Multiple Integration

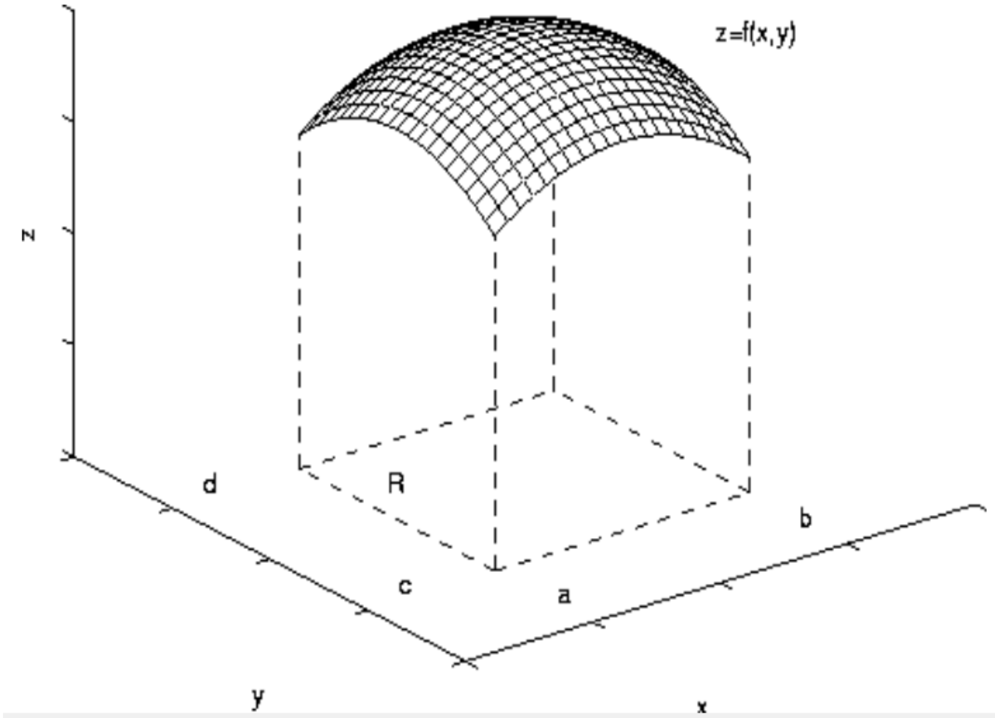
The multiple integral is a generalization of the definite integral to functions of more than one real variable, for example, $f(x, y)$ or $f(x, y, z)$. Integrals of a function of two variables over a region in R^2 are called double integrals, and integrals of a function of three variables over a region of R^3 are called triple integrals. Multiple Integral also known as iterated integrals because we integrate more than once.

Single variable function:

The definite integral $\int_a^b f(x)\,dx$, $f(x) \geq 0$ represents the area under the curve $f(x)$ from $x = a$ and $x = b$ .For general f(x) the definite integral is equal to the area above the x-axis minus the area below the x-axis.

Double variable function:

The definite integral can be extended to functions of more than one variable. Consider a function of two variables z=f(x, y). The definite integral is denoted by $\iint_R f(x, y)\, dA$, where R is the region of integration in the xy-plane. For positive f(x,y), the definite integral is equal to the volume under the surface z=f(x,y) and above xy-plane for x and y in the region R. This is shown in the figure below.



Applications:

Double integrals arise in a number of areas of science and engineering, including computations of

- Area of a 2D region
- Volume
- Mass of 2D plates

- Force on a 2D plate
- Average of a function
- Center of Mass and Moment of Inertia
- Surface Area

Example 01: The volume of a cylinder with height *h* and circular base of radius *R* can be calculated by integrating the constant function *h* over the circular base, using polar coordinates.

$$\text{Volume} = \int_0^{2\pi} d\phi \int_0^R h\rho \, d\rho = h2\pi \left[\frac{\rho^2}{2}\right]_0^R = \pi R^2 h$$

This is in agreement with the formula $\text{Volume} = \text{base area} \times \textit{height}$

Example 02: The volume of a sphere with radius *R* can be calculated by integrating the constant function 1 over the sphere, using spherical coordinates.

$$\begin{aligned}
\text{Volume} &= \iiint_D f(x,y,z)\, dx\, dy\, dz \\
&= \iiint_D 1\, dV \\
&= \iiint_S \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^R \rho^2\, d\rho \\
&= 2\pi \int_0^{\pi} \sin\phi\, d\phi \int_0^R \rho^2\, d\rho \\
&= 2\pi \int_0^{\pi} \sin\phi \frac{R^3}{3}\, d\phi \\
&= \frac{2}{3}\pi R^3[-\cos\phi]_0^{\pi} = \frac{4}{3}\pi R^3.
\end{aligned}$$

Double Integration

Problem-01: Evaluate $\int_1^2\int_0^3 x^2 y\,dx\,dy$

Solution: Let, $I=\int_1^2\int_0^3 x^2 y\,dx\,dy$

$$=\int_1^2\left[\frac{x^3}{3}y\right]_0^3 dy$$

$$=\frac{1}{3}\int_1^2\left[x^3 y\right]_0^3 dy$$

$$=\frac{1}{3}\int_1^2 y\left[x^3\right]_0^3 dy$$

$$=\frac{1}{3}\int_1^2 y\left(3^3-0^3\right)dy$$

$$=\frac{1}{3}\times 27\int_1^2 y\,dy$$

$$=9\int_1^2 y\,dy$$

$$=9\left[\frac{y^2}{2}\right]_1^2$$

$$=\frac{9}{2}\left[y^2\right]_1^2$$

$$=\frac{9}{2}\left(2^2-1^2\right)=\frac{9}{2}\times 3=\frac{27}{2}$$ **(As desired)**

Note: It turns out that the result of two iterated integrals are always equal when the order of integration is altered.

Problem-02: Evaluate $\int_0^2\int_1^2\left(x-3y^2\right)dx\,dy$

Solution: Let, $I=\int_0^2\int_1^2\left(x-3y^2\right)dx\,dy$

$$=\int_0^2\left[\frac{x^2}{2}-3y^2x\right]_1^2 dy$$

$$=\int_0^2\left(2-6y^2-\frac{1}{2}+3y^2\right)dy$$

$$=\int_0^2\left(\frac{3}{2}-3y^2\right)dy$$

$$=\left[\frac{3}{2}y-3.\frac{y^3}{3}\right]_0^2$$

$$=\left[\frac{3y}{2}-y^3\right]_0^2$$

$$=(3-8-0-0)$$

$$=-5$$ **(As desired)**

Problem-03: Evaluate $\int_0^{\ln 2}\int_{-1}^{1} ye^{xy}\,dx\,dy$

Solution: Let, $I=\int_0^{\ln 2}\int_{-1}^{1} ye^{xy}\,dx\,dy$

$$=\int_0^{\ln 2} y\left[\frac{e^{xy}}{y}\right]_{-1}^{1} dy$$

$$=\int_0^{\ln 2} y\times\frac{1}{y}\left[e^{xy}\right]_{-1}^{1} dy$$

$$=\int_0^{\ln 2}\left[e^{xy}\right]_{-1}^{1} dy$$

$$=\int_0^{\ln 2}\left(e^{y}-e^{-y}\right)dy$$

$$=\int_0^{\ln 2} e^{y}\,dy-\int_0^{\ln 2} e^{-y}\,dy$$

$$=\left[e^{y}\right]_0^{\ln 2}-\left[\frac{e^{-y}}{-1}\right]_0^{\ln 2}$$

$$=\left[e^{y}\right]_0^{\ln 2}+\left[e^{-y}\right]_0^{\ln 2}$$

$$=\left(e^{\ln 2}-e^{0}\right)+\left(e^{-\ln 2}-e^{0}\right)$$

$$=(2-1)+\left(e^{\ln 2^{-1}}-1\right)$$

$$=1+\left(2^{-1}-1\right)$$

$$=1+\left(\frac{1}{2}-1\right)$$

$$=\frac{1}{2}$$ **(As desired)**

Problem-04: Evaluate $\int_{0}^{\pi}\int_{1}^{2} y\sin(xy)dx\,dy$

Solution: Let, $I=\int_{0}^{\pi}\int_{1}^{2} y\sin(xy)dx\,dy$

$$=-\int_{0}^{\pi} y\times\frac{1}{y}\left[\cos(\text{xy})\right]_{1}^{2} dy$$

$$=-\int_{0}^{\pi}\left[\cos(\text{xy})\right]_{1}^{2} dy$$

$$=-\int_{0}^{\pi}\left(\cos 2y-\cos y\right)dy$$

$$=\int_{0}^{\pi}\cos y\,dy-\int_{0}^{\pi}\cos 2y\,dy$$

$$=\left[\sin y\right]_{0}^{\pi}-\left[\frac{\sin 2y}{2}\right]_{0}^{\pi}$$

$$=\left(\sin\pi-\sin 0\right)-\frac{1}{2}\left(\sin 2\pi-\sin 0\right)$$

$$=0$$ **(As desired)**

Problem-05: Evaluate $\int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^{2}y\,dy\,dx$

Solution: Let, $I=\int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^{2}y\,dy\,dx$

$$=\int_1^2 x^2\left[\frac{y^2}{2}\right]_{1-x}^{\sqrt{x}}dx$$

$$=\int_1^2 \frac{x^2}{2}\left[y^2\right]_{1-x}^{\sqrt{x}}dx$$

$$=\int_1^2 \frac{x^2}{2}\left(x-(1-x)^2\right)dx$$

$$=\frac{1}{2}\int_1^2 x^2\left(x-(1-x)^2\right)dx$$

$$=\frac{1}{2}\int_1^2 x^2\left(x-\left(1-2x+x^2\right)\right)dx$$

$$=\frac{1}{2}\int_1^2 x^2\left(x-1+2x-x^2\right)dx$$

$$=\frac{1}{2}\int_1^2 \left(x^3-x^2+2x^3-x^4\right)dx$$

$$=\frac{1}{2}\int_1^2 \left(3x^3-x^2-x^4\right)dx$$

$$=\frac{1}{2}\left[3.\frac{x^4}{4}-\frac{x^3}{3}-\frac{x^5}{5}\right]_1^2$$

$$=\frac{1}{2}\left(\frac{3}{4}.16-\frac{8}{3}-\frac{32}{5}-\frac{3}{4}+\frac{1}{3}+\frac{1}{5}\right)$$

$$=\frac{1}{2}\left(12-\frac{8}{3}-\frac{32}{5}-\frac{3}{4}+\frac{1}{3}+\frac{1}{5}\right)$$

$$=\frac{1}{2}\times\frac{163}{60}=\frac{163}{120}$$ **(As desired)**

Problem-06: Evaluate $\int_1^2\int_{y^2}^{2y}\left(4x-2y\right)dxdy$

Solution: Let, $I=\int_1^2\int_{y^2}^{2y}\left(4x-2y\right)dxdy$

$$=\int_1^2\left[4.\frac{x^2}{2}-2yx\right]_{y^2}^{2y}dy$$

$$=\int_1^2 \left[2x^2-2yx\right]_{y^2}^{2y} dy$$

$$=2\int_1^2 \left[x^2-yx\right]_{y^2}^{2y} dy$$

$$=2\int_1^2 \left(4y^2-2y^2-y^4+y^3\right) dy$$

$$=2\int_1^2 \left(2y^2-y^4+y^3\right) dy$$

$$=2\left[2.\frac{y^3}{3}-\frac{y^5}{5}+\frac{y^4}{4}\right]_1^2$$

$$=2\left(2.\frac{8}{3}-\frac{32}{5}+\frac{16}{4}-\frac{2}{3}+\frac{1}{5}-\frac{1}{4}\right)$$

$$=2\left(\frac{16}{3}-\frac{32}{5}+\frac{16}{4}-\frac{2}{3}+\frac{1}{5}-\frac{1}{4}\right)$$

$$=2\times\frac{133}{60}=\frac{133}{30} \qquad \textbf{(As desired)}$$

Problem-07: Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\pi} \cos(x+y)\,dx\,dy$

Solution: Let, $I=\int_0^{\frac{\pi}{2}}\int_0^{\pi} \cos(x+y)\,dx\,dy$

$$=\int_0^{\frac{\pi}{2}} \left[\sin(x+y)\right]_0^{\pi} dy$$

$$=\int_0^{\frac{\pi}{2}} \left[\sin(\pi+y)-\sin(0+y)\right] dy$$

$$=\int_0^{\frac{\pi}{2}} \left[-\sin y-\sin y\right] dy$$

$$=-2\int_0^{\frac{\pi}{2}} \sin y\,dy$$

$$= -2\left[-\cos y\right]_0^{\pi/2}$$

$$= 2\left[\cos y\right]_0^{\pi/2}$$

$$= 2\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= 2\left[0-1\right]$$

$$= -2 \qquad \textbf{(As desired)}$$

H.W:

Problem-01: Evaluate $\int_1^2\int_0^1 (x+y)^2\,dydx$ **Ans:** $\frac{25}{6}$

Problem-02: Evaluate $\int_0^4\int_0^1 \mathrm{xy}(x+y)\,dydx$ **Ans:** 8

Problem-03: Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(x+y)\,dxdy$ **Ans:** 2

Problem-04: Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dydx$ **Ans:** $\frac{3}{35}$

Problem-05: Evaluate $\int_0^1\int_0^{x^2} e^{\frac{y}{x}}\,dydx$ **Ans:** $\frac{1}{2}$

Problem-06: Evaluate $\int_0^1\int_{-x}^{x^2} y^2x\,dydx$ **Ans:** $\frac{13}{120}$

Problem-07: Evaluate $\int_0^1\int_y^1 \frac{1}{1+y^2}\,dxdy$ **Ans:** $\frac{1}{4}\left(\pi - \ln 4\right)$

Problem-08: Evaluate $\int_1^e\int_1^x \ln x\,dydx$ **Ans:** $\frac{1}{4}\left(e^2-5\right)$

Problem-09: Evaluate $\int_0^1\int_x^1 \sin(y^2)\,dydx$ **Ans:** $\frac{1}{2}(1-\cos 1)$

Problem-10: Evaluate $\int_0^1\int_{2x}^2 (x-y)\,dydx$

Triple Integration

Problem-01: Evaluate $\int_0^2\int_0^z\int_0^{x\sqrt{3}} \frac{x}{x^2+y^2}\,dydxdz$

Solution: Let, $I = \int_0^2\int_0^z\int_0^{x\sqrt{3}} \frac{x}{x^2+y^2}\,dydxdz$

$$= \int_0^2\int_0^z x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^{x\sqrt{3}} dxdz$$

$$= \int_0^2\int_0^z \left[\tan^{-1}\frac{y}{x}\right]_0^{x\sqrt{3}} dxdz$$

$$= \int_0^2\int_0^z \left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right)dxdz$$

$$= \int_0^2\int_0^z \left(\frac{\pi}{3} - 0\right)dxdz$$

$$= \int_0^2\int_0^z \frac{\pi}{3}\,dxdz$$

$$= \frac{\pi}{3}\int_0^2\int_0^z dxdz$$

$$= \frac{\pi}{3}\int_0^2 [x]_0^z\,dz$$

$$= \frac{\pi}{3}\int_0^2 (z-0)\,dz$$

$$= \frac{\pi}{3}\int_0^2 z\,dz$$

$$= \frac{\pi}{3}\times\left[\frac{z^2}{2}\right]_0^2$$

$$= \frac{\pi}{3}\times(2-0)$$

$$= \frac{2\pi}{3}$$ **(As desired)**

Problem-02: Evaluate $\int_0^{3a}\int_0^{2a}\int_0^{a}(x+y+z)\,dxdydz$

Solution: Let, $I=\int_0^{3a}\int_0^{2a}\int_0^{a}(x+y+z)\,dxdydz$

$$=\int_0^{3a}\int_0^{2a}\left[\frac{x^2}{2}+yx+zx\right]_0^a dydz$$

$$=\int_0^{3a}\int_0^{2a}\left(\frac{a^2}{2}+ya+za-0\right)dydz$$

$$=\int_0^{3a}\int_0^{2a}\left(\frac{a^2}{2}+ya+za\right)dydz$$

$$=\int_0^{3a}\left[\frac{a^2y}{2}+\frac{ay^2}{2}+zay\right]_0^{2a} dz$$

$$=\int_0^{3a}\left(a^3+2a^3+2a^2z\right)dz$$

$$=\int_0^{3a}\left(3a^3+2a^2z\right)dz$$

$$=\left[3a^3z+2a^2.\frac{z^2}{2}\right]_0^{3a}$$

$$=\left[3a^3z+a^2z^2\right]_0^{3a}$$

$$=\left(9a^4+9a^4-0\right)$$

$$=18a^4$$ **(As desired)**

Problem-03: Evaluate $\int_1^{3}\int_{\frac{1}{x}}^{1}\int_0^{\sqrt{xy}} xyz\,dzdydx$

Solution: Let, $I=\int_1^{3}\int_{\frac{1}{x}}^{1}\int_0^{\sqrt{xy}} xyz\,dzdydx$

$$=\int_1^{3}\int_{\frac{1}{x}}^{1}\left[\frac{xyz^2}{2}\right]_0^{\sqrt{xy}} dydx$$

$$=\frac{1}{2}\int_{1}^{3}\int_{\frac{1}{x}}^{1}\left[xyz^2\right]_0^{\sqrt{xy}}dydx$$

$$=\frac{1}{2}\int_{1}^{3}\int_{\frac{1}{x}}^{1}\left[xy\left(\sqrt{xy}\right)^2-xy.0\right]_0^{\sqrt{xy}}dydx$$

$$=\frac{1}{2}\int_{1}^{3}\int_{\frac{1}{x}}^{1}x^2y^2dydx$$

$$=\frac{1}{2}\int_{1}^{3}\left[\frac{x^2y^3}{3}\right]_{\frac{1}{x}}^{1}dx$$

$$=\frac{1}{6}\int_{1}^{3}\left[x^2y^3\right]_{\frac{1}{x}}^{1}dx$$

$$=\frac{1}{6}\int_{1}^{3}\left[x^2-x^2.\frac{1}{x^3}\right]dx$$

$$=\frac{1}{6}\int_{1}^{3}\left[x^2-\frac{1}{x}\right]dx$$

$$=\frac{1}{6}\left[\frac{x^3}{3}-\ln x\right]_1^3$$

$$=\frac{1}{6}\left[\left(\frac{3^3}{3}-\ln 3\right)-\left(\frac{1}{3}-\ln 1\right)\right]$$

$$=\frac{1}{6}\left[9-\ln 3-\frac{1}{3}+0\right]$$

$$=\frac{1}{6}\left[\frac{27-3\ln 3-1}{3}\right]$$

$$=\frac{1}{18}\left(26-3\ln 3\right)$$ **(As desired)**

Problem-04: Evaluate $\int_{0}^{2}\int_{\sqrt{y}}^{1}\int_{z^2}^{y}xy^2z^3\,dxdzdy$

Solution: Let, $I=\int_{0}^{2}\int_{\sqrt{y}}^{1}\int_{z^2}^{y}xy^2z^3\,dxdzdy$

$$=\int_{0}^{2}\int_{\sqrt{y}}^{1}\left[\frac{x^2y^2z^3}{2}\right]_{z^2}^{y} dzdy$$

$$=\frac{1}{2}\int_{0}^{2}\int_{\sqrt{y}}^{1}\left[x^2y^2z^3\right]_{z^2}^{y} dzdy$$

$$=\frac{1}{2}\int_{0}^{2}\int_{\sqrt{y}}^{1}\left[y^2y^2z^3-z^4y^2z^3\right]dzdy$$

$$=\frac{1}{2}\int_{0}^{2}\int_{\sqrt{y}}^{1}\left[y^4z^3-y^2z^7\right]dzdy$$

$$=\frac{1}{2}\int_{0}^{2}\left[\frac{y^4z^4}{4}-\frac{y^2z^8}{8}\right]_{\sqrt{y}}^{1} dy$$

$$=\frac{1}{16}\int_{0}^{2}\left[2y^4z^4-y^2z^8\right]_{\sqrt{y}}^{1} dy$$

$$=\frac{1}{16}\int_{0}^{2}\left[\left(2y^4-y^2\right)-\left(2y^4.y^2-y^2.y^4\right)\right]dy$$

$$=\frac{1}{16}\int_{0}^{2}\left[2y^4-y^2-\left(2y^6-y^6\right)\right]dy$$

$$=\frac{1}{16}\int_{0}^{2}\left[2y^4-y^2-y^6\right]dy$$

$$=\frac{1}{16}\left[\frac{2y^5}{5}-\frac{y^3}{3}-\frac{y^7}{7}\right]_{0}^{2}$$

$$=\frac{1}{16}\left[\left(\frac{2.2^5}{5}-\frac{2^3}{3}-\frac{2^7}{7}\right)-0\right]$$

$$=\frac{1}{16}\left(\frac{64}{5}-\frac{8}{3}-\frac{128}{7}\right)$$

$$=-\frac{107}{210} \qquad \textbf{(As desired)}$$

Problem-05: Evaluate $\int_{1}^{4}\int_{-1}^{3}\int_{0}^{2}3xy^3z^2\, dzdxdy$

Solution: Let, $I=\int_{1}^{4}\int_{-1}^{3}\int_{0}^{2}3xy^3z^2\, dzdxdy$

$$= \int_{1}^{4}\int_{-1}^{3}\left[\frac{3xy^3z^3}{3}\right]_0^2 dxdy$$

$$= \int_{1}^{4}\int_{-1}^{3}\left[xy^3z^3\right]_0^2 dxdy$$

$$= \int_{1}^{4}\int_{-1}^{3} 8xy^3 dxdy$$

$$= \int_{1}^{4}\left[\frac{8x^2y^3}{2}\right]_{-1}^{3} dy$$

$$= 4\int_{1}^{4}\left[x^2y^3\right]_{-1}^{3} dy$$

$$= 4\int_{1}^{4}\left[9y^3 - y^3\right] dy$$

$$= 32\int_{1}^{4} y^3 dy$$

$$= 32\left[\frac{y^4}{4}\right]_1^4$$

$$= 8\left[y^4\right]_1^4$$

$$= 8(256-1)$$

$$= 8\times 255$$

$$= 2040$$ **(As desired)**

H.W:

Problem-01: Evaluate $\int_{2}^{0}\int_{0}^{z^2}\int_{x}^{z}(x+z)\,dydxdz$ **Ans:** $-\frac{32}{105}$

Problem-02: Evaluate $\int_{-3}^{3}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dzdydx$ **Ans:** 12

Problem-03: Evaluate $\int_{0}^{1}\int_{y^2}^{1}\int_{0}^{1-x} xdzdxdy$ **Ans:** $\frac{4}{35}$