Beta & Gamma Functions

Beta Function or First Eulerian Integral: A function of the form,

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; m,\ n>0$$

is called Beta function or first Eulerian integral and it is denoted by, $\beta(m,n)$.

$$i.e,\ \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; m,\ n>0.$$

Gamma Function or Second Eulerian Integral: A function of the form,

$$\int_0^\infty e^{-x}x^{n-1}dx \quad ;\ n>0$$

is called Gamma function or second Eulerian integral and it is denoted by, $\Gamma(n)$.

$$i.e,\ \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \quad ;\ n>0.$$

Properties of Beta and Gamma functions: The properties are given below:

1. $\beta(m,n)=\beta(n,m)$
2. $\Gamma(1)=1$
3. $\Gamma(n+1)=n\Gamma(n) \quad ; n>0$
4. $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
5. $\displaystyle\int_0^\infty e^{-kx}x^{n-1}dx=\frac{\Gamma(n)}{k^n} \qquad ; k,n>0$
6. $\Gamma(m)\Gamma(1-m)=\dfrac{\pi}{\sin m\pi} \qquad ;0<m<1$
7. $\Gamma(1/2)=\sqrt{\pi}$
8. $\displaystyle\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$
9. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$

Theorem-01: Prove that $\beta(m,n)=\beta(n,m)$.

Proof: We know that the beta function is

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ;\ m,\ n>0 \quad \cdots\cdots\cdots(1)$$

Let $\quad x=1-t \qquad \therefore\ dx=-dt$

$$when\ \ x=0 \qquad then \quad t=1$$

$$when\ \ x=1 \qquad then \quad t=0$$

From (1) we get,

$$\beta(m,n)=\int_1^0 (1-t)^{m-1}\,t^{n-1}\,(-dt) \quad ;\ m,\ n>0$$

$$=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

$=\beta(n,m)$ **(Proved)**

Theorem-02: Prove that $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$.

Proof: We know that the beta function is

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ;\ m,\ n>0$$

If $m=n=\dfrac{1}{2}$ then,

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2})}=\int_0^1 \frac{1}{\sqrt{x}.\sqrt{1-x}}\,dx$$

$$\Rightarrow \frac{\left\{\Gamma(\tfrac{1}{2})\right\}^2}{\Gamma(1)}=\int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$\Rightarrow \frac{\left\{\Gamma(\tfrac{1}{2})\right\}^2}{\Gamma(1)}=\int_0^1 \frac{dx}{\sqrt{-\left(x^2-x\right)}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{-\left[x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left[x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2\right]}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}\frac{\left(x - 1/2\right)}{1/2}\right]_0^1$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}(2x-1)\right]_0^1$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}(2.1-1) - \sin^{-1}(2.0-1)\right]$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \sin^{-1}(1) - \sin^{-1}(-1)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \sin^{-1}(1) + \sin^{-1}(1)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\sin^{-1}(1)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\sin^{-1}.\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \pi$$

$\therefore\ \Gamma(1/2) = \sqrt{\pi}$ **(Proved).**

Theorem-03: Prove that i). $\Gamma(1) = 1$; ii). $\Gamma(n+1) = n\Gamma(n)$; iii). $\Gamma(n) = (n-1)\Gamma(n-1)$

Proof: We know that the Gamma function is

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \quad ; \quad n > 0 \quad \ldots\ldots\ldots (1)$$

If $n=1$ then, from (1) we get,

$$\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}dx$$

$$=\int_0^\infty e^{-x}dx$$

$$=\left[-e^{-x}\right]_0^\infty$$

$$=\left(-e^{-\infty}+e^0\right)$$

$$=(0+1)$$

$$=1$$

$\therefore\ \Gamma(1)=1$ **(Proved)**

Again, replacing n by $(n+1)$ in (1) we get,

$$\Gamma(n+1)=\int_0^\infty e^{-x}x^n dx$$

$$=\left[-x^n e^{-x}\right]_0^\infty+n\int_0^\infty e^{-x}x^{n-1}dx \qquad \text{[Integrating by parts]}$$

$$=0+n\Gamma(n)$$

$$=n\Gamma(n)$$

$\therefore\ \Gamma(n+1)=n\Gamma(n)$ **(Proved)**

Again, from (1) we get,

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$$

$$=\left[-x^{n-1}e^{-x}\right]_0^\infty+(n-1)\int_0^\infty e^{-x}x^{n-2}dx \qquad \text{[Integrating by parts]}$$

$$=0+(n-1)\int_0^\infty e^{-x}x^{(n-1)-1}dx$$

$$=(n-1)\Gamma(n-1)$$

$\therefore\ \Gamma(n)=(n-1)\Gamma(n-1)$ **(Proved)**

Theorem-04: Prove that i). $\Gamma(n+1)=n!$; ii). $\Gamma(n)=(n-1)!$ for n is a +ve integer.

Proof: If n is a positive integer then,

$$\Gamma(n+1)=n\Gamma(n) \qquad \cdots(1)$$

$$=n\Gamma\{(n-1)+1\}$$

$$=n(n-1)\Gamma(n-1) \qquad [by\ using\ (1)]$$

$$=n(n-1)\Gamma\{(n-2)+1\}$$

$$=n(n-1)(n-2)\Gamma(n-2) \qquad [by\ using\ (1)]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$=n(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1\Gamma(1)$$

$$=n(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1$$

$$=n!$$

$\therefore\ \Gamma(n+1)=n!$ **(Proved)**

Again if n is a positive integer then,

$$\Gamma(n)=(n-1)\Gamma(n-1) \qquad \cdots(2)$$

$$=(n-1)\Gamma\{(n-2)+1\}$$

$$=(n-1)(n-2)\Gamma(n-2) \qquad [by\ using\ (2)]$$

$$=(n-1)(n-2)\Gamma\{(n-3)+1\}$$

$$=(n-1)(n-2)(n-3)\Gamma(n-3) \qquad [by\ using\ (2)]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$=(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1\Gamma(1)$$

$$=(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1$$

$$=(n-1)!$$

$\therefore\ \Gamma(n)=(n-1)!$ **(Proved)**

Theorem-05: Prove that $\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$.

Proof: We know that the beta function is

$$\beta(m,n)=\int_0^1 y^{m-1}(1-y)^{n-1}\,dy \quad ; m,\ n>0 \quad \cdots\cdots\cdots(1)$$

Let $\quad y=\dfrac{1}{1+x} \quad or,\ x=\dfrac{1}{y}-1 \qquad \therefore\ dy=-\dfrac{dx}{(1+x)^2}$

$when\ \ y=0 \qquad then \quad x=\infty$

$when\ \ y=1 \qquad then \quad x=0$

From (1) we get,

$$\beta(m,n)=-\int_\infty^0\left(\frac{1}{1+x}\right)^{m-1}\left(1-\frac{1}{1+x}\right)^{n-1}\frac{dx}{(1+x)^2}$$

$$=\int_0^\infty\left(\frac{1}{1+x}\right)^{m-1}\left(\frac{x}{1+x}\right)^{n-1}\frac{dx}{(1+x)^2}$$

$$=\int_0^\infty\frac{1}{(1+x)^{m-1}}\frac{x^{n-1}}{(1+x)^{n-1}}\frac{dx}{(1+x)^2}$$

$$=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \textbf{(Proved)}$$

Ex-01: Prove that $\beta(n,m)=\int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}\,dx$.

Theorem-06: Establish the relation between Gamma and Beta function.

Or, Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof: From the definition of Gamma function we can write

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \qquad ;n>0$$

Assume that $x=\lambda u\ \therefore\ \ dx=\lambda du$.

Limit: when $x=0$, then $u=0$ and when $x=\infty$, then $u=\infty$.

From above relation we have

$$\Gamma(n)=\int_0^\infty e^{-\lambda u}(\lambda u)^{n-1}\lambda du$$

$$=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n-1}\lambda du$$

$$=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n}du \ \cdots\cdots\cdots\cdots(i)$$

Again,

$$\Gamma(m)=\int_0^\infty e^{-\lambda}\lambda^{m-1}d\lambda \quad \cdots\cdots\cdots\cdots(ii)$$

Multiplying (i) and (ii) we get

$$\Gamma(n)\Gamma(m)=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^n du\int_0^\infty e^{-\lambda}\lambda^{m-1}d\lambda$$

$$\Rightarrow \Gamma(m)\Gamma(n)=\int_0^\infty\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^n e^{-\lambda}\lambda^{m-1}d\lambda du$$

$$=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+u)}\lambda^{m+n-1}d\lambda\right]u^{n-1}du$$

$$=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+u)}\lambda^{m+n-1}d\lambda\right]u^{n-1}du$$

$$=\int_0^\infty\left[\frac{\overline{|m+n}}{(1+u)^{m+n}}\right]u^{n-1}du \quad \left[\because \frac{\overline{|n}}{k^n}=\int_0^\infty e^{-kx}x^{n-1}dx\right]$$

$$=\overline{|m+n}\int_0^\infty \frac{u^{n-1}}{(1+u)^{m+n}}du$$

$$=\overline{|m+n}\times\beta(m,n)\left[\because \beta(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx\right]$$

$\therefore \beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ **(Proved)**

Theorem-07: Prove that $\displaystyle\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx=\frac{\overline{\left|\frac{p+1}{2}\right.}\,\overline{\left|\frac{q+1}{2}\right.}}{2\,\overline{\left|\frac{p+q+2}{2}\right.}}$.

OR

Evaluate, in terms of the Gamma function, the integral $\displaystyle\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx, \quad p,q>-1$

Proof: We know that

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Let $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta d\theta$.

Limit: $x=0\Rightarrow\theta=0$ *and* $x=\infty\Rightarrow\theta=\dfrac{\pi}{2}$.

Now,

$$\beta(m\,,\,n)=\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\times 2\sin\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta(\cos^2\theta)^{n-1}\times 2\sin\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\times 2\sin\theta\cos\theta d\theta$$

$$\therefore \beta(m\ ,\ n)=2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Assume $2m-1=p$ *and* $2n-1=q \Rightarrow m=\dfrac{p+1}{2}$ *and* $n=\dfrac{q+1}{2}$.

Now from above equation we get

$$\beta\left(\tfrac{p+1}{2}\ ,\ \tfrac{q+1}{2}\right)=2\int_0^{\frac{\pi}{2}} \sin^{p}\theta\cos^{q}\theta d\theta$$

Using the relation between beta and gamma function $\beta(m\ ,\ n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, we have

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}=2\int_0^{\frac{\pi}{2}} \sin^{p}\theta\cos^{q}\theta d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{p}\theta\cos^{q}\theta d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$ **(Proved)**

Theorem-08: Prove that $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$, where $0<n<1$.

OR

Establish Euler's reflection formula.

Proof: We know that

$$\beta(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \cdots(1)$$

and

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad \cdots(2)$$

Now from (1) and (2), we have

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \cdots(3)$$

Putting $m+n=1$ and so $n=1-m$ in (3), we get

$$\frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)}=\int_0^{\infty}\frac{x^{m-1}}{1+x}\,dx$$

$$or,\ \Gamma(m)\Gamma(1-m)=\int_0^{\infty}\frac{x^{m-1}}{1+x}\,dx \qquad \cdots(4)$$

Again we know that the formula from integral calculus,

$$\int_0^{\infty} \frac{x^{m-1}}{1+x}\,dx = \frac{\pi}{\sin m\pi}$$

The equation (4) can be written as,

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

Now replacing m by n we have

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \qquad \textbf{(Proved)}$$

https://mathoverflow.net/questions/76399/one-line-proof-of-the-eulers-reflection-formula

Problem-01: Evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$

Solution: Let, $I = \int_0^{\frac{\pi}{2}} \cos^7 x\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin^0 x \cos^7 x\;dx$$

$$= \frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\,\Gamma\left(\frac{0+7+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{8}{2}\right)}{2\,\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(4)}{2\,\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3+1)}{2\,\Gamma\left(\frac{7}{2}+1\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)3!}{2\,\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{3.2.1}{2\,\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}}$$

Exer.-01: $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$

Ans: $\frac{3\pi}{16}$

Exer.-02: $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$

Ans: $\frac{8}{15}$

Exer.-03: $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx$

Ans: $\frac{35\pi}{256}$

$$=\frac{16}{35}$$ **(Ans.)**

Problem-02: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\ dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$

$$=\int_0^{\frac{\pi}{2}} \sin^6 x \cos^0 x\ \ dx$$

$$=\frac{\Gamma\frac{6+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{6+0+2}{2}}$$

$$=\frac{\Gamma\frac{7}{2}\ \Gamma\frac{1}{2}}{2\ \ \Gamma\frac{8}{2}}$$

$$=\frac{\Gamma\frac{7}{2}\ \Gamma\frac{1}{2}}{2\ \Gamma 4}$$

$$=\frac{\Gamma\left(\frac{5}{2}+1\right)\ \Gamma\frac{1}{2}}{2\ \Gamma(3+1)}$$

$$=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}.\Gamma\frac{1}{2}}{2.\,3!}$$

$$=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2.\,3.2.1}$$

$$=\frac{5.\pi}{32}$$ **(Ans.)**

Problem-03: Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x\ dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x\ dx$

$$=\frac{\Gamma\frac{4+1}{2}\,\Gamma\frac{3+1}{2}}{2\,\Gamma\frac{4+3+2}{2}}$$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x\, dx$

Ans: $\frac{8}{693}$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x\, dx$

$$=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{4}{2}\right)}{2\,\Gamma\left(\frac{9}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma(2)}{2\,\Gamma\left(\frac{7}{2}+1\right)}$$

$$=\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{2}{35}$$ **(Ans.)**

Problem-04: Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \; dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \; dx$

$$=\int_0^{\frac{\pi}{2}} \cos^3 x\left(\cos^2 x-\sin^2 x\right) dx$$

$$=\int_0^{\frac{\pi}{2}} \left(\cos^5 x-\cos^3 x\sin^2 x\right) dx$$

$$=\int_0^{\frac{\pi}{2}} \cos^5 x dx-\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x \; dx$$

$$=\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{0+5+2}{2}\right)}-\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{2+3+2}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3)}{2\,\Gamma\left(\frac{7}{2}\right)}-\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{2\,\Gamma\left(\frac{7}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+1)}{2\,\Gamma\left(\frac{5}{2}+1\right)}-\frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma(1+1)}{2\,\Gamma\left(\frac{5}{2}+1\right)}$$

Ans: $\frac{7\pi}{2048}$

Exer.-06: $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^3 x\, dx$

Ans: $\frac{2}{63}$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin 2x\cos^4 x\, dx$

Ans: $\frac{1}{3}$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin 2x\sin^2 x\cos^5 x\, dx$

Ans: $\frac{4}{63}$

$$=\frac{\Gamma\left(\frac{1}{2}\right).2\Gamma(1)}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}-\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right).1\Gamma(1)}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{8}{15}-\frac{2}{15}$$

$$=\frac{8-2}{15}$$

$$=\frac{6}{15}$$

$$=\frac{2}{5} \qquad \textbf{(Ans.)}$$

Problem-05: Evaluate $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

Solution: Let, $I = \int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

$$= 2\int_0^{\pi} \sin^4 x \cos^6 x \, dx$$

$$= 4\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx$$

$$= 4.\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{4+6+2}{2}\right)}$$

$$= 2.\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)}$$

$$= 2.\frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma\left(\frac{5}{2}+1\right)}{\Gamma(5+1)}$$

$$= 2.\frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{5.4.3.2.1}$$

$$= \frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{5.4.3}$$

$$= \frac{1}{60}\times\frac{45\pi}{32}$$

$$= \frac{3\pi}{128} \qquad \textbf{(Ans.)}$$

Exer.-06: $\int_0^{\pi} \sin^2 x \cos^4 x \, dx$

Ans: $\frac{\pi}{16}$

Problem-06: Evaluate $\int_0^{\pi} x\sin^6 x\cos^4 x\ dx$

Solution: Let, $I=\int_0^{\pi} x\sin^6 x\cos^4 x\ dx$

$$=\int_0^{\pi}(\pi-x)\sin^6(\pi-x)\cos^4(\pi-x)\ dx$$

$$=\int_0^{\pi}(\pi-x)\sin^6 x\cos^4 x\ dx$$

$$=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx-\int_0^{\pi}x\sin^6 x\cos^4 x\ dx$$

$$=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx-I$$

$$\therefore\ 2I=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^6 x\cos^4 x\ dx$$

$$=2\pi.\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{6+4+2}{2}\right)}$$

$$=\pi.\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)}$$

$$=\pi.\frac{\Gamma\left(\frac{5}{2}+1\right)\Gamma\left(\frac{3}{2}+1\right)}{\Gamma(5+1)}$$

$$=\pi.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{5.4.3.2.1}$$

$$=\pi.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}\ .\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{120}$$

$$=\frac{1}{120}\times\frac{45\pi^2}{32}$$

$$=\frac{3\pi^2}{256}$$

$$\therefore\ I=\frac{3\pi^2}{512}$$ **(Ans.)**

Exer.-07: $\int_0^{\pi} x\sin^2 x\cos^4 x\,dx$

Ans: $\frac{\pi^2}{32}$

Exer.-08: $\int_0^{\pi} x\sin x\cos^2 x\,dx$

Ans: $\frac{\pi}{3}$

Problem-08: Evaluate $\int_0^{1} x^3\left(1-x^2\right)^{\frac{5}{2}}\ dx$

Exer.-09: $\int_0^{1} x^2\left(1-x^2\right)^{\frac{1}{2}}\ dx$

Solution: Let, $I=\int_0^1 x^3\left(1-x^2\right)^{\frac{5}{2}}dx$

Put $\quad x=\sin\theta \quad \therefore\ dx=\cos\theta d\theta$

$Limit: when\ x=0 \quad then\ \theta=0$

$when\ x=1 \quad then\ \theta=\frac{\pi}{2}$

Now, $I=\int_0^{\frac{\pi}{2}}\sin^3\theta\left(1-\sin^2\theta\right)^{\frac{5}{2}}\cos\theta d\theta$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\left(\cos^2\theta\right)^{\frac{5}{2}}\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^5\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^6\theta d\theta$$

$$=\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{3+6+2}{2}\right)}$$

$$=\frac{\Gamma(2)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)}$$

$$=\frac{\Gamma(1+1)\Gamma\left(\frac{5}{2}+1\right)}{2\Gamma\left(\frac{9}{2}+1\right)}$$

$$=\frac{1\Gamma(1).\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{2}{63}$$ **(Ans.)**

Problem-09: Evaluate $\int_0^1\frac{x^5}{\sqrt{\left(1-x^2\right)}}dx$

Solution: Let, $I=\int_0^1\frac{x^3}{\sqrt{\left(1-x^2\right)}}dx$

Put $\quad x=\sin\theta \quad \therefore\ dx=\cos\theta d\theta$

Ans: $\frac{\pi}{16}$

Exer.-10: $\int_0^1 x^4\left(1-x^2\right)^{\frac{3}{2}}dx$

Ans: $\frac{3\pi}{256}$

Exer.-11: $\int_0^1 x^6\left(1-x^2\right)^{\frac{1}{2}}dx$

Ans: $\frac{5\pi}{256}$

Exer.-12: $\int_0^1\frac{x^6}{\sqrt{\left(1-x^2\right)}}dx$

Ans: $\frac{5\pi}{32}$

$Limit: when\ x=0 \quad then\ \theta=0$

$\quad\quad when\ x=1 \quad then\ \theta=\frac{\pi}{2}$

Now, $I=\int_0^{\frac{\pi}{2}}\frac{\sin^5\theta}{\sqrt{\left(1-\sin^2\theta\right)}}.\cos\theta d\theta$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^5\theta}{\sqrt{\cos^2\theta}}.\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^5\theta}{\cos\theta}.\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^5\theta d\theta$$

$$=\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{5+0+2}{2}\right)}$$

$$=\frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$=\frac{\Gamma(2+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{2}+1\right)}$$

$$=\frac{2.1\Gamma(1).\Gamma\left(\frac{1}{2}\right)}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{8}{15}$$ **(Ans.)**

Problem-10: Evaluate $\int_0^{\infty}\frac{x^3}{\left(1+x^2\right)^{\frac{9}{2}}}dx$

Solution: Let, $I=\int_0^{\infty}\frac{x^3}{\left(1+x^2\right)^{\frac{9}{2}}}dx$

Put $x=\tan\theta \quad \therefore\ dx=\sec^2\theta d\theta$

$Limit: when\ x=0 \quad then\ \theta=0$

$\quad\quad when\ x=\infty \quad then\ \theta=\frac{\pi}{2}$

Now, $I=\int_0^{\frac{\pi}{2}}\frac{\tan^3\theta}{\left(1+\tan^2\theta\right)^{\frac{9}{2}}}.\sec^2\theta d\theta$

$$=\int_0^{\frac{\pi}{2}}\frac{\tan^3\theta}{\left(\sec^2\theta\right)^{\frac{9}{2}}}.\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\tan^3\theta}{\sec^9\theta}.\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\tan^3\theta}{\sec^7\theta}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin^3\theta}{\cos^3\theta}.\cos^7\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^4\theta d\theta$$

$$=\frac{\Gamma\frac{3+1}{2}\Gamma\frac{4+1}{2}}{2\Gamma\frac{3+4+2}{2}}$$

$$=\frac{\Gamma 2\,\Gamma\frac{5}{2}}{2\Gamma\frac{9}{2}}$$

$$=\frac{\Gamma(1+1)\,\Gamma\left(\frac{3}{2}+1\right)}{2\Gamma\left(\frac{7}{2}+1\right)}$$

$$=\frac{1\Gamma 1\,.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}{2.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}$$

$$=\frac{2}{35}$$ **(Ans.)**

Problem-11: Evaluate $\int_0^1 x^4\left(1-x\right)^{\frac{3}{2}}dx$

Exer.-13: $\int_0^1 x^3\left(1-x\right)^3 dx$

Ans: $\frac{1}{140}$

Solution: Let, $I=\int_0^1 x^4\left(1-x\right)^{\frac{3}{2}}dx$

$$=\int_0^1 x^{5-1}\left(1-x\right)^{\frac{5}{2}-1}dx$$

$$=\beta\left(5,\frac{5}{2}\right)$$

$$=\frac{\Gamma(5)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(5+\frac{5}{2}\right)}$$

$$=\frac{\Gamma(4+1)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{13}{2}+1\right)}$$

$$=\frac{4.3.2.1\,\Gamma\left(\frac{5}{2}\right)}{\frac{13}{2}.\frac{11}{2}.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)}$$

$$=\frac{256}{15015}$$ **(Ans.)**

Problem-12: Show that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

Exer.-14: Show that $\int_0^\infty e^{-3x^2}dx=\frac{\sqrt{\pi}}{2\sqrt{3}}$

Solution: Let, $I=\int_0^\infty e^{-x^2}dx$

Exer.-15: Show that $\int_0^\infty e^{-a^2x^2}dx=\frac{\sqrt{\pi}}{2a}$

Put, $x^2=z$

$\therefore\quad 2xdx=dz$

$\Rightarrow\ dx=\frac{1}{2\sqrt{z}}dz$

$Limit: when\ x=0\ then\ z=0$

$when\ x=\infty\ then\ z=\infty$

Now, $I=\int_0^\infty e^{-z}.\frac{1}{2\sqrt{z}}dz$

$$=\frac{1}{2}\int_0^\infty e^{-z}z^{-\frac{1}{2}}dz$$

$$=\frac{1}{2}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}dz$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)}{2}$$

$$=\frac{\sqrt{\pi}}{2}$$

$\therefore\ \int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$ **(Showed.)**

Problem-13: Show that $\int_0^\infty e^{-x}x^{\frac{3}{2}}dx=\frac{3\sqrt{\pi}}{4}$

Exer.-16: Show that $\int_0^\infty \sqrt{x}e^{-x^3}dx=\frac{\sqrt{\pi}}{3}$

Solution: Let, $I=\int_0^\infty e^{-x}x^{\frac{3}{2}}dx$

$$=\int_0^\infty e^{-x}x^{\frac{5}{2}-1}dx$$

$$=\left\lceil\frac{5}{2}\right.$$

$$=\left\lceil\frac{3}{2}+1\right.$$

$$=\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right.$$

$$=\frac{3\sqrt{\pi}}{4}$$

$\therefore \int_0^\infty e^{-x}x^{\frac{3}{2}}dx=\frac{3\sqrt{\pi}}{4}$ **(Showed)**

Problem-14: Show that $\int_0^\infty e^{-3x}x^{\frac{3}{2}}dx=\frac{\sqrt{3\pi}}{36}$ **Exer.-17: Show that** $\int_0^\infty e^{-4x^2}dx=\frac{3\sqrt{\pi}}{128}$

Solution: Let, $I=\int_0^\infty e^{-3x}x^{\frac{3}{2}}dx$

$$=\int_0^\infty e^{-3x}x^{\frac{5}{2}-1}dx$$

$$=\frac{\left\lceil\frac{5}{2}\right.}{3^{\frac{5}{2}}}$$

$$=\frac{\left\lceil\frac{3}{2}+1\right.}{\sqrt{3^5}}$$

$$=\frac{\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right.}{9\sqrt{3}}$$

$$=\frac{\sqrt{3\pi}}{36}$$

$\therefore \int_0^\infty e^{-3x}x^{\frac{3}{2}}dx=\frac{\sqrt{3\pi}}{36}$ **(Showed.)**