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<u>Indeterminate forms</u>: If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ then it is called an indeterminate form at $x=a$. The forms $\frac{\propto}{\propto}$, $0\times\propto$, $\propto-\propto$, 0^0, $1^{\propto}$ and $\propto^0$ are also indeterminate forms.

<u>Theorem</u>: State and prove L' Hospital's Rule.

Statement: If two functions $f(x)$ and $g(x)$ are continuous at $x=a$, also their derivatives $f'(x)$, $g'(x)$ are continuous at this point and $f(a)=g(a)=0$ but $g'(a)\neq 0$ then L' Hospital's rule states as,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$$

In case, $f'(a)=g'(a)=0$, the rule maybe extended.

Proof: We have $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f\left(a+\overline{x-a}\right)}{g\left(a+\overline{x-a}\right)}$

Expanding by Taylor's Theorem we get

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+R_1}{g(a)+(x-a)g'(a)+\frac{(x-a)^2}{2!}g''(a)+\cdots+R_2}\quad\cdots(1)$$

where $R_1=\frac{(x-a)^n}{n!}f^n\left(a+\theta_1\overline{x-a}\right)$, $0<\theta_1<1$ and $R_2=\frac{(x-a)^n}{n!}g^n\left(a+\theta_2\overline{x-a}\right)$, $0<\theta_2<1$.

Since $f(a)=g(a)=0$ so from (1) we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+R_1}{(x-a)g'(a)+\frac{(x-a)^2}{2!}g''(a)+\cdots+R_2}$$

$$=\lim_{x\to a}\frac{f'(a)+\frac{(x-a)}{2!}f''(a)+\cdots+R_1}{g'(a)+\frac{(x-a)}{2!}g''(a)+\cdots+R_2}$$

$$=\frac{f'(a)}{g'(a)}$$

$$\therefore \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\frac{f'(a)}{g'(a)}$$ **(Proved).**

Evaluate the following limits:

<u>Problem 01:</u> Find $\lim_{x\to 0}\frac{\tan x}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{\tan x}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\sec^2 x$$
$$=1$$

<u>Problem 02:</u> Find $\lim_{x\to 0}\frac{\sin^{-1} x}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{\sin^{-1} x}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}$$
$$=1$$

<u>Problem 03:</u> Find $\lim_{x\to \propto}\frac{(\ln x)^2}{x}$

<u>Sol:</u> Given that,

$$\lim_{x\to \propto}\frac{(\ln x)^2}{x} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=\lim_{x\to \propto}2\ln x.\frac{1}{x}$$
$$=2\lim_{x\to \propto}\frac{\ln x}{x} \qquad ;\left[Form\ \frac{\propto}{\propto}\right]$$
$$=2\lim_{x\to \propto}\frac{1}{x}$$
$$=2.\frac{1}{\propto}$$
$$=0$$

<u>Problem 04:</u> Find $\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1} x}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\frac{x^2}{\sin x\sin^{-1} x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x}{\cos x\sin^{-1} x+\frac{\sin x}{\sqrt{1-x^2}}} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2x\sqrt{1-x^2}}{\cos x\sin^{-1} x\sqrt{1-x^2}+\sin x} \qquad ;\left[Form\ \frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{2\sqrt{1-x^2}+\frac{2x^2}{\sqrt{1-x^2}}}{-\sin x\sin^{-1} x\sqrt{1-x^2}+\cos x\left(1+\frac{2x}{\sqrt{1-x^2}}\right)+\cos x}$$
$$=\frac{2}{1+1}$$
$$=1$$

<u>Problem 05:</u> Find $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

<u>Sol:</u> Given that,

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x} &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x} &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x} &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}\\
&=\frac{1+1}{1}\\
&=2
\end{aligned}$$

<u>Problem 05:</u> Find $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

<u>Sol:</u> Given that,

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) &&;\left[Form\ \propto-\propto\right]\\
&=\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\\
&=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x\sin x}\right) &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\left(\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right)\\
&=\lim_{x\to 0}\left(\frac{x\sin x}{\sin x+x\cos x}\right) &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\left(\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right)\\
&=\frac{0}{1+1}=0
\end{aligned}$$

<u>Problem 06:</u> Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

<u>Sol:</u> Given that,

$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &&;\left[Form \propto-\propto\right]\\
&=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right) &&;\left[Form\ \frac{0}{0}\right]\\
&=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)\\
&=\frac{0}{1+1-0}\\
&=0
\end{aligned}$$

<u>Problem 07:</u> Find $\lim_{x\to 0}\sin x\ln x^2$

<u>Sol:</u> Given that,

$$\begin{aligned}
&\lim_{x\to 0}\sin x\ln x^2 &&;\left[Form\ 0\times\propto\right]\\
&=\lim_{x\to 0}\frac{2\ln x}{\cos ecx} &&;\left[Form\ \frac{\propto}{\propto}\right]\\
&=2\lim_{x\to 0}\left(\frac{1/x}{-\cos ecx\cot x}\right)\\
&=-2\lim_{x\to 0}\left(\frac{\sin^2 x}{x\cos x}\right) &&;\left[Form\ \frac{0}{0}\right]\\
&=-2\lim_{x\to 0}\left(\frac{2\sin x\cos x}{\cos x-x\sin x}\right)\\
&=-2.\frac{0}{1-0}\\
&=0
\end{aligned}$$

<u>Problem 08:</u> Find $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$

<u>Sol:</u> Given that,

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} \qquad ;\left[Form\ \infty^{\infty}\right]$$

$$Let\ \ y=\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$$

$$\therefore\ \ln y=\frac{1}{x}\ln\left(\frac{\tan x}{x}\right)$$

$$\therefore \lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{\tan x}{x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{\ln\left(\frac{\tan x}{x}\right)}{x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{x}{\tan x}\cdot\frac{x\sec^2 x-\tan x}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{x\sin 2x}\right) \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{2-2\cos 2x}{\sin 2x+2x\cos 2x}\right) \quad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\left(\frac{4\sin 2x}{2\cos 2x+2\cos 2x-4x\sin 2x}\right)$$

$$=\frac{0}{2+2-0}$$

$$=0$$

$$\therefore \lim_{x\to 0} y=e^0$$

$$\therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}=1$$

<u>Problem 09</u>: Find $\lim\limits_{x\to \pi/2}(\sin x)^{\tan x}$

<u>Sol:</u> Given that,

$$\lim_{x\to \pi/2}(\sin x)^{\tan x} \qquad ;\left[Form\ 1^{\infty}\right]$$

$$Let\ \ y=(\sin x)^{\tan x}$$

$$\therefore\ \ln y=\tan x\ln(\sin x)$$

$$\therefore \lim_{x\to \pi/2}\ln y=\lim_{x\to \pi/2}\tan x\ln(\sin x) \quad ;\left[Form\ 0\times\infty\right]$$

$$=\lim_{x\to \pi/2}\frac{\ln(\sin x)}{\cot x} \qquad ;\left[Form\ \frac{0}{0}\right]$$

$$=\lim_{x\to \pi/2}\frac{\cot x}{\cos ec^2 x}$$

$$=0$$

$$\therefore \lim_{x\to \pi/2} y=e^0$$

$$\therefore \lim_{x\to \pi/2}(\sin x)^{\tan x}=1$$

Homework:

<u>Problem 01:</u> Find $\lim\limits_{x\to 0}\dfrac{\sin x\sin^{-1}x}{x^2}$ Ans: 1

<u>Problem 02:</u> Find $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$ Ans: $\dfrac{1}{3}$

<u>Problem 03:</u> Find $\lim\limits_{x\to 0}(\cos x)^{\cos ec^2 x}$ Ans: $e^{-\frac{1}{2}}$

<u>Problem 04:</u> Find $\lim\limits_{x\to 0}\left(\dfrac{x}{x-1}-\dfrac{x}{\ln x}\right)$ Ans: $\dfrac{1}{2}$

<u>Problem 05:</u> Find $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\cot^2 x\right)$ Ans: $\dfrac{2}{3}$

<u>Problem 06:</u> Find $\lim\limits_{x\to 0}(\sin x)^x$ Ans: 1