Maxima and Minima

In this section we will study problems where we wish to find the maximum or minimum values of a functions and also show how differentiation can be used to find the maximum and minimum values of a function. Because the derivative provides information about the gradient or slope of the graph of a function we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality.

Finding a maximum or a minimum of a function is clearly an important in everyday experience such as

- A manufacturer wants to maximize her profits.
- A businessman wish to minimize the cost of production.
- A contractor wants to minimize his costs subject to doing a good job.
- A physicist wants to find the wavelength that produces the maximum intensity of radiation.

There are two types of maxima and minima of a functions:

- Absolute (Global) maxima and Absolute (Global) minima
- Local (Relative) maxima and Local (Relative) minima.

Note:

1. Maxima is the plural form of maximum
2. Minima is also the plural form of maximum
3. Extrema is the plural form of extremun
4. Maxima + Minima=Extrema and Maximum + Minimum=Extremum

Some Special points on a curve:

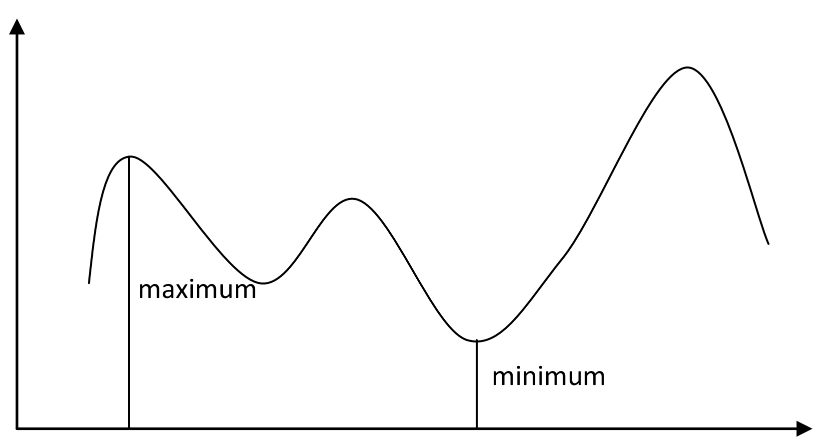
Critical point: A point on a curve in which derivative is zero or function is not differentiable.

Saddle point: A saddle point is a point on a curve where second order derivative zero or undefined.

Stationary point: A stationary point is a point on the curve where gradient of a function is zero. If gradient of the curve changes sign at stationary point then it called turning point otherwise horizontal Inflection.

Definition: A function $f(x)$ is said to have a maximum for a value a of x if $f(a)$ is greater than any other value that the function can have in the small neighborhood of $x=a$.

Similarly, a function $f(x)$ is said to have a minimum for a value a of x if $f(a)$ is less than any other value that the function can have in the small neighborhood of $x=a$.



Theorem (Fermat's Theorem): If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Working rule for finding maxima and minima:

If function $f(x)$ be given, find $f'(x)$ and equate it to zero. Solve this equation for x.

Let its roots be $a_1, a_1, a_3, \ldots \ \ldots \ldots$

Find $f''(x)$ and hence find $f''(a_1), f''(a_2), \ldots \ldots \ldots \ldots$

- ✓ If $f''(a_1)$ is negative we have a maximum at $x = a_1$.
- ✓ If $f''(a_1)$ is positive we have a minimum at $x = a_1$.
- ✓ If $f''(a_1) = 0$ find $f'''(x)$ and then $f'''(a_1)$.
- ✓ If $f'''(a_1) \neq 0$, then there is neither maxima nor minima at $x = a_1$.

If $f'''(a_1) = 0$; find $f^{iv}(x)$ and then $f^{iv}(a_1)$.

If $f^{iv}(a_1)$ is negative, then $f(x)$ is maximum and if $f^{iv}(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.

If $f^{iv}(a_1) = 0$ Then find $f^{v}(a_1)$ and so on.

- ✓ $f'(a_1) = f''(a_1) = \ldots\ldots\ldots\ldots\ldots = f^{n-1}(a_1) = 0 \quad and \ f^{n}(a_1) \neq 0$
- ✓ if n be odd, then there is neither maxima nor minima at $x = a_1$
- ✓ if n be even and if $f^{n}(a_1)$ is negative then $f(x)$ is maximum at $x = a_1$ and
- ✓ if $f^{n}(a_1)$ is positive then $f(x)$ is minimum at $x = a_1$.

Mathematical problems

Problem 01: Find the maximum and minimum values of $y = x^5 - 5x^4 + 5x^3 - 10$.

Solution:

Let $f(x) = y = x^5 - 5x^4 + 5x^3 - 10$

Differentiating with respect to x we get,

$$f'(x) = 5x^4 - 20x^3 + 15x^2 \qquad \cdots\cdots\cdots(1)$$

We know that for maximum and minimum values,

$$f'(x) = 0$$

$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow x^4 - 4x^3 + 3x^2 = 0$$

$$\Rightarrow x^2\left(x^2 - 4x + 3\right) = 0$$

$$\Rightarrow x^2(x-3)(x-1) = 0$$

$$\therefore x = 0, 1, 3$$

Again, differentiating eq.(1) with respect to x we get,

$$f''(x) = 20x^3 - 60x^2 + 30x$$

For $x = 1$ we get,

$$f''(1) = 20 - 60 + 30 = -10 < 0$$

Therefore, the given function is maximum at $x = 1$.

The maximum value is,

$$f(1) = 1 - 5 + 5 - 10 = -9 \qquad (Ans.)$$

For $x = 3$ we get,

$$f''(3) = 540 - 540 + 90 = 90 > 0$$

Therefore, the given function is minimum at $x=3$.

The minimum value is,

$$f\left(3\right) = 243 - 405 + 135 - 10 = -37 \qquad (Ans.)$$

For $x = 0$ we get,

$$f''(0) = 0$$

Therefore the test fails.

$$\therefore f'''(x) = 60x^2 - 120x + 30$$

$$\therefore f'''(0) = 30 \neq 0$$

Therefore, the given function is neither maximum nor minimum at $x = 0$ $\qquad (Ans.)$

Problem 02: Find the extremmum values of $\frac{x^3}{3} + ax^2 - 3a^2 x$.

Solution:

$$Let \quad f(x) = \frac{x^3}{3} + ax^2 - 3a^2 x$$

Differentiating with respect to x we get,

$$f'(x) = \frac{3x^2}{3} + 2ax - 3a^2$$

$$= x^2 + 2ax - 3a^2 \qquad \cdots\cdots\cdots(1)$$

We know that for maximum and minimum values,

$$f'(x) = 0$$

$$\Rightarrow x^2 + 2ax - 3a^2 = 0$$

$$\Rightarrow x^2 + 3ax - ax - 3a^2 = 0$$

$$\Rightarrow x(x + 3a) - a(x + 3a) = 0$$

$$\Rightarrow (x + 3a)(x - a) = 0$$

$$\therefore x = a, \ -3a$$

Again, differentiating eq.(1) with respect to x we get,

$$f''(x) = 2x + 2a$$

For $x = -3a$ we get,

$$f''(-3a) = -6a + 2a = -4a < 0$$

Therefore, the given function is maximum at $x = -3a$.

The maximum value is,

$$f\left(-3a\right) = 9a^3 \qquad (Ans.)$$

For $x = a$ we get,

$$f''(a) = 2a + 2a = 4a > 0$$

Therefore, the given function is minimum at $x = a$

The minimum value is,

$$f\left(a\right) = \frac{5}{3}a^3 \qquad (Ans).$$

Problem 03: Find the maximum and minimum values of $y = 5x^3 - 3x^2 - 2x + 5$.

Solution:

$$Let \quad f(x) = 5x^3 - 3x^2 - 2x + 5$$

Differentiating with respect to x we get,

$$f'(x) = 15x^2 - 6x - 2 \qquad \cdots\cdots\cdots(1)$$

We know that for maximum and minimum values,

$$f'(x) = 0$$

$$\Rightarrow 15x^2 - 6x - 2 = 0$$

$$\Rightarrow x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4.15.(-2)}}{2.15}$$

$$= \frac{6 \pm \sqrt{36 + 120}}{30}$$

$$= \frac{6 \pm \sqrt{156}}{30}$$

$$\therefore x = 0.616, \ -0.216$$

Again, differentiating eq.(1) with respect to x we get,

$$f''(x) = 30x - 6$$

For $x = 0.616$ we get,

$$f''(0.616) = 30 \times 0.616 - 6 = 12.48 > 0$$

Therefore, the given function is minimum at $x = 0.616$.

The minimum value is,

$$f\left(0.616\right) = 5\left(0.616\right)^3 - 3\left(0.616\right)^2 - 2\left(0.616\right) + 5 = 3.8 \qquad (Ans.)$$

For $x = -0.216$ we get,

$$f''(-0.216) = 30\left(-0.216\right) - 6 = -12.48 < 0$$

Therefore, the given function is maximum at $x = -0.216$

The maximum value is,

$$f\left(a\right) = 5\left(-0.216\right)^3 - 3\left(-0.216\right)^2 - 2\left(-0.216\right) + 5 = 5.24 \qquad (Ans.)$$

Try Yourself:

1. Find the maximum and minimum values for $f(x) = 3x^4 - 25x^2 + 60x$.
2. Find the maximum and minimum values for $f(x) = x^6 - 12x^5 + 36x^4 + 4$.
3. Find the maximum and minimum values for $f\left(x\right) = 1 + 2\sin x + 3\cos^2 x$.
4. Find maximum or minimum value of the function $y = 5x^3 - 3x^2 + 2x + 5$.
5. Find the saddle point , critical and point of inflection for $f(x) = 2x^4 + 3x^3 + 4x + 3$
6. Find maxima and minima of the function of the function $f(x) = x^3 - 6x^2 + 11x - 6$
7. Find maxima and minima of the function of the function $f(x) = \frac{x^5}{5} - \frac{x^4}{4}$