General Theory of Irrotational Motion

Flow: If A and B be any two points in a fluid, then the of the integral $\int_A^B \vec{q}\cdot d\vec{r}$ is called the flow along the path from A to B.

When the velocity potential exists i.e. $\vec{q}=-\nabla\phi$, then we have

$$
\begin{aligned}
Flow &= -\int_A^B \nabla\phi\cdot d\vec{r} \\
&= -\int_A^B \left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right) \\
&= -\int_A^B d\phi \\
&= -[\phi]_A^B \\
&= \phi_A-\phi_B.
\end{aligned}
$$

Circulation: The flow round a closed curve is called the circulation round the curve. If C is the closed curve, then the circulation is

$$Circulation=\oint_C \vec{q}\cdot d\vec{r}$$

where $\vec{q}$ is the velocity vector and the line integral is taken round C.

When the velocity potential exists i.e. $\vec{q}=-\nabla\phi$, then the circulation round the curve is zero.

Question-01: State and prove Kelvin's circulation theorem.

Answer: **Statement:** Circulation along any closed circuit moving in the liquid is constant for all times, provided the external forces are conservative and the density of the liquid is either a function of pressure or a constant.

Proof: Let C be any closed circuit moving in the liquid, then

$$Circulation=\oint_C \vec{q}\cdot d\vec{r} \qquad \cdots(1)$$

We shall show that

$$\frac{d}{dt}(Circulation)=0$$

Differentiating (1) with respect to t, we have

$$
\begin{aligned}
\frac{d}{dt}(Circulation) &= \frac{d}{dt}\oint_C \vec{q}\cdot d\vec{r} \\
&= \oint_C \left[\frac{d\vec{q}}{dt}\cdot d\vec{r}+\vec{q}\cdot d\left(\frac{d\vec{r}}{dt}\right)\right]
\end{aligned}
$$

$$=\oint_C\left[\frac{d\vec{q}}{dt}\cdot d\vec{r}+\vec{q}\cdot d\vec{q}\right] \qquad \cdots(2)$$

The equation of motion is

$$\frac{d\vec{q}}{dt}=F-\frac{1}{\rho}\nabla p \qquad \cdots(3)$$

Since the external forces are conservative so we can write

$$F=-\nabla\Omega \qquad \cdots(4)$$

where Ω is any single valued potential function.
From (2), (3) and (4), we get

$$\frac{d}{dt}(Circulation)=\oint_C\left[\left(-\nabla\Omega-\frac{1}{\rho}\nabla p\right)\cdot d\vec{r}+\vec{q}\cdot d\vec{q}\right] \qquad \cdots(5)$$

Putting $\nabla=\frac{\partial}{\partial\vec{r}}$ in (5), we get

$$\frac{d}{dt}(Circulation)=\oint_C\left[\left(-\frac{\partial\Omega}{\partial\vec{r}}-\frac{1}{\rho}\frac{\partial p}{\partial\vec{r}}\right)\cdot d\vec{r}+\vec{q}\cdot d\vec{q}\right]$$

$$=\oint_C\left[-d\Omega-\frac{1}{\rho}dp+d\left(\frac{1}{2}q^2\right)\right] \qquad \left[\because \vec{q}\cdot\vec{q}=q^2\right]$$

$$=\oint_C\left[-d\Omega-d\int\frac{dp}{\rho}+d\left(\frac{1}{2}q^2\right)\right]$$

[Since ρ is either a function of pressure of constant]

$$=\oint_C d\left[-\Omega-\int\frac{dp}{\rho}+\frac{1}{2}q^2\right]$$

$$=\left[-\Omega-\int\frac{dp}{\rho}+\frac{1}{2}q^2\right]_C$$

$$=0$$

$\therefore \frac{d}{dt}(Circulation)=0$. (**Proved**)

Vorticity: If $\vec{q}$ is the velocity vector of a fluid particle, then the quantity, $\omega=\frac{1}{2}\nabla\times\vec{q}$ or $\omega=\nabla\times\vec{q}$ is called the vorticity vector or simply vorticity.

Question-02: Show that the vorticity is the circulation per unit area.

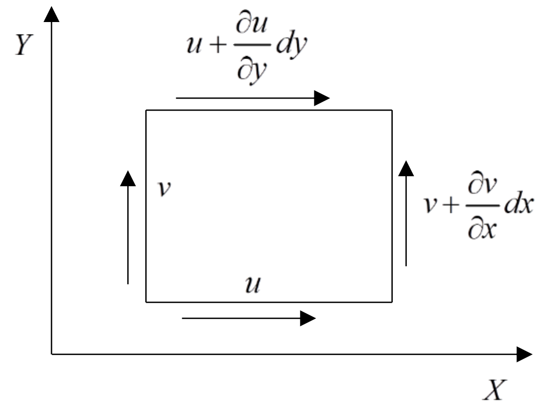
OR

Discuss the relation between circulation and vorticity.

Answer: Consider a two dimensional flow in $xy-$plane. Let dx and dy be the sides of an infinitesimal rectangle of the plane. If the circulation round the rectangular element be $d\tau$, then

$$d\tau = udx + \left(v + \frac{\partial v}{\partial x}dx\right)dy - \left(u + \frac{\partial u}{\partial y}dy\right)dx - vdy$$
$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

Since $dA = dxdy$ is the area of the rectangular element.



So $\quad d\tau = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA$

$$or,\ \frac{d\tau}{dA} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

But we know, vorticity $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Therefore, $\quad \xi = \frac{d\tau}{dA} = \frac{Circulation}{area}$

$$\therefore\ Circulation = Vorticity \times Area$$

This is the required relation between circulation and vorticity.

<u>**Permanence of irrotational motion**</u>**:** When the external forces are conservative and are derivable from a single- valued potential and pressure is a function of density only, then if once the motion of a non-viscous fluid is irrotational, it remains irrotational ever afterwards.

If the motion is irrotational at any instant, the circulation is zero for every closed circuit. But by Kelvin's circulation theorem the circulation in any closed path moving with the fluid is constant for all times. Then by Stoke's theorem

$$Circulation = \oint_C \vec{q} \cdot d\vec{r} = \int_S \left(\nabla \times \vec{q}\right).\vec{n}ds = 0$$

$$or,\ \nabla \times \vec{q} = 0$$

$$or,\ \frac{1}{2}\nabla \times \vec{q} = \omega = 0$$

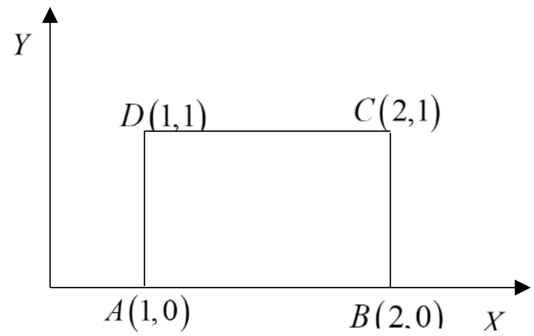
The vorticity ω is zero and therefore the irrotational motion is permanent.

Problem

Problem-01: In case of two dimensional irrotational motion, the velocity field is given by $\vec{q} = \frac{-yi + xj}{x^2 + y^2}$. What will be the circulation round a square having corners at $(1,0), (2,0)$, $(2,1)$ and $(1,1)$? Also what will be the circulation round a unit circle with centre at the origin?

Solution: The given velocity field is

$$\vec{q} = \frac{-yi + xj}{x^2 + y^2}$$



$$Circulation = \oint_C \vec{q} \cdot d\vec{r}$$

$$= \int_A^B \vec{q} \cdot d\vec{r} + \int_B^C \vec{q} \cdot d\vec{r} + \int_C^D \vec{q} \cdot d\vec{r} + \int_D^A \vec{q} \cdot d\vec{r}$$

$$= \int_{x=1}^{2} \frac{xj}{x^2} \cdot idx + \int_{y=0}^{1} \frac{-yi + 2j}{4 + y^2} \cdot jdy + \int_{x=2}^{1} \frac{-i + xj}{1 + x^2} \cdot idx + \int_{y=1}^{0} \frac{-yi + j}{1 + y^2} \cdot jdy$$

$$= 0 + 2\int_{y=0}^{1} \frac{dy}{4 + y^2} - \int_{x=2}^{1} \frac{dx}{1 + x^2} + \int_{y=1}^{0} \frac{dy}{1 + y^2}$$

$$= \left[\tan^{-1}\left(\frac{y}{2}\right)\right]_0^1 - \left[\tan^{-1}(x)\right]_2^1 + \left[\tan^{-1}(y)\right]_1^0$$

$$= \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(1) + \tan^{-1}(2) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(2) - 2\tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2} + 2}{1 - \frac{1}{2} \cdot 2}\right) - 2 \cdot \frac{\pi}{4}$$

$$= \tan^{-1}\left(\frac{\frac{5}{2}}{0}\right) - \frac{\pi}{2}$$

$$= \tan^{-1}(\infty) - \frac{\pi}{2}$$

$$= \frac{\pi}{2} - \frac{\pi}{2}$$

$$= 0$$

Hence the $Circulation = 0$.

2nd part: The velocity components are

$$u = -\frac{y}{x^2 + y^2} \quad \text{and} \quad v = \frac{x}{x^2 + y^2}$$

The z-component of vorticity is

$$\begin{aligned}
\xi_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) \\
&= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} + \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \\
&= 0
\end{aligned}$$

In the case of two dimensional motion,

$$Circulation = Vorticity \times Area$$

$$or,\ Circulation = 0 \times \pi \cdot 1^2 \qquad [\,Area = \pi r^2,\ for\, unit\ circle\ r = 1\,]$$

$$\therefore\ Circulation = 0\,.$$

Problem-02: The velocity components of a fluid flow are $u = 3x + y\,, v = 2x - 3y$. Calculate the circulation around the closed curve $x^2 + y^2 - 2x - 12y + 33 = 0$.

Solution: The velocity components are

$$u = 3x + y \quad \text{and} \quad v = 2x - 3y$$

The z-component of vorticity is

$$\begin{aligned}
\xi_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}(2x - 3y) - \frac{\partial}{\partial y}(3x + y) \\
&= 2 - 1 = 1
\end{aligned}$$

The given curve is

$$x^2 + y^2 - 2x - 12y + 33 = 0$$

$$or,\ (x-1)^2 + (y-6)^2 = 2^2$$

which represents a circle, whose Centre is $(1,6)$ and radius is $r = 2$.

Therefore the area of the circle is

$$\begin{aligned}
Area &= \pi r^2 \\
&= \pi \cdot 2^2 = 4\pi
\end{aligned}$$

The circulation is

$$Circulation = Vorticity \times Area$$

$$or,\ Circulation = 1 \times 4\pi$$

$$\therefore\ Circulation = 4\pi \qquad square\ units/\sec..$$

Problem-03: The velocity components of a fluid flow are $u = 5x^2 + 3y$, $v = 4x - 3y^2$. Calculate the circulation around the closed curve $9x^2 - 18x + 4y^2 - 16y - 11 = 0$.

Solution: The velocity components are

$$u = 5x^2 + 3y \quad \text{and} \quad v = 4x - 3y^2$$

The z-component of vorticity is

$$\begin{aligned} \xi_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\left(4x - 3y^2\right) - \frac{\partial}{\partial y}\left(5x^2 + 3y\right) \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

The given curve is

$$\begin{aligned} &9x^2 - 18x + 4y^2 - 16y - 11 = 0 \\ &or,\ 9\left(x^2 - 2x + 1\right) + 4\left(y^2 - 4y + 4\right) - 36 = 0 \\ &or,\ 9\left(x-1\right)^2 + 4\left(y-2\right)^2 = 36 \\ &or,\ \frac{\left(x-1\right)^2}{4} + \frac{\left(y-2\right)^2}{9} = 1 \\ &\therefore\ \frac{\left(x-1\right)^2}{2^2} + \frac{\left(y-2\right)^2}{3^2} = 1 \end{aligned}$$

which represents an ellipse where $a = 2$ and $b = 3$.

Therefore the area of the ellipse is

$$\begin{aligned} Area &= \pi ab \\ &= \pi \cdot 2 \cdot 3 \\ &= 6\pi \end{aligned}$$

The circulation is

$$\begin{aligned} &Circulation = Vorticity \times Area \\ &or,\ Circulation = 1 \times 6\pi \\ &\therefore\ Circulation = 6\pi \quad square\ units/\sec.. \end{aligned}$$

Problem-04: If Γ be the circulation around any closed circuit moving with the fluid, prove that $\frac{d\Gamma}{dt} = \int pd\left(\frac{1}{\rho}\right)$ if the external forces have a potential and the pressure is a function of the density alone.

Solution: Let C be any closed circuit moving in the liquid, then

$$Circulation = \oint_C \vec{q} \cdot d\vec{r}$$

$$\therefore\ \Gamma = \oint_C \vec{q} \cdot d\vec{r}$$

Differentiating with respect to t, we have

$$\frac{d\Gamma}{dt}=\frac{d}{dt}\oint_C \vec{q}\cdot d\vec{r}$$
$$=\oint_C\left[\frac{d\vec{q}}{dt}\cdot d\vec{r}+\vec{q}\cdot d\left(\frac{d\vec{r}}{dt}\right)\right]$$
$$=\oint_C\left[\frac{d\vec{q}}{dt}\cdot d\vec{r}+\vec{q}\cdot d\vec{q}\right] \qquad \cdots(1)$$

The equation of motion is

$$\frac{d\vec{q}}{dt}=F-\frac{1}{\rho}\nabla p \qquad \cdots(2)$$

Since the external forces have a potential so we can write

$$F=-\nabla\Omega \qquad \cdots(3)$$

From (1), (2) and (3), we get

$$\frac{d\Gamma}{dt}=\oint_C\left[\left(-\nabla\Omega-\frac{1}{\rho}\nabla p\right)\cdot d\vec{r}+\vec{q}\cdot d\vec{q}\right]$$
$$=\oint_C\left[\left(-\frac{\partial\Omega}{\partial\vec{r}}-\frac{1}{\rho}\frac{\partial p}{\partial\vec{r}}\right)\cdot d\vec{r}+d\left(\frac{1}{2}q^2\right)\right] \qquad \left[\because \vec{q}\cdot\vec{q}=q^2 \quad and \quad \nabla=\frac{\partial}{\partial\vec{r}}\right]$$
$$=\oint_C\left[-d\Omega-\frac{1}{\rho}dp+d\left(\frac{1}{2}q^2\right)\right]$$
$$=\oint_C\left[-d\Omega-d\int\frac{dp}{\rho}+d\left(\frac{1}{2}q^2\right)\right] \qquad \text{[Since } P \text{ is a function of density]}$$
$$=\oint_C d\left[-\Omega-\int\frac{dp}{\rho}+\frac{1}{2}q^2\right]$$
$$=\left[-\Omega-\int\frac{dp}{\rho}+\frac{1}{2}q^2\right]_C$$
$$=\left[-\Omega-\frac{p}{\rho}+\int pd\left(\frac{1}{\rho}\right)+\frac{1}{2}q^2\right]_C; \qquad \left[\because \int\frac{dp}{\rho}=\frac{1}{\rho}\int dp-\int\left\{\frac{d}{dp}\left(\frac{1}{\rho}\right)\int dp\right\}dp=\frac{p}{\rho}-\int pd\left(\frac{1}{\rho}\right)\right]$$
$$=\left[-\Omega-\frac{p}{\rho}+\frac{1}{2}q^2\right]_C+\int pd\left(\frac{1}{\rho}\right)$$
$$=0+\int pd\left(\frac{1}{\rho}\right)$$
$$\therefore\ \frac{d\Gamma}{dt}=\int pd\left(\frac{1}{\rho}\right). \qquad \textbf{(Proved)}$$

Problem-05: Find the circulation round a square enclosed by the lines $x=\pm1$, $y=\pm1$ for the flow $u=x+y$, $v=x^2-y$.

Solution: The given velocity components are

$$u=x+y \text{ and } v=x^2-y$$

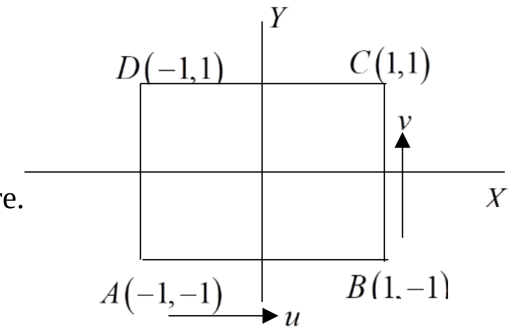
The velocity field is

$$\vec{q}=(x+y)i+(x^2-y)j$$

Also the given lines are

$$x=\pm1,\ y=\pm1$$

The square enclosed by these lines is drawn in figure.



$$Circulation=\oint_C \vec{q}\cdot d\vec{r}$$

$$=\int_A^B \vec{q}\cdot d\vec{r}+\int_B^C \vec{q}\cdot d\vec{r}+\int_C^D \vec{q}\cdot d\vec{r}+\int_D^A \vec{q}\cdot d\vec{r}$$

$$=\int_{x=-1}^{1}(x-1)\,dx+\int_{y=-1}^{1}(1-y)\,dy+\int_{x=1}^{-1}(x+1)\,dx+\int_{y=1}^{-1}(1-y)\,dy$$

$$=\left[\frac{x^2}{2}-x\right]_{-1}^{1}+\left[y-\frac{y^2}{2}\right]_{-1}^{1}+\left[\frac{x^2}{2}+x\right]_{1}^{-1}+\left[y-\frac{y^2}{2}\right]_{1}^{-1}$$

$$=\frac{1}{2}-1-\frac{1}{2}-1+1-\frac{1}{2}+1+\frac{1}{2}+\frac{1}{2}-1-\frac{1}{2}-1-1-\frac{1}{2}-1+\frac{1}{2}$$

$$=-4$$

Hence the $Circulation=4\ \ square\ units/\sec$.

Where neglecting minus sign because circulation cannot be negative.