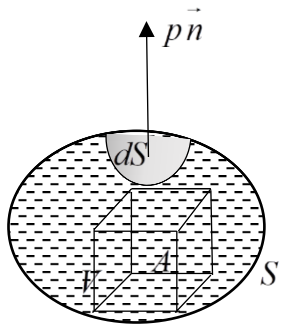
Equation of Motion

Question-01: Derive the Euler's equation of motion.

OR

Established the equation of motion and derive Lamb's hydrodynamical equations.

Answer: Consider a closed surface S in the moving fluid such that it encloses a volume V. Within this surface consider any point A and let ρ be the density of the fluid particle at A and δV be the elementary volume enclosing A.



The mass $\rho\delta V$ of the element at A always remains constant. If $\vec{q}$ be the velocity at A then the momentum M of the volume V is,

$$M = \int_V \rho\vec{q}dV$$

The rate of change of momentum is,

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho\vec{q}dV$$

$$= \int_V \rho\frac{d\vec{q}}{dt}dV + \int_V \vec{q}\frac{d}{dt}(\rho dV)$$

$$= \int_V \rho\frac{d\vec{q}}{dt}dV \quad \cdots (1) \quad [2nd\ integral\ vanishes\ because\ of\ \rho dV = constant]$$

Again let F be the external force per nit mass acting on the fluid. The total force on volume V is,

$$= \int_V \rho F dV \quad \cdots (2)$$

If P be the pressure along the outward drawn unit normal $\vec{n}$ of the element dS then the total surface force is,

$$= -\int_S P\vec{n}dS$$

$$= -\int_V \nabla P dV \quad \cdots (3) \quad [By\ Gauss\ Theorem]$$

Now Newton's second law,

$$\frac{dM}{dt} = \int_V \rho F dV - \int_V \nabla P dV$$

$$or,\ \int_V \rho \frac{d\vec{q}}{dt} dV - \int_V \rho F dV + \int_V \nabla P dV = 0$$

$$or,\ \int_V \left(\rho \frac{d\vec{q}}{dt} - \rho F + \nabla P \right) dV = 0$$

This is true for all volume if

$$or,\ \rho \frac{d\vec{q}}{dt} - \rho F + \nabla P = 0$$

$$or,\ \frac{d\vec{q}}{dt} = F - \frac{1}{\rho} \nabla P \qquad \cdots(4)$$

This is Euler's equation of motion.

Since $\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla\right)\vec{q}$ so equation (4) can be written as,

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla\right)\vec{q} = F - \frac{1}{\rho} \nabla P \qquad \cdots(5)$$

Also we have

$$\nabla\left(\vec{q} \cdot \vec{q}\right) = 2\left[\vec{q} \times \left(\nabla \times \vec{q}\right) + \left(\vec{q} \cdot \nabla\right)\vec{q}\right]$$

$$or,\ \left(\vec{q} \cdot \nabla\right)\vec{q} = \frac{1}{2} \nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right) \qquad \left[\because\ \vec{q} \cdot \vec{q} = q^2\right]$$

The equation (5) reduces to,

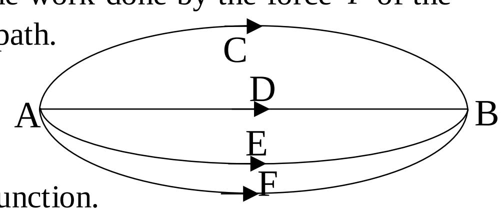
$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2} \nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right) = F - \frac{1}{\rho} \nabla P$$

$$or,\ \frac{\partial \vec{q}}{\partial t} + \frac{1}{2} \nabla q^2 - \vec{q} \times \vec{\xi} = F - \frac{1}{\rho} \nabla P \qquad \left[\because\ \nabla \times \vec{q} = \vec{\xi}\ \text{is a vorticity vector}\right]$$

$$or,\ \frac{\partial \vec{q}}{\partial t} + \frac{1}{2} \nabla q^2 + \vec{\xi} \times \vec{q} = F - \frac{1}{\rho} \nabla P$$

This is called Lamb's hydrodynamical equation.

Conservative Field of Force: In a conservative field, the work-done by the force F of the field in taking a unit mass from A to B is independent of the path.



Thus we have,

$$\int_{ACB} F.dr = \int_{ADB} F.dr = \int_{AEB} F.dr = \int_{AFB} F.dr = -\Omega \quad (say)$$

where Ω is a scalar point function and is known as potential function.

Question-02: Derive pressure equation for irrotational motion of a fluid.

OR

Derive Bernoulli's equation in its most general form.

OR

Derive Bernoulli's equation for irrotational motion of an incompressible fluid.

OR

Discuss the different aspects of motion under conservative body force.

Answer: The Euler's equation of motion is,

$$\frac{d\vec{q}}{dt} = F - \frac{1}{\rho}\nabla P \quad \cdots(1)$$

Since $\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q}$ so equation (1) can be written as,

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q} = F - \frac{1}{\rho}\nabla P \quad \cdots(2)$$

Also we know,

$$\nabla\left(\vec{q}\cdot\vec{q}\right) = 2\left[\vec{q}\times\left(\nabla\times\vec{q}\right) + \left(\vec{q}\cdot\nabla\right)\vec{q}\right]$$

$$or,\ \left(\vec{q}\cdot\nabla\right)\vec{q} = \frac{1}{2}\nabla q^2 - \vec{q}\times\left(\nabla\times\vec{q}\right)$$

The equation (2) reduces to,

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q}\times\left(\nabla\times\vec{q}\right) = F - \frac{1}{\rho}\nabla P \quad \cdots(3)$$

Let us consider the motion is irrotational and the body forces are conservative.

So, $\vec{q} = -\nabla\phi$, $F = -\nabla\Omega$ $\quad and \quad$ $\nabla\times\vec{q} = 0$.

Putting these in (3) we get,

$$\frac{\partial}{\partial t}\left(-\nabla\varphi\right) + \frac{1}{2}\nabla q^2 = -\nabla\Omega - \frac{1}{\rho}\nabla P$$

$$or,\ \frac{\partial}{\partial t}\left(-\nabla\varphi\right) + \frac{1}{2}\nabla q^2 + \nabla\Omega + \frac{1}{\rho}\nabla P = 0 \quad \cdots(4)$$

If the density is a function of pressure only i.e. $\rho = f\left(p\right)$, then we consider following relation,

$$Q = \int \frac{dp}{\rho}$$

Now,
$$\nabla p = \sum i\frac{\partial p}{\partial x}$$
$$= \sum i\frac{\partial p}{\partial Q}\cdot\frac{\partial Q}{\partial x}$$
$$= \sum i\rho\frac{\partial Q}{\partial x}$$

$$or,\ \frac{1}{\rho}\nabla p=\sum i\frac{\partial Q}{\partial x}$$

$$or,\ \frac{1}{\rho}\nabla p=\nabla Q$$

$$\therefore \frac{1}{\rho}\nabla p=\nabla\int\frac{dp}{\rho}$$

Using this value in (4) we get,

$$\frac{\partial}{\partial t}(-\nabla\varphi)+\frac{1}{2}\nabla q^2+\nabla\Omega+\nabla\int\frac{dp}{\rho}=0$$

$$or,\ \nabla\left[-\frac{\partial\varphi}{\partial t}+\frac{1}{2}q^2+\Omega+\int\frac{dp}{\rho}\right]=0$$

which is true if,

$$-\frac{\partial\varphi}{\partial t}+\frac{1}{2}q^2+\Omega+\int\frac{dp}{\rho}=c(t)\qquad\cdots(5)$$

where $c(t)$ denotes an instantaneous constant, i.e. a function of t only and has the same value throughout the fluid.

This equation is called the pressure equation for irrotational motion of a fluid. This is also called Bernoulli's equation in its most general form.

If the density is constant, i.e. $\rho=constant$ then the equation (5) reduces to,

$$-\frac{\partial\varphi}{\partial t}+\frac{1}{2}q^2+\Omega+\frac{p}{\rho}=c(t)\qquad\cdots(6)$$

This is called Bernoulli's equation for the unsteady irrotational motion of an incompressible fluid.

If the motion is steady i.e. $\frac{\partial\varphi}{\partial t}=0$, then equation (6) becomes,

$$\frac{1}{2}q^2+\Omega+\frac{p}{\rho}=c(t)$$

This is called Bernoulli's equation for the steady irrotational motion of an incompressible fluid.

Question-03: State and prove Bernoulli's theorem for a compressible fluid.

Statement: This theorem states that, if the motion of a compressible fluid is steady and the velocity potential does not exists, then

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2+\Omega=c$$

where Ω is the potential function from which the external forces are derivable.

Proof: The Euler's equation of motion is,

$$\frac{d\vec{q}}{dt}=F-\frac{1}{\rho}\nabla P\qquad\cdots(1)$$

Since $\frac{d\vec{q}}{dt}=\frac{\partial\vec{q}}{\partial t}+\left(\vec{q}\cdot\nabla\right)\vec{q}$ so equation (1) can be written as,

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla\right)\vec{q} = F - \frac{1}{\rho}\nabla P \qquad \cdots(2)$$

Also we know,

$$\nabla\left(\vec{q} \cdot \vec{q}\right) = 2\left[\vec{q} \times \left(\nabla \times \vec{q}\right) + \left(\vec{q} \cdot \nabla\right)\vec{q}\right]$$

$$or,\ \left(\vec{q} \cdot \nabla\right)\vec{q} = \frac{1}{2}\nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right)$$

The equation (2) reduces to,

$$\frac{\partial \vec{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right) = F - \frac{1}{\rho}\nabla P \qquad \cdots (3)$$

For steady motion $\frac{\partial \vec{q}}{\partial t} = 0$ so the equation (3) reduces to,

$$\frac{1}{2}\nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right) = F - \frac{1}{\rho}\nabla P \qquad \cdots (4)$$

If F is derivable from some potential function say, Ω then we have,

$$F = -\nabla\Omega$$

Putting these in (4) we get,

$$\frac{1}{2}\nabla q^2 - \vec{q} \times \left(\nabla \times \vec{q}\right) = -\nabla\Omega - \frac{1}{\rho}\nabla P$$

$$or,\ \frac{1}{2}\nabla q^2 + \nabla\Omega + \frac{1}{\rho}\nabla P = \vec{q} \times \left(\nabla \times \vec{q}\right)$$

$$or,\ \frac{1}{2}\nabla q^2 + \nabla\Omega + \nabla\int \frac{dp}{\rho} = \vec{q} \times \left(\nabla \times \vec{q}\right)$$

$$or,\ \nabla\left[\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right] = \vec{q} \times \vec{\xi} \qquad \left[\because \nabla \times \vec{q} = \vec{\xi} \text{ is vorticity vector}\right]$$

$$\therefore\ \vec{q} \cdot \nabla\left[\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right] = \vec{q} \cdot \vec{q} \times \vec{\xi} = 0, \ \left[\because \alpha \cdot \beta \times \gamma = 0 \ if \ any\, of\ the\, two\, vectors\, are\, equal\right]$$

whence

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega = c. \qquad \textbf{(Proved)}$$

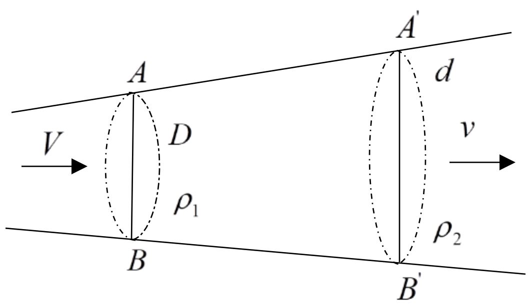
Problem

Problem-01: A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion be supposed to be that of the divergence from the vertex of the come, prove that

$$\frac{v}{V}=\frac{D^2}{d^2}e^{\left(v^2-V^2\right)/2K}$$

where k is the pressure divided by the density and supposed constant.

Answer: Let AB and $A'B'$ be the ends of the conical pipe such that $AB=D$ and $A'B'=d$. Also let ρ_1 and ρ_2 be the densities of the stream at the ends AB and $A'B'$ respectively.



Hence the equation of continuity is

$$\pi\left(\frac{D}{2}\right)^2 V\rho_1=\pi\left(\frac{d}{2}\right)^2 v\rho_2$$

$$or,\ \frac{v}{V}=\frac{D^2}{d^2}\cdot\frac{\rho_1}{\rho_2} \qquad \cdots(1)$$

By Bernoulli's theorem (in absence of external forces like gravity), we have

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2=c \qquad \cdots(2)$$

But $\quad \frac{p}{\rho}=k\Rightarrow dp=kd\rho$.

The equation (2) becomes,

$$k\int\frac{d\rho}{\rho}+\frac{1}{2}q^2=c$$

Integrating, $\quad k\ln\rho+\frac{1}{2}q^2=c \qquad \cdots(4)$

when $q=V$, $\rho=\rho_1$ then equation (4) reduces to,

$$k\ln\rho_1+\frac{1}{2}V^2=c \qquad \cdots(5)$$

when $q=v$, $\rho=\rho_2$ then equation (4) reduces to,

$$k\ln\rho_2+\frac{1}{2}v^2=c \qquad \cdots(6)$$

From(6) and (5), we get

$$k\ln\rho_1+\frac{1}{2}V^2=k\ln\rho_2+\frac{1}{2}v^2$$

$$or,\ k\ln\rho_1-k\ln\rho_2=\frac{1}{2}v^2-\frac{1}{2}V^2$$

$$or,\ k\left(\ln\rho_1-\ln\rho_2\right)=\frac{1}{2}\left(v^2-V^2\right)$$

$$or,\ \ln\frac{\rho_1}{\rho_2}=\frac{1}{2k}\left(v^2-V^2\right)$$

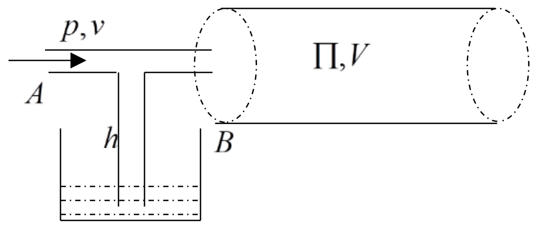
$$or,\ \frac{\rho_1}{\rho_2}=e^{\frac{1}{2k}\left(v^2-V^2\right)}$$

From equation (1), we get

$$\frac{v}{V}=\frac{D^2}{d^2}e^{\frac{1}{2k}\left(v^2-V^2\right)} \qquad \textbf{(Proved)}$$

Problem-02: A stream in a horizontal pipe, after passing a contraction in the pipe at which its area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth $\frac{S^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)$ below the pipe; S being the delivery per second.

Answer: Let v and p be the velocity and pressure at A. Also let V and $\prod$ be the velocity and pressure at B.



Hence the equation of continuity is

$$Av = BV = S$$

$$\therefore\ v = \frac{S}{A},\ \ V = \frac{S}{B}$$

By Bernoulli's theorem (in absence of external forces like gravity) for incompressible fluid, namely

$$\frac{p}{\rho} + \frac{1}{2}q^2 = c \qquad \cdots (1)$$

we obtain

$$\frac{p}{\rho} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} + \frac{1}{2}V^2$$

$$or,\ \frac{\Pi}{\rho} - \frac{p}{\rho} = \frac{1}{2}v^2 - \frac{1}{2}V^2$$

$$or,\ \frac{1}{\rho}(\Pi - P) = \frac{1}{2}\left(v^2 - V^2\right)$$

$$or,\ \frac{1}{\rho}(\Pi - P) = \frac{1}{2}\left(\frac{S}{A^2}^2 - \frac{S}{B^2}^2\right) \qquad \cdots (2)$$

Let h be the height through water is sucked up. If α be the cross section of the tube then

$$\alpha h \rho g = \alpha \Pi - \alpha p$$

$$or,\ \rho g h = difference\, of\ \ pressure = \Pi - p \qquad \cdots (3)$$

From (2) and (3), we have

$$\frac{1}{\rho} \times \rho g h = \frac{1}{2}\left(\frac{S}{A^2}^2 - \frac{S}{B^2}^2\right)$$

$$\therefore\ h = \frac{S^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right) \qquad \textbf{(Showed)}$$

<u>Problem-03</u>: A quantity of liquid occupies a length $2l$ of a straight tube of uniform bore under the action of force which is equal to μx to a point O in the tube, where x is the distance from O. Find the motion and show that if z be the distance of the nearer free surface from O, pressure at any point is given by

$$\frac{p}{\rho} = -\frac{\mu}{2}\left(x^2 - z^2\right) + \mu\left(x - z\right)\left(z + l\right)$$

Answer: Page-159, book Fluid dynamics by Raisinghania.