Motion in two Dimensions

Motion in two Dimensions: When the lines of motion are all parallel to fixed plane (say $z=0$) and velocity at the corresponding points of all planes to $z=0$ plane has the same magnitude and direction, the motion is said to be two dimensional. The fact is shown graphically in Fig-1.

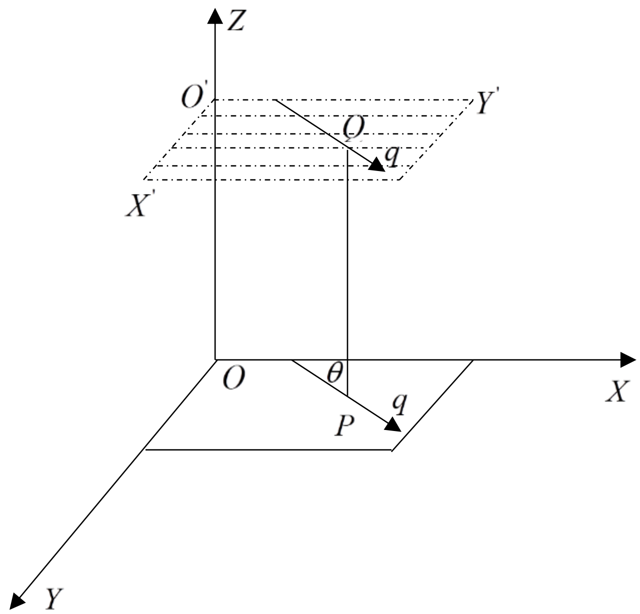


Fig.-1: Two dimensional motion.

Question-01: Define stream function or current function.

OR

Show that the stream function is constant along the stream lines.

Answer: **Stream function or Current function:** In the case of two dimensional motion, the velocity vector $\vec{q}$ is a function of x, y, t but not of z. Hence, the differential equation of the streamlines is given by

$$\frac{dx}{u}=\frac{dy}{v}$$

$$or,\quad vdx-udy=0. \qquad \cdots(1)$$

The equation of continuity for incompressible fluid is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad \cdots(2)$$

But equation (2) expresses the condition that the differential equation (1) should be exact.
Thus we can say that $vdx-udy$ is a complete differential say $d\psi$, so that

$$vdx-udy=d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy. \qquad \cdots(3)$$

The equation (3) gives

$$u=-\frac{\partial \psi}{\partial y} \text{ and } v=\frac{\partial \psi}{\partial x}. \qquad \cdots(4)$$

The function ψ is called the stream function or the current function.

From equations (1) and (3), it is obvious that stream lines are obtained by integrating the equation

$$d\psi = 0 \qquad \cdots(5)$$

which on integrating gives,

$$\psi = constant.$$

This shows that the stream function is constant along stream line. It is therefore to be noted that the current function always exists in all types of two dimensional motion whether rotational or irrotational.

Relation between stream function and velocity potential: In the case of two dimensional irrotational motion of an incompressible inviscid fluid, the velocity potential ϕ always exists such that

$$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y} \qquad \cdots(1)$$

In the case of two dimensional rotational and irrotational motion, the stream function ψ always exists such that

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x} \qquad \cdots(2)$$

From (1) and (2), we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

This are the required relations between stream function and velocity potential.

Question-02: Discuss the physical meaning of $\psi(x_2, y_2) - \psi(x_1, y_1)$.

OR

Prove that the difference between the stream function at two points will express the flow crossing the line joining those points.

Answer: Consider a curve LM lying in the xy-plane and assume that ψ_1 and ψ_2 be the stream functions at L and M respectively. Let ds be the element of the curve at P and θ the angle which the tangent at P makes with OX. Now, if u and v are the velocities parallel to axes of x and y, then we have velocities along inward normal

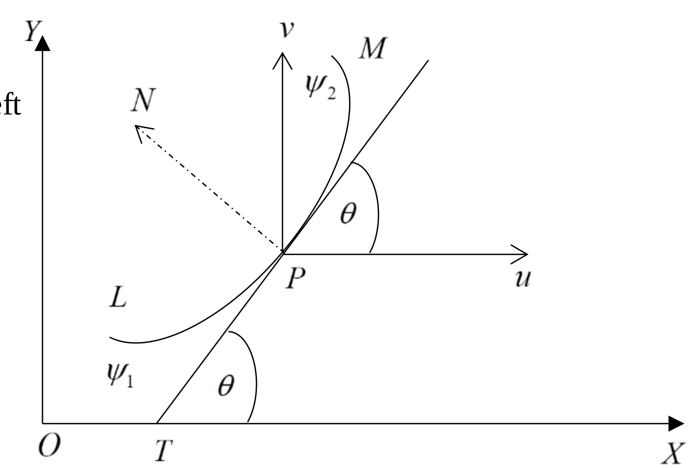
$$PN = v\cos\theta - u\sin\theta.$$

The flow across the curve LM from right to left

$$= \int_L^M (v\cos\theta - u\sin\theta)ds$$

$$= \int_L^M \left(\frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds}\right)ds$$

$$= \int_L^M \left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy\right)$$



$$= \int_L^M d\psi$$

$$= \psi_M - \psi_L$$

$$= \psi_2 - \psi_1 \qquad \textbf{(Proved)}.$$

Irrotational motion in two dimensions: In case of two dimensional irrotational motion, the velocity potential ϕ always exists such that

$$u = -\frac{\partial \phi}{\partial x} \text{ and } v = -\frac{\partial \phi}{\partial y} \qquad \cdots(1)$$

Hence the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \cdots(2)$$

From (1) and (2), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \cdots(3)$$

This shows that ϕ satisfies Laplace's equation.

Further from the relation of stream function and velocity potential, we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\therefore \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial y^2} \text{ and } \frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \psi}{\partial x^2}$$

But in general, $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ which gives,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which shows that ψ satisfies Laplace's equation.

Also $\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0$.

This shows that the families of curves given by $\phi = constant$ and $\psi = constant$ cut orthogonally.

NOTE: In the case of two dimensional motion of an incompressible inviscid fluid,

a. The stream function ψ exists whether the motion is rotational or irrotational.
b. The velocity potential ϕ can exists only where the motion is irrotational.
c. In case of irrotational motion, ϕ and ψ satisfy Laplace's equation and Cauchy Riemann equations.

Question-03: Define complex velocity potential.

Answer: Complex velocity potential: In the case of two dimensional irrotational motion of an incompressible inviscid fluid, the following relations always hold

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \cdots(1)$$

where ϕ and ψ are velocity potential and stream function respectively.

Also if $w = f(z) = \phi(x,y) + i\psi(x,y)$, where $f(z)$ be analytic in a region R, then the relations given by equation (1) also hold, where ϕ and ψ are called conjugate functions.

Here the function $w = f(z) = \phi(x,y) + i\psi(x,y)$, in the case of two dimensional irrotational motion of an incompressible inviscid fluid, is called complex velocity potential.

Complex Velocity: If $w = f(z) = \phi(x,y) + i\psi(x,y)$ and $z = x + iy$, then

$$\frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \qquad \cdots(1) \qquad \left[\because \frac{\partial z}{\partial x} = 1\right]$$

And

$$\frac{dw}{dz}\frac{\partial z}{\partial y} = \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}$$

$$\frac{dw}{dz} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} \qquad \cdots(2) \qquad \left[\because \frac{\partial z}{\partial y} = i\right]$$

From (1) and (2), we get

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \qquad \cdots(3)$$

$$\frac{dw}{dz} = -u + iv$$

$$\because \; -\frac{dw}{dz} = u - iv$$

which is called complex velocity.

The equation (3) gives,

$$\left|\frac{dw}{dz}\right| = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} = \sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2} = \sqrt{u^2 + v^2} = q$$

which gives the speed of the fluid particle at any point in the field of flow.

Stagnation Point: The points where the complex velocity is zero i.e. $\frac{dw}{dz}=0$ are called stagnation points.

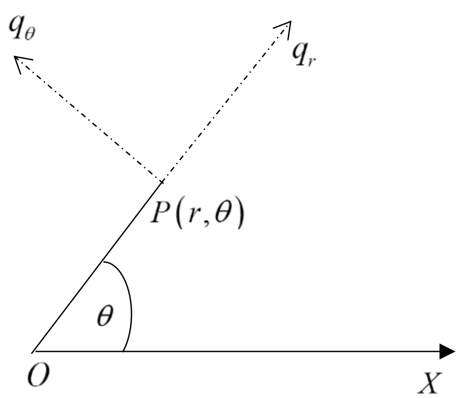
Thus at stagnation points, $\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2=u^2+v^2=0$.

Question-04: Determine the velocity in polar coordinates for two dimensional and irrotational motion.

Answer: Velocity in polar coordinates: Let the Cartesian coordinates of a point P in an incompressible fluid be (x,y), where the motion is assumed to be two dimensional, irrotational. If (r,θ) be the polar coordinates of the point P, then we have,

$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta.$$

$$\therefore \quad \frac{\partial x}{\partial r}=\cos\theta\ ,\ \frac{\partial y}{\partial r}=\sin\theta\ ,\ \frac{\partial x}{\partial \theta}=-\sin\theta\ ,\ \frac{\partial y}{\partial \theta}=\cos\theta \qquad \cdots(1)$$



Also we know that,

$$u=-\frac{\partial\phi}{\partial x}=-\frac{\partial\psi}{\partial y} \quad \text{and} \quad v=-\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x} \qquad \cdots(2)$$

Now $$\frac{\partial\phi}{\partial r}=\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r}$$

$$=\frac{\partial\phi}{\partial x}\cos\theta+\frac{\partial\phi}{\partial y}\sin\theta \quad \text{[by (1)]}$$

$$=\frac{1}{r}\left(\frac{\partial\phi}{\partial x}r\cos\theta+\frac{\partial\phi}{\partial y}r\sin\theta\right)$$

$$=\frac{1}{r}\left(\frac{\partial\phi}{\partial x}\frac{\partial y}{\partial\theta}-\frac{\partial\phi}{\partial y}\frac{\partial x}{\partial\theta}\right)$$

$$=\frac{1}{r}\left(\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\theta}+\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\theta}\right)$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

and $$\frac{\partial \phi}{\partial \theta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial \phi}{\partial x}(-r\sin\theta) + \frac{\partial \phi}{\partial y} r\cos\theta$$

$$= r\left[-\frac{\partial \phi}{\partial x}\frac{\partial y}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial x}{\partial r}\right]$$

$$= r\left[-\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial r} - \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial r}\right]$$

$$= -r\left[\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial r}\right]$$

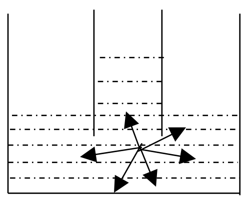
$$= -r\frac{\partial \psi}{\partial r}$$

$$\therefore \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Thus we have, $q_r = -\dfrac{\partial \phi}{\partial r} = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$ and $q_\theta = -\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = \dfrac{\partial \psi}{\partial r}$.

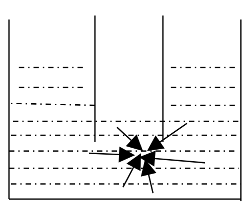
Source: If the motion of a fluid consists of symmetrical outward radial flow in all directions proceeding from a point, the point is known as a simple source. A source is thus a point at which fluid is continously created and distributed.

Example: Airbubble.



Sink: If the motion of a fluid consists of symmetrical inward radial flow to a point from all directions, the point is known as a simple sink. A sink is thus a point at which fluid is continously absorbed and annihilated. Thus a sink is a negative source.

Example: Whirlpool.



Strength of source and sink: If the total flow across a small curve surrounding the source is $2\pi m$, then m is called the strength of the source. Since sink is opposite of source so $-m$ is the strength of the sink.

Doublet: A combination of a source of strength m and a sink of strength $-m$ at a small distance δs apart such that $m\delta s$ is finite is called a doublet.
If $m\delta s = \mu$, a finite quantity, where m is taken infinitely great and δs infinitely small then μ is called the strength of the doublet; the axis of the doublet being in the sense from $-m$ to m.

Question-05: Find complex potential of a source.

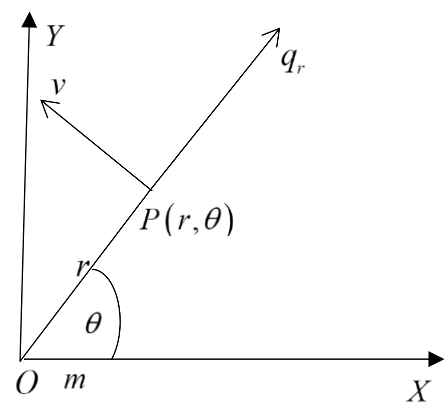
OR

For a source prove that the complex potential is $w = -m_1 \ln(z - z_1)$.

Answer: Consider a source of strength m at the origin O. So the total flow across a small curve surrounding the source is $2\pi m$. Let q_r be the radial velocity at distance r from the source. Then the flow across a circle of radius r is $2\pi r q_r$.
By the conservation of mass, we can write

$$2\pi r q_r = 2\pi m$$

$$or,\ q_r = \frac{m}{r} \qquad \cdots(1)$$



Also we have $$q_r = -\frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \cdots(2)$$

From (1) and (2), we get

$$-\frac{\partial \phi}{\partial r} = \frac{m}{r}$$

$$or,\ \frac{\partial \phi}{\partial r} = -\frac{m}{r}$$

Integrating, $$\phi = -m \ln r$$

Again, $$-\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{r}$$

$$or,\ \frac{\partial \psi}{\partial \theta} = -m$$

Integrating, $$\psi = m\theta$$

Then the complex velocity potential is given by

$$\begin{aligned} w &= \phi + i\psi \\ &= -m \ln r - im\theta \\ &= -m(\ln r + i\theta) \\ &= -m(\ln r + \ln e^{i\theta}) \\ &= -m \ln(re^{i\theta}) \\ &= -m \ln z \end{aligned}$$

This is the complex velocity potential due to a source at the origin.
If the source of strength m_1 at $z = z_1$, then the complex potential of this source is

$$w = -m_1 \ln(z - z_1)$$

Similarly, for source of strengths m_1, m_2, $\cdots$ at points $z = z_1, z_2, \cdots$ the complex velocity potential is

$$w = -m_1 \ln(z - z_1) - m_2 \ln(z - z_2) - \cdots \ \cdots$$

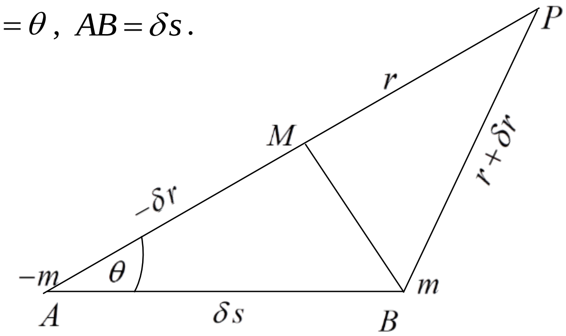
$$= -\sum_{n=1}^{\infty} m_n \ln(z - z_n).$$

Question-06: Derive the complex potential for a doublet.

OR

For a doublet prove that the complex potential is $w = \dfrac{\mu}{z}$.

Answer: Let A and B denote the positions of the sink and source. Also let P be a point such that $AP = r$, $BP = r + \delta r$, $\langle PAB = \theta$, $AB = \delta s$.



Draw a perpendicular line BM from B on AP such that $MP = BP$.

Then $AM = AP - MP = AP - BP = r - (r + \delta r) = -\delta r$.

$$\therefore \quad \cos\theta = -\frac{\delta r}{\delta s}$$

Now the velocity potential at P due to this combination is

$$\phi = m \ln r - m \ln(r + \delta r)$$

$$= -m \ln\left(\frac{r + \delta r}{r}\right)$$

$$= -m \ln\left(1 + \frac{\delta r}{r}\right)$$

$$= -m\frac{\delta r}{r} \qquad \text{[Expanding } \ln\left(1 + \frac{\delta r}{r}\right) \text{ and neglecting higher order terms]}$$

$$= -\frac{m}{r}\left(\frac{\delta r}{\delta s} \cdot \delta s\right)$$

$$= -\frac{m}{r}(-\,\delta s \cos\theta)$$

$$= \frac{m \delta s \cos\theta}{r}$$

$$\therefore \quad \phi = \frac{\mu\cos\theta}{r} \qquad \cdots(1)$$

where $m\delta s = \mu$ is the strength of the doublet.

We know, $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$or,\ -\frac{\mu\cos\theta}{r^2} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$or,\ \frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r}$$

$$\therefore \quad \psi = -\frac{\mu\sin\theta}{r} + F(r) \qquad \ldots(2)$$

Again, $$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

$$or,\ \frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$or,\ \frac{\partial\psi}{\partial r} = \frac{\mu\sin\theta}{r^2}$$

$$\therefore \quad \psi = -\frac{\mu\sin\theta}{r} + f\left(\theta\right) \qquad \cdots(3)$$

Equations (2) and (3) shows that $F(r) = f(\theta)$, which is possible only when $F(r)$ is either zero or constant.

Thus $$\psi = -\frac{\mu\sin\theta}{r}$$

Now the complex potential is

$$\begin{aligned} w &= \phi + i\psi \\ &= \frac{\mu\cos\theta}{r} - i\frac{\mu\sin\theta}{r} \\ &= \frac{\mu}{r}\left(\cos\theta - i\sin\theta\right) \\ &= \frac{\mu e^{-i\theta}}{r} \\ &= \frac{\mu}{re^{i\theta}} \end{aligned}$$

$$\therefore \quad w = \frac{\mu}{z}$$

This is the complex velocity potential due to a doublet.

Further if the doublet makes an angle α with $x-$axis, then we have to write $\theta - \alpha$ for θ and the complex potential will be

$$w = \frac{\mu}{re^{i(\theta-\alpha)}}$$

$$= \frac{\mu e^{i\alpha}}{re^{i\theta}}$$

$$\therefore \quad w = \frac{\mu e^{i\alpha}}{z}$$

If the doublet is at the point $z = z_1$ then

$$w = \frac{\mu e^{i\alpha}}{z - z_1}$$

Similarly at points $z = z_1, z_2, \cdots \cdots,$ we have

$$w = \frac{\mu e^{i\alpha}}{z - z_1} + \frac{\mu e^{i\alpha}}{z - z_2} + \cdots \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\mu e^{i\alpha}}{z - z_1}.$$

Problem

Problem-01: A velocity field is given by $\vec{q} = -xi + (y+t)\,j$. Find the stream function and the streamline for this field at $t = 2$.

Solution: Given that, $\vec{q} = -xi + (y+t)\,j$.

Here $$u = -\frac{\partial \psi}{\partial y} = -x \qquad \cdots(1)$$

$$v = \frac{\partial \psi}{\partial x} = y + t \qquad \cdots(2)$$

From (1), we get

$$-\frac{\partial \psi}{\partial y} = -x$$

$$or,\ \frac{\partial \psi}{\partial y} = x$$

Integrating, $\psi = xy + f(x,t) \qquad \cdots(3)$

$$\therefore\ \frac{\partial \psi}{\partial x} = y + \frac{\partial f}{\partial x} \qquad \cdots(4)$$

Again, from (2), we get

$$\frac{\partial \psi}{\partial x} = y + t$$

$$or,\ y + \frac{\partial f}{\partial x} = y + t$$

$$or,\ \frac{\partial f}{\partial x} = t$$

Integrating, $f(x,t) = tx + g(t) \qquad \cdots(5)$

So equation (3) becomes,

$$\psi = xy + tx + g(t)$$

This is the required stream function.

2nd part: The streamline for $t=2$ is given by

$$x(y+2)+g(2)=\psi=\text{constant}$$

$$\therefore\ x(y+2)=\text{constant}$$

which are rectangular hyperbola.

Problem-02: Find the stream function $\psi(x,y,t)$ for the given velocity field $\vec{q}=Uti+xj$.

Solution: Given that, $\vec{q}=-xi+(y+t)\,j$.

Here $$u=-\frac{\partial\psi}{\partial y}=Ut \qquad \cdots(1)$$

$$v=\frac{\partial\psi}{\partial x}=x \qquad \cdots(2)$$

From (1), we get

$$-\frac{\partial\psi}{\partial y}=Ut$$

$$or,\ \frac{\partial\psi}{\partial y}=-Ut$$

Integrating, $\psi=Uty+f(x,t) \qquad \cdots(3)$

$$\therefore\ \frac{\partial\psi}{\partial x}=\frac{\partial f}{\partial x} \qquad \cdots(4)$$

Again, from (2), we get

$$\frac{\partial\psi}{\partial x}=x$$

$$or,\ \frac{\partial f}{\partial x}=x$$

Integrating, $f(x,t)=\dfrac{x^2}{2}+g(t) \qquad \cdots(5)$

So equation (3) becomes,

$$\psi=Uty+\frac{x^2}{2}+g(t)$$

This is the required stream function.

Problem-03: A two dimensional flow field is given by $\psi=xy$. Then

(a) Show that the flow is irrotational,
(b) Find the velocity potential,
(c) Verify that ψ and ϕ satisfy Laplace equation
(d) Find the streamlines and potential lines.

Solution: Given that, $\psi=xy$

(a) The velocity components are,

$$u=-\frac{\partial\psi}{\partial y}=-x \quad \text{and} \quad v=\frac{\partial\psi}{\partial x}=y$$

The velocity field is

$$\vec{q}=-xi+yj$$

Now $\nabla\times\vec{q}=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix}=0$.

Hence the motion is irrotational. (**Showed**)

(b) We know that, $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\psi}{\partial y} \quad \cdots(1)$

and $$\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \quad \cdots(2)$$

From (1), we get

$$\frac{\partial\phi}{\partial x}=x$$

Integrating, $\phi=\dfrac{x^2}{2}+f(y) \quad \cdots(3)$

Differentiating (3) w.r.to y, we get

$$\frac{\partial\phi}{\partial y}=f'(y)$$

$$or,\ f'(y)=-\frac{\partial\psi}{\partial x}$$

$$or,\ f'(y)=-y$$

Integrating, $f(y)=-\dfrac{y^2}{2}+c \quad \cdots(4)$

So equation (3) becomes,

$$\phi=\frac{x^2}{2}-\frac{y^2}{2}+c$$

This is the required velocity potential.

(c) $$\nabla^2\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0+0=0$$

and $\nabla^2\phi=\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=1-1=0$.

Hence ψ and ϕ satisfy Laplace equation.

(d) The streamlines, $\psi=constant$ and the potential lines, $\phi=constant$, are given by

$$xy=c_1 \quad \text{and} \quad x^2-y^2=c_2$$

where c_1 and c_2 are constants.

Problem-04: Show that the velocity vector $\vec{q}$ is everywhere tangent to lines in the $xy-$plane along which $\psi(x,y)=constant$.

Solution: The stream function ψ in $xy-$plane can be expressed as

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy \qquad \cdots(1)$$

For streamlines, $\psi=constant$

$$\therefore\ d\psi=0$$

$$or,\ \frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=0 \qquad \cdots(2)$$

But from the definition of stream function, we get

$$u=-\frac{\partial\psi}{\partial y} \quad \text{and} \quad v=\frac{\partial\psi}{\partial x} \qquad \cdots(3)$$

Using (2) and (3), we get

$$vdx-udy=0$$

$$or,\ \frac{dx}{u}=\frac{dy}{v}$$

which shows that the velocity vector $\vec{q}$ is tangent to the lines $\psi(x,y)=constant$.

Problem-05: Prove that when the speed is everywhere the same, the streamlines are straight.

Solution: The equation of the streamlines are given by,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$or,\ vdx-udy=0,\ wdy-vdz=0,\ udz-wdx=0$$

Integrating, $vx-uy=constant,\ wy-vz=constant,\ uz-wx=constant$

This implies that the intersection of these planes are straight lines. (**Proved**)

Problem-06: Find the streamlines, lines of equipotential, velocity of the complex potential

$$w=Az^2.$$

Solution: The complex potential is given by

$$w=Az^2$$

$$or,\ \phi+i\psi=A(x+iy)^2$$

$$or,\ \phi+i\psi=A\left(x^2-y^2+2ixy\right)$$

Equating the real and imaginary parts, we get

$$\phi=A\left(x^2-y^2\right)$$

and $$\psi=2Axy$$

which are velocity potential and stream function respectively.

The streamlines are given by

$$\psi = constant$$
$$or,\ 2Axy = constant$$
$$\therefore\ xy = constant$$

Now $\psi = 0$ when $x = 0$ or $y = 0$.

So $x = 0$ and $y = 0$ are streamlines and these give the motion of liquid in the angle between two perpendicular walls.

The lines of equipotential are given by

$$\phi = constant$$
$$or,\ A\left(x^2 - y^2\right) = constant$$
$$\therefore\ x^2 - y^2 = constant$$

The velocity at any point is

$$q = \left|\frac{dw}{dz}\right| = \left|2Az\right| = \left|2Are^{i\theta}\right| = 2Ar\ .$$

Problem-07: Find the streamlines, lines of equipotential, velocity of the complex potential

$$w = Az^n\ .$$

Solution: The complex potential is given by

$$\psi = Az^n$$
$$or,\ \phi + i\psi = A\left(x + iy\right)^n \qquad \cdots(1)$$

Put, $x = r\cos\theta$, $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

The equation (1) becomes,

$$\phi + i\psi = A\left(r\cos\theta + ir\sin\theta\right)^n$$
$$or,\ \phi + i\psi = Ar^n\left(\cos\theta + i\sin\theta\right)^n$$
$$or,\ \phi + i\psi = Ar^n\left(\cos n\theta + i\sin n\theta\right)$$

Equating the real and imaginary parts, we get

$$\phi = Ar^n \cos n\theta$$

and
$$\psi = Ar^n \sin n\theta$$

which are velocity potential and stream function respectively.

The streamlines are given by

$$\psi = constant$$
$$or,\ Ar^n \sin n\theta = constant$$
$$\therefore\ r^n \sin n\theta = constant$$

when $\psi = 0$,then

$$r^n \sin n\theta = 0$$
$$or,\ \sin n\theta = 0$$

$$\therefore\ \theta = 0 \quad or, \quad \theta = \frac{\pi}{n}$$

which give the motion of liquid in the region given by $0 \le \theta \le \frac{\pi}{n}$.

The lines of equipotential are given by

$$\phi = cons\tan t$$

$$or,\ Ar^n \cos n\theta = cons\tan t$$

$$\therefore\ r^n \cos n\theta = cons\tan t$$

The velocity at any point is

$$q = \left|\frac{dw}{dz}\right| = \left|nAz^{n-1}\right| = \left|nAr^{(n-1)}e^{i(n-1)\theta}\right| = nAr^{(n-1)}.$$

Problem-08: Determine the condition for which $u = ax + by$, $v = cx + dy$ will be the velocity components of an incompressible fluid. Show that for this motion the streamlines will be conic section in general and rectangular hyperbolas when the motion is irrotational.

Solution: Given that, $u = ax + by$ and $v = cx + dy$

The necessary condition for possible fluid motion is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$or,\ a + d = 0$$

This is the required condition for the given functions to be velocity components of an incompressible fluid.

2nd part: We know, $u = -\frac{\partial \psi}{\partial y} = ax + by \qquad \cdots(1)$

and $\qquad v = \frac{\partial \psi}{\partial x} = cx + dy \qquad \cdots(2)$

From (2), we get

$$\frac{\partial \psi}{\partial x} = cx + dy$$

Integrating, $\psi = \frac{cx^2}{2} + dxy + f(y) \qquad \cdots(3)$

Differentiating (3) w. r. to y, we get

$$\frac{\partial \psi}{\partial y} = dx + f'(y) \qquad \cdots(4)$$

From (1) and (4), we get

$$dx + f'(y) = -ax - by$$

$$f'(y) = -ax - by - dx$$

Integrating, $\quad f(y) = -axy - \frac{by^2}{2} - dxy + c$

So equation (3) becomes,

$$\psi = \frac{cx^2}{2} + dxy - axy - \frac{by^2}{2} - dxy + c$$

$$\therefore \quad \psi = \frac{cx^2}{2} - axy - \frac{by^2}{2} + c$$

For streamlines, $\psi = constant$

$$or, \ \frac{cx^2}{2} - axy - \frac{by^2}{2} + c = constant$$

$$\therefore \quad cx^2 - 2axy - by^2 + k = 0.$$

This represents a conic (**Showed**).

3rd part: When the motion is irrotational, then

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

$$or, \ b - c = 0$$

$$\therefore \quad b = c$$

Hence the streamline equation in the irrotational flow is

$$bx^2 - 2axy - by^2 + k = 0$$

$$\therefore \quad x^2 - lxy - y^2 + k_1 = 0 \qquad \text{where } \ l = \frac{2a}{b} \quad and \ k_1 = \frac{k}{b}$$

which is clearly the equation of a rectangular hyperbola (**Showed**).

Problem-09: What arrangement of sources and sinks will give rise to the function $w = \ln\left(z - \frac{a^2}{z}\right)$?

Draw a rough sketch of the streamlines in this case and prove that two of them subdivide into the circle $r = a$ and the axis of y.

Solution: Given that, $w = \ln\left(z - \frac{a^2}{z}\right)$

$$= \ln\left(\frac{z^2 - a^2}{z}\right)$$

$$= \ln\left(z^2 - a^2\right) - \ln z$$

$$= \ln(z+a)(z-a) - \ln z$$

$$= \ln(z+a) + \ln(z-a) - \ln z$$

which shows that there are two sinks of unit strength at $z = a$, $z = -a$ and a source of unit strength at origin $z = 0$.

The above equation can be written as

$$w = \ln\left\{(x+a) + iy\right\} + \ln\left\{(x-a) + iy\right\} - \ln(x + iy)$$

$$= \frac{1}{2}\ln\left\{(x+a)^2 + y^2\right\} + i\tan^{-1}\left(\frac{y}{x+a}\right) + \frac{1}{2}\ln\left\{(x-a)^2 + y^2\right\} + i\tan^{-1}\left(\frac{y}{x-a}\right) - \frac{1}{2}\ln\left(x^2 + y^2\right) - i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\left[\because\ \ln(a+ib)=\frac{1}{2}\ln\left(a^2+b^2\right)+i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Separating the real and imaginary part, we get the velocity potential

$$\phi=\frac{1}{2}\ln\left\{(x+a)^2+y^2\right\}+\frac{1}{2}\ln\left\{(x-a)^2+y^2\right\}-\frac{1}{2}\ln\left(x^2+y^2\right)$$

and the stream function

$$\psi=\tan^{-1}\left(\frac{y}{x+a}\right)+\tan^{-1}\left(\frac{y}{x-a}\right)-\tan^{-1}\left(\frac{y}{x}\right)$$

$$=\tan^{-1}\frac{\frac{y}{x+a}+\frac{y}{x-a}}{1-\frac{y}{x+a}\cdot\frac{y}{x-a}}-\tan^{-1}\left(\frac{y}{x}\right)$$

$$=\tan^{-1}\frac{2xy}{x^2-y^2-a^2}-\tan^{-1}\left(\frac{y}{x}\right)$$

$$=\tan^{-1}\frac{\frac{2xy}{x^2-y^2-a^2}-\frac{y}{x}}{1+\frac{2xy}{x^2-y^2-a^2}\cdot\frac{y}{x}}$$

$$=\tan^{-1}\frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}$$

The streamlines are given by

$$\psi=constant$$

$$or,\ \tan^{-1}\frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}=constant$$

$$or,\ \frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}=constant$$

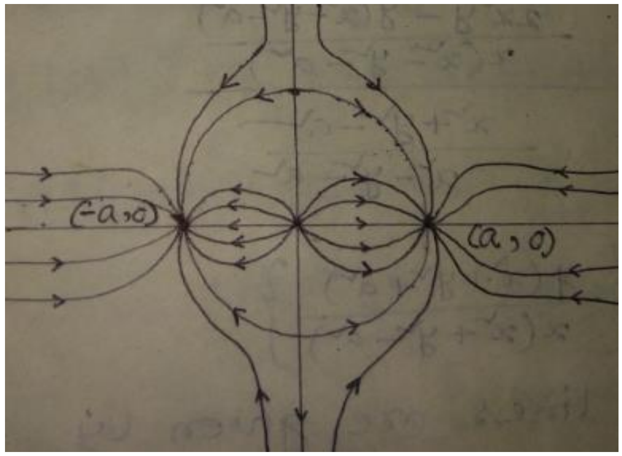
when the constant tends to infinity, the corresponding streamlines are given by

$$x\left(x^2+y^2-a^2\right)=0$$

$$\therefore\ x=0\quad or\quad x^2+y^2=a^2$$

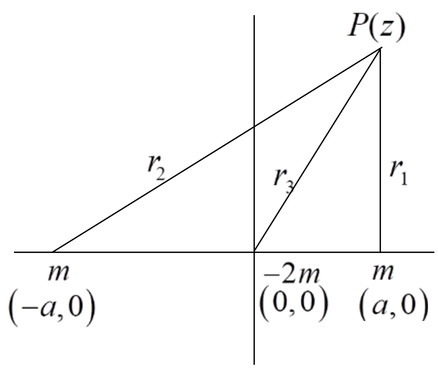
Thus the streamlines subdivide into a circle $x^2+y^2=a^2$ i.e. $r=a$ and the axis of y namely $x=0$. Also if we take constant to be zero, we get $y=0$ i.e. $x-$ axis is a streamline.

Hence the axes and the circle are the streamlines. The other streamlines are sketched in the figure.



Problem-10: Two sources of each strength "m" are placed at the points $(-a,0)$ and $(a,0)$ and a sink of strength 2m is at the origin. Show that the streamlines are curves $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$; where λ is parameter.

Solution: According to the question, we can construct the following figure,



The complex potential is

$$w = -m\ln(z-a) - m\ln(z+a) + 2m\ln z$$

$$= -m\ln\{(x-a)+iy\} - m\ln\{(x+a)+iy\} + 2m\ln(x+iy)$$

$$or,\ \phi + i\psi = -\frac{m}{2}\ln\{(x-a)^2+y^2\} - im\tan^{-1}\left(\frac{y}{x-a}\right) - \frac{m}{2}\ln\{(x+a)^2+y^2\} - im\tan^{-1}\left(\frac{y}{x+a}\right)$$

$$+m\ln\{x^2+y^2\} + 2im\tan^{-1}\left(\frac{y}{x}\right);$$

$$\left[\because\ \ln(a+ib) = \frac{1}{2}\ln(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Separating real and imaginary parts, we get

$$\phi = -\frac{m}{2}\ln\{(x-a)^2+y^2\} - \frac{m}{2}\ln\{(x+a)^2+y^2\} + m\ln(x^2+y^2)$$

and $$\psi = -m\tan^{-1}\left(\frac{y}{x-a}\right) - m\tan^{-1}\left(\frac{y}{x+a}\right) + 2m\tan^{-1}\left(\frac{y}{x}\right)$$

The streamlines are given by

$$\psi = constant$$

$$or,\ -m\tan^{-1}\left(\frac{y}{x-a}\right) - m\tan^{-1}\left(\frac{y}{x+a}\right) + 2m\tan^{-1}\left(\frac{y}{x}\right) = constant$$

$$or,\ -m\left[\tan^{-1}\left(\frac{y}{x-a}\right) + \tan^{-1}\left(\frac{y}{x+a}\right) - 2\tan^{-1}\left(\frac{y}{x}\right)\right] = constant$$

$$or,\ \tan^{-1}\left\{\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y^2}{(x-a)(x+a)}}\right\} - \tan^{-1}\left\{\frac{\frac{2y}{x}}{1-\frac{y^2}{x^2}}\right\} = constant$$

$$or,\ \tan^{-1}\left\{\frac{\frac{2xy}{x^2-a^2}}{1-\frac{y^2}{x^2-a^2}}\right\} - \tan^{-1}\left\{\frac{\frac{2y}{x}}{1-\frac{y^2}{x^2}}\right\} = constant$$

$$or,\ \tan^{-1}\left\{\frac{2xy}{x^2-a^2-y^2}\right\} - \tan^{-1}\left\{\frac{2xy}{x^2-y^2}\right\} = constant$$

$$or,\ \tan^{-1}\frac{\frac{2xy}{x^2-a^2-y^2} - \frac{2xy}{x^2-y^2}}{1+\frac{2xy}{x^2-a^2-y^2}\cdot\frac{2xy}{x^2-y^2}} = constant$$

$$or,\ \tan^{-1}\frac{\frac{2xy\left(x^2-y^2\right)-2xy\left(x^2-a^2-y^2\right)}{\left(x^2-a^2-y^2\right)\left(x^2-y^2\right)}}{1+\frac{4x^2y^2}{\left(x^2-a^2-y^2\right)\left(x^2-y^2\right)}} = constant$$

$$or,\ \tan^{-1}\frac{2xy\left(x^2-y^2\right)-2xy\left(x^2-a^2-y^2\right)}{\left(x^2-a^2-y^2\right)\left(x^2-y^2\right)+4x^2y^2} = constant$$

$$or,\ \frac{2a^2xy}{\left(x^2-a^2-y^2\right)\left(x^2-y^2\right)+4x^2y^2} = constant$$

$$or,\ \frac{2a^2xy}{\left(x^2-y^2\right)^2 - a^2\left(x^2-y^2\right)+4x^2y^2} = constant$$

$$or,\ \frac{2a^2xy}{\left(x^2+y^2\right)^2 - a^2\left(x^2-y^2\right)} = \frac{1}{\lambda_1} \qquad \text{[Putting } constant = \frac{1}{\lambda_1}\text{]}$$

$$or,\ \left(x^2+y^2\right)^2 - a^2\left(x^2-y^2\right) = 2\lambda_1 a^2xy$$

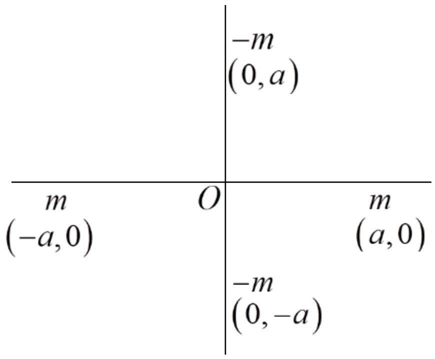
$$or,\ \left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)+2\lambda_1 a^2 xy$$

$$or,\ \left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)+\lambda a^2 xy \qquad \text{where } \lambda=2\lambda_1$$

$$\therefore\ \left(x^2+y^2\right)^2=a^2\left(x^2-y^2+\lambda xy\right) \qquad \textbf{(Showed)}.$$

Problem-11: If there are sources at $(a,0)$ and $(-a,0)$ and sinks at $(0,a)$ and $(0,-a)$, all of equal strength m, then show that the streamlines through these four points is a circle.

Solution: According to the question, we can construct the following figure,



The complex potential is

$$w=-m\ln(z-a)-m\ln(z+a)+m\ln(z-ia)+m\ln(z+ia)$$

$$=-m\ln\{(z-a)(z+a)\}+m\ln\{(z-ia)(z+ia)\}$$

$$=-m\ln\left(z^2-a^2\right)+m\ln\left(z^2+a^2\right)$$

$$=-m\ln\left\{(x+iy)^2-a^2\right\}+m\ln\left\{(x+iy)^2+a^2\right\}$$

$$=-m\ln\left\{\left(x^2-y^2-a^2\right)+2ixy\right\}+m\ln\left\{\left(x^2-y^2+a^2\right)+2ixy\right\}$$

$$=-\frac{m}{2}\ln\left\{\left(x^2-y^2-a^2\right)^2+4x^2y^2\right\}-im\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)+\frac{m}{2}\ln\left\{\left(x^2-y^2+a^2\right)^2+4x^2y^2\right\}$$

$$+im\tan^{-1}\left(\frac{2xy}{x^2-y^2+a^2}\right);$$

$$\left[\because\ \ln(a+ib)=\frac{1}{2}\ln\left(a^2+b^2\right)+i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Separating real and imaginary parts, we get

$$\phi=-\frac{m}{2}\ln\left\{\left(x^2-y^2-a^2\right)^2+4x^2y^2\right\}+\frac{m}{2}\ln\left\{\left(x^2-y^2+a^2\right)^2+4x^2y^2\right\}$$

and
$$\psi=-m\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)+m\tan^{-1}\left(\frac{2xy}{x^2-y^2+a^2}\right)$$

The streamlines are given by

$$\psi=constant$$

$$or,\ -m\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)+m\tan^{-1}\left(\frac{2xy}{x^2-y^2+a^2}\right)=c \qquad [\text{ Putting } c=constant\ ]$$

$$or,\ m\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2+a^2}\right)-\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)\right]=c$$

$$or,\ \tan^{-1}\left\{\frac{\frac{2xy}{x^2-y^2+a^2}-\frac{2xy}{x^2-y^2-a^2}}{1+\frac{2xy}{x^2-y^2+a^2}\cdot\frac{2xy}{x^2-y^2-a^2}}\right\}=c$$

$$or,\ \frac{-4xya^2}{\left(x^2-y^2\right)^2-a^4+4x^2y^2}=c$$

$$or,\ \frac{-4xya^2}{\left(x^2+y^2\right)^2-a^4}=c$$

when $c\to\infty$, we get

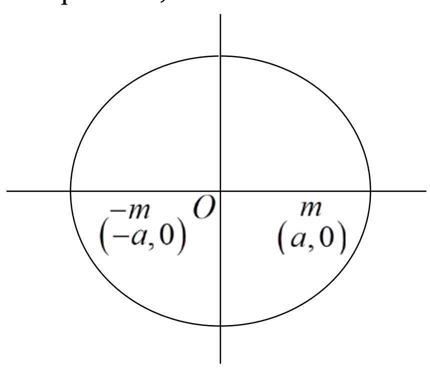
$$or,\ \left(x^2+y^2\right)^2-a^4=c$$

$$\therefore\quad x^2+y^2=a^2$$

Thus the streamline is a circle (**Showed**).

Problem-12: A source and a sink of equal strength are placed at the points $(\pm a,0)$ within a fixed circular cylinder $|z|=2a$. Show that the streamlines are given by $16a^2y^2+\lambda y\left(r^2-4a^2\right)=\left(r^2-16a^2\right)\left(r^2-a^2\right)$; where λ is constant.

Solution: According to the question, we can construct the following figure,



In the absence of circular boundary, the complex potential due to a source of strength "m" (say) at $(a,0)$ and a sink of strength "-m" at $(-a,0)$ is given by

$$w=-m\ln(z-a)+m\ln(z+a)$$

when the circular boundary is inserted into the flow field, then the complex potential due to source and sink inside the boundary is given by

$$w=-m\ln(z-a)+m\ln(z+a)-m\ln\left(\frac{4a^2}{z}-a\right)+m\ln\left(\frac{4a^2}{z}+a\right)$$

[By circle theorem, $w=f(z)+\overline{f}(\overline{z})$ *where* $\overline{z}=\dfrac{4a^2}{z}$]

$$=-m\ln(z-a)+m\ln(z+a)-m\ln\left\{\frac{a}{z}(4a-z)\right\}+m\ln\left\{\frac{a}{z}(4a+z)\right\}$$

$$=-m\ln(z-a)+m\ln(z+a)-m\ln\frac{a}{z}-m\ln(4a-z)+m\ln\frac{a}{z}+m\ln(4a+z)$$

$$=-m\ln(z-a)+m\ln(z+a)-m\ln(4a-z)+m\ln(4a+z)$$

$$=-m\ln\{(x-a)+iy\}+m\ln\{(x+a)+iy\}-m\ln\{(4a-x)-iy\}+m\ln\{(4a+x)+iy\}$$

$$=-\frac{m}{2}\ln\left\{(x-a)^2+y^2\right\}-im\tan^{-1}\left(\frac{y}{x-a}\right)+\frac{m}{2}\ln\left\{(x+a)^2+y^2\right\}+im\tan^{-1}\left(\frac{y}{x+a}\right)$$

$$-\frac{m}{2}\ln\left\{(4a-x)^2+y^2\right\}-im\tan^{-1}\left(\frac{-y}{4a-x}\right)+\frac{m}{2}\ln\left\{(4a+x)^2+y^2\right\}+im\tan^{-1}\left(\frac{y}{4a+x}\right)$$

Separating real and imaginary parts, we get

$$\psi=-m\tan^{-1}\left(\frac{y}{x-a}\right)+m\tan^{-1}\left(\frac{y}{x+a}\right)-m\tan^{-1}\left(\frac{y}{x-4a}\right)+m\tan^{-1}\left(\frac{y}{4a+x}\right)$$

$$=m\left[\tan^{-1}\left(\frac{y}{x+a}\right)-\tan^{-1}\left(\frac{y}{x-a}\right)+\tan^{-1}\left(\frac{y}{4a+x}\right)-\tan^{-1}\left(\frac{y}{x-4a}\right)\right]$$

$$=m\left[\tan^{-1}\frac{\frac{y}{x+a}-\frac{y}{x-a}}{1+\frac{y}{x+a}\cdot\frac{y}{x-a}}+\tan^{-1}\frac{\frac{y}{4a+x}-\frac{y}{x-4a}}{1+\frac{y}{4a+x}\cdot\frac{y}{x-4a}}\right]$$

$$=m\left[\tan^{-1}\frac{-2ay}{x^2+y^2-a^2}+\tan^{-1}\frac{-8ay}{x^2+y^2-16a^2}\right]$$

$$=m\tan^{-1}\frac{\frac{-2ay}{x^2+y^2-a^2}+\frac{-8ay}{x^2+y^2-16a^2}}{1-\frac{-2ay}{x^2+y^2-a^2}\cdot\frac{-8ay}{x^2+y^2-16a^2}}$$

$$=m\tan^{-1}\frac{-10ax^2y-10ay^3+40a^3y}{(x^2+y^2-a^2)(x^2+y^2-16a^2)-16a^2y^2}$$

$$=m\tan^{-1}\frac{-10ay(x^2+y^2-4a^2)}{(x^2+y^2-a^2)(x^2+y^2-16a^2)-16a^2y^2}$$

$$=m\tan^{-1}\frac{-10ay(r^2-4a^2)}{(r^2-a^2)(r^2-16a^2)-16a^2y^2}$$

The streamlines are given by

$$\psi = \text{constant}$$

$$or,\ m\tan^{-1}\frac{-10ay\left(r^2-4a^2\right)}{\left(r^2-a^2\right)\left(r^2-16a^2\right)-16a^2y^2} = \text{constant}$$

$$or,\ \frac{-10ay\left(r^2-4a^2\right)}{\left(r^2-a^2\right)\left(r^2-16a^2\right)-16a^2y^2} = \text{constant}$$

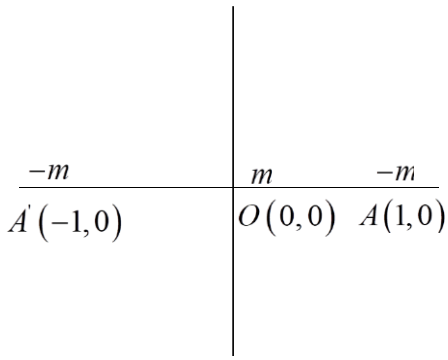
$$or,\ \frac{-10ay\left(r^2-4a^2\right)}{\left(r^2-a^2\right)\left(r^2-16a^2\right)-16a^2y^2} = \frac{-10a}{\lambda}\ ; \qquad \left[\textit{Putting}\quad \text{constant} = \frac{-10a}{\lambda}\right]$$

$$or,\ \lambda y\left(r^2-4a^2\right) = \left(r^2-a^2\right)\left(r^2-16a^2\right)-16a^2y^2$$

$$\therefore\ 16a^2y^2+\lambda y\left(r^2-4a^2\right) = \left(r^2-a^2\right)\left(r^2-16a^2\right)\ \textbf{(Showed)}.$$

Problem-13: There is a source of strength m at the origin and equal sinks at the points $(1,0)$ and $(-1,0)$. Discuss two dimensional motion due to the source and sinks.

Solution: According to the question, we can construct the following figure



Let "m" be the strength of the source at the origin O and "- m" be the strength of sinks at A and A' respectively.

Now the complex potential at any point due to this system of source and sinks is

$$w = m\ln(z-1)+m\ln(z+1)-m\ln z$$

$$= m\ln\{(x-1)+iy\}+m\ln\{(x+1)+iy\}-m\ln(x+iy)$$

$$or,\ \phi+i\psi = \frac{m}{2}\ln\left\{(x-1)^2+y^2\right\}+im\tan^{-1}\left(\frac{y}{x-1}\right)+\frac{m}{2}\ln\left\{(x+1)^2+y^2\right\}+im\tan^{-1}\left(\frac{y}{x+1}\right)$$

$$-\frac{m}{2}\ln\left(x^2+y^2\right)-im\tan^{-1}\left(\frac{y}{x}\right); \qquad \left[\because\ \ln(a+ib) = \frac{1}{2}\ln\left(a^2+b^2\right)+i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

This is the required complex potential.

Separating real and imaginary parts, we get

$$\phi = \frac{m}{2}\ln\left\{(x-1)^2+y^2\right\}+\frac{m}{2}\ln\left\{(x+1)^2+y^2\right\}-\frac{m}{2}\ln\left(x^2+y^2\right)$$

and $$\psi = m\tan^{-1}\left(\frac{y}{x-1}\right) + m\tan^{-1}\left(\frac{y}{x+1}\right) - m\tan^{-1}\left(\frac{y}{x}\right)$$

These are the required velocity potential and stream function respectively.

The lines of equipotential are given by

$$\phi = constant$$

$$or,\ \frac{m}{2}\ln\left\{(x-1)^2+y^2\right\} + \frac{m}{2}\ln\left\{(x+1)^2+y^2\right\} - \frac{m}{2}\ln\left(x^2+y^2\right) = constant$$

$$or,\ \frac{m}{2}\left[\ln\left\{(x-1)^2+y^2\right\} + \ln\left\{(x+1)^2+y^2\right\} - \ln\left(x^2+y^2\right)\right] = constant$$

$$or,\ \ln\frac{\left\{(x-1)^2+y^2\right\}\left\{(x+1)^2+y^2\right\}}{\left(x^2+y^2\right)} = constant.$$

The streamlines are given by

$$\psi = constant$$

$$or,\ m\tan^{-1}\left(\frac{y}{x-1}\right) + m\tan^{-1}\left(\frac{y}{x+1}\right) - m\tan^{-1}\left(\frac{y}{x}\right) = constant$$

$$or,\ m\left[\tan^{-1}\left(\frac{y}{x-1}\right) + \tan^{-1}\left(\frac{y}{x+1}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right] = constant$$

$$or,\ \tan^{-1}\left(\frac{2xy}{x^2-y^2-1}\right) - \tan^{-1}\left(\frac{y}{x}\right) = constant$$

$$or,\ \tan^{-1}\frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)} = constant$$

$$or,\ \frac{y\left(x^2+y^2+1\right)}{x\left(x^2+y^2-1\right)} = constant$$

The fluid speed at any point in the flow field is

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{m}{z-1} + \frac{m}{z+1} - \frac{m}{z}\right|$$

$$= m\left|\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z}\right|$$

$$= m\left|\frac{z(z+1)+z(z-1)-(z-1)(z+1)}{z(z-1)(z+1)}\right|$$

$$= m\left|\frac{z^2+z+z^2-z-z^2+1}{z(z-1)(z+1)}\right|$$

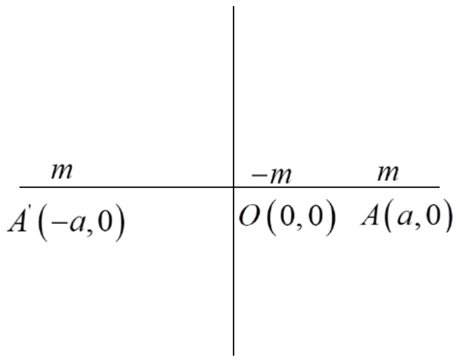
$$= m\left|\frac{z^2+1}{z(z-1)(z+1)}\right|$$

$$= m\left|\frac{(z+i)(z-i)}{z(z-1)(z+1)}\right|$$

$$= \frac{m|z-i||z+i|}{|z||z-1||z+1|}.$$

Problem-14: Find the stream function of two dimensional motion due to two equal sources and an equal sink midway between them. Sketch the streamlines and find the velocity at any point of the flow field.

Solution: According to the question, we can construct the following figure



Let "m" be the strength of sources at A , A' and "-m" be the strength of sink at the middle point of $AA' = 2a$ respectively.

Now the complex potential at any point due to this system of sources and sink is

$$w = -m\ln(z-a) - m\ln(z+a) + m\ln z$$

$$= -m\ln\{(x-a)+iy\} - m\ln\{(x+a)+iy\} + m\ln(x+iy)$$

$$or,\ \phi + i\psi = -\frac{m}{2}\ln\left\{(x-a)^2+y^2\right\} - im\tan^{-1}\left(\frac{y}{x-a}\right) - \frac{m}{2}\ln\left\{(x+a)^2+y^2\right\} - im\tan^{-1}\left(\frac{y}{x+a}\right)$$

$$+\frac{m}{2}\ln\left(x^2+y^2\right) + im\tan^{-1}\left(\frac{y}{x}\right); \quad \left[\because\ \ln(a+ib) = \frac{1}{2}\ln\left(a^2+b^2\right) + i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Separating real and imaginary parts, we get

$$\psi = -m\tan^{-1}\left(\frac{y}{x-a}\right) - m\tan^{-1}\left(\frac{y}{x+a}\right) + m\tan^{-1}\left(\frac{y}{x}\right)$$

This is the required stream function.

The streamlines are given by

$$\psi = cons\tan t$$

$$or,\ -m\tan^{-1}\left(\frac{y}{x-a}\right) - m\tan^{-1}\left(\frac{y}{x+a}\right) + m\tan^{-1}\left(\frac{y}{x}\right) = cons\tan t$$

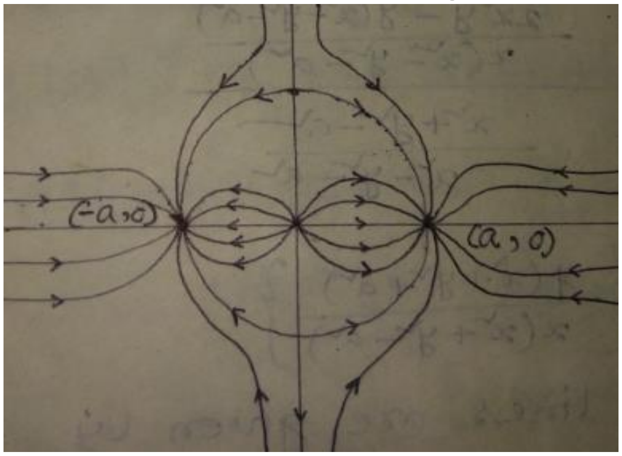
$$or,\ -m\left[\tan^{-1}\left(\frac{y}{x-a}\right) + \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right] = cons\tan t$$

$$or,\ \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right) - \tan^{-1}\left(\frac{y}{x}\right) = cons\tan t$$

$$or,\ \tan^{-1}\frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}=cons\tan t$$

$$or,\ \frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}=c\quad (say)$$

when $c=0$ then $y=0$ is the streamline and when $c\to\infty$ then $x=0$ and $x^2+y^2=a^2$ are the streamlines. The other streamlines are sketched in the figure.



The fluid speed at any point in the flow field is

$$q=\left|\frac{dw}{dz}\right|=\left|-\frac{m}{z-a}-\frac{m}{z+a}+\frac{m}{z}\right|$$

$$=m\left|\frac{1}{z-a}+\frac{1}{z+a}-\frac{1}{z}\right|$$

$$=m\left|\frac{z(z+a)+z(z-a)-(z-a)(z+a)}{z(z-a)(z+a)}\right|$$

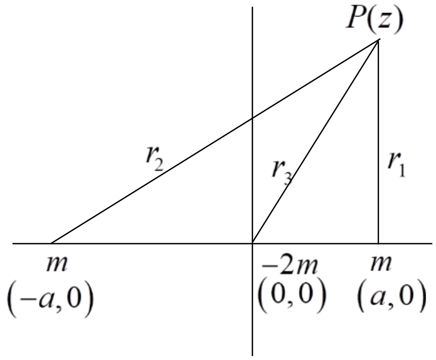
$$=m\left|\frac{z^2+az+z^2-az-z^2+a^2}{z(z-a)(z+a)}\right|$$

$$=m\left|\frac{z^2+a^2}{z(z-a)(z+a)}\right|$$

$$=m\left|\frac{(z+ia)(z-ia)}{z(z-a)(z+a)}\right|$$

$$=\frac{m|z-ia||z+ia|}{|z||z-a||z+a|}.$$

Problem-15: Find Complex potential, Velocity potential, Stream function, Lines of equipotential and the fluid speed at $P(z)$ from the figure.



Solution: According to the given figure, the complex potential is

$$w=-m\ln(z-a)-m\ln(z+a)+2m\ln z$$

$$=-m\ln\{(x-a)+iy\}-m\ln\{(x+a)+iy\}+2m\ln(x+iy)$$

$$or,\ \phi+i\psi=-\frac{m}{2}\ln\{(x-a)^2+y^2\}-im\tan^{-1}\left(\frac{y}{x-a}\right)-\frac{m}{2}\ln\{(x+a)^2+y^2\}-im\tan^{-1}\left(\frac{y}{x+a}\right)$$

$$+m\ln\{x^2+y^2\}+2im\tan^{-1}\left(\frac{y}{x}\right);\quad\left[\because\ \ln(a+ib)=\frac{1}{2}\ln(a^2+b^2)+i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

This is the required complex potential.
Separating real and imaginary parts, we get

$$\phi=-\frac{m}{2}\ln\{(x-a)^2+y^2\}-\frac{m}{2}\ln\{(x+a)^2+y^2\}+m\ln(x^2+y^2)$$

and $$\psi=-m\tan^{-1}\left(\frac{y}{x-a}\right)-m\tan^{-1}\left(\frac{y}{x+a}\right)+2m\tan^{-1}\left(\frac{y}{x}\right)$$

These are the required velocity potential and stream function respectively.
The lines of equipotential are given by

$$\phi=constant$$

$$or,\ -\frac{m}{2}\ln\{(x-a)^2+y^2\}-\frac{m}{2}\ln\{(x+a)^2+y^2\}+m\ln(x^2+y^2)=constant$$

$$or,\ -\frac{m}{2}\left[\ln\{(x-a)^2+y^2\}+\ln\{(x+a)^2+y^2\}-2\ln(x^2+y^2)\right]=constant$$

$$or,\ \ln\frac{\{(x-a)^2+y^2\}\{(x+a)^2+y^2\}}{(x^2+y^2)^2}=constant.$$

The fluid speed at any point in the flow field is

$$q=\left|\frac{dw}{dz}\right|=\left|-\frac{m}{z-a}-\frac{m}{z+a}+\frac{2m}{z}\right|$$

$$= m\left|\frac{1}{z-a}+\frac{1}{z+a}-\frac{2}{z}\right|$$

$$= m\left|\frac{z(z+a)+z(z-a)-2(z-a)(z+a)}{z(z-a)(z+a)}\right|$$

$$= m\left|\frac{z^2+az+z^2-az-2z^2+2a^2}{z(z-a)(z+a)}\right|$$

$$= m\left|\frac{2a^2}{z(z-a)(z+a)}\right|$$

$$= \frac{2ma^2}{|z||z-a||z+a|}$$

$$= \frac{2ma^2}{r_1r_2r_3}.$$

Exercise

Problem-01: A velocity field is given by $\vec{q} = -xi + (y+t)j$. Find the stream function and the streamline for this field at $t = 0$.

Problem-02: Find the stream function $\psi(x, y, t)$ for the given velocity field $u = U,\ v = 0$.

Problem-03: A two dimensional flow field is given by $\psi = \frac{1}{2}(x^2 - y^2)$. Show that the flow is irrotational and find the velocity potential. Also find the equation of the streamlines and potential lines.

Problem-04: If there are sources at $(1,0)$ and $(-1,0)$ and sinks at $(0,1)$ and $(0,-1)$ all of equal strength m, then show that the streamlines through these four points is a circle.

Problem-05: Two sources of each strength "m" are placed at the points $(-1,0)$ and $(1,0)$ and a sink of strength 2m is at the origin. Show that the streamlines are curves $(x^2+y^2)^2 = x^2 - y^2 + \lambda xy$; where λ is parameter.

Problem-06: A source and a sink of equal strength are placed at the points $\left(\pm\frac{a}{2}, 0\right)$ inside the circular cylinder $x^2 + y^2 = a^2$. Show that the streamlines are given by $\left(r^2 - \frac{a^2}{4}\right)(r^2 - 4a^2) - 4a^2y^2 = \lambda y(r^2 - a^2)$; where λ is constant.