Reduction formula: A reduction formula is a formula which connects a given integral with another integral which is of the same type but of a lower degree or of a lower order, or is otherwise easier to evaluate.

Working rule to find a reduction formula for $\int \tan^n x dx$:

1. Separate $\tan^2 x$ from $\tan^n x$. Thus $\tan^n x = \tan^{n-2} x \tan^2 x$.
2. Replace $\tan^2 x$ by $\sec^2 x - 1$.
3. Integrate only the first integral on R.H.S. using $\int [f(x)]^n d\{f(x)\} = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$.

Problem-01: Find a reduction formula of the integral $I_n = \int \tan^n x dx$ and hence find I_5.

Sol^n: Given that

$$I_n = \int \tan^n x dx$$

$$= \int \tan^{n-2} x \tan^2 x dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$= \int \tan^{n-2} x d(\tan x) - \int \tan^{n-2} x dx$$

$$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \qquad \cdots(1)$$

Putting $n = 5$ in (1), we get

$$I_5 = \frac{\tan^4 x}{4} - I_3$$

$$= \frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - I_1\right)$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \int \tan x dx$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2}{2} + \ln(\sec x) + c \qquad \textbf{\textit{Ans.}}$$

Problem-02: If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ then show that $I_n = \frac{1}{n-1} - I_{n-2}$ and hence find the value of

$$\int_0^{\frac{\pi}{4}} \tan^6 x dx.$$

Sol^n : Given that

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x d\left(\tan x\right) - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$= \frac{\tan^{n-1}\left(\frac{\pi}{4}\right) - \tan^{n-1}(0)}{n-1} - I_{n-2}$$

$$= \frac{1-0}{n-1} - I_{n-2}$$

$$\therefore I_n = \frac{1}{n-1} - I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n = \frac{1}{n-1} - I_{n-2} \qquad \cdots(1)$$

Putting $n = 6$ in (1), we get

$$I_6 = \frac{1}{5} - I_4$$

$$=\frac{1}{5}-\left(\frac{1}{3}-I_2\right)$$

$$=\frac{1}{5}-\frac{1}{3}+I_2$$

$$=\frac{1}{5}-\frac{1}{3}+\left(\frac{1}{1}-I_0\right)$$

$$=\frac{1}{5}-\frac{1}{3}+1-\int_0^{\frac{\pi}{4}}\tan^0 xdx$$

$$=\frac{3-5+15}{15}-\int_0^{\frac{\pi}{4}}dx$$

$$=\frac{13}{15}-\left[x\right]_0^{\frac{\pi}{4}}$$

$$=\frac{13}{15}-\left[\frac{\pi}{4}-0\right]$$

$$\therefore\ \int_0^{\frac{\pi}{4}}\tan^6 xdx=\frac{13}{15}-\frac{\pi}{4}\qquad \textbf{\textit{Ans.}}$$

Working rule to find a reduction formula for $\int\sec^n xdx$:

1. Separate $\sec^2 x$ from $\sec^n x$. Thus $\sec^n x=\sec^{n-2}x\sec^2 x$.
2. Integrate by parts taking $\sec^{n-2}x$ as first function.
3. Replace $\tan^2 x$ by $\sec^2 x-1$.
4. Transpose the given integral to L.H.S.

Problem-03: Establish the reduction formula for $I_n=\int\sec^n xdx$ and hence find $\int\sec^6 xdx$.

Sol^n: Given that

$$I_n=\int\sec^n xdx$$

$$=\int\sec^{n-2}x\sec^2 xdx$$

$$=\sec^{n-2}x\int\sec^2 xdx-\int\left\{\frac{d}{dx}\left(\sec^{n-2}x\right)\int\sec^2 xdx\right\}dx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\left(\sec^{n-3}x\cdot\sec x\tan x\right)\tan xdx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\sec^{n-2}x\tan^2 xdx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\sec^{n-2}x\left(\sec^2 x-1\right)dx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\sec^{n}xdx+(n-2)\int\sec^{n-2}xdx$$

$$=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2}$$

$$or,\ I_n+(n-2)I_n=\sec^{n-2}x\tan x+(n-2)I_{n-2}$$

$$or,\ (n-1)I_n=\sec^{n-2}x\tan x+(n-2)I_{n-2}$$

$$\therefore I_n=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{(n-2)}{n-1}I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{(n-2)}{n-1}I_{n-2}\qquad\cdots(1)$$

Putting $n=6$ in (1), we get

$$I_6=\frac{\sec^4 x\tan x}{5}+\frac{4}{5}I_4$$

$$=\frac{\sec^4 x\tan x}{5}+\frac{4}{5}\left[\frac{\sec^2 x\tan x}{3}+\frac{2}{3}I_2\right]$$

$$=\frac{\sec^4 x\tan x}{5}+\frac{4\sec^2 x\tan x}{15}+\frac{8}{15}I_2$$

$$=\frac{\sec^4 x\tan x}{5}+\frac{4\sec^2 x\tan x}{15}+\frac{8}{15}\left[\frac{\sec^0 x\tan x}{1}+\frac{0}{1}I_0\right]$$

$$=\frac{\sec^4 x\tan x}{5}+\frac{4\sec^2 x\tan x}{15}+\frac{8}{15}\tan x+c$$ ***Ans.***

Working rule to find a reduction formula for $\int\sin^n xdx$:

1. Separate $\sin x$ from $\sin^n x$. Thus $\sin^n x=\sin^{n-1}x\sin x$.
2. Integrate by parts taking $\sin^{n-1}x$ as first function.
3. Replace $\cos^2 x$ by $1-\sin^2 x$.
4. Transpose the given integral to L.H.S.

The working rule to find a reduction formula for $\int\cos^n xdx$ is same as above.

Problem-04: Establish the reduction formula for $I_n=\int\sin^n xdx$ and hence find $\int_0^{\frac{\pi}{2}}\sin^4 xdx$.

Sol^n : Given that

$$I_n=\int\sin^n xdx$$

$$=\int \sin^{n-1}x\sin x dx$$

$$=\sin^{n-1}x\int \sin x dx-\int\left\{\frac{d}{dx}\left(\sin^{n-1}x\right)\int \sin x dx\right\}dx$$

$$=-\sin^{n-1}x\cos x-(n-1)\int \sin^{n-2}x\cdot\cos x(-\cos x)dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cdot\cos^2 x dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\left(1-\sin^2 x\right)dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x dx-(n-1)\int \sin^{n}x dx$$

$$=-\sin^{n-1}x\cos x+(n-1)I_{n-2}-(n-1)I_n$$

$$or,\ I_n+(n-1)I_n=(n-1)I_{n-2}-\sin^{n-1}x\cos x$$

$$or,\ nI_n=(n-1)I_{n-2}-\sin^{n-1}x\cos x$$

$$\therefore\ \ I_n=\frac{(n-1)}{n}I_{n-2}-\frac{\sin^{n-1}x\cos x}{n}$$

which is the required reduction formula.

2nd part: We have

$$I_n=\frac{(n-1)}{n}I_{n-2}-\frac{\sin^{n-1}x\cos x}{n}\qquad\cdots(1)$$

Putting $n=4$ in (1), we get

$$I_4=\frac{3}{4}I_2-\frac{\sin^3 x\cos x}{4}$$

$$=\frac{3}{4}\left(\frac{1}{2}I_0-\frac{\sin x\cos x}{2}\right)-\frac{\sin^3 x\cos x}{4}$$

$$=\frac{3}{8}I_0-\frac{3\sin x\cos x}{8}-\frac{\sin^3 x\cos x}{4}$$

$$=\frac{3}{8}\int \sin^0 x dx-\frac{3\sin x\cos x}{8}-\frac{\sin^3 x\cos x}{4}$$

$$=\frac{3}{8}\int dx-\frac{3\sin x\cos x}{8}-\frac{\sin^3 x\cos x}{4}$$

$$=\frac{3}{8}x-\frac{3\sin x\cos x}{8}-\frac{\sin^3 x\cos x}{4}+c$$

$$or,\ I_4=\int \sin^4 x dx=\frac{3}{8}x-\frac{3\sin x\cos x}{8}-\frac{\sin^3 x\cos x}{4}+c$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^4 x dx = \left[\frac{3}{8}x - \frac{3\sin x\cos x}{8} - \frac{\sin^3 x\cos x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\cdot\frac{\pi}{2} - 0$$

$$= \frac{3\pi}{16} \qquad \textit{Ans.}$$

Working rule to find a reduction formula for $\int x^n \sin mx dx$ or $\int x^n \cos mx dx$**:** Integrate twice by parts taking x^n as first function.

Problem-05: Establish the reduction formula for $I_n = \int x^n \sin x dx$ and hence find $\int_0^{\frac{\pi}{6}} x^4 \sin x dx$.

Sol^n : Given that

$$I_n = \int x^n \sin x dx$$

$$= x^n \int \sin x dx - \int\left\{\frac{d}{dx}\left(x^n\right)\int \sin x dx\right\}dx$$

$$= -x^n \cos x - n\int x^{n-1}\left(-\cos x\right)dx$$

$$= -x^n \cos x + n\int x^{n-1}\cos x dx$$

$$= -x^n \cos x + n\left[x^{n-1}\int \cos x dx - \int\left\{\frac{d}{dx}\left(x^{n-1}\right)\int \cos x dx\right\}dx\right]$$

$$= -x^n \cos x + n\left[x^{n-1}\sin x - (n-1)\int x^{n-2}\sin x dx\right]$$

$$\therefore\ I_n = -x^n \cos x + nx^{n-1}\sin x - n(n-1)I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n = -x^n \cos x + nx^{n-1}\sin x - n(n-1)I_{n-2} \qquad \cdots(1)$$

Putting $n=4$ in (1), we get

$$I_4 = -x^4\cos x + 4x^3\sin x - 12I_2$$

$$= -x^4\cos x + 4x^3\sin x - 12\left(-x^2\cos x + 2x\sin x - 2I_0\right)$$

$$= -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x + 24\int x^0 \sin x dx$$

$$= -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c$$

$$or,\ I_4 = \int x^4\sin x dx = -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c$$

$$\therefore \int_0^{\frac{\pi}{6}} x^4 \sin x dx = \left[-x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x\right]_0^{\frac{\pi}{6}}$$

$$= -\left(\frac{\pi}{6}\right)^4 \cdot \frac{\sqrt{3}}{2} + 4\left(\frac{\pi}{6}\right)^3 \cdot \frac{1}{2} + 12\left(\frac{\pi}{6}\right)^2 \cdot \frac{\sqrt{3}}{2} - 24\left(\frac{\pi}{6}\right) \cdot \frac{1}{2} - 24 \cdot \frac{\sqrt{3}}{2} - 0$$

$$= -\frac{\sqrt{3}}{2}\left(\frac{\pi}{6}\right)^4 + 2\left(\frac{\pi}{6}\right)^3 + 6\sqrt{3}\left(\frac{\pi}{6}\right)^2 - 2\pi - 12\sqrt{3} \qquad \textbf{\textit{Ans.}}$$

Working rule to find a reduction formula for $\int x \sin^n x dx$ or $\int x \cos^n x dx$:

1. Write $x \sin^n x$ as $\left(x \sin^{n-1} x\right) \sin x$ or $x \cos^n x$ as $\left(x \cos^{n-1} x\right) \cos x$.
2. Integrate by parts taking $x \sin^{n-1} x$ or $x \cos^{n-1} x$ as first function and replace $\cos^2 x$ by $1 - \sin^2 x$ and $\sin^2 x$ by $1 - \cos^2 x$.
3. Transpose and solve for the given integral.

Problem-06: If $I_n = \int_0^{\frac{\pi}{2}} x \sin^n x dx$ then prove that $I_n = \frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$, $n > 1$. Hence prove that

$$I_5 = \frac{149}{225}.$$

Sol^n : Given that

$$I_n = \int_0^{\frac{\pi}{2}} x \sin^n x dx$$

$$= \int_0^{\frac{\pi}{2}} \left(x \sin^{n-1} x\right) \sin x dx$$

$$= x \sin^{n-1} x \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \left\{\frac{d}{dx}\left(x \sin^{n-1} x\right) \int \sin x dx\right\} dx$$

$$= \left[-x \sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \left\{\sin^{n-1} x + (n-1) x \sin^{n-2} x \cos x\right\} \cos x dx$$

$$= 0 + \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cos x dx + (n-1) \int_0^{\frac{\pi}{2}} x \sin^{n-2} x \cos^2 x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(\sin x) + (n-1) \int_0^{\frac{\pi}{2}} x \sin^{n-2} x \left(1 - \sin^2 x\right) dx$$

$$=\left[\frac{\sin^n x}{n}\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}} x\sin^{n-2} x dx-(n-1)\int_0^{\frac{\pi}{2}} x\sin^n x dx$$

$$=\frac{1}{n}+(n-1)I_{n-2}-(n-1)I_n$$

$$or,\ I_n+(n-1)I_n=\frac{1}{n}+(n-1)I_{n-2}$$

$$or,\ nI_n=\frac{1}{n}+(n-1)I_{n-2}$$

$$\therefore\quad I_n=\frac{1}{n^2}+\frac{n-1}{n}I_{n-2}$$ **(Proved)**

which is the required reduction formula.

2nd part: We have

$$I_n=\frac{1}{n^2}+\frac{n-1}{n}I_{n-2}\qquad\cdots(1)$$

Putting $n=5$ in (1), we get

$$I_5=\frac{1}{25}+\frac{4}{5}I_3$$

$$=\frac{1}{25}+\frac{4}{5}\left(\frac{1}{9}+\frac{2}{3}I_1\right)$$

$$=\frac{1}{25}+\frac{4}{45}+\frac{8}{15}\int_0^{\frac{\pi}{2}} x\sin x dx$$

$$=\frac{1}{25}+\frac{4}{45}+\frac{8}{15}\left[-x\cos x+\sin x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{25}+\frac{4}{45}+\frac{8}{15}\cdot 1$$

$$=\frac{9+20+120}{225}$$

$$=\frac{149}{225}.$$ **(Proved)**

Working rule to find a reduction formula for $\int\sin^m x\cos^n x dx$:

Problem-07: Find the reduction formula for $I_{m,n}=\int\sin^m x\cos^n x dx$.

Sol^n : Given that

$$I_{m,n}=\int\sin^m x\cos^n x dx$$

$$=\int\left(\cos^n x\sin x\right)\sin^{m-1}xdx$$

$$=\sin^{m-1}x\int\cos^n x\sin xdx-\int\left\{\frac{d}{dx}\left(\sin^{m-1}x\right)\int\cos^n x\sin xdx\right\}dx$$

$$=-\sin^{m-1}x\int\cos^n xd\left(\cos x\right)+\left(m-1\right)\int\left\{\sin^{m-2}x\cos x\int\cos^n xd\left(\cos x\right)\right\}dx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\left(m-1\right)\int\left\{\sin^{m-2}x\cos x\cdot\frac{\cos^{n+1}x}{n+1}\right\}dx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x\cdot\cos^2 xdx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n x\left(1-\sin^2 x\right)dx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}\int\sin^{m-2}x\cos^n xdx-\frac{m-1}{n+1}\int\sin^m x\cos^n xdx$$

$$=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n}-\frac{m-1}{n+1}I_{m,n}$$

$$or,\ I_{m,n}+\frac{m-1}{n+1}I_{m,n}=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n}$$

$$or,\ \frac{m+n}{n+1}I_{m,n}=-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}+\frac{m-1}{n+1}I_{m-2,n}$$

$$\therefore\ I_{m,n}=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}I_{m-2,n}$$

which is the required reduction formula.

Problem-08: If $I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m x\cos^n xdx$, then show that $I_{m,n}=\frac{n-1}{m+n}I_{m,n-2}$.

Sol^n : Given that

$$I_{m,n}=\int_0^{\frac{\pi}{2}}\sin^m x\cos^n xdx$$

$$=\int_0^{\frac{\pi}{2}}\left(\sin^m x\cos x\right)\cos^{n-1}xdx$$

$$=\cos^{n-1}x\int_0^{\frac{\pi}{2}}\sin^m x\cos xdx-\int_0^{\frac{\pi}{2}}\left\{\frac{d}{dx}\left(\cos^{n-1}x\right)\int\sin^m x\cos xdx\right\}dx$$

$$=\cos^{n-1}x\int_0^{\frac{\pi}{2}}\sin^m xd\left(\sin x\right)+\left(n-1\right)\int_0^{\frac{\pi}{2}}\left\{\cos^{n-2}x\sin x\int\sin^m xd\left(\sin x\right)\right\}dx$$

$$=\left[\frac{\cos^{n-1}x\sin^{m+1}x}{m+1}\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\left\{\cos^{n-2}x\sin x\cdot\frac{\sin^{m+1}x}{m+1}\right\}dx$$

$$=0+\frac{n-1}{m+1}\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^{m+2}xdx$$

$$=\frac{n-1}{m+1}\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^m x\sin^2 xdx$$

$$=\frac{n-1}{m+1}\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^m x\left(1-\cos^2 x\right)dx$$

$$=\frac{n-1}{m+1}\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^m xdx-\frac{n-1}{m+1}\int_0^{\frac{\pi}{2}}\cos^n x\sin^m xdx$$

$$=\frac{n-1}{m+1}I_{m,n-2}-\frac{n-1}{m+1}I_{m,n}$$

$$or,\ I_{m,n}+\frac{n-1}{m+1}I_{m,n}=\frac{n-1}{m+1}I_{m,n-2}$$

$$or,\ \frac{m+n}{m+1}I_{m,n}=\frac{n-1}{m+1}I_{m,n-2}$$

$$\therefore\ I_{m,n}=\frac{n-1}{m+n}I_{m,n-2}$$ **(Showed)**

Working rule to find a reduction formula for $\int\cos^m x\cos nxdx$ or $\int\cos^m x\sin nxdx$ or $\int\sin^m x\cos nxdx$ or $\int\sin^m x\sin nxdx$ **:**

Problem-09: Find the reduction formula for $I_{m,n}=\int\cos^m x\cos nxdx$ and hence find the value of

$$\int_0^{\frac{\pi}{2}}\cos^3 x\cos 2xdx.$$

Sol^n : Given that

$$I_{m,n}=\int\cos^m x\cos nxdx$$

$$=\cos^m x\int\cos nxdx-\int\left\{\frac{d}{dx}\left(\cos^m x\right)\int\cos nxdx\right\}dx$$

$$=\frac{1}{n}\cos^m x\sin nx+\frac{m}{n}\int\cos^{m-1}x\sin x\sin nxdx$$

$$=\frac{1}{n}\cos^m x\sin nx+\frac{m}{n}\int\cos^{m-1}x\sin nx\sin xdx$$

$$=\frac{1}{n}\cos^m x\sin nx+\frac{m}{n}\int\cos^{m-1}x\left\{\cos(n-1)x-\cos nx\cos x\right\}dx$$

$$= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)xdx - \frac{m}{n}\int \cos^{m-1} x \cos nx \cos xdx$$

$$= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)xdx - \frac{m}{n}\int \cos^{m} x \cos nxdx$$

$$= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}I_{m-1,n-1} - \frac{m}{n}I_{m,n}$$

$$or,\ I_{m,n} + \frac{m}{n}I_{m,n} = \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}I_{m-1,n-1}$$

$$or,\ \frac{m+n}{n}I_{m,n} = \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}I_{m-1,n-1}$$

$$\therefore\ I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}I_{m-1,n-1}$$

which is the required reduction formula.

2nd part: We have

$$I_{m,n} = \int \cos^m x \cos nxdx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}I_{m-1,n-1}$$

$$\therefore \int \cos^3 x \cos 2xdx = \frac{\cos^3 x \sin 2x}{5} + \frac{3}{5}I_{2,1}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^3 x \cos 2xdx = \left[\frac{\cos^3 x \sin 2x}{5}\right]_0^{\frac{\pi}{2}} + \frac{3}{5}\int_0^{\frac{\pi}{2}} \cos^2 x \cos xdx$$

$$= 0 + \frac{3}{5}\left\{\left[\frac{\cos^2 x \sin x}{3}\right]_0^{\frac{\pi}{2}} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^1 x \cos 0 \cdot xdx\right\}$$

$$= \frac{3}{5}\left\{0 + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos xdx\right\}$$

$$= \frac{2}{5}[\sin x]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{5} \qquad \textbf{\textit{Ans.}}$$

Problem-10: Find the reduction formula for $I_{m,n} = \int \cos^m x \sin nxdx$ and deduce the value of

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3xdx.$$

Sol^n : Given that

$$I_{m,n} = \int \cos^m x \sin nxdx$$

$$= \cos^m x \int \sin nx dx - \int \left\{ \frac{d}{dx}\left(\cos^m x\right) \int \sin nx dx \right\} dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x \cos nx \sin x dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x \left\{ \sin nx \cos x - \sin\left(n-1\right)x \right\} dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x \sin nx \cos x dx + \frac{m}{n}\int \cos^{m-1} x \sin\left(n-1\right) x dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m} x \sin nx dx + \frac{m}{n}\int \cos^{m-1} x \sin\left(n-1\right) x dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$or,\ I_{m,n} + \frac{m}{n} I_{m,n} = -\frac{1}{n}\cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

$$or,\ \frac{m+n}{n} I_{m,n} = -\frac{1}{n}\cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore\ I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the required reduction formula.

2nd part: We have

$$I_{m,n} = \int \cos^m x \sin nx dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

$$\therefore \int \cos^5 x \sin 3x dx = -\frac{\cos^5 x \cos 3x}{5+3} + \frac{5}{5+3} I_{4,2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x dx = \left[-\frac{\cos^5 x \cos 3x}{5+3} \right]_0^{\frac{\pi}{2}} + \frac{5}{5+3}\int_0^{\frac{\pi}{2}} \cos^4 x \sin 2x dx$$

$$= \frac{1}{8} + \frac{5}{8}\left\{ \left[-\frac{\cos^4 x \cos 2x}{4+2} \right]_0^{\frac{\pi}{2}} + \frac{4}{4+2}\int_0^{\frac{\pi}{2}} \cos^3 x \sin x dx \right\}$$

$$= \frac{1}{8} + \frac{5}{8}\left\{ \frac{1}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^3 x \sin x dx \right\}$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{12}\left\{ \left[-\frac{\cos^3 x \cos x}{3+1} \right]_0^{\frac{\pi}{2}} + \frac{3}{3+1}\int_0^{\frac{\pi}{2}} \cos^2 x \sin 0 \cdot x dx \right\}$$

$$=\frac{1}{8}+\frac{5}{48}+\frac{5}{12}\left(\frac{1}{4}+\frac{3}{4}\cdot 0\right)$$

$$=\frac{1}{8}+\frac{5}{48}+\frac{5}{48}$$

$$=\frac{6+5+5}{48}$$

$$=\frac{16}{48}$$

$$=\frac{1}{3} \qquad \textbf{\textit{Ans.}}$$

Problem-11: Obtain a reduction formula for $I_n=\int_0^1 x^n \tan^{-1} x dx$ and hence evaluate $\int_0^1 x^4 \tan^{-1} x dx$.

Sol^n : Given that

$$I_n=\int_0^1 x^n \tan^{-1} x dx$$

$$=\tan^{-1} x\int_0^1 x^n dx-\int_0^1\left\{\frac{d}{dx}\left(\tan^{-1} x\right)\int x^n dx\right\}dx$$

$$=\left[\frac{x^{n+1}\tan^{-1} x}{n+1}\right]_0^1-\frac{1}{n+1}\int_0^1\frac{x^{n+1}}{1+x^2}dx$$

$$=\frac{\pi}{4(n+1)}-\frac{1}{n+1}\int_0^1\frac{x^{n+1}}{1+x^2}dx$$

$$or,\ (n+1)I_n=\frac{\pi}{4}-\int_0^1\frac{x^{n+1}}{1+x^2}dx \qquad \cdots(1)$$

Replacing n by $n-2$ in (1), we get

$$or,\ (n-1)I_{n-2}=\frac{\pi}{4}-\int_0^1\frac{x^{n-1}}{1+x^2}dx \qquad \cdots(2)$$

Adding (1) and (2), we get

$$(n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\int_0^1\frac{x^{n+1}}{1+x^2}dx-\int_0^1\frac{x^{n-1}}{1+x^2}dx$$

$$or,\ (n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\int_0^1\frac{x^{n+1}+x^{n-1}}{1+x^2}dx$$

$$or,\ (n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\int_0^1\frac{x^n\cdot x+x^n\cdot\frac{1}{x}}{1+x^2}dx$$

$$or,\ (n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\int_0^1\frac{x^n\cdot(x^2+1)}{x(1+x^2)}dx$$

$$or,\ (n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\int_0^1 x^{n-1}dx$$

$$or,\ (n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\left[\frac{x^n}{n}\right]_0^1$$

$$or,\ (n+1)I_n=\frac{\pi}{2}-\frac{1}{n}-(n-1)I_{n-2}$$

$$\therefore\ I_n=\frac{\pi}{2(n+1)}-\frac{1}{n(n+1)}-\frac{(n-1)}{(n+1)}I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n=\frac{\pi}{2(n+1)}-\frac{1}{n(n+1)}-\frac{(n-1)}{(n+1)}I_{n-2} \qquad \cdots(3)$$

Putting $n=4$ in (3), we get

$$I_4=\frac{\pi}{2(4+1)}-\frac{1}{4(4+1)}-\frac{(4-1)}{(4+1)}I_2$$

$$=\frac{\pi}{10}-\frac{1}{20}-\frac{3}{5}\left[\frac{\pi}{2(2+1)}-\frac{1}{2(2+1)}-\frac{(2-1)}{(2+1)}I_0\right]$$

$$=\frac{\pi}{10}-\frac{1}{20}-\frac{3}{5}\left[\frac{\pi}{6}-\frac{1}{6}-\frac{1}{3}\int_0^1\tan^{-1}x\,dx\right]$$

$$=\frac{\pi}{10}-\frac{1}{20}-\frac{\pi}{10}+\frac{1}{10}+\frac{1}{5}\left[x\tan^{-1}x-\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$$

$$=\frac{1}{20}+\frac{1}{5}\left(\frac{\pi}{4}-\frac{1}{2}\ln 2\right)$$

$$=\frac{1}{20}+\frac{\pi}{20}-\frac{1}{10}\ln 2 \qquad \textbf{\textit{Ans.}}$$

Problem-12: Obtain a reduction formula for $I_n=\int_0^{\frac{\pi}{2}}e^{2x}\sin^n x\,dx,\ n>1$ and hence evaluate I_3.

Sol^n : Given that

$$I_n=\int_0^{\frac{\pi}{2}}e^{2x}\sin^n x\,dx$$

$$=\sin^n x\int_0^{\frac{\pi}{2}} e^{2x}dx-\int_0^{\frac{\pi}{2}}\left\{\frac{d}{dx}\left(\sin^n x\right)\int e^{2x}dx\right\}dx$$

$$=\left[\frac{e^{2x}\sin^n x}{2}\right]_0^{\frac{\pi}{2}}-\frac{n}{2}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-1}x\cos x dx$$

$$=\frac{e^{\pi}}{2}-\frac{n}{2}\left[\sin^{n-1}x\cos x\int_0^{\frac{\pi}{2}} e^{2x}dx-\int_0^{\frac{\pi}{2}}\left\{\frac{d}{dx}\left(\sin^{n-1}x\cos x\right)\int e^{2x}dx\right\}dx\right]$$

$$=\frac{e^{\pi}}{2}-\frac{n}{2}\left[\frac{e^{2x}\sin^{n-1}x\cos x}{2}\right]_0^{\frac{\pi}{2}}+\frac{n}{4}\int_0^{\frac{\pi}{2}}\left\{(n-1)e^{2x}\sin^{n-2}x\cos^2 x-e^{2x}\sin^{n-1}x\sin x\right\}dx$$

$$=\frac{e^{\pi}}{2}-\frac{n}{2}\cdot 0+\frac{n(n-1)}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-2}x\cos^2 x dx-\frac{n}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-1}x\sin x dx$$

$$=\frac{e^{\pi}}{2}-\frac{n}{2}\cdot 0+\frac{n(n-1)}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-2}x\left(1-\sin^2 x\right)dx-\frac{n}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^n x dx$$

$$=\frac{e^{\pi}}{2}+\frac{n(n-1)}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-2}x dx-\frac{n(n-1)}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^n x dx-\frac{n}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^n x dx$$

$$=\frac{e^{\pi}}{2}+\frac{n(n-1)}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^{n-2}x dx-\frac{n^2}{4}\int_0^{\frac{\pi}{2}} e^{2x}\sin^n x dx$$

$$=\frac{e^{\pi}}{2}+\frac{n(n-1)}{4}I_{n-2}-\frac{n^2}{4}I_n$$

$$or,\ I_n+\frac{n^2}{4}I_n=\frac{e^{\pi}}{2}+\frac{n(n-1)}{4}I_{n-2}$$

$$or,\ \frac{n^2+4}{4}I_n=\frac{e^{\pi}}{2}+\frac{n(n-1)}{4}I_{n-2}$$

$$or,\ \left(n^2+4\right)I_n=2e^{\pi}+n(n-1)I_{n-2}$$

$$\therefore\ I_n=\frac{2}{n^2+4}e^{\pi}+\frac{n(n-1)}{n^2+4}I_{n-2}$$

which is the required reduction formula.

2nd part: We have

$$I_n=\frac{2}{n^2+4}e^{\pi}+\frac{n(n-1)}{n^2+4}I_{n-2}\qquad\cdots(1)$$

Putting $n=3$ in (1), we get

$$I_3=\frac{2}{3^2+4}e^{\pi}+\frac{3(3-1)}{3^2+4}I_1$$

$$=\frac{2}{13}e^{\pi}+\frac{6}{13}\int_0^{\frac{\pi}{2}}e^{2x}\sin x dx$$

$$=\frac{2}{13}e^{\pi}+\frac{6}{13}\left[\frac{e^{2x}\left(2\sin x-\cos x\right)}{2^2+1^2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{2}{13}e^{\pi}+\frac{6}{65}\left(2e^{\pi}+1\right)$$

$$=\frac{2}{13}e^{\pi}+\frac{12e^{\pi}}{65}+\frac{6}{65}$$

$$=\frac{22e^{\pi}}{65}+\frac{6}{65}$$ ***Ans.***

Problem-13: Obtain a reduction formula for $I_n=\int x^n e^{ax}dx$.

Sol^n : Given that

$$I_n=\int x^n e^{ax}dx$$

$$=x^n\int e^{ax}dx-\int\left\{\frac{d}{dx}\left(x^n\right)\int e^{ax}dx\right\}dx$$

$$=\frac{1}{a}x^n e^{ax}-\frac{n}{a}\int x^{n-1}e^{ax}dx$$

$$\therefore\ I_n=\frac{1}{a}x^n e^{ax}-\frac{n}{a}I_{n-1}$$

which is the required reduction formula.

Problem-14: Obtain a reduction formula for $I_{m,n}=\int\frac{1}{x^m\left(a+bx\right)^n}dx$.

Sol^n : Given that

$$I_{m,n}=\int\frac{1}{x^m\left(a+bx\right)^n}dx$$

$$=\frac{1}{\left(a+bx\right)^n}\int\frac{1}{x^m}dx-\int\left\{\frac{d}{dx}\left(\frac{1}{\left(a+bx\right)^n}\right)\int\frac{1}{x^m}dx\right\}dx$$

$$=\frac{-1}{\left(m-1\right)x^{m-1}\left(a+bx\right)^n}-\frac{n}{m-1}\int\frac{b}{x^{m-1}\left(a+bx\right)^{n+1}}dx$$

$$=\frac{-1}{\left(m-1\right)x^{m-1}\left(a+bx\right)^n}-\frac{n}{m-1}\int\frac{bx}{x^{m}\left(a+bx\right)^{n+1}}dx$$

$$=\frac{-1}{\left(m-1\right)x^{m-1}\left(a+bx\right)^n}-\frac{n}{m-1}\int\frac{a+bx-a}{x^{m}\left(a+bx\right)^{n+1}}dx$$

$$=\frac{-1}{(m-1)x^{m-1}(a+bx)^n}-\frac{n}{m-1}\int\frac{dx}{x^m(a+bx)^n}+\frac{an}{m-1}\int\frac{dx}{x^m(a+bx)^{n+1}}$$

$$=\frac{-1}{(m-1)x^{m-1}(a+bx)^n}-\frac{n}{m-1}I_{m,n}+\frac{an}{m-1}I_{m,n+1}$$

$$or,\ I_{m,n}+\frac{n}{m-1}I_{m,n}=\frac{-1}{(m-1)x^{m-1}(a+bx)^n}+\frac{an}{m-1}I_{m,n+1}$$

$$or,\ \frac{m+n-1}{m-1}I_{m,n}=\frac{-1}{(m-1)x^{m-1}(a+bx)^n}+\frac{an}{m-1}I_{m,n+1}$$

$$\therefore\ (m+n-1)I_{m,n}=\frac{-1}{x^{m-1}(a+bx)^n}+anI_{m,n+1}$$

Replacing n by $n-1$, we get

$$(m+n-2)I_{m,n-1}=\frac{-1}{x^{m-1}(a+bx)^{n-1}}+a(n-1)I_{m,n}$$

$$or,\ a(n-1)I_{m,n}=\frac{1}{x^{m-1}(a+bx)^{n-1}}+(m+n-2)I_{m,n-1}$$

$$\therefore\ I_{m,n}=\frac{1}{a(n-1)x^{m-1}(a+bx)^{n-1}}+\frac{(m+n-2)}{a(n-1)}I_{m,n-1}$$

which is the required reduction formula.

Theorem-01: State and prove Wallis's formula.

OR

Evaluate $\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ and $\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ for all positive odd and even integral values of n.

Statement: If n is positive integer, then

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & when\ n\ is\ even.\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, & when\ n\ is\ odd.\end{cases}$$

Proof: Let $I_n=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$$=\int_0^{\frac{\pi}{2}}\left\{\cos\left(\frac{\pi}{2}-x\right)\right\}^n dx$$

$$=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

$$=\int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x dx$$

$$=\sin^{n-1} x \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \left\{ \frac{d}{dx}\left(\sin^{n-1} x\right) \int \sin x dx \right\} dx$$

$$=\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$$

$$=0+(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\left(1-\sin^2 x\right) dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$=(n-1) I_{n-2} - (n-1) I_n$$

$$or,\ I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$or,\ nI_n = (n-1) I_{n-2}$$

$$\therefore\ \ I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$\cdots \quad \cdots \quad \cdots$$

$$I_6 = \frac{5}{6} I_4$$

$$I_4 = \frac{3}{4} I_2$$

$$I_2 = \frac{1}{2} I_0$$

$$\therefore\ \ I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2} I_0, \quad \text{when } n \text{ is even}$$

$$or,\ \ I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^0 x dx$$

$$or,\ \ I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$or,\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}}$$

$$or,\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

Again,

$$I_{n-2}=\frac{n-3}{n-2}I_{n-4}$$

$$I_{n-4}=\frac{n-5}{n-4}I_{n-6}$$

$$\cdots\quad\cdots\quad\cdots$$

$$I_5=\frac{4}{5}I_3$$

$$I_3=\frac{2}{3}I_1$$

$$\therefore\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}I_1,\quad \text{when } n \text{ is odd}$$

$$or,\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}}\sin x\,dx$$

$$or,\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$or,\quad I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

Thus $\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when } n \text{ is even.}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, & \text{when } n \text{ is odd.}\end{cases}$ **(Proved)**

Assignment:

Problem-01: If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ then show that $n\left(I_{n+1}+I_{n-1}\right)=1$ and hence find the value of

$$\int_0^{\frac{\pi}{4}}\tan^8 x\,dx.$$

Problem-02: If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ then show that $I_n+I_{n+2}=\frac{1}{n+1}$ and hence find the value of

$$\int_0^{\frac{\pi}{4}}\tan^5 x\,dx.$$

Problem-03: Establish the reduction formula for $I_n = \int \cos^n x dx$ and hence find $\int_0^{\frac{\pi}{2}} \cos^5 x dx$.

Problem-04: If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx$ then show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ and hence find the value of $\int_0^{\frac{\pi}{2}} x^5 \sin x dx$.

Problem-05: If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x dx$ then for $n \ge 2$ show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$.

Problem-06: Find the reduction formula for $I_{m,n} = \int \sin^m x \cos^n x dx$.

Problem-07: If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

Problem-08: Find the reduction formula for $I_{n,n} = \int \sin^n x \cos^n x dx$.

Problem-09: If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$, then show that $I_n = \frac{n-1}{n} I_{n-2}$ and hence find I_n, when n is odd.

Problem-10: If n is a positive integer, then prove that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx = \frac{\pi}{2^{n+1}}$.

Problem-11: If $I_n = \int x^n \sqrt{a-x}\, dx$, then show that $(2n+3)I_n = 2anI_{n-1} - 2x^n (a-x)^{\frac{3}{2}}$.

Problem-12: If $I_n = \int (x^2 + a^2)^n dx$, then show that $I_n = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$.

Problem-13: If $I_n = \int \frac{dx}{(x^2+a^2)^n}$, then show that $I_n = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{(2n-3)}{2(n-1)a^2} I_{n-1}$.