Lecture Sheet

On

Series

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Sequence and Series: A sequence is defined as an arrangement of any objects or a set of numbers in a particular order followed by some rule. On the other hand, a series is defined as the sum of the elements of a sequence.

If $a_1,\ a_2, a_3,\ \cdots$ is a sequence, then the corresponding series is given by

$$S_n = a_1 + a_2 + a_3 + \cdots$$

Note: The series is finite or infinite depending if the sequence is finite or infinite.

Types of Sequence and Series: Some of the most common examples of sequences and series are:

Arithmetic Sequences and Series: A sequence in which every term is created by adding or subtracting a definite number to the preceding number is an arithmetic sequence. The series consists of the terms of arithmetic sequence is called a series in arithmetic progression. If $a,\ a+d, a+2d,\ \cdots$ is an arithmetic sequence, then the corresponding series is given by

$$S_n = a + (a+d) + (a+2d) + \cdots$$

Geometric Sequences and Series: A sequence in which every term is obtained by multiplying or dividing a definite number with the preceding number is known as a geometric sequence. The series consists of the terms of geometric sequence is called a series in geometric progression. If $a,\ ar, ar^2,\ \cdots$ is a geometric sequence, then the corresponding series is given by

$$S_n = a + ar + ar^2 + \cdots$$

Harmonic Sequences and Series: A sequence in which terms are reciprocal of the terms of an arithmetic sequence is called a harmonic sequence. The series consists of the terms of harmonic sequence is called a series in harmonic progression. If $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots$ is a harmonic sequence, then the corresponding series is given by

$$S_n = \frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \cdots$$

Fibonacci sequence and Series: Fibonacci numbers form an interesting sequence of numbers in which each element is obtained by adding two preceding elements and the sequence starts with 0 and 1. Sequence is defined as, $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

If $0, 1, 1, 2, 3, 5, 8, 13, 21,\ \cdots$ is a Fibonacci sequence, then the corresponding series is given by

$$S_n = 0 + 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + \cdots$$

Note:

a) $1+2+3+4+\cdots+n=\dfrac{n(n+1)}{2}$.

b) $1+3+5+7+\cdots+(2n-1)=n^2$.

c) $1^2+2^2+3^2+4^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

d) $1^3+2^3+3^3+4^3+\cdots+n^3=\left\{\dfrac{n(n+1)}{2}\right\}^2$.

e) $$a+ar+ar^2+ar^3+\cdots+ar^{n-1}=\begin{cases} a.\dfrac{r^n-1}{r-1} & if\ \ r>1 \\ a.\dfrac{1-r^n}{1-r} & if\ \ r<1 \end{cases}$$

f) $a+ar+ar^2+ar^3+\cdots+\infty=\dfrac{a}{1-r}\quad where\ \ r<1\ \ and\ \ n\to\infty$.

Problem-01: Sum the series $4+44+444+\cdots$ to n terms.

Solution: Let $S_n=4+44+444+\cdots\ to\ n\ terms$

$$=4[1+11+111+\cdots\ to\ n\ terms]$$

$$=\frac{4}{9}[9+99+999+\cdots\ to\ n\ terms]$$

$$=\frac{4}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\cdots\ to\ n\ terms\right]$$

$$=\frac{4}{9}\left[(10+10^2+10^3+\cdots\ to\ n\ terms)-(1+1+1+\cdots\ to\ n\ terms)\right]$$

$$=\frac{4}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$=\frac{40}{81}(10^n-1)-\frac{4n}{9}.$$

Problem-02: Sum the series $1^2+2^2+3^2+4^2+\cdots+n^2$.

Solution: Let $S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$.

Now consider the following identity to find the value of S_n:

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1 \qquad \cdots(1)$$

Substituting $n = 1, 2, 3, 4, \cdots, n$ in (1), we get

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$

$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$

$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$n^3 - (n-1)^3 = 3.n^2 - 3.n + 1.$$

Adding these we get

$$n^3 - 0^3 = 3\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - 3\left(1 + 2 + 3 + \cdots + n\right) + \left(1 + 1 + 1 + \cdots + n \text{ } times\right)$$

$$or,\ n^3 = 3S_n - 3.\frac{n(n+1)}{2} + n$$

$$or,\ 3S_n = n^3 + \frac{3n(n+1)}{2} - n$$

$$or,\ 3S_n = n\left(n^2 - 1\right) + \frac{3n(n+1)}{2}$$

$$or,\ 3S_n = n(n+1)\left(n - 1 + \frac{3}{2}\right)$$

$$or,\ 3S_n = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore\ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem-03: Sum the series $1^3+2^3+3^3+4^3+\cdots+n^3$.

Solution: Let $S_n=1^3+2^3+3^3+4^3+\cdots+n^3$.

Now consider the following identity to find the value of S_n:

$$n^4-(n-1)^4=4n^3-6n^2+4n-1 \qquad \cdots(1)$$

Substituting $n=1,2,3,4,\cdots,n$ in (1), we get

$$1^4-0^4=4.1^3-6.1^2+4.1-1$$

$$2^4-1^4=4.2^3-6.2^2+4.2-1$$

$$3^4-2^4=4.3^3-6.3^2+4.3-1$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$n^4-(n-1)^4=4n^3-6n^2+4n-1.$$

Adding these we get

$$n^4-0^4=4\left(1^3+2^3+\cdots+n^3\right)-6\left(1^2+2^2+\cdots+n^2\right)+4\left(1+2+\cdots+n\right)-\left(1+1+\cdots n \textit{ times}\right)$$

$$or,\ n^4=4S_n-6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}-n$$

$$or,\ n^4=4S_n-n(n+1)(2n+1)+2n(n+1)-n$$

$$or,\ 4S_n=n^4+n(n+1)(2n+1)-2n(n+1)+n$$

$$or,\ 4S_n=n^4+2n^3+3n^2+n-2n^2-2n+n$$

$$or,\ 4S_n=n^4+2n^3+n^2$$

$$or,\ S_n=\frac{n^2\left(n^2+2n+1\right)}{4}$$

$$\therefore\ 1^3+2^3+3^3+\cdots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Summation by Method of Difference: If we are able to express u_n in the form $v_n - v_{n-1}$, where v_n is some function of n, then we can sum the series to n terms.

For, by hypothesis,

$$u_n = v_n - v_{n-1}$$

$$u_{n-1} = v_{n-1} - v_{n-2}$$

$$u_{n-2} = v_{n-2} - v_{n-3}$$

$$\cdots \quad \cdots \quad \cdots$$

$$u_2 = v_2 - v_1$$

$$u_1 = v_1 - v_0$$

whence by addition

$$S_n = v_n - v_0.$$

Note: To determine $v_r - v_{r-1}$, Firstly, multiply u_r, by the subtraction of the previous term of first term of u_r from the next term of last term of u_r and then divide the subtraction by an appropriate constant for making equal to u_r.

Problem-04: Sum the series $1\cdot 2+2\cdot 3+3\cdot 4+\cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = n(n+1)$$

$$= n(n+1)\left\{\frac{(n+2)-(n-1)}{3}\right\}$$

$$= \frac{1}{3}n(n+1)(n+2) - \frac{1}{3}(n-1)n(n+1)$$

$$= v_n - v_{n-1}$$

where $v_n = \frac{1}{3}n(n+1)(n+2)$ and $v_{n-1} = \frac{1}{3}(n-1)n(n+1)$.

For $n=1$, we get $v_0 = 0$.

Therefore, the sum of the given series is

$$S_n = v_n - v_0 = \frac{1}{3}n(n+1)(n+2) - 0$$

$$\therefore\ S_n = \frac{1}{3}n(n+1)(n+2).$$

Problem-05: Sum the series $1\cdot4\cdot7 + 4\cdot7\cdot10 + 7\cdot10\cdot13 + \cdots$ to *n* terms.

Solution: Here, the nth term of the given series is,

$$u_n = (3n-2)(3n+1)(3n+4)$$

$$= (3n-2)(3n+1)(3n+4)\left\{\frac{(3n+7)-(3n-5)}{12}\right\}$$

$$= \frac{1}{12}(3n-2)(3n+1)(3n+4)(3n+7) - \frac{1}{12}(3n-5)(3n-2)(3n+1)(3n+4)$$

$$= v_n - v_{n-1}$$

where $v_n = \frac{1}{12}(3n-2)(3n+1)(3n+4)(3n+7)$

and $v_{n-1} = \frac{1}{12}(3n-5)(3n-2)(3n+1)(3n+4)$.

For $n=1$, we get $v_0 = -\frac{14}{3}$.

Therefore, the sum of the given series is

$$S_n = v_n - v_0 = \frac{1}{12}(3n-2)(3n+1)(3n+4)(3n+7) + \frac{14}{3}$$

$$\therefore\ S_n = \frac{n}{4}\left(27n^3 + 90n^2 + 45n - 50\right).$$ **ANS.**

If in an arithmetic series, every term contains r factors then the sum of this series will be determined as follows:

Let the nth term of an arithmetic series is,

$$u_n = (a+nb)\left(a+\overline{n+1}\cdot b\right)\cdots\left(a+\overline{n+r-1}\cdot b\right)$$ where a, b, r are constants.

$$= (a+nb)\left(a+\overline{n+1}\cdot b\right)\cdots\left(a+\overline{n+r-1}\cdot b\right)\left\{\frac{\left(a+\overline{n+r}\cdot b\right)-\left(a+\overline{n-1}\cdot b\right)}{(r+1)b}\right\}$$

$$= v_n - v_{n-1}$$

where $v_n = \dfrac{u_n\left(a+\overline{n+r}\cdot b\right)}{(r+1)b}$ and $v_{n-1} = \dfrac{u_n\left(a+\overline{n-1}\cdot b\right)}{(r+1)b}$.

Now putting $n = 1, 2, 3, \cdots, n$, we get

$$u_1 = v_1 - v_0$$

$$u_2 = v_2 - v_1$$

$$u_3 = v_3 - v_2$$

$$\cdots \quad \cdots \quad \cdots$$

$$u_{n-1} = v_{n-1} - v_{n-2}$$

$$u_n = v_n - v_{n-1}$$

whence by addition

$$S_n = v_n - v_0 = v_n + c \quad \text{where } c = -v_0 .$$

$$\therefore\ S_n = \frac{u_n\left(a+\overline{n+r}\cdot b\right)}{(r+1)b} + c = \frac{nth\ term \times Next\ factor}{(Num. of\ \ factors\, in\, numerator+1)\times Common\, difference} + c .$$

Problem-06: Sum the series $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = n(n+1)(n+2)$$

The sum of the given series is

$$S_n = \frac{n(n+1)(n+2)(n+3)}{(3+1)\times1} + c$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} + c \qquad \cdots(1)$$

For $n=1$, we get

$$S_1 = 1\cdot2\cdot3 = \frac{1\cdot2\cdot3\cdot4}{4} + c .$$

$$\therefore\ c = 0$$

Putting the value of c in (1), we get

$$S_n = \frac{n(n+1)(n+2)(n+3)}{4} \qquad \textbf{ANS.}$$

Problem-07: Sum the series $1\cdot4\cdot7+2\cdot5\cdot8+3\cdot6\cdot9+\cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = n(n+3)(n+6)$$

$$= n(n+1+2)(n+6)$$

$$= n(n+1)(n+6) + 2n(n+6)$$

$$= n(n+1)(n+2+4) + 2n(n+1+5)$$

$$= n(n+1)(n+2) + 4n(n+1) + 2n(n+1) + 10n$$

$$= n(n+1)(n+2) + 6n(n+1) + 10n$$

The sum of the given series is

$$S_n = \frac{n(n+1)(n+2)(n+3)}{(3+1)} + \frac{6n(n+1)(n+2)}{(2+1)} + \frac{10n(n+1)}{(1+1)} + c$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} + 2n(n+1)(n+2) + 5n(n+1) + c \qquad \cdots(1)$$

For $n=1$, we get

$$S_1 = 1\cdot 4\cdot 7 = \frac{1\cdot 2\cdot 3\cdot 4}{4} + 2\cdot 1\cdot 2\cdot 3 + 5\cdot 1\cdot 2 + c\,.$$

$$\therefore\ c = 0$$

Putting the value of c in (1), we get

$$S_n = \frac{n(n+1)(n+2)(n+3)}{4} + 2n(n+1)(n+2) + 5n(n+1) \qquad \textbf{ANS.}$$

Problem-08: Sum the series $1\cdot 2^2 + 2\cdot 3^2 + 3\cdot 4^2 + \cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = n(n+1)^2$$

$$= n(n+1)(n+1)$$

$$= n(n+1)(n+2-1)$$

$$= n(n+1)(n+2) - n(n+1)$$

The sum of the given series is

$$S_n = \frac{n(n+1)(n+2)(n+3)}{(3+1)} - \frac{n(n+1)(n+2)}{(2+1)} + c$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - \frac{n(n+1)(n+2)}{3} + c \qquad \cdots(1)$$

For $n=1$, we get

$$S_1 = 1\cdot 2^2 = \frac{1\cdot 2\cdot 3\cdot 4}{4} - \frac{1\cdot 2\cdot 3}{3} + c\ .$$

$$\therefore\ c = 0$$

Putting the value of c in (1), we get

$$S_n = \frac{n(n+1)(n+2)(n+3)}{4} - \frac{n(n+1)(n+2)}{3}$$

$$= \frac{n(n+1)(n+2)}{12}(3n+9-4)$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \qquad \textbf{ANS.}$$

Problem-09: Sum the series $1^2\cdot 2\cdot 3 + 2^2\cdot 3\cdot 4 + 3^2\cdot 4\cdot 5 + \cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = n^2(n+1)(n+2)$$

$$= n(n+1)(n+2)(n+3-3)$$

$$= n(n+1)(n+2)(n+3) - 3n(n+1)(n+2)$$

The sum of the given series is

$$S_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{(4+1)\times 1} - \frac{3n(n+1)(n+2)(n+3)}{(3+1)\times 1} + c$$

$$= \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{3n(n+1)(n+2)(n+3)}{4} + c \qquad \cdots(1)$$

For $n=1$, we get

$$S_1 = 1^2 \cdot 2 \cdot 3 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{5} - \frac{3\cdot 1\cdot 2\cdot 3\cdot 4}{4} + c\,.$$

$$\therefore\ c = 0$$

Putting the value of c in (1), we get

$$S_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{3n(n+1)(n+2)(n+3)}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{20}(4n+16-15)$$

$$= \frac{n(n+1)(n+2)(n+3)(4n+1)}{20}$$ **ANS.**

Again, if in an arithmetic series, every term contains r reciprocal factors then the sum of this series will be determined as follows:

Let the nth term of an arithmetic series is,

$$u_n = \frac{1}{(a+nb)\left(a+\overline{n+1}\cdot b\right)\cdots\left(a+\overline{n+r-1}\cdot b\right)}$$ where a, b, r are constants.

$$= \frac{1}{(r-1)b}\,\frac{\left(a+\overline{n+r-1}\cdot b\right)-(a+nb)}{(a+nb)\left(a+\overline{n+1}\cdot b\right)\cdots\left(a+\overline{n+r-1}\cdot b\right)}$$

$$= v_{n-1} - v_n$$

where $v_n = \dfrac{u_n \times (a+nb)}{(r-1)b}$ and $v_{n-1} = \dfrac{u_n \times \left(a+\overline{n+r-1}\cdot b\right)}{(r-1)b}$

Now putting $n = 1, 2, 3, \cdots, (n-1), n$, we get

$$u_1 = v_0 - v_1$$

$$u_2 = v_1 - v_2$$

$$u_3 = v_2 - v_3$$

$$\cdots \quad \cdots \quad \cdots$$

$$u_{n-1} = v_{n-2} - v_{n-1}$$

$$u_n = v_{n-1} - v_n$$

whence by addition

$$S_n = v_0 - v_n = c - v_n \quad \text{where } c = v_0 .$$

$$\therefore\ S_n = c - \frac{u_n(a+nb)}{(r-1)b} = c - \frac{nth\ term \times First\ factor}{(Num. of\ \ factors\, in\, numerator - 1) \times Common\, difference} .$$

Problem-10: Sum the series $\frac{1}{1\cdot 4\cdot 7} + \frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = \frac{1}{(3n-2)(3n+1)(3n+4)}$$

$$= \frac{1}{6}\, \frac{(3n+4)-(3n-2)}{(3n-2)(3n+1)(3n+4)}$$

$$= v_{n-1} - v_n$$

where $v_n = \frac{1}{6}\, \frac{1}{(3n+1)(3n+4)}$

and $v_{n-1} = \frac{1}{6}\, \frac{1}{(3n-2)(3n+1)}$.

For $n=0$, we get $v_0 = \frac{1}{24}$.

Therefore, the sum of the given series is

$$S_n = v_0 - v_n = \frac{1}{24} - \frac{1}{6}\frac{1}{(3n+1)(3n+4)} \quad . \quad \textbf{ANS.}$$

The sum up to infinity is,

$$\lim_{n\to\infty} S_n = \frac{1}{24}. \qquad \textbf{ANS.}$$

Problem-11: Sum the series $\frac{4}{2\cdot3\cdot4} + \frac{7}{3\cdot4\cdot5} + \frac{10}{4\cdot5\cdot6} + \cdots$ to n terms.

Solution: Here, the nth term of the given series is,

$$u_n = \frac{3n+1}{(n+1)(n+2)(n+3)}$$

$$= \frac{3(n+1)-2}{(n+1)(n+2)(n+3)}$$

$$= \frac{3}{(n+2)(n+3)} - \frac{2}{(n+1)(n+2)(n+3)}$$

The sum of the given series is

$$S_n = c - \left[\frac{3}{(2-1)\times1\times(n+3)} - \frac{2}{(3-1)\times1\times(n+2)(n+3)}\right]$$

$$= c - \frac{3}{(n+3)} + \frac{1}{(n+2)(n+3)} \qquad \cdots(1)$$

For $n=1$, we get

$$S_1 = \frac{4}{2\cdot3\cdot4} = c - \frac{3}{4} + \frac{1}{3\cdot4}.$$

$$\therefore \quad c = \frac{5}{6}$$

Putting the value of c in (1), we get

$$S_n = \frac{5}{6} - \frac{3}{(n+3)} + \frac{1}{(n+2)(n+3)}$$

$$= \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)} \qquad \textbf{ANS.}$$

Exercise:

Problem-01: Sum the series $5+55+555+\cdots$ to n terms.

Problem-02: Sum the series $1\cdot2\cdot4+2\cdot3\cdot5+3\cdot4\cdot6+\cdots$ to n terms.

Problem-03: Sum the series $1\cdot5\cdot9+2\cdot6\cdot10+3\cdot7\cdot11+\cdots$ to n terms.

Problem-04: Sum the series $1\cdot2^2\cdot3+2\cdot3^2\cdot4+3\cdot4^2\cdot5+\cdots$ to n terms.

Problem-05: Sum the series $2\cdot4\cdot6^2+4\cdot6\cdot8^2+6\cdot8\cdot10^2+\cdots$ to n terms.

Problem-06: Sum the series $\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\frac{1}{7\cdot9\cdot11}+\cdots$ to n terms.

Problem-07: Sum the series $\frac{1}{2\cdot5\cdot8}+\frac{1}{5\cdot8\cdot11}+\frac{1}{8\cdot11\cdot14}+\cdots$ to n terms.

Problem-08: Sum the series $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$ to n terms.