Integration: The process of finding an anti-derivative or integral of a function is called integration. It is the inverse process of differentiation. If $f(x)$ be a function of x related with another function $F(x)$ in such a way that

$$\frac{d}{dx}\left[F(x)\right]=f(x)$$

then

$$\int f(x)dx=F(x)+c$$

which is called an indefinite integral of $f(x)$ with respect to x where $f(x), F(x)$ and c are called integrand, integral and constant of integration respectively.

And

$$\int_a^b f(x)dx=F(b)-F(a)$$

which is called the definite integral of $f(x)$ from a to b, and 'a' is called the lower limit and 'b' the upper limit of the definite integral.

Fundamental Properties:

1. $\int\left[f_1(x)\pm f_2(x)\pm \ldots\ldots\ldots \text{ to } n \text{ terms}\right]dx=\int f_1(x)dx\pm\int f_2(x)dx\pm\ldots\ldots \text{ to } n \text{ terms}$.
2. $\int cf(x)dx=c\int f(x)dx$

where c is a constant.

General rules of integration:

1. $\int x^n dx=\frac{x^{n+1}}{n+1}$ *where* $(n\neq -1)$.
2. $\int \frac{dx}{x^n}=-\frac{1}{(n-1)x^{n-1}}$ *where* $(n\neq 1)$.
3. $\int \frac{dx}{\sqrt{x}}=2\sqrt{x}+c$.
4. $\int dx=x+c$.
5. $\int \frac{dx}{x}=\ln x$.
6. $\int e^x dx=e^x$.
7. $\int e^{mx}dx=\frac{e^{mx}}{m}$.
8. $\int a^x dx=\frac{a^x}{\ln a}$ *where* $a>0$.
9. $\int \sin mx dx=-\frac{\cos mx}{m}$.
10. $\int \sin x dx=-\cos x$.

11. $\int \cos mx dx = \frac{\sin mx}{m}$.

12. $\int \cos x dx = \sin x$.

13. $\int \sec^2 x dx = \tan x$.

14. $\int \sec x \tan x dx = \sec x$.

15. $\int co\sec^2 x dx = -\cot x$.

16. $\int co\sec x \cot x dx = -co\sec x$.

17. $\int \tan x dx = \ln|\sec x|$.

18. $\int \cot x dx = \ln|\sin x|$.

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ *where* $a \neq 0$.

20. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$.

21. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$.

22. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)$.

23. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right)$.

24. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$.

25. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

26. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$

27. $\int co\sec x dx = \ln|co\sec x - \cot x|$.

28. $\int \sec x dx = \ln|\sec x + \tan x|$.

29. $\int uv dx = u\int v dx - \int\left(\frac{du}{dx}.\int v dx\right)dx$.

30. $\int \sqrt{x^2 + a^2} dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right|$.

31. $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right|$.

32. $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$.

33. $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$.

34. $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x)$.

35. $\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$.

36. $\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$.

Indefinite Integration

Illustrative Examples:

Problem-01: $\int \sin^2 x dx$

$Sol^n : Let\ I = \int \sin^2 x dx$

$$= \frac{1}{2}\int 2\sin^2 x dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

where c is an integrating constant.

Exercise-01: $\int \cos^2 x dx$.

Ans: $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c.$

Problem-02: $\int \tan^2 x dx$

$Sol^n : Let\ I = \int \tan^2 x dx$

$$= \int \left(\sec^2 x - 1\right) dx$$

$$= (\tan x - x) + c.$$

where c is an integrating constant.

Exercise-02: $\int \cot^2 x dx$

Ans: $-\cot x - x + c.$

Problem-03: $\int \frac{a\sin^2 x + b\cos^2 x}{\sin^2 x \cos^2 x} dx$

$Sol^n : Let\ I = \int \frac{a\sin^2 x + b\cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int \left(\frac{a}{\cos^2 x} + \frac{b}{\sin^2 x}\right) dx$$

$$= \int \left(a\sec^2 x + b\cos ec^2 x\right) dx$$

$$= a\tan x - b\cot x + c$$

where c is an integrating constant.

Exercise-03: $\int \frac{\sin^3 x + \cos^3 x}{\cos x + \sin x} dx$

Ans: $x + \frac{1}{4}\cos 2x + c$.

Problem-04: $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

$Sol^n : Let\ I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

$$= \int \frac{\cos x - \left(\cos^2 x - \sin^2 x\right)}{1 - \cos x} dx$$

Exercise-04: $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

Ans: $x + c$.

$$=\int \frac{\cos x-\cos^2 x+\sin^2 x}{1-\cos x}dx$$

$$=\int \frac{\cos x(1-\cos x)+(1-\cos^2 x)}{1-\cos x}dx$$

$$=\int \left\{\frac{\cos x(1-\cos x)}{1-\cos x}+\frac{(1-\cos x)(1+\cos x)}{1-\cos x}\right\}dx$$

$$=\int (\cos x+1+\cos x)dx$$

$$=\int (1+2\cos x)dx$$

$$=x+2\sin x+c$$

where c is an integrating constant.

Problem-05: $\int \sqrt{1-\sin 2x}dx$

$Sol^n : Let\ I=\int \sqrt{1-\sin 2x}dx$

$$=\int \sqrt{\cos^2 x+\sin^2 x-2\sin x\cos x}dx$$

$$=\int \sqrt{(\cos x-\sin x)^2}dx$$

$$=\int (\cos x-\sin x)dx$$

$$=\sin x+\cos x+c$$

where c is an integrating constant.

Exercise-05: $\int \sqrt{1+\sin x}dx$.

Ans: $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$.

Problem-06: $\int \sqrt{1+\cos x}dx$

$Sol^n : Let\ I=\int \sqrt{1+\cos x}dx$

$$=\int \sqrt{2\cos^2\frac{x}{2}}dx$$

$$=\sqrt{2}\int \cos\frac{x}{2}dx$$

$$=\sqrt{2}\,\frac{\sin\frac{x}{2}}{1/2}+c$$

$$=2\sqrt{2}\sin\frac{x}{2}+c$$

where c is an integrating constant.

Exercise-06: $\int \sqrt{1-\cos 2x}dx$.

Ans: $-\sqrt{2}\cos x+c$.

Problem-07: $\int \frac{dx}{1+\sin x}$

Exercise-07: $\int \frac{dx}{1+\cos x}$.

Ans: $-\cot x + \operatorname{cosec} x + c$.

Sol^n : *Let* $I = \int \frac{dx}{1+\sin x}$

$$= \int \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}dx$$

$$= \int \frac{1-\sin x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx$$

$$= \tan x - \sec x + c$$

where c is an integrating constant.

Problem-08: $\int \cos^4 x dx$

Exercise-08: 1. $\int \sin^4 x dx$.

Ans: $\frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + c$.

Sol^n : *Let* $I = \int \cos^4 x dx$

$$= \frac{1}{4}\int \left(2\cos^2 x\right)^2 dx$$

$$= \frac{1}{4}\int (1+\cos 2x)^2 dx$$

$$= \frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left\{1+2\cos 2x+\frac{1}{2}\left(2\cos^2 2x\right)\right\}dx$$

$$= \frac{1}{4}\int \left\{1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right\}dx$$

$$= \frac{1}{4}\left[x+\sin 2x+\frac{x}{2}+\frac{\sin 4x}{8}\right]+c$$

$$= \frac{1}{4}\left[\frac{3x}{2}+\sin 2x+\frac{\sin 4x}{8}\right]+c$$

$$= \frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+c$$

where c is an integrating constant.

Problem-09: $\int \frac{\sin x}{\sqrt{1+\cos x}}dx$

Exercise-09: $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}dx$

Ans: $\sqrt{2}\sin x + c.$

$Sol^n : Let\ I = \int \frac{\sin x}{\sqrt{1+\cos x}}dx$

$$= \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}}}dx$$

$$= \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2}\cos\frac{x}{2}}dx$$

$$= \sqrt{2}\int \sin\frac{x}{2}dx$$

$$= -\sqrt{2}\frac{\cos\frac{x}{2}}{\frac{1}{2}} + c$$

$$= -2\sqrt{2}\cos\frac{x}{2} + c$$

where c is an integrating constant.

Problem-10: $\int \frac{dx}{\sin^2 x\cos^2 x}$

$sol^n : Let\ I = \int \frac{dx}{\sin^2 x\cos^2 x}$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx$$

$$= \int \left(\sec^2 x + co\sec^2 x\right)dx$$

$$= \tan x - \cot x + c.$$

where c is an integrating constant.

Problem-11: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x}dx$

Exercise-09: $\int \frac{\sin^8 x - \cos^8 x}{1-2sin^2 x\cos^2 x}dx$

$Sol^n : Let\ I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x}dx$

Ans: $-\frac{1}{2}\sin 2x + c$.

$$=\int \frac{\left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3}{\sin^2 x\cos^2 x}dx$$

$$=\int \frac{\left(\sin^2 x+\cos^2 x\right)^3-3\sin^2 x\cos^2 x\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x\cos^2 x}dx$$

$$=\int \frac{1-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\left(\frac{1}{\sin^2 x\cos^2 x}-3\right)dx$$

$$=\int\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-3\right)dx$$

$$=\int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}-3\right)dx$$

$$=\int\left(\sec^2 x+\cos ec^2 x-3\right)dx$$

$$=\tan x-\cot x-3x+c.$$

where c is an integrating constant.

Problem-12: $\int \frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}}dx$

Sol^n : *Let* $I=\int \frac{\cos^4 x-\sin^4 x}{\sqrt{1+\cos 4x}}dx$

$$=\int \frac{\left(\cos^2 x\right)^2-\left(\sin^2 x\right)^2}{\sqrt{2\cos^2 2x}}dx$$

$$=\frac{1}{\sqrt{2}}\int \frac{\left(\cos^2 x+\sin^2 x\right)\left(\cos^2 x-\sin^2 x\right)}{\cos 2x}dx$$

$$=\frac{1}{\sqrt{2}}\int \frac{\cos 2x}{\cos 2x}dx$$

$$=\frac{1}{\sqrt{2}}\int dx$$

$$=\frac{x}{\sqrt{2}}+c$$

where c is an integrating constant.

Problem-13: $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

Exercise-10: $\int \frac{\sin x}{\sin(x-a)} dx$.

Ans: $x\cos a + \sin a \ln\left[\sin(x-a)\right] + c$.

$sol^n : Let\ I = \int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

$$= \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x} dx$$

$$= \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$$

$$= \int \left(\frac{\cos 3x}{\sin 3x} - \frac{\cos 5x}{\sin 5x} \right) dx$$

$$= \int \frac{\cos 3x}{\sin 3x} dx - \int \frac{\cos 5x}{\sin 5x} dx$$

$$= \int (\cot 3x - \cot 5x) dx$$

$$= \frac{1}{3}\ln(\sin 3x) - \frac{1}{5}\ln(\sin 5x) + c.$$

where c is an integrating constant.

Problem-14: $\int \sin x \sin 2x \sin 3x dx$

Exercise-11: $\int \sin 3x \sin 4x dx$

Ans: $\frac{1}{2}\sin x - \frac{1}{14}\sin 7x + c$.

$sol^n : Let\ I = \int \sin x \sin 2x \sin 3x dx$

$$= \frac{1}{2}\int (2\sin x \sin 2x)\sin 3x dx$$

$$= \frac{1}{2}\int \{\cos(x-2x) - \cos(x+2x)\}\sin 3x dx$$

$$= \frac{1}{2}\int \{\cos x - \cos 3x\}\sin 3x dx$$

$$= \frac{1}{4}\int \{2\sin 3x \cos x - 2\sin 3x \cos 3x\} dx$$

$$= \frac{1}{4}\int \{2\sin 3x \cos x - \sin 6x\} dx$$

$$= \frac{1}{4}\int \{\sin(3x+x) + \sin(3x-x) - \sin 6x\} dx$$

$$= \frac{1}{4}\int \{\sin 4x + \sin 2x - \sin 6x\} dx$$

$$= \frac{1}{4}\left\{ -\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right\} + c$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + c .$$

where c is an integrating constant.

Problem-15: $\int \frac{dx}{\sqrt{x}+\sqrt{x+1}}$

$Sol^n : Let\ I = \int \frac{dx}{\sqrt{x}+\sqrt{x+1}}$

$$= \int \frac{\left(\sqrt{x}-\sqrt{x+1}\right)}{\left(\sqrt{x}+\sqrt{x+1}\right)\left(\sqrt{x}-\sqrt{x+1}\right)}dx$$

$$= \int \frac{\left(\sqrt{x}-\sqrt{x+1}\right)}{\left(\sqrt{x}\right)^2-\left(\sqrt{x+1}\right)^2}dx$$

$$= \int \frac{\left(\sqrt{x}-\sqrt{x+1}\right)}{x-\left(x+1\right)}dx$$

$$= \int \frac{\left(\sqrt{x}-\sqrt{x+1}\right)}{x-x-1}dx$$

$$= -\int \left(\sqrt{x}-\sqrt{x+1}\right)dx$$

$$= -\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\left(x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{2}{3}\left(x+1\right)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c$$

where c is an integrating constant.

Exercise-12: $\int \frac{dx}{\sqrt{x}-\sqrt{x-1}}$

Ans: $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}\left(x-1\right)^{\frac{3}{2}} + c$

Problem-15: $\int \frac{2-\sin 2x}{1-\cos 2x}dx$ **Exercise-13:** $\int \frac{dx}{\sqrt{x}-\sqrt{x-1}}$

$Sol^n : Let\ I = \int \frac{2-\sin 2x}{1-\cos 2x}dx$ Ans: $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}\left(x-1\right)^{\frac{3}{2}} + c$

$$= \int \frac{2-2\sin x\cos x}{2\sin^2 x}dx$$

$$= \int \left(\frac{2}{2\sin^2 x} - \frac{2\sin x\cos x}{2\sin^2 x}\right)dx$$

$$= \int \left(\cos ec^2 x - \cot x\right)dx$$

$$= \int \cos ec^2 x dx - \int \cot x dx$$

$$= -\cot x - \ln(\sin x) + c$$

where c is an integrating constant.

Problem-17: $\int \frac{dx}{a\cos x + b\sin x}$

Sol^n : $Let\ I = \int \frac{dx}{a\cos x + b\sin x}$

$Let, \quad a = r\cos\theta \quad and \quad b = r\sin\theta$

$\therefore \quad r = \sqrt{a^2 + b^2} \quad and \quad \theta = \tan^{-1}\frac{y}{x}$

$$= \frac{1}{r}\int \frac{dx}{\cos x\cos\theta + \sin x\sin\theta}$$

$$= \frac{1}{r}\int \frac{dx}{\cos(x-\theta)}$$

$$= \frac{1}{r}\int \sec(x-\theta)\,dx$$

$$= \frac{1}{r}\ln\{\sec(x-\theta) + \tan(x-\theta)\} + c$$

$$= \frac{1}{\sqrt{a^2+b^2}}\ln\left\{\sec\left(x - \tan^{-1}\frac{y}{x}\right) + \tan\left(x - \tan^{-1}\frac{y}{x}\right)\right\} + c$$

where c is an integrating constant.

Method of substitution

Sometimes, the integration of given integral $\int f(x)\,dx$ is relatively difficult. In this case, we can replace x by $\varphi(z)$ and dx by $\varphi'(z)dz$ for integrating easily. This process is known as method of substitution.

Illustrative Examples:

Problem-01: $\int (a+bx)^n\,dx$

sol^n : $Let\ I = \int (a+bx)^n dx$

Put $\quad z = a + bx \therefore dz = bdx$

$$\Rightarrow \frac{1}{b}dz = dx$$

Now $\ I = \int z^n \frac{1}{b}dz$

$$= \frac{1}{b}\int z^n dz$$

Exercise-01: $\int \frac{2\sin x}{5+3\cos x}dx$

Ans: $-\frac{2}{3}\ln(5+3\cos x) + c$.

$$=\frac{1}{b}\frac{z^{n+1}}{n+1}+c$$

$$=\frac{\left(a+bx\right)^{n+1}}{b\left(n+1\right)}+c$$

where c is an integrating constant.

Problem-02: $\int \frac{dx}{x\sqrt{\left(x^2-a^2\right)}}$

Exercise-02: $\int \frac{dx}{x\sqrt{x^2-1}}$

Ans: $\sec^{-1}x+c$.

sol^n : $Let\ I=\int \frac{dx}{x\sqrt{\left(x^2-a^2\right)}}$

Put $x=a\sec\theta \therefore\ dx=a\sec\theta\tan\theta d\theta$

Now $I=\int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\sqrt{\left(a^2\sec^2\theta-a^2\right)}}$

$$=\int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta a\sqrt{\left(\sec^2\theta-1\right)}}$$

$$=\frac{1}{a}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\tan^2\theta}}$$

$$=\frac{1}{a}\int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\tan\theta}$$

$$=\frac{1}{a}\int d\theta$$

$$=\frac{1}{a}\theta+c$$

$$=\frac{1}{a}\sec^{-1}\frac{x}{a}+c$$

where c is an integrating constant.

Problem-03: $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Exercise-03: $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

Ans: $\frac{e^{m\tan^{-1}x}}{m}+c$.

sol^n : $Let\ I=\int \frac{\sin^{-1}x dx}{\sqrt{1-x^2}}$

Put $z=\sin^{-1}x\ \therefore\ dz=\frac{dx}{\sqrt{1-x^2}}$

Now $I=\int zdz$

$$=\frac{z^2}{2}+c$$

$$=\frac{\left(\sin^{-1}x\right)^2}{2}+c$$

where c is an integrating constant.

Problem-04: $\int\frac{(1+x)e^x}{\cos^2\left(xe^x\right)}dx$

$sol^n : Let\ I=\int\frac{(1+x)e^x}{\cos^2\left(xe^x\right)}dx$

Put $xe^x=z\ \therefore\ (1+x)e^x dx=dz$

Now $I=\int\frac{dz}{\cos^2 z}$

$$=\int\sec^2 z dz$$

$$=\tan z+c$$

$$=\tan\left(xe^x\right)+c$$

where c is an integrating constant.

Exercise-04: $\int\frac{(x+1)(x+\ln x)^2}{x}dx$

Ans: $\frac{1}{3}(x+\ln x)^3+c$.

Problem-05: $\int\frac{dx}{e^x+1}$

$sol^n : Let\ I=\int\frac{dx}{e^x+1}$

$$=\int\frac{e^{-x}}{1+e^{-x}}dx$$

Put $1+e^{-x}=z\ \therefore -e^{-x}dx=dz$

Now $I=-\int\frac{dz}{z}$

$$=-\ln z+c$$

$$=-\ln\left(1+e^{-x}\right)+c$$

where c is an integrating constant.

Exercise-05: $\int\frac{\sin x}{(1-\cos x)^2}dx$

Ans: $-\frac{1}{(1-\cos x)}+c$.

Problem-06: $\int\frac{\sqrt{x}+\ln x}{x}dx$

$sol^n : Let\ I=\int\frac{\sqrt{x}+\ln x}{x}dx$

$$=\int\left(\frac{1}{\sqrt{x}}+\frac{\ln x}{x}\right)dx$$

$$=\int x^{-\frac{1}{2}}dx+\int\frac{\ln x}{x}dx$$

$$=\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+\frac{1}{2}(\ln x)^2+c\ \left[putting\ \ln x=z\ in the\, 2nd\ part\right]$$

$$=2\sqrt{x}+\frac{1}{2}(\ln x)^2+c$$

where c is an integrating constant.

Problem-07: $\int x\sqrt{1+x}dx$

Exercise-06: $\int x\sqrt[3]{\left(1-x^2\right)^5}dx$

Ans: $-\frac{3}{16}\left(1-x^2\right)^{\frac{8}{3}}+c$.

$sol^n: Let\ I=\int x\sqrt{1+x}dx$

Put $\quad 1+x=z\ \therefore\ dx=dz$

Now $\ I=\int(z-1)z^{\frac{1}{2}}dz$

$$=\int(z-1)z^{\frac{1}{2}}dz$$

$$=\int\left(z^{\frac{3}{2}}-z^{\frac{1}{2}}\right)dz$$

$$=\frac{2}{5}z^{5/2}-\frac{2}{3}z^{3/2}+c$$

$$=\frac{2}{5}(1+x)^{5/2}-\frac{2}{3}(1+x)^{3/2}+c$$

where c is an integrating constant.

Problem-08: $\int\frac{x}{\sqrt{x}+1}dx$

Exercise-07: $\int\frac{dx}{\sqrt{x}-1}$

Ans: $2\sqrt{x}+2\ln\left(\sqrt{x}-1\right)+c$.

$sol^n: Let\ I=\int\frac{x}{\sqrt{x}+1}dx$

Put $\quad x=z^2\ \ \therefore 2zdz=dx$

Now $\ I=2\int\frac{z^2z}{z+1}dz$

$$=2\int\frac{z^3}{z+1}dz$$

$$= 2\int \frac{z^3 + z^2 - z^2 - z + z + 1 - 1}{z+1} dz$$

$$= 2\int \frac{z^2(z+1) - z(z+1) + (z+1) - 1}{z+1} dz$$

$$= 2\int \left[z^2 - z + 1 - \frac{1}{z+1} \right] dz$$

$$= 2\left[\frac{z^3}{3} - \frac{z^2}{2} + z - \ln(z+1) \right] + c$$

$$= 2\left[\frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x} - \ln(\sqrt{x} + 1) \right] + c$$

where c is an integrating constant.

Problem-09: $\int \frac{x^3}{\sqrt{x-1}} dx$

Exercise-08: $\int \frac{x^2}{\sqrt{x+1}} dx$

Ans: $\frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2\sqrt{(x+1)} + c$.

sol^n : *Let* $I = \int \frac{x^3}{\sqrt{x-1}} dx$

Put $\quad x - 1 = z \;\; \therefore\; dx = dz$

Now $\;I = \int \frac{(z+1)^3}{\sqrt{z}} dz$

$$= \int \frac{(z^3 + 3z^2 + 3z + 1)}{\sqrt{z}} dz$$

$$= \int \left(z^{\frac{5}{2}} + 3z^{\frac{3}{2}} + 3z^{\frac{1}{2}} + z^{-\frac{1}{2}} \right) dz$$

$$= \frac{2}{7} z^{7/2} + \frac{6}{5} z^{5/2} + 2z^{3/2} + 2z^{1/2} + c$$

$$= \frac{2}{7}(x-1)^{7/2} + \frac{6}{5}(x-1)^{5/2} + 2(x-1)^{3/2} + 2(x-1)^{1/2} + c.$$

where c is an integrating constant.

Problem-10: $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

sol^n : *Let* $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

Put $\tan x = z \quad \therefore \ \sec^2 x dx = dz$

Now $I = \int \frac{dz}{\sqrt{z}}$

$$= \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{z^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{\tan x} + c$$

where c is an integrating constant.

Problem-11: $\int \frac{1}{\sqrt{x}} \cos\sqrt{x} dx$

$sol^n : Let \ I = \int \frac{1}{\sqrt{x}} \cos\sqrt{x} dx$

$Let, \sqrt{x} = z$

$\therefore \frac{1}{2\sqrt{x}} dx = dz$

$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dz$

$Now \quad I = 2\int \cos z dz$

$$= 2\sin z + c$$

$$= 2\sin\sqrt{x} + c$$

where c is an integrating constant.

Problem-12: $\int \frac{1-\sin x}{x+\cos x} dx$

$sol^n : Let \ I = \int \frac{1-\sin x}{x+\cos x} dx$

$Let, \ x + \cos x = z$

$\therefore (1-\sin x) dx = dz$

Exercise-09: $\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x + d} dx$

Ans: $\ln(a\sin x + b\cos x + d) + c$

$Now \quad I = \int \frac{dz}{z}$

$= \ln z + c$

$= \ln(x + \cos x) + c$

where c is an integrating constant.

Problem-13: $\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}dx$

$sol^n : Let \ I = \int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x}dx$

$Let,\ a\sin^2 x + b\cos^2 x = z$

$\therefore (2a\sin x\cos x - 2b\sin x\cos x)dx = dz$

$\Rightarrow (a-b)2\sin x\cos x dx = dz$

$\Rightarrow \sin 2x dx = \frac{1}{(a-b)}dz$

$Now \quad I = \frac{1}{(a-b)}\int \frac{dz}{z}$

$= \frac{1}{(a-b)}\ln z + c$

$= \frac{1}{(a-b)}\ln\left(a\sin^2 x + b\cos^2 x\right) + c$

where c is an integrating constant.

Exercise-10: $\int \frac{\sin 2x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}dx$

Ans: $\frac{1}{\left(a^2 - b^2\right)}\left\{\frac{1}{a^2\cos^2 x + b^2\sin^2 x}\right\} + c$

Problem-14: $\int \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}dx$

$sol^n : Let \ I = \int \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}dx$

$= \int \frac{\sin x\cos x}{\cos^4 x\left(1 + \frac{\sin^4 x}{\cos^4 x}\right)}dx$

$= \int \frac{\tan x\sec^2 x}{\left(1 + \tan^4 x\right)}dx$

Put $\quad \tan^2 x = z \ \therefore 2\tan x\sec^2 x dx = dz$

Now $\quad I = \frac{1}{2}\int \frac{dz}{1+z^2}$

$$=\frac{1}{2}\tan^{-1}z+c$$

$$=\frac{1}{2}\tan^{-1}\left(\tan^2 x\right)+c$$

where c is an integrating constant.

Problem-15: $\int\frac{x^2+\sin^2 x}{1+x^2}\sec^2 x dx$

sol^n : $Let\ I=\int\frac{x^2+\sin^2 x}{1+x^2}\sec^2 x dx$

$$=\int\frac{x^2+1-\cos^2 x}{1+x^2}\sec^2 x dx$$

$$=\int\frac{\left(1+x^2\right)\sec^2 x-1}{1+x^2}dx$$

$$=\int\sec^2 x dx-\int\frac{dx}{1+x^2}$$

$$=\tan x-\tan^{-1}x+c$$

where c is an integrating constant.

Problem-16: $\int\frac{dx}{\left(1+x^2\right)^2}$

sol^n : $Let\ I=\int\frac{dx}{\left(1+x^2\right)^2}$

Put $x=\tan\theta\quad\therefore dx=\sec^2\theta d\theta$

Now $I=\int\frac{\sec^2\theta d\theta}{\left(1+\tan^2\theta\right)^2}$

$$=\int\frac{\sec^2\theta d\theta}{\left(\sec^2\theta\right)^2}$$

$$=\int\frac{\sec^2\theta d\theta}{\sec^4\theta}$$

$$=\int\frac{d\theta}{\sec^2\theta}$$

$$=\int\cos^2\theta d\theta$$

$$=\frac{1}{2}\int 2\cos^2\theta d\theta$$

$$=\frac{1}{2}\int\left(1+\cos 2\theta\right)d\theta$$

Exercise-11: $\int\frac{dx}{\left(1+x^2\right)^{3/2}}$

Ans: $\frac{x}{\sqrt{1+x^2}}+c$.

$$=\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$=\frac{1}{2}\left(\theta+\frac{1}{2}\frac{2\tan\theta}{\left(1+\tan^2\theta\right)}\right)+c$$

$$=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{\left(1+x^2\right)}\right)+c$$

where c is an integrating constant.

Problem-17: $\int\sqrt{\frac{x}{a-x}}dx$

$sol^n : Let\, I=\int\sqrt{\frac{x}{a-x}}dx$

Put $\quad x=a\sin^2\theta \quad \therefore\, dx=2a\sin\theta\cos\theta d\theta$

Now $\quad I=\int\sqrt{\frac{a\sin^2\theta}{a-a\sin^2\theta}}\;2a\sin\theta\cos\theta d\theta$

$$=\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\;2a\sin\theta\cos\theta d\theta$$

$$=\int\sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\;2a\sin\theta\cos\theta d\theta$$

$$=\int\frac{\sin\theta}{\cos\theta}2a\sin\theta\cos\theta d\theta$$

$$=a\int 2\sin^2\theta d\theta$$

$$=a\int\left(1-\cos 2\theta\right)d\theta$$

$$=a\left(\theta-\frac{\sin 2\theta}{2}\right)+c$$

$$=a\left(\theta-\sin\theta\cos\theta\right)+c$$

$$=a\left(\theta-\sin\theta\sqrt{1-\sin^2\theta}\right)+c$$

$$=a\left(\sin^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right)+c\,.$$

where c is an integrating constant.

Problem-18: $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

$sol^n : Let\ I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

$$= \int \frac{\sqrt{x}}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2}} dx$$

Put $\quad x^{\frac{3}{2}} = z \therefore \frac{3}{2}\sqrt{x}dx = dz$

Now $\quad I = \frac{2}{3}\int \frac{dz}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - z^2}}$

$$= \frac{2}{3}\sin^{-1}\left(\frac{z}{a^{\frac{3}{2}}}\right) + c$$

$$= \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}} + c$$

where c is an integrating constant.

Problem-19: $\int \frac{x^2 + 1}{x^4 + 1} dx$

$sol^n : Let\ I = \int \frac{x^2 + 1}{x^4 + 1} dx$

$$= \int \frac{x^2\left(1 + \frac{1}{x^2}\right)}{x^2\left(x^2 + \frac{1}{x^2}\right)} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{2 + \left(x - \frac{1}{x}\right)^2} dx$$

Exercise-12: $\int \frac{x^2 - 1}{x^4 + 1} dx$

Ans: $\frac{1}{2\sqrt{2}} \ln\left(\frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x}\right) + c$

$$=\int \frac{1+\frac{1}{x^2}}{\left(\sqrt{2}\right)^2+\left(x-\frac{1}{x}\right)^2}dx$$

Put $\quad x-\frac{1}{x}=z \therefore \left(1+\frac{1}{x^2}\right)dx=dz$

Now $\quad I=\int \frac{dz}{\left(\sqrt{2}\right)^2+z^2}$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+c$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+c$$

where c is an integrating constant.

Problem-20: $\int \sqrt{1+\sec x}dx$

$sol^n : Let\, I=\int \sqrt{1+\sec x}dx$

$$=\int \sqrt{1+\frac{1}{\cos x}}dx$$

$$=\int \sqrt{\frac{1+\cos x}{\cos x}}dx$$

$$=\int \sqrt{\frac{2\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}dx$$

$$=\int \sqrt{\frac{2\cos^2\frac{x}{2}}{1-\sin^2\frac{x}{2}-\sin^2\frac{x}{2}}}dx$$

$$=\int \frac{\sqrt{2}\cos\frac{x}{2}}{\sqrt{1-2\sin^2\frac{x}{2}}}dx$$

$$=\int \frac{\sqrt{2}\cos\frac{x}{2}}{\sqrt{1-\left(\sqrt{2}\sin\frac{x}{2}\right)^2}}dx$$

Put $\sqrt{2}\sin\frac{x}{2}=z \quad \therefore \sqrt{2}\cos\frac{x}{2}dx=2dz$

Now $I=\int\frac{2dz}{\sqrt{1-z^2}}$

$$=2\int\frac{dz}{\sqrt{1-z^2}}$$

$$=2\sin^{-1}z+c$$

$$=2\sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)+c\,.$$

where c is an integrating constant.

Problem-21: $\int\sqrt{\tan x}dx$

sol^n : *Let* $I=\int\sqrt{\tan x}dx$

put $\tan x=z^2 \quad \therefore \sec^2 xdx=2zdz$

$$\Rightarrow dx=\frac{2zdz}{\sec^2 x}$$

$$=\frac{2zdz}{1+\tan^2 x}$$

$$=\frac{2zdz}{1+z^4}$$

Now $I=\int\frac{2z^2dz}{1+z^4}$

$$=\int\frac{\left(z^2+1\right)+\left(z^2-1\right)}{z^4+1}dz$$

$$=\int\frac{z^2+1}{z^4+1}dz+\int\frac{z^2-1}{z^4+1}dz$$

$$=\int\frac{1+\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz+\int\frac{1-\frac{1}{z^2}}{z^2+\frac{1}{z^2}}dz$$

$$=\int\frac{1+\frac{1}{z^2}}{2+\left(z-\frac{1}{z}\right)^2}dz+\int\frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2-2}dz$$

Exercise-13: $\int\frac{dx}{x^4+1}$

Ans: $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)-\frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right)+c$

$$= \int \frac{1+\frac{1}{z^2}}{\left(\sqrt{2}\right)^2 + \left(z-\frac{1}{z}\right)^2} dz + \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2 - \left(\sqrt{2}\right)^2} dz$$

$$= I_1 + I_2 \qquad \cdots\cdots\cdots(1)$$

where, $I_1 = \int \frac{1+\frac{1}{z^2}}{\left(\sqrt{2}\right)^2 + \left(z-\frac{1}{z}\right)^2} dz$

Put $\quad z - \frac{1}{z} = t \;\therefore \left(1+\frac{1}{z^2}\right) dz = dt$

Now $\quad I_1 = \int \frac{dt}{\left(\sqrt{2}\right)^2 + t^2}$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c_1$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right) + c_1$$

and $I_2 = \int \frac{1-\frac{1}{z^2}}{\left(z+\frac{1}{z}\right)^2 - \left(\sqrt{2}\right)^2} dz$

Put $\quad z + \frac{1}{z} = t \;\therefore \left(1-\frac{1}{z^2}\right) dz = dt$

Now $\quad I_2 = \int \frac{dt}{t^2 - \left(\sqrt{2}\right)^2}$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{t-\sqrt{2}}{t+\sqrt{2}}\right) + c_2$$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{z^2+1-\sqrt{2}z}{z^2+1+\sqrt{2}z}\right) + c_2$$

From (1) we have,

$$I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{z^2+1-\sqrt{2}z}{z^2+1+\sqrt{2}z}\right)+c \qquad ;\ putting,\ c=c_1+c_2$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{\tan x+1-\sqrt{2\tan x}}{\tan x+1+\sqrt{2\tan x}}\right)+c$$

where c is an integrating constant.

Problem-22: $\int\frac{dx}{\sqrt[3]{x}\sqrt[3]{(1+x)^5}}$

Exercise-14: $\int\frac{dx}{x^{\frac{1}{2}}(1+x)^{\frac{5}{2}}}$

Ans: $2\sqrt{\left(\frac{x}{1+x}\right)}-\frac{2}{3}\left(\frac{x}{1+x}\right)^{3/2}+c$

sol^n : *Let* $I=\int\frac{dx}{\sqrt[3]{x}\sqrt[3]{(1+x)^5}}$

$$=\int\frac{dx}{x^{\frac{1}{3}}(1+x)^{\frac{5}{3}}}$$

Put $\quad 1+x=zx\Rightarrow x=\frac{1}{z-1}$

or, $z=1+\frac{1}{x}$

$\therefore\ dz=-\frac{1}{x^2}dx\Rightarrow dx=-x^2dz$

Now $\quad I=-\int\frac{x^2dz}{x^{\frac{1}{3}}z^{\frac{5}{3}}x^{\frac{5}{3}}}$

$$=-\int\frac{x^2dz}{x^2z^{\frac{5}{3}}}$$

$$=-\int z^{-5/3}dz$$

$$=-\frac{z^{-\frac{5}{3}+1}}{-\frac{5}{3}+1}+c$$

$$=-\frac{z^{-\frac{2}{3}}}{-\frac{2}{3}}+c$$

$$=\frac{3}{2}\left(\frac{1+x}{x}\right)^{-2/3}+c$$

$$=\frac{3}{2}\left(\frac{x}{1+x}\right)^{2/3}+c$$

where c is an integrating constant.

NOTE: Integrals of the type $\int \frac{dx}{x^m (a+bx)^n}$ *where* $m \neq 0, n \neq 0$ can be evaluated exactly in the same way.

Some Important Standard Integrals

Problem-01: $\int \frac{dx}{4x^2+4x+5}$

sol^n : *Let* $I = \int \frac{dx}{4x^2+4x+5}$

$$= \frac{1}{4}\int \frac{dx}{x^2+x+\frac{5}{4}}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+2.\frac{1}{2}x+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}$$

$$= \frac{1}{4}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+1}$$

$$= \frac{1}{4}\tan^{-1}\left(x+\frac{1}{2}\right)+c$$

where c is an integrating constant.

Problem-02: $\int \frac{dx}{1+x-x^2}$

sol^n : *Let* $I = \int \frac{dx}{1+x-x^2}$

$$= \int \frac{dx}{-x^2+x+1}$$

$$= \int \frac{dx}{-(x^2-x-1)}$$

$$= \int \frac{dx}{-\left(x^2-2x.\frac{1}{2}+\frac{1}{4}-1-\frac{1}{4}\right)}$$

Exercise-01: $\int \frac{dx}{1+x+x^2}$

Ans: $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$.

Exercise-02: $\int \frac{dx}{x^2-x-6}$

Ans: $\frac{1}{5}\ln\left|\frac{x-3}{x+2}\right|+c$.

Exercise-03: $\int \frac{dx}{x^2+7x-18}$

Ans: $\frac{1}{11}\ln\left|\frac{x-2}{x+9}\right|+c$.

$$=\int\frac{dx}{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$=\int\frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$=\frac{1}{\sqrt{5}}\ln\left(\frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)}\right)+c$$

$$=\frac{1}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right)+c$$

where c is an integrating constant.

Problem-03: $\int\frac{dx}{\sqrt{x^2-7x+12}}$

sol^n : Let $I=\int\frac{dx}{\sqrt{x^2-7x+12}}$

$$=\int\frac{dx}{\sqrt{x^2-2.\frac{7}{2}x+\left(\frac{7}{2}\right)^2-\frac{49}{4}+12}}$$

$$=\int\frac{dx}{\sqrt{\left(x-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$=\ln\left(\left(x-\frac{7}{2}\right)+\sqrt{\left(x-\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right)+c$$

$$=\ln\left(\left(\frac{2x-7}{2}\right)+\sqrt{x^2-7x+12}\right)+c$$

where c is an integrating constant.

Exercise-04: $\int\frac{dx}{\sqrt{1-x-x^2}}$

Ans: $\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+c$.

Exercise-05: $\int\frac{dx}{\sqrt{2ax-x^2}}$

Ans: $\sin^{-1}\left(\frac{x-a}{a}\right)+c$.

Problem-04: $\int\frac{dx}{\sqrt{x^2-4x+3}}$

sol^n : Let $I=\int\frac{dx}{\sqrt{x^2-4x+3}}$

Exercise-06: $\int\frac{dx}{\sqrt{3x-x^2-2}}$

Ans: $\sin^{-1}(2x-3)+c$.

$$=\int \frac{dx}{\sqrt{x^2-2.x.2+4-1}}$$

$$=\int \frac{dx}{\sqrt{(x-2)^2-(1)^2}}$$

$$=\ln\left((x-2)+\sqrt{(x-2)^2-1}\right)+c$$

$$=\ln\left((x-2)+\sqrt{x^2-4x+3}\right)+c\,.$$

where c is an integrating constant.

Problem-05: $\int\sqrt{4-3x-2x^2}\,dx$

Exercise-07: $\int\sqrt{18x-65-x^2}\,dx$

sol^n : $Let\ I=\int\sqrt{4-3x-2x^2}\,dx$

$$=\int\sqrt{4-2\left(x^2+\frac{3}{2}x\right)}dx$$

$$=\sqrt{2}\int\sqrt{2-\left(x^2+2.x.\frac{3}{4}+\frac{9}{16}-\frac{9}{16}\right)}dx$$

$$=\sqrt{2}\int\sqrt{2+\frac{9}{16}-\left(x^2+2.x.\frac{3}{4}+\frac{9}{16}\right)}dx$$

$$=\sqrt{2}\int\sqrt{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$=\sqrt{2}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$=\sqrt{2}\left\{\frac{\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}}{2}+\frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\sin^{-1}\left(\frac{\left(x+\frac{3}{4}\right)}{\frac{\sqrt{41}}{4}}\right)\right\}+c$$

$$=\sqrt{2}\left\{\frac{(4x+3)\sqrt{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2}}{8}+\frac{41}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\}+c$$

$$=\sqrt{2}\left\{\frac{(4x+3)\sqrt{2-\frac{3}{2}x-x^2}}{8}+\frac{41}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\}+c$$

$$=\frac{(4x+3)\sqrt{4-3x-2x^2}}{8}+\frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)+c$$

where c is an integrating constant.

Problem-06: $\int\sqrt{2ax-x^2}\,dx$

Exercise-08: $\int\sqrt{3x-x^2}\,dx$

sol^n : Let $I=\int\sqrt{2ax-x^2}\,dx$

$$=\int\sqrt{a^2-\left(x^2-2ax+a^2\right)}\,dx$$

$$=\int\sqrt{a^2-(x-a)^2}\,dx$$

$$=\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$

$$=\frac{(x-a)\sqrt{2ax-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$

where c is an integrating constant.

NOTE: Integrals of the type $\int\frac{dx}{ax^2+bx+c}$, $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, $\int\sqrt{ax^2+bx+c}\,dx$ *where* $a\neq 0$ can be evaluated exactly in the same way.

Problem-07: $\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Exercise-09: $\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Ans: $2\sin^{-1}\left(\frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}}\right)+c$.

sol^n : Let $I=\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Put $x-\alpha=z^2 \therefore dx=2zdz$ and $x=z^2+\alpha$

Now $I=\int\frac{2zdz}{\sqrt{z^2\left(z^2+\alpha-\beta\right)}}$

$$=2\int\frac{dz}{\sqrt{z^2+\left(\sqrt{\alpha-\beta}\right)^2}}$$

$$=2\ln\left|z+\sqrt{z^2+\left(\sqrt{\alpha-\beta}\right)^2}\right|+c$$

$$=2\ln\left|\sqrt{x-\alpha}+\sqrt{x-\alpha+\alpha-\beta}\right|+c$$

$$=2\ln\left|\sqrt{x-\alpha}+\sqrt{x-\beta}\right|+c.$$

where c is an integrating constant.

Problem-08: $\int \frac{x+1}{x^2+4x+5}dx$

Exercise-10: $\int \frac{4x+15}{x^2+6x+10}dx$

Ans: $2\ln\left(x^2+6x+10\right)+3\tan^{-1}(x+3)+c$.

sol^n : Let $I=\int \frac{x+1}{x^2+4x+5}dx$

Put $\quad x+1=l(2x+4)+m \quad$; $\left[Let,\ px+q=l\times diff.coeff\ of\ \left(ax^2+bx+c\right)+m\right]$

$\therefore 1=2l,\ 1=4l+m$

$\Rightarrow l=\frac{1}{2}, m=-1$

Now $\quad I=\int \frac{\frac{1}{2}(2x+4)-1}{x^2+4x+5}dx$

$=\frac{1}{2}\int \frac{(2x+4)}{x^2+4x+5}dx-\int \frac{1}{x^2+4x+5}dx$

$=\frac{1}{2}\int \frac{(2x+4)}{x^2+4x+5}dx-\int \frac{1}{(x+2)^2+1}dx$

$=\frac{1}{2}\ln\left(x^2+4x+5\right)-\tan^{-1}(x+2)+c$.

where c is an integrating constant.

Problem-09: $\int \frac{x-2}{\sqrt{2x^2-8x+5}}dx$

Exercise-11: $\int \frac{x-1}{\sqrt{4+x^2-2x}}dx$

Ans: $\sqrt{x^2-2x+4}+c$.

sol^n : Let $I=\int \frac{x-2}{\sqrt{2x^2-8x+5}}dx$

Put $\quad x-2=l(4x-8)+m \quad$; $\left[Let,\ px+q=l\times diff.coeff\ of\ \left(ax^2+bx+c\right)+m\right]$

$\therefore 1=4l, -2=-8l+m$

$\Rightarrow l=\frac{1}{4},\ m=0$

Exercise-12: $\int \frac{2x+5}{\sqrt{x^2-2x+2}}dx$

Now $I=\int \frac{\frac{1}{4}(4x-8)}{\sqrt{2x^2-8x+5}}dx$

$=\frac{1}{4}\int \frac{dz}{\sqrt{z}} \quad$; putting $2x^2-8x+5=z$

$=\frac{1}{4}\int z^{-\frac{1}{2}}dz$

$=\frac{1}{4}\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$

$$= \tfrac{1}{2} z^{1/2} + c$$

$$= \tfrac{1}{2}\sqrt{2x^2 - 8x + 5} + c$$

where c is an integrating constant.

NOTE: Integrals of the type $\int \frac{px+q}{ax^2+bx+c}dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ *where* $a \neq 0, p \neq 0$ can be evaluated exactly in the same way.

Problem-10: $\int \frac{dx}{(2x+1)\sqrt{(4x+3)}}$

$sol^n : Let\ I = \int \frac{dx}{(2x+1)\sqrt{(4x+3)}}$

Put $4x+3 = z^2$ or $x = \frac{z^2-3}{4}$

$$\therefore dx = \frac{1}{2} zdz$$

Now $I = \int \frac{\tfrac{1}{2} zdz}{\left(\frac{z^2-3}{2}+1\right)z}$

$$= \tfrac{1}{2}\int \frac{dz}{\left(\frac{z^2-3+2}{2}\right)}$$

$$= \int \frac{dz}{z^2-1}$$

$$= \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right| + c$$

where c is an integrating constant.

Exercise-13: $\int \frac{dx}{(2+x)\sqrt{(1+x)}}$

Ans: $2\tan^{-1}\left(\sqrt{1+x}\right) + c$.

Exercise-14: $\int \frac{dx}{(x-3)\sqrt{(x-2)}}$

Ans: $\ln\left(\frac{\sqrt{x-2}-1}{\sqrt{x-2}+1}\right) + c$.

Exercise-15: $\int \frac{dx}{(1-x)\sqrt{(1+x)}}$

Ans: $\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{2}-\sqrt{1-x}}\right) + c$.

Problem-11: $\int \frac{xdx}{(1+x^2)\sqrt{(x^2-1)}}$

$sol^n : Let\ I = \int \frac{xdx}{(1+x^2)\sqrt{(x^2-1)}}$

Put $x^2 - 1 = z^2$ or $x^2 = z^2 + 1$

Exercise-16: $\int \frac{xdx}{(x^2+2)\sqrt{(x^2+3)}}$

Ans: $\frac{1}{2}\ln\left(\frac{\sqrt{x^2+3}-1}{\sqrt{x^2+3}+1}\right) + c$.

$$\therefore xdx = zdz$$

Now $I = \int \frac{zdz}{(z^2+1+1)z}$

$$= \int \frac{dz}{z^2+2}$$

$$= \int \frac{dz}{(\sqrt{2})^2 + z^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{x^2-1}}{\sqrt{2}}\right) + c$$

where c is an integrating constant.

NOTE: Integrals of the type $\int \frac{dx}{(ax+b)\sqrt{(cx+d)}}$, $\int \frac{xdx}{(ax^2+b)\sqrt{(cx^2+d)}}$ *where* $a \neq 0, c \neq 0$ can be evaluated exactly in the same way.

Problem-12: $\int \frac{dx}{(1+x^2)\sqrt{(x^2+4)}}$

$sol^n: Let\ I = \int \frac{dx}{(1+x^2)\sqrt{(x^2+4)}}$

Put $x = \frac{1}{z}$

$$\therefore dx = -\frac{1}{z^2} dz$$

Now $I = \int \frac{-\frac{1}{z^2} dz}{\left(\frac{1}{z^2}+1\right)\sqrt{\frac{1}{z^2}+4}}$

$$= -\int \frac{zdz}{(z^2+1)\sqrt{4z^2+1}}$$

Again let $4z^2+1 = t^2$ $or,$ $z^2 = \frac{t^2-1}{4}$

$$\therefore \quad zdz = \frac{1}{4} tdt$$

Exercise-17: $\int \frac{dx}{(x^2+1)\sqrt{(1-x^2)}}$

Ans: $-\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right) + c$.

$$\therefore\ I=-\frac{1}{4}\int\frac{tdt}{t\left(\frac{t^2-1}{4}+1\right)}$$

$$=-\frac{1}{4}\int\frac{dt}{\left(\frac{t^2+3}{4}\right)}$$

$$=-\int\frac{dt}{3+t^2}$$

$$=-\int\frac{dt}{\left(\sqrt{3}\right)^2+t^2}$$

$$=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c$$

$$=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{4z^2+1}}{\sqrt{3}}\right)+c$$

$$=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{4+x^2}}{x\sqrt{3}}\right)+c$$

where c is an integrating constant.

NOTE: Integrals of the type $\int\frac{dx}{\left(ax^2+b\right)\sqrt{\left(cx^2+d\right)}}$ *where* $a\neq 0, c\neq 0$ can be evaluated exactly in the same way.

Problem-13: $\int\frac{dx}{(1+x)\sqrt{\left(1+2x-x^2\right)}}$

$sol^n: Let\ I=\int\frac{dx}{(1+x)\sqrt{\left(1+2x-x^2\right)}}$

Put $1+x=\frac{1}{z}\ \therefore\ dx=-\frac{1}{z^2}dz$

Now $I=\int\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\left[1+2\left(\frac{1}{z}-1\right)-\left(\frac{1}{z}-1\right)^2\right]}}$

$$=-\int\frac{dz}{z\sqrt{\left[1+\frac{2}{z}-2-\left(\frac{1}{z^2}-\frac{2}{z}+1\right)\right]}}$$

Exercise-18: $\int\frac{dx}{(x-3)\sqrt{\left(x^2-6x+8\right)}}$

Ans: $\sec^{-1}(x-3)+c$.

Exercise-19: $\int\frac{dx}{(2x+3)\sqrt{\left(x^2+3x+2\right)}}$

Ans: $\sec^{-1}(2x+3)+c$.

Exercise-20: $\int\frac{dx}{(x-1)\sqrt{\left(x^2+2x+2\right)}}$

$$= -\int \frac{dz}{z\sqrt{\left(-\frac{1}{z^2}+\frac{4}{z}-2\right)}}$$

$$= -\int \frac{dz}{\sqrt{\left(-1+4z-2z^2\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\left(-\frac{1}{2}+2z-z^2\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-\frac{1}{2}-\left(z^2-2z\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-\frac{1}{2}-\left(z^2-2z+1\right)+1}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\left[\frac{1}{2}-\left(z-1\right)^2\right]}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\left[\left(\frac{1}{\sqrt{2}}\right)^2-\left(z-1\right)^2\right]}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{z-1}{\frac{1}{\sqrt{2}}}\right)+c$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\frac{1}{(x+1)}-1}{\frac{1}{\sqrt{2}}}\right)+c$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(-\frac{x\sqrt{2}}{x+1}\right)+c$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{x+1}\right)+c.$$

where c is an integrating constant.

NOTE: Integrals of the type $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$ *where* $p \neq 0, a \neq 0$ can be evaluated exactly in the same way.

Integration by Parts

The formula for the integration of a product of two functions is referred to as integration by parts. *i.e,*

$$\int (uv)dx = u\int vdx - \int\left\{\frac{du}{dx}\int vdx\right\}dx.$$

While applying the above rule for integration by parts to the product of two functions, care should be taken to choose properly the first function, i.e., the function not to be integrated.

Illustrative Examples:

Problem-01: $\int xe^x dx$

$sol^n : Let\ I = \int xe^x dx$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx$$

$$= xe^x - \int 1.e^x dx$$

$$= xe^x - e^x + c$$

where c is an integrating constant.

Exercise-01: $\int x^2 \cos x dx$

Ans: $x^2 \sin x + 2x\cos x - 2\sin x + c$

Exercise-02: $\int x^n \ln x dx$

Ans: $\frac{x^{n+1}}{(n+1)^2}\{(n+1)\ln x - 1\} + c$

Problem-02: $\int x^3 e^{-x} dx$

$sol^n : Let\ I = \int x^3 e^{-x} dx$

$$= x^3\int e^{-x}dx - \int\left\{\frac{d}{dx}(x^3)\int e^{-x}dx\right\}dx$$

$$= -x^3e^{-x} - \int\left\{3x^2(-e^{-x})\right\}dx$$

$$= -x^3e^{-x} + 3\int x^2e^{-x}dx$$

$$= -x^3e^{-x} + 3\left[x^2\int e^{-x}dx - \int\left\{\frac{d}{dx}(x^2)\int e^{-x}dx\right\}dx\right]$$

$$= -x^3e^{-x} + 3\left[-x^2e^{-x} - \int\left\{2x(-e^{-x})\right\}dx\right]$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\int xe^{-x}dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\left[x\int e^{-x}dx - \int\left\{\frac{dx}{dx}\int e^{-x}dx\right\}dx\right]$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\left[-xe^{-x} - \int 1.\left(-e^{-x}\right)dx\right]$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + 6\int e^{-x}dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + c$$

where c is an integrating constant.

Problem-03: $\int \tan^{-1} x dx$

$sol^n : Let\ I = \int \tan^{-1} x dx$

$$= \tan^{-1} x \int dx - \int \left\{ \frac{d}{dx}\left(\tan^{-1} x\right) \int dx \right\} dx$$

$$= x\tan^{-1} x - \int \frac{1}{1+x^2}.xdx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + c$$

where c is an integrating constant.

Exercise-03: $\int \cos^{-1} x dx$

Ans: $x\cos^{-1} x - \sqrt{1-x^2} + c$

Problem-04: $\int \frac{xe^x}{(1+x)^2}dx$

$sol^n : Let\ I = \int \frac{xe^x}{(1+x)^2}dx$

$$= \int \frac{(x+1-1)e^x}{(1+x)^2}dx$$

$$= \int \frac{e^x}{1+x}dx - \int \frac{e^x}{(1+x)^2}dx$$

$$= \frac{1}{1+x}\int e^x dx - \int \left\{ \frac{d}{dx}\left(\frac{1}{1+x}\right)\int e^x dx \right\} dx - \int \frac{e^x}{(1+x)^2}dx + c$$

$$= \frac{e^x}{1+x} - \int \left\{ \frac{-1}{(1+x)^2}e^x \right\} dx - \int \frac{e^x}{(1+x)^2}dx + c$$

$$= \frac{e^x}{1+x} + \int \frac{e^x}{(1+x)^2}dx - \int \frac{e^x}{(1+x)^2}dx + c$$

$$= \frac{e^x}{1+x} + c$$

where c is an integrating constant.

Exercise- 04: $\int e^x \frac{x^2+1}{(1+x)^2}dx$

Ans: $e^x\left(\frac{x-1}{x+1}\right) + c$

Problem-05: $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

Exercise- 05: $\int e^x \frac{x-1}{(1+x)^3} dx$

Ans: $e^x \frac{1}{(x+1)^2}+c$

$sol^n : Let\ I=\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

$$=\int e^x \frac{1-2x+x^2}{\left(1+x^2\right)^2} dx$$

$$=\int e^x \left[\frac{\left(1+x^2\right)-2x}{\left(1+x^2\right)^2}\right] dx$$

$$=\int e^x \left[\frac{1}{1+x^2}-\frac{2x}{\left(1+x^2\right)^2}\right] dx$$

$$=\int \frac{e^x}{1+x^2} dx-\int e^x \frac{2x}{\left(1+x^2\right)^2} dx$$

$$=\frac{1}{1+x^2}\int e^x dx-\int\left\{\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\int e^x dx\right\} dx-\int e^x \frac{2x}{\left(1+x^2\right)^2} dx$$

$$=\frac{e^x}{1+x^2}+\int e^x \frac{2x}{\left(1+x^2\right)^2} dx-\int e^x \frac{2x}{\left(1+x^2\right)^2} dx$$

$$=\frac{e^x}{1+x^2}+c$$

where c is an integrating constant.

Problem-06: $\int \cos\sqrt{x}dx$

Exercise-06: $\int x^2 \sin^2 x dx$

Ans: $\frac{x^3}{6}-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+c$

$sol^n : Let\ I=\int \cos\sqrt{x}dx$

Put $\sqrt{x}=z \therefore \frac{1}{2\sqrt{x}}dx=dz \Rightarrow dx=2zdz$

Now $I=\int \cos z.2zdz$

$$=2\int z\cos zdz$$

$$=2\left[z\int \cos zdz-\int\left\{\frac{dz}{dz}\int \cos zdz\right\}dz\right]$$

$$=2\left[z\sin z-\int \sin zdz\right]$$

$$= 2\left[z\sin z + \cos z\right] + c$$

$$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$$

where c is an integrating constant.

Problem-07: $\int x^2 \sin x \cos x dx$

$sol^n : Let\ I = \int x^2 \sin x \cos x dx$

$$= \frac{1}{2}\int x^2 .2\sin x \cos x dx$$

$$= \frac{1}{2}\int x^2 \sin 2x dx$$

$$= \frac{1}{2}\left[x^2 \int \sin 2x dx - \int\left\{\frac{d}{dx}\left(x^2\right)\int \sin 2x dx\right\}dx\right]$$

$$= \frac{1}{2}\left[-\frac{x^2}{2}\cos 2x + \int x\cos 2x dx\right]$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2}\left[x\int \cos 2x dx - \int\left\{\frac{d}{dx}(x)\int \cos 2x dx\right\}dx\right]$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2}\left[\frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x dx\right]$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2}\left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right] + c$$

$$= -\frac{x^2}{4}\cos 2x + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c$$

where c is an integrating constant.

Problem-08: $\int \frac{\ln(\ln x)}{x}dx$

$sol^n : Let\ I = \int \frac{\ln(\ln x)}{x}dx$

Put $\ln x = z \quad \therefore \frac{1}{x}dx = dz$

Now $I = \int \ln z dz$

$$= \ln z\int dz - \int\left\{\frac{d}{dz}(\ln z)\int dz\right\}dz$$

$$= z\ln z - \int \frac{1}{z}.z dz$$

$$= z\ln z - \int dz$$

$$= z\ln z - z + c$$

$$= \ln x \ln(\ln x) - \ln x + c$$

where c is an integrating constant.

Problem-09: $\int \frac{x}{1+\cos x}dx$

$Sol^n : Let\ I = \int \frac{x}{1+\cos x}dx$

$$= \int \frac{x}{2\cos^2 \frac{x}{2}}dx$$

$$= \frac{1}{2}\int x\sec^2 \frac{x}{2}dx$$

$$= \frac{1}{2}\left[x\int \sec^2 \frac{x}{2}dx - \int \left\{\frac{d}{dx}(x)\int \sec^2 \frac{x}{2}dx\right\}dx\right]$$

$$= \frac{1}{2}\left[x\frac{\tan\frac{x}{2}}{1/2} - \int \frac{\tan\frac{x}{2}}{1/2}dx\right] + c$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}dx + c$$

$$= x\tan\frac{x}{2} - \frac{\ln\left|\sec\frac{x}{2}\right|}{1/2} + c$$

$$= x\tan\frac{x}{2} - 2\ln\left|\sec\frac{x}{2}\right| + c$$

where c is an integrating constant.

Exercise-07: $\int \frac{x+\sin x}{1+\cos x}dx$

Ans: $x\tan\frac{x}{2} + c$.

Problem-10: $\int e^{ax}\cos bx dx$

$Sol^n : Let\ I = \int e^{ax}\cos bx dx$

$$= e^{ax}\int \cos bx dx - \int \left\{\frac{d}{dx}(e^{ax})\int \cos bx dx\right\}dx$$

$$= \frac{e^{ax}\sin bx}{b} - \int \left\{ae^{ax}\frac{\sin bx}{b}\right\}dx$$

Exercise-08: $\int e^{ax}\sin(bx+d)dx$

Ans: $\frac{e^{ax}\left[a\sin(bx+d) - b\cos(bx+d)\right]}{a^2+b^2} + c$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx dx$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[e^{ax}\int \sin bx dx - \int\left\{\frac{d}{dx}\left(e^{ax}\right)\int \sin bx dx\right\}dx\right]$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[\frac{-e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx dx\right]$$

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$\therefore\ I + \frac{a^2}{b^2}I = \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx$$

$$\Rightarrow \frac{I\left(a^2+b^2\right)}{b^2} = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{b^2}$$

$$\Rightarrow I = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2}$$

$$\therefore\ I = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2} + c$$

where c is an integrating constant.

Integration of Trigonometric Functions

Problem-01: $\int \frac{dx}{5+4\cos x}$

Exercise-01: $\int \frac{dx}{2+\cos x}$

Ans: $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+c$

$$Sol^n : Let\ I = \int \frac{dx}{5+4\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{dx}{\frac{5+5\tan^2 x/2 + 4 - 4\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{1+\tan^2 x/2\, dx}{9+\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2\, dx}{3^2+\tan^2 x/2}$$

Put $\tan\frac{x}{2} = z \therefore \sec^2\frac{x}{2}dx = 2dz$

Now $I = 2\int\frac{dz}{3^2+z^2}$

$= \frac{2}{3}\tan^{-1}\frac{z}{3} + c$

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$

where c is an integrating constant.

Problem-02: $\int\frac{dx}{2+3\cos 2x}$

Exercise-02: $\int\frac{dx}{3+5\cos x}$

Ans: $\frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + c$

$Sol^n: Let\ I = \int\frac{dx}{2+3\frac{1-\tan^2 x}{1+\tan^2 x}}$

$= \int\frac{dx}{\frac{2+2\tan^2 x+3-3\tan^2 x}{1+\tan^2 x}}$

$= \int\frac{1+\tan^2 x}{5-\tan^2 x}dx$

$= \int\frac{\sec^2 x}{5-\tan^2 x}dx$

Put $\tan x = z \therefore \sec^2 x dx = dz$

Now $I = \int\frac{dz}{5-z^2}$

$= \int\frac{dz}{\left(\sqrt{5}\right)^2 - z^2}$

$= \frac{1}{2\sqrt{5}}\ln\left(\frac{\sqrt{5}+z}{\sqrt{5}-z}\right) + c$

$= \frac{1}{2\sqrt{5}}\ln\left(\frac{\sqrt{5}+\tan x}{\sqrt{5}-\tan x}\right) + c$

where c is an integrating constant.

Problem-03: $\int \frac{dx}{4+5\sin x}$

Exercise-03: $\int \frac{dx}{3+2\cos x}$

Ans: $\frac{2}{\sqrt{5}}\tan^{-1}\left\{\frac{3\tan\left(\frac{x}{2}\right)+2}{\sqrt{5}}\right\}+c$

Sol^n : Let $I=\int \frac{dx}{4+5\sin x}$

$$=\int \frac{dx}{4+5.\frac{2\tan x/2}{1+\tan^2 x/2}}$$

$$=\int \frac{dx}{\frac{4+4\tan^2 x/2+10\tan x/2}{1+\tan^2 x/2}}$$

$$=\int \frac{1+\tan^2 x/2}{4\tan^2 x/2+10\tan x/2+4}$$

$$=\int \frac{\sec^2 x/2}{4\tan^2 x/2+10\tan x/2+4}$$

Put $\quad \tan\frac{x}{2}=z \qquad \therefore\ \sec^2\frac{x}{2}dx=2dz$

Now $\quad I=\int \frac{2dz}{4z^2+10z+4}$

$$=\frac{1}{2}\int \frac{dz}{\left(z^2+\frac{5}{2}z+1\right)}$$

$$=\frac{1}{2}\int \frac{dz}{z^2+2.\frac{5}{4}z+\left(\frac{5}{4}\right)^2-\frac{9}{16}}$$

$$=\frac{1}{2}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$=\frac{1}{2}\left[\frac{1}{2.\frac{3}{4}}\ln\left|\frac{\left(z+\frac{5}{4}\right)-\frac{3}{4}}{\left(z+\frac{5}{4}\right)+\frac{3}{4}}\right|\right]+c$$

$$=\frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}+\frac{1}{2}}{\tan\frac{x}{2}+2}\right|+c$$

where c is an integrating constant.

Problem-04: $\int\frac{11\cos x-16\sin x}{2\cos x+5\sin x}dx$

Sol^n : *Let* $I=\int\frac{11\cos x-16\sin x}{2\cos x+5\sin x}dx$

Put $\quad 11\cos x-16\sin x=l\left(2\cos x+5\sin x\right)+m\left(-2\sin x+5\cos x\right)+n$

Comparing coefficient of $\cos x$, $\sin x$ and constant terms, we get

$$2l+5m=11;\;\; 5l-2m=-16;\;\; n=0$$

Solving,

$$l=-2;m=3;n=0$$

Now $\quad I=\int\frac{-2\left(2\cos x+5\sin x\right)+3\left(-2\sin x+5\cos x\right)}{2\cos x+5\sin x}dx$

$$=-2\int\frac{2\cos x+5\sin x}{2\cos x+5\sin x}dx+3\int\frac{-2\sin x+5\cos x}{2\cos x+5\sin x}dx$$

$$=-2\int dx+3\int\frac{-2\sin x+5\cos x}{2\cos x+5\sin x}dx$$

$$=-2x+3\ln\left(2\cos x+5\sin x\right)+c$$

where c is an integrating constant.

Exercise-04: $\int\frac{2\sin x+3\cos x}{7\sin x-2\cos x}dx$

Ans: $\frac{8x}{53}+\frac{25}{53}\ln\left|7\sin x-2\cos x\right|+c$

Exercise-05: $\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$

Ans: $\frac{18x}{25}+\frac{1}{25}\ln\left|3\sin x+4\cos x\right|+c$