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# Pair of Straight lines

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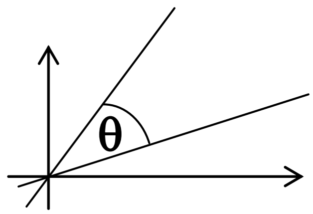
# Pair of straight lines

**Pair of straight lines:** A pair of straight lines is the locus of a point whose coordinates satisfy a second degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$. A collection of combined two straight lines is called a pair of straight lines.

**Homogeneous equation:** An equation, in which degree of each term is equal, is called a homogeneous equation. Such as $ax^2+2hxy+by^2=0$ is a homogeneous equation of degree or order 2 because degree of its each term is two. It is noted that homogeneous equation always represents straight lines passing through the origin.

**Non-homogeneous equation:** An equation, in which degree of each term is not equal, is called a non-homogeneous equation. Such as $ax^2+2hxy+by^2+2gx+2fy+c=0$ is a non- homogeneous equation of degree or order 2.




**Theorem-01:** Prove that a homogeneous equation of the second degree always represents a pair of straight lines through the origin.

**Proof:** The homogeneous equation of second degree is,

$$ax^2+2hxy+by^2=0 \qquad \cdots(1)$$

Dividing both sides of (1) by $x^2$ and $b$ (if $b\neq 0$), we have

$$\left(\frac{y}{x}\right)^2+\frac{2h}{b}\frac{y}{x}+\frac{a}{b}=0 \qquad \cdots(2)$$

This represents a quadratic equation in $\frac{y}{x}$. Let $m_1$ and $m_2$ be the roots of this equation.

Sum of the roots is

$$m_1+m_2=\frac{-2h}{b}$$

and product of the roots is

$$m_1m_2=\frac{a}{b}.$$

The equation (2) must be equivalent to

$$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0 \qquad \cdots(3)$$

The two lines represented by (2) i.e. (1) are given

$$\frac{y}{x}-m_1=0\text{, and }\frac{y}{x}-m_2=0$$

i.e. $y=m_1x$, and $y=m_2x$.

Which pass through the origin.

Thus, the homogeneous quadratic equation $ax^2+2hxy+by^2=0$ always represents a pair of straight lines, real or imaginary, through the origin. (**Proved**).

**Alternatively**, Multiplying both sides of (1) by $a$ (if $a\neq 0$), we have

$$a^2x^2+2ahxy+aby^2=0$$

$$or,\ (ax)^2+2ax.hy+(hy)^2-(h^2-ab)y^2=0$$

$$or,\ (ax+hy)^2-\left(\sqrt{h^2-ab}\,y\right)^2=0$$

$$or,\ \left\{ax+\left(h+\sqrt{h^2-ab}\right)y\right\}\left\{ax+\left(h-\sqrt{h^2-ab}\right)y\right\}=0$$

$$\left.\begin{aligned}\therefore\ either\ \ ax+\left(h+\sqrt{h^2-ab}\right)y=0\\ or,\ \ ax+\left(h-\sqrt{h^2-ab}\right)y=0\end{aligned}\right\}\qquad\cdots(4),$$

which represent two straight lines, real or imaginary through the origin. (**Proved**).

**Theorem-02:** Find the angle between the straight lines represented by the homogeneous equation

$$ax^2+2hxy+by^2=0.$$

**Proof:** Given homogeneous equation of second degree is,

$$ax^2+2hxy+by^2=0\qquad\cdots(1)$$

Suppose, $y=m_1x$ and $y=m_2x$ be the lines represented by (1).

So,

$$(y-m_1x)(y-m_2x)=0$$

$$or,\ y^2-m_2xy-m_1xy+m_1m_2x^2=0$$

$$or,\ m_1m_2x^2-(m_1+m_2)xy+y^2=0\qquad\cdots(2)$$

This equation is same as to the equation (1), so the ratios of the coefficient of like terms are equal. Now comparing the coefficients

$$\frac{m_1m_2}{a}=\frac{-(m_1+m_2)}{2h}=\frac{1}{b}$$

From 2nd and 3rd parts, we get $\frac{-(m_1+m_2)}{2h}=\frac{1}{b}\qquad\Rightarrow(m_1+m_2)=-\frac{2h}{b}$

From 1nd and 3rd parts, we get $\frac{m_1m_2}{a}=\frac{1}{b}\qquad\Rightarrow m_1m_2=\frac{a}{b}$

If $\theta$ be the angle between two straight lines represented by the given equation then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

$$\text{or, }\tan\theta=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$\text{or, }\tan\theta=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}$$

$$\text{or, }\tan\theta=\frac{\sqrt{\frac{4h^2-4ab}{b^2}}}{\frac{a+b}{b}}$$

$$\text{or, }\tan\theta=\frac{\frac{2}{b}\sqrt{h^2-ab}}{\frac{a+b}{b}}$$

$$\therefore\ \tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}\qquad \textbf{(Proved).}$$

**Theorem-03:** Find the condition that general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represents two straight lines.

**Proof:** Given general equation of second degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ \cdots\cdots\cdots\cdots(1)$$

The above equation can be written as,

$$ax^2+2(hy+g)x+by^2+2fy+c=0$$

Solving we get,

$$x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$\boxed{\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$or,\ x=\frac{-2(hy+g)\pm2\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{2a}$$

$$or,\ x=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}$$

$$or,\ ax=-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}$$

$$or,\ ax+hy+g=\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}\ \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Equation (1) represents two straight lines if it is possible to factorize the left hand side of (1) as a product of two linear factors.

It will be done if the quantity of under the square root sign in the equation (2) be a perfect square.

That means $(hy+g)^2-a(by^2+2fy+c)$ must be perfect square. We know, $(hy+g)^2-a(by^2+2fy+c)$ be perfect square if the roots of the equation $(hy+g)^2-a(by^2+2fy+c)=0$ are equal.

Now,
$$(hy+g)^2-a(by^2+2fy+c)=0$$
$$or,\ (h^2y^2+2hyg+g^2)-(aby^2+2afy+ca)=0$$
$$or,\ h^2y^2+2hyg+g^2-aby^2-2afy-ca=0$$
$$or,\ (h^2-ab)y^2+2(gh-af)y+g^2-ca=0$$

Roots of the above quadratic equation in y are equal if the discriminant of the equation $B^2-4AC=0$.

Here,
$$\{2(gh-af)\}^2-4(h^2-ab)(g^2-ca)=0$$
$$or,\ 4(gh-af)^2-4(h^2-ab)(g^2-ca)=0$$
$$or,\ (gh-af)^2-(h^2-ab)(g^2-ca)=0$$
$$or,\ g^2h^2-2ghaf+a^2f^2-(h^2g^2-h^2ca-abg^2+a^2bc)=0$$
$$or,\ g^2h^2-2ghaf+a^2f^2-h^2g^2+h^2ca+abg^2-a^2bc=0$$
$$or,\ -2ghaf+a^2f^2+h^2ca+abg^2-a^2bc=0$$
$$or,\ -2ghf+af^2+h^2c+bg^2-abc=0$$
$$or,\ abc+2ghf-af^2-bg^2-ch^2=0$$

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

This is the required condition to represents the straight lines. **(Proved).**

**Theorem-04:** Find the equation of the bisectors of the angles between the straight lines represented by the homogeneous equation $ax^2 + 2hxy + by^2 = 0$.

**Proof:** Given homogeneous equation of second degree is,

$$ax^2 + 2hxy + by^2 = 0 \qquad \cdots(1)$$

If $m_1$ and $m_2$ be the roots of this equation, then the lines represented by the equation (1) are

$$y - m_1 x = 0 \qquad \cdots(2)$$

and

$$y - m_2 x = 0 \qquad \cdots(3)$$

Sum of the roots is

$$m_1 + m_2 = \frac{-2h}{b}$$

and product of the roots is

$$m_1 m_2 = \frac{a}{b}.$$

The equations of the required bisectors are

$$\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \pm \frac{y - m_2 x}{\sqrt{1 + m_2^2}}$$

$$or,\ \frac{(y - m_1 x)^2}{1 + m_1^2} = \frac{(y - m_2 x)^2}{1 + m_2^2} \qquad [Squaring]$$

$$or,\ \left(y^2 - 2m_1 xy + m_1^2 x^2\right)\left(1 + m_2^2\right) = \left(y^2 - 2m_2 xy + m_2^2 x^2\right)\left(1 + m_1^2\right)$$

$$or,\ y^2 - 2m_1 xy + m_1^2 x^2 + m_2^2 y^2 - 2m_1 m_2^2 xy + m_1^2 m_2^2 x^2 = y^2 - 2m_2 xy + m_2^2 x^2 + m_1^2 y^2 - 2m_1^2 m_2 xy + m_1^2 m_2^2 x^2$$

$$or,\ m_1^2 x^2 + m_2^2 y^2 - m_2^2 x^2 - m_1^2 y^2 = 2m_1 xy - 2m_2 xy + 2m_1 m_2^2 xy - 2m_1^2 m_2 xy$$

$$or,\ \left(m_1^2 - m_2^2\right)\left(x^2 - y^2\right) = 2xy\left\{(m_1 - m_2) - m_1 m_2 (m_1 - m_2)\right\}$$

$$or,\ (m_1 + m_2)(m_1 - m_2)\left(x^2 - y^2\right) = 2xy(m_1 - m_2)(1 - m_1 m_2)$$

$$or,\ (m_1 + m_2)\left(x^2 - y^2\right) = 2xy(1 - m_1 m_2)$$

$$or,\ \frac{-2h}{b}\left(x^2 - y^2\right) = 2xy\left(1 - \frac{a}{b}\right)$$

$$\therefore\ \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

This is the required equation of the bisector.

## Note:

- Angle between the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ *or*, $ax^2+2hxy+by^2=0$ is calculated by formula $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$.
  - Lines be perpendicular if $a+b=0$
  - Lines be parallel if $h^2-ab=0$
  - Lines represented by homogeneous or non-homogeneous equation are real if $h^2>ab$.
  - Lines represented by homogeneous or non-homogeneous equation are imaginary if $h^2<ab$.
- The equation of the bisectors of an angle produced by the pair of straight line represented by the homogeneous equation $ax^2+2hxy+by^2=0$ is $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$.
- The equation of the bisectors of an angle produced by the pair of straight line represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $\dfrac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\dfrac{(x-\alpha)(y-\beta)}{h}$, where $(\alpha,\beta)$ is the intersection point of those lines.
- The intersecting point $(\alpha,\beta)$ of the straight lines represented by the equation

  $ax^2+2hxy+by^2+2gx+2fy+c=0$

  Let $f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c$

  Then set $\dfrac{\partial f}{\partial x}=2ax+2hy+2g=0\Rightarrow ax+hy+g=0\ \cdots\cdots\cdots(1)$

  $\dfrac{\partial f}{\partial y}=2hx+2by+2f=0\Rightarrow hx+by+f=0\ \cdots\cdots\cdots(2)$

  Solving Eq. (1) & Eq. (2) we get,

  $$\alpha=\frac{bg-hf}{h^2-ab}\ \&\ \beta=\frac{af-gh}{h^2-ab}$$

  This is the point of intersection of the straight lines.

**Problem-01:** Find the angle between the lines represented by the equation $3x^2-16xy+5y^2=0$. Also find the equations of the straight lines and equation of the bisector's of the angle.

**Solution: 1st part:** Given that,

$$3x^2-16xy+5y^2=0\ \cdots\cdots\cdots(1)$$

The general equation of second degree in homogeneous form is,

$$ax^2+2hxy+by^2=0\ \cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we have,

$$a=3, h=-8\ \ and\ \ b=5.$$

Let $\theta$ be the angle between the lines. So, the angle is,

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\text{or, }\tan\theta=\frac{2\sqrt{(-8)^2-3.5}}{3+5}$$

$$\text{or, }\tan\theta=\frac{2\sqrt{64-15}}{8}$$

$$\text{or, }\tan\theta=\frac{2\sqrt{49}}{8}=\frac{2.7}{8}=\frac{14}{8}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{14}{8}\right) = 60.26^0$$

Therefore, the angle between the lines is, $60.26^0$.

**2nd part:** The given equation can be written as,

$$3x^2 - 16y.x + 5y^2 = 0$$

$$\boxed{\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$or,\ x = \frac{16y \pm \sqrt{(-16y)^2 - 4.3.5y^2}}{2.3}$$

$$or,\ x = \frac{16y \pm \sqrt{256y^2 - 60y^2}}{6}$$

$$or,\ x = \frac{16y \pm \sqrt{196y^2}}{6}$$

$$or,\ x = \frac{16y \pm 14y}{6}$$

Taking positive sign we get,

$$x = \frac{16y + 14y}{6}$$

$$or, x = 5y$$

$$\therefore\ x - 5y = 0$$

Again taking negative sign we get,

$$x = \frac{16y - 14y}{6}$$

$$or,\ x = \frac{y}{3}$$

$$\therefore\ 3x - y = 0$$

Therefore, $x - 5y = 0$ and $3x - y = 0$ are the required straight lines passing through the origin.

**3rd part:** The equation of bisector's of the angle produced by the straight lines is,

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$or,\ \frac{x^2 - y^2}{3 - 5} = \frac{xy}{-8}$$

$$or,\ \frac{x^2 - y^2}{-2} = \frac{xy}{-8}$$

$$or,\ x^2 - y^2 = \frac{xy}{4}$$

$$\therefore\ 4(x^2 - y^2) = xy$$

This is the required equation of the bisector's.

**H.W:** Find the angle between the lines represented by the following equations. Also find the equations of the straight lines and equation of the bisector's of the angle.

**1)** $3x^2 + 8xy - 3y^2 = 0$.

**2)** $2x^2 + 5xy + 3y^2 = 0$.

**3)** $8x^2 - 42xy - 11y^2 = 0$.

**4)** $5x^2 - 12xy + 3y^2 = 0$.

**5)** $3x^2 - 16xy + 5y^2 = 0$.

**6)** $33x^2 - 71xy - 14y^2 = 0$.

**7)** $x^2 + 2xy\sec\theta + y^2 = 0$.

**8)** $x^2\cos 2\theta + 4xy\cos\theta + 2y^2 + x^2 = 0$.

**9)** $x^2 + 2xy\cot\theta + y^2 = 0$.

**Problem-02:** Show that $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents pair of straight lines. Also find their equations, the point of intersection, the angle and the equation of the bisector's of angle.

**Solution: 1st part:** Given that,

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad \cdots\cdots\cdots\cdots (1)$$

The general equation of second degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \cdots\cdots\cdots\cdots (2)$$

Comparing Eq. (1) & Eq. (2) we have,

$$a = 6,\ h = -\frac{5}{2}, b = -6, g = 7,\ f = \frac{5}{2}\ \&\ c = 4$$

Now, $$\Delta = \begin{vmatrix} 6 & -\frac{5}{2} & 7 \\ -\frac{5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix}$$

$$= 6\left(-24 - \frac{25}{4}\right) - \left(-\frac{5}{2}\right)\left(-10 - \frac{35}{2}\right) + 7\left(-\frac{25}{4} + 42\right)$$

$$= \left(-144 - \frac{75}{2}\right) - \left(25 + \frac{175}{4}\right) + \left(-\frac{175}{4} + 294\right)$$

$$= -144 - \frac{75}{2} - 25 - \frac{175}{4} - \frac{175}{4} + 294$$

$$= 125 - \frac{75}{2} - \frac{175}{4} - \frac{175}{4}$$

$$= \frac{500 - 150 - 175 - 175}{4}$$

$$= \frac{500 - 500}{4}$$

$$= 0$$

Since, $\Delta = 0$ so the given equation represents a pair of straight lines. (***Showed***)

**2nd part:** The given equation can be written as the following quadratic equation in $x$,

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$$

$$or,\ x^2 + (6y + 4)x + 9y^2 + 12y - 5 = 0$$

$$\therefore\ x = \frac{-(6y+4) \pm \sqrt{(6y+4)^2 - 4.1.(9y^2 + 12y - 5)}}{2.1}$$

$$\boxed{\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$or,\ x = \frac{-(6y+4) \pm \sqrt{(6y+4)^2 - 4(9y^2 + 12y - 5)}}{2}$$

$$or,\ x = \frac{-(6y+4) \pm \sqrt{36y^2 + 48y + 16 - (36y^2 + 48y - 20)}}{2}$$

$$or,\ x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-36y^2-48y+20}}{2}$$
$$or,\ x=\frac{-(6y+4)\pm\sqrt{16+20}}{2}$$
$$or,\ x=\frac{-(6y+4)\pm\sqrt{36}}{2}$$
$$or,\ x=\frac{-(6y+4)\pm 6}{2}$$

Taking positive sign we get,

$$x=\frac{-(6y+4)+6}{2}$$
$$or,\ x=\frac{-6y-4+6}{2}$$
$$or,\ x=\frac{-6y+2}{2}$$
$$or,\ 2x=-6y+2$$
$$or,\ x=-3y+1$$
$$\therefore\quad x+3y-1=0$$

Taking negative sign we get,

$$x=\frac{-(6y+4)-6}{2}$$
$$or,\ x=\frac{-6y-4-6}{2}$$
$$or,\ x=\frac{-6y-10}{2}$$
$$or,\ 2x=-6y-10$$
$$or,\ x=-3y-5$$
$$\therefore\quad x+3y+5=0$$

Therefore, required equations of the straight lines $x+3y-1=0$ and $x+3y+5=0$.

**3rd part:** Suppose, $f(x,y)=6x^2-5xy-6y^2+14x+5y+4$

Differentiating the function with respect to $x$ and $y$ partially and equating with zero, we get

$$\frac{\partial f}{\partial x}=12x-5y+14$$
$$\Rightarrow 12x-5y+14=0 \cdots\cdots\cdots\cdots\cdots (3)$$

And

$$\frac{\partial f}{\partial x}=-5x-12y+5$$
$$\Rightarrow -5x-12y+5=0$$
$$\Rightarrow\ 5x+12y-5=0 \cdots\cdots\cdots\cdots\cdots (4)$$

Solving Eq. (3) &Eq.(4) we get the point of intersection of lines represented by the given equation.
Using cross multiplication method on Eq. (3) & Eq.(4)

$$\frac{x}{25-168}=\frac{y}{70+60}=\frac{1}{144+25}$$

$$or,\ \frac{x}{-143}=\frac{y}{130}=\frac{1}{169}$$

$$\therefore\ x=-\frac{143}{169}=-\frac{11}{13} \quad \& \quad y=\frac{130}{169}=\frac{10}{13}$$

Therefore, the coordinates of the point of intersection is $(x,y)=\left(-\frac{11}{13},\frac{10}{13}\right)$ i.e. $(\alpha,\beta)=\left(-\frac{11}{13},\frac{10}{13}\right)$.

**4th part:** If $\theta$ be the angle between the lines then,

$$\theta=\tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

$$=\tan^{-1}\left(\frac{2\sqrt{\left(-5/2\right)^2-6.(-6)}}{6+(-6)}\right)$$

$$=\tan^{-1}\left(\frac{2\sqrt{25/4+36}}{0}\right)$$

$$=\tan^{-1}\infty$$

$$=\tan^{-1}\tan\left(\pi/2\right)$$

$$=\pi/2$$

$$=90^\circ$$

Since, the angle is $\theta=90^\circ$ so the lines are perpendicular.

**5th part:** The equation of the bisector's is,

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

$$or,\ \frac{\left(x+11/13\right)^2-\left(y-10/13\right)^2}{6+6}=\frac{\left(x+11/13\right)\left(y-10/13\right)}{-5/2}$$

$$or,\ \frac{\left(x^2+22x/13+121/169\right)-\left(y^2-20y/13+100/169\right)}{12}=\frac{2\left(xy-10x/13+11y/13-110/169\right)}{-5}$$

$$or,\ \frac{x^2+22x/13+121/169-y^2+20y/13-100/169}{12}=\frac{2\left(xy-10x/13+11y/13-110/169\right)}{-5}$$

$$or,\ -5\left(x^2+22x/13+121/169-y^2+20y/13-100/169\right)=24\left(xy-10x/13+11y/13-110/169\right)$$

$$or,\ \left(-5x^2-110x/13-605/169+5y^2+100y/13+500/169\right)=\left(24xy-240x/13+264y/13-2640/169\right)$$

$$or,\ -845x^2-1430x-605+845y^2+1300y+500=4056xy-3120x+3432y-2640$$

$$or,\ -845x^2-4056xy+845y^2-1430x+3120x+1300y-3432y-605+500+2640=0$$

$$or,\ 845x^2+4056xy-845y^2-1690x+2132y-2535=0$$ **(As desired).**

**Problem-03:** Show that $2x^2-7xy+3y^2+x+7y-6=0$ represents pair of straight lines. Also find their equations, the point of intersection, the angle and the equation of the bisector's of angle.

**Solution: 1st part:** Given that,

$$2x^2-7xy+3y^2+x+7y-6=0 \quad \cdots\cdots\cdots\cdots(1)$$

The general equation of second degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \cdots\cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we have,

$$a=2\ ,h=-7/2,b=3,g=1/2,\ f=7/2\ \&c=-6$$

Now, $\Delta=\begin{vmatrix} 2 & -7/2 & 1/2 \\ -7/2 & 3 & 7/2 \\ 1/2 & 7/2 & -6 \end{vmatrix}$

$$=2\left(-18-49/4\right)-\left(-7/2\right)\left(21-7/4\right)+1/2\left(-49/4-3/2\right)$$

$$=-36-49/2+147/2-49/8-49/8-3/4$$

$$=\frac{-288-196+588-49-49-6}{8}$$

$$=\frac{588-588}{8}$$

$$=0$$

Since, $\Delta=0$ so the given equation represents a pair of straight lines. (***Showed***)

**2$^{nd}$ part:** The given equation can be written as the following quadratic equation in $x$,

$$2x^2-7xy+3y^2+x+7y-6=0$$

$$or,\ 2x^2-(7y-1)x+3y^2+7y-6=0$$

$$\therefore\ x=\frac{-\{-(7y-1)\}\pm\sqrt{(7y-1)^2-4.2.(3y^2+7y-6)}}{2.2}$$

$$\boxed{\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$or,\ x=\frac{(7y-1)\pm\sqrt{49y^2-14y+1-24y^2-56y+48}}{4}$$

$$or,\ x=\frac{(7y-1)\pm\sqrt{25y^2-70y+49}}{4}$$

$$or,\ x=\frac{(7y-1)\pm\sqrt{(5y-7)^2}}{4}$$

$$or,\ x=\frac{(7y-1)\pm(5y-7)}{4}$$

Taking positive sign we get,

$$x=\frac{(7y-1)+(5y-7)}{4}$$

$$or, 4x=7y-1+5y-7$$

$$or,\ 4x=12y-8$$

$$or,\ 4x-12y+8=0$$

$$\therefore\ x-3y+2=0$$

Taking negative sign we get,

$$x=\frac{(7y-1)-(5y-7)}{4}$$

$$or, 4x = 7y - 1 - 5y + 7$$
$$or,\ 4x = 2y + 6$$
$$or, 4x - 2y - 6 = 0$$
$$\therefore\ 2x - y - 3 = 0$$

Therefore, required equations of the straight lines $x - 3y + 2 = 0$ and $2x - y - 3 = 0$.

**3rd part:** Suppose, $(\alpha, \beta)$ be the point of intersection of the lines.

$$\therefore\ \alpha = \frac{bg - hf}{h^2 - ab} \qquad \& \quad \beta = \frac{af - gh}{h^2 - ab}$$

$$= \frac{3.\,{}^{1}\!/_{2} - \left(-{}^{7}\!/_{2}\right).{}^{7}\!/_{2}}{\left(-{}^{7}\!/_{2}\right)^2 - 2.3} \qquad = \frac{2.\,{}^{7}\!/_{2} - {}^{1}\!/_{2}.\left(-{}^{7}\!/_{2}\right)}{\left(-{}^{7}\!/_{2}\right)^2 - 2.3}$$

$$= \frac{{}^{3}\!/_{2} + {}^{49}\!/_{4}}{{}^{49}\!/_{4} - 6} \qquad = \frac{7 + {}^{7}\!/_{4}}{{}^{49}\!/_{4} - 6}$$

$$= \frac{{}^{55}\!/_{4}}{{}^{25}\!/_{4}} \qquad = \frac{{}^{35}\!/_{4}}{{}^{25}\!/_{4}}$$

$$= \frac{11}{5} \qquad = \frac{7}{5}$$

Therefore, the point of intersection is, $(\alpha, \beta) = \left(\frac{11}{5}, \frac{7}{5}\right)$.

**4th part:** If $\theta$ be the angle between the lines then,

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{\left(-{}^{7}\!/_{2}\right)^2 - 2.3}}{2 + 3}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{{}^{49}\!/_{4} - 6}}{5}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{{}^{25}\!/_{4}}}{5}\right)$$

$$= \tan^{-1}\left(\frac{2.\,{}^{5}\!/_{2}}{5}\right)$$

$$= \tan^{-1}(1)$$

$$= \tan^{-1}\tan\left({}^{\pi}\!/_{4}\right)$$

$$= {}^{\pi}\!/_{4}$$

$$= 45^\circ$$

**5th part:** The equation of the bisector's is,

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

$$or,\ \frac{\left(x-\frac{11}{5}\right)^2-\left(y-\frac{7}{5}\right)^2}{2-3}=\frac{\left(x-\frac{11}{5}\right)\left(y-\frac{7}{5}\right)}{-\frac{7}{2}}$$

$$or,\ \frac{\left(x^2-\frac{22x}{5}+\frac{121}{25}\right)-\left(y^2-\frac{14y}{5}+\frac{49}{25}\right)}{-1}=\frac{2\left(xy-\frac{7x}{5}-\frac{11y}{5}+\frac{77}{25}\right)}{-7}$$

$$or,\ \frac{x^2-\frac{22x}{5}+\frac{121}{25}-y^2+\frac{14y}{5}-\frac{49}{25}}{-1}=\frac{2\left(xy-\frac{7x}{5}-\frac{11y}{5}+\frac{77}{25}\right)}{-7}$$

$$or,\ 7\left(x^2-y^2-\frac{22x}{5}+\frac{14y}{5}+\frac{72}{25}\right)=2\left(xy-\frac{7x}{5}-\frac{11y}{5}+\frac{77}{25}\right)$$

$$or,\ 7x^2-7y^2-\frac{154x}{5}+\frac{98y}{5}+\frac{504}{25}=2xy-\frac{14x}{5}-\frac{22y}{5}+\frac{154}{25}$$

$$or,\ 7x^2-7y^2-\frac{154x}{5}+\frac{98y}{5}+\frac{504}{25}-2xy+\frac{14x}{5}+\frac{22y}{5}-\frac{154}{25}=0$$

$$or,\ 7x^2-2xy-7y^2-\frac{140x}{5}+\frac{120y}{5}+\frac{350}{25}=0$$

$or, 7x^2-2xy-7y^2-28x+24y+14=0$ **(As desired).**

**Problem-04:** Show that $x^2+6xy+9y^2+4x+12y-5=0$ represents pair of straight lines. Also find their equations and the angle.

**Solution: 1st part:** Given that,

$$x^2+6xy+9y^2+4x+12y-5=0\ \cdots\cdots\cdots\cdots(1)$$

The general equation of second degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ \cdots\cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we have,

$a=1, h=3, b=9, g=2, f=6\ \&\ c=-5$

Now, $\Delta=\begin{vmatrix}1&3&2\\3&9&6\\2&6&-5\end{vmatrix}$

$=1(-45-36)-3(-15-12)+2(18-18)$

$=-81+81+0$

$=0$

Since, $\Delta=0$ so the given equation represents a pair of straight lines. (***Showed***)

**2nd part:** The given equation can be written as the following quadratic equation in *x*,

$$x^2+6xy+9y^2+4x+12y-5=0$$

$$or,\ x^2+(6y+4)x+9y^2+12y-5=0$$

$$\therefore\ x=\frac{-(6y+4)\pm\sqrt{(6y+4)^2-4.1.(9y^2+12y-5)}}{2.1}$$

$$\boxed{\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$or,\ x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-36y^2-48y+20}}{2}$$

$$or,\ x=\frac{-(6y+4)\pm\sqrt{36}}{2}$$
$$or,\ x=\frac{-(6y+4)\pm 6}{2}$$
Taking positive sign we get,
$$x=\frac{-(6y+4)+6}{2}$$
$$or, 2x=-6y-4+6$$
$$or,\ 2x=-6y+2$$
$$or,\ 2x+6y-2=0$$
$$\therefore\ x+3y-1=0$$
Taking negative sign we get,
$$x=\frac{-(6y+4)-6}{2}$$
$$or, 2x=-6y-4-6$$
$$or,\ 2x=-6y-10$$
$$or,\ 2x+6y+10=0$$
$$\therefore\ x+3y+5=0$$
Therefore, required equations of the straight lines $x+3y-1=0$ and $x+3y+5=0$.

**3rd part:** If $\theta$ be the angle between the lines then,
$$\theta=\tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$
$$=\tan^{-1}\left(\frac{2\sqrt{(3)^2-1.9}}{1+9}\right)$$
$$=\tan^{-1}\left(\frac{2\sqrt{9-9}}{10}\right)$$
$$=\tan^{-1}(0)$$
$$=\tan^{-1}\tan 0^\circ$$
$=0^\circ$ **(As desired).**

**H.W:** Show that the following equations represent pair of straight lines. Also find their equations, the point of intersection, the angle and the equation of the bisector's of angle.

1. $2y^2-xy-x^2+2x+y-1=0$
2. $2y^2+3xy+5y-6x+2=0$
3. $3y^2-8xy-3y^2-29x+3y-18=0$
4. $x^2+6xy+9y^2+4x+12y-5=0$
5. $2x^2-7xy+3y^2+x+7y-6=0$

**Problem-05:** For what value of $\lambda$ the equation $12x^2+36xy+\lambda y^2+6x+6y+3=0$ represents a pair of straight lines.

**Solution:** Given that,
$$12x^2+36xy+\lambda y^2+6x+6y+3=0\ \cdots\cdots\cdots\cdots(1)$$
The general equation of second degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \cdots\cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we have,

$$a = 12\ , h = 18, b = \lambda, g = 3, f = 3 \ \& \ c = 3$$

Here, the given equation represents a pair of straight lines if $\Delta = 0$.

Now, $\Delta = 0$

$$or, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$or, \begin{vmatrix} 12 & 18 & 3 \\ 18 & \lambda & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0$$

$$or, 12(3\lambda - 9) - 18(54 - 9) + 3(54 - 3\lambda) = 0$$

$$or, 36\lambda - 108 - 810 + 162 - 9\lambda = 0$$

$$or, 27\lambda - 756 = 0$$

$$or, 27\lambda = 756$$

$$\therefore\ \lambda = 28$$

This is the required value of $\lambda$ .**(Ans)**

**Problem-06:** For what value of $\lambda$ the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines.

**Solution:** Given that,

$$x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0 \ \cdots\cdots\cdots\cdots(1)$$

The general equation of second degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \cdots\cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we have,

$$a = 1\ , h = -\lambda/2, b = 2, g = 3/2, f = -5/2 \ \& c = 2$$

Here, the given equation represents a pair of straight lines if $\Delta = 0$.

Now, $\Delta = 0$

$$or, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$or, \begin{vmatrix} 1 & -\lambda/2 & 3/2 \\ -\lambda/2 & 2 & -5/2 \\ 3/2 & -5/2 & 2 \end{vmatrix} = 0$$

$$or, 1\left(4 - 25/4\right) - \left(-\lambda/2\right)\left(-\lambda + 15/4\right) + 3/2\left(5\lambda/4 - 3\right) = 0$$

$$or, -9/4 - \lambda^2/2 + 15\lambda/8 + 15\lambda/8 - 9/2 = 0$$

$$or, -18 - 4\lambda^2 + 15\lambda + 15\lambda - 36 = 0$$

$$or, -4\lambda^2 + 30\lambda - 54 = 0$$

$$or, -2\left(2\lambda^2 - 15\lambda + 27\right) = 0$$

$$or,\ 2\lambda^2 - 15\lambda + 27 = 0$$

$or,\ 2\lambda^2 - 9\lambda - 6\lambda + 27 = 0$

$or,\ 2\lambda^2 - 9\lambda - 6\lambda + 27 = 0$

$or, \lambda(2\lambda - 9) - 3(2\lambda - 9) = 0$

$or,\ (2\lambda - 9)(\lambda - 3) = 0$

$\therefore\ 2\lambda - 9 = 0 \quad \& \quad \lambda - 3 = 0$

$\Rightarrow\ \lambda = 9/2 \quad \& \quad \lambda = 3$

These are the required values of $\lambda$ .**(Ans)**

## H.W:

1. For what value of $\mu$ the equation $x^2 - \mu xy + 2y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines.
2. For what value of $\lambda$ the equation $\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$ represents a pair of straight lines.
3. For what value of $\mu$ the equation $2x^2 + xy - y^2 - 2x - 5y + \mu = 0$ represents a pair of straight lines.
4. For what value of $\eta$ the equation $\eta xy - 8x + 9y - 12 = 0$ represents a pair of straight lines.

**Question-01:** Describe various conditions of general equation of second degree.

**Answer:** The general equation of second degree is,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

This will represent the followings,

1. A pair of straight lines if the determinant, $\Delta = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0.$

   Two parallel lines if$\Delta = 0, h^2 = \text{ab}.$

   Two perpendicular lines if $\Delta = 0, \text{a} + \text{b} = 0.$
2. A circle if $a = b, h = 0.$
3. A parabola if $\Delta \neq 0, h^2 = \text{ ab}$
4. An ellipse if $\Delta \neq 0, h^2 - \text{ab} < 0.$
5. A hyperbola if $\Delta \neq 0, h^2 - \text{ab} > 0.$
6. A rectangular hyperbola if $a + b = 0, h^2 - ab > 0, \Delta \neq 0.$

**Problem-07:** Test the nature of the equation $3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$ and also find its centre.

***Solution: 1st part:*** Given that,

$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$

Also the general equation of second degree is,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$

Comparing $(i)$ and $(ii)$ we have,

$a = 3, h = -4, b = -3, g = 5, f = -\frac{13}{2}, c = 8.$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 3 \times (-3) \times 8 + 2 \times \left(-\frac{13}{2}\right) \times 5 \times (-4) - 3 \times (-\frac{13}{2})^2 - (-3) \times 25 - 8 \times 16$

$= -72 + 260 - \frac{507}{4} + 75 - 128$

$= \frac{33}{4}$ .

Since, $\Delta = \frac{33}{4} \neq 0$ so the given equation represents a conic.

Again, $h^2 - \text{ab} = 16 + 9 = 25 > 0$

And, $a + b = 3 - 3 = 0$

Since, $a + b = 0, h^2 - ab > 0, \Delta = 0$. so the given equation represents a rectangular hyperbola.

***2^nd part:*** Let, $f(x, y) = 3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$

$\therefore \frac{\partial f}{\partial x} = 6x - 8y + 10 = 0$

And $\frac{\partial f}{\partial y} = 8x + 6y + 13 = 0$

The centre of the conic is the intersection of two lines,

$6x - 8y + 10 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

$8x + 6y + 13 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$

Solving $(iii)$ and $(iv)$ we have,

$x = -\frac{41}{25}, y = \frac{1}{50}$

Hence the centre is at $\left(-\frac{41}{25}, \frac{1}{50}\right)$. (**As desired** )

**Problem-08:** Test the nature of the equation $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$.

***Solution:*** Given that,

$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (i)

Also the general equation of second degree is,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

Comparing $(i)$ and $(ii)$ we have,

$a = 9, h = -12, b = 16, g = -9, f = -\frac{101}{2}, c = 19.$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9 \times (-12) \times 19 + 2 \times \left(-\frac{101}{2}\right) \times (-9) \times (-12) - 9 \times \left(-\frac{101}{2}\right)^2 - 16 \times (-9)^2 - 19 \times (-12)^2$

$= -2052 - 10908 - \frac{91809}{4} - 1296 - 2736$

$= -\frac{159777}{4}$ .

Since, $\Delta = -\frac{159777}{4} \neq 0$ so the given equation represents a conic.

Again, $h^2 - \text{ab} = (-12)^2 - 9 \times 16 = 144 - 144 = 0$

Since, $h^2 - ab = 0, \Delta \neq 0$. so the given equation represents a parabola.(**As desired** )

## H.W:

Test the nature of the following equations and find its centre.

*a.* $2x^2 - 3\text{xy} + \text{y}^2 - 5x + 4y + 6 = 0$ $\quad Ans: Hyperbola; (2,1).$

*b.* $4x^2 + 9y^2 - 8x + 36y - 31 = 0$ $\quad Ans: Ellipse$

*c.* $2x^2 - 3y^2 + 8x + 30y - 27 = 0.$ $\quad Ans: Hyperbola; (-2,5).$

*d.* $x^2 - xy - 2y^2 - x - 4y - 2 = 0$ $\quad Ans: Pair\ of\ straight\ lines; \left(0, \frac{7}{9}\right).$

*e.* $x^2 + 2xy + y^2 + 2x - 1 = 0.$ $\quad Ans: Parabola.$

# Reduction of equation to a standard form

**Problem-09:** Reduce the equation $8x^2+4xy+5y^2-24x-24y=0$ to the standard form.

**Solution:** Given that,

$$8x^2+4xy+5y^2-24x-24y=0 \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

The general equation of second degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \cdots\cdots\cdots(2)$$

Comparing Eq. (1) & Eq. (2) we get,

$$a=8,\ h=2, b=5, g=-12, f=-12 \ \& \ c=0$$

Now,

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} 8 & 2 & -12 \\ 2 & 5 & -12 \\ -12 & -12 & 0 \end{vmatrix}=8(0-144)-2(0-144)-12(-24+60)$$

$$=8(-144)-2(-144)-12(-24+60)$$

$$=-1152+288-432$$

$$=-1296\neq 0$$

and

$$h^2-ab=2^2-40=4-40=-36<0$$

Since $\Delta\neq 0$ and $h^2-ab<0$. So the equation represents an ellipse.

Let, $(\alpha,\beta)$ be the centre of conic.

$$\alpha=\frac{bg-hf}{h^2-ab}=\frac{-60+24}{4-40}=\frac{-40}{-40}=1$$

and $$\beta=\frac{af-gh}{h^2-ab}=\frac{-96+24}{4-40}=\frac{-72}{-36}=2$$

Therefore, the coordinates of centre is $(\alpha,\beta)=(1,2)$.

Therefore, the equation of the conic referred to centre as origin is,

$$8x^2+4xy+5y^2+c_1=0 \cdots\cdots\cdots\cdots(3)$$

where,

$$c_1=g\alpha+f\beta+c=-12-24+0=-36$$

So the equation $(3)$ becomes,

$$8x^2+4xy+5y^2-36=0 \cdots\cdots\cdots\cdots(4)$$

When the xy term is removed by the rotation of axes then the reduced equation is,

$$a_1x^2+b_1y^2=36 \cdots\cdots\cdots\cdots(5)$$

Then by invariants we have

$$a_1+b_1=a+b=8+5=13 \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

and, $$h_1^2-a_1b_1=h^2-ab$$

$$or,\ 0-a_1b_1=4-40$$

$$or,\ a_1b_1=36$$

We know,

$$(a_1-b_1)^2=(a_1+b_1)^2-4a_1b_1$$

$$or,\ (a_1-b_1)^2=13^2-4\times 36$$

$$or,\ (a_1-b_1)^2=169-144=25$$

$$or,\ (a_1-b_1)^2=25$$

$$or,\ a_1-b_1=5\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Solving equations $(6)$ and $(7)$ we have $a_1=9$ and $b_1=4$

The equation $(5)$ becomes $9x^2+4y^2=36$

$$or,\ \frac{x^2}{4}+\frac{y^2}{9}=1$$

This is required equations.(**As desired)**

**Problem-10:** Reduce the equation $32x^2+52xy-7y^2-64x-52y-148=0$ to the standard form.

**Solution:** Given that,

$$32x^2+52xy-7y^2-64x-52y-148=0\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

The general equation of 2nd degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Comparing Eq. $(1)$ & Eq. $(2)$ we get,

$$a=32\ ,h=26,b=-7,g=-32,\ f=-26\ \&\ c=-148$$

Now,

$$\Delta=\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=\begin{vmatrix}32 & 26 & -32\\ 26 & -7 & -26\\ -32 & -26 & -148\end{vmatrix}=162000\neq 0$$

and $\quad h^2-ab=26^2+224=900>0$

Since, $\Delta\neq 0$ and $h^2-ab>0$ . So the equation represents hyperbola.

Let, $(\alpha,\beta)$ be the centre of conic.

$$\alpha=\frac{bg-hf}{h^2-ab}=\frac{224+676}{676+224}=\frac{900}{900}=1$$

and $\quad \beta=\dfrac{af-gh}{h^2-ab}=\dfrac{-832++832}{676+224}=0$

Therefore, the coordinates of centre is $(\alpha,\beta)=(1,0)$.

Therefore, the equation of the conic referred to centre as origin is,

$$32x^2+52xy-7y^2+c_1=0\cdots\cdots\cdots\cdots(3)$$

where,

$$c_1=g\alpha+f\beta+c=-32+0-148=-180$$

So the equation $(3)$ becomes,

$$32x^2+52xy-7y^2-180=0\cdots\cdots\cdots\cdots(4)$$

When the xy term is removed by the rotation of axes then the reduced equation is

$$a_1x^2+b_1y^2=180\cdots\cdots\cdots\cdots(5)$$

Then by invariants we have

$$a_1+b_1=a+b=32-7=25\cdots\cdots\cdots\cdots\cdots\cdots(6)$$

and, $\quad h_1^2-a_1b_1=h^2-ab$

$$or,\ 0-a_1b_1=676+224$$

$$or,\ a_1b_1=-900$$

We know,

$$(a_1-b_1)^2=(a_1+b_1)^2-4a_1b_1$$
$$or,\ (a_1-b_1)^2=(25)^2-4\times(-900)$$
$$or,\ (a_1-b_1)^2=625+3600$$
$$or,\ (a_1-b_1)^2=4225$$
$$or,\ a_1-b_1=65\ \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Solving equations (6) and (7) we have $a_1=45$ and $b_1=-20$

The equation (5) becomes, $45x^2-20y^2=180$

$$or,\ \frac{x^2}{4}-\frac{y^2}{9}=1$$

This is required equation. **(As desired)**

**Problem-05:** Reduce the equation $x^2+2xy+y^2+2x-1=0$ to the standard form.

**Solution:** Given that,

$$x^2+2xy+y^2+2x-1=0\ \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

The general equation of 2nd degree is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Comparing Eq. (1) &Eq. (2) we get,

$$a=1,\ h=1, b=1, g=1,\ f=0\ \&\ c=-1$$

Now,

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}=1\neq 0$$

and

$$h^2-ab=1-1=0$$

Since, $\Delta\neq 0$ and $h^2-ab=0$. So the equation representsa parabola.
From given equation we have,

$$x^2+2xy+y^2+2x-1=0$$
$$or,\ (x+y)^2=-2x+1$$
$$or,\ (x+y+\lambda)^2=-2x+1+\lambda^2+2\lambda x+2\lambda y$$
$$or,\ (x+y+\lambda)^2=(-2+2\lambda)x+2\lambda y+(1+\lambda^2)\cdots\cdots\cdots\cdots\cdots\cdots\ (3)$$

The lines $x+y+\lambda=0$ and $(-2+2\lambda)x+2\lambda y+(1+\lambda^2)=0$ are perpendicular if

$$a_1a_2+b_1b_2=0.$$

i.e. $1.(-2+2\lambda)+1.2\lambda=0$

$$or,\ -2+2\lambda+2\lambda=0$$
$$or,\ -2+4\lambda=0\Rightarrow\lambda=\frac{1}{2}$$

Putting the value of $\lambda=\frac{1}{2}$ in (3) we get

$$\left(x+y+\frac{1}{2}\right)^2=-2x+1+\frac{1}{4}+x+y$$

$$or, \left(x+y+\frac{1}{2}\right)^2 = -x+y+\frac{5}{4}$$

$$or, \left(\frac{x+y+\frac{1}{2}}{\sqrt{1^2+1^2}}\right)^2 \left(1^2+1^2\right) = \frac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{1^2+1^2}}.\sqrt{1^2+1^2}$$

$$or, \left(\frac{x+y+\frac{1}{2}}{\sqrt{2}}\right)^2 .2 = \frac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{2}}.\sqrt{2}$$

$$or, \left(\frac{x+y+\frac{1}{2}}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}}\frac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{2}}$$

$$or, \left(\frac{x+y+\frac{1}{2}}{\sqrt{2}}\right)^2 = 4.\frac{1}{4\sqrt{2}}\frac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{2}}$$

$$or, \left(\frac{x+y+\frac{1}{2}}{\sqrt{2}}\right)^2 = 4.\frac{1}{4\sqrt{2}}\frac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{2}}$$

$$\therefore\ Y^2 = 4AX$$

where, $Y = \dfrac{x+y+\frac{1}{2}}{\sqrt{2^2}}, A = \dfrac{1}{4\sqrt{2}}\ \&\ X = \dfrac{\left(-x+y+\frac{5}{4}\right)}{\sqrt{2}}$

That is the standard form of Parabola.

**H.W:**

Reduce the following equations to the standard forms

1. $x^2 - 6xy + 9y^2 + 4x + 8y + 15 = 0$     Ans: $y^2 = \dfrac{\sqrt{10}}{5}x$
2. $9x^2 - 4xy + 6y^2 - 10x - 7 = 0$     Ans: $\dfrac{x^2}{2} + \dfrac{y^2}{1} = 1$
3. $x^2 - 4xy - 2y^2 + 10x + 4y = 0$     Ans: $2x^2 - 3y^2 = 1$
4. $x^2 - 4xy + 2x - 16y + 1 = 0$     Ans: $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$
5. $9x^2 + 24xy + 16y^2 + 22x + 16y + 9 = 0$     Ans: $y^2 = \dfrac{2}{5}x$