Lecture Sheet

On

Analytical/Coordinate Geometry

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Coordinate Geometry

Coordinate geometry is the branch of mathematics in which geometry is studied with the help of algebra. The great France mathematician and philosopher Rene Descartes (1596-1650) first applied algebraic formulae in geometry. A system of geometry where the position of points on the plane is described using two numbers called an ordered pair of numbers or coordinates. The first element of the ordered pair represents the distance of that point on x-axis called abscissa and second element on y-axis called ordinate. This abscissa and ordinate makes coordinates of that point.

René Descartes

The method of describing the location of points in this way was proposed by the French mathematician René Descartes (1596 - 1650). (Pronounced "day CART"). He proposed further that curves and lines could be described by equations using this technique, thus being the first to link algebra and geometry. In honor of his work, the coordinates of a point are often referred to as its Cartesian coordinates and the coordinate plane as the Cartesian Coordinate Plane and coordinate geometry sometimes called Cartesian geometry.

Classification of geometry :

Generally geometry is classified into two following categories such as:

1'Ordinary Geometry:

Ordinary geometry is a branch of mathematics concerned with questions of shape, size and relative position of figures and the properties of space. A mathematician who works in the field of geometry is called a geometer. Geometry arose independently in a number of early cultures as a body of practical knowledge concerning lengths, areas, and volumes etc. The father of ordinary Geometry is Euclid who is famous Greek mathematician for his writing elements book which has 13 volumes.

2′Coordinate/Analytical Geometry:

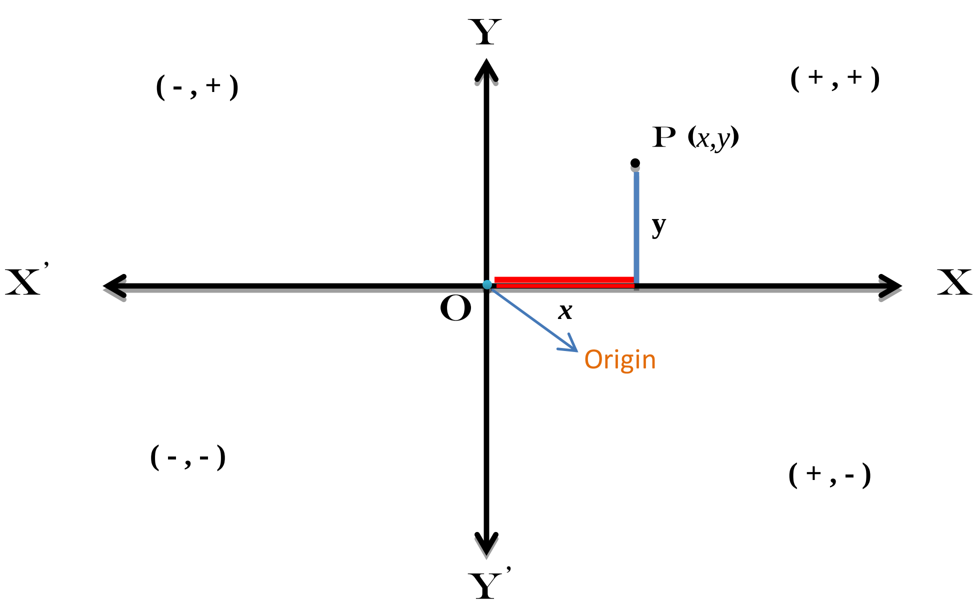
Analytical geometry is the study of geometry using a coordinate system and the principles of algebra and analysis. It is also known as Coordinate geometry or Cartesian geometry. The father of Coordinate/Analytical geometry is French mathematician René Descartes.

Coordinates: The sets of numbers which describe the position along a line, on a surface or in space are called coordinates. So the coordinates of a point in two dimensions are a pair of numbers that define its exact location.

Coordinates Systems in Two Dimensions: There are two systems in two dimensions such as:

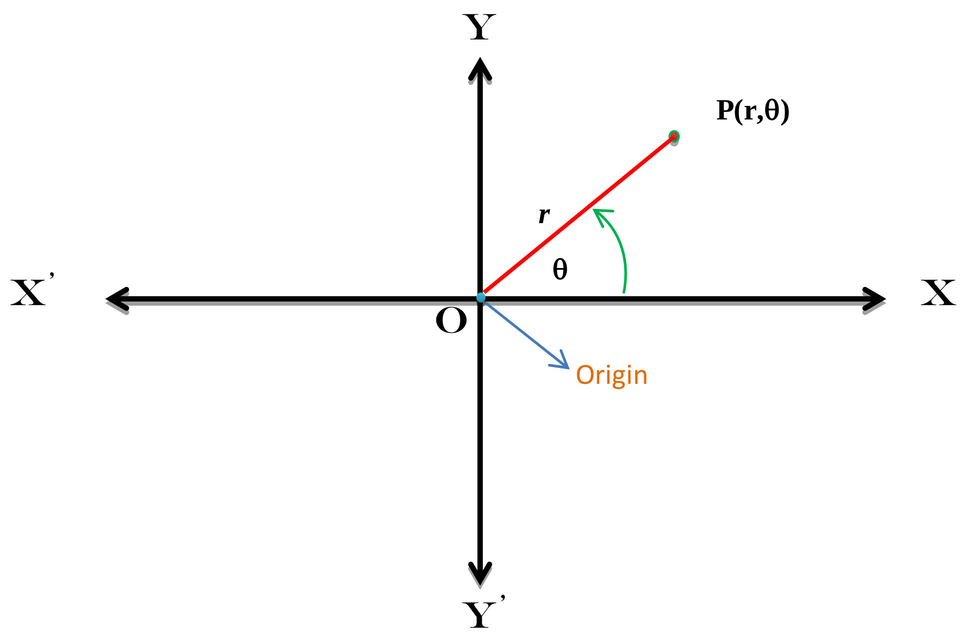
1. **Cartesian/Rectangular Coordinates System:**

In two dimensional planes, to locate the position of a point, there is needed two coordinates. The mathematician Rene Descartes first considered two perpendicular intersecting fixed straight lines in a plane as axes of coordinates. These two straight lines are named as rectangular axes and intersecting point as the origin denotes by the symbol **O** and the symbol **O** comes from the first letter of the word origin. The Cartesian coordinate system is named after the inventor name Rene Descartes. The Cartesian coordinate system is also known as Rectangular coordinates system as the axes are in right angle. In Cartesian coordinate system, the position of a point is measured by the distance on both axes. First one is on x-axis called abscissa or x-coordinate denoted by the symbol ***x*** and second one is on y-axis called ordinate or y-coordinate denoted by the symbol *y*. We express the coordinate of a point **P** in Cartesian plane by the ordered pair *P(x,y) or P(abscissa, ordinate)*.



The horizontal line XOX' is called x-axis and the vertical line YOY' is called y-axis. Both axes divide the whole plane into four parts called Quadrants. Four Quadrants XOY, $X'OY$, $X'OY'$ and XOY' are called anti-clock-wisely 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quadrant respectively. The coordinate of the origin is **O**(0,0) because all distances measured considering origin as starting point.

2. Polar coordinates System:

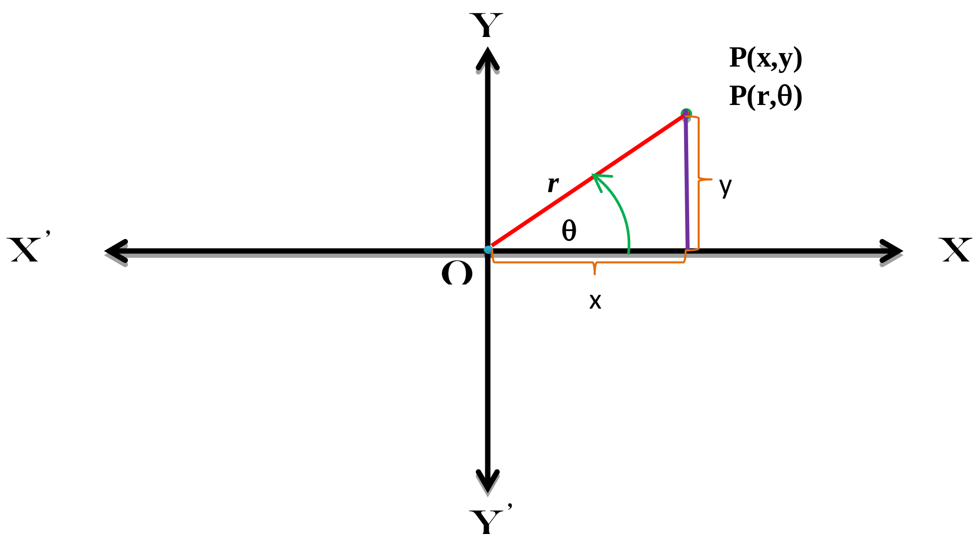


In similar manner of Cartesian system for fixing or locating the point **P** in a plane we take a fixed point **O** called the pole and a fixed straight line **OX** called the initial line. Joining line of the points P and O is called radius vector and length of radius vector **OP** = r and the positive angle $\angle XOP = \theta$ is called vectorial angle. It is sometimes convenient to locate the position of a point **P** in terms of its distances from a fixed point and its direction from a fixed line through this point. So the coordinates of locating points in this system is called Polar coordinates system. The coordinates of point in this system are called Polar coordinates. The polar coordinates of the point P are expressed as $P(r,\theta)$.In expressing the polar coordinates of the point P the radius vector is always written as the first coordinate. It is considered positive if measured from the pole along the line bounding the vectorial angle otherwise negative. In a polar system the same point has an infinite number of representations and it is the demerits of polar coordinate system to Cartesian system.

For example: The point P has the coordinates $(r,\theta),(-r,\theta+\pi),(-r,\theta-\pi),(r,\theta-2\pi)$ etc.

Relation between Cartesian and Polar Coordinate System:

Suppose that the coordinates of the point P in Cartesian system is $P(x,y)$ and in Polar system is $P(r,\theta)$. Our target here to establish the relation between two coordinates systems. From the triangle with the help of trigonometry we can find the relation between the Cartesian system & the polar system.



From the pictorial triangle
We get,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \qquad \text{and} \qquad \sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

Again, applying Pythagorean Theorem from geometry we have a relation

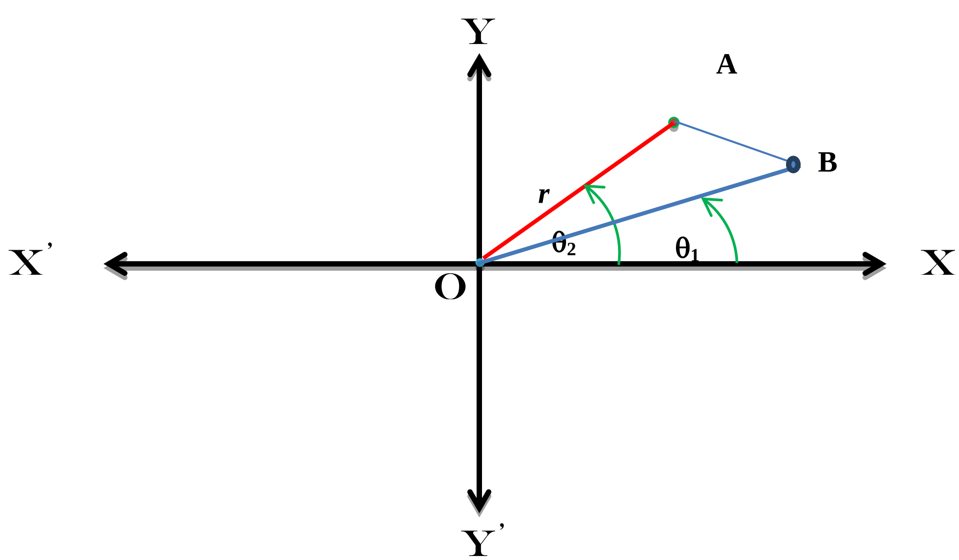
$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2} \quad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

Therefore the relations are:

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}} \quad \text{and} \quad \boxed{\begin{array}{l} r = \sqrt{x^2 + y^2} \\ \tan\theta = \dfrac{y}{x} \end{array}}$$

Distance between two Points:

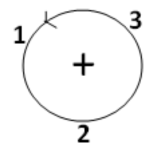
If the coordinates of two points in Cartesian system are $A(x_1, y_1)$ and $B(x_2, y_2)$, then the distance between two points is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Again,
If the coordinates of two points in Polar system are $A(r_1,\theta_1)$ and $B(r_2,\theta_2)$, then the distance between two points is $AB=\sqrt{r_1^{\ 2}+r_2^{\ 2}-2r_1r_2\cos(\theta_2-\theta_1)}$.

Area of a triangle:
If the coordinates of the vertices of the triangle in Cartesian system are respectively $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$, then the area of the triangle is

$$\Delta ABC=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{Sq. Units}$$



$$\boldsymbol{\Delta ABC=\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}}$$

Area by Sarrus Diagram Method

$$\triangle ABC=\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$=\frac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)\}$$

An alternative representation of Sarrus diagram Method:

$$\triangle ABC=\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$=\frac{1}{2}\{(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)\}$$

Note:

1. In the determinant point must be choose in anti-clock-wise direction.
2. In Sarrus Diagram method first point is repeated.
3. Applying Sarrus Diagram Method we find the area of a polygon.

Again,

If the coordinates of the vertices of the triangle in Cartesian system are respectively $A(r_1,\theta_1)$, $B(r_2,\theta_2)$ and $C(r_3,\theta_3)$, then the area of the triangle is

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} r_1\cos\theta_1 & r_1\sin\theta_1 & 1 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 1 \\ r_3\cos\theta_3 & r_3\sin\theta_3 & 1 \end{vmatrix} \quad \text{Sq. Units}$$

$$\Delta ABC = \frac{1}{2}\left\{r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)-r_1r_3\sin(\theta_3-\theta_1)\right\}$$

Mathematical Problem

Problem-01: Determine the polar coordinates of the point $\left(\sqrt{3},-1\right)$.

Solution:

We have given $(x,y)=\left(\sqrt{3},-1\right)$.

Therefore $x=\sqrt{3}$ and $y=-1$

We know that,

$$r=\sqrt{x^2+y^2}=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2$$

And

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)=\tan^{-1}\tan\left(2\pi-\frac{\pi}{6}\right)=\frac{12\pi-\pi}{6}=\frac{11\pi}{6}$$

Therefore, the polar form of the given point is $(r,\theta)=\left(2,\frac{11\pi}{6}\right)$ or $(r,\theta)=\left(2,-\frac{\pi}{6}\right)$.

Problem-02: Determine the Cartesian coordinates of the point $\left(2\sqrt{2},\frac{5\pi}{4}\right)$.

Solution:

We have given $(r,\theta)=\left(2\sqrt{2},\frac{5\pi}{4}\right)$.

Therefore $r=2\sqrt{2}$ and $\theta=\frac{5\pi}{4}$

We know that,

$$x=r\cos\theta=2\sqrt{2}\cos\frac{5\pi}{4}=2\sqrt{2}\cos\left(\pi+\frac{\pi}{4}\right)=-2\sqrt{2}\cos\frac{\pi}{4}=-2\sqrt{2}\times\frac{1}{\sqrt{2}}=-2$$

And

$$y=r\sin\theta=2\sqrt{2}\sin\frac{5\pi}{4}=2\sqrt{2}\sin\left(\pi+\frac{\pi}{4}\right)=-2\sqrt{2}\sin\frac{\pi}{4}=-2\sqrt{2}\times\frac{1}{\sqrt{2}}=-2$$

Therefore the Cartesian form of the given point is $(x, y) = (-2, -2)$.

H.W:

1. Convert the following points to the polar form:

i) $\left(1, -\sqrt{3}\right)$ **ii)** $\left(2\sqrt{3}, -2\right)$ **iii)** $(-1, -1)$ **iv)** $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

v) $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ **vi)** $\left(a, a\sqrt{3}\right)$ **vii)** $\left(\frac{5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2}\right)$

2. Convert the following points to the Cartesian form:

i) $\left(3, \frac{\pi}{6}\right)$ **ii)** $\left(5, -\frac{\pi}{4}\right)$ **iii)** $\left(-2a, -\frac{2\pi}{3}\right)$ **iv)** $\left(2, \frac{2\pi}{3}\right)$ **v)** $\left(1, \frac{\pi}{6}\right)$

vi) $\left(2, \frac{\pi}{3}\right)$ **vii)** $\left(3, \frac{\pi}{2}\right)$ **viii)** $\left(2, -\frac{\pi}{6}\right)$ **ix)** $\left(4, \frac{11\pi}{6}\right)$ **x)** $\left(\sqrt{2}, \frac{5\pi}{4}\right)$

Problem-03: Transform the equation $x^3 + y^3 = 3axy$ to Polar equation.

Solution:

Given Cartesian Equation is, $x^3 + y^3 = 3axy$.

We have

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Now replacing x and y from the above equation by its values given equation reduces to the following form

$$r^3\cos^3\theta + r^3\sin^3\theta = 3a.r\cos\theta.r\sin\theta$$

$$or,\ r^3\left(\cos^3\theta + \sin^3\theta\right) = 3a.r^2\cos\theta\sin\theta$$

$$or,\ r\left(\cos^3\theta + \sin^3\theta\right) = 3a\cos\theta\sin\theta$$

$$or,\ r\left(\cos^3\theta + \sin^3\theta\right) = \frac{3}{2}a \times 2\cos\theta\sin\theta$$

$$or,\ r\left(\cos^3\theta + \sin^3\theta\right) = \frac{3}{2}a\sin 2\theta \textbf{ (As desired)}$$

Problem-04: If x, y be related by means of the equation $\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right)$, find the corresponding relation between r and θ.

Solution:

Given Cartesian Equation is, $\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right)$.

We have

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$ the above relation is transformed into the following form

$$\left(r^2\cos^2\theta + r^2\sin^2\theta\right)^2 = a^2\left(r^2\cos^2\theta - r^2\sin^2\theta\right)$$

$$or,\ r^4\left(\cos^2\theta + \sin^2\theta\right)^2 = a^2r^2\left(\cos^2\theta - \sin^2\theta\right)$$

$$or,\ r^2(1)^2 = a^2\left(\cos^2\theta - \sin^2\theta\right)$$

$$or,\ r^2 = a^2\cos 2\theta \textbf{ (As desired)}$$

H.W:

Convert the followings to the polar form:

1. $x^2\left(x^2+y^2\right)=a^2\left(x^2-y^2\right)$
2. $xy^3+yx^3=a^2$
3. $\left(x^2-y^2\right)^2-y^2(2x+1)+2x^3=0$
4. $4\left(x^3-y^3\right)-3(x-y)\left(x^2+y^2\right)=5kxy$
5. $x^3=y^2(2a-x)$
6. $x^4+x^2y-(x+y)^2=0$

Problem-05: Transform the equation $2a\sin^2\theta-r\cos\theta=0$ to Cartesian equation.

Solution:

Given Polar Equation is, $2a\sin^2\theta-r\cos\theta=0$.

We have

$$x=r\cos\theta,\ y=r\sin\theta \ \ and \ \ x^2+y^2=r^2$$

Putting $x=r\cos\theta,\ y=r\sin\theta \ \ and \ \ x^2+y^2=r^2$ the above relation is transformed into the following form

$$2a\sin^2\theta-r\cos\theta=0$$

$$or,\ \frac{2a\,r^2\sin^2\theta}{r^2}-r\cos\theta=0$$

$$or,\ \frac{2a\left(r\sin\theta\right)^2}{r^2}-r\cos\theta=0$$

$$or,\ \frac{2a\,y^2}{x^2+y^2}-x=0$$

$$or,\ 2a\,y^2-x\left(x^2+y^2\right)=0$$

$$or,\ 2a\,y^2=x^3+xy^2$$

$$or,\ x^3=2a\,y^2-xy^2$$

$$or,\ x^3=y^2\left(2a-x\right)$$ **(As desired)**

H.W:

Convert the followings to the Cartesian form:

1. $r^2\cos^2\theta=a^2\cos 2\theta$
2. $r^4=2a^2\cos ec2\theta$
3. $r\cos 2\theta=2\sin^2\dfrac{\theta}{2}$
4. $r(\cos 3\theta+\sin 3\theta)=5k\sin\theta\cos\theta$
5. $2a\sin^2\theta=r\cos\theta$
6. $r=\pm\left(1+\tan\theta\right)$

Parameter: A parameter is a symbol which is used to identify or classify the behavior of a mathematical object. For example, $f(x) = ax^2 + bx + c$, here a, b and c are parameters that determine the behavior of the function f. For each value of the parameters, we get different functions which are parabolas.

Parametric equation: A parametric equation defines a group of quantities as functions of one or more independent variables called parameters. If $y = f(x)$ represents a curve and both variables x and y are expressed in terms of a new variable t such as $x = \varphi(t)$, $y = \psi(t)$ then these functions are called a parametric equation or a parametric representation of the curve.

For example, the equations $x = \cos t$ and $y = \sin t$ form a parametric representation of the unit circle $x^2 + y^2 = 1$, where t is the parameter.

Similarly, the equations $x = t$ and $y = t^2$ is a parametric representation of the parabola $y = x^2$, where t is the parameter.