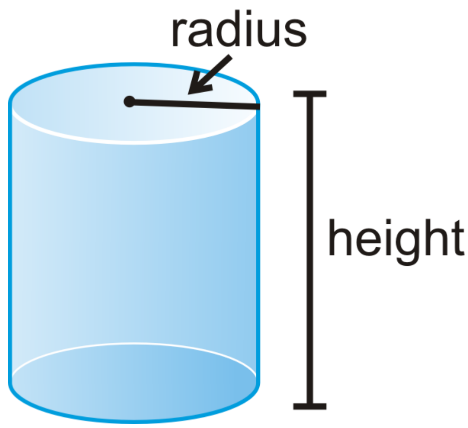
Partial differentiation

Introduction: In this section we concentrate on the mathematical term partial differentiation, so to understand this term we should have a knowledge about function of several variables. Now I am trying to clear the term function of several variables by choosing the term volume of a cylinder.



Volume of a Cylinder is $V = \pi r^2 h$ where r is the radius of the Cylinder and h is the height of the Cylinder. We observe that if r changes then no change of h in the above figure besides of this if h changes then no change of r in the above figure. That means r and h are independent variables in $V = \pi r^2 h$. So we call $V = \pi r^2 h = f(r, h)$ is a function of two independent variables r and h it means V is a function of several variables.

Function of Several variables: A function that contains more than one independent variables is called several variables function. For example $u = f(\mathrm{x,y,z}) = \mathrm{x}^2 + y^2 + z^2$ is a function of three variables x, y and z.

Partial Differentiation: The differentiation of a function $u = f(x, y)$, with respect to x **,** treating *y* as constant, is called the partial derivative of *u* with respect to *x*, and it is denoted as,

$$\frac{\partial u}{\partial x},\ u_x,\ \frac{\partial f}{\partial x},\ f_x.$$

Analytically, $\dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

when this limit exists.

Similarly, the differentiation of a function $u = f(x, y)$, with respect to y, treating x as constant, is called the partial derivative of u with respect to y, and it is denoted as, $\frac{\partial u}{\partial y}, u_y, \frac{\partial f}{\partial y}, f_y$.

Analytically, $\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$

provided this limit exists.

Successive Partial Derivatives: Consider a function $u = f(x, y)$, which has the partial derivatives $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ with respect to the independent variables x and y respectively. Also each of them may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of u, and these are denoted as,

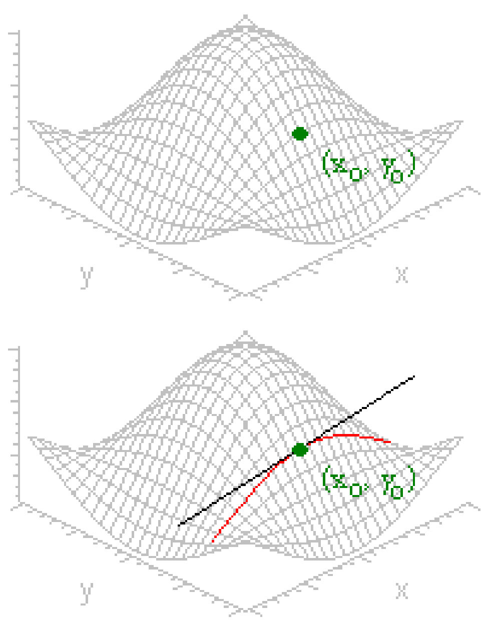
$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y \partial x}.$$

Similarly, the third order partial derivatives of u are denoted as,

$$\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial y^3}, \frac{\partial^3 u}{\partial x^2 \partial y}, \frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial^3 u}{\partial y \partial x^2} \text{ and } \frac{\partial^3 u}{\partial y^2 \partial x}.$$

and so on for higher order derivatives.

Geometrical Meaning:



Suppose the graph of z =*f*(*x*, *y*) is the surface shown in the above mentioned figure. Consider the partial derivative of z =*f*(*x*, *y*) with respect to x at a point$(x_0\ , y_0)$.Holding y as constant and varying x we trace out a curve that intersection of the surface with vertical plane $y = y_0$.

The partial derivative $f_x(x_0, y_0)$ measures the change in z per unit increase in x along this curve. That is, $f_x(x_0, y_0)$ is just the slope of the curve at (x_0, y_0).The geometrical interpretation $f_y(x_0, y_0)$ is analogous. That is $\dfrac{\partial z}{\partial x}$ means slope of tangent with x-axis of the function z=*f*(*x*, *y*) at the point (x, y, z) and $\dfrac{\partial z}{\partial y}$ means slope of tangent with y-axis of the function z=*f*(*x*, *y*) at the point (x, y, z).

Symmetric Function: A function $u = f\left(x, y\right)$ is called a symmetric function if it satisfies the condition $f\left(x, y\right) = f\left(y, x\right)$.

Example: $u = x^2 + y^2$ is a symmetric function.

Problem-01: *If* $u = x^3 + 3x^2y + 3xy^2 + y^3$ *then find* $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$ *and* $\dfrac{\partial^2 u}{\partial y \partial x}$.

Sol : *Giventhat*, $u = x^3 + 3x^2y + 3xy^2 + y^3 \cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + 3x^2y + 3xy^2 + y^3\right)$$

$$= 3x^2 + 6xy + 3y^2 + 0$$

$$\therefore\ \frac{\partial u}{\partial x} = 3x^2 + 6xy + 3y^2 \cdots\cdots(2)$$

Now differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 6xy + 3y^2\ \right)$$

$$= 6x + 6y + 0$$

$$= 6x + 6y \quad \textbf{(Ans.)}$$

Again Differentiating (1) partially with respect to *y* we get,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + 3x^2y + 3xy^2 + y^3\right)$$

$$= 0 + 3x^2 + 6xy + 3y^2$$

$$\therefore\ \frac{\partial u}{\partial y}=3x^2+6xy+3y^2\cdots\cdots(3)$$

Now differentiating (3) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(3x^2+6xy+3y^2\right)$$

$$=0+6x+6y$$

$$=6x+6y \quad \textbf{(Ans.)}$$

Again Differentiating (3) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(3x^2+6xy+3y^2\right)$$

$$=6x+6y+0$$

$$=6x+6y \quad \textbf{(Ans.)}$$

Again Differentiating (2) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(3x^2+6xy+3y^2\right)$$

$$=0+6x+6y$$

$$=6x+6y \quad \textbf{(Ans.)}$$

Problem-02: *If* $u=x^2+y^2\ln x+2e^{-x}y$ *then find* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial y^2}$.

Sol : *Giventhat*, $u=x^2+y^2\ln x+2e^{-x}y\ \cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2\ln x+2e^{-x}y\right)$$

$$=2x+\frac{y^2}{x}-2e^{-x}y\ \cdots\cdots(2)$$

Now differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(2x+\frac{y^2}{x}-2e^{-x}y\right)$$

$$= 2 - \frac{y^2}{x^2} + 2e^{-x}y \quad \textbf{(Ans.)}$$

Again Differentiating (1) partially with respect to *y* we get,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 \ln x + 2e^{-x}y\right)$$

$$= 0 + 2y\ln x + 2e^{-x}$$

$$= 2y\ln x + 2e^{-x} \cdots\cdots(3)$$

Now differentiating (3) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(2y\ln x + 2e^{-x}\right)$$

$$= 2\ln x + 0$$

$$= 2\ln x \quad \textbf{(Ans.)}$$

Problem-03: *If* $u = e^x\left(x\cos y - y\sin y\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial y^2}$. Also show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Sol : *Giventhat*, $u = e^x\left(x\cos y - y\sin y\right)$

$$= xe^x \cos y - ye^x \sin y \cdots\cdots(1)$$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(xe^x \cos y - ye^x \sin y\right)$$

$$= \cos y \frac{\partial}{\partial x}\left(xe^x\right) - y\sin y \frac{\partial}{\partial x}\left(e^x\right)$$

$$= \cos y\left(xe^x + e^x\right) - ye^x \sin y$$

$$= xe^x \cos y + e^x \cos y - ye^x \sin y \cdots\cdots(2)$$

Now differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(xe^x \cos y + e^x \cos y - ye^x \sin y\right)$$

$$= \left(xe^x + e^x\right)\cos y + e^x \cos y - ye^x \sin y$$

$$= xe^x \cos y + 2e^x \cos y - ye^x \sin y \cdots\cdots(3) \qquad \textbf{(Ans.)}$$

Again Differentiating (1) partially with respect to *y* we get,

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(xe^x\cos y-ye^x\sin y\right)$$

$$=xe^x\frac{\partial}{\partial y}(\cos y)-e^x\frac{\partial}{\partial y}(y\sin y)$$

$$=xe^x(-\sin y)-e^x(y\cos y+\sin y)$$

$$=-xe^x\sin y-ye^x\cos y-e^x\sin y\ \cdots\cdots(4)$$

Now differentiating (4) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(-xe^x\sin y-ye^x\cos y-e^x\sin y\right)$$

$$=-xe^x\cos y-e^x(-y\sin y+\cos y)-e^x\cos y$$

$$=-xe^x\cos y+e^x y\sin y-e^x\cos y-e^x\cos y$$

$$=-xe^x\cos y+e^x y\sin y-2e^x\cos y\ \cdots\cdots(5)\qquad \textbf{(Ans.)}$$

Finally, adding (3) and (5) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(xe^x\cos y+2e^x\cos y-ye^x\sin y\right)+\left(-xe^x\cos y+e^x y\sin y-2e^x\cos y\right)$$

$$=xe^x\cos y+2e^x\cos y-ye^x\sin y-xe^x\cos y+e^x y\sin y-2e^x\cos y$$

$$=0\quad \textbf{(Showed).}$$

Problem-04: *If* $u=\tan^{-1}\left(\frac{y}{x}\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Sol : *Giventhat*, $u=\tan^{-1}\left(\frac{y}{x}\right)\cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$=\frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(-\frac{y}{x^2}\right)$$

$$= -\frac{x^2}{x^2+y^2} \cdot \frac{y}{x^2}$$

$$= -\frac{y}{x^2+y^2} \cdots\cdots (2)$$

Now differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\left\{-\frac{y}{\left(x^2+y^2\right)^2} \cdot (2x+0)\right\}$$

$$= \frac{2xy}{\left(x^2+y^2\right)^2} \qquad \textbf{(Ans.)}$$

Again Differentiating (1) partially with respect to *y* we get,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2} \cdots\cdots (3)$$

Now differentiating (3) partially with respect to *y* we get,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= -\frac{x}{\left(x^2+y^2\right)^2} \cdot (0+2y)$$

$$= -\frac{2xy}{\left(x^2+y^2\right)^2} \qquad \textbf{(Ans.)}$$

Again Differentiating (2) partially with respect to y we get,

$$\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\left\{\frac{\left(x^2+y^2\right)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}\right\}$$

$$= -\left\{\frac{\left(x^2+y^2\right) - y\left(2x+0\right)}{\left(x^2+y^2\right)^2}\right\}$$

$$= -\frac{x^2+y^2-2xy}{\left(x^2+y^2\right)^2} \quad \textbf{(Ans.)}$$

Problem-05: *If* $u = x^2 + y^2 + z^2$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$

Sol : *Given that*, $u = x^2 + y^2 + z^2 \cdots\cdots(1)$

Differentiating (1) partially with respect to x we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$= 2x + 0 + 0$$

$$= 2x$$

$$\therefore x\frac{\partial u}{\partial x} = 2x^2 \cdots\cdots(2)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to y and z we get,

$$y\frac{\partial u}{\partial y} = 2y^2 \cdots\cdots(3)$$

and $$z\frac{\partial u}{\partial z} = 2z^2 \cdots\cdots(4)$$

Finally adding (2), (3) and (4) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2x^2 + 2y^2 + 2z^2$$

$$=2\left(x^2+y^2+z^2\right)$$

$$=2u \qquad \textbf{(Showed.)}$$

Problem-06: *If* $u=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$ *then show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u.$

Sol : *Given that*, $u=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\}$$

$$=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\left(2x+0+0\right)$$

$$=-x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$\therefore x\frac{\partial u}{\partial x}=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(2)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to *y* and *z* we get,

$$y\frac{\partial u}{\partial y}=-y^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(3)$$

and $$z\frac{\partial u}{\partial z}=-z^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\cdots\cdots(4)$$

Finally adding (2), (3) and (4) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}-y^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}-z^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=-\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$=-u \qquad \textbf{(Showed.)}$$

Problem-07: *If* $u=\left(x^2+y^2+z^2\right)^{\frac{1}{2}}$ *then show that* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2}{u}$.

Sol : *Given that*, $u=\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\cdots\cdots(1)$

Differentiating (1) partially with respect to *x* we get,

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right\}$$

$$=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$=\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}.\left(2x+0+0\right)$$

$$=x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\ \cdots\cdots(2)$$

Again Differentiating (2) partially with respect to *x* we get,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\right\}$$

$$=x.\left\{-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}.\left(2x+0+0\right)\right\}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$=-x^2\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}+\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$=-\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

$$=\frac{-x^2+x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$=\frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\ \cdots\cdots(3)$$

Since the given function (1) is a symmetric function, so similarly differentiating (1) with respect to *y* and *z* we get,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \cdots\cdots(4)$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \cdots\cdots(5)$$

Finally adding (3), (4) and (5) we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}+\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$= \frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$= \frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

$$= \frac{2}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

$$= \frac{2}{u}$$ **(Showed.)**

Exercise:

Problem-01: *If* $u = e^{xy}\sin x\cos x$ *then find* $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-02: *If* $u = x\cos y + y\cos x$ *then find* $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-03: *If* $u = \ln\left(x^2y + xy^2\right)$ *then find* $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x\partial y}$ *and* $\frac{\partial^2 u}{\partial y\partial x}$.

Problem-04: *If* $u = \ln\left(x^3+y^3+z^3-3xyz\right)$ *then show that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

Problem-05: *If* $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Problem-06: *If* $u = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$ *then show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Problem-07: *If* $u = z\tan^{-1}\left(\frac{y}{x}\right)$ *then show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Problem-08: *If* $u = \ln\sqrt{\left(x^2 + y^2 + z^2\right)}$ *then show that* $\left(x^2 + y^2 + z^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$

Homogeneous function: A function $f(x, y)$ is said to be homogeneous of degree *n* in the variables *x* and *y* if it can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ *or* $y^n\phi\left(\frac{x}{y}\right)$.

Alternatively, a function $f(x, y)$ is said to be homogeneous of degree *n* in the variables *x* and *y* if $f(tx, ty) = t^n f(x, y)$ for all values of *t*, where *t* is independent of *x* and *y*.

Example: $f(x, y) = \sqrt{x} + \sqrt{y}$ is a homogeneous function of degree $\frac{1}{2}$.

Euler's theorem on Homogeneous functions: If $f(x, y)$ be a homogeneous function of *x* and *y* of degree *n*, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

Problem-01: *If* $u = \tan^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Sol: *Given that,* $u = \tan^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$

$$\Rightarrow \tan u = \left(\frac{x^3 + y^3}{x + y}\right)$$

$$\Rightarrow \tan u = \frac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{x\left(1 + \frac{y}{x}\right)}$$

$$\Rightarrow \tan u = x^2\phi\left(\frac{y}{x}\right) \ (say)$$

Here, $\tan u$ is a homogeneous function of degree 2.

By Euler's Theorem we get,

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \textbf{(Showed).}$$

Problem-02: *If* $u = \ln\left(\frac{x^3+y^3}{x^2+y^2}\right)$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$

Sol : *Given that*, $u = \ln\left(\frac{x^3+y^3}{x^2+y^2}\right)$

$$\Rightarrow e^u = \frac{x^3+y^3}{x^2+y^2}$$

$$\Rightarrow e^u = \frac{x^3\left\{1+\left(\frac{y}{x}\right)^3\right\}}{x^2\left\{1+\left(\frac{y}{x}\right)^2\right\}}$$

$$\Rightarrow e^u = x\phi\left(\frac{y}{x}\right) \ (say)$$

Here, e^u is a homogeneous function of degree 1.

By Euler's Theorem we get,

$$x\frac{\partial}{\partial x}\left(e^u\right)+y\frac{\partial}{\partial y}\left(e^u\right)=1$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=1.e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1 \quad \textbf{(Showed).}$$

Problem-03: *If* $u=\sin^{-1}\left(\dfrac{x}{y+z}\right)$ *then show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0$.

Sol : *Giventhat*, $u=\sin^{-1}\left(\dfrac{x}{y+z}\right)$

$$\Rightarrow \sin u=\frac{x}{y+z}$$

$$\Rightarrow \sin u=\left(\frac{y+z}{x}\right)^{-1}$$

$$\Rightarrow \sin u=\left(\frac{y}{x}+\frac{z}{x}\right)^{-1}$$

$$\Rightarrow \sin u=x^0\phi\left(\frac{y}{x},\frac{z}{x}\right)\ (say)$$

Here, $\sin u$ is a homogeneous function of degree 0.

By Euler's Theorem we get,

$$x\frac{\partial}{\partial x}\left(\sin u\right)+y\frac{\partial}{\partial y}\left(\sin u\right)+z\frac{\partial}{\partial z}\left(\sin u\right)=0.\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z}=0$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0 \quad \textbf{(Showed).}$$

Problem-04: *If* $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ *then show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{2}\cot u=0$.

Sol : *Given that*, $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\Rightarrow \cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos u=\frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\dfrac{y}{x}}\right)}$$

$$\Rightarrow \cos u=\frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\dfrac{y}{x}}\right)}$$

$$\Rightarrow \cos u=x^{\frac{1}{2}}\phi\left(\frac{y}{x}\right)\ (say)$$

Here, $\cos u$ is a homogeneous function of degree $\dfrac{1}{2}$.

By Euler's Theorem we get,

$$x\frac{\partial}{\partial x}(\cos u)+y\frac{\partial}{\partial y}(\cos u)=\frac{1}{2}\cos u$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}+=\frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0 \quad \textbf{(Showed).}$$

Exercise:

Problem-01: *If* $u=\tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$ *then show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\sin 2u$.

Problem-02: *If* $u=\sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ *then show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

Problem-03: *If* $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Problem-04: *If* $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Problem-05: *If* $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ *then show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.