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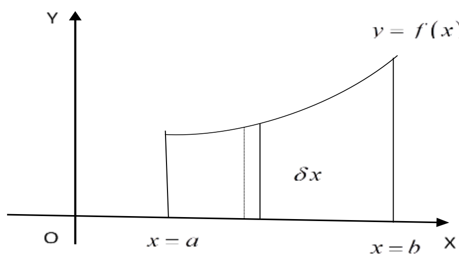
Area under curves (Quadrature)

Our concentration in this Chapter is to find the area bounded by curves with a general formula or with the help of definite integration. This process is called Quadrature.

Area formula for Cartesian equation:

(1). The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is,

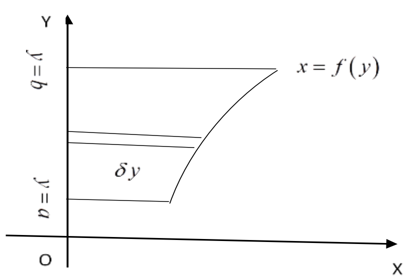
$$A = \int_a^b y\,dx$$



Where, $y = f(x)$ is a continuous single valued function and it does not change sign for $a \le x \le b$.

(2). The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$ is,

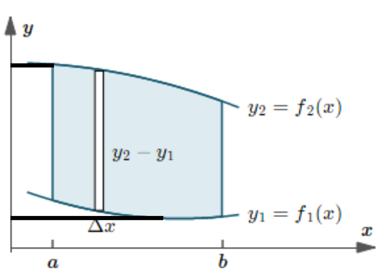
$$A = \int_a^b x\,dy$$



Where, $x = f(y)$ is a continuous single valued function and it does not change sign for $a \le y \le b$.

(3). The area bounded by two curves $y_1 = f_1(x)$, $y_2 = f_2(x)$ and two vertical lines $x = a$ & $x = b$ is

$$A = \int_a^b (y_2 - y_1)\,dx.$$



(4). The area bounded by the curve Symmetry about the x-axis is,

$$A = 2\int_0^a y\,dx$$

(5). The area bounded by the curve Symmetry about the y-axis is,

$$A = 2\int_0^a x dy$$

Symmetry about the x-axis: If all the powers of y occurring in an equation are even then it is symmetry about the x-axis. For example, $y^2 = 4ax$ is symmetry about the x-axis.

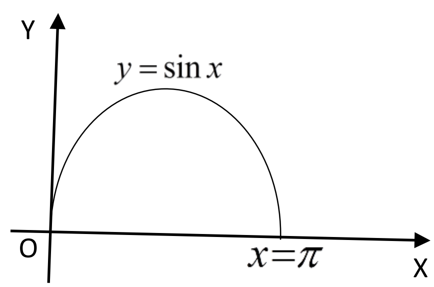
Symmetry about the y-axis: If all the powers of x occurring in an equation are even then it is symmetry about the y-axis. For example, $x^2 = 4ay$ is symmetry about the y-axis.

Mathematical Problems

Problem 01: Find the area bounded by the curve $y = \sin x$, the $x-axis$ and the straight lines $x = 0$ and $x = \pi$.

Solution: We have, $y = \sin x$ and $x = 0$; $x = \pi$.

The graph of the given curve is,



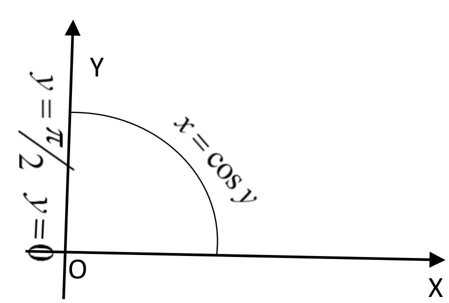
The area of the region is,

$$A = \int_0^\pi y\,dx$$
$$= \int_0^\pi \sin x\,dx$$
$$= [-\cos x]_0^\pi$$
$$= -\cos\pi - (-\cos 0)$$
$$= -(-1) + 1$$
$$= 1 + 1$$
$$= 2$$
$\therefore A = 2$ **Sq. Units.**

Problem 02: Find the area bounded by the curve $x = \cos y$, the $y-axis$ and the straight lines $y = 0$ and $y = \pi/2$.

Solution: We have, $x = \cos y$ and $y = 0$; $y = \pi/2$.

The graph of the given curve is,



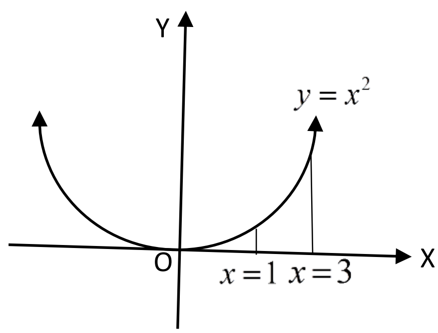
The area of the region is,

$$A = \int_0^{\pi/2} x\,dy$$
$$= \int_0^{\pi/2} \cos y\,dy$$
$$= [\sin y]_0^{\pi/2}$$
$$= \sin\left(\pi/2\right) - \sin 0$$
$$= 1 - 0$$
$$= 1$$
$\therefore A = 1$ **Sq. Units.**

Problem 03: Find the area bounded by the curve $y = x^2$, the $x-axis$ and the straight lines $x = 1$ and $x = 3$.

Solution: We have, $y = x^2$ and $x = 1$; $x = 3$.

The graph of the given curve is,



The area of the region is,

$$A = \int_1^3 y\,dx$$
$$= \int_1^3 x^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_1^3$$
$$= \frac{1}{3}\left[x^3\right]_1^3$$
$$= \frac{1}{3}\left(3^3 - 1\right)$$
$$= \frac{1}{3}(27 - 1)$$
$\therefore A = \dfrac{26}{3}$ **Sq. Units.**

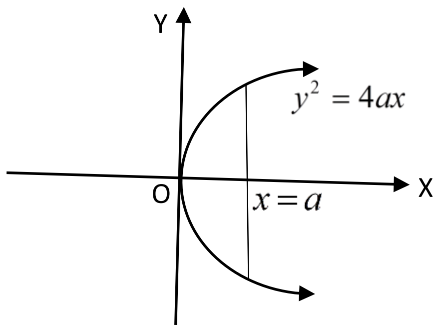
H.W:

1. Find the area bounded by the curve $x = \sin y$, the $y - axis$ and the straight lines $y = 0$ and $y = \pi$.
2. Find the area bounded by the curve $y = \sin x$, the $x - axis$ and the straight lines $x = 0$ and $x = \pi$.
3. Find the area bounded by the curve $y = x^3$, the $x - axis$ and the straight lines $x = 1$ and $x = 4$.

Problem 04: Find the area of the region bounded by the curve $y^2 = 4ax$; from $x = 0$ and $x = a$.

Solution: We have, $y^2 = 4ax$ and $x = 0$; $x = a$.

Since, only even power of *y* occurs in the given curve so the curve is symmetric about the *x*-axis.
The graph of the given curve is,



Also, the given curve can be written as,

$$y^2 = 4ax$$

$$\Rightarrow y = \pm 2\sqrt{ax}$$

The area of the region is,

$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a 2\sqrt{ax}\,dx \qquad [Neglecting\ negative\ sign]$$

$$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8\sqrt{a}}{3}\left[x^{3/2}\right]_0^a$$

$$= \frac{8\sqrt{a}}{3}\left(a^{3/2} - 0\right)$$

$$= \frac{8\sqrt{a} \times a^{3/2}}{3}$$

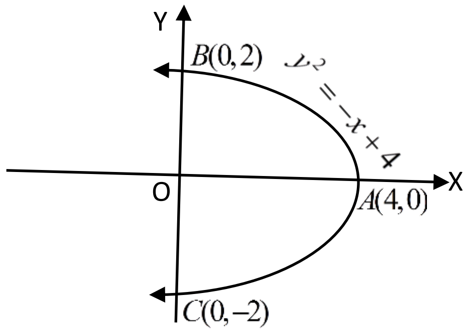
$$= \frac{8a^2}{3}$$

$$\therefore A=\frac{8a^2}{3} \qquad \textbf{Sq. Units.}$$

Problem 05: Find the area of the region bounded by the curve $y^2=-x+4$ and y-axis.

Solution: We have, $y^2=-x+4 \quad \cdots\cdots\cdots(1)$

Since, only even power of y occurs in the given curve so the curve is symmetric about the x-axis.
The graph of the given curve is,



Putting $y=0$ in (1) then we have $x=4$, so the vertex is at $A(4,0)$.

Also putting $x=0$ in (1) then we have $y=\pm 2$. So the curve crosses the y-axis at $B(0,2)$ and $C(0,-2)$. The given curve can be written as,

$$y^2=-x+4$$
$$\Rightarrow y=\pm\sqrt{4-x}$$

The area of the region is,

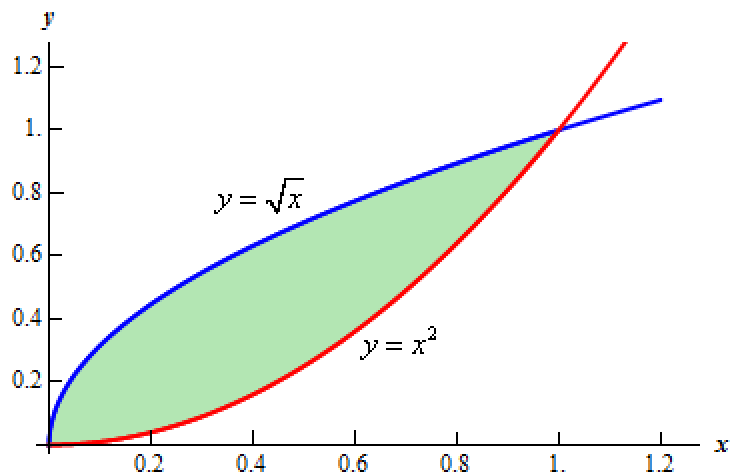
$$A=2\int_0^4 y\,dx$$
$$=2\int_0^4 \sqrt{4-x}\,dx \qquad [Neglecting\ negative\ sign]$$
$$=2\left[\frac{(4-x)^{3/2}}{(-1).3/2}\right]_0^4$$
$$=-2.\frac{2}{3}.\left[(4-x)^{3/2}\right]_0^4$$
$$=-\frac{4}{3}.\left[(4-4)^{3/2}-(4-0)^{3/2}\right]$$
$$=-\frac{4}{3}.\left[0-(4)^{3/2}\right]$$
$$=\frac{4}{3}.\left(2^2\right)^{3/2}$$
$$=\frac{4}{3}.\,2^3$$
$$\therefore A=\frac{32}{3} \qquad \textbf{Sq. Units.}$$

H.W:

1. Find the area of the region bounded by the curve $x^2 = 4ay$; from $y = 0$ and $y = a$.
2. Find the area of the region bounded by the curve $y^2 = 12x$; from $x = 0$ and $x = 3$.

Problem 06: Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

Solution: The equation of the given curves are $y = x^2$ and $y = \sqrt{x}$.
The graph of the given curves are as follows:



We have

$$y = x^2 \text{ and } y = \sqrt{x}$$

Now,

$$x^2 = \sqrt{x}$$

$$or,\ \left(x^2\right)^2 = \left(\sqrt{x}\right)^2 \qquad \text{[Squaring both sides]}$$

$$or,\ x^4 = x$$

$$or,\ x^4 - x = 0$$

$$or,\ x\left(x^3 - 1\right) = 0$$

Therefore, $x = 0$ and $x^3 - 1 = 0$

$$\Rightarrow (x-1)(x^2 + x + 1) = 0$$

$$\therefore\ x - 1 = 0 \qquad or\ \ x^2 + x + 1 = 0$$

$$\Rightarrow x = 1 \qquad or\ x^2 + x + 1 = 0$$

$$or,\ x = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$or,\ x = \frac{-1 \pm \sqrt{-3}}{2}$$

For real $x = 0\,\&\,1$ we get respectively $y = 0\,\&\,1$

Therefore, the given curves intersect each other in two point at $(0,0)$ and $(1,1)$.

In the question, $a=0$, $b=1, y_2=\sqrt{x}$ *and* $y_1=x^2$.
So, the area of the region is,

$$A=\int_a^b (y_2-y_1)\,dx$$

$$=\int_0^1 \left(\sqrt{x}-x^2\right)dx$$

$$=\int_0^1 \sqrt{x}\,dx-\int_0^1 x^2\,dx$$

$$=\int_0^1 x^{\frac{1}{2}}\,dx-\int_0^1 x^2\,dx$$

$$=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1-\left[\frac{x^3}{3}\right]_0^1$$

$$=\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1-\frac{1}{3}\left[x^3\right]_0^1$$

$$=\frac{2}{3}(1-0)-\frac{1}{3}(1-0)$$

$$=\frac{2}{3}(1)-\frac{1}{3}(1)$$

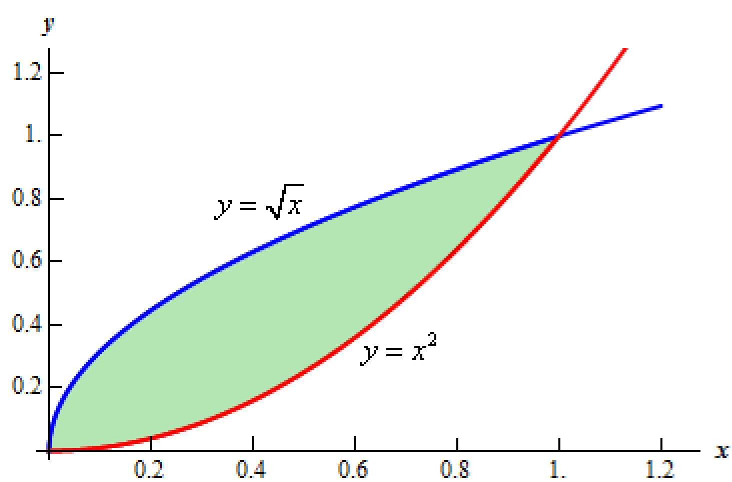
$$=\frac{2}{3}-\frac{1}{3}$$

$$\therefore A=\frac{1}{3}$$ **Sq. Units.** **(As desired)**

Second Process:

Solution:

The equation of the given curves are $y=x^2$ and $y=\sqrt{x}$.
The graph of the given curves are as follows:



We have

$$y = x^2 \text{ and } y = \sqrt{x}$$

Now,

$$x^2 = \sqrt{x}$$

$$or,\ \left(x^2\right)^2 = \left(\sqrt{x}\right)^2 \qquad \text{[Squaring both sides]}$$

$$or,\ x^4 = x$$

$$or,\ x^4 - x = 0$$

$$or,\ x\left(x^3 - 1\right) = 0$$

Therefore, $x = 0$ and $x^3 - 1 = 0$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\therefore\ x - 1 = 0 \qquad or\ \ x^2 + x + 1 = 0$$

$$\Rightarrow x = 1 \qquad or\ x^2 + x + 1 = 0$$

$$or,\ x = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$or,\ x = \frac{-1 \pm \sqrt{-3}}{2}$$

For real $x = 0\&1$ we get respectively $y = 0\&1$

Therefore, the given curves intersect each other in two point at $(0,0)$ and $(1,1)$.

In the question, $c = 0$, $d = 1, x_2 = \sqrt{y}$ and $x_1 = y^2$.

So, the area of the region is,

$$A = \int_c^d \left(x_2 - x_1\right) dy$$

$$= \int_0^1 \left(\sqrt{y} - y^2\right) dy$$

$$= \int_0^1 \sqrt{y}\, dy - \int_0^1 y^2\, dy$$

$$= \int_0^1 y^{\frac{1}{2}}\, dy - \int_0^1 y^2\, dy$$

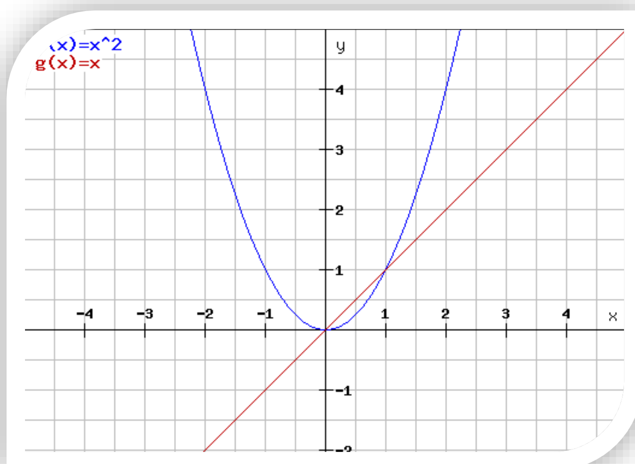
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^1 - \frac{1}{3}\left[y^3\right]_0^1$$

$$=\frac{2}{3}(1-0)-\frac{1}{3}(1-0)$$
$$=\frac{2}{3}(1)-\frac{1}{3}(1)$$
$$=\frac{2}{3}-\frac{1}{3}$$
$\therefore\ A=\frac{1}{3}$ **Sq. Units.** **(As desired)**

Note: It is noted that when we calculate the area with respect to x or y axis we get the same result.

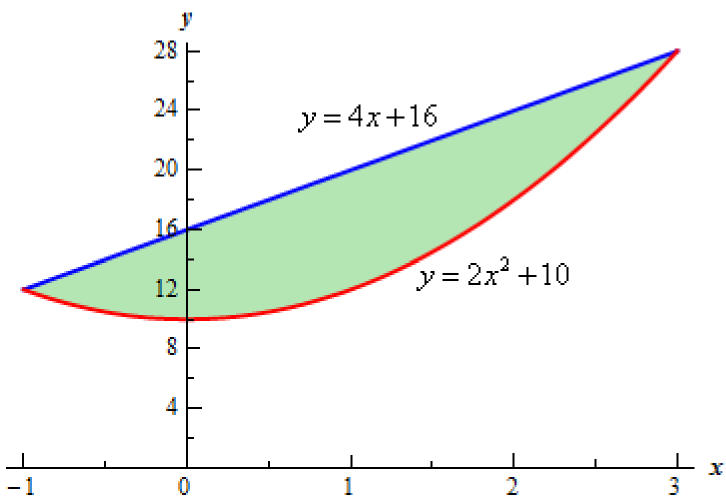
Problem 07: Obtain the area of the region enclosed by $y=x^2$ and $y=x$.
Solution:
The equation of the given curve is $y=x^2$ and also the straight line is $y=x$.
The graph of the given curve and straight lines are as follows:



We have
$$y=x^2 \text{ and } y=x$$
Now,
$$x=x^2$$
$$or,\ x^2-x=0$$
$$or,\ x(x-1)=0$$
Therefore, $x=0 \quad or \quad (x-1)=0$
$x=0 \quad or \quad x=1$
For real $x=0\,\&\,1$ we get respectively $y=0\,\&\,1$.
Therefore, the given point of intersection of curve and straight lines are $(0,0)$ and $(1,1)$.
In the question , $a=0$, $b=1, y_2=x \quad and \quad y_1=x^2$.
So, the area of the region is

$$A = \int_a^b (y_2 - y_1)\,dx$$
$$= \int_0^1 (x - x^2)\,dx$$
$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$
$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{2}\left[x^2\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1$$
$$= \frac{1}{2}(1-0) - \frac{1}{3}(1-0)$$
$$= \frac{1}{2}(1) - \frac{1}{3}(1)$$
$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$ **Sq. Units. (As desired)**

Problem 08: Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.
Solution:
The equation of the given curve is $y = 2x^2 + 10$ and also the straight line is $y = 4x + 16$.
The graph of the given curve and straight lines are as follows:



We have

$$y = 2x^2 + 10 \text{ and } y = 4x + 16$$

Now,

$$4x + 16 = 2x^2 + 10$$
$$or,\ 2x + 8 = x^2 + 5$$
$$or,\ x^2 + 5 - 2x - 8 = 0$$
$$or,\ x^2 - 2x - 3 = 0$$

$$or,\ x^2-3x+x-3=0$$

$$or,\ x(x-3)+1(x-3)=0$$

$$or,\ (x-3)(x+1)=0$$

Therefore $(x-3)=0$ $\quad or \quad$ $(x+1)=0$

$x=3$ $\quad or \quad$ $x=-1$

For $x=-1\,\&\,3$ we get respectively $y=12\,\&\,28$.
Therefore, the given point of intersection of curve and straight lines are $(-1,12)$ and $(3,28)$.
In the question, $a=-1$, $b=3, y_2=4x+16$ $\ and \ $ $y_1=2x^2+10$.

So, the area of the region is

$$A=\int_a^b (y_2-y_1)\,dx$$

$$=\int_{-1}^{3}\left(4x+16-2x^2-10\right)dx$$

$$=\int_{-1}^{3}\left(4x-2x^2+6\right)dx$$

$$=\int_{-1}^{3}\left(4x-2x^2+6\right)dx$$

$$=4\int_{-1}^{3}x\,dx-2\int_{-1}^{3}x^2\,dx+6\int_{-1}^{3}dx$$

$$=4\left[\frac{x^2}{2}\right]_{-1}^{3}-2\left[\frac{x^3}{3}\right]_{-1}^{3}+6[x]_{-1}^{3}$$

$$=4\left[\frac{x^2}{2}\right]_{-1}^{3}-2\left[\frac{x^3}{3}\right]_{-1}^{3}+6[x]_{-1}^{3}$$

$$=2\left[x^2\right]_{-1}^{3}-\frac{2}{3}\left[x^3\right]_{-1}^{3}+6[x]_{-1}^{3}$$

$$=2(9-1)-\frac{2}{3}(27+1)+6(3+1)$$

$$=16-\frac{56}{3}+24$$

$$=\frac{48-56+72}{3}$$

$$\therefore A=\frac{64}{3}$$ **Sq. Units. (As desired)**