Successive derivative: If $y = f(x)$ be a function of x then the first order derivative of y with respect to x is denoted by $\frac{dy}{dx}$, $f^{'}(x)$, y_1 , $y^{(1)}$, $f^{(1)}(x)$, $f_x^{'}(x)$ etc.

Again the derivative of first ordered derivative of y with respect to x is called second order derivative and is denoted by $\frac{d^2y}{dx^2}$, $f^{''}(x)$, y_2 , $y^{(2)}$, $f^{(2)}(x)$, $f_x^{''}(x)$ etc.

Similarly, the nth derivative of y with respect to x is denoted by $\frac{d^ny}{dx^n}$, $f^n(x)$, y_n , $y^{(n)}$, $f^{(n)}(x)$, $f_x^n(x)$ etc.

- Find the nth derivative of the following functions:

1. $y = x^n$

$sol:$ Given that, $y = x^n$

Differentiating with respect to x we get,

$$y_1 = nx^{n-1}$$
$$\therefore\ y_2 = n(n-1)x^{n-2}$$
$$\therefore\ y_3 = n(n-1)(n-2)x^{n-3}$$

Similarly,

$$y_r = n(n-1)(n-2)\cdots\cdots\{n-(r-1)\}x^{n-r} \quad ;\text{ where, } r<n$$
$$\therefore\ y_n = n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}x^{n-n}$$
$$= n(n-1)(n-2)\cdots\cdots 3.2.1 = n! \quad Ans.$$

2. $y = e^{ax}$

$sol:$ Given that, $y = e^{ax}$

Differentiating with respect to x we get,

$$y_1 = ae^{ax}$$
$$\therefore\ y_2 = a^2e^{ax}$$
$$\therefore\ y_3 = a^3e^{ax}$$

Similarly,

$$y_r = a^re^{ax} \quad ;\text{ where, } r<n$$
$$\therefore\ y_n = a^ne^{ax} \quad Ans.$$

3. $y = (ax+b)^m$, $m > n$

$sol:$ Given that, $y = (ax+b)^m$

Differentiating with respect to x we get,

$$y_1 = am(ax+b)^{m-1}$$
$$\therefore\ y_2 = a^2m(m-1)(ax+b)^{m-2}$$
$$\therefore\ y_3 = a^3m(m-1)(m-2)(ax+b)^{m-3}$$

Similarly,

$$y_r = a^rm(m-1)(m-2)\cdots\cdots\{m-(r-1)\}(ax+b)^{m-r} \quad ;\text{ where, } r<n$$
$$\therefore\ y_n = a^nm(m-1)(m-2)\cdots\cdots\{m-(n-1)\}(ax+b)^{m-n}$$
$$= \frac{a^nm(m-1)(m-2)\cdots\cdots\{m-(n-1)\}(m-n)!}{(m-n)!}(ax+b)^{m-n}$$
$$= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n} \quad Ans.$$

4. $y=(ax+b)^{-m}$, $m>n$

$sol: Given\,that,\ y=(ax+b)^{-m}$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=a(-m)(ax+b)^{-m-1}$$

$$\therefore\ y_2=a^2(-m)(-m-1)(ax+b)^{-m-2}$$

$$\therefore\ y_3=a^3(-m)(-m-1)(-m-2)(ax+b)^{-m-3}$$

$Similarly,$

$$y_r=a^r(-m)(-m-1)(-m-2)\cdots\cdots\{-m-(r-1)\}(ax+b)^{-m-r}\quad ;\ where,\ r<n$$

$$\therefore\ y_n=a^n(-m)(-m-1)(-m-2)\cdots\cdots\{-m-(n-1)\}(ax+b)^{-m-n}$$

$$=a^n(-1)^n m(m+1)(m+2)\cdots\cdots\{m+(n-1)\}(ax+b)^{-m-n}$$

$$=\frac{a^n(-1)^n 1.2.3\cdots\cdots(m-1)m(m+1)(m+2)\cdots\cdots\{m+(n-1)\}}{1.2.3\cdots\cdots(m-1)}(ax+b)^{-m-n}$$

$$=\frac{a^n(-1)^n(m+n-1)!}{(m-1)!}(ax+b)^{-m-n}\ \ Ans.$$

5. $y=a^x$

$sol: Given\,that,\ y=a^x$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=a^x\ln a$$

$$\therefore\ y_2=(\ln a)^2 a^x$$

$$\therefore\ y_3=(\ln a)^3 a^x$$

$Similarly,$

$$y_r=(\ln a)^r a^x\quad ;\ where,\ r<n$$

$$\therefore\ y_n=(\ln a)^n a^x\ \ Ans.$$

6. $y=\sin(ax+b)$

$sol: Given\ that,\ y=\sin(ax+b)$

Differentiating with respect to x we get,

$$y_1 = a\cos(ax+b)$$
$$= a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2 = a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\sin\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3 = a^3\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\sin\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\sin\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r = a^r\sin\left\{\frac{r\pi}{2}+(ax+b)\right\}\ ;\ where,\ r<n$$
$$\therefore\ y_n = a^n\sin\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

7. $y=\cos(ax+b)$

$sol: Given\ that,\ y=\cos(ax+b)$

Differentiating with respect to x we get,

$$y_1 = -a\sin(ax+b)$$
$$= a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2 = -a^2\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\cos\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$= a^2\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3 = -a^3\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\cos\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$= a^3\cos\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r = a^r\cos\left\{\frac{r\pi}{2}+(ax+b)\right\}\ ;\ where,\ r<n$$
$$\therefore\ y_n = a^n\cos\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

8. $y=e^{ax}\sin(bx+c)$

$sol: Given\ that,\ y=e^{ax}\sin(bx+c)$

Differentiating with respect to x we get,

$$y_1 = ae^{ax}\sin(bx+c)+be^{ax}\cos(bx+c)$$
$$= e^{ax}\{a\sin(bx+c)+b\cos(bx+c)\}$$

$put\ \ a=r\cos\varphi\ \ and\ b=r\sin\varphi$

$$\therefore\ r=\sqrt{a^2+b^2}\ \ and\ \ \varphi=\tan^{-1}\left(\frac{b}{a}\right)$$

$$Now,\ y_1 = e^{ax}\{r\cos\varphi\sin(bx+c)+r\sin\varphi\cos(bx+c)\}$$
$$= re^{ax}\sin(bx+c+\varphi)$$
$$\therefore\ y_2 = re^{ax}\{a\sin(bx+c+\varphi)+b\cos(bx+c+\varphi)\}$$
$$= re^{ax}\{r\cos\varphi\sin(bx+c+\varphi)+r\sin\varphi\cos(bx+c+\varphi)\}$$
$$= r^2e^{ax}\sin(bx+c+2\varphi)$$
$$\therefore\ y_3 = r^3e^{ax}\sin(bx+c+3\varphi)$$
$$Similarly,\ y_n = r^ne^{ax}\sin(bx+c+n\varphi)$$
$$= \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)\quad Ans.$$

9. $y=\ln(ax+b)$

$sol: Given\ that,\ y=\ln(ax+b)$

Differentiating with respect to x we get,

$$y_1 = \frac{a}{(ax+b)}$$
$$\therefore\ y_2 = -\frac{1.a^2}{(ax+b)^2}$$
$$\therefore\ y_3 = \frac{1.2\,a^3}{(ax+b)^3}$$
$$\therefore\ y_4 = -\frac{1.2.3\,a^4}{(ax+b)^4}$$

Similarly,

$$\therefore\ y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}\quad Ans.$$

10. $y = e^{ax}\cos(bx+c)$

$sol: Given\,that,\ y = e^{ax}\cos(bx+c)$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$$
$$= e^{ax}\{a\cos(bx+c) - b\sin(bx+c)\}$$

$put\ \ a = r\cos\varphi\ \ and\ b = r\sin\varphi$

$$\therefore\ r = \sqrt{a^2+b^2}\ \ and\ \ \varphi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$Now,\ y_1 = e^{ax}\{r\cos\varphi\cos(bx+c) - r\sin\varphi\sin(bx+c)\}$$
$$= re^{ax}\cos(bx+c+\varphi)$$
$$\therefore\ \ y_2 = re^{ax}\{a\cos(bx+c+\varphi) - b\sin(bx+c+\varphi)\}$$
$$= re^{ax}\{r\cos\varphi\cos(bx+c+\varphi) - r\sin\varphi\sin(bx+c+\varphi)\}$$
$$= r^2e^{ax}\sin(bx+c+2\varphi)$$
$$\therefore\ \ y_3 = r^3e^{ax}\cos(bx+c+3\varphi)$$

$Similarly,$

$$y_n = r^n e^{ax}\cos(bx+c+n\varphi)$$
$$= \left(\sqrt{a^2+b^2}\right)^n e^{ax}\cos\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)\ \ Ans.$$

11. $y = \sin 2x\sin 3x$

$sol: Given\,that,\ y = \sin 2x\sin 3x$

$$= \frac{1}{2}\left[\cos(2x-3x) - \cos(2x+3x)\right] = \frac{1}{2}\left[\cos x - \cos 5x\right]$$

$Differentiating\ successively\ with\ respect\ to\ x\ we\ get,$

$$y_1 = \frac{1}{2}\left[-\sin x + 5\sin 5x\right] = \frac{1}{2}\left[\cos\left(\frac{\pi}{2}+x\right) - 5\cos\left(\frac{\pi}{2}+5x\right)\right]$$
$$y_2 = \frac{1}{2}\left[-\sin\left(\frac{\pi}{2}+x\right) + 5^2\sin\left(\frac{\pi}{2}+5x\right)\right] = \frac{1}{2}\left[\cos\left(\frac{2\pi}{2}+x\right) - 5^2\cos\left(\frac{2\pi}{2}+5x\right)\right]$$
$$y_3 = \frac{1}{2}\left[-\sin\left(\frac{2\pi}{2}+x\right) + 5^3\sin\left(\frac{2\pi}{2}+5x\right)\right]$$
$$= \frac{1}{2}\left[\cos\left(\frac{3\pi}{2}+x\right) - 5^3\sin\left(\frac{3\pi}{2}+5x\right)\right]$$

$Similarly,$

$$y_n = \frac{1}{2}\left[\cos\left(\frac{n\pi}{2}+x\right) - 5^n\sin\left(\frac{n\pi}{2}+5x\right)\right]\ Ans.$$

12. $y=\sin^2 x\cos 2x$

$sol: Given\,that,\ y=\sin^2 x\cos 2x=\frac{1}{2}\left[(1-\cos 2x)\cos 2x\right]=\frac{1}{2}\left[\cos 2x-\frac{1}{2}(1+\cos 4x)\right]$

$$=\frac{1}{2}\cos 2x-\frac{1}{4}\cos 4x-\frac{1}{4}$$

Differentiating successively with respect to x we get,

$$y_1=\frac{1}{2}[-2\sin 2x]-\frac{1}{4}[-4\sin 4x]-0=-\sin 2x+\sin 4x=\cos\left(\frac{\pi}{2}+2x\right)-\cos\left(\frac{\pi}{2}+4x\right)$$

$$y_2=-2\sin\left(\frac{\pi}{2}+2x\right)+4\sin\left(\frac{\pi}{2}+4x\right)=2\cos\left(\frac{2\pi}{2}+2x\right)-4\cos\left(\frac{2\pi}{2}+4x\right)$$

$$y_3=-2^2\sin\left(\frac{2\pi}{2}+2x\right)+4^2\sin\left(\frac{2\pi}{2}+4x\right)$$

$$=2^2\cos\left(\frac{3\pi}{2}+2x\right)-4^2\cos\left(\frac{3\pi}{2}+4x\right)$$

Similarly,

$$y_n=2^{n-1}\cos\left(\frac{n\pi}{2}+2x\right)-4^{n-1}\cos\left(\frac{n\pi}{2}+4x\right)\ Ans.$$

13. *If* $y=\sin nx+\cos nx$ *then show that* $y_r=n^r\left[1+(-1)^r\sin 2nx\right]^{1/2}$.

$sol: Given\,that,\ y=\sin nx+\cos nx$

Differentiating with respect to x we get,

$$y_1=n\cos nx-n\sin nx$$

$$=n\sin\left(\frac{\pi}{2}+nx\right)+n\cos\left(\frac{\pi}{2}+nx\right)$$

$$\therefore\ y_2=n^2\cos\left(\frac{\pi}{2}+nx\right)-n^2\sin\left(\frac{\pi}{2}+nx\right)$$

$$=n^2\sin\left(\frac{2\pi}{2}+nx\right)+n^2\cos\left(\frac{2\pi}{2}+nx\right)$$

$$\therefore\ y_3=n^3\cos\left(\frac{2\pi}{2}+nx\right)-n^3\sin\left(\frac{2\pi}{2}+nx\right)$$

$$=n^3\sin\left(\frac{3\pi}{2}+nx\right)+n^3\cos\left(\frac{3\pi}{2}+nx\right)$$

Similarly,

$$y_r=n^r\sin\left(\frac{r\pi}{2}+nx\right)+n^3\cos\left(\frac{r\pi}{2}+nx\right)$$

$$=n^r\left[\left\{\sin\left(\frac{r\pi}{2}+nx\right)+\cos\left(\frac{r\pi}{2}+nx\right)\right\}^2\right]^{1/2}$$

$$=n^r\left[\sin^2\left(\frac{r\pi}{2}+nx\right)+\cos^2\left(\frac{r\pi}{2}+nx\right)+2\sin\left(\frac{r\pi}{2}+nx\right)\cos\left(\frac{r\pi}{2}+nx\right)\right]^{1/2}$$

$$=n^r\left[1+\sin 2\left(\frac{r\pi}{2}+nx\right)\right]^{1/2}$$

$$=n^r\left[1+\sin(r\pi+2nx)\right]^{1/2}$$

$$=n^r\left[1+(-1)^r\sin 2nx\right]^{1/2}\quad showed.$$

14. $y = x^{2n}$

$sol: Given\, that,\ y = x^{2n}$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = 2nx^{2n-1}$$

$$\therefore\ y_2 = 2n(2n-1)x^{2n-2}$$

$$\therefore\ y_3 = 2n(2n-1)(2n-2)x^{2n-3}$$

$Similarly,$

$$y_r = 2n(2n-1)(2n-2)\cdots\cdots\{2n-(r-1)\}x^{2n-r}\quad ;\ where,\ r<n$$

$$\therefore\ y_n = 2n(2n-1)(2n-2)\cdots\cdots\{2n-(n-1)\}x^{2n-n}$$

$$= \frac{2n(2n-1)(2n-2)\cdots\cdots(n+1)n(n-1)(n-2)\cdots\cdots 3.2.1}{n(n-1)(n-2)\cdots\cdots 3.2.1}x^n$$

$$= \frac{\{2n(2n-2)(2n-4)\cdots\cdots 6.4.2\}\{(2n-1)(2n-3)\cdots\cdots 5.3.1\}}{n!}x^n$$

$$= \frac{2^n\{n(n-1)(n-2)\cdots\cdots 3.2.1\}\{1.3.5\cdots\cdots(2n-1)\}}{n!}x^n$$

$$= \frac{2^n n!\{1.3.5\cdots\cdots(2n-1)\}}{n!}x^n$$

$$= 2^n\{1.3.5\cdots\cdots(2n-1)\}x^n\quad Ans.$$

15. $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$

$$sol: Given\, that,\ y = \frac{x^2+x-1}{x^3+x^2-6x} = \frac{x^2+x-1}{x(x^2+x-6)} = \frac{x^2+x-1}{x(x-2)(x+3)}$$

$$= \frac{1}{6}.\frac{1}{x} + \frac{1}{2}.\frac{1}{(x-2)} + \frac{1}{3}.\frac{1}{(x+3)}$$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = (-1)\frac{1}{6}.\frac{1}{x^2} + (-1)\frac{1}{2}.\frac{1}{(x-2)^2} + (-1)\frac{1}{3}.\frac{1}{(x+3)^2}$$

$$\therefore\ y_2 = (-1)(-2)\frac{1}{6}.\frac{1}{x^3} + (-1)(-2)\frac{1}{2}.\frac{1}{(x-2)^3} + (-1)(-2)\frac{1}{3}.\frac{1}{(x+3)^3}$$

$$\therefore\ y_3 = (-1)(-2)(-3)\frac{1}{6}.\frac{1}{x^4} + (-1)(-2)(-3)\frac{1}{2}.\frac{1}{(x-2)^4} + (-1)(-2)(-3)\frac{1}{3}.\frac{1}{(x+3)^4}$$

$Similarly,$

$$\therefore\ y_n = (-1)^n n!\left[\frac{1}{6}.\frac{1}{x^{n+1}} + \frac{1}{2}.\frac{1}{(x-2)^{n+1}} + \frac{1}{3}.\frac{1}{(x+3)^{n+1}}\right] Ans.$$

16. $y=\dfrac{1}{x^2+a^2}$

sol : Given that, $y=\dfrac{1}{x^2+a^2}$

$$=\frac{1}{x^2-(ia)^2}$$

$$=\frac{1}{(x-ia)(x+ia)}$$

$$=\frac{1}{2ia}\left[\frac{1}{(x-ia)}-\frac{1}{(x+ia)}\right]$$

$$=\frac{1}{2ia}\left[(x-ia)^{-1}-(x+ia)^{-1}\right]$$

Differentiating with respect to x we get,

$$y_1=\frac{1}{2ia}\left[(-1)(x-ia)^{-2}-(-1)(x+ia)^{-2}\right]$$

$$\therefore\ y_2=\frac{1}{2ia}\left[(-1)(-2)(x-ia)^{-3}-(-1)(-2)(x+ia)^{-3}\right]$$

$$\therefore\ y_3=\frac{1}{2ia}\left[(-1)(-2)(-3)(x-ia)^{-4}-(-1)(-2)(-3)(x+ia)^{-4}\right]$$

Similarly,

$$\therefore\ y_n=\frac{1}{2ia}\left[(-1)(-2)(-3)\cdots(-n)(x-ia)^{-(n+1)}-(-1)(-2)(-3)\cdots(-n)(x+ia)^{-(n+1)}\right]$$

$$=\frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)}-(x+ia)^{-(n+1)}\right]\ \cdots(1)$$

Putting $x=r\cos\theta$ *and* $a=r\sin\theta$ $\therefore r=\dfrac{a}{\sin\theta}$ *and* $\theta=\tan^{-1}\left(\dfrac{a}{x}\right)$

Now $x+ia=r(\cos\theta+i\sin\theta)=re^{i\theta}$

and $x-ia=r(\cos\theta-i\sin\theta)=re^{-i\theta}$

Now from (1) *we have*

$$y_n=\frac{(-1)^n n!}{2ia}\left[\left(re^{-i\theta}\right)^{-(n+1)}-\left(re^{i\theta}\right)^{-(n+1)}\right]$$

$$=\frac{(-1)^n n!}{2iar^{n+1}}\left[\cos(n+1)\theta+i\sin(n+1)\theta-\cos(n+1)\theta+i\sin(n+1)\theta\right]$$

$$=\frac{(-1)^n n!}{2iar^{n+1}}\left[2i\sin(n+1)\theta\right]$$

$$=\frac{(-1)^n n!}{a\left(\dfrac{a}{\sin\theta}\right)^{n+1}}\sin(n+1)\theta$$

$$=\frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta.\sin(n+1)\theta,\ \ where\ \theta=\tan^{-1}\left(\frac{a}{x}\right).$$

17. $y=\tan^{-1}\left(\frac{x}{a}\right)$

sol: *Given that*, $y=\tan^{-1}\left(\frac{x}{a}\right)$

Differentiating successively with respect to x we get

$$y_1=\frac{a}{x^2+a^2}$$
$$=\frac{a}{x^2-(ia)^2}$$
$$=\frac{a}{(x-ia)(x+ia)}$$
$$=\frac{1}{2i}\left[\frac{1}{(x-ia)}-\frac{1}{(x+ia)}\right]$$
$$=\frac{1}{2i}\left[(x-ia)^{-1}-(x+ia)^{-1}\right]$$
$$y_2=\frac{1}{2i}\left[(-1)(x-ia)^{-2}-(-1)(x+ia)^{-2}\right]$$
$$\therefore\ y_3=\frac{1}{2i}\left[(-1)(-2)(x-ia)^{-3}-(-1)(-2)(x+ia)^{-3}\right]$$
$$\therefore\ y_4=\frac{1}{2i}\left[(-1)(-2)(-3)(x-ia)^{-4}-(-1)(-2)(-3)(x+ia)^{-4}\right]$$

Similarly,

$$\therefore\ y_n=\frac{1}{2i}\left[(-1)(-2)(-3)\cdots\{-(n-1)\}(x-ia)^{-n}-(-1)(-2)(-3)\cdots\{-(n-1)\}(x+ia)^{-n}\right]$$
$$=\frac{(-1)^{n-1}(n-1)!}{2i}\left[(x-ia)^{-n}-(x+ia)^{-n}\right]\ \cdots(1)$$

Putting $x=r\cos\theta$ *and* $a=r\sin\theta$ $\therefore r=\frac{a}{\sin\theta}$ *and* $\theta=\tan^{-1}\left(\frac{a}{x}\right)$

Now $x+ia=r(\cos\theta+i\sin\theta)=re^{i\theta}$

and $x-ia=r(\cos\theta-i\sin\theta)=re^{-i\theta}$

Now from (1) *we have*

$$y_n=\frac{(-1)^{n-1}(n-1)!}{2i}\left[\left(re^{-i\theta}\right)^{-n}-\left(re^{i\theta}\right)^{-n}\right]$$
$$=\frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta\right]$$
$$=\frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[2i\sin n\theta\right]$$
$$=\frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta$$
$$=\frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta.\sin n\theta,\ \ where\ \theta=\tan^{-1}\left(\frac{a}{x}\right).$$

Homework:

1. Find the nth derivative of the following functions:

a. $y = x^m \ , \ m \in N$ Ans: $y_n = \begin{cases} 0 & if \ n > m \\ \dfrac{m!}{(m-n)!} x^{m-n} & if \ n \le m \end{cases}$

b. $y = (ax+b)^{-m}, \ m \in N$ Ans: $y_n = \dfrac{(-1)^n (m+n-1)! a^n}{(m-1)!} (ax+b)^{-m}$

c. $y = e^{ax} \cos bx$ Ans: $y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n \tan^{-1} \dfrac{b}{a}\right)$

d. $y = \dfrac{1}{x^2+5x+6}$ Ans: $y_n = (-1)^n n! \left[\dfrac{1}{(x+2)^{n+1}} - \dfrac{1}{(x+3)^{n+1}}\right]$

e. $y = \dfrac{2x+3}{x^2+3x+2}$. Ans: $y_n = (-1)^n n! \left[\dfrac{1}{(x+1)^{n+1}} + \dfrac{1}{(x+2)^{n+1}}\right]$

f. $y = \dfrac{x}{x^2+a^2}$ Ans: $y_n = \dfrac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cdot \cos(n+1)\theta$

g. $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ Ans: $y_n = 2(-1)^{n-1} (n-1)! \sin^n \theta \cdot \sin n\theta, \ where \ \theta = \tan^{-1}\left(\dfrac{a}{x}\right)$

h. $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

i. $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

j. $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

k. $y = \cot^{-1}\left(\dfrac{x}{a}\right)$

l. $y = e^x \sin^2 x$

m. $y = e^x \sin x \sin 2x$

Theorem: State and prove Leibnitz's theorem.

<u>**Answer**</u>**:** **Statement:** If u and v are two functions of x, then the nth derivative of their product is,

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \cdots\cdots + {}^n c_r u_{n-r} v_r + \cdots\cdots + u v_n$$

where the suffixes in u and v denote the order of differentiations of u and v with respect to x.

Proof: We shall prove the theorem by mathematical induction.

Step-1: Let $y = uv$

By actual differentiation on both sides with respect to x, we have

$$y_1 = u_1 v + u v_1$$

Thus the theorem is true for $n=1$.

Step-2: Let us assume that the theorem is true for $n=m$

i.e. $$y_m = u_m v + {}^m c_1 u_{m-1} v_1 + {}^m c_2 u_{m-2} v_2 + \cdots\cdots + {}^m c_r u_{m-r} v_r + \cdots\cdots + u v_m \qquad \cdots(1)$$

Step-3: Theorem will be true for $n=m+1$ if

$$y_{m+1} = u_{m+1} v + {}^{m+1} c_1 u_m v_1 + {}^{m+1} c_2 u_{m-1} v_2 + \cdots\cdots + {}^{m+1} c_r u_{m-r+1} v_r + \cdots\cdots + u v_{m+1} \qquad \cdots(2)$$

is true.

Now differentiating on both sides of (1) with respect to x, we get

$$y_{m+1} = \left(u_{m+1} v + u_m v_1\right) + {}^m c_1 \left(u_m v_1 + u_{m-1} v_2\right) + {}^m c_2 \left(u_{m-1} v_2 + u_{m-2} v_3\right) + \cdots\cdots$$

$$+ {}^m c_r \left(u_{m-r+1} v_r + u_{m-r} v_{r+1}\right) + \cdots\cdots + \left(u_1 v_m + u v_{m+1}\right)$$

$$= u_{m+1} v + \left(1 + {}^m c_1\right) u_m v_1 + \left({}^m c_1 + {}^m c_2\right) u_{m-1} v_2 + \cdots\cdots + \left({}^m c_{r-1} + {}^m c_r\right) u_{m-r+1} v_r + \cdots\cdots + u v_{m+1}$$

$$= u_{m+1} v + \left({}^m c_0 + {}^m c_1\right) u_m v_1 + \left({}^m c_1 + {}^m c_2\right) u_{m-1} v_2 + \cdots\cdots + \left({}^m c_{r-1} + {}^m c_r\right) u_{m-r+1} v_r + \cdots\cdots + u v_{m+1}$$

$$= u_{m+1} v + {}^{m+1} c_1 u_m v_1 + {}^{m+1} c_2 u_{m-1} v_2 + \cdots\cdots + {}^{m+1} c_r u_{m-r+1} v_r + \cdots\cdots + u v_{m+1}$$

$$\left[\because {}^m c_{r-1} + {}^m c_r = {}^{m+1} c_r\right]$$

which is exactly same of the form as (2).

Since the theorem hold for $n=m$ hence it also hold for $n=m+1$.

Hence, by the principle of mathematical induction, the theorem is true for every positive integer n.

(Proved)

Using Leibnitz's theorem find y_n of the following functions:

1. $y = x^3 \sin x$

Sol : Given that, $y = x^3 \sin x$

Differentiating n times by Leibnitz's theorem we get,

$$y_n = \left(x^3 \sin x\right)_n$$

$$= \left(\sin x\right)_n x^3 + {}^n c_1 \left(\sin x\right)_{n-1} \left(x^3\right)_1 + {}^n c_2 \left(\sin x\right)_{n-2} \left(x^3\right)_2 + {}^n c_3 \left(\sin x\right)_{n-3} \left(x^3\right)_3 + {}^n c_4 \left(\sin x\right)_{n-4} \left(x^3\right)_4 + \cdots\cdots + \sin x \left(x^3\right)_n$$

$$= \sin\left(\frac{n\pi}{2} + x\right).x^3 + n \sin\left\{\frac{(n-1)\pi}{2} + x\right\}.3x^2 + \frac{n(n-1)}{2} \sin\left\{\frac{(n-2)\pi}{2} + x\right\}.6x + \frac{n(n-1)(n-2)}{6} \sin\left\{\frac{(n-3)\pi}{2} + x\right\}.6 + 0$$

$$= x^3 \sin\left(\frac{n\pi}{2} + x\right) - 3nx^2 \sin\left\{\frac{\pi}{2} - \left(\frac{n\pi}{2} + x\right)\right\} - 3n(n-1)x \sin\left\{\pi - \left(\frac{n\pi}{2} + x\right)\right\} - n(n-1)(n-2) \sin\left\{\frac{3\pi}{2} - \left(\frac{n\pi}{2} + x\right)\right\}$$

$$= x^3 \sin\left(\frac{n\pi}{2} + x\right) - 3nx^2 \cos\left(\frac{n\pi}{2} + x\right) - 3n(n-1)x \sin\left(\frac{n\pi}{2} + x\right) + n(n-1)(n-2) \cos\left(\frac{n\pi}{2} + x\right)$$

$$= \left\{x^3 - 3n(n-1)x\right\} \sin\left(\frac{n\pi}{2} + x\right) - \left\{3nx^2 - n(n-1)(n-2)\right\} \cos\left(\frac{n\pi}{2} + x\right) \quad Ans.$$

2. $y = x^2 \ln x$

Sol : Given that, $y = x^2 \ln x$

Differentiating n times by Leibnitz's theorem we get,

$$y_n = \left(x^2 \ln x\right)_n$$
$$= (\ln x)_n x^2 + {}^n c_1 (\ln x)_{n-1} \left(x^2\right)_1 + {}^n c_2 (\ln x)_{n-2} \left(x^2\right)_2 + {}^n c_3 (\ln x)_{n-3} \left(x^2\right)_3 + \cdots\cdots + \ln x.\left(x^2\right)_n$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^n}.x^2 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.2x + \frac{n(n-1)}{2}.\frac{(-1)^{n-3}(n-2)!}{x^{n-2}}.2 + 0$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + \frac{2(-1)^{n-2} n(n-2)!}{x^{n-2}} + \frac{(-1)^{n-3} n(n-1)(n-2)!}{x^{n-2}} \quad Ans.$$

3. $y = e^x \ln x$

Sol : Given that, $y = e^x \ln x$

Differentiating n times by Leibnitz's theorem we get,

$$y_n = \left(e^x \ln x\right)_n$$
$$= (\ln x)_n e^x + {}^n c_1 (\ln x)_{n-1} \left(e^x\right)_1 + {}^n c_2 (\ln x)_{n-2} \left(e^{x2}\right)_2 + \cdots\cdots + \ln x.\left(e^x\right)_n$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^n}.e^x + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.e^x + \frac{n(n-1)}{2}.\frac{(-1)^{n-3}(n-2)!}{x^{n-2}}.e^x + \cdots\cdots + e^x \ln x. \quad Ans.$$

P−01: *If* $y = \tan^{-1} x$ *then show that* $\left(1+x^2\right) y_{n+2} + 2(n+1) x y_{n+1} + \left(n^2+n\right) y_n = 0$

Sol : Given that, $y = \tan^{-1} x$

Differentiating with respect to x we get,

$$y_1 = \frac{1}{1+x^2}$$

$$or, \left(1+x^2\right) y_1 = 1$$

Again, differentiating with respect to x we get,

$$\left(1+x^2\right) y_2 + 2xy_1 = 0$$

By Leibnitz's theorem we get,

$$\left(1+x^2\right) y_{n+2} + {}^n c_1.2x.y_{n+1} + {}^n c_2.2.y_n + 2xy_{n+1} + {}^n c_1.2.y_n = 0$$
$$or,\ \left(1+x^2\right) y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}.2y_n + 2xy_{n+1} + 2ny_n = 0$$
$$or,\ \left(1+x^2\right) y_{n+2} + 2nxy_{n+1} + \left(n^2-n\right) y_n + 2xy_{n+1} + 2ny_n = 0$$
$$or,\ \left(1+x^2\right) y_{n+2} + 2(n+1) xy_{n+1} + \left(n^2-n+2n\right) y_n = 0$$
$$or,\ \left(1+x^2\right) y_{n+2} + 2(n+1) xy_{n+1} + \left(n^2+n\right) y_n = 0 \quad showed.$$

$\text{P}-02: \textit{If } y=\left(\sin^{-1}x\right)^2 \textit{ then show that } \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-n^2y_n=0.$

$Sol: Given that, \ y=\left(\sin^{-1}x\right)^2$

Differentiating with respect to x we get,

$$y_1=2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$$

$$or,\ y_1^{\ 2}=4\left(\sin^{-1}x\right)^2.\frac{1}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^{\ 2}=4y$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2+\left(-2x\right).y_1^{\ 2}=4y_1$$

$$or,\ \left(1-x^2\right)y_2-xy_1=2$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2}+{}^nc_1.\left(-2x\right).y_{n+1}+{}^nc_2.\left(-2\right).y_n-\left\{xy_{n+1}+{}^nc_1.1.y_n\right\}=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\frac{n\left(n-1\right)}{2}.2y_n-xy_{n+1}-ny_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\left(n^2-n\right)y_n-xy_{n+1}-ny_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-n+n\right)y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-n^2y_n=0 \qquad showed.$$

$\text{P}-03: \textit{If } y=\sin\left(a\sin^{-1}x\right) \textit{ then show that } \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-a^2\right)y_n=0.$

$Sol: Given that, \ y=\sin\left(a\sin^{-1}x\right)$

Differentiating with respect to x we get,

$$y_1=\cos\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}}$$

$$or,\ y_1^{\ 2}=\cos^2\left(a\sin^{-1}x\right).\frac{a^2}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^{\ 2}=a^2\left\{1-\sin^2\left(a\sin^{-1}x\right)\right\}$$

$$or, \left(1-x^2\right)y_1^{\ 2}=a^2\left(1-y^2\right)$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2+\left(-2x\right).y_1^{\ 2}=a^2\left(-2yy_1\right)$$

$$or,\ \left(1-x^2\right)y_2-xy_1=-a^2y$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2}+{}^nc_1.\left(-2x\right).y_{n+1}+{}^nc_2.\left(-2\right).y_n-\left\{xy_{n+1}+{}^nc_1.1.y_n\right\}=-a^2y_n$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\frac{n\left(n-1\right)}{2}.2y_n-xy_{n+1}-ny_n+a^2y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\left(n^2-n\right)y_n-xy_{n+1}-ny_n+a^2y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-n+n-a^2\right)y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-a^2\right)y_n=0 \qquad showed.$$

P−04: *If* $y = e^{a\sin^{-1}x}$ *then show that* $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0.$

Also, find the value of y_n *when* $x = 0$.

Sol : Given that, $y = e^{a\sin^{-1}x} \quad \cdots(1)$

Differentiating with respect to x we get,

$$y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$$

$$or,\; y_1^{\,2} = \left(e^{a\sin^{-1}x}\right)^2.\frac{a^2}{\left(1-x^2\right)} \quad ;\left[Squaring\ both\ sides\right]$$

$$or, \left(1-x^2\right)y_1^{\,2} = a^2y^2 \quad \cdots(2)$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2 + (-2x)\,y_1^{\,2} = 2a^2yy_1$$

$$or,\; \left(1-x^2\right)y_2 - xy_1 = a^2y \quad \cdots(3)$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2} + {}^nc_1.(-2x).y_{n+1} + {}^nc_2.(-2).y_n - \left\{xy_{n+1} + {}^nc_1.1.y_n\right\} = a^2y_n$$

$$or,\; \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$or,\; \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \left(n^2-n\right)y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$or,\; \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-n+n+a^2\right)y_n = 0$$

$$or,\; \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0 \quad \cdots(4) \quad (Showed).$$

2nd Part : From (1), (2), (3), *we have* $y = 1, y_1 = a, y_2 = a^2,$ *when* $x = 0.$

Putting $n = 1, 3, 5, \cdots,$ *successively in* (4), *we get*

$$y_3 = \left(1^2+a^2\right)y_1 = \left(1^2+a^2\right)a$$

$$y_5 = \left(3^2+a^2\right)y_3 = \left(3^2+a^2\right)\left(1^2+a^2\right)a$$

$$y_7 = \left(5^2+a^2\right)y_5 = \left(5^2+a^2\right)\left(3^2+a^2\right)\left(1^2+a^2\right)a$$

$$\cdots \quad \cdots\cdots\cdots\cdots\cdots$$

$$y_n = \left\{(n-2)^2+a^2\right\}\cdots\ \cdots\left(5^2+a^2\right)\left(3^2+a^2\right)\left(1^2+a^2\right)a \quad , when\ n\ is\ odd.$$

Putting $n = 2, 4, 6, \cdots,$ *successively in* (4), *we get*

$$y_4 = \left(2^2+a^2\right)y_2 = \left(2^2+a^2\right)a^2$$

$$y_6 = \left(4^2+a^2\right)y_4 = \left(4^2+a^2\right)\left(2^2+a^2\right)a^2$$

$$y_8 = \left(6^2+a^2\right)y_6 = \left(6^2+a^2\right)\left(4^2+a^2\right)\left(2^2+a^2\right)a^2$$

$$\cdots \quad \cdots\cdots\cdots\cdots\cdots$$

$$y_n = \left\{(n-2)^2+a^2\right\}\cdots\ \cdots\left(6^2+a^2\right)\left(4^2+a^2\right)\left(2^2+a^2\right)a^2 \;, when\ n\ is\ even.$$

$\text{P}-05: If\ y=\cos\left(m\sin^{-1}x\right)\ then\ show\ that\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-m^2\right)y_n=0.$

$Also,\ find\ the\ value\ of\ y_n\ when\ x=0.$

$Sol: Given\ that,\ y=\cos\left(m\sin^{-1}x\right) \quad \cdots(1)$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1=-\sin\left(m\sin^{-1}x\right).\frac{m}{\sqrt{1-x^2}}$$

$$or,\ y_1^{\,2}=\sin^2\left(m\sin^{-1}x\right).\frac{m^2}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^{\,2}=m^2\left\{1-\cos^2\left(a\sin^{-1}x\right)\right\}$$

$$or, \left(1-x^2\right)y_1^{\,2}=m^2\left(1-y^2\right) \quad \cdots(2)$$

$Again,\ differentiating\ with\ respect\ to\ x\ we\ get,$

$$\left(1-x^2\right).2y_1y_2+\left(-2x\right).y_1^{\,2}=m^2\left(-2yy_1\right)$$

$$or,\ \left(1-x^2\right)y_2-xy_1+m^2y=0 \quad \cdots(3)$$

$By\ Leibnitz's\ theorem\ we\ get,$

$$\left(1-x^2\right)y_{n+2}+{}^nc_1.(-2x).y_{n+1}+{}^nc_2.(-2).y_n-\left\{xy_{n+1}+{}^nc_1.1.y_n\right\}+m^2y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\frac{n(n-1)}{2}.2y_n-xy_{n+1}-ny_n+m^2y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\left(n^2-n\right)y_n-xy_{n+1}-ny_n+m^2y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-n+n-m^2\right)y_n=0$$

$$or,\ \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-m^2\right)y_n=0 \quad \cdots(4) \quad (Showed).$$

$2nd\ Part: From\ (1),(2),(3),\ we\ have\ y=1,\ y_1=0,\ y_2=-m^2,\ \ when\ \ x=0.$

$Putting\ n=1,3,5,\cdots,\ successively\ in\ (4),\ we\ get$

$$y_3=\left(1^2-m^2\right)y_1=\left(1^2-m^2\right)\times 0=0$$

$$y_5=\left(3^2-m^2\right)y_3=0$$

$$y_7=\left(5^2-m^2\right)y_5=0$$

$$\cdots\ \cdots\cdots\cdots\cdots\cdots$$

$$y_n=0 \quad ,when\ n\ is\ odd.$$

$Putting\ n=2,4,6,\cdots,\ successively\ in\ (4),\ we\ get$

$$y_4=\left(2^2-m^2\right)y_2=-\left(2^2-m^2\right)m^2$$

$$y_6=\left(4^2-m^2\right)y_4=-\left(4^2-m^2\right)\left(2^2-m^2\right)m^2$$

$$y_8=\left(6^2-m^2\right)y_6=-\left(6^2-m^2\right)\left(4^2-m^2\right)\left(2^2-m^2\right)m^2$$

$$\cdots\ \cdots\cdots\cdots\cdots\cdots$$

$$y_n=-\left\{\left(n-2\right)^2-m^2\right\}\cdots\ \cdots\left(6^2-m^2\right)\left(4^2-m^2\right)\left(2^2-m^2\right)m^2 \quad ,when\ n\ is\ even.$$

$\text{P}-06: If\ y=\cos\{\ln(1+x)\}\ then\ show\ that\ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0.$

$Sol: Given\ that,\ y=\cos\{\ln(1+x)\}$

Differentiating with respect to x we get,

$$y_1=-\sin\{\ln(1+x)\}.\frac{1}{(1+x)}$$

$$or,\ (1+x)y_1=-\sin\{\ln(1+x)\}$$

Again, differentiating with respect to x we get,

$$(1+x)y_2+y_1=-\cos\{\ln(1+x)\}.\frac{1}{(1+x)}$$

$$or,\ (1+x)^2 y_2+(1+x)y_1=-y$$

By Leibnitz's theorem we get,

$$(1+x)^2 y_{n+2}+{}^nc_1.2(1+x).y_{n+1}+{}^nc_2.2.y_n+(1+x)y_{n+1}+{}^nc_1.1.y_n=-y_n$$

$$or,\ \ (1+x)^2 y_{n+2}+2n(1+x)y_{n+1}+\frac{n(n-1)}{2}.2y_n+(1+x)y_{n+1}+ny_n+y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2-n)y_n+ny_n+y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2-n+n+1)y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0\quad showed.$$

$\text{P}-07: If\ y=(x^2-1)^n\ then\ show\ that\ (x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$

$Sol: Given\ that,\ y=(x^2-1)^n$

Differentiating with respect to x we get,

$$y_1=n(x^2-1)^{n-1}.2x$$

$$or,\ (x^2-1)y_1=2nx(x^2-1)^n\quad ;[Multiplying\ by\ (x^2-1)]$$

$$or,\ (x^2-1)y_1=2nxy$$

Again, differentiating with respect to x we get,

$$(x^2-1)y_2+2xy_1=2ny+2nxy_1$$

$$or,\ (x^2-1)y_2+2(1-n)xy_1=2ny$$

By Leibnitz's theorem we get,

$$(1+x)^2 y_{n+2}+{}^nc_1.2(1+x).y_{n+1}+{}^nc_2.2.y_n+(1+x)y_{n+1}+{}^nc_1.1.y_n=-y_n$$

$$or,\ \ (1+x)^2 y_{n+2}+2n(1+x)y_{n+1}+\frac{n(n-1)}{2}.2y_n+(1+x)y_{n+1}+ny_n+y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2-n)y_n+ny_n+y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2-n+n+1)y_n=0$$

$$or,\ \ (1+x)^2 y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0\quad showed.$$

$\mathrm{P-08}: If\ x=\sin t,\ y=\sin kt\ where\, k\, is\, a\, constant\, t,\ then\, show\, that\ \left(1-x^2\right)y_2-xy_1+k^2y=0$

$Sol: Given\, that,\ x=\sin t,\ y=\sin kt$

$Differentiating\ with\ respect\ to\ t\ we\ get,$

$$\frac{dx}{dt}=\cos t\quad and\quad \frac{dy}{dt}=k\cos kt$$

$$Now\ \ y_1=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{k\cos kt}{\cos t}$$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_2=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{k\cos kt}{\cos t}\right).\frac{dt}{dx}$$

$$=\left(\frac{-k^2\cos t\sin kt+k\sin t\cos kt}{\cos^2 t}\right).\frac{1}{\cos t}$$

$$=\frac{-k^2\sin kt+k\sin t\frac{\cos kt}{\cos t}}{1-\sin^2 t}=\frac{-k^2y+xy_1}{1-x^2}$$

$$or,\left(1-x^2\right)y_2-xy_1+k^2y=0\quad (Showed).$$

$\mathrm{P-09}: If\ y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x,\ then\ prove\ that\ \left(x^2-1\right)y_2+xy_1-m^2y=0$

$Sol: Given\, that,\ y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x\quad \cdots(1)$

$$let\ \ u=y^{\frac{1}{m}}$$

$Then\ from\,(1)\,we\ have$

$$u^2-2ux+1=0$$

$$\therefore u=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$$

$$or\ y^{\frac{1}{m}}=x\pm\sqrt{x^2-1}$$

$Taking\ \log arithm\ of\ the\ both\ sides$

$$\frac{1}{m}\ln y=\ln\left(x\pm\sqrt{x^2-1}\right)$$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$\frac{1}{m}\cdot\frac{1}{y}\cdot y_1=\frac{1}{x\pm\sqrt{x^2-1}}\cdot\left(1\pm\frac{x}{\sqrt{x^2-1}}\right)$$

$$or,\ \frac{y_1}{my}=\frac{1}{\sqrt{x^2-1}}$$

$$or,\ \left(x^2-1\right)y_1^2=m^2y^2$$

$Differentiating\ again\ with\ respect\ to\ x\ we\ get,$

$$\left(x^2-1\right)\cdot 2y_1y_2+2xy_1^2=2m^2yy_1$$

$$or,\left(x^2-1\right)y_2+xy_1=m^2y$$

$$\therefore\ \left(x^2-1\right)y_2+xy_1-m^2y=0\quad (\mathrm{Pr}oved)$$

Homework:-

2. $If\ y=\cot^{-1}x\ then\, show\, that\ \left(1+x^2\right)y_{n+2}+2\left(n+1\right)xy_{n+1}+n\left(n+1\right)y_n=0.$
3. $If\ y=a\cos\left(\ln x\right)+b\sin\left(\ln x\right)\ then\, show\, that\ x^2y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2+1\right)y_n=0.$
4. $If\ y=\sin\left\{a\ln\left(x+b\right)\right\}\ then\, show\, that\ \left(x+b\right)^2y_{n+2}+\left(2n+1\right)\left(x+b\right)y_{n+1}+\left(n^2+a^2\right)y_n=0.$
5. $If\ \ln y=\tan^{-1}x\ then\, show\, that\left(1+x^2\right)y_{n+2}+\left(2nx+2x-1\right)y_{n+1}+n\left(n+1\right)y_n=0.$
6. $If\ y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}\ then\, show\, that\left(1-x^2\right)y_{n+2}-\left(2n+3\right)xy_{n+1}-\left(n+1\right)^2y_n=0.$
7. $If\ x=\sin\theta,\ y=\sin p\theta,\ then\, show\, that\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}+\left(p^2-n^2\right)y_n=0.$