Complex Differentiation

Derivatives: If $f(z)$ is single-valued in some region R of the z plane, then the derivative of $f(z)$ is defined as

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which $\Delta z\to 0$.

Alternatively, if $f(z)$ is defined in some neighbourhood of z_0, then the derivative of $f(z)$ at z_0 is defined as

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

provided that the limit exists independent of the manner in which $z\to z_0$.

Analytic (regular or holomorphic) function: A complex function $f(z)$ is said to be analytic at a point z_0, if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

Cauchy-Riemann equations: A necessary condition is that if $w=f(z)=u(x,y)+iv(x,y)$ be analytic in a region R, then u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad (1)$$

If the partial derivatives of (1) are continuous in R, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in R.

Harmonic Function: Function such as $u(x,y)$ which satisfies the Laplace's equation $\nabla^2 u=0$ in a region R is called harmonic function and is said to be harmonic in R.

Harmonic conjugate: The function v is said to be a harmonic conjugate of u if u and v are harmonic and satisfy Cauchy-Riemann equations.

L'Hospital's rule: If $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0)=g(z_0)=0$ but $g'(z_0)\neq 0$. Then L'Hospital's rule states that

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}.$$

In case $f'(z_0)=g'(z_0)=0$, the rule may be extended.

Singular point: A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. Various types of singularities exist such as:

1. **Isolated singularity:** The point z_0 is called an isolated singular point of $f(z)$ if we can find $\delta>0$ such that the circle $|z-z_0|=\delta$ encloses no singular point other than z_0. That is, there exists a deleted neighbourhood $0<|z-z_0|<\delta$ in which $f(z)$ is analytic. There are three types isolated singular points such as pole, removable singular point and essential singular point.

   Example: $f(z)=\dfrac{1}{z-1}$ has an isolated singularity at $z=1$.

2. **Pole:** If we can find a positive integer n such that $\lim\limits_{z\to z_0}(z-z_0)^n f(z)=A\neq 0$, then z_0 is called a pole of order n. If $n=1$, z_0 is called a simple pole.

   Example: (a). $f(z)=\dfrac{1}{(z-2)^3}$ has a pole of order 3 at $z=2$.

   (b). $f(z)=\dfrac{1}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z=1$, and simple poles at $z=-1$ and $z=4$.

3. **Branch point:** The branch point of multiple-valued function have already been studied which are also singular points.

   Example: (a). $f(z)=(z-3)^{1/2}$ has a branch point at $z=3$.

   (b). $f(z)=\ln(z^2+z-2)$ has branch points where $z^2+z-2=0$, i.e. at $z=1$ and $z=-2$.

4. **Removable singularity:** The singular point z_0 is called a removable singularity of $f(z)$ if $\lim\limits_{z\to z_0} f(z)$ exists.

   Example: (a). The singular point $z=0$ is a removable singularity of $f(z)=\dfrac{\sin z}{z}$ since $\lim\limits_{z\to 0}\dfrac{\sin z}{z}=1$.

5. **Essential singularity:** A singularity which is not a pole, branch point or removable singularity is called an essential singularity.

   Example: (a). $f(z)=e^{1/(z-2)}$ has an essential singularity at $z=2$.

6. **Singularity at Infinity:** The function $f(z)$ has a singularity at $z=\infty$ if $w=0$ is a singularity of $f\left(\dfrac{1}{w}\right)$.

   Example: The function $f(z)=z^3$ has a pole of order 3 at $z=\infty$, since $f\left(\dfrac{1}{w}\right)=\dfrac{1}{w^3}$ has a pole of order 3 at $w=0$.

Complex differential operators: Let $F(x,y)$ is a real continuously differentiable function of x and y while $A(x,y)=P(x,y)+iQ(x,y)$ is a complex continuously differentiable function of x and y.

Since $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$. So in terms of conjugate coordinates, we have

$$F(x,y)=F\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)=G(z,\bar{z})$$

and $A(x,y)=B(z,\bar{z})$

Since $F(x,y)$ is any continuously differentiable function so

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}=\frac{\partial F}{\partial z}+\frac{\partial F}{\partial \bar{z}}$$

$$\therefore \frac{\partial}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}$$

and
$$\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}=i\left(\frac{\partial F}{\partial z}-\frac{\partial F}{\partial \bar{z}}\right)$$

$$\therefore \frac{\partial}{\partial y}=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)$$

Now
$$\nabla=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}+i^2\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)=2\frac{\partial}{\partial \bar{z}}$$

and
$$\bar{\nabla}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}-i^2\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)=2\frac{\partial}{\partial z}$$

Here, ∇ and $\bar{\nabla}$ are complex differential operators.

Gradient, Divergence, Curl and Laplacian: Let $F(x,y)$ is a real continuously differentiable function of x and y while $A(x,y)=P(x,y)+iQ(x,y)$ is a complex continuously differentiable function of x and y. Since $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$. So in terms of conjugate coordinates, we have

$$F(x,y)=F\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)=G(z,\bar{z})$$

and $A(x,y)=B(z,\bar{z})$

(1) Gradient: We define the gradient of a real function F (scalar) by

$$grad\, F=\nabla F=\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}=2\frac{\partial G}{\partial \bar{z}}$$

Geometrically, this represents a vector normal to the curve $F(x,y)=c$ where c is a constant.

Similarly, the gradient of a complex function $A=P+iQ$ (vector) is defined by

$$grad\, A=\nabla A=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(P+iQ)$$

$$=\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}+i\left(\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)=2\frac{\partial B}{\partial \bar{z}}$$

In particular if B is an analytic function of z then $\frac{\partial B}{\partial \bar{z}}=0$ and so the gradient is zero, i.e. $\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}$ and $\frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}$, which shows that the Cauchy-Riemann equations are satisfied in this case.

(2) Divergence: We define the divergence of a complex function (vector) by

$$\begin{aligned} div\, A = \nabla \cdot A &= \operatorname{Re}\left\{\bar{\nabla} A\right\} \qquad \left[\because z_1 \cdot z_2 = \operatorname{Re}\left(\bar{z}_1 z_2\right)\right] \\ &= \operatorname{Re}\left\{\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(P+iQ)\right\} \\ &= \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} \\ &= 2\operatorname{Re}\left\{\frac{\partial B}{\partial z}\right\} \end{aligned}$$

Similarly we can define the divergence of a real function. It should be noted that the divergence of a complex or real function (vector or scalar) is always a real function (scalar).

(3) Curl: We define the curl of a complex function (vector) by

$$\begin{aligned} curl\, A = \nabla \times A &= \operatorname{Im}\left\{\bar{\nabla} A\right\} \qquad \left[\because z_1 \times z_2 = \operatorname{Im}\left(\bar{z}_1 z_2\right)\right] \\ &= \operatorname{Im}\left\{\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(P+iQ)\right\} \\ &= \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} \\ &= 2\operatorname{Im}\left\{\frac{\partial B}{\partial z}\right\} \end{aligned}$$

Similarly we can define the curl of a real function.

(4) Laplacian: The Laplacian operator is defined as the dot or scalar product of ∇ with itself.

$$\begin{aligned} \text{i.e. } \nabla \cdot \nabla = \nabla^2 &= \operatorname{Re}\left\{\bar{\nabla}\nabla\right\} \qquad \left[\because z_1 \cdot z_2 = \operatorname{Re}\left(\bar{z}_1 z_2\right)\right] \\ &= \operatorname{Re}\left\{\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right\} \\ &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \\ &= 4\frac{\partial^2}{\partial z \partial \bar{z}} \end{aligned}$$

Note that if A is analytic, $\nabla^2 A=0$ so that $\nabla^2 P=0$ and $\nabla^2 Q=0$, i.e. P and Q are harmonic.

Problems

Problem-01: Show that $\frac{d}{dz}\left(\bar{z}\right)$ does not exist anywhere.

Solution: Here $f\left(z\right)=\bar{z}$

By definition we have

$$\frac{d}{dz}f\left(z\right)=\lim_{\Delta z\to 0}\frac{f\left(z+\Delta z\right)-f\left(z\right)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z=\Delta x+i\Delta y$ approaches zero.

Now

$$\begin{aligned}\frac{d}{dz}f\left(z\right)&=\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}\\&=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\overline{x+iy+\Delta x+i\Delta y}-\overline{x+iy}}{\Delta x+i\Delta y}\\&=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\overline{\left(x+\Delta x\right)+i\left(y+\Delta y\right)}-\overline{x+iy}}{\Delta x+i\Delta y}\\&=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\left(x+\Delta x\right)-i\left(y+\Delta y\right)-\left(x-iy\right)}{\Delta x+i\Delta y}\\&=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{x+\Delta x-iy-i\Delta y-x+iy}{\Delta x+i\Delta y}\\&=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}\end{aligned}$$

Taking limit along real axis $\left(\Delta x\to 0,\Delta y=0\right)$, we get

$$\frac{d}{dz}f\left(z\right)=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

Again, taking limit along imaginary axis $\left(\Delta x=0,\Delta y\to 0\right)$, we get

$$\frac{d}{dz}f\left(z\right)=\lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y}=-1$$

The above two limits are not equal, that is, the limit depends on manner in which $\Delta z\to 0$.

Hence $\frac{d}{dz}\left(\bar{z}\right)$ does not exist anywhere. **(Showed)**

Problem-02: Show that $\frac{d}{dz}\left(z^2\bar{z}\right)$ does not exist anywhere.

Solution: Here $f\left(z\right)=z^2\bar{z}$

By definition we have

$$\frac{d}{dz}f\left(z\right)=\lim_{\Delta z\to 0}\frac{f\left(z+\Delta z\right)-f\left(z\right)}{\Delta z}$$

if this limit exists and independent of the manner in which $\Delta z=\Delta x+i\Delta y$ approaches zero.

Now $\dfrac{d}{dz}f(z)=\lim\limits_{\Delta z\to 0}\dfrac{(z+\Delta z)^2\,\overline{z+\Delta z}-z^2\bar{z}}{\Delta z}$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2\,\overline{z+\Delta z}-z^2\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\left\{z^2+2z\Delta z+(\Delta z)^2\right\}\left(\bar{z}+\overline{\Delta z}\right)-z^2\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{z^2\bar{z}+2z\bar{z}\Delta z+\bar{z}(\Delta z)^2+z^2\overline{\Delta z}+2z\Delta z\overline{\Delta z}+(\Delta z)^2\,\overline{\Delta z}-z^2\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\left\{2z\bar{z}+\bar{z}\Delta z\right\}\Delta z+\left\{z^2+2z\Delta z+(\Delta z)^2\right\}\overline{\Delta z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\left\{2z\bar{z}+\bar{z}\Delta z\right\}\Delta z}{\Delta z}+\lim_{\Delta z\to 0}\frac{\left\{z^2+2z\Delta z+(\Delta z)^2\right\}\overline{\Delta z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\left\{2z\bar{z}+\bar{z}\Delta z\right\}+\lim_{\Delta z\to 0}\left\{z^2+2z\Delta z+(\Delta z)^2\right\}\cdot\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$

$$=2z\bar{z}+z^2\cdot\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}$$

In different way $\lim\limits_{\Delta z\to 0}\dfrac{\overline{\Delta z}}{\Delta z}$ gives different values, so that $\dfrac{d}{dz}\left(z^2\bar{z}\right)$ does not exist anywhere. **(Proved)**

Problem-03: State and prove Cauchy-Riemann Equations.

OR

State and prove necessary condition for a function to be analytic.

OR

State and prove sufficient condition for a function to be analytic.

OR

Prove (a) necessary and (b) sufficient condition that $w=f(z)=u+iv$ is analytic, iff it satisfy the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$.

<u>Solution:</u> Sufficient condition: Let, $w=f(z)=u(x,y)+iv(x,y)$ be a function defined in a region R. If in R, the Cauchy Riemann equations are satisfied and $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}$ are continuous then $f(z)$ is analytic in R.

Proof: Since $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are continuous in R.

Then we have

$$\Delta u=u(x+\Delta x,y+\Delta y)-u(x,y)$$

$$=u(x+\Delta x,y+\Delta y)-u(x,y+\Delta y)+u(x,y+\Delta y)-u(x,y)$$

$$=\left(\frac{\partial u}{\partial x}+\varepsilon_1\right)\Delta x+\left(\frac{\partial u}{\partial y}+\eta_1\right)\Delta y$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$ and $\eta_1 \to 0$ as $\Delta y \to 0$.

Again $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in R.

Then we have

$$\Delta v = v(x+\Delta x, y+\Delta y)-v(x,y)$$

$$= v(x+\Delta x, y+\Delta y)-v(x,y+\Delta y)+v(x,y+\Delta y)-v(x,y)$$

$$=\left(\frac{\partial v}{\partial x}+\varepsilon_2\right)\Delta x+\left(\frac{\partial v}{\partial y}+\eta_2\right)\Delta y$$

where $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\eta_2 \to 0$ as $\Delta y \to 0$.

Now, $\Delta w = \Delta u + i\Delta v$

$$=\left(\frac{\partial u}{\partial x}+\varepsilon_1\right)\Delta x+\left(\frac{\partial u}{\partial y}+\eta_1\right)\Delta y+i\left(\frac{\partial v}{\partial x}+\varepsilon_2\right)\Delta x+i\left(\frac{\partial v}{\partial y}+\eta_2\right)\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+(\varepsilon_1+i\varepsilon_2)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y+(\eta_1+i\eta_2)\Delta y$$

where $\varepsilon=\varepsilon_1+i\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\eta=\eta_1+i\eta_2 \to 0$ as $\Delta y \to 0$.

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y+\varepsilon\Delta x+\eta\Delta y$$

By using Cauchy-Riemann equations, we get

$$\Delta w=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y+\varepsilon\Delta x+\eta\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+i\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta y+\varepsilon\Delta x+\eta\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)+\varepsilon\Delta x+\eta\Delta y$$

Then on dividing by $\Delta z=\Delta x+i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+\lim_{\Delta z\to 0}\frac{\varepsilon\Delta x+\eta\Delta y}{\Delta z}$$

$$or,\ \frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+0$$

$$\therefore\ f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Since the derivative exists. Hence $f(z)$ is analytic. **(Proved)**

Necessary condition: : Let, $w = f(z) = u(x,y) + iv(x,y)$ be a function defined in a region R. The necessary condition for $f(z)$ to be analytic in R is that the Cauchy Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in R.

Proof: Let $f(z)$ be analytic in R.

Then at any point $z \in R$,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero.

Taking limit along real axis ($\Delta x \to 0,\ \Delta y = 0$), we get

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (1)$$

Again, taking limit along imaginary axis ($\Delta x = 0,\ \Delta y \to 0$), we get

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (2)$$

Since $f(z)$ is analytic, then two limits (1) and (2) must be equal.

Hence, $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Now, equating real and imaginary part on both sides, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy-Riemann equations. **(Proved)**

Problem-04: Prove that in polar form the Cauchy-Riemann equations can be written as

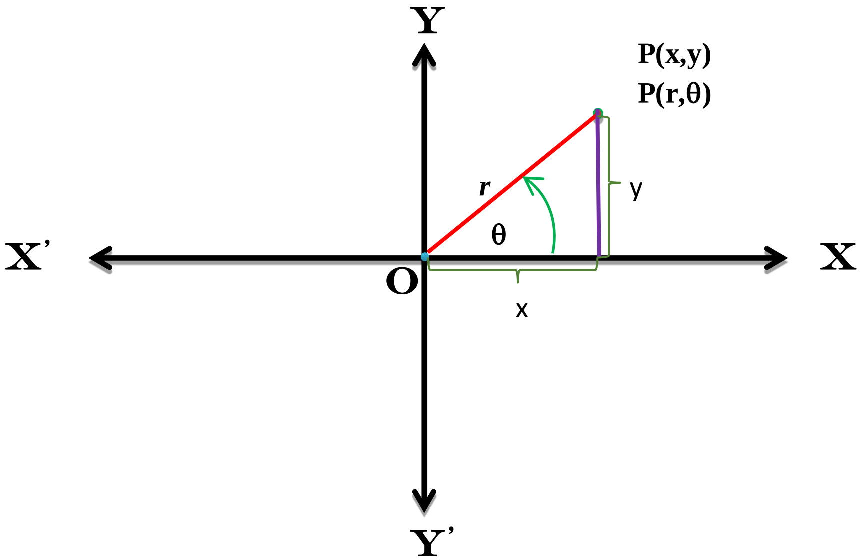
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$$

Solution: By relation of Cartesian coordinate (x,y) and Polar coordinate (r,θ), we have

$$x = r\cos\theta \quad \cdots(1) \quad \text{and} \quad y = r\sin\theta \quad \cdots(2)$$

From (1) and (2), we get

$$r = \sqrt{x^2 + y^2} \quad \cdots(3) \text{ and } \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \cdots(4)$$



Differentiating (3) and (4) with respect to x and y, we get

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x}$

$$= \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta}$$

Again, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x}$

$$= \cos\theta \frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

By Cauchy- Riemann equations, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$or,\ \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

$$or,\ \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad \cdots(5)$$

Again, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$or,\ \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta \frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}$$

$$or,\ \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta + \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta = 0 \qquad \cdots(6)$$

Multiplying (5) by $\cos\theta$ and (6) by $\sin\theta$ and adding, we get

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos^2\theta + \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin^2\theta = 0$$

$$or,\ \frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0$$

$$\therefore\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Again, multiplying (5) by $\sin\theta$ and (6) by $\cos\theta$ and subtracting, we get

$$\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos^2\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin^2\theta = 0$$

$$or,\ \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

$$\therefore\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence, the Cauchy- Riemann equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$$ **(Proved)**

Problem-05: Prove that $f(z) = z\bar{z}$ is nowhere analytic

Solution: Given that $f(z) = z\bar{z}$

Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$$f(z) = (x+iy)(x-iy)$$
$$= x^2 + y^2$$

Here $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$

Now $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = 2y$, $\dfrac{\partial v}{\partial x} = 0$ and $\dfrac{\partial v}{\partial y} = 0$

The above equations show that $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are continuous everywhere. But the Cauchy-Riemann equations are satisfied only at the origin. Hence $z = 0$ is only the point at which $f'(z)$ exists. Thus $f(z) = z\bar{z}$ is nowhere analytic. **(Proved)**

Problem-06: Prove that an analytic function with constant modulus is a constant.

Solution: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \left|f(z)\right| = c_1 \quad \text{where } c_1 \text{ is a constant}$$
$$or,\ \sqrt{u^2+v^2} = c_1$$
$$or,\ u^2 + v^2 = c_1^2$$

Differentiating with respect to *x*, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$
$$or,\ u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad (1)$$

Similarly, $u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} = 0 \qquad (2)$

Using Cauchy-Riemann equations in (2), we get

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \qquad (3)$$

Squaring (1) and (3) and then adding, we have

$$\left(u^2+v^2\right)\left(\frac{\partial u}{\partial x}\right)^2 + \left(u^2+v^2\right)\left(\frac{\partial v}{\partial x}\right)^2 = 0$$
$$or,\ \left(u^2+v^2\right)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$
$$or,\ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad (4)$$

The equation (4) will be valid if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$$

which implies

$$f'(z)=0 \qquad \because\ f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Integrating, we get

$$f(z)=c_2 \qquad \text{where } c_2 \text{ is a constant}$$

Hence $f(z)$ is a constant function. **(Proved)**

Problem-07: If u and v are harmonic in a region R, then prove that $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ is analytic.

OR

If $\varphi(x,y)$ and $\psi(x,y)$ satisfy Laplace's equation, then show $s+it$ is analytic.

where $s=\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}$ and $t=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}$.

Solution: Since u and v are harmonic function in a region *R*.

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (1)$$

and

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (2)$$

Let $$f(z)=\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Here $$U=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} \qquad (3)$$

and $$V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \qquad (4)$$

Differentiating (3) and (4) with respect to *x* and *y* respectively, we get

$$\frac{\partial U}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2} \qquad (5)$$

$$\frac{\partial U}{\partial y}=\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial y\partial x} \qquad (6)$$

$$\frac{\partial V}{\partial x}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y} \qquad (7)$$

$$\frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2} \qquad (8)$$

Subtracting (8) from (5), we get

$$\frac{\partial U}{\partial x}-\frac{\partial V}{\partial y}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial^2 v}{\partial y^2}$$

$$=-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$= 0$$

$$\therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

Adding (6) and (7), we get

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= 0$$

$$\therefore \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Since, the Cauchy-Riemann equations are satisfied so $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic in R. **(Proved)**

Problem-08: If $f(z)$ is an analytic function of z, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left|\operatorname{Re}\{f(z)\}\right|^2 = 2\left|f'(z)\right|^2$.

Solution: Let $f(z)$ be an analytic function.

Then $\quad \operatorname{Re}\{f(z)\} = u$

and $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\therefore \left|f'(z)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left|\operatorname{Re}\{f(z)\}\right|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2$$

Now, $\frac{\partial^2}{\partial x^2}\left(u^2\right) = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] \qquad (1)$

$$\frac{\partial^2}{\partial y^2}\left(u^2\right) = \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right) = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right] \qquad (2)$$

Adding (1) and (2), we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] + 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$$

$$= 2\left[\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + \left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\}u\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad \text{Since } u \text{ is harmonic } \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$=2\left|f^{'}(z)\right|^{2}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|\operatorname{Re}\{f(z)\}\right|^2=2\left|f^{'}(z)\right|^2 \qquad \textbf{(Proved)}$$

Problem-09: Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation $\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2}=0$.

Solution: We know that the Cauchy-Riemann equations in polar form are

$$\frac{\partial v}{\partial\theta}=r\frac{\partial u}{\partial r} \qquad (1)$$

$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta} \qquad (2)$$

Differentiating (1) with respect to r, we get

$$\frac{\partial^2 v}{\partial r\partial\theta}=\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2} \qquad (3)$$

Differentiating (2) with respect to θ, we get

$$\frac{\partial^2 v}{\partial\theta\partial r}=-\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2} \qquad (4)$$

We know that

$$\frac{\partial^2 v}{\partial r\partial\theta}=\frac{\partial^2 v}{\partial\theta\partial r}$$

$$or,\ \frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}$$

$$or,\ \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$$

Similarly, $$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0$$

$$\therefore \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0 \qquad \textbf{(Proved)}$$

Problem-10: Show that $f(z)=|z|^2$ is differentiable at $z=0$.

Solution: Here $f(z)=|z|^2$

By definition we have

$$f^{'}(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z=\Delta x+i\Delta y$ approaches zero.

Now $$f^{'}(0)=\lim_{\Delta z\to 0}\frac{f(0+\Delta z)-f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|0+\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (\Delta x - i\Delta y)$$

Taking limit along real axis $(\Delta x \to 0, \Delta y = 0)$, we get

$$f^{'}(0) = \lim_{\Delta x \to 0} \Delta x = 0$$

Again, taking limit along imaginary axis $(\Delta x = 0, \Delta y \to 0)$, we get

$$f^{'}(0) = \lim_{\Delta y \to 0} (-i\Delta y) = 0$$

The above two limits are equal, that is, the limit does not depend on manner in which $\Delta z \to 0$.

Hence $f(z) = |z|^2$ is differentiable at $z = 0$. (**Showed**)

Problem-11: If $f(z) = u + iv$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then show that u and v are harmonic in R.

Solution: Given $f(z) = u + iv$ is analytic in the region R. By Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (2)$$

Again given u and v have continuous second order partial derivatives in R. So we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \qquad (3)$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \qquad (4)$$

Now from (3) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$or,\ \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$or,\ -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$or,\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus, v satisfies Laplace equation and hence it is harmonic.

Again, from (4) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$or,\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

$$or,\ \frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$$

$$or,\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Thus, u satisfies Laplace equation and hence it is harmonic. (**Showed**)

Problem-12: Prove that, if a function $f(z)$ is differentiable at a point, then $f(z)$ is continuous at that point, but the converse is not necessarily true.

Solution: Let the function $f(z)$ is differentiable at z_0.

Now $$f(z)-f(z_0)=\frac{f(z)-f(z_0)}{z-z_0}\cdot(z-z_0)$$

$$or,\ \lim_{z\to z_0} f(z)-f(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0)$$

$$=f^{'}(z_0)\cdot 0$$

$$=0$$

$$\therefore \lim_{z\to z_0} f(z)=f(z_0)$$

Hence $f(z)$ is continuous at z_0. Thus every differentiable function is continuous.

Converse part: The converse of the given statement is not true. We shall prove this by the following counter example.

Let $f(z)=\bar{z}$ $\quad\therefore\ f(0)=\bar{0}=0$

Now $\lim_{z\to 0} f(z)=\lim_{z\to 0}\bar{z}$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\overline{x+iy}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}(x-iy)$$

Taking the limit along real axis $(x\to 0, y=0)$, we have

$$\lim_{z\to 0} f(z)=\lim_{x\to 0}(x)=0$$

Taking the limit along imaginary axis $(x=0, y\to 0)$, we have

$$\lim_{z\to 0} f(z)=\lim_{y\to 0}(-iy)=0$$

Since the above two limit are equal so $\lim_{z\to 0} f(z)$ exists and equal to the functional value at $z=0$,

$$\text{i.e. } \lim_{z\to 0} f(z) = f(0)$$

Hence $f(z)$ is continuous at $z=0$.

Again at $z=0$, we have $f^{'}(0) = \lim_{\Delta z\to 0} \dfrac{\overline{0+\Delta z}-0}{\Delta z}$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Taking limit along real axis $(\Delta x \to 0, \Delta y = 0)$, we get

$$f^{'}(z) = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1$$

Taking limit along imaginary axis $(\Delta x = 0, \Delta y \to 0)$, we get

$$f^{'}(z) = \lim_{\Delta y\to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

The above two limits are not equal, that is, the limit depends on manner in which $\Delta z \to 0$.
Hence $f(z)$ is not differentiable at $z=0$. **(Showed)**

Problem-13: Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find its harmonic conjugate v and express $u+iv$ as an analytic function of z.

Solution: Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ (1)

1st part: Differentiating (1) with respect to x, we get

$$\frac{\partial u}{\partial x} = 6xy + 4x \qquad (2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 6y + 4 \qquad (3)$$

Again, differentiating (1) with respect to y, we get

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y \qquad (4)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -6y - 4 \qquad (5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4$$

$$= 0.$$

Since u satisfies the Laplace's equation so it is harmonic. **(Proved)**

2nd part: If v is harmonic conjugate of u, then by Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$or, \ \frac{\partial v}{\partial x} = -3x^2 + 3y^2 + 4y \qquad (6)$$

Integrating (6) with respect to x, we get

$$v = -x^3 + 3xy^2 + 4xy + f(y) \qquad (7)$$

Differentiating (7) with respect to y, we get

$$\frac{\partial v}{\partial y} = 6xy + 4x + f'(y)$$

$$or, \ \frac{\partial u}{\partial x} = 6xy + 4x + f'(y)$$

$$or, \ 6xy + 4x = 6xy + 4x + f'(y)$$

$$or, \ f'(y) = 0 \qquad (8)$$

Integrating (8) with respect to y, we get

$$f(y) = c$$

From (7), we have

$$v = -x^3 + 3xy^2 + 4xy + c$$

This is the required harmonic conjugate of u. **(Ans)**

3rd part: Let $f(z) = u + iv$

$$= 3x^2y + 2x^2 - y^3 - 2y^2 + i\left(-x^3 + 3xy^2 + 4xy + c\right)$$

$$= -ix^3 + 3x^2y + 3ixy^2 - y^3 + 2x^2 + 4ixy - 2y^2 + ic$$

$$= -i\left(x^3 + i3x^2y + 3i^2xy^2 + i^3y^3\right) + 2\left(x^2 + 2ixy + i^2y^2\right) + ic$$

$$= -i(x+iy)^3 + 2(x+iy)^2 + c_1 \qquad \text{where } c_1 = ic$$

$$\therefore \ f(z) = -iz^3 + 2z^2 + c_1 \qquad \textbf{(Ans)}$$

Problem-14: Prove that the function $u = 2x - x^3 + 3xy^2$ is harmonic. Find its harmonic conjugate.

Solution: Given that $u = 2x - x^3 + 3xy^2$ (1)

1st part: Differentiating (1) with respect to x, we get

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \qquad (2)$$

$$\therefore \ \frac{\partial^2 u}{\partial x^2} = -6x \qquad (3)$$

Again, differentiating (1) with respect to y, we get

$$\frac{\partial u}{\partial y} = 6xy \qquad (4)$$

$$\therefore \ \frac{\partial^2 u}{\partial y^2} = 6x \qquad (5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x$$

$$= 0.$$

Since u satisfies the Laplace's equation so it is harmonic. **(Proved)**

2nd part: If v is harmonic conjugate of u, then by Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$or, \ \frac{\partial v}{\partial x} = -6xy \qquad (6)$$

Integrating (6) with respect to x, we get

$$v = -3x^2 y + f(y) \qquad (7)$$

Differentiating (7) with respect to y, we get

$$\frac{\partial v}{\partial y} = -3x^2 + f'(y)$$

$$or, \ \frac{\partial u}{\partial x} = -3x^2 + f'(y)$$

$$or, \ 2 - 3x^2 + 3y^2 = -3x^2 + f'(y)$$

$$or, \ f'(y) = 2 + 3y^2 \qquad (8)$$

Integrating (8) with respect to y, we get

$$f(y) = 2y + y^3 + c$$

From (7), we have

$$v = y^3 - 3x^2 y + 2y + c$$

This is the required harmonic conjugate of u. **(Ans)**

Problem-15: Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. Find v such that $f(z) = u + iv$ is analytic.

Solution: Given that $u = e^{-x}(x\sin y - y\cos y)$ (1)

1st part: Differentiating (1) with respect to x, we get

$$\frac{\partial u}{\partial x} = -e^{-x}(x\sin y - y\cos y) + e^{-x}\sin y \qquad (2)$$

$$\therefore \ \frac{\partial^2 u}{\partial x^2} = e^{-x}(x\sin y - y\cos y) - e^{-x}\sin y - e^{-x}\sin y$$

$$= xe^{-x}\sin y - e^{-x}y\cos y - 2e^{-x}\sin y \qquad (3)$$

Again, differentiating (1) with respect to y, we get

$$\frac{\partial u}{\partial y}=e^{-x}\left(x\cos y-\cos y+y\sin y\right) \qquad (4)$$

$$\therefore\ \frac{\partial^2 u}{\partial y^2}=e^{-x}\left(-x\sin y+\sin y+\sin y+y\cos y\right)$$

$$=-xe^{-x}\sin y+2e^{-x}\sin y+e^{-x}y\cos y \qquad (5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xe^{-x}\sin y-e^{-x}y\cos y-2e^{-x}\sin y-xe^{-x}\sin y+2e^{-x}\sin y+e^{-x}y\cos y$$

$$=0.$$

Since u satisfies the Laplace's equation so it is harmonic. **(Showed)**

2nd part: If v is harmonic conjugate of u, then by Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$$or,\ \frac{\partial v}{\partial x}=-xe^{-x}\cos y+e^{-x}\cos y-e^{-x}y\sin y \qquad (6)$$

Integrating (6) with respect to x, we get

$$v=-\left(-xe^{-x}-e^{-x}\right)\cos y-e^{-x}\cos y+e^{-x}y\sin y+f\left(y\right)$$

$$or,\ v=xe^{-x}\cos y+e^{-x}\cos y-e^{-x}\cos y+e^{-x}y\sin y+f\left(y\right)$$

$$or,\ v=xe^{-x}\cos y+e^{-x}y\sin y+f\left(y\right) \qquad (7)$$

Differentiating (7) with respect to y, we get

$$\frac{\partial v}{\partial y}=-xe^{-x}\sin y+e^{-x}\sin y+e^{-x}y\cos y+f'\left(y\right)$$

$$or,\ \frac{\partial u}{\partial x}=-xe^{-x}\sin y+e^{-x}\sin y+e^{-x}y\cos y+f'\left(y\right)$$

$$or,\ -xe^{-x}\sin y+e^{-x}y\cos y+e^{-x}\sin y=-xe^{-x}\sin y+e^{-x}\sin y+e^{-x}y\cos y+f'\left(y\right)$$

$$or,\ f'\left(y\right)=0 \qquad (8)$$

Integrating (8) with respect to y, we get

$$f\left(y\right)=c$$

From (7), we have

$$v=xe^{-x}\cos y+e^{-x}y\sin y+c$$

This is the required harmonic conjugate of u. **(Ans)**

3rd part: Let $f\left(z\right)=u+iv$

$$=xe^{-x}\sin y-e^{-x}y\cos y+i\left(xe^{-x}\cos y+e^{-x}y\sin y+c\right)$$

$$=xe^{-x}\sin y-e^{-x}y\cos y+ixe^{-x}\cos y+ie^{-x}y\sin y+ic$$

$$=e^{-x}\left(x\sin y-y\cos y+ix\cos y+iy\sin y\right)+ic$$

$$=e^{-x}\left\{x\left(\sin y+i\cos y\right)-y\left(\cos y-i\sin y\right)\right\}+ic$$

$$= e^{-x}\left\{xi\left(\cos y - i\sin y\right) - y\left(\cos y - i\sin y\right)\right\} + ic$$
$$= e^{-x}\left\{i\left(x+iy\right)\left(\cos y - i\sin y\right)\right\} + ic$$
$$= e^{-x}\left\{ize^{-iy}\right\} + ic$$
$$= ize^{-x-iy} + ic$$
$$= ize^{-z} + c_1 \qquad \text{where } c_1 = ic$$
$$\therefore\ f\left(z\right) = ize^{-z} + c_1 \qquad \textbf{(Ans)}$$

Problem-16: Prove that the function $f\left(z\right) = \begin{cases} \dfrac{x^3\left(1+i\right) - y^3\left(1-i\right)}{x^2+y^2} & when\ \ z \neq 0 \\ 0 & when\ z = 0 \end{cases}$ is not analytic at origin but the Cauchy-Riemann equations are satisfied.

Solution: Given that $f\left(z\right) = \begin{cases} \dfrac{x^3\left(1+i\right) - y^3\left(1-i\right)}{x^2+y^2} & when\ \ z \neq 0 \\ 0 & when\ z = 0 \end{cases}$

$$or,\quad f\left(z\right) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} + i\dfrac{x^3-y^3}{x^2+y^2} & when\ \left(x,y\right) \neq 0 \\ 0 & when\ \left(x,y\right) = 0 \end{cases}$$

Here, $u\left(x,y\right) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & when\ \left(x,y\right) \neq 0 \\ 0 & when\ \left(x,y\right) = 0 \end{cases}$

and $v\left(x,y\right) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & when\ \left(x,y\right) \neq 0 \\ 0 & when\ \left(x,y\right) = 0 \end{cases}$

Now $\dfrac{\partial u}{\partial x} = \lim\limits_{h \to 0} \dfrac{u\left(x+h,y\right) - u\left(x,y\right)}{h}$ and $\dfrac{\partial u}{\partial y} = \lim\limits_{k \to 0} \dfrac{u\left(x,y+k\right) - u\left(x,y\right)}{k}$

At $\left(0,0\right)$, we get

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u\left(0+h,0\right) - u\left(0,0\right)}{h}$$
$$= \lim_{h \to 0} \frac{h-0}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= 1$$

and $\dfrac{\partial u}{\partial y} = \lim\limits_{k \to 0} \dfrac{u\left(0,0+k\right) - u\left(0,0\right)}{k}$

$$= \lim_{k \to 0} \frac{-k-0}{k}$$

$$= \lim_{k \to 0} \frac{-k}{k}$$

$$= -1$$

Similarly, at $(0,0)$, we get

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,0)-v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h-0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

and $$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,0+k)-v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{k-0}{k}$$

$$= \lim_{k \to 0} \frac{k}{k}$$

$$= 1$$

From the above relations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider $$f^{'}(0) = \lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z \to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{(x,y) \to (0,0)} \frac{x^3(1+i)-y^3(1-i)}{(x+iy)(x^2+y^2)}$$

Taking the limit along real axis $(x \to 0, y = 0)$, we have

$$= \lim_{x \to 0} \frac{x^3(1+i)}{x^3}$$

$$= 1+i$$

Taking the limit along imaginary axis $(x = 0, y \to 0)$, we have

$$= \lim_{y \to 0} \frac{-y^3(1-i)}{iy^3}$$

$$= 1+i$$

Taking the limit along the path $y = x$, we have

$$=\lim_{y\to 0}\frac{y^3(1+i)-y^3(1-i)}{(y+iy)(y^2+y^2)}$$

$$=\lim_{y\to 0}\frac{2iy^3}{2y^3(1+i)}$$

$$=\frac{i}{1+i}$$

which is different from the above limits.

Therefore $f'(0)$ does not exist and so $f(z)$ is not analytic at origin. **(Proved)**

Problem-17: Prove that the function $f(z)=\begin{cases}\dfrac{xy^2(x+iy)}{x^2+y^4} & when\ \ z\neq 0\\ 0 & when\ z=0\end{cases}$ is not analytic at origin but the Cauchy-Riemann equations are satisfied.

Solution: Given that $f(z)=\begin{cases}\dfrac{xy^2(x+iy)}{x^2+y^4} & when\ \ z\neq 0\\ 0 & when\ z=0\end{cases}$

$$or,\quad f(z)=\begin{cases}\dfrac{x^2y^2}{x^2+y^4}+i\dfrac{xy^3}{x^2+y^4} & when\ (x,y)\neq 0\\ 0 & when\ (x,y)=0\end{cases}$$

Here, $u(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^4} & when\ (x,y)\neq 0\\ 0 & when\ (x,y)=0\end{cases}$

and $v(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^4} & when\ (x,y)\neq 0\\ 0 & when\ (x,y)=0\end{cases}$

Now $\dfrac{\partial u}{\partial x}=\lim\limits_{h\to 0}\dfrac{u(x+h,y)-u(x,y)}{h}$ and $\dfrac{\partial u}{\partial y}=\lim\limits_{k\to 0}\dfrac{u(x,y+k)-u(x,y)}{k}$

At $(0,0)$, we get

$$\frac{\partial u}{\partial x}=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{0-0}{h}$$

$$=0$$

and $$\frac{\partial u}{\partial y}=\lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{0-0}{k}$$

$$= 0$$

Similarly, at $(0,0)$, we get

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h,0) - v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0$$

and $$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,0+k) - v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k}$$

$$= 0$$

From the above relations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$$

Taking the limit along real axis $(x \to 0, y = 0)$, we have

$$= \lim_{x \to 0} \frac{0}{x^2}$$

$$= 0$$

Taking the limit along imaginary axis $(x = 0, y \to 0)$, we have

$$= \lim_{y \to 0} \frac{0}{y^4}$$

$$= 0$$

Taking the limit along the path $x = y^2$, we have

$$= \lim_{y \to 0} \frac{y^4}{2y^4}$$

$$= \frac{1}{2}$$

which is different from the above limits.

Therefore $f'(0)$ does not exist and so $f(z)$ is not analytic at origin. **(Proved)**

Problem-18: Prove that the function $f(z)=\begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & when \;\; z\neq 0\\ 0 & when \;\; z=0\end{cases}$ is not analytic at origin but the Cauchy-Riemann equations are satisfied.

Solution: Given that $f(z)=\begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & when \;\; z\neq 0\\ 0 & when \;\; z=0\end{cases}$

$$or, \quad f(z)=\begin{cases}\dfrac{x^3y^5}{x^4+y^{10}}+i\dfrac{x^2y^6}{x^4+y^{10}} & when \;\; (x,y)\neq 0\\ 0 & when \;\; (x,y)=0\end{cases}$$

Here, $u(x,y)=\begin{cases}\dfrac{x^3y^5}{x^4+y^{10}} & when \;\; (x,y)\neq 0\\ 0 & when \;\; (x,y)=0\end{cases}$

and $v(x,y)=\begin{cases}\dfrac{x^2y^6}{x^4+y^{10}} & when \;\; (x,y)\neq 0\\ 0 & when \;\; (x,y)=0\end{cases}$

Now $\dfrac{\partial u}{\partial x}=\lim\limits_{h\to 0}\dfrac{u(x+h,y)-u(x,y)}{h}$ and $\dfrac{\partial u}{\partial y}=\lim\limits_{k\to 0}\dfrac{u(x,y+k)-u(x,y)}{k}$

At $(0,0)$, we get

$$\begin{aligned}\frac{\partial u}{\partial x}&=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h}\\&=\lim_{h\to 0}\frac{0-0}{h}\\&=0\end{aligned}$$

and
$$\begin{aligned}\frac{\partial u}{\partial y}&=\lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k}\\&=\lim_{k\to 0}\frac{0-0}{k}\\&=0\end{aligned}$$

Similarly, at $(0,0)$, we get

$$\begin{aligned}\frac{\partial v}{\partial x}&=\lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h}\\&=\lim_{h\to 0}\frac{0-0}{h}\\&=0\end{aligned}$$

and $$\frac{\partial v}{\partial y}=\lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k}$$
$$=\lim_{k\to 0}\frac{0-0}{k}$$
$$=0$$

From the above relations, we have
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Hence, the Cauchy-Riemann equations are satisfied at origin.

Consider $$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$$
$$=\lim_{z\to 0}\frac{f(z)-f(0)}{z}$$
$$=\lim_{(x,y)\to(0,0)}\frac{x^2y^5}{x^4+y^{10}}$$

Taking the limit along real axis $(x\to 0, y=0)$, we have
$$=\lim_{x\to 0}\frac{0}{x^4}$$
$$=0$$

Taking the limit along imaginary axis $(x=0, y\to 0)$, we have
$$=\lim_{y\to 0}\frac{0}{y^{10}}$$
$$=0$$

Taking the limit along the path $x^2=y^5$, we have
$$=\lim_{y\to 0}\frac{y^{10}}{2y^{10}}$$
$$=\frac{1}{2}$$

which is different from the above limits.

Therefore $f'(0)$ does not exist and so $f(z)$ is not analytic at origin. **(Proved)**

Problem-19: If $w=f(z)=\dfrac{1+z}{1-z}$, find (a) $\dfrac{dw}{dz}$ and (b) determine where $f(z)$ is non-analytic.

Solution: We have $w=f(z)=\dfrac{1+z}{1-z}$

(a) $$\frac{dw}{dz}=\frac{(1-z)\frac{d}{dz}(1+z)-(1+z)\frac{d}{dz}(1-z)}{(1-z)^2}$$

$$=\frac{(1-z)+(1+z)}{(1-z)^2}$$

$$=\frac{2}{(1-z)^2}$$

(b) The function $f(z)$ is analytic for all finite values of z except $z=1$ where the derivative does not exist and the function is non-analytic. The point $z=1$ is a singular point of $f(z)$.

Problem-20: For the function $f(z)=\dfrac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$, locate and name all the singularities in the finite $z-plane$ and also determine where $f(z)$ is analytic.

Solution: Given that $f(z)=\dfrac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$

In the finite $z-plane$ the singularities will be obtained by solving the equation

$$(z-1)^3(3z+2)^2=0$$

$$\therefore (z-1)^3=0 \qquad or,\quad (3z+2)^2=0$$

$$\therefore z=1,1,1 \qquad or,\quad z=-\frac{2}{3},-\frac{2}{3}$$

In the finite z-plane, the singular point $z=1$ is a pole of order 3 and $z=-\dfrac{2}{3}$ is a pole of order 2.

In the finite $z-plane$ $f(z)$ is analytic everywhere except the points $z=1$ and $z=-\dfrac{2}{3}$.

Problem-21: Determine the singular points of $f(z)=\dfrac{z^3+7}{(z^2-2z+2)(z-3)}$ in the finite z-plane

Solution: Given that $f(z)=\dfrac{z^3+7}{(z^2-2z+2)(z-3)}$

The singular points are obtained by solving the equation

$$(z^2-2z+2)(z-3)=0$$

$$\therefore z-3=0 \qquad or,\quad z^2-2z+2=0$$

$$\therefore z=3 \qquad or,\quad z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i$$

In the finite z-plane, the singular point $z=3$ and $z=1\pm i$ are simple poles.

Problem-22: For the function $f(z)=\dfrac{(z+3i)^5}{(z^2-2z+5)^2}$, locate and name all the singularities.

Solution: Given that $f(z)=\dfrac{(z+3i)^5}{(z^2-2z+5)^2}$

In the finite $z-plane$ the singularities will be obtained by solving the equation

$$(z^2-2z+5)^2=0$$

$$or,\quad z=\frac{2\pm\sqrt{4-20}}{2},\ \frac{2\pm\sqrt{4-20}}{2}$$

$$\therefore\ z=1\pm 2i, 1\pm 2i$$

In the finite z-plane, the singular point $z=1\pm 2i$ is a pole of order 2.

To determine whether there is a singularity at $z=\infty$ (the point at infinity), let $z=\dfrac{1}{w}$.

Then $f\left(\dfrac{1}{w}\right)=\dfrac{\left(\dfrac{1}{w}+3i\right)^5}{\left(\dfrac{1}{w^2}-\dfrac{2}{w}+5\right)^2}$

$$=\frac{(1+3iw)^5}{w\left(1-2w+5w^2\right)^2}$$

Since $w=0$ is a simple pole for the function $f\left(\dfrac{1}{w}\right)$ so $z=\infty$ is a simple pole at infinity for the function $f(z)$.

Exercise

Problem-01: Prove that the function $u=x^3+6x^2y-3xy^2-2y^3$ is harmonic. Find its harmonic conjugate.

Problem-02: Show that $u=\dfrac{1}{2}\ln\left(x^2+y^2\right)$ satisfies the Laplace's equation. Find its harmonic conjugate v such that $f(z)=u+iv$ is analytic.

Problem-03: Prove that the function $u=e^x(x\cos y-y\sin y)$ is harmonic. Find its harmonic conjugate v and express $u+iv$ as an analytic function of z.

Problem-04: Prove that the function $u=x^2-y^2+2e^{-x}\sin y$ is harmonic. Find its harmonic conjugate v and express $u+iv$ as an analytic function of z.

Problem-05: If $u=x^2-y^2$ and $v=\dfrac{-y}{x^2+y^2}$, then show that both u and v satisfy the Laplace's equation but $u+iv$ is not an analytic function.

Problem-06: For each of the following functions locate and name the singularities in the finite z-plane:

(a) $f(z)=\dfrac{z^2-3z}{z^2+2z+2}$, (b) $f(z)=\dfrac{\ln(z+3i)}{z^2}$, (c) $\sin^{-1}\left(\dfrac{1}{z}\right)$, (d) $\sqrt{z\left(z^2+1\right)}$

(e) $f(z)=\dfrac{\cos z}{(z+i)^3}$, (f) $f(z)=\dfrac{\ln(z-2)}{(z^2+2z+2)^4}$

Ans: (a) $z=-1\pm i$; simple pole, (b) $z=-3i$; branch point, $z=0$; pole of order 2, **(c)** $z=0$; essential singularity. (d) $z=0$, $\pm i$; branch points. (e) $z=-i$; pole of order, (f) $z=2$; branch point, $z=-1\pm i$; pole of order 4.

Problem-07: Determine which of the following functions are harmonic. For each harmonic function find the conjugate harmonic function v and express $u+iv$ as an analytic function of z.

(a) $u=3x^2y+2x^2-y^3-2y^2$

(b) $u=2xy+3xy^2-2y^3$

(c) $u=xe^x\cos y-ye^x\sin y$

(d) $u=e^{-2xy}\sin(x^2-y^2)$

Ans: (a) $v=4xy-x^3+3xy^2+c,\quad f(z)=2z^2-iz^3+ic$

(b). Not harmonic

(c) $v=ye^x\cos y+xe^x\sin y+c,\quad f(z)=ze^z+ic$

(d) $v=-e^{-2xy}\cos(x^2-y^2)+c,\quad f(z)=-ie^{iz^2}+ic$

Problem-08: Verify that the Cauchy-Riemann equations are satisfied for the following functions:

(a) $f(z)=e^{z^2}$

(b) $f(z)=\cos 2z$; $\left[Note:\cos(ix)=\cosh x,\sin(ix)=i\sinh x,\dfrac{d}{dx}(\cosh x)=\sinh x,\dfrac{d}{dx}(\sinh x)=\cosh x\right]$

(c) $f(z)=\sin 2z$

(d) $f(z)=\cosh(4z)$ $\left[Note:\cosh x=\cos(ix),\ i\sinh x=\sin(ix)\right]$

(e) $f(z)=\sinh(4z)$

(f) $f(z)=e^y(\cos x+i\sin x)$

(g) $f(z)=e^x(\cos y+i\sin y)$

(h) $f(z)=e^{-y}(\sin x-i\cos x)$

(i) $f(z)=\ln z$ $\left[Note:z=re^{i\theta},r=\sqrt{x^2+y^2},\theta=\tan^{-1}\dfrac{y}{x}\right]$

N.T: For solution see the book Complex analysis- A.K.M. Shahidullah