Functions, Limits and Continuity

Function of a complex variable: If to each value of a complex variable z there corresponds one or more values of a complex variable w, then w is called a complex function of z and it is written as,

$$w = f(z)$$

where z is independent variable and w is dependent variable.

Every complex function $w = f(z)$ can be expressed as,

$$w = f(z) = u + iv$$

where u and v are two real functions of real variables x and y.

Example: (1). $w = f(z) = z^2$

(2). $w = f(z) = \sqrt{z}$

Single-valued function: Let $w = f(z)$ be a complex function. If only one value of w corresponds to each value of z, then it is called a single-valued function.

Example: $w = f(z) = z^2$ is a single-valued function.

Multiple-valued function: Let $w = f(z)$ be a complex function. If more than one value of w correspond to each value of z, then it is called a multiple-valued function.

Example: $w = f(z) = \sqrt{z}$ is a multiple-valued function.

Inverse function: If $w = f(z)$ be a complex function of z, then $z = f^{-1}(w)$ is also a complex function of w. The function f^{-1} is often called the inverse function corresponding to f.

Neighbourhoods: The neighbourhood of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any positive number. The deleted δ neighbourhood of z_0 is a neighbourhood of z_0 in which the point z_0 is omitted, i.e. $0 < |z - z_0| < \delta$.

Limit: Let $f(z)$ be defined and single-valued in a neighbourhood of z_0. The number l is called limit of $f(z)$ as z tends to z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any positive number ε (however small) we can find some positive number δ (usually depending on ε) such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Alternatively, the number l is called limit of $f(z)$ if $f(z)$ approaches to l as z approaches to z_0.

Theorem-01: If $\lim\limits_{z\to z_0} f(z)$ exists, then prove that it must be unique.

Proof: We must show that if $\lim\limits_{z\to z_0} f(z)=l_1$ and $\lim\limits_{z\to z_0} f(z)=l_2$, then $l_1=l_2$.

By hypothesis, for any given $\varepsilon>0$, there exists $\delta>0$ such that

$$\left|f(z)-l_1\right|<\frac{\varepsilon}{2} \quad \text{when } 0<|z-z_0|<\delta$$

$$\text{and } \left|f(z)-l_2\right|<\frac{\varepsilon}{2} \quad \text{when } 0<|z-z_0|<\delta.$$

$$\begin{aligned}
\text{Now } |l_1-l_2| &= |l_1-f(z)+f(z)-l_2| \\
&\le |l_1-f(z)|+|f(z)-l_2| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

$$\therefore |l_1-l_2|<\varepsilon$$

This means $|l_1-l_2|$ is less than any positive number ε (however small) and it must be equal to zero.

i.e. $|l_1-l_2|=0$

$\therefore\ l_1=l_2$

Thus if $\lim\limits_{z\to z_0} f(z)$ exists, then it must be unique. **(Proved)**

Continuity: Let $f(z)$ be defined and single-valued in a neighbourhood of z_0 and $f(z_0)$ is the functional value of it at z_0. The function $f(z)$ is said to be continuous at z_0, if for any $\varepsilon>0$, we can find $\delta>0$ such that $\left|f(z)-f(z_0)\right|<\varepsilon$ whenever $|z-z_0|<\delta$.

Alternatively, the function $f(z)$ is said to be continuous at z_0 if the following conditions are satisfied:

(1). $\lim\limits_{z\to z_0} f(z)=l$ must exist

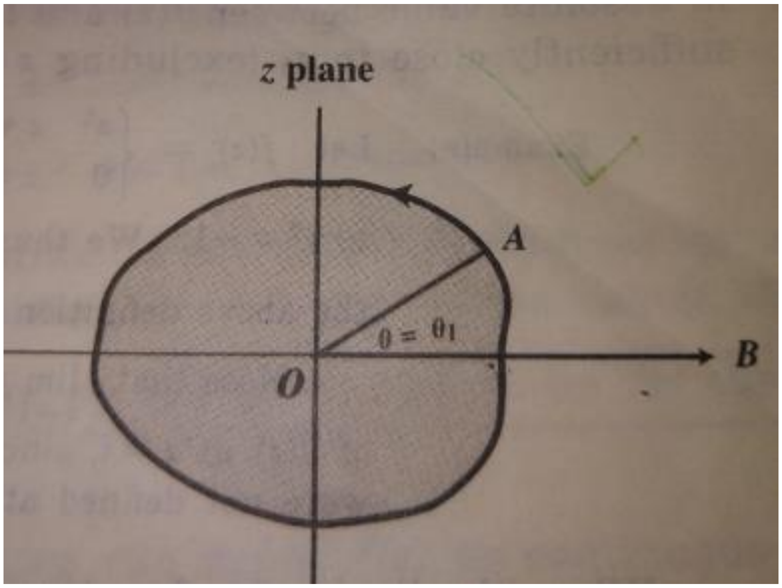
(2). $f(z_0)$ must exist, i.e. $f(z)$ is defined at z_0

(3). $f(z_0)=l$.

A function $f(z)$ is said to be continuous in a region if it is continuous at all points of the region.

Uniform continuity: The function $f(z)$ is uniformly continuous in a region if for any $\varepsilon>0$, we can find $\delta>0$ such that $\left|f(z_1)-f(z_2)\right|<\varepsilon$ whenever $|z_1-z_2|<\delta$ where z_1 and z_2 are any two points of the region.

Branch points and Branch lines: Suppose we have the function $w=\sqrt{z}$ and we allow z to make a complete circuit (counterclockwise) around the origin starting from point A. We have $z=re^{i\theta}$, $w=\sqrt{r}e^{i\theta/2}$ so that at A, $\theta=\theta_1$ and $w=\sqrt{r}e^{i\theta_1/2}$. After a complete circuit back to A, $\theta=\theta_1+2\pi$ and $w=\sqrt{r}e^{i(\theta_1+2\pi)/2}=-\sqrt{r}e^{i\theta_1/2}$. Thus we have not achieved the same value of w with which we started. However, by making a second complete circuit back to A, i.e. $\theta=\theta_1+4\pi$, $w=\sqrt{r}e^{i(\theta_1+4\pi)/2}=\sqrt{r}e^{i\theta_1/2}$ and we then do obtain the same value of w with which we started. We can describe the above by stating that if $0\le\theta<2\pi$ we are on one branch of the multiple-valued function $w=\sqrt{z}$, while if $2\pi\le\theta<4\pi$ we are on another branch of the function.



It is clear that each branch of the function is single-valued. In order to keep the function single –valued, we set up an artificial barrier such as OB where B is at infinity which we agree not to cross. This barrier is called a branch line or branch cut and point O is called a branch point. It should be noted that a circuit around any point other than $z=0$ does not lead to different values; thus $z=0$ is the only finite branch point.

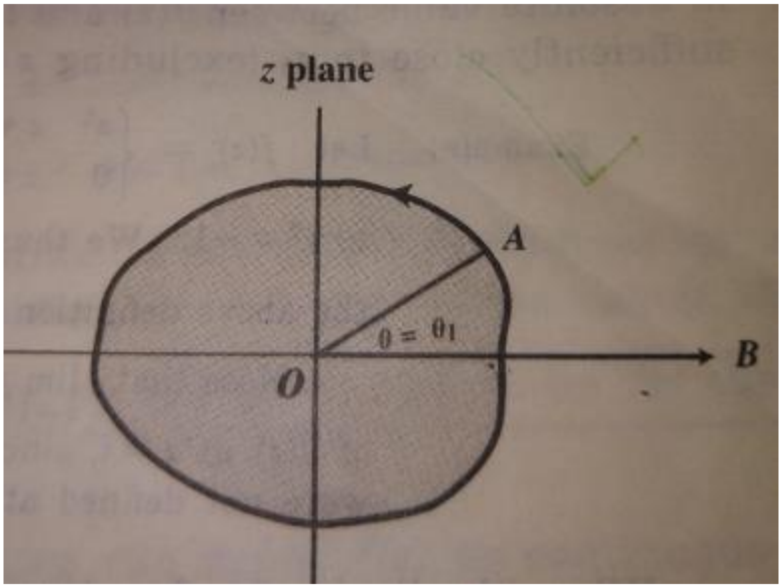
Problems

Problem-01: Prove that $f(z) = \ln z$ has a branch point at $z = 0$.

Solution: We have $f(z) = \ln z$

$$= \ln\left(re^{i\theta}\right)$$

$$= \ln r + i\theta$$



Suppose we start from A at which $r = r_1$, $\theta = \theta_1$

$$\therefore f(z) = \ln r_1 + i\theta_1$$

After making a complete circuit in the counter clockwise direction and back to A, we have

$$r = r_1,\ \theta = \theta_1 + 2\pi$$

$$\therefore f(z) = \ln r_1 + i(\theta_1 + 2\pi).$$

We have not achieved the same value with which we have started.

Thus, we have another branch of $f(z)$ and so $z = 0$ is a branch point. **(Proved)**

Problem-02: Prove that $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ does not exist.

Solution: Let $z = x + iy$. Then $z \to 0 \Rightarrow x \to 0$, $y \to 0$.

$$\therefore \lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x - iy}{x + iy}$$

Taking limit along the real axis $(x \to 0,\ y = 0)$, we have

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1.$$

Again, taking limit along the imaginary axis $(x = 0,\ y \to 0)$, we have

$$\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{y\to 0}\frac{-iy}{iy}=-1.$$

The above two limits are not equal, that is, the limit depends on manner in which $z\to 0$.

Hence $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist. **(Proved)**

Problem-03: If $f(z)=z^2$, then prove that $\lim_{z\to 0} f(z)=z_0^2$.

Solution: We must show that for any given $\varepsilon>0$, we can find $\delta>0$ (depending on ε) such that

$$\left|z^2-z_0^2\right|<\varepsilon \quad \text{whenever} \quad 0<\left|z-z_0\right|<\delta.$$

If $\delta\le 1$, then $0<\left|z-z_0\right|<\delta$ implies that

$$\begin{aligned}\left|z^2-z_0^2\right| &= \left|z-z_0\right|\left|z+z_0\right| \\ &< \delta\left|z-z_0+2z_0\right| \\ &< \delta\left\{\left|z-z_0\right|+\left|2z_0\right|\right\} \\ &< \delta\left(1+2\left|z_0\right|\right)\end{aligned}$$

Take δ as 1 or $\frac{\varepsilon}{\left(1+2\left|z_0\right|\right)}$, whichever is smaller.

Then we have,

$$\left|z^2-z_0^2\right|<\varepsilon \quad \text{whenever} \quad 0<\left|z-z_0\right|<\delta.$$

Hence the required result is proved.

Problem-04: Prove that $f(z)=\begin{cases} z^2, & z\ne z_0 \\ 0, & z=z_0 \end{cases}$, where , is discontinuous at $z=z_0$.

Solution: We have $f(z)=\begin{cases} z^2, & z\ne z_0 \\ 0, & z=z_0 \end{cases}$

Now, $\lim_{z\to z_0} f(z)=z_0^2$

and $f(z_0)=0$.

Since $\lim_{z\to z_0} f(z)\ne f(z_0)$, so $f(z)$ is discontinuous at $z=z_0$ if $z_0\ne 0$. **(Proved)**

Problem-05: Prove that $f(z)=z^2$ is uniformly continuous in the region $|z|<1$.

Solution: We must show that for any given $\varepsilon>0$, we can find $\delta>0$ (depending only on ε but not on the any particular point z_0 of the region) such that

$$\left|z^2-z_0^2\right|<\varepsilon \quad \text{when} \quad \left|z-z_0\right|<\delta.$$

If z and z_0 are any points in $|z|<1$, then

$$\begin{aligned}\left|z^2-z_0^2\right| &= \left|z-z_0\right|\left|z+z_0\right| \\ &\le \left|z-z_0\right|\left\{|z|+\left|z_0\right|\right\}\end{aligned}$$

$$< 2|z - z_0|$$

Thus if $|z - z_0| < \delta$, it follows that

$$|z^2 - z_0^2| < 2\delta$$

Choosing $\delta = \dfrac{\varepsilon}{2}$, we see that

$$|z^2 - z_0^2| < \varepsilon .$$

Hence $f(z) = z^2$ is uniformly continuous in the region. **(Proved)**

Problem-06: Prove that $f(z) = \dfrac{1}{z}$ is not uniformly continuous in the region $|z| < 1$.

Solution: Let z_0 and $z_0 + \xi$ be any two points of the region such that

$$|z_0 + \xi - z_0| = |\xi| = \delta .$$

Then
$$\begin{aligned}
|f(z) - f(z_0)| &= \left|\frac{1}{z_0} - \frac{1}{z_0 + \xi}\right| \\
&= \left|\frac{\xi}{z_0(z_0 + \xi)}\right| \\
&= \frac{|\xi|}{|z_0||z_0 + \xi|} \\
&= \frac{\delta}{|z_0||z_0 + \xi|} .
\end{aligned}$$

This can be made larger than any positive number by choosing z_0 sufficiently close to 0.

Hence the function can not be uniformly continuous in the region. **(Proved)**