Lecture Sheet

On

Inequality

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Inequality

Inequality: An inequality is a statement which expresses a non-equal relationship between two mathematical expressions.

The followings indicate the meaning of inequality signs:

1. $a > b$ means a is greater than b.
2. $a < b$ means a is less than b.
3. $a \geq b$ means a is greater than or equal to b.
4. $a \leq b$ means a is less than or equal to b.

Some rules of inequality: For all real numbers a, b and c:

1. If $a, b \in R$ then $a > b$ or $a = b$ or $a < b$.
2. If $a, b > 0$ then $a + b > 0$.
3. If $a, b > 0$ or $a, b < 0$ then $ab > 0$.
4. If $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$ then $ab < 0$.
5. If $a > b$ and $c > 0$ then $a + c > b + c$, $a - c > b - c$, $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
6. If $a > b$ and $c < 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
7. If $a > b > 0$ then $a^n > b^n$, $e^a > e^b$ and $\frac{1}{a} < \frac{1}{b}$ where $n \in N$.

Problem-01: If $a, b > 0$ and $a \neq b$, then show that $a^{m+n} + b^{m+n} > a^m b^n + a^n b^m$.

Solution: Since $a, b > 0$ and $a \neq b$, so the relation between a and b must be $a > b$ or $a < b$.

Now
$$\begin{aligned} a^{m+n} + b^{m+n} - \left(a^m b^n + a^n b^m\right) &= a^{m+n} - a^m b^n + b^{m+n} - a^n b^m \\ &= a^m\left(a^n - b^n\right) - b^m\left(a^n - b^n\right) \\ &= \left(a^n - b^n\right)\left(a^m - b^m\right) \end{aligned}$$

Here, it follows that if $a > b$ or $a < b$ then $\left(a^n - b^n\right)\left(a^m - b^m\right) > 0$.

$$\therefore\ a^{m+n} + b^{m+n} > a^m b^n + a^n b^m$$ **(Showed).**

Problem-02: If $a > 0$ and $a \neq 1$, then show that $a^3 - a^2 > a^{-2} - a^{-3}$.

Solution: Here,
$$\begin{aligned} a^3 - a^2 - \left(a^{-2} - a^{-3}\right) &= a^2(a-1) - a^{-3}(a-1) \\ &= \left(a^2 - a^{-3}\right)(a-1) \\ &= a^{-3}\left(a^5 - 1\right)(a-1) \end{aligned}$$

Here, it follows that if $a > 0$ and $a \neq 1$ then $a^{-3}\left(a^5 - 1\right)(a-1) > 0$.

i.e. $a^3 - a^2 - \left(a^{-2} - a^{-3}\right) > 0$

$$\therefore \quad a^3 - a^2 > a^{-2} - a^{-3}$$ **(Showed).**

Problem-03: Suppose a, b, c, d are positive real numbers. If $\frac{a-b}{a+b} < \frac{c-d}{c+d}$, then show that $\frac{a+b}{b} < \frac{c+d}{d}$.

Solution: Here , $\frac{a-b}{a+b} < \frac{c-d}{c+d}$

$$or,\ \frac{a-b}{a+b} - 1 < \frac{c-d}{c+d} - 1$$

$$or,\ \frac{a-b-a-b}{a+b} < \frac{c-d-c-d}{c+d}$$

$$or,\ \frac{-2b}{a+b} < \frac{-2d}{c+d}$$

$$or,\ \frac{b}{a+b} > \frac{d}{c+d}$$

$$\therefore\ \frac{a+b}{b} < \frac{c+d}{d}$$ **(Showed).**

Problem-04: If $a_1, a_2, \cdots, a_n > 1$ and $p > q$, then show that $n^{p-q}\left(a_1^{\,p} + a_2^{\,p} + \cdots + a_n^{\,p}\right) > \left(a_1^{\,q} + a_2^{\,q} + \cdots + a_n^{\,q}\right)$.

Solution: If $a_1, a_2, \cdots, a_n > 1$ and $p > q$, then $a_1^{\,p} > a_1^{\,q}, a_2^{\,p} > a_2^{\,q}, \cdots, a_n^{\,p} > a_n^{\,q}$.

Now adding these we get,

$$a_1^{\,p} + a_2^{\,p} + \cdots + a_n^{\,p} > a_1^{\,q} + a_2^{\,q} + \cdots + a_n^{\,q} \qquad \cdots(1)$$

Again since $n \in N$ and $p > q$, so

$$n^p > n^q \qquad \cdots(2)$$

Multiplying (1) and (2), we get

$$n^p\left(a_1^{\,p} + a_2^{\,p} + \cdots + a_n^{\,p}\right) > n^q\left(a_1^{\,q} + a_2^{\,q} + \cdots + a_n^{\,q}\right)$$

$$or,\ n^{p-q}\left(a_1^{\,p} + a_2^{\,p} + \cdots + a_n^{\,p}\right) > \left(a_1^{\,q} + a_2^{\,q} + \cdots + a_n^{\,q}\right)$$ **(Showed).**

Problem-05: If $n \in N$ and $0 < x < 1$, then show that $\frac{1-x^{n+1}}{n+1} < \frac{1-x^n}{n}$.

Solution: Here, $\frac{1-x^{n+1}}{1-x^n} = \frac{1-x^n+x^n-x^{n+1}}{1-x^n}$

$$= 1 + \frac{x^n(1-x)}{1-x^n}$$

$$=1+\frac{x^n(1-x)}{(1-x)\left(1+x+x^2+\cdots+x^{n-1}\right)}$$

$$=1+\frac{x^n}{1+x+x^2+\cdots+x^{n-1}}$$

$$=1+\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\cdots+\frac{1}{x}} \qquad \cdots(1)$$

Since $0<x<1$ so we get,

$$\frac{1}{x^n}>1,\ \frac{1}{x^{n-1}}>1,\ \cdots,\ \frac{1}{x}>1.$$

Now the sum of these inequalities gives us,

$$\frac{1}{x^n}+\frac{1}{x^{n-1}}+\cdots+\frac{1}{x}>n$$

$$or,\ \frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\cdots+\frac{1}{x}}<\frac{1}{n} \qquad \cdots(2)$$

From (1) and (2), we get

$$\frac{1-x^{n+1}}{1-x^n}<1+\frac{1}{n}$$

$$or,\ \frac{1-x^{n+1}}{1-x^n}<\frac{n+1}{n}$$

$$or,\ \frac{1-x^{n+1}}{1+n}<\frac{1-x^n}{n} \qquad \textbf{(Showed).}$$

Problem-06: If $x>0$, then show that $\frac{x}{1+x}<\ln(1+x)<x$.

Solution: We know, $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

Then $\qquad e^x>1+x$

$$\therefore\ x>\ln(1+x) \qquad \cdots(1)$$

Again, we can write,

$$e^{\frac{x}{1+x}}=1+\frac{x}{1+x}+\frac{1}{2!}\left(\frac{x}{1+x}\right)^2+\frac{1}{3!}\left(\frac{x}{1+x}\right)^3+\cdots$$

$$<1+\frac{x}{1+x}+\left(\frac{x}{1+x}\right)^2+\left(\frac{x}{1+x}\right)^3+\cdots$$

$$=\left(1-\frac{x}{1+x}\right)^{-1}$$

$$=\frac{1}{1-\frac{x}{1+x}}$$
$$=\frac{1+x}{1+x-x}$$
$$=1+x$$

$$\therefore \quad \frac{x}{1+x}<\ln(1+x) \qquad \cdots(2)$$

From (1) and (2), we get

$$\frac{x}{1+x}<\ln(1+x)<x \qquad \textbf{(Showed).}$$

Arithmetic and Geometric Means: Let $a, b, c, \cdots, k$ represent n positive numbers. The arithmetic and geometric mean of these numbers are defined as follows,

$$Arithmetic\ Mean\,(A.M.)=\frac{a+b+c+\cdots+k}{n}$$
$$Geometric\ Mean\,(G.M.)=\sqrt[n]{abc\cdots k}\,.$$

Theorem-01: Prove that the arithmetic mean (A) of n positive numbers is greater than or equal to their geometric mean (G).

Proof: Let $a, b, c, \cdots, k$ are n positive numbers. The arithmetic and geometric mean of these numbers are defined as follows,

$$Arithmetic\ Mean\,(A)=\frac{a+b+c+\cdots+k}{n}$$
$$Geometric\ Mean\,(G)=\sqrt[n]{abc\cdots k}\,.$$

Case-01: Suppose the numbers are not all equal to one another. Then we know,

$$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$$
$$or,\ ab<\left(\frac{a+b}{2}\right)^2$$

Similarly, $$cd<\left(\frac{c+d}{2}\right)^2.$$

Therefore, $$abcd<\left(\frac{a+b}{2}\right)^2\left(\frac{c+d}{2}\right)^2<\left(\frac{a+b+c+d}{4}\right)^4.$$

Proceeding in this way , we can show that if n is a power of 2, then

$$abc\cdots k<\left(\frac{a+b+c+\cdots+k}{n}\right)^n$$
$$or,\ \sqrt[n]{abc\cdots k}<\frac{a+b+c+\cdots+k}{n}$$

$$or,\ G < A$$
$$\therefore\ A > G$$

If n is not a power of 2, consider the set $a, b, c, \cdots, k, A, A, \cdots$, where A occurs r times and $n+r$ is a power of 2. By the preceding,

$$abc\cdots k.A^r < \left(\frac{a+b+c+\cdots+k+rA}{n+r}\right)^{n+r}$$
$$or,\ abc\cdots k.A^r < \left(\frac{nA+rA}{n+r}\right)^{n+r}$$
$$or,\ abc\cdots k.A^r < A^{n+r}$$
$$or,\ abc\cdots k < A^n$$
$$or,\ \sqrt[n]{abc\cdots k} < A$$
$$or,\ G < A$$
$$\therefore\ A > G$$

This implies that the arithmetic mean of the numbers is greater than their geometric mean.
Case-02: Suppose the numbers are all equal to one another i.e. $a = b = c = \cdots = k$. Then we have,

$$Arithmetic\ Mean\,(A) = \frac{a+b+c+\cdots+k}{n} = \frac{na}{n} = a$$
$$Geometric\ Mean\,(G) = \sqrt[n]{abc\cdots k} = \sqrt[n]{(a)^n} = a\,.$$

This implies that the arithmetic mean of the numbers is equal to their geometric mean.
Hence, it is concluded that the arithmetic mean (A) of n positive numbers is greater than or equal to their geometric mean (G). (**Proved**)

Problem-07: Show that $a^2b + b^2c + c^2a \geq 3abc$.

Solution: We know, $A.M. \geq G.M.$

$$\therefore\ \frac{a^2b+b^2c+c^2a}{3} \geq \left(a^2b.b^2c.c^2a\right)^{\frac{1}{3}}$$
$$or,\ \frac{a^2b+b^2c+c^2a}{3} \geq abc$$
$$or,\ a^2b+b^2c+c^2a \geq 3abc$$ **(Showed).**

Problem-08: Show that $a^2\left(1+b^2\right)+b^2\left(1+c^2\right)+c^2\left(1+a^2\right) \geq 6abc$.

Solution: We know, $A.M. \geq G.M.$

So

$$\frac{a^2+b^2c^2}{2} \geq \left(a^2b^2c^2\right)^{\frac{1}{2}}$$
$$\Rightarrow a^2+b^2c^2 \geq 2abc \qquad \cdots(1)$$

Again, $\dfrac{b^2+c^2a^2}{2}\geq\left(a^2b^2c^2\right)^{\frac{1}{2}}$

$\Rightarrow b^2+c^2a^2\geq 2abc \qquad \cdots(2)$

Again, $\dfrac{c^2+a^2b^2}{2}\geq\left(a^2b^2c^2\right)^{\frac{1}{2}}$

$\Rightarrow c^2+a^2b^2\geq 2abc \qquad \cdots(3)$

Adding (1), (2) and (3), we get

$$a^2+b^2c^2+b^2+c^2a^2+c^2+a^2b^2\geq 6abc$$

$$\therefore\ a^2\left(1+b^2\right)+b^2\left(1+c^2\right)+c^2\left(1+a^2\right)\geq 6abc$$ **(Showed).**

Exercise:

Problem-01: If $a,\ b,\ c>0$, then show that

i. $(b+c)(c+d)(a+b)\geq 8abc$.

ii. $a^2\left(1+b^2c^2\right)+b^2\left(1+c^2d^2\right)+c^2\left(1+d^2a^2\right)+d^2\left(1+a^2b^2\right)\geq 8abcd$.

iii. $a\left(b^2+c^2\right)+b\left(c^2+a^2\right)+c\left(a^2+b^2\right)\geq 6abc$.

iv. $ab(a+b)+bc(b+c)+ca(c+a)\geq 6abc$.

v. $\dfrac{1}{2}(a+b+c)\geq\dfrac{bc}{b+c}+\dfrac{ca}{c+a}+\dfrac{ab}{a+b}$.

vi. $\left(\dfrac{a+b+c}{3}\right)^3\geq(a+b-c)(b+c-a)(c+a-b)$.

Problem-09: If $a,b>0$ and $a\neq b$, then show that $a^b b^a<\left(\dfrac{a+b}{2}\right)^{a+b}$.

Answer: We know, $A.M.>G.M.$

Here, $\dfrac{(a+a+\cdots+b\ number)+(b+b+\cdots+a\ number)}{a+b}>\left(a^b b^a\right)^{\frac{1}{a+b}}$

$$or,\ \frac{ba+ab}{a+b}>\left(a^b b^a\right)^{\frac{1}{a+b}}$$

$$or,\ \left(\frac{2ab}{a+b}\right)^{a+b}>a^b b^a \qquad \cdots(1)$$

Again we know,

$$\frac{a+b}{2}>(ab)^{\frac{1}{2}}$$

$$or,\ \left(\frac{a+b}{2}\right)^2>ab$$

$$or, \quad \frac{a+b}{2} > \frac{2ab}{a+b}$$

$$or, \quad \left(\frac{a+b}{2}\right)^{a+b} > \left(\frac{2ab}{a+b}\right)^{a+b} \qquad \cdots(2)$$

Now from (1) and (2), we get

$$\left(\frac{a+b}{2}\right)^{a+b} > a^b b^a$$

$$\therefore \; a^b b^a < \left(\frac{a+b}{2}\right)^{a+b} \qquad \textbf{(Showed).}$$

Theorem-02: If $a,b>0$ and $a \neq b$, then show that $\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$ when $m \notin [0,1]$.

Answer: We know,

$$a^m+b^m = \left(\frac{a+b}{2}+\frac{a-b}{2}\right)^m + \left(\frac{a+b}{2}-\frac{a-b}{2}\right)^m \qquad \cdots(1)$$

Since $\frac{a-b}{2} < \frac{a+b}{2}$, so expanding (1) as a power series of $\frac{a-b}{2}$ and then dividing by 2 , we get

$$\frac{a^m+b^m}{2} = \left(\frac{a+b}{2}\right)^m + \frac{m(m-1)}{2!}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2 + \frac{m(m-1)(m-2)(m-3)}{4!}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4 + \cdots \qquad \cdots(2)$$

Case-01: If $m \in N$ but $m \neq 1$, or if m is negative number, then the right hand side of (2) is always positive.
So in this case,

$$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

Case-02: If $0<m<1$, then in the right hand side of (2) all terms are negative except first term.
So in this case,

$$\frac{a^m+b^m}{2} < \left(\frac{a+b}{2}\right)^m.$$

Case-03: If $m>1$ and let $m=\frac{1}{n}$ where $n<1$, then

$$\left(\frac{a^m+b^m}{2}\right)^{\frac{1}{m}} = \left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n > \frac{\left(a^{\frac{1}{n}}\right)^n + \left(b^{\frac{1}{n}}\right)^n}{2}$$

$$or,\quad \left(\frac{a^m+b^m}{2}\right)^{\frac{1}{m}} > \frac{a+b}{2}$$

$$\therefore\quad \left(\frac{a^m+b^m}{2}\right) > \left(\frac{a+b}{2}\right)^m. \qquad \textbf{(Showed).}$$

Problem-10: If $a_1, a_2, \cdots, a_n > 0$ and all are not equal to one another, then show that

a) $\dfrac{a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m}}{n} > \left(\dfrac{a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m}}{n}\right)^m$ when $m \notin [0,1]$.

b) $\dfrac{a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m}}{n} < \left(\dfrac{a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m}}{n}\right)^m$ when $m \in [0,1]$.

Answer: We know, if $x>0$ and $x \neq 1$, then

$$x^p - 1 > p(x-1) \qquad when\ p \notin [0,1] \qquad \cdots(1)$$

and

$$x^p - 1 < p(x-1) \qquad when\ p \in [0,1] \qquad \cdots(2)$$

(a) Suppose, $A = \dfrac{a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m}}{n}$

$$or,\ a_1^{\ m}+a_2^{\ m}+\cdots+a_n^{\ m} = nA \qquad \cdots(3)$$

Putting $p=m$ and $x = \dfrac{a_1}{A}, \dfrac{a_2}{A}, \cdots, \dfrac{a_n}{A}$ consecutively in (1) and then multiplying each time by A_m

, we get

$$a_1^{\ m} - A^m > mA^{m-1}(a_1 - A)$$

$$a_2^{\ m} - A^m > mA^{m-1}(a_2 - A)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_n^{\ m} - A^m > mA^{m-1}(a_n - A)$$

Now adding these, we get

$$\sum_{r=1}^{n} a_r^{\ m} - nA^m > mA^{m-1}\left(\sum_{r=1}^{n} a_r - nA\right)$$

$$or,\ \sum_{r=1}^{n} a_r^{\ m} - nA^m > mA^{m-1}(nA - nA)$$

$$or,\ \sum_{r=1}^{n} a_r^{\ m} - nA^m > 0$$

$$or,\ \frac{\sum_{r=1}^{n} a_r^{\ m}}{n} > A^m$$

$$\therefore \frac{a_1^m + a_2^m + \cdots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^m. \qquad \textbf{(Showed).}$$

(b) Similarly from (2) we can show that

$$\frac{a_1^m + a_2^m + \cdots + a_n^m}{n} < \left(\frac{a_1^m + a_2^m + \cdots + a_n^m}{n}\right)^m. \qquad \textbf{(Showed).}$$

Probem-11: If $a,b,c>0$, then show that $\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq \frac{9}{a+b+c}$.

Answer: We know,

$$\frac{\left(\frac{a+b}{2}\right)^{-1}+\left(\frac{b+c}{2}\right)^{-1}+\left(\frac{c+a}{2}\right)^{-1}}{3} \geq \left(\frac{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}}{3}\right)^{-1}$$

$$or,\ \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq 3\left(\frac{a+b+c}{3}\right)^{-1}$$

$$\therefore \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq \frac{9}{a+b+c}. \qquad \textbf{(Showed).}$$

Theorem-03: State and Prove Weiestras's Inequality.

Answer: Statement: If $a_1, a_2, \cdots, a_n > 0$ and $S = a_1 + a_2 + \cdots + a_n$, then

a) $(1+a_1)(1+a_2)\cdots(1+a_n) > 1+S$.

b) $(1-a_1)(1-a_2)\cdots(1-a_n) > 1-S$ where $a_1, a_2, \cdots, a_n < 1$.

c) $(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{1-S}$ where $S<1$.

d) $(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+S}$.

Proof: (a). Here, $(1+a_1)(1+a_2)\cdots(1+a_n) = 1+\sum a_1 + \sum a_1a_2 + \cdots + a_1a_2\cdots a_n$

$$or,\ (1+a_1)(1+a_2)\cdots(1+a_n) > 1+\sum a_1$$

Since $\sum a_1 = a_1 + a_2 + \cdots + a_n = S$, so we can write,

$$(1+a_1)(1+a_2)\cdots(1+a_n) > 1+S \qquad \cdots(1). \qquad \textbf{(Proved).}$$

(b) Here, $(1-a_1)(1-a_2) = 1-a_1-a_2+a_1a_2$.

$$\therefore (1-a_1)(1-a_2) > 1-a_1-a_2 \qquad \text{where } 0<a_1,a_2<1.$$

Similarly, $(1-a_1)(1-a_2)(1-a_3) > (1-a_1-a_2)(1-a_3)$

$$or,\ (1-a_1)(1-a_2)(1-a_3) > 1-a_1-a_2-a_3 \qquad \text{where } 0<a_1,a_2,a_3<1.$$

Proceeding in the same way, we can write

$$(1-a_1)(1-a_2)\cdots(1-a_n)>1-\sum a_1 \qquad \text{where } a_1,a_2,\cdots,a_n<1$$

$$\therefore (1-a_1)(1-a_2)\cdots(1-a_n)>1-S \qquad \cdots(2) .$$

(c) Here, $(1-a_1)(1+a_1)=1-a_1^2<1$

$$\therefore 1+a_1<\frac{1}{1-a_1}.$$

Similarly, $1+a_2<\dfrac{1}{1-a_2}.$

$$1+a_3<\frac{1}{1-a_3}.$$

$$\cdots \quad \cdots \quad \cdots$$

$$1+a_n<\frac{1}{1-a_n}.$$

Now multiplying these, we get

$$(1+a_1)(1+a_2)\cdots(1+a_n)<\frac{1}{(1-a_1)(1-a_2)\cdots(1-a_n)} \qquad \cdots(3)$$

If $S<1$, then using (2) in (3), we have

$$(1+a_1)(1+a_2)\cdots(1+a_n)<\frac{1}{1-S}.$$ **(Proved).**

(d) Here, $(1-a_1)(1+a_1)=1-a_1^2<1$

$$\therefore 1-a_1<\frac{1}{1+a_1}.$$

Similarly, $1-a_2<\dfrac{1}{1+a_2}.$

$$1-a_3<\frac{1}{1+a_3}.$$

$$\cdots \quad \cdots \quad \cdots$$

$$1-a_n<\frac{1}{1+a_n}.$$

Now multiplying these, we get

$$(1-a_1)(1-a_2)\cdots(1-a_n)<\frac{1}{(1+a_1)(1+a_2)\cdots(1+a_n)} \qquad \cdots(4)$$

Using (1) in (4), we have

$$(1-a_1)(1-a_2)\cdots(1-a_n)<\frac{1}{1+S}.$$ **(Proved).**

Theorem-04: State and Prove Cauchy-Schwarz's Inequality.

Answer: Statement: If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are two sets of real numbers, then

$$\left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

the sign of equality occurring only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Proof: For all real numbers, $\lambda, a_i, b_i \quad (i = 1, 2, \cdots, n)$, we can write,

$$\left(a_1\lambda + b_1\right)^2 + \left(a_2\lambda + b_2\right)^2 + \cdots + \left(a_n\lambda + b_n\right)^2 \ge 0$$

$$or,\ \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\lambda^2 + 2\left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)\lambda + \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \ge 0$$

$$or,\ A\lambda^2 + 2C\lambda + B \ge 0 \qquad \cdots(1)$$

where $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = b_1^2 + b_2^2 + \cdots + b_n^2$, and $C = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Dividing both sides of (1) by A, we have

$$\lambda^2 + 2\frac{C}{A}\lambda + \frac{B}{A} \ge 0 \qquad ;[A \ne 0]$$

$$or,\ \lambda^2 + 2\frac{C}{A}\lambda + \frac{C^2}{A^2} + \frac{B}{A} - \frac{C^2}{A^2} \ge 0$$

$$or,\ \left(\lambda + \frac{C}{A}\right)^2 + \frac{AB - C^2}{A^2} \ge 0 \qquad \cdots(2)$$

For all value of λ, (2) is true if

$$\frac{AB - C^2}{A^2} \ge 0$$

$$or,\ AB - C^2 \ge 0 \qquad \left[\because A^2 > 0\right]$$

$$or,\ AB \ge C^2$$

$$or,\ C^2 \le AB$$

$$\therefore \left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \qquad \cdots(3)$$

Suppose the sets are proportional i.e.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = k \ (say) \qquad \cdots(4).$$

Now using (4) in (3), we get

$$L.H.S = \left(kb_1^2 + kb_2^2 + \cdots + kb_n^2\right)^2$$

$$= k^2\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^2$$

$$R.H.S = \left(k^2b_1^2 + k^2b_2^2 + \cdots + k^2b_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$$

$$= k^2 \left(b_1^{\ 2} + b_2^{\ 2} + \cdots + b_n^{\ 2}\right)^2$$

$\therefore \ L.H.S = R.H.S$. **(Proved).**