Linear Partial Differential Equations Of Order One.

Partial Differential Equation: An equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation.

Example: 1. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

2. $\dfrac{\partial v}{\partial s}+\dfrac{\partial v}{\partial t}=v$

Order and Degree: The order of the highest ordered derivative involving in a partial differential equation is called the order of the partial differential equation.

Again, the exponent of the highest ordered derivative involving in a partial differential equation is called the degree of the partial differential equation, after freed from radicals and fractions in its derivatives.

Example: Consider the following partial differential equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

The order of this equation is 2 and the degree is 1.

NOTE: In the case of two independent variables, we usually assume that, $z=f(x,y)$ where x and y are independent variables and z is the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p=\frac{\partial z}{\partial x}\ ,\ q=\frac{\partial z}{\partial y}\ ,\ r=\frac{\partial^2 z}{\partial x^2}\ ,\ s=\frac{\partial^2 z}{\partial x\partial y}\ ,\ t=\frac{\partial^2 z}{\partial y^2}\ .$$

Formation of Partial Differential Equation: A partial differential equation can be formed by the elimination of arbitrary constants or arbitrary functions.

Problem-01: Find a partial differential equation by eliminating a and b from the equation $z=ax+by+a^2+b^2$.

Solution: Given that, $z=ax+by+a^2+b^2 \qquad \cdots(1)$

Differentiating (1) partially with respect to *x*, we get

$$\frac{\partial z}{\partial x} = a \qquad \cdots(2)$$

And differentiating (1) partially with respect to *y*, we get

$$\frac{\partial z}{\partial y} = b \qquad \cdots(3)$$

Substituting these values of *a* and *b* in (1), we get

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required partial differential equation.

Problem-02: Eliminate arbitrary constants from the equation, $z = (x-a)^2 + (y-b)^2$.

Solution: Given that, $z = (x-a)^2 + (y-b)^2 \qquad \cdots(1)$

Differentiating (1) partially with respect to *x*, we get

$$\frac{\partial z}{\partial x} = 2(x-a)$$

$$or, \left(\frac{\partial z}{\partial x}\right)^2 = 4(x-a)^2 \qquad [Squaring] \qquad \cdots(2)$$

And differentiating (1) partially with respect to *y*, we get

$$\frac{\partial z}{\partial y} = 2(y-b)$$

$$or, \left(\frac{\partial z}{\partial y}\right)^2 = 4(y-b)^2 \qquad [Squaring] \qquad \cdots(3)$$

Adding equations (1) and (2), we get

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x-a)^2 + 4(y-b)^2$$

$$= 4\left\{(x-a)^2 + (y-b)^2\right\}$$

$$= 4z \qquad \text{[using eq.(1)]}$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4z$$

which is the required partial differential equation.

Problem-03: Find the differential equation of all spheres of radius λ, having centre in the *xy* - plane.

Solution: From the coordinate geometry of three dimensions, the equation of any sphere of radius λ, having centre (*h*, *k*, *0*) in the *xy* – plane is,

$$(x-h)^2+(y-k)^2+z^2=\lambda^2 \qquad \cdots(1)$$

Differentiating (1) partially with respect to *x*, we get

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

$$or,(x-h)=-z\frac{\partial z}{\partial x}$$

$$or,(x-h)^2=z^2\left(\frac{\partial z}{\partial x}\right)^2 \qquad [Squaring] \qquad \cdots(2)$$

And differentiating (1) partially with respect to *y*, we get

$$2(y-k)+2z\frac{\partial z}{\partial y}=0$$

$$or,(y-k)=-z\frac{\partial z}{\partial y}$$

$$or,(y-k)^2=z^2\left(\frac{\partial z}{\partial y}\right)^2 \qquad [Squaring] \qquad \cdots(3)$$

From equations (1), (2) and (3), we get

$$z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$$

which is the required partial differential equation.

Problem-04: Eliminate the arbitrary constants *a*, *b* and *c* from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Given that, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $\cdots(1)$

Differentiating (1) partially with respect to *x*, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$or,\; c^2x+a^2z\frac{\partial z}{\partial x}=0 \qquad \cdots(2)$$

Again, differentiating (2) partially with respect to *x*, we get

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \qquad \cdots(3)$$

From (2), we get

$$c^2=-\frac{a^2z}{x}\frac{\partial z}{\partial x}$$

Putting this value of c^2 in (4), we obtain

$$-\frac{a^2z}{x}\frac{\partial z}{\partial x}+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\; -\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\; -z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+zx\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\; zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0$$

which is the required partial differential equation.

Problem-05: Form partial differential equation by eliminating constant *A* and *p* from $z=Ae^{pt}\sin px$.

Solution: Given that, $z=Ae^{pt}\sin px$ $\cdots(1)$

Differentiating (1) partially with respect to *x*, we get

$$\frac{\partial z}{\partial x} = Ape^{pt} \cos px \qquad \cdots(2)$$

Again, differentiating (2) partially with respect to *x*, we get

$$\frac{\partial^2 z}{\partial x^2} = -Ap^2 e^{pt} \sin px \qquad \cdots(3)$$

Similarly, differentiating (1) partially with respect to *t*, we get

$$\frac{\partial z}{\partial t} = Ape^{pt} \sin px \qquad \cdots(4)$$

Again, differentiating (4) partially with respect to *t*, we get

$$\frac{\partial^2 z}{\partial t^2} = Ap^2 e^{pt} \sin px \qquad \cdots(5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = -Ap^2 e^{pt} \sin px + Ap^2 e^{pt} \sin px$$

$$\therefore \ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is the required partial differential equation.

Exercise: Try yourself: Find the partial differential equation by eliminating constants from the following relations:

1. $z = Ae^{-p^2 t} \cos px$ (*constants A , p*)
2. $ax^2 + by^2 + cz^2 = 1$ (*constants a , b*)
3. $z = ax + by + cxy$ (*constants a , b, c*)
4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (*constants a , b*)

Linear Partial Differential Equations of Order One: A differential equation involving derivatives *p* and *q* only and no higher is called of order one. If, in addition, the degree or power of *p* and *q* is unity, then it is a linear partial differential equation of order one.

Example: 1. $3xp + 9yq = z$

2. $px^3 + qy^4 = z^2$

The standard form of linear partial differential equation of order one is,

$$Pp + Qq = R \qquad \cdots(A)$$

where, P, Q and R being functions of x, y and z. This is also known as Lagrange equation.

The general solution of (1) is,

$$\phi(u, v) = 0$$

where ϕ is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are solutions of equations,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \cdots(B)$$

which are called Lagrange auxiliary or subsidiary equations for (1).

Working procedure for solving $Pp + Qq = R$ by Lagrange's method:

Step-1: Put the given linear partial differential equation in the standard form,

$$Pp + Qq = R \qquad \cdots(A)$$

Step-2: Write down Lagrange's auxiliary equations for (1) namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \cdots(B)$$

Step-3: Solve (2) by well-known methods. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step-4: The general solution (or integral) of (1) is then written in one of the following three equivalent forms:

$$\phi(u, v) = 0 \ , \ u = \phi(v) \ \text{and} \ v = \phi(u).$$

There are four types of problems based on $Pp + Qq = R$:

<u>Type-I</u>: If one of the variables is either absent or cancels out from any two fractions of given equations (*B*). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of the given equations (*B*).

Problem-01: Solve $\left(y^2z/x\right)p + xzq = y^2$

Solution: Given that, $\left(y^2z/x\right)p + xzq = y^2$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{\left(y^2z/x\right)} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad \cdots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{\left(y^2z/x\right)} = \frac{dy}{xz}$$

$$or,\ \frac{xdx}{y^2z} = \frac{dy}{xz}$$

$$or,\ x^2dx = y^2dy$$

$$or,\ x^2dx - y^2dy = 0$$

Integrating,

$$\frac{x^3}{3} - \frac{y^3}{3} = \frac{c_1}{3}$$

$$or,\ x^3 - y^3 = c_1 \qquad \cdots(3)$$

Next, taking the first and the last fractions of (2), we get

$$\frac{dx}{\left(y^2z/x\right)} = \frac{dz}{y^2}$$

$$or, \frac{xdx}{y^2z} = \frac{dz}{y^2}$$

$$or, xdx = zdz$$

$$or,\ xdx - zdz = 0$$

Integrating,

$$\frac{x^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}$$

$$or,\ x^2 - z^2 = c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$$

where, ϕ is an arbitrary constant.

Problem-02: Solve $p \tan x + q \tan y = \tan z$

Solution: Given that, $p \tan x + q \tan y = \tan z$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \qquad \cdots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$or,\ \cot x dx = \cot y dy$$

$$or,\ \cot x dx - \cot y dy = 0$$

Integrating,

$$\ln(\sin x) - \ln(\sin y) = \ln c_1$$

$$or,\ \ln\left(\frac{\sin x}{\sin y}\right) = \ln c_1$$

$$or,\ \frac{\sin x}{\sin y} = c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$or,\ \cot y dy = \cot z dz$$

$$or,\ \cot y dy - \cot z dz = 0$$

Integrating,

$$\ln(\sin y) - \ln(\sin z) = \ln c_2$$

$$or,\ \ln\left(\frac{\sin y}{\sin z}\right)=\ln c_2$$

$$or,\ \frac{\sin y}{\sin z}=c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\frac{\sin x}{\sin y}=\phi\left(\frac{\sin y}{\sin z}\right)$$

where, ϕ is an arbitrary constant.

Problem-03: Solve $zp=-x$

Solution: Given that, $zp=-x$ $\qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{z}=\frac{dy}{0}=\frac{dz}{-x} \qquad \cdots(2)$$

Taking the first and the last fractions of (2), we get

$$\frac{dx}{z}=\frac{dz}{-x}$$

$$or,-xdx=zdz$$

$$or,xdx+zdz=0$$

Integrating,

$$\frac{x^2}{2}-\frac{y^3}{2}=\frac{c_1}{2}$$

$$or,\ x^2+y^2=c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{0}=\frac{dz}{-x}$$

$$or,\ dy=0$$

Integrating,

$$y=c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$x^2 + y^2 = \phi(y)$$

where, ϕ is an arbitrary constant.

Problem-04: Solve $y^2 p - xyq = x(z-2y)$

Solution: Given that, $y^2 p - xyq = x(z-2y)$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad \cdots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$or, -xdx = ydy$$

$$or, xdx + zdz = 0$$

Integrating,

$$\frac{x^2}{2} - \frac{y^3}{2} = \frac{c_1}{2}$$

$$or,\ x^2 + y^2 = c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$or,\ -\frac{dy}{y} = \frac{dz}{(z-2y)}$$

$$or,\ \frac{dz}{dy} = -\frac{(z-2y)}{y}$$

$$or,\ \frac{dz}{dy} = 2 - \frac{z}{y}$$

$$or,\ \frac{dz}{dy}+\frac{z}{y}=2 \qquad \cdots(4)$$

which is a linear equation.

$$I.F=e^{\int\frac{dy}{y}}=e^{\ln y}=y$$

From (4), we get

$$y\frac{dz}{dy}+z=2y$$

$$or,\ \frac{d}{dy}(yz)=2y$$

Integrating,

$$yz=\int 2y\,dy$$

$$or,\ yz=y^2+c_2$$

$$or,\ yz-y^2=c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\phi\left(x^2+y^2,\ yz-y^2\right)=0$$

where, ϕ is an arbitrary constant.

Problem-05: Solve $\left(x^2-yz\right)p+\left(y^2-zx\right)q=\left(z^2-xy\right)$

Solution: Given that, $\left(x^2-yz\right)p+\left(y^2-zx\right)q=\left(z^2-xy\right) \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{\left(x^2-yz\right)}=\frac{dy}{\left(y^2-zx\right)}=\frac{dz}{\left(z^2-xy\right)} \qquad \cdots(2)$$

Equations of (2) can be written as,

$$\frac{dx-dy}{\left(x^2-yz\right)}=\frac{dy-dz}{\left(y^2-zx\right)}=\frac{dz-dx}{\left(z^2-xy\right)}$$

$$or,-xdx=ydy$$

$$or, xdx + zdz = 0$$

Integrating,

$$\frac{x^2}{2} - \frac{y^3}{2} = \frac{c_1}{2}$$

$$or,\ x^2 + y^2 = c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$or,\ -\frac{dy}{y} = \frac{dz}{(z-2y)}$$

$$or,\ \frac{dz}{dy} = -\frac{(z-2y)}{y}$$

$$or,\ \frac{dz}{dy} = 2 - \frac{z}{y}$$

$$or,\ \frac{dz}{dy} + \frac{z}{y} = 2 \qquad \cdots(4)$$

which is a linear equation.

$$I.F = e^{\int \frac{dy}{y}} = e^{\ln y} = y$$

From (4), we get

$$y\frac{dz}{dy} + z = 2y$$

$$or,\ \frac{d}{dy}(yz) = 2y$$

Integrating,

$$yz = \int 2y dy$$

$$or,\ yz = y^2 + c_2$$

$$or,\ yz - y^2 = c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\phi\left(x^2+y^2,\ yz-y^2\right)=0$$

where, ϕ is an arbitrary constant.

Exercise:

1. $2p+3q=1$
2. $yzp+2xq=xy$
3. $x^2p+y^2q+z^2=0$
4. $xp+yq=z$

Problem-05: Solve $\left(x^2-yz\right)p+\left(y^2-zx\right)q=\left(z^2-xy\right)$

Solution: Given that, $\left(x^2-yz\right)p+\left(y^2-zx\right)q=\left(z^2-xy\right)$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{\left(x^2-yz\right)}=\frac{dy}{\left(y^2-zx\right)}=\frac{dz}{\left(z^2-xy\right)} \qquad \cdots(2)$$

Equations of (2) can be written as,

$$\frac{dx-dy}{\left(x^2-yz\right)-\left(y^2-zx\right)}=\frac{dy-dz}{\left(y^2-zx\right)-\left(z^2-xy\right)}=\frac{dz-dx}{\left(z^2-xy\right)-\left(x^2-yz\right)}$$

$$or,\ \frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dz-dx}{z^2-xy-x^2+yz}$$

$$or,\ \frac{dx-dy}{(x+y)(x-y)+z(x-y)}=\frac{dy-dz}{(y+z)(y-z)+x(y-z)}=\frac{dz-dx}{(z+x)(z-x)+y(z-x)}$$

$$or,\ \frac{dx-dy}{(x+y+z)(x-y)}=\frac{dy-dz}{(x+y+z)(y-z)}=\frac{dz-dx}{(x+y+y)(z-x)}$$

$$or,\ \frac{dx-dy}{(x-y)}=\frac{dy-dz}{(y-z)}=\frac{dz-dx}{(z-x)} \qquad \cdots(3)$$

Taking first two fractions of (3) we get

$$\frac{dx-dy}{(x-y)}=\frac{dy-dz}{(y-z)}$$

$$or,\ \frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

$$or,\ \frac{d(x-y)}{(x-y)} - \frac{d(y-z)}{(y-z)} = 0$$

Integrating we get

$$\ln(x-y) - \ln(y-z) = \ln c_1$$

$$or,\ \ln\left(\frac{x-y}{y-z}\right) = \ln c_1$$

$$or,\ \frac{x-y}{y-z} = c_1$$

Next, taking last two fractions of (3) we get

$$\frac{dy-dz}{(y-z)} = \frac{dz-dx}{(z-x)}$$

$$or,\ \frac{d(y-z)}{(y-z)} = \frac{d(z-x)}{(z-x)}$$

$$or,\ \frac{d(y-z)}{(y-z)} - \frac{d(z-x)}{(z-x)} = 0$$

Integrating we get

$$\ln(y-z) - \ln(z-x) = \ln c_2$$

$$or,\ \ln\left(\frac{y-z}{z-x}\right) = \ln c_2$$

$$or,\ \frac{y-z}{z-x} = c_2$$

The general solution of (1) is,

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

where, ϕ is an arbitrary constant.

<u>Type-II</u>: If one integral of (B) is known by using rule-I but another integral cannot be obtained. Then the integral known to us is used to find another integral.

Problem-01: Solve $p+3q=5z+\tan(y-3x)$

Solution: Given that, $p+3q=5z+\tan(y-3x) \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)} \qquad \cdots(2)$$

Taking first two fractions of (2) we get

$$\frac{dx}{1}=\frac{dy}{3}$$

$$or,\ dy-3dx=0$$

Integrating, we get

$$y-3x=c_1$$

Next, taking the first and last fractions of (2) we get

$$\frac{dx}{1}=\frac{dz}{5z+\tan(y-3x)}$$

$$or,\ dx=\frac{dz}{5z+\tan c_1}$$

$$or,\ dx-\frac{dz}{5z+\tan c_1}=0$$

Integrating we get

$$x-\frac{1}{5}\ln(5z+\tan c_1)=\frac{1}{5}c_2$$

$$or,\ 5x-\ln\left[5z+\tan(y-3x)\right]=c_2$$

The general solution of (1) is,

$$\phi\left[5x-\ln\left[5z+\tan(y-3x),\ y-3x\ \right]\right]=0$$

where, ϕ is an arbitrary constant.

Problem-02: Solve $xyp + y^2 q = zxy - 2x^2$

Solution: Given that, $xyp + y^2 q = zxy - 2x^2$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2} \qquad \cdots(2)$$

Taking first two fractions of (2) we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$or,\ \frac{dx}{x} = \frac{dy}{y}$$

$$or,\ \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating, we get

$$\ln x - \ln y = \ln c_1$$

$$or,\ \ln\left(\frac{x}{y}\right) = \ln c_1$$

$$or,\ \frac{x}{y} = c_1$$

Next, taking the last two fractions of (2) we get

$$\frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$$

$$or,\ \frac{dy}{y^2} = \frac{dz}{zc_1 y^2 - 2c_1{}^2 y^2}$$

$$or,\ dy = \frac{dz}{zc_1 - 2c_1{}^2}$$

$$or,\ dy - \frac{dz}{zc_1 - 2c_1{}^2} = 0$$

$$or,\ c_1 dy - \frac{dz}{z - 2c_1} = 0$$

Integrating we get

$$c_1 y - \ln(z - 2c_1) = c_2$$

$$or,\ x - \ln\left[z - 2\left(x/y\right)\right] = c_2$$

The general solution of (1) is,

$$x - \ln\left[z - 2\left(x/y\right)\right] = \phi\left(x/y\right)$$

where, ϕ is an arbitrary constant.

Problem-03: Solve $xz\left(z^2 + xy\right)p - yz\left(z^2 + xy\right)q = x^4$

Solution: Given that, $xz\left(z^2 + xy\right)p - yz\left(z^2 + xy\right)q = x^4$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{xz\left(z^2 + xy\right)} = \frac{dy}{-yz\left(z^2 + xy\right)} = \frac{dz}{x^4} \qquad \cdots(2)$$

Taking first two fractions of (2) we get

$$\frac{dx}{xz\left(z^2 + xy\right)} = \frac{dy}{-yz\left(z^2 + xy\right)}$$

$$or,\ \frac{dx}{x} = \frac{dy}{-y}$$

$$or,\ \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, we get

$$\ln x + \ln y = \ln c_1$$

$$or,\ \ln(xy) = \ln c_1$$

$$or,\ xy = c_1$$

Next, taking the first and last fractions of (2) we get

$$\frac{dx}{xz\left(z^2+xy\right)}=\frac{dz}{x^4}$$

$$or,\ \frac{dx}{z\left(z^2+c_1\right)}=\frac{dz}{x^3}$$

$$or,\ x^3dx=z\left(z^2+c_1\right)dz$$

$$or,\ x^3dx-\left(z^3+zc_1\right)dz=0$$

Integrating we get

$$\frac{x^4}{4}-\left(\frac{z^4}{4}+\frac{z^2}{2}c_1\right)=\frac{c_2}{4}$$

$$or,\ x^4-\left(z^4+2c_1z^2\right)=c_2$$

$$or,\ x^4-z^4-2xyz^2=c_2$$

The general solution of (1) is,

$$\phi\left(xy,\ x^4-z^4-2xyz^2\right)=0$$

where, ϕ is an arbitrary constant.

Exercise:

1. **Solve** $xzp+yzq=xy$
2. **Solve** $py+qx=xyz^2\left(x^2-y^2\right)$
3. **Solve** $p-2q=3x^2\sin\left(y+2x\right)$
4. **Solve** $\left(x^2-y^2-z^2\right)p+2xyq=2xz$
5. **Solve** $z\left(p-q\right)=z^2+\left(x+y\right)^2$

<u>Type-III</u>: If P_1 , Q_1 , R_1 be functions of *x*, *y* and *z* or constants, which are called multipliers. Then, by a well-known principle of algebra, each fraction in (*B*) will be equal to,

$$\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$$

If $P_1P+Q_1Q+R_1R=0$ then we know that $P_1dx+Q_1dy+R_1dz=0$ and which can be integrated easily. This procedure can be repeated to obtain another integral. If another integral cannot be obtained by this process then the rule-I can be applied.

Problem-01: Solve $(mz-ny)p+(nx-lz)q=ly-mx$

Solution: Given that, $(mz-ny)p+(nx-lz)q=ly-mx$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx} \qquad \cdots(2)$$

Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to

$$=\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore\ xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$or,\ x^2+y^2+z^2=c_1$$

Again, choosing *l*, *m*, *n* as multipliers, each fraction of (2) is equal to

$$=\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}$$

$$\therefore\ ldx+mdy+ndz=0$$

Integrating, we get

$$lx+my+nz=c_2$$

The general solution of (1) is,

$$\phi\left(x^2+y^2+z^2\,,\ lx+my+nz\right)=0$$

where, ϕ is an arbitrary constant.

Problem-02: Solve $\left(z^2-2yz-y^2\right)p+\left(xy+xz\right)q=xy-xz$

Solution: Given that, $\left(z^2-2yz-y^2\right)p+\left(xy+xz\right)q=xy-xz$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{xy+xz}=\frac{dz}{xy-xz} \qquad \cdots(2)$$

Choosing x, y, z as multipliers, each fraction of (2) is equal to

$$=\frac{xdx+ydy+zdz}{x\left(z^2-2yz-y^2\right)+y\left(xy+xz\right)+z\left(xy-xz\right)}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore\ xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$or,\ x^2+y^2+z^2=c_1$$

Again, taking last two fractions of (2) we get

$$\frac{dy}{xy+xz}=\frac{dz}{xy-xz}$$

$$or,\ \frac{dy}{y+z}=\frac{dz}{y-z}$$

$$or,\ ydy-zdy=ydz+zdz$$

$$or,\ ydy-zdy-ydz-zdz=0$$

$$or,\ ydy-\left(zdy+ydz\right)-zdz=0$$

$$or,\ ydy-d\left(yz\right)-zdz=0$$

Integrating, we get

$$\frac{y^2}{2}-yz-\frac{z^2}{2}=\frac{c_2}{2}$$

$$or,\ y^2-2yz-z^2=c_2$$

The general solution of (1) is,

$$\phi\left(x^2+y^2+z^2\,,\;y^2-2yz-z^2\right)=0$$

where, ϕ is an arbitrary constant.

Problem-03: Solve $\left(y^2+z^2-x^2\right)p-2xyq+2xz=0$

Solution: Given that, $\left(y^2+z^2-x^2\right)p-2xyq+2xz=0$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz} \qquad \cdots(2)$$

Taking last two fractions of (2) we get

$$\frac{dy}{-2xy}=\frac{dz}{-2xz}$$

$$or,\ \frac{dy}{y}=\frac{dz}{z}$$

$$or,\ \frac{dy}{y}-\frac{dz}{z}=0$$

Integrating, we get

$$\ln x-\ln y=\ln c_1$$

$$or,\ \ln\left(\frac{x}{y}\right)=\ln c_1$$

$$or,\ \frac{x}{y}=c_1$$

Again, choosing x, y, z as multipliers, we get

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}=\frac{xdx+ydy+zdz}{-x\left(x^2+y^2+z^2\right)}$$

From the last two of this, we get

$$\frac{dz}{-2xz}=\frac{xdx+ydy+zdz}{-x\left(x^2+y^2+z^2\right)}$$

$$or,\ \frac{dz}{z}=\frac{2\left(xdx+ydy+zdz\right)}{\left(x^2+y^2+z^2\right)}$$

$$or,\ \frac{2\left(xdx+ydy+zdz\right)}{\left(x^2+y^2+z^2\right)}-\frac{dz}{z}=0$$

Integrating, we get

$$\ln\left(x^2+y^2+z^2\right)-\ln z=\ln c_2$$

$$or,\ \ln\left(\frac{y^2-2yz-z^2}{z}\right)=\ln c_2$$

$$or,\ \frac{y^2-2yz-z^2}{z}=c_2$$

The general solution of (1) is,

$$\phi\left(\frac{x}{y},\ \frac{y^2-2yz-z^2}{z}\right)=0$$

where, ϕ is an arbitrary constant.

Problem-04: Solve $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$

Solution: Given that, $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right) \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)} \qquad \cdots(2)$$

Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to

$$\frac{xdx+ydy+zdz}{x^2\left(y^2-z^2\right)+y^2\left(z^2-x^2\right)+z^2\left(x^2-y^2\right)}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore\ xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$or,\ x^2+y^2+z^2=c_1$$

Again, choosing $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers, each fraction of (2) is equal to

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\therefore\ \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

Integrating, we get

$$\ln x+\ln y+\ln z=\ln c_2$$

$$or,\ \ln(xyz)=\ln c_2$$

$$or,\ xyz=c_2$$

The general solution of (1) is,

$$\phi\left(x^2+y^2+z^2,\ xyz\right)=0$$

where, ϕ is an arbitrary constant.

Problem-05: Solve $x\left(y^2-z^2\right)p-y\left(z^2+x^2\right)q=z\left(x^2+y^2\right)$

Solution: Given that, $x\left(y^2-z^2\right)p-y\left(z^2+x^2\right)q=z\left(x^2+y^2\right)$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{-y\left(z^2+x^2\right)}=\frac{dz}{z\left(x^2+y^2\right)} \qquad \cdots(2)$$

Choosing *x*, *y*, *z* as multipliers, each fraction of (2) is equal to

$$\frac{xdx+ydy+zdz}{x^2\left(y^2-z^2\right)-y^2\left(z^2+x^2\right)+z^2\left(x^2+y^2\right)}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore\ xdx+ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$or,\ x^2+y^2+z^2=c_1$$

Again, choosing $\frac{1}{x},-\frac{1}{y},-\frac{1}{z}$ as multipliers, each fraction of (2) is equal to

$$\frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{y^2-z^2+z^2+x^2-x^2-y^2}=\frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{0}$$

$$\therefore\ \frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}=0$$

Integrating, we get

$$\ln x-\ln y-\ln z=\ln c_2$$

$$or,\ \ln\left(\frac{x}{yz}\right)=\ln c_2$$

$$or,\ xyz=c_2$$

The general solution of (1) is,

$$\phi\left(x^2+y^2+z^2\,,\,\frac{x}{yz}\right)=0$$

where, ϕ is an arbitrary constant.

Problem-06: Solve $(y-zx)p+(x+yz)q=x^2+y^2$

Solution: Given that, $(y-zx)p+(x+yz)q=x^2+y^2 \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y-zx}=\frac{dy}{x+yz}=\frac{dz}{x^2+y^2} \qquad \cdots(2)$$

Choosing $x,-y,z$ as multipliers, each fraction of (2) is equal to

$$\frac{xdx-ydy+zdz}{x(y-zx)-y(x+yz)+z(x^2+y^2)}=\frac{xdx-ydy+zdz}{0}$$

$$\therefore\ xdx-ydy+zdz=0$$

Integrating, we get

$$\frac{x^2}{2}-\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}$$

$$or,\ x^2-y^2+z^2=c_1$$

Again, choosing $y, x, -1$ as multipliers, each fraction of (2) is equal to

$$\frac{ydx+xdy-dz}{y(y-zx)+x(x+yz)-(x^2+y^2)}=\frac{ydx+xdy-dz}{0}$$

$$\therefore\ ydx+xdy-dz=0$$

$$or,\ d(xy)-dz=0$$

Integrating, we get

$$xy-z=c_2$$

The general solution of (1) is,

$$\phi\left(x^2-y^2+z^2,\ xy-z\right)=0$$

where, ϕ is an arbitrary constant.