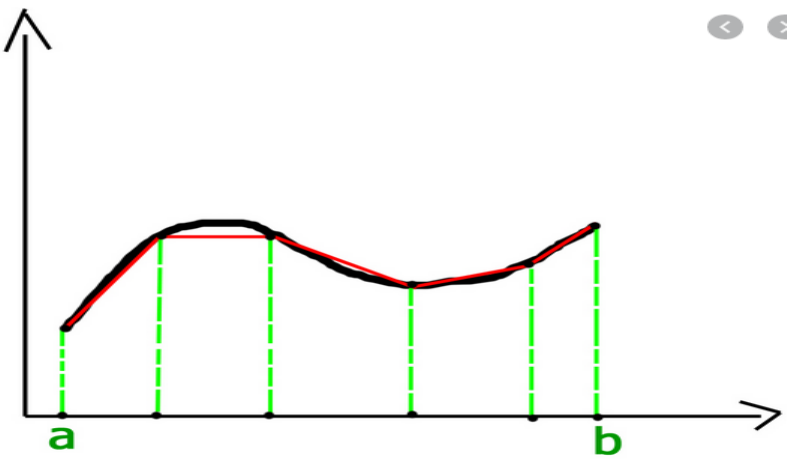
Definite Integration

Let f be a function which is continuous on the closed interval $[a,b]$. The definite integral of f from a and b is defined to be the limit

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i$$

where $\sum_{i=1}^{n} f(x_i)\Delta x_i$ is a Riemann sum of f on $[a,b]$. So a definite integral is an integral $\int_a^b f(x)dx$ with upper and lower limits. If x is restricted to lie on the real line, the definite integral is known as a Riemann integral. However, a general definite integral is taken in the complex plane, resulting in the contour integral $\int_a^b f(z)dz$ with a,b and z in general being complex numbers and the path of integration from a to b known as a contour.

Integration as the limit of a sum: Let, $f(x)$ be a continuous, bounded and single-valued function defined in the interval $[a, b]$ where a, b are finite quantities and $b > a$.



If the interval $[a, b]$ be divided into n equal sub-intervals, each of length $h(h\to 0)$, by the points $a+h,\ a+2h,\cdots\cdots\ a+(n-1)h$ so that $nh=b-a$, then the area enclosed by $f(x)$ is defined as

$$S = \lim_{h\to 0}\left[hf(a)+hf(a+h)+hf(a+2h)+\cdots\cdots+hf\{a+(n-1)h\}\right]$$

$$=\lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) \qquad where,\ nh=b-a$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} f\left(a + rh\right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f\left(a + \frac{r}{n}\right) \qquad \text{where } h = \frac{1}{n} \text{ if } h \to 0 \text{ then } n \to \infty.$$

Which is also defined as the definite integral of $f(x)$ with respect to x between the limits a and b, and is denoted by the symbol,

$$\int_a^b f(x)dx$$

where, a is called the lower limit and b is called the upper limit.

Therefore, $\int_a^b f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f\left(a + \frac{r}{n}\right)$ where $nh = b - a$.

NOTE:

1. $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$;*OR*, $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$;*OR*, $\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

2. $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n-1} f\left(\frac{r}{n}\right) = \int_0^2 f(x)dx$ *OR*, $\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)dx$

3. $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{3n-1} f\left(\frac{r}{n}\right) = \int_0^3 f(x)dx$ *OR*, $\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{3n} f\left(\frac{r}{n}\right) = \int_0^3 f(x)dx$.

Problem-01: Evaluate $\int_a^b xdx$ from the definition of the integral as the limit of a sum.

Solution: We have $I = \int_a^b xdx$

Here $f(x) = x$

$$\therefore f(a) = a,\ f(a+h) = a+h,\ f(a+2h) = a+2h, \cdots, f\{a+(n-1)h\} = a+(n-1)h$$

Since $\int_a^b f(x)dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots + f\{a+(n-1)h\}\right]$

where $nh = b - a$

$$\therefore I = \lim_{h \to 0} h\left[a + (a+h) + (a+2h) + \cdots + \{a+(n-1)h\}\right]$$

$$=\lim_{h\to 0} h\left[na+h\left\{1+2+\cdots+(n-1)\right\}\right]$$

$$=\lim_{h\to 0} h\left[na+h.\frac{n(n-1)}{2}\right]$$

$$=\lim_{h\to 0}\left[nha+\frac{nh(nh-h)}{2}\right]$$

$$=\lim_{h\to 0}\left[(b-a)a+\frac{(b-a)(b-a-h)}{2}\right]$$

$$=(b-a)a+\frac{(b-a)(b-a)}{2}=\frac{b^2-a^2}{2}.$$

Problem-02: Evaluate $\int_a^b x^2dx$ from the definition of the integral as the limit of a sum.

Solution: We have $I=\int_a^b x^2dx$

Here $f(x)=x^2$

$\therefore f(a)=a^2,\ f(a+h)=(a+h)^2,\ f(a+2h)=(a+2h)^2,\ \cdots,\ f\left\{a+(n-1)h\right\}=\left\{a+(n-1)h\right\}^2$

Since $\int_a^b f(x)dx=\lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\cdots+f\left\{a+(n-1)h\right\}\right]$

where $nh=b-a$

$$\therefore I=\lim_{h\to 0} h\left[a^2+(a+h)^2+(a+2h)^2+\cdots+\left\{a+(n-1)h\right\}^2\right]$$

$$=\lim_{h\to 0} h\left[a^2+\left(a^2+2ah+h^2\right)+\left(a^2+4ah+4h^2\right)+\cdots+\left\{a^2+2a(n-1)h+(n-1)^2h^2\right\}\right]$$

$$=\lim_{h\to 0} h\left[na^2+2ah\left\{1+2+\cdots+(n-1)\right\}+h^2\left\{1^2+2^2+3^2+\cdots+(n-1)^2\right\}\right]$$

$$=\lim_{h\to 0} h\left[na^2+2ah.\frac{n(n-1)}{2}+h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

$$=\lim_{h\to 0}\left[nha^2+anh(nh-h)+\frac{1}{6}(nh-h)nh(2nh-h)\right]$$

$$=\lim_{h\to 0}\left[(b-a)a^2+a(b-a)(b-a-h)+\frac{1}{6}(b-a-h)(b-a)(2b-2a-h)\right]$$

$$=(b-a)a^2+a(b-a)(b-a)+\frac{1}{6}(b-a)(b-a)(2b-2a)$$

$$=(b-a)a^2+a(b-a)^2+\frac{1}{3}(b-a)^3$$

$$=(b-a)\left[a^2+a(b-a)+\frac{1}{3}(b-a)^2\right]$$

$$=(b-a)\left[a^2+ab-a^2+\frac{1}{3}\left(b^2-2ab+a^2\right)\right]$$

$$=\frac{1}{3}(b-a)\left(3ab+b^2-2ab+a^2\right)$$

$$=\frac{1}{3}(b-a)\left(b^2+ab+a^2\right)$$

$$=\frac{1}{3}\left(b^3-a^3\right) \qquad \textit{Ans.}$$

Problem-03: Evaluate $\int_a^b \sin x dx$ from the definition of the integral as the limit of a sum.

Solution: We have $I=\int_a^b \sin x dx$

Here $f(x)=\sin x$

$$\therefore f(a)=\sin a,\ f(a+h)=\sin(a+h),\ \cdots,\ f\{a+(n-1)h\}=\sin\{a+(n-1)\}h$$

Since $\int_a^b f(x)dx=\lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\cdots+f\{a+(n-1)h\}\right]$

where $nh=b-a$

$$\therefore I=\lim_{h\to 0} h\left[\sin a+\sin(a+h)+\sin(a+2h)+\cdots+\sin\{a+(n-1)h\}\right]$$

$$=\lim_{h\to 0} h\left[\frac{\sin\left(a+\frac{n-1}{2}h\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$=2\lim_{\frac{h}{2}\to 0}\frac{\frac{h}{2}}{\sin\left(\frac{h}{2}\right)}.\lim_{h\to 0}\sin\left(a+\frac{nh-h}{2}\right)\sin\left(\frac{nh}{2}\right)$$

$$= 2.1.\lim_{h\to 0}\sin\left(a+\frac{b-a-h}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= 2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= \cos a - \cos b. \qquad \textit{Ans.}.$$

NOTE: $\sin a + \sin(a+h) + \sin(a+2h) + \cdots + \sin\{a+(n-1)h\} = \dfrac{\sin\left(a+\frac{n-1}{2}h\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}.$

Problem-04: Evaluate $\int_a^b \cos x dx$ from the definition of the integral as the limit of a sum.

Solution: We have $I = \int_a^b \cos x dx$

Here $\quad f(x) = \cos x$

$$\therefore f(a) = \cos a,\ f(a+h) = \cos(a+h),\ \cdots,\ f\{a+(n-1)h\} = \cos\{a+(n-1)\}h$$

Since $\displaystyle\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots + f\{a+(n-1)h\}\right]$

where $nh = b-a$

$$\therefore I = \lim_{h\to 0} h\left[\cos a + \cos(a+h) + \cos(a+2h) + \cdots + \cos\{a+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[\frac{\cos\left(a+\frac{n-1}{2}h\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}\right]$$

$$= 2\lim_{\frac{h}{2}\to 0}\frac{\frac{h}{2}}{\sin\left(\frac{h}{2}\right)}.\lim_{h\to 0}\cos\left(a+\frac{nh-h}{2}\right)\sin\left(\frac{nh}{2}\right)$$

$$= 2.1.\lim_{h\to 0}\cos\left(a+\frac{b-a-h}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= 2\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= \sin b - \sin a. \qquad \textit{Ans.}$$

NOTE: $\cos a+\cos(a+h)+\cos(a+2h)+\cdots+\cos\{a+(n-1)h\}=\dfrac{\cos\left(a+\frac{n-1}{2}h\right)\sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$

Problem-05: Evaluate $\int_a^b e^x dx$ from the definition of the integral as the limit of a sum.

Solution: We have $I=\int_a^b e^x dx$

Here $f(x)=e^x$

$$\therefore f(a)=e^a,\ f(a+h)=e^{(a+h)},\ \cdots,\ f\{a+(n-1)h\}=e^{\{a+(n-1)\}h}$$

Since $\int_a^b f(x)dx=\lim_{h\to 0}h\left[f(a)+f(a+h)+f(a+2h)+\cdots+f\{a+(n-1)h\}\right]$

where $nh=b-a$

$$\therefore I=\lim_{h\to 0}h\left[e^a+e^{(a+h)}+e^{(a+2h)}+\cdots+e^{\{a+(n-1)\}h}\right]$$

$$=\lim_{h\to 0}h\left[e^a+e^a\cdot e^h+e^a\cdot e^{2h}+\cdots+e^a\cdot e^{(n-1)h}\right];\qquad\left[\because\ 1+r+r^2+\cdots+r^{n-1}=\frac{r^n-1}{r-1}\right]$$

$$=e^a\lim_{h\to 0}h\left[1+e^h+e^{2h}+\cdots+e^{(n-1)h}\right]$$

$$=e^a\lim_{h\to 0}h\left\{\frac{e^{nh}-1}{e^h-1}\right\}$$

$$=e^a\lim_{h\to 0}h\left\{\frac{e^{b-a}-1}{e^h-1}\right\}$$

$$=\lim_{h\to 0}h\left\{\frac{e^b-e^a}{e^h-1}\right\}$$

$$=\left(e^b-e^a\right)\lim_{h\to 0}\left\{\frac{h}{e^h-1}\right\}$$

$$=\left(e^b-e^a\right)\lim_{h\to 0}\left\{\frac{1}{e^h}\right\}\qquad[\textit{by L.Hospital rule}]$$

$$=\left(e^b-e^a\right)\cdot 1$$

$$=e^b-e^a.\qquad \textit{Ans.}$$

There is another definition of a finite integral as the limit of a sum and is generally used for evaluating by summation $\int_a^b x^m dx$, where m is a positive integer ≥ 3 or a negative integer or a positive or negative fraction.

If $f(x)$ is continuous and single valued in the closed interval $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{r\to 1}(r-1)\left[af(a) + arf(ar) + ar^2 f\left(ar^2\right) + \cdots + ar^{n-1} f\left(ar^{n-1}\right)\right]$$

where $r^n = \dfrac{b}{a}$.

Problem-06: Evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$ from the definition of the integral as the limit of a sum.

Solution: We have $I = \int_0^1 \frac{dx}{\sqrt{x}}$

Here $\quad f(x) = \frac{1}{\sqrt{x}}$, $a = 0$ and $b = 1$.

$$\therefore f(a) = \frac{1}{\sqrt{a}},\ f(ar) = \frac{1}{\sqrt{ar}},\ f\left(ar^2\right) = \frac{1}{\sqrt{ar^2}} \cdots,\ f\left(ar^{n-1}\right) = \frac{1}{\sqrt{ar^{n-1}}}$$

Since $\int_a^b f(x)\,dx = \lim_{r\to 1}(r-1)\left[af(a) + arf(ar) + ar^2 f\left(ar^2\right) + \cdots + ar^{n-1} f\left(ar^{n-1}\right)\right]$

where, $r^n = \dfrac{b}{a}$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{r\to 1}(r-1)\left[a\cdot\frac{1}{\sqrt{a}} + ar\cdot\frac{1}{\sqrt{ar}} + ar^2\cdot\frac{1}{\sqrt{ar^2}} + \cdots + ar^{n-1}\cdot\frac{1}{\sqrt{ar^{n-1}}}\right]$$

$$= \lim_{r\to 1}(r-1)\left[\sqrt{a} + \sqrt{ar} + \sqrt{ar^2} + \cdots + \sqrt{ar^{n-1}}\right]$$

$$= \sqrt{a}\lim_{r\to 1}(r-1)\left[1 + \sqrt{r} + \left(\sqrt{r}\right)^2 + \cdots + \left(\sqrt{r}\right)^{n-1}\right]$$

$$= \sqrt{a}\lim_{r\to 1}(r-1)\left[\frac{\left(\sqrt{r}\right)^n - 1}{\sqrt{r} - 1}\right]$$

$$= \sqrt{a}\lim_{r\to 1}\left(\sqrt{r} + 1\right)\left(\sqrt{r^n} - 1\right)$$

$$= \sqrt{a}\lim_{r\to 1}\left(\sqrt{r}+1\right)\left(\sqrt{\frac{b}{a}}-1\right)$$

$$= \sqrt{a}\left(\sqrt{1}+1\right)\left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}}\right)$$

$$= 2\left(\sqrt{b}-\sqrt{a}\right)$$

$$= 2\left(\sqrt{1}-\sqrt{0}\right)$$

$$= 2 \qquad\qquad \textit{Ans.}$$

Problem-07: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

Solution: Given that, $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)}$$

$$= \int_0^1\frac{dx}{1+x}$$

$$= \left[\ln(1+x)\right]_0^1$$

$$= \ln(1+1)-\ln(1+0)$$

$$= \ln 2$$

Problem-08: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Solution: Given that, $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$=\left[\sin^{-1}x\right]_0^1$$

$$=\sin^{-1}.1-\sin^{-1}.0$$

$$=\sin^{-1}.\sin\frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

Problem-09: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

$$=\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{n^2}{(n+n)^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}$$

$$=\left[-\frac{1}{2}\frac{1}{(1+x)^2}\right]_0^1$$

$$=\left[-\frac{1}{2}\frac{1}{(1+1)^2}+\frac{1}{2}\frac{1}{(1+0)^2}\right]$$

$$=-\frac{1}{8}+\frac{1}{2}$$

$$=\frac{3}{8}$$

Problem-10: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\sqrt{1-x^2}\,dx$$

$$=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=\left[\frac{1.\sqrt{1-1^2}}{2}+\frac{1}{2}\sin^{-1}.1-\frac{0.\sqrt{1-0^2}}{2}-\frac{1}{2}\sin^{-1}.0\right]$$

$$=\frac{1}{2}\sin^{-1}.\sin\frac{\pi}{2}$$

$$=\frac{\pi}{4}$$

Problem-11: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\frac{dx}{1+x^2}$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\tan^{-1}.1-\tan^{-1}.0$$

$$=\tan^{-1}.\tan\frac{\pi}{4}-\tan^{-1}.\tan 0$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$

Problem-12: Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n.1-1^2}}+\frac{1}{\sqrt{2n.2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{2n.n-n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2nr-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\left\{\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2\right\}}}$$

$$=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(1-x)^2}}$$

$$=-\left[\sin^{-1}(1-x)\right]_0^1$$

$$=-\left[\sin^{-1}(1-1)-\sin^{-1}(1-0)\right]$$

$$=-\sin^{-1}.0+\sin^{-1}.1$$

$$=-\sin^{-1}.\sin 0+\sin^{-1}.\sin\frac{\pi}{2}$$

$$=0+\frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

Problem-13: Evaluate $\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{\sqrt{4n^2+r^2}}$

Solution: Given that, $\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{\sqrt{4n^2+r^2}}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{\sqrt{4+\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^2\frac{dx}{\sqrt{4+x^2}}$$

$$=\int_0^2\frac{dx}{\sqrt{2^2+x^2}}$$

$$=\left[\ln\left(x+\sqrt{2^2+x^2}\right)\right]_0^2$$

$$=\left[\ln\left(2+\sqrt{2^2+2^2}\right)-\ln\left(0+\sqrt{2^2+0^2}\right)\right]$$

$$= \ln\left(2+\sqrt{8}\right) - \ln 2$$

$$= \ln\left(\frac{2+\sqrt{8}}{2}\right)$$

$$= \ln\left(1+\sqrt{2}\right)$$

Problem-14: Evaluate $\lim_{n\to\infty}\sum_{r=0}^{3n}\frac{n}{3^2 n^2 + r^2}$

Solution: Given that, $\lim_{n\to\infty}\sum_{r=0}^{3n}\frac{n}{3^2 n^2 + r^2}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n}\frac{1}{3^2+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^3 \frac{dx}{3^2+x^2}$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{3}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$= \frac{1}{3}\tan^{-1}.1$$

$$= \frac{1}{3}\tan^{-1}.\tan\frac{\pi}{4}$$

$$= \frac{1}{3}.\frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Assignment:

Problem-01: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

Problem-02: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Problem-03: Evaluate $\int_0^{\frac{\pi}{2}} \cos x dx$ from the definition of the integral as the limit of a sum.

Problem-04: Evaluate $\int_a^b e^{-x} dx$ from the definition of the integral as the limit of a sum.

Problem-05: Evaluate $\int_0^{\frac{\pi}{2}} \sin x dx$ from the definition of the integral as the limit of a sum.

Problem-06: Evaluate $\int_0^1 x^{\frac{3}{2}} dx$ from the definition of the integral as the limit of a sum.

Theorem-01: State and prove Fundamental theorem of Integral Calculus.

OR

State and prove the first Fundamental theorem of Calculus.

Statement: If $f(x)$ be a bounded and continuous function defined in the interval $[a, b]$ where, $b > a$ and there exists a function $\varphi(x)$ such that $\varphi'(x) = f(x)$, then

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$$

This is called the fundamental theorem of integral calculus.

Proof: Let $x_1, x_2, x_3, \cdots, x_{n-1}$ be any points in $[a, b]$ such that

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

Since $x_0 = a$ and $x_n = b$, so $\varphi(x_0) = \varphi(a)$ and $\varphi(x_n) = \varphi(b)$. These points divide $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\cdots$, $[x_{n-1}, x_n]$ whose lengths are denoted by $\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4, \cdots, \Delta x_n$.

i.e. $\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \Delta x_3 = x_3 - x_2, \cdots, \Delta x_n = x_n - x_{n-1}$.

Since $\varphi(x)$ is an anti-derivative of $f(x)$ on (a,b) i.e $\varphi'(x) = f(x)$ for all x on (a,b), so $\varphi(x)$ satisfies the hypothesis of the mean value theorem of differential calculus on each n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\cdots$, $[x_{n-1}, x_n]$.

Then by the Lagrange's Mean value theorem of differential calculus we can find points $\xi_1, \xi_2, \xi_3, \cdots, \xi_{n-1}$ in the respective subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\cdots$, $[x_{n-1}, x_n]$.

i.e. $x_0 < \xi_1 < x_1, x_2 < \xi_2 < x_3, \cdots, x_{n-1} < \xi_n < x_n$, such that

$$\varphi'(\xi_1)=\frac{\varphi(x_1)-\varphi(x_0)}{x_1-x_0}$$

$$or,\ \varphi(x_1)-\varphi(x_0)=\varphi'(\xi_1)(x_1-x_0)$$

$$\therefore\ \varphi(x_1)-\varphi(x_0)=f(\xi_1)\Delta x_1, \qquad \because \varphi'(x)=f(x) \qquad \cdots(1)$$

$$\varphi'(\xi_2)=\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}$$

$$or,\ \varphi(x_2)-\varphi(x_1)=\varphi'(\xi_2)(x_2-x_1)$$

$$\therefore\ \varphi(x_2)-\varphi(x_1)=f(\xi_2)\Delta x_2, \qquad \cdots(2)$$

$$\varphi'(\xi_3)=\frac{\varphi(x_3)-\varphi(x_2)}{x_3-x_2}$$

$$or,\ \varphi(x_3)-\varphi(x_2)=\varphi'(\xi_3)(x_3-x_2)$$

$$\therefore\ \varphi(x_3)-\varphi(x_2)=f(\xi_3)\Delta x_2, \qquad \cdots(3)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\varphi'(\xi_{n-1})=\frac{\varphi(x_{n-1})-\varphi(x_{n-2})}{x_{n-1}-x_{n-2}}$$

$$or,\ \varphi(x_{n-1})-\varphi(x_{n-2})=\varphi'(\xi_{n-1})(x_{n-1}-x_{n-2})$$

$$\therefore\ \varphi(x_{n-1})-\varphi(x_{n-2})=f(\xi_{n-1})\Delta x_{n-1} \qquad \cdots(n-1)$$

$$\varphi'(\xi_n)=\frac{\varphi(x_n)-\varphi(x_{n-1})}{x_n-x_{n-1}}$$

$$or,\ \varphi(x_n)-\varphi(x_{n-1})=\varphi'(\xi_n)(x_n-x_{n-1})$$

$$\therefore\ \varphi(x_n)-\varphi(x_{n-1})=f(\xi_n)\Delta x_n \qquad \cdots(n)$$

Adding (1) to (n), we get

$$\varphi(x_n)-\varphi(x_0)=\sum_{k=1}^{n} f(\xi_k)\Delta x_k \qquad \cdots(i)$$

We now allow $n\to\infty$ i.e. the numbers of sub-intervals is infinity in such a way that $\Delta x_k \to 0$ and $\xi_k \in \Delta x_k$, then by the definition of definite integrals we have

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\xi_k\right)\Delta x_k = \int_a^b f\left(x\right)dx \qquad \cdots(ii)$$

Now taking limit $n \to \infty$ on both sides of (i) we get

$$\lim_{n\to\infty}\left[\varphi\left(x_n\right)-\varphi\left(x_0\right)\right]=\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\xi_k\right)\Delta x_k$$

$$or,\ \varphi\left(x_n\right)-\varphi\left(x_0\right)=\int_a^b f\left(x\right)dx$$

$$\therefore \int_a^b f\left(x\right)dx=\varphi\left(b\right)-\varphi\left(a\right)$$ **(Hence proved)**

Some Definite integration

Problem-01: Evaluate $\int_0^{\pi/2} \cos^2 x dx$

Solution: Let, $I=\int_0^{\pi/2} \cos^2 x dx$

$$=\frac{1}{2}\int_0^{\pi/2} 2\cos^2 x dx$$

$$=\frac{1}{2}\int_0^{\pi/2} \left(1+\cos 2x\right)dx$$

$$=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin 2.\frac{\pi}{2}}{2}\right)-\left(0+\frac{\sin 2.0}{2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)$$

$$=\frac{1}{2}\left(\frac{\pi}{2}+0\right)$$

$$=\frac{\pi}{4}$$

Problem-02: Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{dx}{1+\cos x}$

$$= \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2}\left[\frac{\tan \frac{x}{2}}{1/2}\right]_0^{\pi/2}$$

$$= \left[\tan \frac{x}{2}\right]_0^{\pi/2}$$

$$= \tan \frac{\pi}{4} - \tan \frac{0}{2}$$

$$= 1$$

Problem-03: Evaluate $\int_0^{\ln 2} \frac{e^x}{1+e^x} dx$

Solution: Let, $I = \int_0^{\ln 2} \frac{e^x}{1+e^x} dx$

$$= \left[\ln\left(1+e^x\right)\right]_0^{\ln 2}$$

$$= \ln\left(1+e^{\ln 2}\right) - \ln\left(1+e^0\right)$$

$$= \ln(1+2) - \ln(1+1)$$

$$= \ln 3 - \ln 2$$

$$= \ln \frac{3}{2}$$

Problem-04: Evaluate $\int_0^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

Solution: Let, $I = \int_0^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

$$= \frac{1}{4}\int_0^{\pi/3} \frac{4\cos x dx}{3+4\sin x}$$

$$= \frac{1}{4}\left[\ln(3+4\sin x)\right]_0^{\pi/3}$$

$$= \frac{1}{4}\left[\ln\left(3+4\sin\frac{\pi}{3}\right) - \ln(3+4\sin 0)\right]$$

$$= \frac{1}{4}\left[\ln\left(3+4.\frac{\sqrt{3}}{2}\right) - \ln 3\right]$$

$$= \frac{1}{4}\left[\ln\left(3+2\sqrt{3}\right) - \ln 3\right]$$

$$= \frac{1}{4}\ln\left(\frac{3+2\sqrt{3}}{3}\right)$$

Problem-05: Evaluate $\int_0^{\pi/2} (\sec\theta - \tan\theta) d\theta$

Solution: Let, $I = \int_0^{\pi/2} (\sec\theta - \tan\theta) d\theta$

$$= \int_0^{\pi/2} \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)} d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} d\theta$$

$$= 2\int_0^{\pi/2} \frac{\frac{1}{2}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} d\theta$$

$$= 2\left[\ln\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

$$= 2\left[\ln\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) - \ln\left(\cos\frac{0}{2} + \sin\frac{0}{2}\right)\right]$$

$$= 2\left[\ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \ln 1\right]$$

$$= 2\left[\ln\left(\frac{2}{\sqrt{2}}\right) - 0\right]$$

$$= 2\ln\sqrt{2}$$

$$= \ln 2$$

Problem-06: Evaluate $\int_0^{\pi/2} \cos 2x \cos 3x dx$

Solution: Let, $I = \int_0^{\pi/2} \cos 2x \cos 3x dx$

$$= \frac{1}{2}\int_0^{\pi/2} 2\cos 2x \cos 3x dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\cos(2x+3x) + \cos(2x-3x)\right] dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\cos 5x+\cos x\right]dx$$

$$=\frac{1}{2}\left[\frac{\sin 5x}{5}+\sin x\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\sin 5.\frac{\pi}{2}}{5}+\sin\frac{\pi}{2}\right)-\left(\frac{\sin 0}{5}+\sin 0\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{5}\sin\frac{5\pi}{2}+1\right)$$

$$=\frac{1}{10}\sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{2}$$

$$=\frac{1}{10}\sin\frac{\pi}{2}+\frac{1}{2}$$

$$=\frac{1}{10}+\frac{1}{2}$$

$$=\frac{3}{5}$$

Problem-07: Evaluate $\int_0^{\pi/2}\cos^7 x dx$

Solution: Let, $I=\int_0^{\pi/2}\cos^7 x dx$

$$=\int_0^{\pi/2}\cos^6 x\cos x dx$$

$$=\int_0^{\pi/2}\left(\cos^2 x\right)^3\cos x dx$$

$$=\int_0^{\pi/2}\left(1-\sin^2 x\right)^3\cos x dx$$

put $\sin x=t \therefore \cos x dx=dt$

when $x=0$ then $t=0$

when $\quad x=\dfrac{\pi}{2}$ then $t=1$

Now, $\quad I=\int_0^1\left(1-t^2\right)^3 dt$

$$=\int_0^1\left(1-3t^2+3t^4-t^6\right)dt$$

$$=\left[t-t^3+3\frac{t^5}{5}-\frac{t^7}{7}\right]_0^1$$

$$=1-1+\frac{3}{5}-\frac{1}{7}$$

$$=\frac{16}{35}$$

Problem-08: Evaluate $\displaystyle\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$

Solution: Let, $\displaystyle I=\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$

$$=\frac{1}{b^2}\int_0^{\pi/2}\frac{dx}{\cos^2 x\left\{\left(\frac{a}{b}\right)^2+\tan^2 x\right\}}$$

$$=\frac{1}{b^2}\int_0^{\pi/2}\frac{\sec^2 x dx}{\left(\frac{a}{b}\right)^2+\tan^2 x}$$

put, $\quad \tan x=t \therefore \sec^2 x dx=dt$

when $\quad x=0$ then $t=0$

when $\quad x=\dfrac{\pi}{2}$ then $t=\infty$

Now, $\quad \displaystyle I=\frac{1}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+\mathrm{t}^2}$

$$=\frac{1}{b^2}\left[\frac{1}{a/b}\tan^{-1}\frac{t}{a/b}\right]_0^\infty$$

$$=\frac{1}{b^2}\left[\frac{b}{a}\tan^{-1}\frac{bt}{a}\right]_0^\infty$$

$$=\frac{1}{ab}\left(\tan^{-1}\infty-\tan^{-1}0\right)$$

$$=\frac{1}{ab}\left(\tan^{-1}\tan\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2ab}$$

Problem-09: Evaluate $\int_0^{\pi/2}\frac{dx}{4+5\sin x}$

Exercise-01: $\int_0^{\pi/2}\frac{dx}{5+4\sin x}$

Ans: $\frac{2}{3}\tan^{-1}\frac{1}{3}$

Exercise-02: $\int_0^{\pi}\frac{dx}{2+\cos x}$

Ans: $\frac{\pi}{\sqrt{3}}$

Solution: Let, $I=\int_0^{\pi/2}\frac{dx}{4+5\sin x}$

$$=\int_0^{\pi/2}\frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{dx}{\frac{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{1+\tan^2\frac{x}{2}}{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}dx$$

$$=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{4\tan^2\frac{x}{2}+10\tan\frac{x}{2}+4}dx$$

put, $\tan\frac{x}{2}=t\ \therefore\ \sec^2\frac{x}{2}dx=2dt$

when $x=0$ then $t=0$

when $x=\frac{\pi}{2}$ then $t=1$

Now, $I=\int_0^1 \frac{2dt}{4t^2+10t+4}$

$$=\frac{1}{2}\int_0^1 \frac{dt}{t^2+5t/2+1}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{t^2+2.t.5/4+\left(5/4\right)^2+1-25/16}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{\left(t+5/4\right)^2-9/16}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{\left(t+5/4\right)^2-\left(3/4\right)^2}$$

$$=\frac{1}{2}\left[\frac{1}{2\times 3/4}\ln\left(\frac{t+5/4-3/4}{t+5/4+3/4}\right)\right]_0^1$$

$$=\frac{1}{2}\left[\frac{2}{3}\ln\left(\frac{t+1/2}{t+2}\right)\right]_0^1$$

$$=\frac{1}{3}\left[\ln\left(\frac{1+1/2}{1+2}\right)-\ln\left(\frac{1/2}{2}\right)\right]$$

$$=\frac{1}{3}\left[\ln\left(\frac{1}{2}\right)-\ln\left(\frac{1}{4}\right)\right]$$

$$=\frac{1}{3}\ln\left(\frac{1/2}{1/4}\right)$$

$$=\frac{1}{3}\ln 2$$

Problem-10: Evaluate $\int_0^{\pi/2} \frac{dx}{5+3\cos x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{dx}{5+3\cos x}$

$$= \int_0^{\pi/2} \frac{dx}{5+3\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int_0^{\pi/2} \frac{dx}{\frac{5+5\tan^2\frac{x}{2}+3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{8+2\tan^2\frac{x}{2}}dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{4+\tan^2\frac{x}{2}}dx$$

put, $\tan\frac{x}{2} = t \therefore \sec^2\frac{x}{2}dx = 2dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = 1$

Now, $I = \frac{1}{2}\int_0^1 \frac{2dt}{4+t^2}$

$$= \int_0^1 \frac{dt}{2^2+t^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^1$$

Exercise-03: $\int_0^{\pi/2} \frac{dx}{3+5\cos x}$

Ans: $\frac{1}{4}\ln 3$

Exercise-04: $\int_0^{\pi/2} \frac{dx}{1+2\cos x}$

Ans: $\frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right)$

$$=\frac{1}{2}\left(\tan^{-1}\frac{1}{2}-\tan^{-1}.0\right)$$

$$=\frac{1}{2}\tan^{-1}\frac{1}{2}$$

Problem-11: Evaluate $\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Solution: Let, $I=\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

put, $1+x=\frac{1}{t}\ \therefore\ dx=-\frac{1}{t^2}dt$

when $x=0$ then $t=1$

when $x=1$ then $t=\frac{1}{2}$

Now, $I=\int_1^{1/2}\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+2\left(\frac{1}{t}-1\right)-\left(\frac{1}{t}-1\right)^2}}$

$$=-\int_1^{1/2}\frac{dt}{t\sqrt{1+\frac{2}{t}-2-\left(\frac{1}{t^2}-\frac{2}{t}+1\right)}}$$

$$=-\int_1^{1/2}\frac{dt}{t\sqrt{\frac{2}{t}-1-\frac{1}{t^2}+\frac{2}{t}-1}}$$

$$=-\int_1^{1/2}\frac{dt}{t\sqrt{\frac{4}{t}-\frac{1}{t^2}-2}}$$

$$=-\int_1^{1/2}\frac{dt}{t\sqrt{\frac{4t-1-2t^2}{t^2}}}$$

$$=-\int_1^{1/2}\frac{dt}{\sqrt{-1-2t^2+4t}}$$

$$= -\int_1^{1/2} \frac{dt}{\sqrt{2}\sqrt{-1/2 - t^2 + 2t}}$$

$$= -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dt}{\sqrt{-1/2 - \left(t^2 - 2t\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dt}{\sqrt{1 - 1/2 - \left(t^2 - 2t + 1\right)}}$$

$$= -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dt}{\sqrt{1/2 - (t-1)^2}}$$

$$= -\frac{1}{\sqrt{2}}\int_1^{1/2} \frac{dt}{\sqrt{\left(1/\sqrt{2}\right)^2 - (t-1)^2}}$$

$$= -\frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{t-1}{1/\sqrt{2}}\right)\right]_1^{1/2}$$

$$= -\frac{1}{\sqrt{2}}\left[\sin^{-1}\sqrt{2}(t-1)\right]_1^{1/2}$$

$$= -\frac{1}{\sqrt{2}}\left[\sin^{-1}\sqrt{2}\left(1/2 - 1\right) - \sin^{-1}\sqrt{2}(1-1)\right]$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(-1/2\right)$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(1/2\right)$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}}$$

Problem-12: Evaluate $\int_0^1 \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}}$

Solution: Let, $I = \int_0^1 \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}}$

Put $x=\frac{1}{z}\ \therefore dx=-\frac{1}{z^2}dz$

when $x=0$ then $z=\infty$

when $x=1$ then $z=1$

Now $I=\int_{\infty}^{1}\frac{-\frac{1}{z^2}dz}{\left(1+\frac{1}{z^2}\right)\sqrt{1-\frac{1}{z^2}}}$

$$=\int_{1}^{\infty}\frac{zdz}{\left(z^2+1\right)\sqrt{z^2-1}}$$

Again let $z^2-1=t^2\ \ or,\ z^2=t^2+1$

$$\therefore\ zdz=tdt$$

when $z=1$ then $t=0$

when $z=\infty$ then $t=\infty$

$$\therefore\ I=\int_{0}^{\infty}\frac{tdt}{\left(t^2+1+1\right)\sqrt{t^2}}$$

$$=\int_{0}^{\infty}\frac{dt}{2+t^2}$$

$$=\int_{0}^{\infty}\frac{dt}{\left(\sqrt{2}\right)^2+t^2}$$

$$=\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^{\infty}$$

$$=\frac{1}{\sqrt{2}}\left(\tan^{-1}.\infty-\tan^{-1}.0\right)$$

$$=\frac{1}{\sqrt{2}}\left(\tan^{-1}.\tan\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2\sqrt{2}}$$

General Properties of Definite Integrals: The general properties are,

1. $\int_a^b f(x)dx=\int_a^b f(z)dz$
2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$
3. $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$
4. $\int_0^a f(x)dx=\int_0^a f(a-x)dx$
5. $\int_0^{2a} f(x)dx=2\int_0^a f(x)dx \quad if \ f(2a-x)=f(x)$
6. $\int_{-a}^a f(x)dx=\begin{cases} 2\int_0^a f(x)\,dx & if \ f(-x)=f(x) \\ 0 & if \ f(-x)=-f(x) \end{cases}$

Question-01: Prove that $\int_a^b f(x)dx=\int_a^b f(z)dz$.

Proof: Let $\int f(x)\,dx=F(x)$ and $\int f(z)\,dx=F(z)$

$$\therefore \int_a^b f(x)dx=\left[F(x)\right]_a^b$$

$$=F(b)-F(a) \qquad \cdots(i)$$

Again, $\int_a^b f(z)dz=\left[F(z)\right]_a^b$

$$=F(b)-F(a) \qquad \cdots(ii)$$

From (i) and (ii) we have

$$\int_a^b f(x)dx=\int_a^b f(z)dz \qquad \textbf{(Thus proved)}$$

Question-02: Prove that $\int_a^b f(x)dx=-\int_b^a f(x)dx$.

Proof: Let $\int f(x)\,dx=F(x)$

$$\therefore \int_a^b f(x)dx = \left[F(x)\right]_a^b$$

$$= F(b) - F(a) \qquad \cdots(i)$$

Again, $-\int_b^a f(x)dx = -\left[F(x)\right]_b^a$

$$= F(b) - F(a) \qquad \cdots(ii)$$

From (i) and (ii) we have

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$ **(Thus proved)**

Question-03: Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, when $a < c < b$.

Proof: Let $\int f(x)dx = F(x)$

$$\therefore \int_a^b f(x)dx = \left[F(x)\right]_a^b$$

$$= F(b) - F(a) \qquad \cdots(i)$$

Again, $\int_a^c f(x)dx + \int_c^b f(x)dx = \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) \qquad \cdots(ii)$$

From (i) and (ii) we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$ **(Thus proved)**

Question-04: Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

Proof: Let $a - x = z$, then $dx = -dz$

when $x = 0$ then $z = a$

when $x = a$ then $z = 0$

Now $\int_0^a f(a-x)dx = -\int_a^0 f(z)dz$

$$= \int_0^a f(z)dz$$

$$= \int_0^a f(x)dx$$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$ **(Thus proved)**

Question-05: Prove that $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ if $f(a+x) = f(x)$.

Proof: Here, $\int_0^{na} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx + \int_{2a}^{3a} f(x)dx + \cdots + \int_{(n-1)a}^{na} f(x)dx$

$$\therefore \int_0^{na} f(x)dx = I_1 + I_2 + I_3 + \cdots + I_n \qquad (say) \qquad \cdots(i)$$

Now $I_2 = \int_a^{2a} f(x)dx$

Let $x = a+z$, then $dx = dz$

when $x = a$ then $z = 0$

when $x = 2a$ then $z = a$

$$\therefore I_2 = \int_0^a f(a+z)dz$$

$$= \int_0^a f(a+x)dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(z)dz\right]$$

$$= \int_0^a f(x)dx \qquad [\because f(a+x) = f(x)]$$

$$= I_1$$

Similarly we have,

$$I_3 = I_4 = I_5 = \cdots = I_n = I_1$$

From (i) we have,

$$\int_0^{na} f(x)dx = I_1 + I_1 + I_1 + \cdots + I_1$$

$$or, \int_0^{na} f(x)dx = nI_1$$

$$\therefore \int_0^{na} f(x)dx = n\int_0^{a} f(x)dx \qquad \textbf{(Thus proved)}$$

Question-06: Prove that $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ f(2a-x) = f(x) \\ 0 & if\ f(2a-x) = -f(x) \end{cases}$

Proof: Here, $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_a^{2a} f(x)dx$

$$\therefore \int_0^{2a} f(x)dx = I_1 + I_2 \qquad (say) \qquad \cdots(i)$$

Now $\quad I_2 = \int_a^{2a} f(x)dx$

Let $x = 2a - z$, then $dx = -dz$

when $x = a$ then $z = a$

when $x = 2a$ then $z = 0$

$$\therefore I_2 = -\int_a^0 f(2a-z)dz$$

$$= \int_0^a f(2a-z)dz$$

$$= \int_0^a f(2a-x)dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(z)dz\right]$$

From (i) we have,

$$\int_0^{2a} f(x)dx = I_1 + \int_0^a f(2a-x)dx \qquad \cdots(ii)$$

If $f(2a-x) = f(x)$ then (ii) reduces as

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

Again, if $f(2a-x) = -f(x)$ then (ii) reduces as

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx$$

$$= 0$$

$$\therefore \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if \; f(2a-x) = f(x) \\ 0 & if \; f(2a-x) = -f(x) \end{cases}$$ **(Thus proved)**

Question-07: Prove that $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if \; f(x) \, is \, an \, even \;\; function \\ 0 & if \; f(x) \; is \, an \, odd \;\; function \end{cases}$

Proof: Here, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$$\therefore \int_{-a}^{a} f(x)dx = I_1 + I_2 \qquad (say) \qquad \cdots(i)$$

Now $\quad I_1 = \int_{-a}^{0} f(x)dx$

Let $x = -z$, then $dx = -dz$

when $x = -a$ then $z = a$

when $x = 0$ then $z = 0$

$$\therefore I_1 = -\int_a^0 f(-z)dz$$

$$= \int_0^a f(-z)dz$$

$$= \int_0^a f(-x)dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(z)dz\right]$$

From (i) we have,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx \qquad \cdots(ii)$$

If $f(-x) = f(x)$ i.e. $f(x)$ is an even function, then (ii) reduces as

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(x)dx \ .$$

Again, if $f(-x) = -f(x)$ i.e. $f(x)$ is an odd function, then (ii) reduces as

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 0 .$$

$$\therefore \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx \ \ if \ f(x) \ is\ an\ even \ \ function \\ 0 \qquad if \ f(x) \ is\ an\ odd \ \ function \end{cases}$$ **(Thus proved)**

Problem-01: Evaluate $\int_{0}^{\pi/2} \frac{dx}{1+\cot x}$

Solution: Let, $I = \int_{0}^{\pi/2} \frac{dx}{1+\cot x}$

$$= \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$= \int_{0}^{\pi/2} \frac{\sin\left(\pi/2 - x\right)}{\sin\left(\pi/2 - x\right) + \cos\left(\pi/2 - x\right)} dx$$

$$= \int_{0}^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$$

Now $2I = \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_{0}^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$

$$=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}dx$$

$$=\int_0^{\pi/2}dx$$

$$=[x]_0^{\pi/2}$$

$$=\pi/2$$

$$\therefore\ I=\pi/4$$

Problem-02: Evaluate $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx\ OR,\ \int_0^{\pi/2}\frac{dx}{1+\sqrt{\cot x}}\ OR,\ \int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}$

Solution: Let, $I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$$=\int_0^{\pi/2}\frac{\sqrt{\sin\left(\pi/2-x\right)}}{\sqrt{\sin\left(\pi/2-x\right)}+\sqrt{\cos\left(\pi/2-x\right)}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

Now $2I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx+\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$$=\int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$=\int_0^{\pi/2}dx$$

$$=[x]_0^{\pi/2}$$

$$=\pi/2$$

$$\therefore\ I = \pi/4$$

Problem-03: Evaluate $\int_0^{\pi} \frac{xdx}{1+\sin x}$

Solution: Let, $I = \int_0^{\pi} \frac{xdx}{1+\sin x}$

$$= \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx$$

Now $\quad 2I = \int_0^{\pi} \frac{x}{1+\sin x} dx + \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx$

$$= \int_0^{\pi} \frac{x + \pi - x}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

$$= \int_0^{\pi} \frac{\pi(1-\sin x)}{1-\sin^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{(1-\sin x)}{\cos^2 x} dx$$

$$= \pi \int_0^{\pi} \sec^2 x (1-\sin x) dx$$

$$= \pi \int_0^{\pi} \left(\sec^2 x - \sec^2 x \sin x\right) dx$$

$$= \pi \int_0^{\pi} \left(\sec^2 x - \sec x \tan x\right) dx$$

$$= \pi \left[\tan x - \sec x\right]_0^{\pi}$$

$$= \pi\left[0+1-0+1\right]$$

$$= 2\pi$$

$$\therefore \quad I = \pi$$

Problem-04: Evaluate $\int_0^{\pi} \frac{x\sin x dx}{1+\cos^2 x}$

Solution: Let, $I = \int_0^{\pi} \frac{x\sin x dx}{1+\cos^2 x}$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

Now $\quad 2I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$

$$= \int_0^{\pi} \frac{(x+\pi - x)\sin x}{1+\cos^2 x}dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$$

put $\cos x = t \quad \therefore \ -\sin x dx = dt$

when $x = 0$ then $t = 1$

when $x = \pi$ then $t = -1$

$$\therefore \quad 2I = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= \pi\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= \pi\left[\tan^{-1}.1 - \tan^{-1}(-1)\right]$$

$$= \pi\left[\tan^{-1}.1 + \tan^{-1}.1\right]$$

$$= \pi\left[\tan^{-1}.\tan\frac{\pi}{4} + \tan^{-1}.\tan\frac{\pi}{4}\right]$$

$$= \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}$$

Problem-05: Evaluate $\int_0^{\pi/2} \frac{xdx}{\sin x + \cos x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{xdx}{\sin x + \cos x}$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin x + \cos x}$$

Now $\qquad 2I = \int_0^{\pi/2} \frac{xdx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin x + \cos x}$

$$= \int_0^{\pi/2} \frac{\left(x + \frac{\pi}{2} - x\right)dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2}dx}{\sin x + \cos x}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \left[\ln\left\{ \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right\} \right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}} \left[\ln\left\{ \sec\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \right\} - \ln\left\{ \sec\frac{\pi}{4} - \tan\frac{\pi}{4} \right\} \right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\ln\left\{ \sec\frac{\pi}{4} + \tan\frac{\pi}{4} \right\} - \ln\left\{ \sec\frac{\pi}{4} - \tan\frac{\pi}{4} \right\} \right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) \right]$$

$$= \frac{\pi}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

$$= \frac{\pi}{2\sqrt{2}} \ln\left(\sqrt{2}+1\right)^2$$

$$= \frac{\pi}{\sqrt{2}} \ln\left(\sqrt{2}+1\right)$$

$$\therefore \; I = \frac{\pi}{2\sqrt{2}} \ln\left(\sqrt{2}+1\right)$$

Problem-06: Evaluate $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

Solution: Let, $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

put $x = \tan\theta \quad \therefore\ dx = \sec^2\theta d\theta$

when $x = 0$ then $\theta = 0$

when $x = 1$ then $\theta = \frac{\pi}{4}$

$$\therefore\ I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \ln(1+\tan\theta)d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta$$

$$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right)d\theta$$

Now $$2I = \int_0^{\pi/4} \ln(1+\tan\theta)d\theta + \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \ln\left\{(1+\tan\theta).\frac{2}{(1+\tan\theta)}\right\}d\theta$$

$$= \int_0^{\pi/4} \ln 2 d\theta$$

$$= \ln 2[\theta]_0^{\pi/4}$$

$$= \frac{\pi}{4}\ln 2$$

$$\therefore \quad I = \frac{\pi}{8}\ln 2$$

Problem-07: Evaluate $\int_0^{\pi/2} \ln\sin x dx \ OR \ \int_0^{\pi/2} \ln\cos x dx$

Solution: Let, $I = \int_0^{\pi/2} \ln\sin x dx$

$$= \int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} \ln\cos x dx$$

Now $\qquad 2I = \int_0^{\pi/2} \ln\sin x dx + \int_0^{\pi/2} \ln\cos x dx$

$$= \int_0^{\pi/2} (\ln\sin x + \ln\cos x)dx$$

$$= \int_0^{\pi/2} \ln(\sin x\cos x)dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{1}{2}\sin 2x\right)dx$$

$$= \int_0^{\pi/2} \ln \sin 2x dx - \ln 2 \int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \ln \sin 2x dx - \ln 2 [x]_0^{\pi/2}$$

$$= I_1 - \frac{\pi}{2} \ln 2 \quad \ldots \ldots \ldots (1)$$

where, $I_1 = \int_0^{\pi/2} \ln \sin 2x dx$

put $2x = t \quad \therefore \; dx = \frac{1}{2} dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = \pi$

$$\therefore \quad I_1 = \frac{1}{2} \int_0^{\pi} \ln \sin t dt$$

$$= \int_0^{\pi/2} \ln \sin t dt$$

$$= \int_0^{\pi/2} \ln \sin x dx$$

$$= I$$

From (1) we get

$$2I = I - \frac{\pi}{2} \ln 2$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2$$

$$\Rightarrow I = \frac{\pi}{2} \ln \frac{1}{2}$$

Problem-08: Evaluate $\int_0^{\pi/2} \ln \tan x dx$

Solution: Let, $I = \int_0^{\pi/2} \ln \tan x dx$

$$= \int_0^{\pi/2} \ln \tan\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} \ln \cot x dx$$

Now $2I = \int_0^{\pi/2} \ln \tan x dx + \int_0^{\pi/2} \ln \cot x dx$

$$= \int_0^{\pi/2} (\ln \tan x + \ln \cot x)dx$$

$$= \int_0^{\pi/2} \ln(\tan x \cot x)dx$$

$$= \int_0^{\pi/2} \ln 1 dx$$

$$= 0$$

$\therefore\ I = 0$