Lecture Sheet

On

Basic Concepts of Algebra

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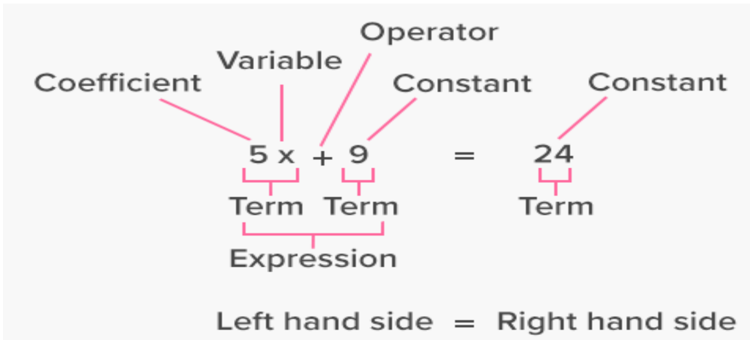
Equation: An equation is a mathematical statement, which defines the equality of two expressions connected by an equal sign "=". The most common type of equation is an algebraic equation containing one or more variables (unknown).

For instance, $2x+3y=6$ is an equation, in which $2x+3y$ and 6 are two expressions separated by an 'equal' sign. In an algebraic equation, the left-hand side is equal to the right-hand side.



Solving an equation containing variables determines the values of the variables which make the equality true and these values are called the solutions of the equation.

There are two kinds of equations such as:

1. **Identity:** An identity is an equation that is always true for any value substituted into the variable. For example, $2x+3x=5x$ is an identity.
2. **Conditional equation:** A conditional equation is an equation that is only true for particular values of the variables. For example, $2x+3=5$ is a conditional equation.

Algebraic Expression: An algebraic expression is a combination of integer constants, variables, exponents and algebraic operations such as addition, subtraction, multiplication and division. 5*x*, *x* + *y*, *x*-3 and more are examples of algebraic expression.

Binary Relation: A binary relation R between sets A and B is a subset of the Cartesian product $A\times B$. Suppose $A=\{1,2,3\}$ and $B=\{4,5,6\}$. Further suppose $R=\{(1,4),(2,5),(1,6)\}$. This is a subset of $A\times B$ so it is a binary relation between A and B.

Relation: A relation R consists of the following:

1. a set A
2. a set B
3. an open sentence P(*x*, *y*) in which P(a, b) is either true or false for any ordered pair (a, b) belonging to $A\times B$.

We then call R a relation from A to B and denote it by

$$R=(A,\ B,P(x,y)).$$

Furthermore, if P(a, b) is true we write

$$a\,R\,b$$

which reads " a is related to b". On the other hand, if P(a, b) is not true we write

$$a\,\not{R}\,b$$

which reads " a is not related to b".

Example: 1. Let $R=\left(A,\ B,P\left(x,y\right)\right)$ where $A=\{2,3,4\}$, $B=\{3,4,5,6\}$, and $P\left(x,y\right)$ reads " x divides y". Then the solution set of R is

$$R=\{(2,4),(2,6),(3,3),(3,6),(4,4)\}.$$

Example: 2. Let $R=\left(A,\ B,P\left(x,y\right)\right)$ where A is the set of men, B is the set of women, and $P\left(x,y\right)$ reads " x is the husband of y". Then R is a relation.

Example: 3. Let $R=\left(A,\ B,P\left(x,y\right)\right)$ where A is the set of men, B is the set of women, and $P\left(x,y\right)$ reads " x divides y". Then R is not a relation a relation since P(a, b) has no meaning if a is a men and b is a women.

Inverse Relation: Every relation R from A to B has an inverse relation R^{-1} from B to A which is defined by

$$R^{-1}=\{(b,a)\,|\,(a,b)\in R\}.$$

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed, i.e. permuted belong to R.

Example: Let $A=\{2,3,4\}$ and $B=\{3,4,5,6\}$. Then $R=\{(2,4),(2,6),(3,3),(3,6),(4,4)\}$ is a relation from A to B. The inverse relation of R is $R^{-1}=\{(4,2),(6,2),(3,3),(6,3),(4,4)\}$.

Reflexive Relation: Let $R=\left(A,\ A,P\left(x,y\right)\right)$ be a relation in a set A, i.e. let R be a subset of $A\times A$. Then R is called a reflexive relation if, for $a\in A$, $(a,a)\in R$. In other words, R is reflexive if every element in A is related to itself.

Example: Let $A=\{2,3,4\}$ and $R=\{(2,3),(2,4),(3,3),(3,4),(4,4)\}$. Then R is not a reflexive relation since (2, 2) does not belong to R. Notice that all ordered pairs (a, a) must belong to R in order to R to be reflexive.

Symmetric Relation: Let R be a subset of $A \times A$, i.e. let R be a relation in A. Then R is called a symmetric relation if $(a,b) \in R$ implies $(b,a) \in R$ that is, if a is related to b then b is related to a.

Example: Let $A = \{2,3,4\}$ and $R = \{(2,3),(2,4),(4,2),(3,4),(4,3)\}$. Then R is not a symmetric relation since $(2,3) \in R$ but $(3,2) \notin R$.

Anti-symmetric Relation: Let R be a subset of $A \times A$, i.e. let R be a relation in A. Then R is called an anti-symmetric relation if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$. In other words, if $a \neq b$ then possibly a is related to b or possibly b is related to a, but never both.

Example: Let $A = \{2,3,4\}$ and $R = \{(2,3),(2,4),(4,2),(3,4)\}$. Then R is not an anti-symmetric relation since $(2,4) \in R$ and $(4,2) \in R$.

Transitive Relation: Let R be a subset of $A \times A$, i.e. let R be a relation in A. Then R is called a transitive relation if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$. In other words, if a is related to b and b is related to c, then a is related to c.

Example: Let $A = \{2,3,4\}$ and $R = \{(2,3),(4,3),(3,4)\}$. Then R is not a transitive relation since $(2,3) \in R$ and $(3,4) \in R$ but $(2,4) \notin R$.

Equivalence Relation: Let R be a subset of $A \times A$, i.e. let R be a relation in A. Then R is called an equivalence relation if

1. R is reflexive, that is, for every $a \in A$, $(a,a) \in R$.
2. R is symmetric, that is, $(a,b) \in R$ implies $(b,a) \in R$.
3. R is transitive, that is, $(a,b) \in R$, and $(b,c) \in R$ implies $(a,c) \in R$.

Example: The most important example of an equivalence relation is that of “equality”. For any elements in any set:

1. $a = a$.
2. $a = b$ implies $b = a$.
3. $a = b$ and $b = c$ implies $a = c$.

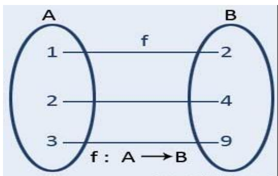
Function: If a variable *y* depends on a variable *x* in such a way that each value of *x* determines exactly one value of *y*, then *y* is called a function of *x* and it is denoted by the following symbol,

$$y = f(x)$$

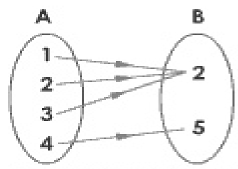
where x is independent variable and y is dependent variable. The inverse of this function is denoted by $f^{-1}(y)=x$.

Example: $y=x^2+x+1$; $y=sinx$; $y=e^x$; $y=lnx$ etc.

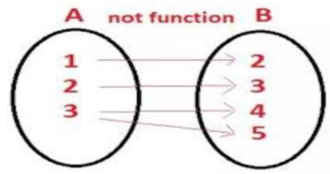
Alternatively, let A and B be two non empty sets. A mapping $f: A\to B$ is called function if each element of A is assigned by unique element of B.



Function



Function



It is not function

Types of functions: There are many types of functions. These have been discussed as:

Even function: A function $y=f(x)$ is called an even function if it satisfies the condition

$f(-x)=f(x)$.

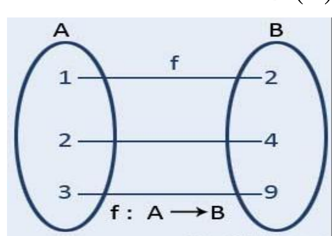
Example:$y=cosx,\ y=x^4, etc.$ are even functions.

Odd function: A function $y=f(x)$ is called an odd function if it satisfies the condition

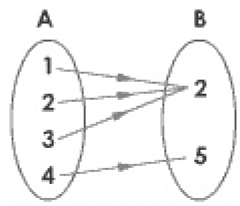
$f(-x)=-f(x)$.

Example:$y=sinx,\ y=x^3, etc.$ are odd functions.

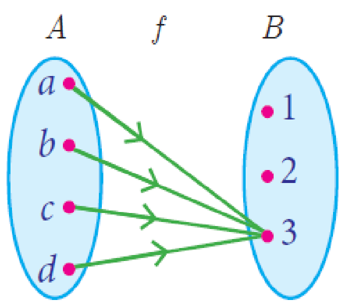
One–one Function: Let f map A into B, i.e, $f: A\to B$. Then f is called a one-one function if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, $f: A\to B$ is one-one if $f(a)=f(b)$ implies $a=b$ or, equivalently, $a\neq b$ implies $f(a)\neq f(b)$.



Onto Function: Let f be a function of A into B. Then f is called a onto function if every element of B appears as the image of at least one element of A. More briefly, $f: A\to B$ is onto function if $f(A)=B$.



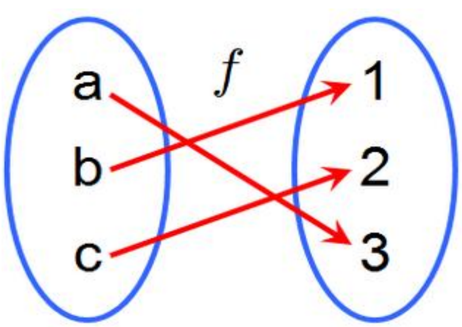
Constant Function: A function f of A into B is called a constant function if the same element in B is assigned to every element in A. More briefly, $f: A \to B$ is a constant function if the range of f consists of only one element.



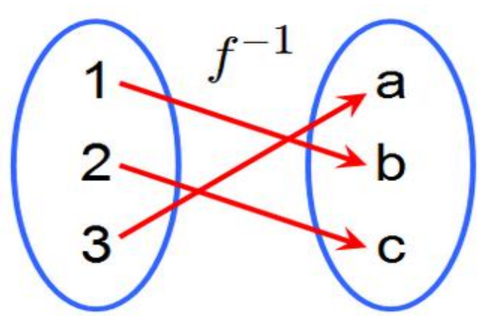
Inverse Function: Let f be a function of A into B. In general, $f^{-1}(b)$ could consist of more than one element or might even be the empty set $\varnothing$. Now if $f: A \to B$ is a one-one function and an onto function, then for each $b \in B$ the inverse $f^{-1}(b)$ will consist of a single element in A. We therefore have a rule that assigns to each $b \in B$ a unique element $f^{-1}(b)$ in A. Accordingly, f^{-1} is a function of B into A and we can write

$$f^{-1}: B \to A$$

In this situation, when $f: A \to B$ is one-one and onto, we call f^{-1} the inverse function of f.



Function, $f: A \to B$



Inverse function, $f^{-1}: B \to A$

Real number: Numbers are the foundation of Mathematics. The most common numbers in Mathematics are the real numbers. These numbers are closed under the operations addition and multiplication. The set of rational and irrational numbers is called the set of real number and it is denoted by R.

Properties of real numbers: For all real numbers a, b, and c the following properties hold:

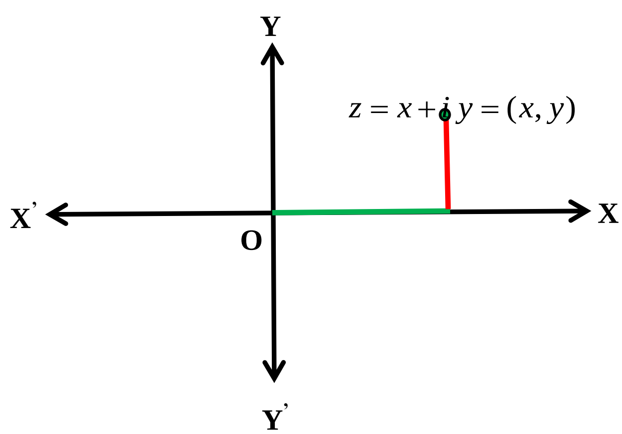
1. Closure properties: $a+b$ and ab are real numbers.
2. Commutative properties: $a+b=b+a$ and $ab=ba$.
3. Associative properties: $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
4. Distributive properties: $a(b+c)=ab+ac$.
5. Identity properties: There exists a unique real number 0 with respect to addition such that $a+0=0+a=a$.
   There exists a unique real number 1 with respect to multiplication such that $1\cdot a=a\cdot 1=a$.
6. Inverse properties: There exists a unique real number – a such that $a+(-a)=0$ and $(-a)+a=0$.
   If $a\neq 0$, there exists a unique real number $1/a$.

In general, any set which satisfy above conditions is called a field.

Complex number: Any number of the form $x+iy$, where $x\in R, y\in R$ and $i=\sqrt{-1}$, is called a complex number and it is denoted by z.

i.e. $z=x+i\,y$

In complex number z, x is the real part of z denoted by the symbol $\text{Re}(z)=x$ and y is the imaginary part of z denoted by the symbol $\text{Im}(z)=y$ and also i is called imaginary unit. Geometrically, a complex number represents a unique point in the complex plane/Argand Plane/Argand diagram/Gaussian Plane. Also geometrically, $\text{Re}(z)$ is the projection of $z=(x,y)$ on to the x axis, and $\text{Im}(z)$ is the projection of z on to the y axis.



Properties of complex numbers: If z_1, z_2, z_3 belong to the set S of complex numbers, the following properties hold.

7. z_1+z_2 *and* $z_1z_2 \in S$ Closure law
8. $z_1+z_2=z_2+z_1$ Commutative law of addition
9. $z_1+(z_2+z_3)=(z_1+z_2)+z_3$ Associative law of addition
10. $z_1z_2=z_2z_1$ Commutative law of multiplication
11. $z_1(z_2z_3)=(z_1z_2)z_3$ Associative law of multiplication
12. $z_1(z_2+z_3)=z_1z_2+z_1z_3$ Distributive law
13. $z_1+0=0+z_1=z_1$, $1\cdot z_1=z_1\cdot 1=z_1$ 0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication.
14. For any complex number z_1 there is a unique number z in S such that $z+z_1=0$; z is called the inverse of z_1 with respect to addition and is denoted by $-z_1$.
15. For any $z_1\neq 0$ there is a unique number z in S such that $z_1z=zz_1=1$; z is called the inverse of z_1 with respect to multiplication and is denoted by $1/z_1$.

In general, any set which satisfy above conditions is called a field.

Conjugate of complex number: The conjugate of a complex number $z=x+iy$ is obtained by changing the sign of y and is denoted by the symbol $\bar{z}$ i.e. $\bar{z}=x-iy$.Geometrically, the conjugate of a complex number represents the reflection or image of the complex number z about the real axis x.