right)^2 + \tan^2 x}$$

put, $\tan x = t \quad \therefore \quad \sec^2 x dx = dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = \infty$

Now, $I = \frac{1}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$

$$= \frac{1}{b^2}\left[\frac{1}{a/b}\tan^{-1}\frac{t}{a/b}\right]_0^{\infty}$$

$$= \frac{1}{b^2}\left[\frac{b}{a}\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$

$$= \frac{1}{ab}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{1}{ab}\left(\tan^{-1}\tan\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2ab}$$

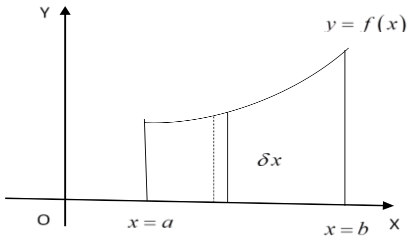
Area under curves (Quadrature)

Our concentration in this Chapter is to find the area bounded by curves with a general formula or with the help of definite integration. This process is called Quadrature.

Area formula for Cartesian equation:

(1). The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is,

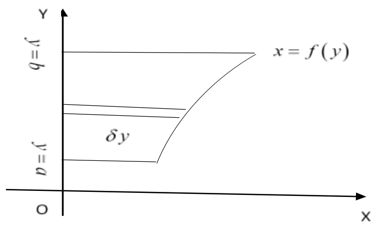
$$A = \int_a^b y dx$$



Where, $y = f(x)$ is a continuous single valued function and it does not change sign for $a \le x \le b$.

(2). The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$ is,

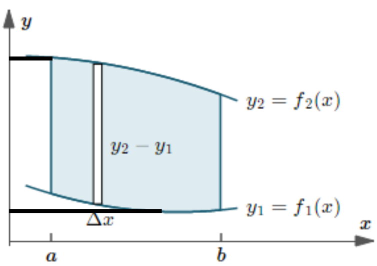
$$A = \int_a^b x dy$$



Where, $x = f(y)$ is a continuous single valued function and it does not change sign for $a \le y \le b$.

(3).The area bounded by two curves $y_1 = f_1(x)$, $y_2 = f_2(x)$ and two vertical lines $x = a$ & $x = b$ is

$$A = \int_a^b (y_2 - y_1) dx.$$



(4). The area bounded by the curve Symmetry about the x -axis is,

$$A = 2\int_0^a y dx$$

(5). The area bounded by the curve Symmetry about the y -axis is,

$$A = 2\int_0^a x dy$$

Symmetry about the x -axis: If all the powers of y occurring in an equation are even then it is symmetry about the x -axis. For example, $y^2 = 4ax$ is symmetry about the x -axis.

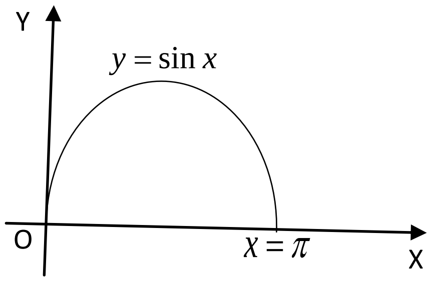
Symmetry about the y -axis: If all the powers of x occurring in an equation are even then it is symmetry about the y -axis. For example, $x^2 = 4ay$ is symmetry about the y -axis.

Mathematical Problems

Problem 01: Find the area bounded by the curve $y = \sin x$, the $x - axis$ and the straight lines $x = 0$ and $x = \pi$.

Solution: We have, $y = \sin x$ and $x = 0$; $x = \pi$.

The graph of the given curve is,



The area of the region is,

$$A = \int_0^\pi y\, dx$$

$$= \int_0^\pi \sin x\, dx = \left[-\cos x\right]_0^\pi$$

$$= -\cos\pi - (-\cos 0) = -(-1) + 1$$

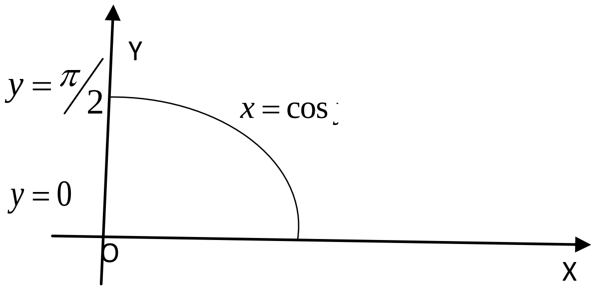
$$= 1 + 1 = 2$$

$\therefore A = 2$ **Sq. Units.**

Problem 02: Find the area bounded by the curve $x = \cos y$, the $y - axis$ and the straight lines $y = 0$ and $y = \pi/2$.

Solution: We have, $x = \cos y$ and $y = 0$; $y = \pi/2$.

The graph of the given curve is,



The area of the region is,

$$A=\int_0^{\pi/2} x\,dy$$

$$=\int_0^{\pi/2} \cos y\,dy=[\sin y]_0^{\pi/2}$$

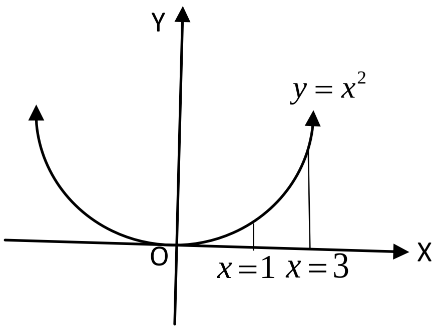
$$=\sin\left(\pi/2\right)-\sin 0=1-0=1$$

$\therefore\ A=1$ **Sq. Units.**

Problem 03: Find the area bounded by the curve $y=x^2$, the $x-axis$ and the straight lines $x=1$ and $x=3$.

Solution: We have, $y=x^2$ and $x=1$; $x=3$.

The graph of the given curve is,



The area of the region is,

$$A=\int_1^3 y\,dx$$

$$=\int_1^3 x^2\,dx=\left[\frac{x^3}{3}\right]_1^3=\frac{1}{3}\left[x^3\right]_1^3=\frac{1}{3}\left(3^3-1\right)=\frac{1}{3}\left(27-1\right)$$

$\therefore\ A=\dfrac{26}{3}$ **Sq. Units.**

H.W:

1. Find the area bounded by the curve $x=\sin y$, the $y-axis$ and the straight lines $y=0$ and $y=\pi$.

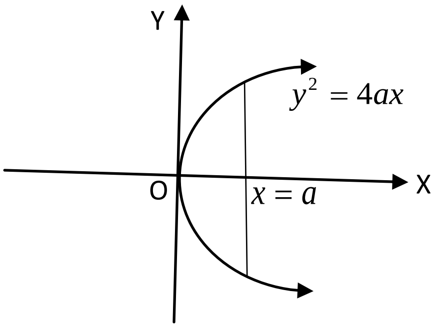
2. Find the area bounded by the curve $y=\sin x$, the $x-axis$ and the straight lines $x=0$ and $x=\pi$.

3. Find the area bounded by the curve $y=x^3$, the $x-axis$ and the straight lines $x=1$ and $x=4$.

Problem 04: Find the area of the region bounded by the curve $y^2=4ax$; from $x=0$ and $x=a$.

Solution: We have, $y^2=4ax$ and $x=0$; $x=a$.

Since, only even power of y occurs in the given curve so the curve is symmetric about the x-axis.
The graph of the given curve is,



Also, the given curve can be written as,

$$y^2=4ax$$
$$\Rightarrow y=\pm 2\sqrt{ax}$$

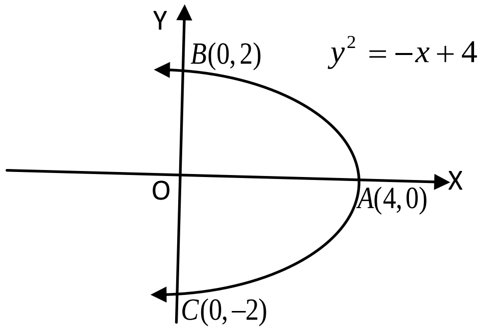
The area of the region is,

$$A=2\int_0^a y\,dx$$
$$=2\int_0^a 2\sqrt{ax}\,dx \qquad [Neglecting\ negative\ sign]$$
$$=4\sqrt{a}\int_0^a \sqrt{x}\,dx$$
$$=4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$
$$=\frac{8\sqrt{a}}{3}\left[x^{3/2}\right]_0^a$$
$$=\frac{8\sqrt{a}}{3}\left(a^{3/2}-0\right)$$
$$=\frac{8\sqrt{a}\times a^{3/2}}{3}=\frac{8a^2}{3}$$
$$\therefore A=\frac{8a^2}{3} \qquad \textbf{Sq. Units.}$$

Problem 05: Find the area of the region bounded by the curve $y^2=-x+4$ and y-axis.

Solution: We have, $y^2=-x+4 \quad \cdots\cdots\cdots(1)$

Since, only even power of y occurs in the given curve so the curve is symmetric about the x-axis.
The graph of the given curve is,



Putting $y = 0$ in (1) then we have $x = 4$, so the vertex is at $A(4,0)$.
Also putting $x = 0$ in (1) then we have $y = \pm 2$. So the curve crosses the y-axis at $B(0,2)$ and $C(0,-2)$. The given curve can be written as,

$$y^2 = -x+4$$
$$\Rightarrow y = \pm\sqrt{4-x}$$

The area of the region is,

$$A = 2\int_0^4 y\,dx$$
$$= 2\int_0^4 \sqrt{4-x}\,dx \qquad \left[Neglecting\ negative\ sign\right]$$
$$= 2\left[\frac{(4-x)^{3/2}}{(-1).\,3/2}\right]_0^4$$
$$= -2.\,\frac{2}{3}.\left[(4-x)^{3/2}\right]_0^4$$
$$= -\frac{4}{3}.\left[(4-4)^{3/2} - (4-0)^{3/2}\right]$$
$$= -\frac{4}{3}.\left[0-(4)^{3/2}\right]$$
$$= \frac{4}{3}.(2^2)^{3/2}$$
$$= \frac{4}{3}.\ 2^3$$

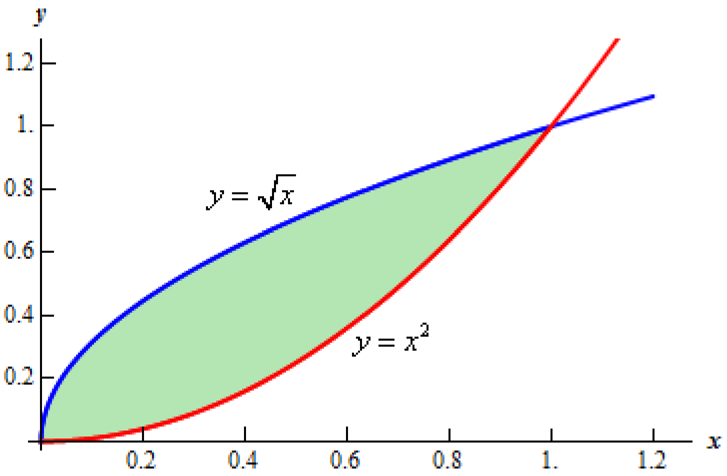
$\therefore A = \dfrac{32}{3}$ **Sq. Units.**

H.W:

1. Find the area of the region bounded by the curve $x^2 = 4ay$; from $y = 0$ and $y = a$.
2. Find the area of the region bounded by the curve $y^2 = 12x$; from $x = 0$ and $x = 3$.

Problem 06: Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution: The equation of the given curves are $y = x^2$ and $y = \sqrt{x}$.
The graph of the given curves is as follows:



We have

$$y = x^2 \text{ and } y = \sqrt{x}$$

Now,

$$x^2 = \sqrt{x}$$

$$or,\ \left(x^2\right)^2 = \left(\sqrt{x}\right)^2 \qquad \text{[Squaring both sides]}$$

$$or,\ x^4 = x$$

$$or,\ x^4 - x = 0$$

$$or,\ x\left(x^3 - 1\right) = 0$$

Therefore, $x = 0$ and $x^3 - 1 = 0$

$$\Rightarrow (x-1)(x^2 + x + 1) = 0$$

$\therefore\ x - 1 = 0 \qquad or\ \ x^2 + x + 1 = 0$

$$\Rightarrow x = 1 \qquad or\ \ x^2 + x + 1 = 0$$

$$or,\ x = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$or,\ x = \frac{-1 \pm \sqrt{-3}}{2}$$

For real $x = 0 \,\&\, 1$ we get respectively $y = 0 \,\&\, 1$

Therefore, the given curves intersect each other in two point at $(0,0)$ and $(1,1)$.

In the question, $a = 0$,$b = 1, y_2 = \sqrt{x}$ *and* $y_1 = x^2$.

So, the area of the region is,

$$A = \int_a^b (y_2 - y_1)\,dx$$

$$= \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$

$$= \int_0^1 x^{\frac{1}{2}}\, dx - \int_0^1 x^2\, dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 - \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}} \right]_0^1 - \frac{1}{3}\left[x^3 \right]_0^1$$

$$= \frac{2}{3}(1-0) - \frac{1}{3}(1-0)$$

$$= \frac{2}{3}(1) - \frac{1}{3}(1) = \frac{2}{3} - \frac{1}{3}$$

$\therefore A = \frac{1}{3}$ **Sq. Units.** **(As desired)**

Integration by Partial Fraction

Rational Fraction: If $P(x)$ & $Q(x)$ are two polynomials in x and $Q(x) \neq 0$ then the quotient $\frac{P(x)}{Q(x)}$ is called a rational fraction.

Example: $\frac{x^2+1}{x^3-2x+3}$ is a rational fraction.

Proper Fraction: A fraction in which the degree of the numerator is less than the degree of denominator is called a proper fraction.

Example: $\frac{x^2+1}{x^3-2x+3}$ is a proper fraction.

Improper Fraction: A fraction in which the degree of the numerator is greater or equal to the degree of denominator is called an improper fraction.

Example: $\frac{x^2+1}{x^2-2x+3}$ & $\frac{x^3+1}{x^2-2x+3}$ are improper fractions.

Partial Fraction: A given fraction may be written as a sum of other fractions (called partial fractions) whose denominator is less than the denominator of the given fraction.

Fundamental theorem: Any fraction may be written as the sum of partial fractions according the following rules:

Case-1: When the fraction is **Proper fraction**:

a. When all factors are linear and different
   i.e.

$$\frac{f(x)}{(x \pm a)(x \pm b)} = \frac{(?)}{x \pm a} + \frac{(?)}{x \pm a} \cdots\cdots (1)$$

where the coefficients of the blank spaces cannot be zero.

NOTE: Using the **Cover up method** we can find the values of the blank spaces of (1).

Cover up method: This method is applicable only for linear factors.

If $\dfrac{f(x)}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$ then

For A: Cover $(x-a)$ term in the denominator of the left-hand side and substitute $x = a$ in the remaining expression.

For B: Cover $(x-b)$ term in the denominator of the left-hand side and substitute $x = b$ in the remaining expression.

b. When all factors are linear and some are repeated
i.e.

$$\frac{f(x)}{(x\pm a)(x\pm b)^n} = \frac{(?)}{(x\pm a)} + \frac{(?)}{(x\pm b)^n} + \frac{A}{(x\pm b)^{n-1}} + \cdots\cdots + \frac{B}{(x\pm b)} \quad \cdots\cdots (2)$$

NOTE: Find the coefficients of the blank spaces by using **Cover up method** and then to find A substitute any value for x except $x = \pm\, a$ & $x = \pm\, b$.

c. When all factors are quadratic and different
i.e.

$$\frac{f(x)}{(x^2\pm a)(x^2\pm b)} = \frac{Ax+B}{x^2\pm a} + \frac{Cx+D}{x^2\pm b} \quad \cdots\cdots (3)$$

NOTE: To find the values of A, B, C & D multiplying both sides of (3) by $(x^2\pm a)(x^2\pm b)$ and then substitute the appropriate values for x.

d. When all factors are quadratic and some are repeated
i.e.,

$$\frac{f(x)}{(x^2\pm a)(x^2\pm b)^2} = \frac{Ax+B}{(x^2\pm a)} + \frac{Cx+D}{(x^2\pm b)^2} + \frac{C_1x+D_1}{(x^2\pm b)} \quad \cdots\cdots (4)$$

NOTE: To find the values of A, B, C, D, C_1 & D_1 multiplying both sides of (4) by $(x^2\pm a)(x^2\pm b)^2$ and then substitute the appropriate value for x.

Case-2: When the fraction is **improper fraction**: To split an improper fraction into a partial fraction, we will have to divide the numerator by denominator.

Example: If $\dfrac{3x^2-2x-2}{x^2-3x+2}$ then

$$x^2-3x+2\overline{\big)\,3x^2-3x-2\,\big(}3$$
$$\underline{3x^2-9x+6}$$
$$6x-8$$

$$Since,\ Dividend=\left(Divisor\times Quotient\right)+\mathrm{Re}\,mainder$$

Rewriting the given improper fraction we get

$$\frac{3x^2-2x-2}{x^2-3x+2}=3+\frac{6x-8}{x^2-3x+2}$$

Now using the Cover up method anyone can solve the fraction.

Problem-01: Evaluate $\int\frac{5x-11}{2x^2+x-6}dx$.

Solution: Let $I=\int\frac{5x-11}{2x^2+x-6}dx$

$$=\int\frac{5x-11}{2x^2+4x-3x-6}dx$$

$$=\int\frac{5x-11}{2x(x+2)-3(x+2)}dx$$

$$=\int\frac{5x-11}{(x+2)(2x-3)}dx$$

$$=\int\left(\frac{3}{x+2}-\frac{1}{2x-3}\right)dx$$

$$=3\ln(x+2)-\frac{1}{2}\ln(2x-3)+c$$

where c is an integrating constant.

Problem-02: Evaluate $\int\frac{3x^2+x-2}{(x-2)^2(1-2x)}dx$.

Solution: Let $I=\int\frac{3x^2+x-2}{(x-2)^2(1-2x)}dx$

Here $\frac{3x^2+x-2}{(x-2)^2(1-2x)}=\frac{-4}{(x-2)^2}+\frac{-\frac{1}{3}}{(1-2x)}+\frac{A}{(x-2)}\qquad\cdots(1)$

Putting $x=0$ in (1) we get,

$$\frac{3(0)^2+0-2}{(0-2)^2(1-2\times0)}=\frac{-4}{(0-2)^2}+\frac{-\frac{1}{3}}{(1-2\times0)}+\frac{A}{(0-2)}$$

$$or,\ \frac{-2}{4}=\frac{-4}{4}+\frac{-\frac{1}{3}}{1}+\frac{A}{-2}$$

$$or,\ -\frac{1}{2}=-1-\frac{1}{3}-\frac{A}{2}$$

$$or,\ \frac{A}{2}=-1-\frac{1}{3}+\frac{1}{2}$$

$$or,\ \frac{A}{2}=\frac{-6-2+3}{6}$$

$$or,\ A=-\frac{5}{3}$$

From (1) we get,

$$\frac{3x^2+x-2}{(x-2)^2(1-2x)}=-\frac{4}{(x-2)^2}-\frac{1}{3}\frac{1}{(1-2x)}-\frac{5}{3}\frac{1}{(x-2)}$$

Now $I=\int\left[-\frac{4}{(x-2)^2}-\frac{1}{3}\frac{1}{(1-2x)}-\frac{5}{3}\frac{1}{(x-2)}\right]dx$

$$=\frac{4}{x-2}+\frac{1}{6}\ln(1-2x)-\frac{5}{3}\ln(x-2)+c$$

Problem-03: Evaluate $\int\frac{7+x}{(1+x)\left(1+x^2\right)}dx$.

Solution: Let $I=\int\frac{7+x}{(1+x)\left(1+x^2\right)}dx$

Here $\frac{7+x}{(1+x)\left(1+x^2\right)}=\frac{3}{(1+x)}+\frac{Ax+B}{\left(1+x^2\right)}\ \cdots\cdots$ (1)

Multiplying both sides by $(1+x)\left(1+x^2\right)$, we get

$$7+x=3\left(1+x^2\right)+(Ax+B)(1+x)$$

$$or,\ 7+x=3+3x^2+Ax^2+Bx+Ax+B$$

$$or,\ 7+x=3+3x^2+Ax^2+Bx+Ax+B$$

$$or,\ 7+x=(3+A)x^2+(A+B)x+B+3$$

Equating the coefficients of x^2, x and constant terms we get,

$$A+3=0 \quad ; \quad A+B=1 \quad ; B+3=7$$

$$or,\ A=-3 \quad ; \quad B=4$$

From (1) we get,

$$\frac{7+x}{(1+x)\left(1+x^2\right)}=\frac{3}{1+x}+\frac{-3x+4}{1+x^2}$$

$$=\frac{3}{1+x}+\frac{4-3x}{1+x^2}$$

Now $I=\int\left[\frac{3}{1+x}+\frac{4-3x}{1+x^2}\right]dx$

$$=\int\left[\frac{3}{1+x}+\frac{4}{1+x^2}-\frac{3x}{1+x^2}\right]dx$$

$$=3\ln(1+x)+4\tan^{-1}x-\frac{3}{2}\ln\left(1+x^2\right)+c$$

where c is an integrating constant.

Problem-04: Evaluate $\int\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}\,dx$.

Solution: Let $I=\int\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}dx$

Here $\quad \frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}=\frac{Ax+B}{\left(x^2+5\right)}+\frac{Cx+D}{\left(x^2-3\right)} \cdots\cdots (1)$

Multiplying both sides of (1) by $\left(x^2+5\right)\left(x^2-3\right)$ we get,

$$x+1=(Ax+B)\left(x^2-3\right)+(Cx+D)\left(x^2+5\right)$$

$$or,\ x+1=Ax^3-3Ax+Bx^2-3B+Cx^3+5Cx+Dx^2+5D$$

$$or,\ x+1=(A+C)x^3+(B+D)x^2+(5C-3A)x-3B+5D$$

Equating the coefficients of like term we get,

$$A+C=0\ ; B+D=0\ ;\ 5C-3A=1\ ;\ -3B+5D=1$$

$$A=-C\ ; B=-D\ ;\ 5C-3A=1\ ;\ -3B+5D=1$$

Since $A=-C$ so $5C-3A=1 \Rightarrow 5C-3(-C)=1$

$$or,\ 5C+3C=1$$

$$or,\ 8C=1$$

$$or,\ C=\frac{1}{8} \text{ and } A=-\frac{1}{8}$$

Again $B=-D$ so $-3B+5D=1 \Rightarrow -3(-D)+5D=1$

$$or,\ 3D+5D=1$$

$$or,\ 8D=1$$

$$or,\ D=\frac{1}{8} \text{ and } B=-\frac{1}{8}$$

From (1) we get,

$$\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}=\frac{-\frac{1}{8}x-\frac{1}{8}}{\left(x^2+5\right)}+\frac{\frac{1}{8}x+\frac{1}{8}}{\left(x^2-3\right)}$$

$$=\frac{1}{8}\cdot\frac{x+1}{\left(x^2-3\right)}-\frac{1}{8}\cdot\frac{x+1}{\left(x^2+5\right)}$$

Now $I=\int\left[\frac{1}{8}\cdot\frac{x+1}{\left(x^2-3\right)}-\frac{1}{8}\cdot\frac{x+1}{\left(x^2+5\right)}\right]dx$

$$=\frac{1}{8}\int\left[\frac{x}{x^2-3}+\frac{1}{x^2-3}-\frac{x}{x^2+5}-\frac{1}{x^2+5}\right]dx$$

$$=\frac{1}{8}\left[\frac{1}{2}\ln\left(x^2-3\right)+\frac{1}{2\sqrt{3}}\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}-\frac{1}{2}\ln\left(x^2+5\right)-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]+c$$

$$=\frac{1}{16}\ln\left(x^2-3\right)+\frac{1}{16\sqrt{3}}\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}-\frac{1}{16}\ln\left(x^2+5\right)-\frac{1}{8\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

where c is an integrating constant.

Problem-05: Evaluate $\int\frac{2x^2+x+1}{x^2+2x-3}\,dx$.

Solution: Let $I=\int\frac{2x^2+x+1}{x^2+2x-3}\,dx$

Here $\frac{2x^2+x+1}{x^2+2x-3}=2+\frac{7-3x}{x^2+2x-3}$

$$=2+\frac{7-3x}{x^2+3x-x-3}$$

$$= 2 + \frac{7-3x}{x(x+3)-1(x+3)}$$

$$= 2 + \frac{7-3x}{(x+3)(x-1)}$$

$$= 2 + \frac{1}{x-1} + \frac{-4}{x+3}$$

$$= 2 + \frac{1}{x-1} - \frac{4}{x+3}$$

Now $I = \int \left[2 + \frac{1}{x-1} - \frac{4}{x+3} \right] dx$

$$= 2x + \ln(x-1) - 4\ln(x+3) + c$$

where c is an integrating constant.

Exercise:

1. Evaluate $\int \frac{x+2}{(x-1)(x+3)}\, dx$.
2. Evaluate $\int \frac{1}{(x+2)(x+1)}\, dx$.
3. Evaluate $\int \frac{x}{(x-2)(x+1)^2}\, dx$.
4. Evaluate $\int \frac{42-19x}{\left(x^2+1\right)(x-4)}\, dx$.
5. Evaluate $\int \frac{1}{\left(x^2+5\right)\left(x^2-3\right)}\, dx$.
6. Evaluate $\int \frac{x^2+5x-7}{x^2-x-2}\, dx$.
7. Evaluate $\int \frac{6x^3+5x^2-7}{3x^2-2x-1}\, dx$.
8. Evaluate $\int \frac{x^4+5x^3-7}{x^2+5x+6}\, dx$.