Proper Integral: The definite integral $\int_a^b f(x)\,dx$ is called a proper integral if

(i). the interval of integration $[a,b]$ is finite or bounded

(ii). the integrand $f(x)$ is bounded on $[a,b]$.

Improper Integral: The definite integral $\int_a^b f(x)\,dx$ is called an improper integral if either or both the above conditions are not satisfied. Thus $\int_a^b f(x)\,dx$ is an improper integral if either the interval of integration $[a,b]$ is not finite and the integrand $f(x)$ is not bounded on $[a,b]$ or neither the interval $[a,b]$ is finite nor the integrand $f(x)$ is bounded over it.

The improper integrals are of three kinds:

(i). improper integral of first kind.

(ii). improper integral of second kind and

(iii). improper integral of third kind.

Improper integral of first kind: In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite, so that the interval of integration is unbounded but the integrand $f(x)$ is bounded, then the definite integral $\int_a^b f(x)\,dx$ is called an improper integral of first kind.

Example: $\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx$, $\int_{-\infty}^{0} e^{2x}\,dx$, $\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dx$ are improper integrals of first kind.

Improper integral of second kind: In the definite integral $\int_a^b f(x)\,dx$, if both a and b are finite, so that the interval of integration is finite but the integrand $f(x)$ has one or more points of infinite discontinuity, i.e. the integrand $f(x)$ is not bounded on $[a,b]$, then the definite integral $\int_a^b f(x)\,dx$ is called an improper integral of second kind.

Example: $\int_0^1 \frac{1}{x} dx$, $\int_1^2 \frac{1}{x-2} dx$ are improper integrals of second kind.

Improper integral of third kind: In the definite integral $\int_a^b f(x)dx$, if either a or b or both a and b are infinite, so that the interval of integration is unbounded and the integrand $f(x)$ is also unbounded, then the definite integral $\int_a^b f(x)dx$ is called an improper integral of third kind.

Example: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$ is an improper integral of third kind.

Convergence and divergence of improper integral: Determine whether the limit exists. If the limit exists or is finite, then the improper integral is said to be convergent and take the limit for its value. If the limit does not exist or is infinite, then the improper integral is said to be divergent.

Improper integral of the first kind as the limit of a proper integral: When the improper integral is of the first kind, either a or b or both a and b are infinite but the integrand $f(x)$ is bounded.

(i). $\int_a^\infty f(x)dx = \lim_{p\to\infty} \int_a^p f(x)dx, \ (p > a).$

The improper integral $\int_a^\infty f(x)dx$ is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(ii). $\int_{-\infty}^b f(x)dx = \lim_{p\to-\infty} \int_p^b f(x)dx, \ (p < b).$

The improper integral $\int_{-\infty}^b f(x)dx$ is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(iii). $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx = \lim_{p_1\to-\infty} \int_{p_1}^b f(x)dx + \lim_{p_2\to\infty} \int_c^{p_2} f(x)dx$

where c is any real number. The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if both the limit on the R.H.S. exists finitely and independent of each other, otherwise the integral is said to be divergent.

Note: If the integral is neither convergent nor divergent, then it is said to be oscillating.

Problem-01: Test for convergence of $\int_0^{\infty} \frac{dx}{2+x^2}$.

Solution: Let $I = \int_0^{\infty} \frac{dx}{2+x^2}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_0^{\infty} \frac{dx}{2+x^2} = \lim_{p\to\infty} \int_0^{p} \frac{dx}{2+x^2}$$

$$= \lim_{p\to\infty} \int_0^{p} \frac{dx}{\left(\sqrt{2}\right)^2 + x^2}$$

$$= \lim_{p\to\infty} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} \right]_0^p$$

$$= \frac{1}{\sqrt{2}} \cdot \lim_{p\to\infty} \left[\tan^{-1} \frac{p}{\sqrt{2}} - \tan^{-1} \frac{0}{\sqrt{2}} \right]$$

$$= \frac{1}{\sqrt{2}} \cdot \lim_{p\to\infty} \left[\tan^{-1} \frac{p}{\sqrt{2}} - 0 \right]$$

$$= \frac{1}{\sqrt{2}} \cdot \lim_{p\to\infty} \tan^{-1} \frac{p}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \tan^{-1}(\infty)$$

$$= \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\tan \frac{\pi}{2} \right)$$

$$= \frac{\pi}{2\sqrt{2}}.$$

The given integral is convergent and its value is $\frac{\pi}{2\sqrt{2}}$.

Problem-02: Test for convergence of $\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)}$.

Solution: Let $I = \int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)} = \lim_{p\to\infty} \int_1^{p} \frac{dx}{\sqrt{x}(1+x)} \qquad \cdots(1)$$

Putting $\sqrt{x} = t \quad \therefore \frac{1}{2\sqrt{x}}dx = dt$

when $x=1$ then $t=1$

when $x = p$ then $t = \sqrt{p}$.

Now from (1), we have

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)} = 2\lim_{p\to\infty} \int_1^{\sqrt{p}} \frac{dx}{1+t^2}$$

$$= 2\lim_{p\to\infty} \left[\tan^{-1} t\right]_1^{\sqrt{p}}$$

$$= 2\lim_{p\to\infty} \left[\tan^{-1}\sqrt{p} - \tan^{-1} 1\right]$$

$$= 2\lim_{p\to\infty} \left[\tan^{-1}\sqrt{p} - \frac{\pi}{4}\right]$$

$$= 2\left[\tan^{-1}(\infty) - \frac{\pi}{4}\right]$$

$$= 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}.$$

The given integral is convergent and its value is $\frac{\pi}{2}$.

Problem-03: Test for convergence of $\int_{-\infty}^{\infty} \frac{xdx}{1+x^4}$.

Solution: Let $I = \int_{-\infty}^{\infty} \frac{xdx}{1+x^4}$

$$\therefore I=\int_{-\infty}^{0}\frac{xdx}{1+x^4}+\int_{0}^{\infty}\frac{xdx}{1+x^4} \qquad \cdots(1)$$

Now by the definition, we have

$$\int_{-\infty}^{0}\frac{xdx}{1+x^4}=\lim_{p\to-\infty}\int_{p}^{0}\frac{xdx}{1+\left(x^2\right)^2} \qquad \cdots(2)$$

Putting $x^2=t \quad \therefore\ 2xdx=dt$

when $x=p$ then $t=p^2$

when $x=0$ then $t=0$.

Now from (2), we have

$$\int_{-\infty}^{0}\frac{xdx}{1+x^4}=\frac{1}{2}\lim_{p\to-\infty}\int_{p^2}^{0}\frac{dt}{1+t^2}$$

$$=\frac{1}{2}\lim_{p\to-\infty}\left[\tan^{-1}t\right]_{p^2}^{0}$$

$$=\frac{1}{2}\lim_{p\to-\infty}\left[\tan^{-1}0-\tan^{-1}p^2\right]$$

$$=\frac{1}{2}\lim_{p\to-\infty}\left[0-\tan^{-1}p^2\right]$$

$$=-\frac{1}{2}\tan^{-1}(\infty)$$

$$=-\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=-\frac{\pi}{4}$$

Also by the definition, we have

$$\int_{0}^{\infty}\frac{xdx}{1+x^4}=\lim_{p\to\infty}\int_{0}^{p}\frac{xdx}{1+\left(x^2\right)^2} \qquad \cdots(3)$$

Putting $x^2=t \quad \therefore\ 2xdx=dt$

when $x=0$ then $t=0$

when $x=p$ then $t=p^2$.

Now from (3), we have

$$\int_{0}^{p}\frac{xdx}{1+x^4}=\frac{1}{2}\lim_{p\to\infty}\int_{0}^{p^2}\frac{dt}{1+t^2}$$

$$=\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}t\right]_0^{p^2}$$

$$=\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}p^2-\tan^{-1}0\right]$$

$$=\frac{1}{2}\lim_{p\to\infty}\tan^{-1}p^2$$

$$=\frac{1}{2}\tan^{-1}(\infty)$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\frac{\pi}{4}$$

So (1) becomes,

$$\int_{-\infty}^{\infty}\frac{xdx}{1+x^4}=-\frac{\pi}{4}+\frac{\pi}{4}=0.$$

This is finite. So the given integral is convergent and its value is 0.

Problem-04: Test for convergence of $\int_0^{\infty}\frac{x^3dx}{16+x^4}$.

Solution: Let $I=\int_0^{\infty}\frac{x^3dx}{16+x^4}$

Now by the definition, we have

$$\int_0^{\infty}\frac{xdx}{1+x^4}=\lim_{p\to\infty}\int_0^{p}\frac{x^3dx}{16+x^4}\qquad\cdots(1)$$

$$=\frac{1}{4}\lim_{p\to\infty}\int_0^{p}\frac{4x^3dx}{16+x^4}$$

$$=\frac{1}{4}\lim_{p\to\infty}\left[\ln\left(16+x^4\right)\right]_0^{p}$$

$$=\frac{1}{4}\lim_{p\to\infty}\left[\ln\left(16+p^4\right)-\ln 16\right]$$

$$=\frac{1}{4}\left[\ln(\infty)-\ln 16\right]$$

$$=\frac{1}{4}\left[\infty-\ln 16\right]$$

$$=\infty.$$

This is infinite. So the given integral is divergent.

Problem-05: Test for convergence of $\int_0^\infty \frac{\sqrt{x}}{1+x}dx$.

Solution: Let $I = \int_0^\infty \frac{\sqrt{x}}{1+x}dx$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_0^\infty \frac{\sqrt{x}}{1+x}dx = \lim_{p\to\infty}\int_0^p \frac{\sqrt{x}}{1+x}dx \qquad \cdots(1)$$

Putting $\sqrt{x} = t \Rightarrow x = t^2 \quad \therefore\ dx = 2tdt$

when $x = 0$ then $t = 0$

when $x = p$ then $t = \sqrt{p}$.

Now from (1), we have

$$\int_0^\infty \frac{\sqrt{x}}{1+x}dx = \lim_{p\to\infty}\int_0^{\sqrt{p}} \frac{t}{1+t^2}\cdot 2tdt$$

$$= 2\lim_{p\to\infty}\int_0^{\sqrt{p}} \frac{t^2}{1+t^2}dt$$

$$= 2\lim_{p\to\infty}\int_0^{\sqrt{p}} \frac{1+t^2-1}{1+t^2}dt$$

$$= 2\lim_{p\to\infty}\int_0^{\sqrt{p}} dt - 2\lim_{p\to\infty}\int_0^{\sqrt{p}} \frac{1}{1+t^2}dt$$

$$= 2\lim_{p\to\infty}\left[t\right]_0^{\sqrt{p}} - 2\lim_{p\to\infty}\left[\tan^{-1}t\right]_0^{\sqrt{p}}$$

$$= 2\lim_{p\to\infty}\left(\sqrt{p}-0\right) - 2\lim_{p\to\infty}\left(\tan^{-1}\sqrt{p} - \tan^{-1}0\right)$$

$$= 2(\infty) - 2\tan^{-1}(\infty)$$

$$= \infty - 2\cdot\frac{\pi}{2}$$

$$= \infty$$

This is infinite. The given integral is divergent.

Problem-06: Evaluate $\int_{-\infty}^\infty \frac{dx}{\left(1+x^2\right)^2}$.

Solution: Let $I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

$$= \int_{-\infty}^{0} \frac{dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{dx}{(1+x^2)^2}$$

$$\therefore I = \lim_{p\to-\infty} \int_{p}^{0} \frac{dx}{(1+x^2)^2} + \lim_{p\to\infty} \int_{0}^{p} \frac{dx}{(1+x^2)^2} \qquad \cdots(1)$$

Now,

$$\int \frac{dx}{(1+x^2)^2}$$

Putting $x = \tan\theta \Rightarrow \theta = \tan^{-1} x \quad \therefore\ dx = \sec^2\theta d\theta$

We have

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2}$$

$$= \int \frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$= \int \frac{d\theta}{\sec^2\theta}$$

$$= \int \cos^2\theta d\theta$$

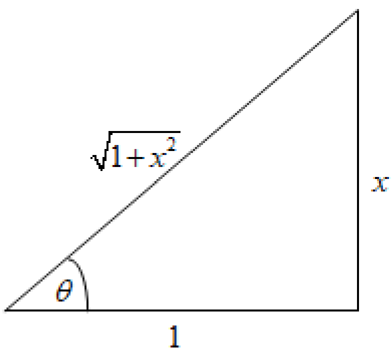
$$= \frac{1}{2}\int 2\cos^2\theta d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right)$$



$$=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{1+x^2}\right)$$

From (1), we get

$$I=\frac{1}{2}\lim_{p\to-\infty}\left[\tan^{-1}x+\frac{x}{1+x^2}\right]_p^0+\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}x+\frac{x}{1+x^2}\right]_0^p$$

$$=\frac{1}{2}\lim_{p\to-\infty}\left[0-\tan^{-1}p-\frac{p}{1+p^2}\right]+\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}p+\frac{p}{1+p^2}-0\right]$$

$$=-\frac{1}{2}\lim_{p\to-\infty}\left(\tan^{-1}p+\frac{p}{1+p^2}\right)+\frac{1}{2}\lim_{p\to\infty}\left(\tan^{-1}p+\frac{p}{1+p^2}\right)$$

$$=-\frac{1}{2}\tan^{-1}(-\infty)-\frac{1}{2}\lim_{p\to-\infty}\frac{p}{1+p^2}+\frac{1}{2}\tan^{-1}\infty+\frac{1}{2}\lim_{p\to\infty}\frac{p}{1+p^2}$$

$$=\frac{1}{2}\tan^{-1}(\infty)-\frac{1}{2}\lim_{p\to-\infty}\frac{p}{1+p^2}+\frac{1}{2}\tan^{-1}\infty+\frac{1}{2}\lim_{p\to\infty}\frac{p}{1+p^2}$$

$$=\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\lim_{p\to-\infty}\frac{p}{1+p^2}+\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\lim_{p\to\infty}\frac{p}{1+p^2}$$

$$=\frac{\pi}{4}-\frac{1}{2}\lim_{p\to-\infty}\frac{p}{1+p^2}+\frac{\pi}{4}+\frac{1}{2}\lim_{p\to\infty}\frac{p}{1+p^2}$$

$$=\frac{\pi}{2}-\frac{1}{2}\lim_{p\to-\infty}\frac{1}{2p}+\frac{1}{2}\lim_{p\to\infty}\frac{1}{2p} \qquad \text{By L. Hospital's Rule}$$

$$=\frac{\pi}{2}-\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0$$

$$=\frac{\pi}{2}.$$

This is finite. So the given integral is convergent and its value is $\frac{\pi}{2}$.

Problem-07: Show that $\int_1^\infty \frac{x^2dx}{\left(1+x^2\right)^2}=\frac{1}{8}(\pi+2)$.

Solution: Let $I=\int_1^\infty \frac{x^2dx}{\left(1+x^2\right)^2}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_1^\infty \frac{x^2dx}{\left(1+x^2\right)^2}=\lim_{p\to\infty}\int_1^p \frac{x^2dx}{\left(1+x^2\right)^2} \qquad \cdots(1)$$

Now,

$$\int \frac{x^2 dx}{\left(1+x^2\right)^2}$$

Putting $x = \tan\theta \Rightarrow \theta = \tan^{-1} x \quad \therefore\ dx = \sec^2\theta d\theta$

We have

$$\int \frac{x^2 dx}{\left(1+x^2\right)^2} = \int \frac{\tan^2\theta \sec^2\theta d\theta}{\left(1+\tan^2\theta\right)^2}$$

$$= \int \frac{\tan^2\theta \sec^2\theta d\theta}{\left(\sec^2\theta\right)^2}$$

$$= \int \frac{\tan^2\theta \sec^2\theta d\theta}{\sec^4\theta}$$

$$= \int \frac{\tan^2\theta}{\sec^2\theta} d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta d\theta$$

$$= \int \sin^2\theta d\theta$$

$$= \frac{1}{2}\int 2\sin^2\theta d\theta$$

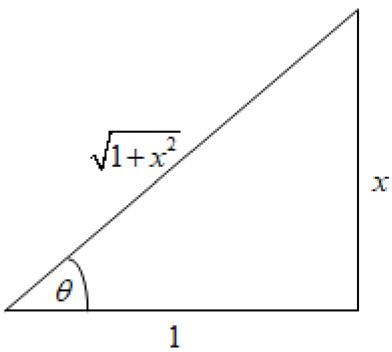
$$= \frac{1}{2}\int \left(1-\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$= \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right)$$

$$= \frac{1}{2}\left(\tan^{-1} x - \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{2}\left(\tan^{-1} x - \frac{x}{1+x^2}\right)$$



From (1), we get

$$I = \frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1} x - \frac{x}{1+x^2}\right]_1^p$$

$$=\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}p-\frac{p}{1+p^2}-\left(\tan^{-1}1-\frac{1}{1+1^2}\right)\right]$$

$$=\frac{1}{2}\tan^{-1}(\infty)-\frac{1}{2}\lim_{p\to\infty}\frac{p}{1+p^2}-\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\lim_{p\to\infty}\frac{1}{2p}-\frac{\pi}{8}+\frac{1}{4}$$ By L. Hospital's Rule

$$=\frac{\pi}{4}-\frac{1}{2}\cdot 0-\frac{\pi}{8}+\frac{1}{4}$$

$$=\frac{\pi}{8}+\frac{1}{4}$$

$$=\frac{1}{8}(\pi+2).$$ **(Showed)**

Problem-08: Evaluate $\int_2^\infty \frac{2x^2dx}{x^4-1}$.

Solution: Let $I=\int_2^\infty \frac{2x^2dx}{x^4-1}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_2^\infty \frac{2x^2}{x^4-1}dx=\lim_{p\to\infty}\int_2^p \frac{2x^2}{x^4-1}dx$$

$$=\lim_{p\to\infty}\int_2^p \frac{(x^2+1)+(x^2-1)}{(x^2)^2-1}dx$$

$$=\lim_{p\to\infty}\int_2^p \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}dx$$

$$=\lim_{p\to\infty}\int_2^p \frac{1}{x^2-1}dx+\lim_{p\to\infty}\int_2^p \frac{1}{x^2+1}dx$$

$$=\lim_{p\to\infty}\left[\frac{1}{2}\ln\frac{x-1}{x+1}\right]_2^p+\lim_{p\to\infty}\left[\tan^{-1}x\right]_2^p$$

$$=\frac{1}{2}\lim_{p\to\infty}\left(\ln\frac{p-1}{p+1}-\ln\frac{2-1}{2+1}\right)+\lim_{p\to\infty}\left(\tan^{-1}p-\tan^{-1}2\right)$$

$$=\frac{1}{2}\lim_{p\to\infty}\ln\frac{p-1}{p+1}-\frac{1}{2}\ln\frac{1}{3}+\lim_{p\to\infty}\tan^{-1}p-\tan^{-1}2$$

$$=\frac{1}{2}\lim_{p\to\infty}\ln\frac{1-\frac{1}{p}}{1+\frac{1}{p}}-\frac{1}{2}\ln\frac{1}{3}+\tan^{-1}(\infty)-\tan^{-1}2$$

$$=\frac{1}{2}\ln\frac{1-0}{1+0}-\frac{1}{2}\ln\frac{1}{3}+\frac{\pi}{2}-\tan^{-1}2$$

$$=\frac{1}{2}\ln 1-\frac{1}{2}\ln\frac{1}{3}+\frac{\pi}{2}-\tan^{-1}2$$

$$=\frac{1}{2}\cdot 0-\frac{1}{2}\ln\frac{1}{3}+\frac{\pi}{2}-\tan^{-1}2$$

$$=\frac{1}{2}\ln 3+\frac{\pi}{2}-\tan^{-1}2\,.$$

Problem-09: Evaluate $\int_0^\infty \frac{dx}{a^2e^x+b^2e^{-x}}$.

Solution: Let $I=\int_0^\infty \frac{dx}{a^2e^x+b^2e^{-x}}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_0^\infty \frac{dx}{a^2e^x+b^2e^{-x}}=\lim_{p\to\infty}\int_0^p \frac{dx}{a^2e^x+b^2e^{-x}}$$

$$=\lim_{p\to\infty}\int_0^p \frac{e^x dx}{a^2e^{2x}+b^2}$$

$$=\frac{1}{a^2}\lim_{p\to\infty}\int_0^p \frac{e^x dx}{\left(e^x\right)^2+\left(\frac{b}{a}\right)^2}$$

$$=\frac{1}{a^2}\lim_{p\to\infty}\int_0^\infty \frac{d\left(e^x\right)}{\left(e^x\right)^2+\left(\frac{b}{a}\right)^2}$$

$$=\frac{1}{a^2}\lim_{p\to\infty}\left[\frac{a}{b}\tan^{-1}\frac{ae^x}{b}\right]_0^p$$

$$=\frac{1}{ab}\lim_{p\to\infty}\left(\tan^{-1}\frac{ae^p}{b}-\tan^{-1}\frac{ae^0}{b}\right)$$

$$=\frac{1}{ab}\left(\tan^{-1}\infty-\tan^{-1}\frac{a}{b}\right)$$

$$=\frac{1}{ab}\left(\frac{\pi}{2}-\tan^{-1}\frac{a}{b}\right).$$

Problem-10: Show that $\int_1^\infty \frac{dx}{x^p}$ converges if $p>1$, it diverges otherwise.

Solution: Let $I=\int_1^\infty \frac{dx}{x^p}$

Since the upper limit of the given integral is ∞, so by the definition we have

$$\int_1^\infty \frac{dx}{x^p}=\lim_{t\to\infty}\int_1^t \frac{dx}{x^p} \qquad \cdots(1)$$

when $p=-1$, then

$$\lim_{t\to\infty}\int_1^t \frac{dx}{x^{-1}}=\lim_{t\to\infty}\int_1^t xdx=\lim_{t\to\infty}\left[\frac{x^2}{2}\right]_1^t=\lim_{t\to\infty}\left(\frac{t^2}{2}-\frac{1}{2}\right)=\left(\infty-\frac{1}{2}\right)=\infty$$

i.e. when $p=-1$, then the given integral is divergent.

when $p\neq -1$, then

$$\lim_{t\to\infty}\int_1^t \frac{dx}{x^p}=\lim_{t\to\infty}\int_1^t x^{-p}dx=\lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t=\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p}-\frac{1}{1-p}\right).$$

Here the sign of $1-p$ is important. When $1-p>0$ that is $p<1$ then t^{1-p} is in the numerator.

Therefore

$$\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p}-\frac{1}{1-p}\right)=\infty$$

Thus the given integral diverges.

When $1-p<0$ that is $p>1$ then t^{1-p} is in the denominator.

Therefore

$$\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p}-\frac{1}{1-p}\right)=-\frac{1}{1-p}$$

Thus the given integral converges.

Hence, the given integral converges if $p>1$ and it diverges otherwise. **(Showed)**

Improper integral of the second kind as the limit of a proper integral: When the improper integral is of the second kind, both a and b are infinite but the integrand $f(x)$ has one or more points of infinite discontinuity.

(i). If $f(x)$ becomes infinite at $x=b$ only, we define $\int_a^b f(x)\,dx = \lim_{\in\to 0+}\int_a^{b-\in} f(x)dx$.

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(ii). If $f(x)$ becomes infinite at $x=a$ only, we define $\int_a^b f(x)\,dx = \lim_{\in\to 0+}\int_{a+\in}^{b} f(x)dx$.

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(iii). If $f(x)$ becomes infinite at $x=c$ only where $a<c<b$, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\in_1\to 0+}\int_a^{c-\in_1} f(x)dx + \lim_{\in_2\to 0+}\int_{c+\in_2}^{b} f(x)dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limit on the R.H.S. exists finitely and independent of each other, otherwise the integral is said to be divergent.

Problem-11: Test for convergence of $\int_0^1 \frac{dx}{x+\sqrt{x}}$.

Solution: Let $I = \int_0^1 \frac{dx}{x+\sqrt{x}}$

Since 0 is the only point of infinite discontinuity of the integrand $f(x)=\frac{1}{x+\sqrt{x}}$ on $[0,1]$, so by the definition we have

$$\int_0^1 \frac{dx}{x+\sqrt{x}} = \lim_{\in\to 0+}\int_{0+\in}^{1}\frac{dx}{x+\sqrt{x}} \qquad \cdots(1)$$

Putting $\sqrt{x}=t \Rightarrow x=t^2 \quad \therefore dx = 2tdt$

when $x=\in$ then $t=\sqrt{\in}$

when $x=1$ then $t=1$

Now from (1), we have

$$\int_0^1 \frac{dx}{x+\sqrt{x}} = \lim_{\in\to 0+} \int_{\sqrt{\in}}^1 \frac{2tdt}{t^2+t}$$

$$= 2\lim_{\in\to 0+} \int_{\sqrt{\in}}^1 \frac{dt}{t+1}$$

$$= 2\lim_{\in\to 0+} \left[\ln(t+1)\right]_{\sqrt{\in}}^1$$

$$= 2\lim_{\in\to 0+} \left[\ln(1+1) - \ln\left(\sqrt{\in}+1\right)\right]$$

$$= 2\left[\ln 2 - \ln(0+1)\right]$$

$$= 2\ln 2\,.$$

The given integral is convergent and its value is $2\ln 2$.

Problem-12: Test for convergence of $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$.

Solution: Let $I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

Since 0 and 1 are the points of infinite discontinuity of the integrand $f(x) = \frac{1}{\sqrt{x(1-x)}}$ on $[0,1]$, so by the definition we have

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\in_1\to 0+} \int_{0+\in_1}^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \lim_{\in_2\to 0+} \int_{\frac{1}{2}}^{1-\in_2} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{\in_1\to 0+} \int_{0+\in_1}^{\frac{1}{2}} \frac{dx}{\sqrt{\frac{1}{4}-\frac{1}{4}+2\cdot\frac{1}{2}\cdot x - x^2}} + \lim_{\in_2\to 0+} \int_{\frac{1}{2}}^{1-\in_2} \frac{dx}{\sqrt{\frac{1}{4}-\frac{1}{4}+2\cdot\frac{1}{2}\cdot x - x^2}}$$

$$= \lim_{\in_1\to 0+} \int_{0+\in_1}^{\frac{1}{2}} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}} + \lim_{\in_2\to 0+} \int_{\frac{1}{2}}^{1-\in_2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}$$

$$= \lim_{\epsilon_1 \to 0+} \left[\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}} \right) \right]_{0+\epsilon_1}^{\frac{1}{2}} + \lim_{\epsilon_2 \to 0+} \left[\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}} \right) \right]_{\frac{1}{2}}^{1-\epsilon_2}$$

$$= \lim_{\epsilon_1 \to 0+} \left[\sin^{-1}(2x-1) \right]_{0+\epsilon_1}^{\frac{1}{2}} + \lim_{\epsilon_2 \to 0+} \left[\sin^{-1}(2x-1) \right]_{\frac{1}{2}}^{1-\epsilon_2}$$

$$= \lim_{\epsilon_1 \to 0+} \left[\sin^{-1}(0) - \sin^{-1}(2\epsilon_1 - 1) \right] + \lim_{\epsilon_2 \to 0+} \left[\sin^{-1}(1-2\epsilon_2) - \sin^{-1}(0) \right]$$

$$= 0 - \sin^{-1}(-1) + \sin^{-1}(1) - 0$$

$$= \sin^{-1}(1) + \sin^{-1}(1)$$

$$= 2\sin^{-1}(1)$$

$$= 2\sin^{-1}\left(\sin\frac{\pi}{2} \right)$$

$$= 2 \cdot \frac{\pi}{2}$$

$$= \pi \,.$$

The given integral is convergent and its value is π.

Problem-13: Test for convergence of $\int_0^5 \frac{dx}{(x-1)^3}$.

Solution: Let $I = \int_0^5 \frac{dx}{(x-1)^3}$

Since 1 is the only point of infinite discontinuity of the integrand $f(x) = \frac{1}{(x-1)^3}$ on $[0,5]$, so by the definition we have

$$\int_0^5 \frac{dx}{(x-1)^3} = \int_0^1 \frac{dx}{(x-1)^3} + \int_1^5 \frac{dx}{(x-1)^3}$$

$$\therefore \int_0^5 \frac{dx}{(x-1)^3} = \lim_{\epsilon_1 \to 0+} \int_0^{1-\epsilon_1} \frac{dx}{(x-1)^3} + \lim_{\epsilon_2 \to 0+} \int_{1+\epsilon_2}^5 \frac{dx}{(x-1)^3}$$

$$= \lim_{\epsilon_1 \to 0+} \left[-\frac{1}{2(x-1)^2} \right]_0^{1-\epsilon_1} + \lim_{\epsilon_2 \to 0+} \left[-\frac{1}{2(x-1)^2} \right]_{1+\epsilon_2}^{1}$$

$$= \lim_{\in_1 \to 0+}\left[-\frac{1}{2(1-\in_1-1)^2}+\frac{1}{2(0-1)^2}\right]+\lim_{\in_2 \to 0+}\left[-\frac{1}{2(5-1)^2}+\frac{1}{2(1+\in_2-1)^2}\right]$$

$$= \lim_{\in_1 \to 0+}\left(-\frac{1}{2\in_1^2}+\frac{1}{2}\right)+\lim_{\in_2 \to 0+}\left(-\frac{1}{32}+\frac{1}{2\in_2^2}\right)$$

$$= \infty + \infty$$

$$\infty.$$

The given integral is divergent.

Problem-14: Test for convergence of $\int_{-a}^{a}\frac{xdx}{\sqrt{a^2-x^2}}$.

Solution: Let $I=\int_{-a}^{a}\frac{xdx}{\sqrt{a^2-x^2}}$

where $f(x)=\frac{x}{\sqrt{a^2-x^2}}$

Now $f(-x)=-\frac{x}{\sqrt{a^2-x^2}}=f(x)$

So this is an odd function.

$$\therefore I=\int_{-a}^{a}\frac{xdx}{\sqrt{a^2-x^2}}=0$$

Which is finite. So the given integral is convergent.

Comparison Test: Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a know integral. The test below shows us how to do this.

Let $f(x)$ and $g(x)$ be functions which are continuous on the interval $[a,\infty)$. Suppose that $0<f(x)\le g(x)$ for all x in $[a,\infty)$.

(i) If $\int_a^\infty g(x)dx$ is convergent then $\int_a^\infty f(x)dx$ is also convergent.

(ii) If $\int_a^\infty g(x)dx$ is divergent then $\int_a^\infty f(x)dx$ is also divergent.

Dirichlet's Test: If $f(x)$ is bounded and monotonic decreasing in the interval $[a,\infty)$, $\lim_{x\to\infty} f(x)=0$ and $\left|\int_a^x g(x)dx\right|\le A$ (a finite number) for finite values of x, then $\int_a^\infty f(x)g(x)dx$ is convergent.

Problem-15: Test for convergence of $\int_0^\infty \frac{\cos x}{1+x^2}dx$.

Solution: Let $I=\int_0^\infty \frac{\cos x}{1+x^2}dx$

Now $\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}$, $\quad \because |\cos x| \le 1 \ \forall x \ge 0$

Again $\int_0^\infty \frac{dx}{1+x^2} = \lim_{p\to\infty}\int_0^p \frac{dx}{1+x^2} = \lim_{p\to\infty}\left[\tan^{-1}x\right]_0^p = \lim_{p\to\infty}\left[\tan^{-1}p-\tan^{-1}0\right] = \tan^{-1}\infty = \frac{\pi}{2}$

which is finite. So $\int_0^\infty \frac{dx}{1+x^2}$ is convergent.

So by comparison test, we have $\int_0^\infty \frac{\cos x}{1+x^2}dx$ is convergent.

Problem-16: Test for convergence of $\int_0^\infty \frac{\sin x}{x}dx$.

Solution: Let $I=\int_0^\infty \frac{\sin x}{x}dx$

$$\therefore \int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx \qquad \cdots(1)$$

Since $\frac{\sin x}{x}$ is continuous in $0<x\le 1$ and $\lim_{x\to 0+}\frac{\sin x}{x}=1$, so $\int_0^1 \frac{\sin x}{x}dx$ is a proper integral.

We know that the proper integral being always convergent. So $\int_0^1 \frac{\sin x}{x}dx$ is convergent.

Now we need only to test the convergence of $\int_1^\infty \frac{\sin x}{x}dx$.

Now by applying Dirichlet's Test we will examine the convergence of $\int_1^\infty \frac{\sin x}{x}dx$.

Let $f(x)=\frac{1}{x}$ and $g(x)=\sin x$

$\therefore f(1)=1$ and $\lim_{x\to\infty} f(x)=0$, so $f(x)=\frac{1}{x}$ is monotonic decreasing for all $x\ge 1$.

Now

$$\left|\int_1^x g(x)\,dx\right| = \left|\int_1^x \sin x\,dx\right| = \left|\left[-\cos x\right]_1^x\right| = \left|-\cos x + \cos 1\right| \le \left|-\cos x\right| + \left|\cos 1\right| \le 1+1=2$$

$$\therefore \left|\int_1^x g(x)\,dx\right| \le 2$$

Therefore, for all finite value x, $\int_1^x g(x)\,dx$ is bounded.

Thus, by Dirichlet's Test we have $\int_1^\infty f(x)g(x)dx$, i.e. $\int_1^\infty \frac{\sin x}{x}dx$ is convergent.

So that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

Problem-17: Test for convergence of $\int_a^\infty \frac{e^{-x}\sin x}{x^2}dx$.

Solution: Let $I = \int_a^\infty \frac{e^{-x}\sin x}{x^2}dx$

Now by applying Dirichlet's Test we will examine the convergence of $\int_a^\infty \frac{e^{-x}\sin x}{x^2}dx$.

Let $f(x) = \frac{e^{-x}}{x^2}$ and $g(x) = \sin x$

$\therefore f(a) = \frac{e^{-a}}{a^2}$ and $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\frac{e^{-x}}{x^2} = \lim\limits_{x\to\infty}\frac{-e^{-x}}{2x} = \lim\limits_{x\to\infty}\frac{e^{-x}}{2} = 0$, so $f(x) = \frac{e^{-x}}{x^2}$ is monotonic decreasing for all $x \ge a$.

Now

$$\left|\int_a^x g(x)\,dx\right| = \left|\int_a^x \sin x\,dx\right| = \left|\left[-\cos x\right]_a^x\right| = \left|-\cos x + \cos a\right| \le \left|-\cos x\right| + \left|\cos a\right| \le 1+1=2$$

$$\therefore \left|\int_a^x g(x)\,dx\right| \le 2$$

Therefore, for all finite value x, $\int_a^x g(x)\,dx$ is bounded.

Thus, by Dirichlet's Test we have $\int_a^\infty f(x)g(x)dx$, i.e. $\int_a^\infty \frac{e^{-x}\sin x}{x^2}dx$ is convergent.

❖ If the integrand is a function of one or more parameters in addition to the variable of integration, then the integral between the limits which may be constants or functions of the parameters is a function of these parameters.

$$\int_1^2 (x+\alpha)^2 dx = \left[\frac{(x+\alpha)^3}{3}\right]_1^2 = \frac{1}{3}\left[(2+\alpha)^3-(1+\alpha)^3\right] = \frac{1}{3}\left(3\alpha^2+9\alpha+7\right) = F(\alpha)$$

Thus, in general $\int_a^b f(x,\alpha)dx = F(\alpha) \quad \cdots(i)$, $\int_a^b f(x,\alpha,\beta)dx = F(\alpha,\beta)$, where a, b may be constants or functions of parameters. Sometimes $f(x,\alpha)$ is such that the evaluation of the integral is very complicated for impossible. However the integral with integrand $\frac{\partial f}{\partial \alpha}$ may be easily evaluated. Hence we discuss here how to differentiate the integral (i) w.r.to the parameter α.

Leibnitz's rule for differentiation under the integral sign: If $f(x,\alpha)$ and $\frac{\partial f}{\partial \alpha}$ are continuous functions of x and α for $a \le x \le b$, $c \le \alpha \le d$, a, b being independent of α, then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx.$$

Working rules for evaluate a given integral $\int_a^b f(x,\alpha)dx$**:**

(i) Let $F(\alpha) = \int_a^b f(x,\alpha)dx$

(ii) Differentiate both sides w.r.to α using Leibnitz's Rule.

Then $F'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$.

(iii) Evaluate the integral on R.H.S.

(iv) Integrate both sides w.r.to α, adding the constant of integration on R.H.S.

(v) Evaluate the constant of integration by giving suitable value to the parameter α.

Problem-18: Prove that $\int_0^1 \frac{x^a-1}{\ln x}dx = \ln(a+1)$.

Solution: Let $F(a) = \int_0^1 \frac{x^a-1}{\ln x}dx \qquad \cdots(1)$

Differentiating (1) w. r. to a under the integral sign, we have

$$\frac{d}{da}F(a)=\int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-1}{\ln x}\right)dx$$

$$or,\ F'(a)=\int_0^1 \frac{x^a\ln x-0}{\ln x}dx$$

$$=\int_0^1 x^a\,dx$$

$$=\left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$\therefore F'(a)=\frac{1}{a+1} \qquad \cdots(2)$$

Now integrating (2) w. r. to *a* we get

$$F(a)=\int\frac{da}{a+1}$$

$$\therefore F(a)=\ln(a+1)+c \qquad \cdots(3)$$

where *c* is a constant of integration.

Putting $a=0$ in (3), we get

$$F(0)=\ln(0+1)+c$$

$$\therefore \int_0^1 \frac{x^0-1}{\ln x}dx=\ln 1+c$$

$$or,\ \int_0^1 0\,dx=0+c$$

$$or,\ c=0$$

Using the value *c* in (3), we get

$$\int_0^1 \frac{x^a-1}{\ln x}dx=\ln(a+1) \qquad \textbf{(Proved).}$$

Problem-19: Prove that $\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx=\tan^{-1}\frac{b}{a}$ and hence show that $\int_0^\infty \frac{\sin bx}{x}dx=\frac{\pi}{2}$.

Solution: Let $F(a,b)=\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx \qquad \cdots(1)$

Differentiating (1) w. r. to *b* under the integral sign, we have

$$\frac{d}{db}F(a,b)=\int_0^\infty \frac{\partial}{\partial b}\left(\frac{e^{-ax}\sin bx}{x}\right)dx$$

$$or,\ F'(a,b)=\int_0^\infty \frac{e^{-ax}x\cos bx}{x}dx$$

$$=\int_0^\infty e^{-ax}\cos bx\,dx$$

$$=\left[\frac{e^{-ax}(-a\cos bx+b\sin bx)}{a^2+b^2}\right]_0^\infty$$

$$=0+\frac{a}{a^2+b^2}$$

$$\therefore F'(a,b)=\frac{a}{a^2+b^2} \qquad \cdots(2)$$

Now integrating (2) w. r. to *b* we get

$$F(a,b)=a\int\frac{db}{a^2+b^2}$$

$$=a\cdot\frac{1}{a}\tan^{-1}\frac{b}{a}+c$$

$$\therefore F(a,b)=\tan^{-1}\frac{b}{a}+c \qquad \cdots(3)$$

where *c* is a constant of integration.

Putting $b=0$ in (3), we get

$$F(a,0)=\tan^{-1}0+c$$

$$\therefore \int_0^\infty \frac{e^{-ax}\sin 0.x}{x}dx=0+c$$

$$or,\ c=0$$

Using the value *c* in (3), we get

$$\therefore \int_0^\infty \frac{e^{-ax}\sin bx}{x}dx=\tan^{-1}\frac{b}{a}$$ **(Proved).**

2nd part: Putting $a=0$, we get

$$\int_0^\infty \frac{\sin bx}{x}dx=\tan^{-1}\infty$$

$$\therefore \int_0^\infty \frac{\sin bx}{x}dx=\frac{\pi}{2}$$ **(Proved).**

Assignment:

Problem-01: Test for convergence of $\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$.

Problem-02: Test for convergence of $\int_1^\infty \frac{dx}{\sqrt{x}(2+x)}$.

Problem-03: Test for convergence of $\int_0^\infty \frac{dx}{x\ln x}$.

Problem-04: Test for convergence of $\int_{-\infty}^0 e^{2x}dx$.

Problem-05: Evaluate $\int_0^\infty \frac{x^2dx}{(1+x^2)^2}$.

Problem-06: Evaluate $\int_0^\infty \frac{xdx}{(1+x)(1+x^2)}$.

Problem-07: Test for convergence of $\int_0^1 \frac{dx}{x(1+x)}$.

Problem-08: Test for convergence of $\int_0^1 \frac{dx}{x^3(1+x^2)}$.

Problem-09: Evaluate $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx$.

Problem-10: Evaluate $\int_0^\pi \frac{\ln(1+\sin\alpha\cos x)}{\cos x}dx$.

Problem-11: Evaluate $\int_0^\pi \frac{\ln(1+a\cos x)}{\cos x}dx$.

Problem-12: Evaluate $\int_0^\infty \frac{\ln(1+a^2x^2)}{1+b^2x^2}dx$.