Existence, Uniqueness and Approximation

Differential Equation: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Example: For examples of differential equation we list the following**:**

1. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0$
2. $\dfrac{d^4u}{dt^4}+5\dfrac{d^2u}{dt^2}+3u=\sin t$
3. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$
4. $\dfrac{\partial u}{\partial s}+\dfrac{\partial u}{\partial t}=v$

Classification: Differential equations are classified on the basis of type as follows**:**

1. Ordinary Differential Equation (ODE),
2. Partial Differential Equation (PDE).

Ordinary Differential Equation (ODE): A differential equation involving derivatives of one or more dependent variables with respect to only one independent variable is called an ordinary differential equation.

Example: 1. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=0$

2. $\dfrac{d^3y}{dx^3}+5\dfrac{d^2y}{dx^2}+2y=\sin x$

Partial Differential Equation (PDE): A differential equation involving derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Example: 1. $\dfrac{\partial u}{\partial s}+\dfrac{\partial u}{\partial t}=v$

2. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

Order of a differential equation: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Example: 1. $\dfrac{dy}{dx}+y=0$ is a first order differential equation.

2. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0$ is a second order differential equation**.**

Degree of a differential equation: The power of the highest ordered derivative involved in a differential equation is called the degree of the differential equation, after the equation is freed from radicals and fractions in its derivatives.

Example: 1. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$ is a differential equation of first degree.

2. $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^4+y=0$ is a differential equation of second degree.

3. $\sqrt{\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}+y}=\left(\frac{dy}{dx}\right)^2$ is a differential equation of first degree.

Linear ordinary differential equation: An ordinary differential equation of order n is called a linear ordinary differential equation of order n if it does not contain,

1. the transcendental functions of dependent variable,
2. the product of dependent variable and
3. the product of the derivatives of dependent variable.

It can be expressed as

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)$$

where, a_0 is not identically zero.

Example: 1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

2. $\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+xy=xe^x$

Nonlinear ordinary differential equation: A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

Example: 1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y^2=0$

2. $\frac{d^3y}{dx^3}+e^y\frac{d^2y}{dx^2}+xy=xe^x$

3. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+6y=0$

Solution: Any relation of dependent and independent variable which satisfies a differential equation is called a solution of that differential equation.

General or Complete Solution: If the solution of a differential equation of order n contains n arbitrary constants, then the solution is called a general solution of that differential equation.

Example: The general solution of the equation $\frac{dy}{dx}=y$ is $y=Ae^x$.

Particular Solution: If a solution is obtained from the general solution of a differential equation for definite value of arbitrary constant, then the solution is called a particular solution of the differential equation.

Example: The particular solution of the equation $\frac{dy}{dx} = y$ is $y = e^x$.

Singular Solution: A solution of differential equation which is not obtained from general solution for definite value of arbitrary constants and which is also not particular solution is called a singular solution.

Example: The general solution of $\left(\frac{dy}{dx}\right)^2 = y$ is $y = \frac{(x+c)^2}{4}$ but it's singular solution is $y = 0$.

Initial Value Problem: A differential equation with one or more supplementary conditions for same value of independent variable, which must satisfy the solution of the differential equation, is called an initial value problem.

Example: 1. $\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$

Boundary Value Problem: A differential equation with more than one supplementary conditions for different values of independent variable, which must satisfy the solution of the differential equation, is called a boundary value problem.

Example: 1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad y(x_0) = y_0, \quad y'(x_1) = y_1$.

NOTE: There are many general method for solving linear equations but no general method is known for solving nonlinear equations. The concept of general solution for linear equations differs from that for nonlinear equations. A first order linear equation has only one general solution, where as a nonlinear equation may have a general solution as well as singular solutions.

An initial value problem of first order may have

i). no solution

ii). one and only one solution

iii). more than one solution.

Now the question arises whether the initial value problem of first order has a solution. If yes, is the solution unique? The existence and uniqueness theorem gives the answer of the question.

Lipschitz condition and Lipschitz constant: A function $f(x, y)$ is said to satisfy the Lipschitz condition in a rectangular region R, where $R = \{(x, y): |x - x_0| \leq a, |y - y_0| \leq b\}$ in *xy*-plane if there exists a positive constant k such that

$$|f(x, y_2) - f(x, y_1)| \leq K|y_2 - y_1| \ \forall\ (x, y_2), (x, y_1) \in R$$

The constant K is called a Lipschitz constant for the function $f(x,y)$.

Question-01: State the existence and uniqueness theorem for $y'(x)=f(x,y)$, $y(x_0)=y_0$ in a bounded domain and prove the existence for it.

Answer: **Statement:** Let $f(x,y)$ be a real- valued continuous function on the rectangle

$$R=\{(x,y):|x-x_0|\le a, |y-y_0|\le b\} \qquad (1)$$

where $a,b>0$, $|f(x,y)|\le M \ \forall (x,y)\in R$ and $Ma<b$.

Further let $f(x,y)$ satisfies Lipschitz condition with Lipschitz constant K in R.

i.e. $|f(x,y_2)-f(x,y_1)|\le K|y_2-y_1| \ \forall (x,y_2),(x,y_1)\in R$ (2)

where K is a constant independent of x, y_1, y_2.

Then, the differential equation $y'(x)=f(x,y)$ has a unique solution $y=y(x)$ for which $y(x_0)=y_0$.

Proof: We shall prove the theorem by method of successive approximations. Let x be such that $|x-x_0|\le a$. By successive approximations or Picard iterations we have

$$\left.\begin{aligned} y_1(x) &= y_0+\int_{x_0}^{x} f(x,y_0)dx \\ y_2(x) &= y_0+\int_{x_0}^{x} f(x,y_1)dx \\ &\cdots \ \cdots \ \cdots \ \cdots \ \cdots \ \cdots \\ y_n(x) &= y_0+\int_{x_0}^{x} f(x,y_{n-1})dx \end{aligned}\right\} \qquad (3)$$

This proof consists of four parts:

Step-1: We prove that, for $x_0-a\le x\le a+x_0$, the curve $y=y_n(x)$ lies in the rectangle R.

Now, $$|y_1-y_0|=\left|\int_{x_0}^{x} f(x,y_0)dx\right|$$

$$or,\ |y_1-y_0|\le \int_{x_0}^{x} |f(x,y_0)|\cdot|dx|$$

$$or,\ |y_1-y_0|\le M|x-x_0|$$

$$or,\ |y_1 - y_0| \le Ma$$

$$or,\ |y_1 - y_0| < b$$

This proves that the results holds for $n = 1$.

Assume that $y = y_{n-1}(x)$ lies in R and so $f(x, y_{n-1})$ is defined and continuous and satisfies

$$|f(x, y_{n-1})| \le M \text{ on } [x_0 - a,\ x_0 + a].$$

Therefore from (4), we get

$$|y_n - y_0| = \left| \int_{x_0}^{x} f(x, y_{n-1}) dx \right|$$

$$or,\ |y_n - y_0| \le \int_{x_0}^{x} |f(x, y_{n-1})| \cdot |dx|$$

$$or,\ |y_n - y_0| \le M|x - x_0|$$

$$or,\ |y_n - y_0| \le Ma$$

$$or,\ |y_n - y_0| < b$$

This shows that $y_n(x)$ lies in R and hence $f(x, y_n)$ is defined and continuous on $[x_0 - a,\ x_0 + a]$

.

Thus by induction the result holds for all $n \in N$.

Step-2: We prove again by induction that

$$|y_n - y_{n-1}| \le \frac{MK^{n-1}}{\underline{|n}} |x - x_0|^n \qquad (5)$$

We have already verified (5) for $n = 1$ in first step where we have shown that

$$|y_1 - y_0| \le M|x - x_0|.$$

Assume that (5) holds for $n - 1$ in place of n

i.e. $$|y_{n-1} - y_{n-2}| \le \frac{MK^{n-2}}{\underline{|n-1}} |x - x_0|^{n-1} \qquad (6)$$

Then $$|y_n - y_{n-1}| \le \left| y_0 + \int_{x_0}^{x} f(x, y_{n-1}) dx - y_0 - \int_{x_0}^{x} f(x, y_{n-2}) dx \right|$$

$$or,\ |y_n - y_{n-1}| \le \left| \int_{x_0}^{x} \{ f(x, y_{n-1}) - f(x, y_{n-2}) \} dx \right|$$

$$or,\ |y_n - y_{n-1}| \le \int_{x_0}^{x} \left|\left\{ f\left(x, y_{n-1}\right) - f\left(x, y_{n-2}\right)\right\}\right| \cdot |dx| \qquad (7)$$

$$or,\ |y_n - y_{n-1}| \le \int_{x_0}^{x} K\,|y_{n-1} - y_{n-2}| \cdot |dx|$$

$$or,\ |y_n - y_{n-1}| \le K \cdot \frac{MK^{n-2}}{\lfloor n-1} \int_{x_0}^{x} |x - x_0|^{n-1} \cdot |dx|$$

$$or,\ |y_n - y_{n-1}| \le \frac{MK^{n-1}}{\lfloor n-1} \frac{|x - x_0|^n}{n}$$

$$or,\ |y_n - y_{n-1}| \le \frac{MK^{n-1}}{\lfloor n} |x - x_0|^n$$

Hence by mathematical induction, we conclude that (5) is true for all $n \in N$.

Step-3: We shall prove that the sequence y_n converges uniformly to a limit for $x_0 - a \le x \le x_0 + a$.

Using the result (5), we have, the infinite series

$$y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) + \cdots$$

$$\le y_0 + \frac{M}{\lfloor 1}|x_1 - x_0| + \frac{MK}{\lfloor 2}|x_1 - x_0|^2 + \cdots + \frac{MK^{n-1}}{\lfloor n}|x_1 - x_0|^n + \cdots$$

$$\le y_0 + Ma + \frac{MK}{\lfloor 2}a^2 + \cdots + \frac{MK^{n-1}}{\lfloor n}a^n + \cdots$$

$$= y_0 + \frac{M}{K}\left(Ka + \frac{K^2a^2}{\lfloor 2} + \cdots + \frac{K^n a^n}{\lfloor n} + \cdots \right)$$

$$= y_0 + \frac{M}{K}\left(e^{Ka} - 1\right)$$

Which is convergent for all values of K, h and M. Consequently the above series is surely convergent.

Step-4: We now show that $y = y(x)$ satisfies the differential equation $\frac{dy}{dx} = f(x, y)$, that is ,

the existence of a solution of the given differential equation.

From (4) by taking limit $n \to \infty$ we have

$$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f(x, y_{n-1})\,dx$$

$$or,\ \ y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \qquad (8)$$

The integrand on the right hand side of (8) being a continuous function of x, we conclude that the integral has the derivative.

Thus, the limit function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = f(x,y)$ on $[x_0 - a,\ x_0 + a]$ and is such that $y(x_0) = y_0$.

Question-02: State the existence and uniqueness theorem for $y'(x) = f(x,y), y(x_0) = y_0$ in a bounded domain and prove the uniqueness for it.

Answer: **Statement:** Let $f(x,y)$ be a real- valued continuous function on the rectangle

$$R = \{(x,y) : |x - x_0| \le a, |y - y_0| \le b\} \qquad (1)$$

where $a,b > 0$, $|f(x,y)| \le M\ \forall\ (x,y) \in R$ and $Ma < b$.

Further let $f(x,y)$ satisfies Lipschitz condition with Lipschitz constant K in R.

i.e. $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|\ \forall\ (x,y_2),(x,y_1) \in R \qquad (2)$

where K is a constant independent of x, y_1, y_2.

Then, the differential equation $y'(x) = f(x,y)$ has a unique solution $y = y(x)$ for which $y(x_0) = y_0$.

2nd part: Now we need to prove that the solution $y(x)$ is the only solution for which $y(x_0) = y_0$. By successive approximations or Picard iterations we have

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))dx \qquad \cdots (3)$$

If possible, let $y = Y(x)$ is another solution of the given problem.

Let $\quad |Y(x)| \le B$, where $x_0 - a \le x \le x_0 + a \qquad \cdots (4)$

Then from (3) we have

$$Y(x) = y_0 + \int_{x_0}^{x} f(x, Y(x))dx \qquad \cdots (5)$$

Now from (3), (4) and (5) we have

$$|Y(x)-y(x)| = \left| y_0 + \int_{x_0}^{x} f\left(x, Y(x)\right)dx - y_0 - \int_{x_0}^{x} f\left(x, y(x)\right)dx \right|$$

$$\leq \int_{x_0}^{x} \left| f\left(x, Y(x)\right) - f\left(x, y(x)\right) \right| . |dx|$$

$$\leq K \int_{x_0}^{x} |Y(x)-y(x)| . |dx| \qquad \text{[by (2)]} \qquad \cdots (6)$$

Again by (4) we get

$$|Y(x)-y(x)| \leq KB|x-x_0| \qquad \cdots (7)$$

Now substituting (7) for the integrand in (6) we get

$$|Y(x)-y(x)| \leq K \int_{x_0}^{x} KB|x-x_0| . |dx| \leq K^2B \int_{x_0}^{x} |x-x_0| . dx \leq \frac{K^2B|x-x_0|^2}{2!} \qquad \cdots (8)$$

Again substituting (8) for the integrand in (6) we get

$$|Y(x)-y(x)| \leq \frac{K^3B}{2!} \int_{x_0}^{x} |x-x_0|^2 \; dx \leq \frac{K^3B|x-x_0|^3}{3!} \qquad \cdots (9)$$

Continuing in this way, we shall surely get

$$|Y(x)-y(x)| \leq \frac{K^nB|x-x_0|^n}{n!} \leq \frac{(Ka)^nB}{n!} \qquad \because \; |x-x_0| \leq a \qquad \cdots (10)$$

Now the series $\sum_{i=0}^{n} \frac{(Ka)^iB}{i!}$ converges, and so $\lim_{n\to\infty} \frac{(Ka)^nB}{n!} = 0$

Thus, $|Y(x)-y(x)|$ can be made less than any number however small and consequently we conclude that

$$Y(x)-y(x) = 0 \quad or, \; Y(x) = y(x)$$

This shows that the solution $y(x)$ is always unique.

Question-03: State and prove Cauchy- Peano existence theorem.

<u>Solution</u>: Statement: Let $f(t,x)$ be a continuous function on

$$B_0 = \{(t,x) \in \Omega : t_0 \leq t \leq t_0 + a, \; \|x - x_0\| \leq b\} \qquad \cdots (1)$$

and $\quad M = \max_{(t,x)\in B_0} \|f(t,x)\|, \;\; \alpha = min(a, b/M).$

Then there exists a solution $x(t)$ of $x' = f(t,x)$ on the interval $t_0 \leq t \leq t_0 + \alpha$ such that $x(t_0) = x_0$.

<u>Proof</u>: Consider a monotonically decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers such that $\varepsilon_n \to 0$ as $\to \infty$. Then for each $\varepsilon_n > 0$, there exists an ε_n- approximate solution $x_n(t)$ of $x' = f(t,x)$ on the interval $t_0 \leq t \leq t_0 + \alpha$ such that $x_0(t) = x_0$. If $x_n(t)$ be one such solution for each ε_n , then

$$\|x_n(t) - x_n(\tilde{t})\| \leq M|t - \tilde{t}\ |, \text{ where } t\,, \tilde{t} \in [t_0\,, t_0 + \alpha\,] \qquad \cdots (2)$$

This in turns imply that the sequence $\{x_n(t)\}$ is equicontinuous on $[t_0\,, t_0 + \alpha\,]$.

Putting $\tilde{t} = t_0$ in (2) we get,

$$\|x_n(t) - x_n(t_0)\| \leq M|t - t_0\ | \qquad \cdots (3)$$

$$\Rightarrow \|x_n(t) - x_n(t_0)\| \leq M\alpha \qquad since\ \ t \leq t_0 + \alpha\ \Rightarrow t - t_0 \leq \alpha$$

$$\Rightarrow \|x_n(t) - x_n(t_0)\| \leq b \qquad since\ \ \alpha \leq \frac{b}{M}.$$

This shows that the sequence $\{x_n(t)\}$ is uniformly bounded by $\|x_0\| + b$. Hence there exists a subsequence $\{x_{n_k}(t)\}$ converges uniformly to a continuous function $x(t)$ on $[t_0\,, t_0 + \alpha\,]$ satisfying $\|x(t) - x(\tilde{t})\| \leq M|t - \tilde{t}\ |$.

Now we define a function $\Delta_n(t)$ in the following way:

$$\Delta_n(t) = \begin{cases} x_n'(t) - f\big(t, x_n(t)\big), & at\ the\ points\ where\ x_n' \\ \qquad\qquad 0 & otherwise \end{cases}$$

Then we have $\Delta_n(t) = x_n'(t) - f\big(t, x_n(t)\big)$

$$\Rightarrow x_n'(t) = f\big(t, x_n(t)\big) + \Delta_n(t) \qquad \cdots (4)$$

Integrating (4) from t_0 to t we get,

$$[x_n(t)]_{t_0}^{t} = \int_{t_0}^{t} \big[f\big(s, x_n(s)\big) + \Delta_n(s)\big]\, ds$$

$$\Rightarrow x_n(t) = x_0 + \int_{t_0}^{t} \big[f\big(s, x_n(s)\big) + \Delta_n(s)\big]\, ds \qquad since\ x_n(t_0) = x_0 \qquad \cdots (5)$$

Also, since $x_n(t)$ is an ε_n- approximate solution, so

$$\|\Delta_n(t)\| = \big\|x_n'(t) - f\big(t, x_n(t)\big)\big\|$$

$$\Rightarrow \|\Delta_n(t)\| \leq \varepsilon$$

Since $x_{n_k}(t) \to x(t)$ uniformly on $[t_0\,, t_0 + \alpha\,]$ as $k \to \infty$, so from the uniform continuity of f on B_0 we have $f\big(t,\ x_{n_k}(t)\big) \to f(t,\ x(t))$ uniformly on $[t_0\,, t_0 + \alpha\,]$ as $k \to \infty$. In fact $\varepsilon_{n_k} \to 0$ as $k \to 0 \Rightarrow \Delta_{n_k} \to 0$ as $k \to 0$ uniformly on $[t_0\,, t_0 + \alpha\,]$.

Replacing n by n_k in (5) we get,

$$x_{n_k}(t) = x_0 + \int_{t_0}^{t} \left[f\left(s, x_{n_k}(s)\right) + \Delta_{n_k}(s)\right] ds$$

Now putting $k \to \infty$ we have

$$x(t) = x_0 + \int_{t_0}^{t} \big[f\big(s, x(s)\big) + 0\big]\, ds$$

$$\therefore x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big)\, ds$$

This complete the proof.

Question-04: Explain Picard's method of successive approximation.

<u>Solution</u>: Picard's method: Consider the following initial value problem

$$\frac{dy}{dx} = f(x, y) \qquad \cdots (1)$$

and $$y(x_0) = y_0 \qquad \cdots (2)$$

For Picard's method of successive approximation, we choose a function ϕ_0 as a zeroth approximation of the solution. Then a first approximation $\phi_1 = \phi_1(x)$ is determined in such a way that it satisfies the equations (1) and (2). i.e.

$$\frac{d}{dx}[\phi_1(x)] = f(x, \phi_0(x)) \qquad \cdots (3)$$

and $$\phi_1(x_0) = y_0 \qquad \cdots (4)$$

Now ϕ_1 satisfies (3) and (4) if and only if

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt \qquad \cdots (5)$$

The second approximation ϕ_2 , which satisfies (1) and (2) is as follows:

$$\frac{d}{dx}[\phi_2(x)] = f(x, \phi_1(x)) \qquad \cdots (6)$$

and $$\phi_2(x_0) = y_0 \qquad \cdots (7)$$

Now ϕ_2 satisfies (6) and (7) if and only if

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))dt \qquad \cdots (8)$$

Similarly we can determine the third approximation ϕ_3 , the fourth approximation ϕ_4 and so on.

The nth approximation ϕ_n is determined as

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt$$

where ϕ_{n-1} is the $(n-1)$th approximation.

The exact solution of the initial value problem is

$$\phi(x) = \lim_{n \to \infty} \phi_n(x).$$

Problem

Problem-01: Find the Lipschitz constant for $f(x,y) = y^2$ in $|x| \le a, |y| \le b$. Show that $F(x,y) = x|y|$ satisfies Lipschitz condition in $|x| \le a, |y| \le b$.

Solution: The given function is $(x,y) = y^2$, which is defined in the rectangular domain $|x| \le a$, $|y| \le b$. Let $(x, y_1), (x, y_2) \in R$ then

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &= |y_1^2 - y_2^2| \\
&= |(y_1 + y_2)(y_1 - y_2)| \\
&= |y_1 + y_2||(y_1 - y_2)| \\
&\le (|y_1| + |y_2|)|(y_1 - y_2)| \\
&\le (b + b)|(y_1 - y_2)| \\
&\le 2b|(y_1 - y_2)|
\end{aligned}
$$

This shows that the function $f(x,y) = y^2$ satisfies Lipschitz condition and the Lipschitz constant is $k = 2b$.

2nd part: The given function is $(x,y) = x|y|$, which is defined in the rectangular domain $|x| \le a$, $|y| \le b$. Let $(x, y_1), (x, y_2) \in R$ then

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &= |(x|y_1|) - (x|y_2|)| \\
&= |x(|y_1| - |y_2|)| \\
&= |x|\big||y_1| - |y_2|\big| \\
&\le a|y_1 - y_2|
\end{aligned}
$$

This shows that the function $F(x,y) = x|y|$ satisfies Lipschitz condition in $|x| \le a, |y| \le b$.

Problem-02: Find the Lipschitz constant for $f(x,y) = xy^2 + y^4$, $|x| < 1$, $|y - 2| \le 3$. Is the solution of $\frac{dy}{dx} = f(x,y)$ unique?

Solution: The given function is $f(x,y) = xy^2 + y^4$, which is defined in the rectangular domain R.i.e.

$$\begin{aligned} R &= \{(x,y): |x| < 1, |y-2| \le 3\} \\ &= \{(x,y): -1 < x < 1, -3 \le y-2 \le 3\} \\ &= \{(x,y): -1 < x < 1, -1 \le y \le 5\} \end{aligned}$$

Let $(x,y_1), (x,y_2) \in R$ then

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= |xy_1^2 + y_1^4 - xy_2^2 - y_2^4| \\ &= |x(y_1^2 - y_2^2) + (y_1^4 - y_2^4)| \\ &= |x(y_1^2 - y_2^2) + (y_1^2 + y_2^2)(y_1^2 - y_2^2)| \\ &= |(x + y_1^2 + y_2^2)(y_1^2 - y_2^2)| \\ &= |(x + y_1^2 + y_2^2)(y_1 + y_2)(y_1 - y_2)| \\ &= |x + y_1^2 + y_2^2||y_1 + y_2||y_1 - y_2| \\ &\le |1 + 5^2 + 5^2||5 + 5||y_1 - y_2| \\ &\le 510|y_1 - y_2| \end{aligned}$$

This shows that the function $f(x,y) = xy^2 + y^4$ satisfies Lipschitz condition and the Lipschitz constant is $L = 510$.

2nd part: Here $\frac{dy}{dx} = f(x,y) = xy^2 + y^4$

$$\therefore \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^2 + y^4) = 2xy + 4y^3$$

It is evident that both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in the rectangular region

$$R = \{(x,y): -1 < x < 1, -1 \le y \le 5\}$$

Hence the given equation has a unique solution.

Problem-03: Discuss the existence and uniqueness of a solution of the IVP $\frac{dy}{dx} = xy^3$, $y(0) = 1$. Solve the IVP and the interval of existence.

Solution: Given initial value problem is $\frac{dy}{dx} = xy^3$, $y(0) = 1$ $\cdots(1)$

Comparing (1) with $y'(x) = f(x,y), y(x_0) = y_0$ we get

$$f(x,y) = xy^3, x_0 = 0 \text{ and } y_0 = 1.$$

We put $a = 1$ and $b = 1$then the rectangular region is

$$R = \{(x,y): |x - x_0| \le a, |y - y_0| \le b\}$$

$$= \{(x, y): |x - 0| \leq 1, |y - 1| \leq 1\}$$
$$= \{(x, y): |x| \leq 1, |y - 1| \leq 1\}$$
$$= \{(x, y): -1 \leq x \leq 1, -1 \leq y - 1 \leq 1\}$$
$$= \{(x, y): -1 \leq x \leq 1, 0 \leq y \leq 2\} \qquad \cdots (2)$$

It is evident that $f(x, y) = xy^3$ is continuous in rectangular region R.

Let $(x, y_1), (x, y_2) \in R$ then

$$|f(x, y_1) - f(x, y_2)| = |xy_1^3 - xy_2^3|$$
$$= |x(y_1^3 - y_2^3)|$$
$$= |x(y_1 - y_2)(y_1^2 + y_1y_2 + y_2^2)|$$
$$= |x||(y_1 - y_2)||(y_1^2 + y_1y_2 + y_2^2)|$$
$$\leq 1|(y_1 - y_2)||(2^2 + 2.2 + 2^2)|$$
$$\leq 12|(y_1 - y_2)|$$

$|f(x, y_1) - f(x, y_2)| \leq A|(y_1 - y_2)|$, where $A = 12$

Hence the function $f(x, y) = xy^3, \forall\, (x, y)$ satisfies Lipschitz condition. Since $f(x, y) = xy^3$ is continuous and satisfies Lipschitz condition so the given initial value problem has a unique solution.

2nd part: We have $\frac{dy}{dx} = xy^3$

$$or, y^{-3}dy = xdx$$
$$or, \int y^{-3}dy = \int xdx$$
$$or, \frac{y^{-2}}{-2} = \frac{x^2}{2} + c \qquad \cdots (3)$$

Using the initial condition $y(0) = 1$ we get

$$\frac{1}{-2} = \frac{0}{2} + c \qquad or, c = -\frac{1}{2}$$

then (3) becomes

$$\frac{y^{-2}}{-2} = \frac{x^2}{2} - \frac{1}{2} \qquad or, \frac{1}{y^2} = 1 - x^2 \qquad or, y = \frac{1}{\sqrt{1 - x^2}}$$

which is the required solution.

The solution is valid when

$$1 - x^2 > 0 \;\; or, x^2 < 1 \;\; or, |x| < 1$$

Thus the interval of the existence of the solution is

$|x| < 1$ i.e. $-1 < x < 1$.

Problem-04: Discuss the existence and uniqueness of solution of the initial value problem

$\frac{dx}{dt} = 1 + x^2$, $x(0) = 0$ and solve it.

<u>Solution</u>: Given initial value problem is $\frac{dx}{dt} = 1 + x^2$, $x(0) = 0$ $\cdots (1)$

We know that the differential equation

$$x'(t) = f(t,x) \text{ with } x(t_0) = x_0 \quad \cdots (2)$$

has a unique solution through the point (t_0, x_0) if both $f(t,x)$ and $\frac{\partial f(t,x)}{\partial x}$ are continuous in a rectangular domain

$$R = \{(t,x): |t - t_0| \leq a, |x - x_0| \leq b\}.$$

Comparing (1) with (2) we get,

$$f(t,x) = 1 + x^2, t_0 = 0 \text{ and } x_0 = 0.$$

Here $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1 + x^2) = 2x$

It is evident that both $f(t,x)$ and $\frac{\partial f}{\partial x}$ are continuous in domain R containing the point. Hence the given equation has a unique solution with the initial condition.

<u>2nd part</u>: We have $\frac{dx}{dt} = 1 + x^2$

$$or, \frac{dx}{1 + x^2} = dt$$

$$or, \int \frac{dx}{1 + x^2} = \int dt$$

$$or, tan^{-1}x = t + c$$

$$or, x = \tan(t + c) \quad \cdots (3)$$

Using the initial condition $x(0) = 0$ in (4), we get

$$0 = \tan(0 + c) \Rightarrow 0 = tanc \Rightarrow c = 0$$

then (4) becomes

$$x = \tan(t + 0) \qquad or, x = tant$$

which is the required solution.

Problem-05: Examine the uniqueness of solution of $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$ and find the interval of existence.

Solution: Given initial value problem is $\frac{dy}{dx} = x^2 + y^2\,,\ y(0) = 0$ $\cdots(1)$

Comparing (1) with $y'(x) = f(x,y), y(x_0) = y_0$ we get

$$f(x,y) = x^2 + y^2, x_0 = 0 \text{ and } y_0 = 0.$$

The rectangular region is

$$\begin{aligned} R &= \{(x,y): |x - x_0| \le a, |y - y_0| \le b\} \\ &= \{(x,y): |x - 0| \le a, |y - 0| \le b\} \\ &= \{(x,y): |x| \le a, |y| \le b\} \\ &= \{(x,y): -a \le x \le a, -b \le y \le b\} \end{aligned} \quad \cdots(2)$$

Here $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$

It is evident that both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in domain R.

Hence the given equation has a unique solution with the initial condition.

2nd part: Suppose $M = max|f(x,y)|$, for $(x,y) \in R$ and $h = min(a, b/M)$.

Then the existence and uniqueness theorem asserts that the given problem possesses a unique solution on $|x| \le h$.

Now $M = max|f(x,y)| = |a^2 + b^2| = a^2 + b^2$

and so $h = min\left(a, \frac{b}{a^2+b^2}\right)$

Let $F(b) = \frac{b}{a^2+b^2}$

$$\therefore F'(b) = \frac{(a^2+b^2).1 - b.2b}{(a^2+b^2)^2} = \frac{a^2 - b^2}{(a^2+b^2)^2}$$

$$\begin{aligned} \therefore F''(b) &= \frac{(a^2+b^2)^2.(-2b) - (a^2-b^2).2(a^2+b^2).2b}{(a^2+b^2)^4} \\ &= \frac{-2b(a^2+b^2) - 4b(a^2-b^2)}{(a^2+b^2)^3} \\ &= \frac{-2b(a^2+b^2+2a^2-2b^2)}{(a^2+b^2)^3} \\ &= \frac{-2b(3a^2-b^2)}{(a^2+b^2)^3} \end{aligned}$$

For maximum or minimum value of b we know

$$F'(b) = 0$$

$$\Rightarrow \frac{a^2 - b^2}{(a^2+b^2)^2} = 0$$

$$\Rightarrow a^2 - b^2 = 0$$

$$\Rightarrow b = \pm a$$

$$\Rightarrow b = a \quad since\ b > 0.$$

When $b = a$ then

$$F''(b) = \frac{-2a(3a^2 - a^2)}{(a^2 + a^2)^3} = \frac{-4a^3}{8a^6} = -ve$$

Therefore $F(b)$ has a maximum value at $b = a$ and the maximum value is

$$F(a) = \frac{a}{a^2 + a^2} = \frac{a}{2a^2} = \frac{1}{2a}$$

Thus if $a \geq \frac{1}{2a}$ then $\frac{b}{a^2+b^2} \leq a$, for all $b > 0$

and so $h = \frac{b}{a^2+b^2} \leq a$

If $a < \frac{1}{2a}$ then $h < \frac{1}{2a}$.

Thus, in any case $h \leq \frac{1}{2a}$

For $b = a$, $a \geq \frac{1}{2a}$ we have $h = min\left(a, \frac{b}{a^2+b^2}\right) = min\left(a, \frac{1}{2a}\right) = \frac{1}{2a}$.

The equality of $a \geq \frac{1}{2a}$ gives $a = \frac{1}{2a}$

$$\Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}}, \qquad since\ a > 0.$$

$$\therefore h = \frac{1}{2a} \Rightarrow \frac{1}{\sqrt{2}}.$$

Thus, the given equation has a unique solution on the interval

$$|x| \leq h$$

$$\Rightarrow |x| \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}.$$

Problem-06: Examine the existence and uniqueness of solution of the IVP $\frac{dy}{dx} = x^2y^2 - \frac{3y}{x}$, $y(1) = 1$. Find also the solution and its region of validity.

<u>Solution</u>: Given initial value problem is $\frac{dy}{dx} = x^2y^2 - \frac{3y}{x}$, $y(1) = 1$ $\cdots(1)$

Comparing (1) with $y'(x) = f(x,y), y(x_0) = y_0$ we get

$$f(x,y) = x^2y^2 - \frac{3y}{x},\ x_0 = 1 \text{ and } y_0 = 1.$$

The rectangular region is

$$
\begin{aligned}
R &= \{(x,y): |x - x_0| \le a, |y - y_0| \le b\} \\
&= \{(x,y): |x-1| \le a, |y-1| \le b\} \\
&= \{(x,y): -a \le x - 1 \le a, -b \le y - 1 \le b\} \\
&= \{(x,y): 1 - a \le x \le 1 + a, 1 - b \le y \le 1 + b\} \qquad \cdots (2)
\end{aligned}
$$

It is evident that $f(x,y)$ is continuous in rectangular region R.

Let $(x, y_1), (x, y_2) \in R$ then

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &= \left|x^2y_1^2 - \frac{3y_1}{x} - x^2y_2^2 + \frac{3y_2}{x}\right| \\
&= \left|x^2(y_1^2 - y_2^2) - \frac{3}{x}(y_1 - y_2)\right| \\
&= \left|x^2(y_1 + y_2)(y_1 - y_2) - \frac{3}{x}(y_1 - y_2)\right| \\
&\le |x^2(y_1 + y_2)(y_1 - y_2)| + \left|\frac{3}{x}(y_1 - y_2)\right| \\
&\le x^2|y_1 + y_2||y_1 - y_2| + \frac{3}{x}|y_1 - y_2| \\
&\le \left\{x^2|y_1 + y_2| + \frac{3}{x}\right\}|y_1 - y_2| \qquad \cdots (3)
\end{aligned}
$$

Since $1 - a \le x \le 1 + a$

$$\Rightarrow 1 - a \le x \quad and\ x \le 1 + a$$

$$\Rightarrow \frac{1}{x} \le \frac{1}{1-a}$$

and $1 - b \le y \le 1 + b$

$$\Rightarrow |y| \le 1 + b$$

$$\Rightarrow |y_1| \le 1 + b, \qquad |y_2| \le 1 + b$$

$$\therefore |y_1 + y_2| \le |y_1| + |y_2| \le 1 + b + 1 + b = 2(1 + b)$$

Using these values in (3) we get

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &\le \left\{2(1+a)^2(1+b) + \frac{3}{1-a}\right\}|y_1 - y_2| \\
&\le L|y_1 - y_2|
\end{aligned}
$$

where $L = 2(1+a)^2(1+b) + \frac{3}{1-a}$

Hence the function $f(x,y) = x^2y^2 - \frac{3y}{x}$ satisfies Lipschitz condition. Since $f(x,y)$ is continuous and satisfies Lipschitz condition so the given initial value problem has a unique solution.

2nd part: The equation (1) can be written as

$$\frac{dy}{dx} + \frac{3y}{x} = x^2y^2 \qquad \cdots(4)$$

This is a Bernoulli's equation.
Dividing the equation (4) by y^2 we get

$$y^{-2}\frac{dy}{dx} + \frac{3}{x}y^{-1} = x^2 \qquad \cdots(5)$$

Put $v = y^{-1}$

$$\therefore \frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$$

Now the equation (5) becomes,

$$-\frac{dv}{dx} + \frac{3}{x}v = x^2$$

$$\Rightarrow \frac{dv}{dx} - \frac{3}{x}v = -x^2 \qquad \cdots(6)$$

This is a linear equation.

$$I.F = e^{\int -\frac{3}{x}dx} = e^{-3lnx} = e^{lnx^{-3}} = x^{-3}$$

Multiply both sides of equation (6) by x^{-3} we get

$$x^{-3}\frac{dv}{dx} - x^{-3}\frac{3}{x}v = -x^{-3}x^2$$

$$\Rightarrow x^{-3}\frac{dv}{dx} - 3x^{-4}v = -x^{-1}$$

$$\Rightarrow \frac{d}{dx}(vx^{-3}) = -x^{-1} \qquad \cdots(7)$$

Integrating both sides of (7) we get

$$vx^{-3} = -\int \frac{dx}{x}$$

$$\Rightarrow vx^{-3} = -lnx + c$$

$$\Rightarrow v = -x^3lnx + cx^3$$

$$\Rightarrow \frac{1}{y} = -x^3lnx + cx^3 \qquad \cdots(8)$$

Using the initial condition $y(1) = 1$ in (8) we get

$$\frac{1}{1} = -1^3.ln1 + c.1^3$$

$$\Rightarrow 1 = -1^3.0 + c$$

$$\Rightarrow c = 1$$

Putting the value of c in (8) we get

$$\frac{1}{y} = -x^3lnx + x^3$$

$$\Rightarrow y = \frac{1}{(x^3 - x^3lnx)}$$

3rd part: when $x = 0$ and $x = e$ then y is undefined.
Therefore the solution is valid in the interval $0 < x < e$.

Problem-07: Show that the IVP $x' = 4x^{3/4}$, $x(0) = 1$ has a unique solution but the IVP $x' = 4x^{3/4}$, $x(0) = 0$ has infinitely many solutions.

<u>Solution</u>: Given that $x' = 4x^{3/4}$, $x(0) = 1$ $\cdots(1)$

We know that the differential equation

$$x'(t) = f(t,x) \text{ with } x(t_0) = x_0 \quad \cdots(2)$$

has a unique solution through the point (t_0, x_0) if both $f(t,x)$ and $\frac{\partial f(t,x)}{\partial x}$ are continuous in a rectangular domain

$$R = \{(t,x): |t - t_0| \leq a, |x - x_0| \leq b\}.$$

Comparing (1) with (2) we get,

$$f(t,x) = 4x^{3/4}, t_0 = 0 \text{ and } x_0 = 1.$$

Here $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(4x^{3/4}\right) = 3x^{-\frac{1}{4}}$

It is evident that both $f(t,x)$ and $\frac{\partial f}{\partial x}$ are continuous in domain R containing the point $(t_0, x_0) = (0,1)$. Hence the given equation has a unique solution with the initial condition.

<u>2nd part</u>: Given that $x' = 4x^{3/4}$, $x(0) = 0$ $\cdots(3)$

Comparing (3) with (2) we get,

$$f(t,x) = 4x^{3/4}, t_0 = 0 \text{ and } x_0 = 0.$$

Here $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(4x^{3/4}\right) = 3x^{-\frac{1}{4}}$

$\therefore f(t,x) = 4x^{3/4}$ which is continuous in the whole (t,x) plane but $\frac{\partial f}{\partial x} = 3x^{-\frac{1}{4}}$ is continuous except $x = 0$.

Therefore the given equation has unique solution through any point (t_0, x_0) with $x \neq 0$.

Also, $\frac{dx}{dt} = 4x^{3/4}$

$$\Rightarrow \frac{1}{4} \cdot \frac{dx}{x^{3/4}} = dt \quad \cdots(4)$$

Integrating (4) we get

$$\frac{1}{4}\int x^{-3/4}\, dx = \int dt$$

$$\Rightarrow \frac{1}{4} \cdot \frac{x^{1/4}}{1/4} = t - c$$

$$\Rightarrow x^{1/4} = t - c$$

$$\Rightarrow x = (t - c)^4$$

$$\Rightarrow x(t) = \begin{cases} 0 & if \ -\infty < t \le c \\ (t-c)^4 & if \ c \le t \le \infty \end{cases}$$

which has a continuous derivative for $-\infty < t < \infty$ and is a solution through $(0,0)$ for every value of $c > 0$. Thus the IVP $x' = 4x^{3/4}$, $x(0) = 0$ has infinitely many solutions.

Question-08: Using Picard's method of successive approximation find the first three approximations to the solution of $\frac{dy}{dx} = x + y^2,\ \ y(0) = 0$.

<u>Solution</u>: Given initial value problem is $\frac{dy}{dx} = x + y^2,\ \ y(0) = 0$ $\cdots(1)$

Comparing (1) with $y'(x) = f(x,y),\ y(x_0) = y_0$ we get

$$f(x,y) = x + y^2,\ x_0 = 0 \text{ and } y_0 = 0.$$

By picard's method of successive approximation the zero approximation of the actual solution is

$$\phi_0(x) = y_0(x) = y_0(x_0) = 0.$$

The 1st approximation is

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f\,(t, \phi_0(t))dt = 0 + \int_0^x (t+0)\,dt = \int_0^x t\,dt = \left[\frac{t^2}{2}\right]_0^x = \frac{1}{2}x^2$$

The 2nd approximation is

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f\,\big(t, \phi_1(t)\big)dt$$

$$= 0 + \int_0^x \left[t + \left(\frac{1}{2}t^2\right)^2\right]dt = \int_0^x \left[t + \frac{1}{4}t^4\right]dt = \left[\frac{t^2}{2} + \frac{t^5}{20}\right]_0^x = \frac{x^2}{2} + \frac{x^5}{20}$$

The 3rd approximation is

$$\phi_3(x) = y_0 + \int_{x_0}^{x} f\,(t, \phi_2(t))dt$$

$$= 0 + \int_0^x \left[t + \left(\frac{t^2}{2} + \frac{t^5}{20}\right)^2\right]dt$$

$$= \int_0^x \left[t + \frac{t^4}{4} + \frac{t^7}{20} + \frac{t^{10}}{400}\right]dt$$

$$= \left[\frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400}\right]_0^x$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

Question09: Use Picard's method of successive approximations to find the solution of the IVP

$$x' = 4t + 2tx\ ,\ x(0) = 2 \text{ and verify your result.}$$

<u>Solution</u>: Given initial value problem is $x' = 4t + 2tx\ ,\ x(0) = 2$ $\cdots(1)$

Comparing (1) with $x'(t) = f(t,x),\ x(t_0) = x_0$ we get

$$f(t,x) = 4t + 2tx,\ t_0 = 0 \text{ and } x_0 = 2.$$

By picard's method of successive approximations the zero approximation of the actual solution is

$$\phi_0(t) = x_0(t) = x_0(t_0) = 2.$$

The 1st approximation is

$$\phi_1(t) = x_0 + \int_{t_0}^{t} f\left(s, \phi_0(s)\right) ds$$

$$= 2 + \int_0^t (4s + 2s.2)\, ds$$

$$= 2 + \int_0^t 8s\, ds = 2 + [4s^2]_0^t = 2 + 4t^2 = 4(1 + t^2) - 2$$

The 2nd approximation is

$$\phi_2(t) = x_0 + \int_{t_0}^{t} f\left(s, \phi_1(s)\right) ds$$

$$= 2 + \int_0^t [4s + 2s(2 + 4s^2)] ds$$

$$= 2 + \int_0^t [8s + 8s^3] ds$$

$$= 2 + [4s^2 + 2s^4]_0^t$$

$$= 2 + 4t^2 + 2t^4$$

$$= 4\left\{1 + t^2 + \frac{t^4}{2}\right\} - 2$$

The 3rd approximation is

$$\phi_3(t) = x_0 + \int_{t_0}^{t} f\,(s, \phi_2(s))ds$$

$$= 2 + \int_0^t [4s + 2s(2 + 4s^2 + 2s^4)]\,ds$$

$$= 2 + \int_0^t [4s + 4s + 8s^3 + 4s^5]\,ds$$

$$= 2 + \int_0^t [8s + 8s^3 + 4s^5]\,ds$$

$$= 2 + \left[4s^2 + 2s^4 + \frac{2s^6}{3}\right]_0^t$$

$$= 2 + 4t^2 + 2t^4 + \frac{2t^6}{3}$$

$$= 4\left\{1 + t^2 + \frac{t^4}{2} + \frac{t^6}{6}\right\} - 2$$

$$= 4\left\{1 + t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!}\right\} - 2$$

Proceeding in this way we have

$$\phi_n(t) = x_n(t) = 4\left\{1 + t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots + \frac{(t^2)^n}{n!}\right\} - 2$$

Therefore the required solution is,

$$x(t) = \lim_{n\to\infty} x_n(t)$$

$$= \lim_{n\to\infty}\left[4\left\{1 + t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots + \frac{(t^2)^n}{n!}\right\} - 2\right]$$

$$= 4\left\{1 + t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \cdots\right\} - 2$$

$$= 4e^{t^2} - 2$$

2nd part: Verification: The solution is $x(t) = 4e^{t^2} - 2$ $\cdots(2)$

Differentiating (2) w.r.to ‘t’ we get

$$x'(t) = 8te^{t^2} \qquad \cdots(3)$$

From (2) and (3) we get

$$x'(t) = 2t\{2 + x(t)\}$$

$$\Rightarrow x'(t) = 4t + 2tx \qquad \because x(t) = x$$

Also putting $t = 0$ in (2) we get

$$x(0) = 4e^0 - 2$$
$$\Rightarrow x(0) = 4 - 2$$
$$\Rightarrow x(0) = 2.$$

Thus, the solution satisfies the given differential equation with the given initial condition.

Question-10: Use Picard's method of successive approximations to find the third approximate solution of $\frac{dy}{dx} = 1 + xy^2,\ \ y(0) = 0$.

<u>Solution</u>: Given initial value problem is $\frac{dy}{dx} = 1 + xy^2,\ \ y(0) = 0$ $\cdots (1)$

Comparing (1) with $y'(x) = f(x,y),\ y(x_0) = y_0$ we get

$$f(x,y) = 1 + xy^2,\ x_0 = 0 \text{ and } y_0 = 0.$$

By picard's method of successive approximation the zero approximation of the actual solution is

$$\phi_0(x) = y_0(x) = y_0(x_0) = 0.$$

The 1st approximation is

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f\,(t,\phi_0(t))dt = 0 + \int_0^x (1 + t.0)\,dt = \int_0^x dt = [t]_0^x = x$$

The 2nd approximation is

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f\,\big(t,\phi_1(t)\big)dt$$
$$= 0 + \int_0^x [1 + t.t^2]dt = \int_0^x [1 + t^3]dt = \left[t + \frac{t^4}{4}\right]_0^x = x + \frac{x^4}{4}$$

The 3rd approximation is

$$\phi_3(x) = y_0 + \int_{x_0}^{x} f\,(t,\phi_2(t))dt$$
$$= 0 + \int_0^x \left[1 + t\left(t + \frac{t^4}{4}\right)^2\right]dt$$
$$= \int_0^x \left[1 + t^3 + \frac{t^6}{2} + \frac{t^9}{16}\right]dt$$
$$= \left[t + \frac{t^4}{4} + \frac{t^7}{14} + \frac{t^{10}}{160}\right]_0^x$$

$$= x + \frac{x^4}{4} + \frac{x^7}{14} + \frac{x^{10}}{160}.$$

This is the third approximate solution.