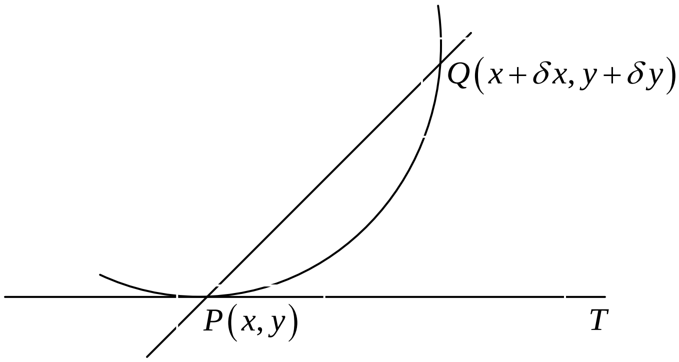
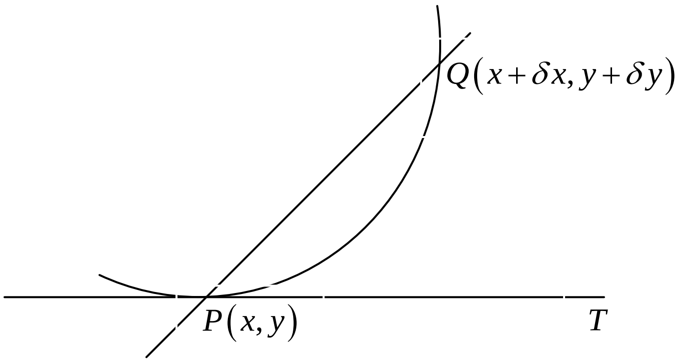
Tangent: Let P and Q be two neighboring points on a curve. As Q tends to P (i.e. approaches to P) along the curve, the straight line joining P and Q tends to definite straight line PT, called the tangent to the curve at P.



Simply, we can say that a tangent is a straight line that touches a curve at a single point and does not cross through it. The point where the curve and the tangent meet is called the point of tangency.

Theorem-01: Derive the equation of the tangent to a curve $y = f(x)$ at any point of it.

Answer: Let $P(x, y)$ be the given point on the curve $y = f(x)$. Let $Q(x+\delta x, y+\delta y)$ be a point neighboring to P on the curve.



Then the equation of the line PQ is

$$\frac{Y-y}{y-(y+\delta y)} = \frac{X-x}{x-(x+\delta x)}$$

$$or,\ Y-y = \frac{\delta y}{\delta x}(X-x) \qquad \cdots(1)$$

where X and Y are the current co-ordinates.

Now as $Q \to P$, $\delta y \to 0$, $\delta x \to 0$ then $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ and the line PQ tends to the tangent PT to the curve at P, so the equation of the tangent to the curve $y = f(x)$ at $P(x, y)$ is

$$Y-y = \frac{dy}{dx}(X-x).$$

Theorem-02: Derive the equation of the tangent to a curve $f(x,y)=0$ at any point of it.

Answer: The equation of the curve is $f(x,y)=0$

The total differential of this curve is

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$$

$$or,\ f_x dx+f_y dy=0$$

$$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}\ ,\ where\ f_y\neq 0 \qquad \cdots(1)$$

We know the equation of the tangent at the point (x,y) is

$$Y-y=\frac{dy}{dx}(X-x) \qquad \cdots(2)$$

From (1) and (2), we get

$$Y-y=-\frac{f_x}{f_y}(X-x)$$

$$or,(X-x)f_x+(Y-y)f_y=0.$$

This is the equation of the tangent to the given curve at any point of it.

Theorem-03: Derive the equation of the tangent to a parametric curve $x=f(t),\ y=\varphi(t)$ at any point t of it.

Answer: The equation of the curve is $x=f(t),\ y=\varphi(t)$ $\cdots(1)$

Differentiating with respect to t, we get

$$\frac{dx}{dt}=f'(t) \text{ and } \frac{dy}{dt}=\varphi'(t)$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\varphi'(t)}{f'(t)} \qquad \cdots(2)$$

We know the equation of the tangent at the point (x,y) is

$$Y-y=\frac{dy}{dx}(X-x) \qquad \cdots(3)$$

From (1), (2) and (3), we get

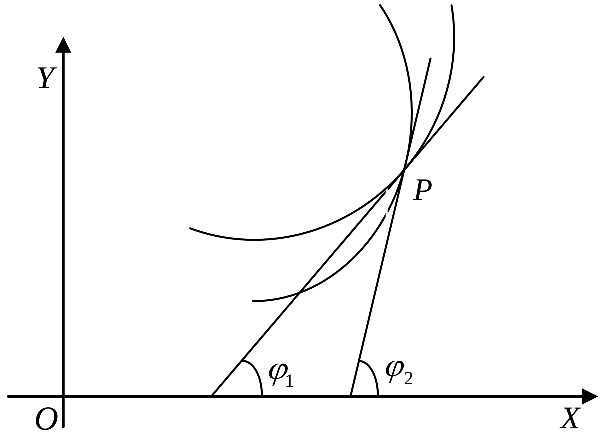
$$Y-\varphi(t)=\frac{\varphi'(t)}{f'(t)}\left(X-f(t)\right).$$

This is the equation of the tangent to the given curve at any point t of it.

Tangents parallel to the co-ordinates axes: If a tangent is parallel to the X-axis , then $\varphi=0$ i.e. $\tan\varphi=0$ and so we have $\frac{dy}{dx}=0$ at that point.

If a tangent is parallel to the Y-axis, then $\varphi=\frac{\pi}{2}$ i.e. $\tan\varphi=\infty$ and so we have $\frac{dy}{dx}=\infty$ or $\frac{dx}{dy}=0$ at that point.

Angle of intersection of two curves: If two curves intersect each other at P, then the angle of intersection of curves is defined as the angle between the tangents to the curves at P.



Let φ_1 and φ_2 be the angles which the tangents to the curve at P make with x- axis. Then if θ be the required angle, it is evident from the figure,

$$\theta = \varphi_1 - \varphi_2$$

$$or, \tan\theta = \tan(\varphi_1 - \varphi_2)$$

$$or, \tan\theta = \frac{\tan\varphi_1 - \tan\varphi_2}{1+\tan\varphi_1 \tan\varphi_2}$$

$$or,\ \theta = \tan^{-1}\left(\frac{\tan\varphi_1 - \tan\varphi_2}{1+\tan\varphi_1 \tan\varphi_2}\right)$$

$$\therefore\ \theta = \tan^{-1}\left[\frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}\right].$$

This is the required angle.

If the two curves cut orthogonally, i.e. $\theta = \frac{\pi}{2}$, then

$$\tan\left(\frac{\pi}{2}\right) = \frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}$$

$$or,\ \frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2} = \infty$$

$$or,\ 1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = 0$$

$$\therefore \left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1.$$

If the equation of the curves be $f(x, y) = 0$ and $\varphi(x, y) = 0$, then

$$\theta = \tan^{-1}\left(\frac{f_x\varphi_y - f_y\varphi_x}{f_x\varphi_x + f_y\varphi_y}\right).$$

This curves cut orthogonally i.e. $\theta = \frac{\pi}{2}$ if $f_x\varphi_x + f_y\varphi_y = 0$.

Problem-01: Find the equation of the tangent at (x, y) to the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$

Solution: The equation of the curve is $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

$$\text{Here} \quad f(x, y) = \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} - 1 = 0 \qquad \cdots(1)$$

Differentiating (1) partially with respect to x and y, we get

$$f_x = \frac{2}{3}\left(\frac{x}{a}\right)^{-\frac{1}{3}} \cdot \frac{1}{a}$$

$$\text{and} \quad f_y = \frac{2}{3}\left(\frac{y}{b}\right)^{-\frac{1}{3}} \cdot \frac{1}{b}$$

The equation of the tangent is,

$$(X - x) f_x + (Y - y) f_y = 0$$

$$or,\ (X - x) \cdot \frac{2}{3}\left(\frac{x}{a}\right)^{-\frac{1}{3}} \cdot \frac{1}{a} + (Y - y) \cdot \frac{2}{3}\left(\frac{y}{b}\right)^{-\frac{1}{3}} \cdot \frac{1}{b} = 0$$

$$or,\ (X - x)\frac{x^{-\frac{1}{3}}}{a^{\frac{2}{3}}} + (Y - y)\frac{y^{-\frac{1}{3}}}{b^{\frac{2}{3}}} = 0$$

$$or,\ \frac{Xx^{-\frac{1}{3}}}{a^{\frac{2}{3}}} - \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{Yy^{-\frac{1}{3}}}{b^{\frac{2}{3}}} - \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 0$$

$$or,\ \frac{Xx^{-\frac{1}{3}}}{a^{\frac{2}{3}}} + \frac{Yy^{-\frac{1}{3}}}{b^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}}$$

$$\therefore \frac{Xx^{-\frac{1}{3}}}{a^{\frac{2}{3}}} + \frac{Yy^{-\frac{1}{3}}}{b^{\frac{2}{3}}} = 1.$$

Problem-02: Find the equation of the tangent to the curve $y = x^2 + 2x + 1$ at $(1, 4)$.

Solution: The equation of the curve is $y = x^2 + 2x + 1$. $\cdots(1)$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = 2x + 2$$

At $(1,4)$, the slope of the given curve is

$$\frac{dy}{dx} = 2 \cdot 1 + 2 = 4$$

The equation of the tangent at $(1,4)$ is,

$$Y - y = \frac{dy}{dx}(X - x)$$
$$or,\ Y - 4 = 4(X - 1)$$
$$or,\ Y - 4 = 4X - 4$$
$$or,\ Y = 4X$$

Replacing $X \text{ and } Y$ by $x \text{ and } y$, we get

$$y = 4x.$$

Problem-03: Find the equation of the tangent to the curve $x = a\sin^3 t,\ y = b\cos^3 t$ at the point t.

Solution: The equation of the curve is $x = a\sin^3 t,\ y = b\cos^3 t$.

$$\text{Here } x = f(t) = a\sin^3 t,\ y = \varphi(t) = b\cos^3 t \qquad \cdots(1)$$

Differentiating (1) with respect to t, we get

$$f'(t) = 3a\sin^2 t\cos t$$

and $$\varphi'(t) = -3b\cos^2 t\sin t$$

The equation of the tangent at t is,

$$Y - \varphi(t) = \frac{\varphi'(t)}{f'(t)}(X - f(t))$$
$$or,\ Y - b\cos^3 t = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t}(X - a\sin^3 t)$$
$$or,\ Y - b\cos^3 t = \frac{-b\cos t}{a\sin t}(X - a\sin^3 t)$$
$$or,\ aY\sin t - ab\cos^3 t\sin t = -bX\cos t + ab\sin^3 t\cos t$$
$$or,\ bX\cos t + aY\sin t = ab\cos^3 t\sin t + ab\sin^3 t\cos t$$
$$or,\ bX\cos t + aY\sin t = ab\sin t\cos t(\cos^2 t + \sin^2 t)$$
$$\therefore\ \left(\frac{X}{a}\right)\operatorname{cosec} t + \left(\frac{Y}{b}\right)\sec t = 1.$$

Problem-04: Find the angle of intersection of the curves $x^2 - y^2 = a^2 \text{ and } x^2 + y^2 = a^2\sqrt{2}$.

Solution: The equations of the curves are,

$$x^2 - y^2 = a^2 \qquad \cdots(1)$$

and $$x^2 + y^2 = a^2\sqrt{2} \qquad \cdots(2)$$

Adding (1) and (2), we get

$$2x^2 = a^2 + a^2\sqrt{2}$$

$$\therefore\ x = \pm \frac{a\sqrt{1+\sqrt{2}}}{\sqrt{2}}$$

Subtracting (1) from (2), we get

$$2y^2 = a^2\sqrt{2} - a^2$$

$$\therefore\ y = \pm \frac{a\sqrt{\sqrt{2}-1}}{\sqrt{2}}$$

Therefore the point of intersection of the curves is $(x, y) = \left(\pm \frac{a\sqrt{1+\sqrt{2}}}{\sqrt{2}}, \pm \frac{a\sqrt{\sqrt{2}-1}}{\sqrt{2}} \right)$.

The equations (1) and (2) can be written as,

$$f(x, y) = x^2 - y^2 - a^2 = 0 \qquad \cdots(3)$$

and $$\varphi(x, y) = x^2 + y^2 - a^2\sqrt{2} = 0 \qquad \cdots(4)$$

Differentiating (3) and (4) with respect to x *and* y, we get

$$f_x = 2x,\ f_y = -2y,\ \varphi_x = 2x \text{ and } \varphi_y = 2y$$

If θ be the angle of intersection of the curves, then

$$\theta = \tan^{-1}\left(\frac{f_x\varphi_y - f_y\varphi_x}{f_x\varphi_x + f_y\varphi_y} \right)$$

$$or,\ \theta = \tan^{-1}\left\{ \frac{2x \cdot 2y - (-2y) \cdot 2x}{2x \cdot 2x + (-2y) \cdot 2y} \right\}$$

$$or,\ \theta = \tan^{-1}\left\{ \frac{2xy}{x^2 - y^2} \right\}$$

$$or,\ \theta = \tan^{-1}\left(\frac{2 \cdot \frac{a\sqrt{1+\sqrt{2}}}{\sqrt{2}} \cdot \frac{b\sqrt{\sqrt{2}-1}}{\sqrt{2}}}{a^2} \right)$$

$$or,\ \theta = \tan^{-1}(1)$$

$$or,\ \theta = \tan^{-1}\left(\tan\frac{\pi}{4} \right)$$

$$\therefore\ \theta = 45^\circ.$$

Problem-05: Find the condition that the conics $ax^2 + by^2 = 1$ *and* $a_1x^2 + b_1y^2 = 1$ will cut orthogonally.

Solution: The equations of the conics are,

$$f(x, y) = ax^2 + by^2 - 1 = 0 \qquad \cdots(1)$$

and $$\varphi(x, y) = a_1x^2 + b_1y^2 - 1 = 0 \qquad \cdots(2)$$

Differentiating (1) and (2) with respect to x *and* y, we get

$$f_x = 2ax,\ f_y = 2by,\ \varphi_x = 2a_1x \text{ and } \varphi_y = 2b_1y$$

The curves will cut orthogonally at (x, y) if

$$f_x\varphi_x + f_y\varphi_y = 0$$

$$or,\ 2ax\cdot 2a_1x+2by\cdot 2b_1y=0$$

$$\therefore aa_1x^2+bb_1y^2=0 \qquad \cdots(3)$$

Since the point (x,y) is common to both (1) and (2), the required condition is obtained by eliminating x,y from (1), (2) and (3).

Subtracting (2) from (1), we get

$$(a-a_1)x^2+(b-b_1)y^2=0 \qquad \cdots(4)$$

Comparing (3) and (4), we get

$$\frac{(a-a_1)}{aa_1}=\frac{(b-b_1)}{bb_1}$$

$$\therefore \frac{1}{a_1}-\frac{1}{a}=\frac{1}{b_1}-\frac{1}{b}$$

which is the required condition.

Problem-06: Find the condition that the straight line $x\cos\alpha+y\sin\alpha=p$ may touch the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$.

OR

If $x\cos\alpha+y\sin\alpha=p$ touches the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$, then show that $(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}$.

Solution: The equations of the line and curve are,

$$x\cos\alpha+y\sin\alpha=p \qquad \cdots(1)$$

and $$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1 \qquad \cdots(2)$$

The curve (2) can be written as,

$$f(x,y)=\frac{x^m}{a^m}+\frac{y^m}{b^m}-1=0 \qquad \cdots(3)$$

Differentiating (3) partially with respect to x *and* y, we get

$$f_x=\frac{m}{a^m}\cdot x^{m-1}$$

and $$f_y=\frac{m}{b^m}\cdot y^{m-1}$$

The equation of the tangent at (x,y) is,

$$(X-x)f_x+(Y-y)f_y=0$$

$$or,\ (X-x)\cdot\frac{m}{a^m}\cdot x^{m-1}+(Y-y)\cdot\frac{m}{b^m}\cdot y^{m-1}=0$$

$$or,\ \frac{Xx^{m-1}}{a^m}-\frac{x^m}{a^m}+\frac{Yy^{m-1}}{b^m}-\frac{y^m}{b^m}=0$$

$$or,\ \frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=\frac{x^m}{a^m}+\frac{y^m}{b^m}$$

$$\therefore \frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=1. \qquad \cdots(4)$$

If (1) touches (2), then (1) and (4) must be identical. So from (1) and (4) we can write,

$$\frac{\frac{x^{m-1}}{a^m}}{\cos\alpha}=\frac{\frac{y^{m-1}}{b^m}}{\sin\alpha}=\frac{1}{p}$$

$$or,\ \frac{\frac{x^{m-1}}{a^{m-1}}}{a\cos\alpha}=\frac{\frac{y^{m-1}}{b^{m-1}}}{b\sin\alpha}=\frac{1}{p}$$

$$\therefore\ \left(\frac{x}{a}\right)^{m-1}=\frac{a\cos\alpha}{p}\quad and\ \left(\frac{y}{b}\right)^{m-1}=\frac{b\sin\alpha}{p}$$

$$\therefore\ \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m$$

$$or,\ \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=1$$

$$i.e.\ (a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}.$$

This is the required condition. **(Showed)**

Problem-07: Show that, the condition that the curves $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may touch if $a+b=c$.

Solution: The equations of the curves are,

$$f(x,y)=x^{\frac{2}{3}}+y^{\frac{2}{3}}-c^{\frac{2}{3}}=0 \qquad \cdots(1)$$

and $$\varphi(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \qquad \cdots(2)$$

Differentiating (1) partially with respect to $x\ and\ y$, we get

$$f_x=\frac{2}{3}x^{-\frac{1}{3}}\ \text{ and }\ f_y=\frac{2}{3}y^{-\frac{1}{3}}$$

The equation of the tangent at (x,y) is,

$$(X-x)f_x+(Y-y)f_y=0$$

$$or,\ (X-x)\cdot\frac{2}{3}x^{-\frac{1}{3}}+(Y-y)\cdot\frac{2}{3}y^{-\frac{1}{3}}=0$$

$$or,\ Xx^{-\frac{1}{3}}-x^{\frac{2}{3}}+Yy^{-\frac{1}{3}}-y^{\frac{2}{3}}=0$$

$$or,\ Xx^{-\frac{1}{3}}+Yy^{-\frac{1}{3}}=x^{\frac{2}{3}}+y^{\frac{2}{3}}$$

$$\therefore\ Xx^{-\frac{1}{3}}+Yy^{-\frac{1}{3}}=c^{\frac{2}{3}}. \qquad \cdots(3)$$

Again differentiating (2) partially with respect to $x\ and\ y$, we get

$$f_x=\frac{2x}{a^2}\ \text{ and }\ f_y=\frac{2y}{b^2}$$

The equation of the tangent at (x,y) is,

$$(X-x)f_x+(Y-y)f_y=0$$

$$or,\ (X-x)\cdot\frac{2x}{a^2}+(Y-y)\cdot\frac{2y}{b^2}=0$$

$$or,\ (X-x)\cdot\frac{x}{a^2}+(Y-y)\cdot\frac{y}{b^2}=0$$

$$or,\ \frac{Xx}{a^2}+\frac{Yy}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

$$\therefore\ \frac{Xx}{a^2}+\frac{Yy}{b^2}=1. \qquad \cdots(4)$$

Since the given curves (1) and (2) touch to each other at the point (x,y) so that the tangents (3) and (4) represent the same straight lines. Hence from (3) and (4), we get

$$\frac{\frac{x}{a^2}}{x^{-\frac{1}{3}}}=\frac{\frac{y}{b^2}}{y^{-\frac{1}{3}}}=\frac{1}{c^{\frac{2}{3}}}$$

$$\therefore\ x^{\frac{4}{3}}=\frac{a^2}{c^{\frac{2}{3}}}\ \ and\ \ y^{\frac{4}{3}}=\frac{b^2}{c^{\frac{2}{3}}}$$

$$or,\ x^{\frac{2}{3}}=\frac{a}{c^{\frac{1}{3}}}\ \ and\ \ y^{\frac{2}{3}}=\frac{b}{c^{\frac{1}{3}}}$$

Adding these terms, we have

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{a}{c^{\frac{1}{3}}}+\frac{b}{c^{\frac{1}{3}}}$$

$$or,\ c^{\frac{2}{3}}=\frac{a}{c^{\frac{1}{3}}}+\frac{b}{c^{\frac{1}{3}}}$$

$i.e.\ a+b=c.$ **(Showed)**

Homework:

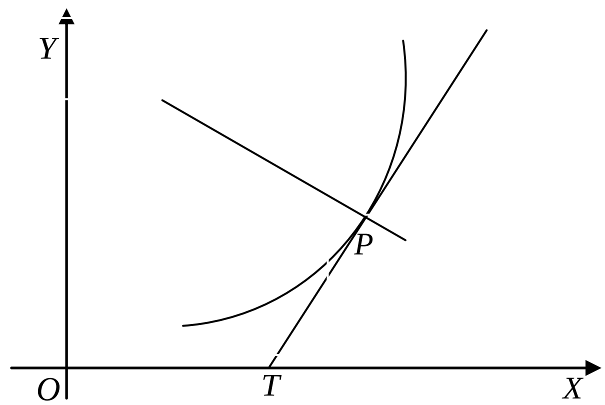
Problem-01: Find the equation of the tangent at (x,y) to the curve $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1.$

Problem-02: Find the equation of the tangent to the curve $x=a(t+\sin t),\ y=a(1-\cos t)$ at the point t.

Problem-03: Prove the condition that the straight line $x\cos\alpha+y\sin\alpha=p$ touches the curve $x^m y^n=a^{m+n}$ is

$$p^{m+n}m^m n^n=(m+n)^{m+n}\cos^m\alpha\sin^n\alpha.$$

Normal: The normal to a curve at a point is the line perpendicular to the tangent line at the point.



Theorem-04: To find the equation of the normal to a curve $y = f(x)$ at any point of it.

Answer: We know that the tangent to a curve $y = f(x)$ at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \quad \cdots(1)$$

where $\frac{dy}{dx}$ is the gradient of tangent.

Also any line through the point (x, y) is given by

$$Y - y = m(X - x) \quad \cdots(2)$$

If this line is normal to the curve $y = f(x)$ at (x, y) then this must be perpendicular to the tangent to the curve, so that

$$m \cdot \frac{dy}{dx} = -1$$

$$\therefore\ m = -\frac{1}{\frac{dy}{dx}} \quad \cdots(3)$$

Now from (2) and (3), we get

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x).$$

This is the equation of the normal to the given curve at any point of it.

Theorem-05: To find the equation of the normal to a curve $f(x, y) = 0$ at any point of it.

Answer: The equation of the curve is $f(x, y) = 0$

The total differential of this curve is

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$or,\ f_x dx + f_y dy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y}, \text{ where } f_y \neq 0 \qquad \cdots(1)$$

We know the equation of the normal at the point (x, y) is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \qquad \cdots(2)$$

From (1) and (2), we get

$$Y - y = \frac{f_y}{fx}(X - x)$$

$$or, (X - x) f_y - (Y - y) f_x = 0.$$

This is the equation of the normal to the given curve at any point of it.

Theorem-06: Derive the equation of the tangent to a parametric curve $x = f(t),\ y = \varphi(t)$ at any point t of it.

Answer: The equation of the curve is $x = f(t),\ y = \varphi(t)$ $\cdots(1)$

Differentiating with respect to t, we get

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = \varphi'(t)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\varphi'(t)}{f'(t)} \qquad \cdots(2)$$

We know the equation of the normal at the point (x, y) is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \qquad \cdots(3)$$

From (1), (2) and (3), we get

$$Y - \varphi(t) = -\frac{1}{\frac{\varphi'(t)}{f'(t)}}(X - f(t)).$$

This is the equation of the normal to the given curve at any point t of it.

Problem-01: Find the equation of the normal at (a, b) to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$

Solution: The equation of the curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$

Here $\quad f(x, y) = \left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n - 2 = 0 \qquad \cdots(1)$

Differentiating (1) partially with respect to x and y, we get

$$f_x = n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} \quad \text{and} \quad f_y = n\left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b}$$

The equation of the normal at (x,y) is,

$$(X-x)f_y-(Y-y)f_x=0$$

$$or,\ (X-x)\cdot n\left(\frac{y}{b}\right)^{n-1}\cdot\frac{1}{b}-(Y-y)\cdot n\left(\frac{x}{a}\right)^{n-1}\cdot\frac{1}{a}=0$$

$$or,\ \frac{Xy^{n-1}}{b^n}-\frac{xy^{n-1}}{b^n}-\frac{Yx^{n-1}}{a^n}+\frac{yx^{n-1}}{a^n}=0$$

$$\therefore\ \frac{Xy^{n-1}}{b^n}-\frac{Yx^{n-1}}{a^n}=\frac{xy^{n-1}}{b^n}-\frac{yx^{n-1}}{a^n}$$

At (a,b) the equation of the normal is,

$$\frac{Xb^{n-1}}{b^n}-\frac{Ya^{n-1}}{a^n}=\frac{ab^{n-1}}{b^n}-\frac{ba^{n-1}}{a^n}$$

$$or,\ \frac{X}{b}-\frac{Y}{a}=\frac{a}{b}-\frac{b}{a}$$

Replacing X and Y by x and y, we get

$$\frac{x}{b}-\frac{y}{a}=\frac{a}{b}-\frac{b}{a}.$$

This is the required equation.

Problem-02: Show that the normal at any point of the curve $x=a(\cos t+t\sin t),\ y=a(\sin t-t\cos t)$ is at a constant distance from the origin.

Solution: The equation of the curve is $x=a(\cos t+t\sin t),\ y=a(\sin t-t\cos t)$.

$$\text{Here}\quad x=f(t)=a(\cos t+t\sin t)\quad and\quad y=\varphi(t)=a(\sin t-t\cos t)\qquad\cdots(1)$$

Differentiating (1) with respect to t, we get

$$f'(t)=a(-\sin t+t\cos t+\sin t)=at\cos t$$

$$\text{and}\qquad \varphi'(t)=a(\cos t-\cos t+t\sin t)=at\sin t$$

The equation of the normal at t is,

$$Y-\varphi(t)=-\frac{1}{\dfrac{\varphi'(t)}{f'(t)}}\left(X-f(t)\right)$$

$$or,\ Y-a(\sin t-t\cos t)=-\frac{1}{\dfrac{at\sin t}{at\cos t}}\left(X-a(\cos t+t\sin t)\right)$$

$$or,\ Y-a(\sin t-t\cos t)=-\frac{\cos t}{\sin t}\left(X-a(\cos t+t\sin t)\right)$$

$$or,\ Y\sin t-a\sin^2 t+at\sin t\cos t=-X\cos t+a\cos^2 t+at\sin t\cos t$$

$$or,\ X\cos t+Y\sin t=a\sin^2 t+a\cos^2 t$$

$$or,\ X\cos t+Y\sin t=a\left(\sin^2 t+\cos^2 t\right)$$

$$or,\ X\cos t+Y\sin t=a$$

Replacing X and Y by x and y, we get

$$x\cos t+y\sin t=a$$

Now the distance of this normal from the origin is

$$d = \left| \frac{0 \cdot \cos t + 0 \cdot \sin t - a}{\sqrt{\cos^2 t + \sin^2 t}} \right| = a$$

Thus the distance of the normal from the origin is constant. **(Showed)**

Homework:

Problem-01: Find the equation of the normal at (x, y) to the curve $y = be^{-\frac{x}{a}}$. Also find normal at the point where it cuts y-axis.

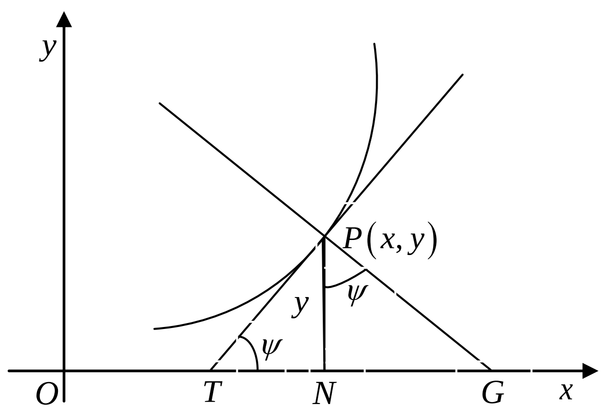
Problem-02: Find the tangent and the normal to the curve $x = e^t \sin t,\ y = e^{-t} \cos t$ at the point $t = \pi$.

Problem-03: Find the tangent and the normal to the curve $x = a\cos^3 t,\ y = b\sin^3 t$ at the point $t = \frac{\pi}{4}$.

Problem-04: Find the tangent and the normal to the curve $x = e^{-t} \cos t,\ y = e^t \sin t$ at the point $t = \pi$.

Question: Define Cartesian subtangent, subnormal, lengths of tangent and normal.

Answer: Let $P(x, y)$ be any point on the curve. Let the tangent and normal at P to the curve meet x -axis in T and G respectively. Let PN be the ordinate, then $PN = y$. Then TN is called the subtangent, NG is called the subnormal and the lengths of tangent intercepted between the point of contact and x -axis, i.e. PT is called the length of tangent. Similarly, the length of the normal intercepted between the point of contact and x -axis, i.e. PG is called the length of normal.



If the angle which the tangent at P makes with x -axis be ψ, then $\frac{dy}{dx} = \tan\psi$ and from the figure we get the following results:

(i) Subtangent $= TN = y\cot\psi = y\frac{1}{\tan\psi} = \frac{y}{\frac{dy}{dx}}$

(ii) Subnormal $= NG = y\tan\psi = y\frac{dy}{dx}$

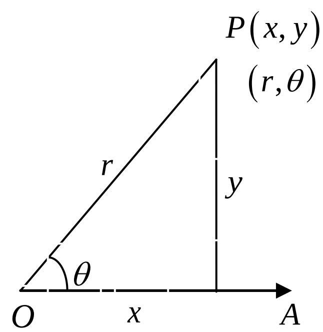
(iii) Length of the tangent= $PT = y\cos ec\psi = y\sqrt{1+\cot^2\psi} = y\frac{\sqrt{1+\tan^2\psi}}{\tan\psi} = \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$

(iv) Length of the normal= $PG = y\sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Polar co-ordinates: Let O be a fixed point and OA be a fixed straight line through O. This fixed point O is called pole and the fixed straight line OA is called initial line. Then the position of a point P on the plane is given by:

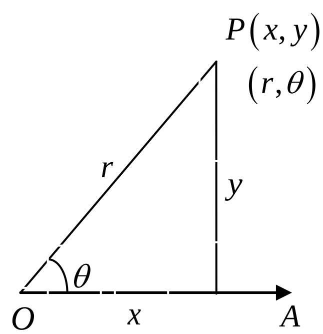
(i) its distance r from the pole O and

(ii) the inclination θ of OP to the initial line OA.



Here (r,θ) is called the polar co-ordinate of the point P, r is called the radius vector and θ is called the vectorial angle of the point P.

Relation between Polar and Cartesian co-ordinates: Let the Cartesian co-ordinates of the point P be (x,y). Also the polar co-ordinates of the same point be (r,θ).



From the figure we have

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

and $$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

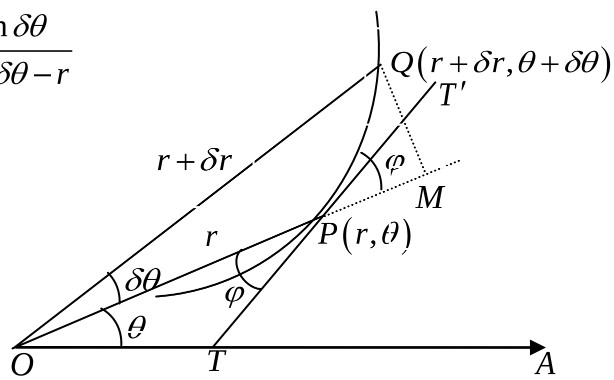
Angle between radius vector and tangent: Let $P(r,\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be two neighboring points on the curve $r = f(\theta)$. Let O is the pole and OA is the initial line. Join OP, OQ and PQ. From Q draw QM perpendicular to OP. TPT' is the tangent to the curve at the point P.

Let φ be the angle between the tangent PT and the radius vector OP.

Now as $Q \to P$, $\delta r \to 0$, $\delta\theta \to 0$, the secant $PQ \to$ the tangent PT and $\angle QPM \to \varphi$.

$$\therefore \tan\varphi = \tan \lim_{\delta\theta\to 0} \angle QPM = \lim_{\delta\theta\to 0} \tan\angle QPM = \lim_{\delta\theta\to 0} \frac{QM}{PM} = \lim_{\delta\theta\to 0} \frac{OQ\cdot\frac{QM}{OQ}}{OM-OP}$$

$$= \lim_{\delta\theta\to 0} \frac{OQ\cdot\frac{QM}{OQ}}{OQ\cdot\frac{OM}{OQ}-OP} = \lim_{\delta\theta\to 0} \frac{(r+\delta r)\sin\delta\theta}{(r+\delta r)\cos\delta\theta - r}$$



$$= \lim_{\delta\theta\to 0} \frac{(r+\delta r)\left\{\delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots\right\}}{(r+\delta r)\left\{1 - \frac{(\delta\theta)^2}{2!} + \cdots\right\} - r}$$

$$= \lim_{\delta\theta\to 0} \frac{r\delta\theta}{\delta r}$$, Neglecting higher powers of infinitesimals.

$$\therefore \tan\varphi = r\frac{d\theta}{dr} \;\; or,\; \cot\varphi = \frac{1}{r}\cdot\frac{dr}{d\theta}.$$

Question: If p be the length of perpendicular from pole, then show that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

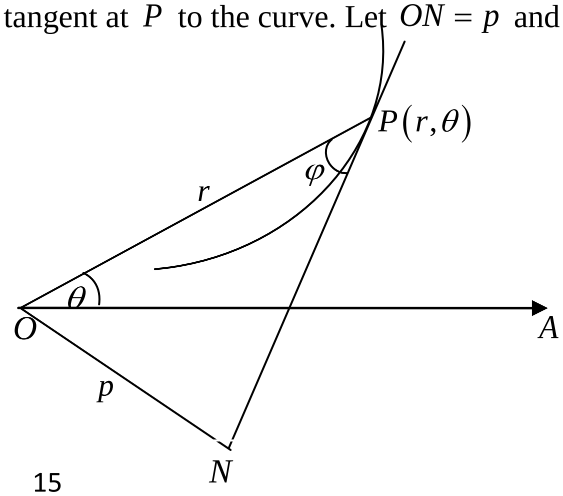
Answer: Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$. Let O is the pole and OA is the initial line. Join OP. From O draw ON perpendicular to the tangent at P to the curve. Let $ON = p$ and $\angle OPN = \varphi$.

From ΔOPN, we get

$$\frac{ON}{OP} = \sin\varphi$$

$$or,\; \frac{p}{r} = \sin\varphi$$

$$\therefore\; p = r\sin\varphi$$



which gives the length of the perpendicular from pole on the tangent.

If we require the value of p in terms of r and θ, then we can proceed as follows:

$$\frac{1}{p^2}=\frac{1}{r^2\sin^2\varphi}$$

$$or,\ \frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\varphi$$

$$or,\ \frac{1}{p^2}=\frac{1}{r^2}\left(1+\cot^2\varphi\right)$$

$$or,\ \frac{1}{p^2}=\frac{1}{r^2}\left\{1+\left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2\right\}$$

$$or,\ \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad \textbf{(Showed)}$$

Polar equation: An equation for a curve written in terms of the polar coordinates r and θ.

Pedal equation: The relation between p and r for a given curve is called its pedal equation, where p is the length of the perpendicular from the pole on the tangent to the curve at any point of it and r is the radius vector of this point.

Question: To find the pedal equation from Cartesian equation and polar equation.

Answer: 1st part (Cartesian equation): The equation of the tangent at any point (x,y) is

$$Y-y=\frac{dy}{dx}(X-x)$$

$$or,\ X\frac{dy}{dx}-Y+\left(y-x\frac{dy}{dx}\right)=0$$

Since p is the length of the perpendicular from the pole on the tangent to the curve, so

$$p=\left|\frac{0-0+y-x\dfrac{dy}{dx}}{\sqrt{\left(\dfrac{dy}{dx}\right)^2+(-1)^2}}\right|=\frac{x\dfrac{dy}{dx}-y}{\sqrt{\left(\dfrac{dy}{dx}\right)^2+1}} \qquad \cdots(1)$$

Also since r is the radius vector of the point (x,y), so

$$r^2 = x^2 + y^2 \qquad \cdots(2)$$

And let the Cartesian equation of the curve be

$$f(x, y) = 0 \qquad \cdots(3)$$

Now eliminating x and y from (1), (2) and (3), we get a relation p and r, which is the required pedal equation of the curve.

2nd part (Polar equation): Let the curve be $r = f(\theta)$ $\cdots(1)$

Also we know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \cdots(2)$$

Now eliminating θ from (1) and (2), we get a relation p and r, which is the required pedal equation of the curve.

Alternatively, Let the curve be $r = f(\theta)$ $\cdots(1)$

Also we know that

$$\tan\varphi = r\frac{d\theta}{dr} \qquad \cdots(2)$$

$$\text{and} \quad p = r\sin\varphi \qquad \cdots(3)$$

Now eliminating θ and φ from (1), (2) and (3), we get a relation p and r, which is the required pedal equation of the curve.

Problem-01: Find the pedal equation of the parabola $y^2 = 4a(x+a)$.

Solution: The equation of the parabola is $y^2 = 4a(x+a)$. $\cdots(1)$

Differentiating (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

The equation of the tangent at (x, y) is,

$$Y - y = \frac{dy}{dx}(X - x)$$

$$or,\ Y - y = \frac{2a}{y}(X - x)$$

$$or,\ Yy - y^2 = 2aX - 2ax$$

$$\therefore\ 2aX - Yy + (y^2 - 2ax) = 0 \qquad \cdots(2)$$

Since p is the length of perpendicular from $(0,0)$ on the tangent (2), so

$$p=\left|\frac{2a\cdot 0-0\cdot y+\left(y^2-2ax\right)}{\sqrt{(2a)^2+(-y)^2}}\right|=\frac{y^2-2ax}{\sqrt{4a^2+y^2}}=\frac{4a(x+a)-2ax}{\sqrt{4a^2+4a(x+a)}}$$

$$=\frac{2a(x+2a)}{\sqrt{4a(x+2a)}}=\sqrt{a(x+2a)}$$

$$or,\ p^2=a(x+2a)$$

$$\therefore\ x+2a=\frac{p^2}{a}$$

Also we have $r^2=x^2+y^2=x^2+4a(x+a)=(x+2a)^2=\left(\frac{p^2}{a}\right)^2$

$$\therefore\ r^2=\frac{p^4}{a^2}.$$

This is the required pedal equation.

Problem-02: Find the pedal equation of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Solution: The equation of the parabola is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Here $f(x,y)=x^{\frac{2}{3}}+y^{\frac{2}{3}}-a^{\frac{2}{3}}=0 \qquad \cdots(1)$

Differentiating (1) partially with respect to x and y, we get

$$f_x=\frac{2}{3}x^{-\frac{1}{3}} \text{ and } f_y=\frac{2}{3}y^{-\frac{1}{3}}$$

The equation of the tangent at (x,y) is,

$$(X-x)f_x+(Y-y)f_y=0$$

$$or,\ (X-x)\cdot\frac{2}{3}x^{-\frac{1}{3}}+(Y-y)\cdot\frac{2}{3}y^{-\frac{1}{3}}=0$$

$$or,\ (X-x)x^{-\frac{1}{3}}+(Y-y)y^{-\frac{1}{3}}=0$$

$$or,\ Xx^{-\frac{1}{3}}-x^{\frac{2}{3}}+Yy^{-\frac{1}{3}}-y^{\frac{2}{3}}=0$$

$$or,\ Xx^{-\frac{1}{3}}+Yy^{-\frac{1}{3}}=x^{\frac{2}{3}}+y^{\frac{2}{3}}$$

$$\therefore Xx^{-\frac{1}{3}}+Yy^{-\frac{1}{3}}=a^{\frac{2}{3}}. \qquad \cdots(2)$$

Since p is the length of perpendicular from $(0,0)$ on the tangent (2), so

$$p=\left|\frac{0\cdot x^{-\frac{1}{3}}+0\cdot y^{-\frac{1}{3}}-a^{\frac{2}{3}}}{\sqrt{\left(x^{-\frac{1}{3}}\right)^2+\left(y^{-\frac{1}{3}}\right)^2}}\right|=\frac{a^{\frac{2}{3}}}{\sqrt{x^{-\frac{2}{3}}+y^{-\frac{2}{3}}}}=\frac{a^{\frac{2}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}$$

$$= \frac{a^{\frac{2}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}}}$$

$$\therefore\ p = a^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}} \qquad \cdots(3)$$

Also we have $r^2 = x^2 + y^2 = \left(x^{\frac{2}{3}}\right)^3 + \left(y^{\frac{2}{3}}\right)^3$

$$= \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3 - 3x^{\frac{2}{3}} y^{\frac{2}{3}} \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}}\right)^3 - 3x^{\frac{2}{3}} y^{\frac{2}{3}} \left(a^{\frac{2}{3}}\right)$$

$$= a^2 - 3\left(a^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}\right)^2$$

$$\therefore\ r^2 = a^2 - 3p^2.$$

This is the required pedal equation.

Problem-03: Show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$.

Solution: The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ $\cdots(1)$

Here $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ $\cdots(2)$

Differentiating (2) partially with respect to x and y, we get

$$f_x = \frac{2x}{a^2} \text{ and } f_y = \frac{2y}{b^2}$$

The equation of the tangent at (x, y) is,

$$(X - x) f_x + (Y - y) f_y = 0$$

$$or,\ (X - x)\cdot\frac{2x}{a^2} + (Y - y)\cdot\frac{2y}{b^2} = 0$$

$$or,\ \frac{Xx}{a^2} - \frac{x^2}{a^2} + \frac{Yy}{b^2} - \frac{y^2}{b^2} = 0$$

$$or,\ \frac{Xx}{a^2} + \frac{Yy}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\therefore \frac{Xx}{a^2} + \frac{Yy}{b^2} = 1. \qquad \cdots(3)$$

Since p is the length of perpendicular from $(0,0)$ on the tangent (3), so

$$p=\left|\frac{0+0-1}{\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}}\right|=\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}}}$$

$$or,\ p^2=\frac{1}{\frac{x^2}{a^4}+\frac{y^2}{b^4}}$$

$$\therefore\ \frac{x^2}{a^4}+\frac{y^2}{b^4}-\frac{1}{p^2}=0 \qquad \cdots(4)$$

Also we have

$$r^2=x^2+y^2$$

$$\therefore x^2+y^2-r^2=0 \qquad \cdots(5)$$

Now the pedal equation of the given ellipse will be obtained by eliminating x^2 and y^2 from (1) , (4) and (5). Eliminating x^2 and y^2 from (1) , (4) and (5), we get

$$\begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & -1 \\ \frac{1}{a^4} & \frac{1}{b^4} & \frac{-1}{p^2} \\ 1 & 1 & -r^2 \end{vmatrix}=0$$

$$\therefore\ \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}. \qquad \textbf{(Showed)}$$

Problem-04: Show that the pedal equation of the cardioid $r=a(1-\cos\theta)$ with respect to pole is $2ap^2=r^3$.

Solution: The given curve is $r=a(1-\cos\theta)$. $\cdots(1)$

Differentiating (1) with respect to θ, we get

$$\frac{dr}{d\theta}=a\sin\theta$$

We know,

$$\tan\varphi=r\frac{d\theta}{dr}$$

$$or,\quad \tan\varphi=r\frac{1}{\frac{dr}{d\theta}}$$

$$or,\quad \tan\varphi=a(1-\cos\theta)\cdot\frac{1}{a\sin\theta}$$

$$or,\quad \tan\varphi = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$or,\quad \tan\varphi = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$or,\quad \tan\varphi = \tan\frac{\theta}{2}$$

$$\therefore\ \varphi = \frac{\theta}{2}.$$

Also we know that

$$p = r\sin\varphi$$

$$or,\ p = r\sin\frac{\theta}{2}$$

$$or,\ p^2 = r^2\sin^2\frac{\theta}{2}$$

$$or, 2ap^2 = r^2a\cdot 2\sin^2\frac{\theta}{2}$$

$$or, 2ap^2 = r^2a\cdot(1-\cos\theta)$$

$$\therefore\ 2ap^2 = r^3. \qquad \textbf{(Showed)}$$

Problem-05: Find the pedal equation of $r^m = a^m\cos m\theta$.

Solution: The given curve is $r^m = a^m\cos m\theta$. $\cdots(1)$

Taking logarithm on both side, we have

$$m\ln r = m\ln a + \ln\cos m\theta \qquad \cdots(2)$$

Differentiating (2) with respect to θ, we get

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{\sin m\theta}{\cos m\theta}\cdot m$$

$$or,\ \frac{m}{r}\frac{dr}{d\theta} = -m\frac{\sin m\theta}{\cos m\theta}$$

$$\therefore \frac{1}{r}\frac{dr}{d\theta} = -\tan m\theta$$

We know,

$$\cot\varphi = \frac{1}{r}\frac{dr}{d\theta}$$

$$or,\ \cot\varphi = -\tan m\theta$$

$$or,\ \cot\varphi = \cot\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore\ \varphi = \frac{\pi}{2} + m\theta$$

Also we know that

$$\begin{aligned} p &= r\sin\varphi \\ or,\ p &= r\sin\left(\frac{\pi}{2}+m\theta\right) \\ or,\ p &= r\cos m\theta \\ or,\ p &= r\cdot\frac{r^m}{a^m} \\ \therefore\ r^{m+1} &= a^m p\,. \end{aligned}$$

This is the required pedal equation.

Homework:

Problem-01: Find the pedal equation of the circle $x^2+y^2=2ax$.

Problem-02: Find the pedal equation of the curve $x^2-y^2=a^2$.

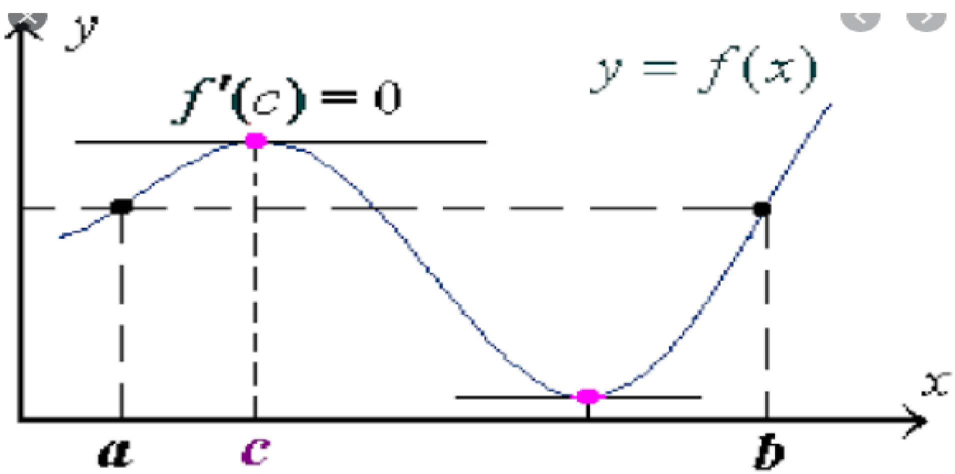
Problem-03: Find the pedal equation of $r^n=a^n\sin n\theta$.

Theorem-07: State and prove Rolle's Theorem.

Statement: If a function $f(x)$ is such that

1. it is continuous in the closed interval $[a,b]$
2. it is differentiable in the open interval (a,b)
3. $f(a)=f(b)$

then there exists at least one value $'c'$ in the interval (a,b) such that $f'(c)=0$.



Proof: Since the function $f(x)$ is continuous on the closed interval $[a,b]$ and $f(a)=f(b)$, so when $x>a$ then the function $f(x)$ should either increase or decrease unless $f(x)$ is constant.

If the function $f(x)$ is constant function on $[a,b]$ then we have

$$f'(x)=0 \quad \forall\, x\in[a,b].$$

In particular $f'(c)=0$ when $c\in(a,b)$.

Hence the theorem is true for any $c\in(a,b)$.

Again suppose the function $f(x)$ increases when $x>a$ and $f(a)=f(b)$, so the function $f(x)$ increase at the point C where $x=c$ such that $a<c<b$ and the function $f(x)$ decreases thereafter.

At the point C where $x=c$, there is a maximum value of the function $f(x)$, which is $f(c)$. So by the definition of maximum value, we have

$$f(c+h)-f(c)<0 \text{ and } f(c-h)-f(c)<0, \text{ where } h>0 \text{ and } h\to 0$$

$$\Rightarrow \frac{f(c+h)-f(c)}{h}<0 \text{ and } \frac{f(c-h)-f(c)}{-h}>0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}<0 \text{ and } \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}>0$$

$$\Rightarrow Rf'(c)<0 \text{ and } Lf'(c)>0.$$

Since $f(x)$ derivable in the open interval (a,b), so by the definition of differentiability we have

$$Rf'(c)=Lf'(c)$$

Here this statement will be valid if $f'(c)=0$. Hence the theorem is true for any $c\in(a,b)$. **(Proved)**

Algebraic interpretation of Rolle's Theorem: If $f(x)$ be a polynomial in x and $x=a$, $x=b$ be the two roots of the equation $f(x)=0$, then from Rolle's Theorem we find that at least one root of the equation $f'(x)=0$ lies between a and b.

Theorem-08: State and prove Mean value Theorem.

Statement: If a function $f(x)$ is such that

1. it is continuous in the closed interval $[a,b]$
2. it is differentiable in the open interval (a,b),

then there exists at least one value $'c'$ in the interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Proof: Consider the function $\varphi(x)=f(x)+Ax$ $\cdots(1)$

where A is a constant to be determined such that

$$\varphi(a)=\varphi(b)$$

$$\therefore\ f(a)+Aa=f(b)+Ab$$

$$i.e.\ f(b)-f(a)=-A(b-a)$$

$$\therefore\ -A=\frac{f(b)-f(a)}{b-a} \qquad \cdots(2)$$

Now $f(x)$ is given to be continuous in the closed interval $[a,b]$ and is differentiable in the open interval (a,b).

Also, A being a constant, Ax is also continuous in the closed interval $[a,b]$ and is differentiable in the open interval (a,b).

So $\varphi(x)=f(x)+Ax$ is

1. continuous in the closed interval $[a,b]$
2. differentiable in the open interval (a,b) and also
3. $\varphi(a)=\varphi(b)$.

Therefore, $\varphi(x)$ satisfies all the three conditions of Rolle's Theorem. So there must exist at least one value $'c'$ in the interval (a,b) such that

$$\varphi'(c)=0$$

$$or,\ f'(c)+A=0$$

$$\therefore f'(c)=-A \qquad \cdots(3)$$

Now from (2) and (3) we get

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$ **(Proved)**

Problem-01: Verify Rolle's Theorem for the function $f(x)=\frac{\sin x}{e^x}$ in $[0,\pi]$.

Solution: The given function is $f(x)=\frac{\sin x}{e^x}$

i.e. $$f(x)=e^{-x}\sin x \qquad \cdots(1)$$

Differentiating (1) with respect to x, we get

$$f'(x)=e^{-x}(\cos x-\sin x)$$

Since $\sin x$ and e^x are continuous for all x and $e^x\neq 0$ for any finite value of x, so $f(x)$ is continuous in $[0,\pi]$ and $f'(x)$ is derivable for all values of x in $(0,\pi)$.

Also $f(\pi)=e^{-\pi}\sin\pi=0$ and $f(0)=e^0\sin 0=0$ so that $f(\pi)=f(0)$.

Since $f(x)$ satisfies all the three conditions of Rolle's Theorem, so there exists at least one number 'c' in $(0,\pi)$ such that

$$f'(c)=0$$
$$or,\ e^{-c}(\cos c-\sin c)=0$$
$$or,\ \cos c-\sin c=0,\quad [\because e^{-c}\neq 0]$$
$$or,\ \tan c=1$$
$$\therefore\ c=\frac{\pi}{4}\in(0,\pi).$$

Hence the Rolle's Theorem is verified.

Problem-02: Verify the truth of Rolle's Theorem for the function $f(x)=x^2-3x+2$ in $[1,2]$.

Solution: The given function is $f(x)=x^2-3x+2$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$f'(x)=2x-3$$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the interval $(-\infty,\infty)$.

Therefore, $f(x)$ is continuous in the interval $[1,2]$ and differentiable in the interval $(1,2)$.

Here $f(1)=1^2-3\cdot1+2=1-3+2=0$

and $f(2)=2^2-3\cdot2+2=4-6+2=0$

$$\therefore\ f(1)=f(2).$$

Since $f(x)$ satisfies all the three conditions of Rolle's Theorem, so there exists at least one number 'c' in $(1,2)$ such that

$$f'(c)=0$$
$$or,\ 2c-3=0$$
$$\therefore\ c=\frac{3}{2}\in(1,2).$$

Hence the Rolle's theorem is verified.

Problem-03: Verify the truth of Rolle's Theorem for the function $f(x)=x^2-8x+15$.

Solution: The given function is $f(x)=x^2-8x+15$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$f'(x)=2x-8.$$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the interval $(-\infty,\infty)$.

Now $f(x)=0$ gives,

$$x^2-8x+15=0$$
$$or,\ (x-3)(x-5)=0$$
$$\therefore\ x=3,5.$$

i.e. $f(3)=0$ and $f(5)=0$.

Thus we find that $f(3)=f(5)$.

Also, $f(x)$ is continuous in the interval $[3,5]$ and differentiable in the interval $(3,5)$.

Since $f(x)$ satisfies all the three conditions of Rolle's Theorem, so there exists at least one number 'c' in $(3,5)$ such that

$$f'(c)=0$$
$$or,\ 2c-8=0$$
$$\therefore\ c=4\in(3,5).$$

Hence the Rolle's theorem is verified.

Problem-04: Justify the validity of the Mean Value Theorem for the function $f(x)=3+2x-x^2$ in the interval $[0,1]$.

Solution: The given function is $f(x)=3+2x-x^2$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$f'(x)=2-2x$$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable everywhere.

Therefore, $f(x)$ is continuous in the interval $[0,1]$ and differentiable in the interval $(0,1)$. Thus the hypotheses of the Mean Value Theorem with $a=0$ and $b=1$.

Here $f(a)=f(0)=3+2\cdot 0-0=3$

and $f(b)=f(1)=3+2\cdot 1-1^2=3+2-1=4$

Now by Mean Value Theorem there exists at least one number 'c' in the interval $(0,1)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$
$$or,\ 2-2c=\frac{4-3}{1-0}$$
$$or,\ 2-2c=1$$
$$or,\ c=\frac{1}{2}$$
$$\therefore\ c=\frac{1}{2}\in(0,1).$$

Hence the Mean Value Theorem is valid for the given function.

Problem-05: Justify the validity of the Mean Value Theorem for the function $f(x)=(x-1)(x-2)(x-3)$ in the interval $[0,4]$.

Solution: The given function is $f(x)=(x-1)(x-2)(x-3)$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$f'(x)=3x^2-12x+11$$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable everywhere.

Therefore, $f(x)$ is continuous in the interval $[0,4]$ and differentiable in the interval $(0,4)$. Thus the hypotheses of the Mean Value Theorem with $a=0$ and $b=4$.

Here $f(a) = f(0) = (0-1)(0-2)(0-3) = -6$

and $f(b) = f(4) = (4-1)(4-2)(4-3) = 6$

Now by Mean Value Theorem there exists at least one number 'c' in the interval $(0,4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$or,\ 3c^2 - 12c + 11 = \frac{6+6}{4-0}$$

$$or,\ 3c^2 - 12c + 11 = 3$$

$$or,\ 3c^2 - 12c + 8 = 0$$

$$or,\ c = 3.15,\ 0.85$$

$$\therefore\ c = 3.15,\ 0.85 \in (0,4).$$

Hence the Mean Value Theorem is valid for the given function.

Homework:

Problem-01: Verify Rolle's Theorem for the function $f(x) = e^x(\sin x + \cos x)$ in $\left[\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Problem-02: Verify Rolle's Theorem for the function $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$.

Problem-03: Verify Rolle's Theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in $[0,2]$.

Problem-04: Justify the validity of the Mean Value Theorem for the function $f(x) = x^2 - 3x + 2$ in the interval $[-2,3]$.

Increasing function: A function $y = f(x)$ defined on an interval (a,b) where $a < b$, is called an increasing function over the interval if $f(a) < f(b)$.

Example: $y = x^2,\ 0 \le x \le 5$is an increasing function.

Decreasing function: A function $y = f(x)$ defined on an interval (a,b) where $a < b$, is called a decreasing function over the interval if $f(a) > f(b)$.

Example: $y = \frac{1}{x},\ 1 \le x \le 5$ is a decreasing function.

Concavity and Convexity (with respect to a given point):

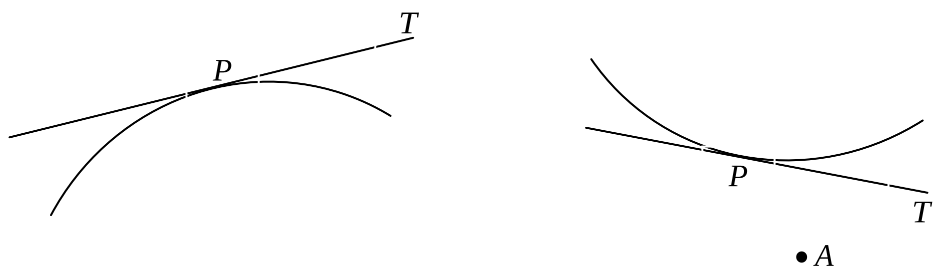


Fig. (i) **Fig. (ii)**

Let PT be the tangent to a curve at P. Then the curve at P is said to be concave or convex with respect to a point A not lying on PT, according as a small portion of the curve in the immediate neighbourhood of P (on both side of it) lies entirely on the same side of PT as A [as in Fig. (i)], or on opposite sides of PT with respect to A [as in Fig. (ii)].

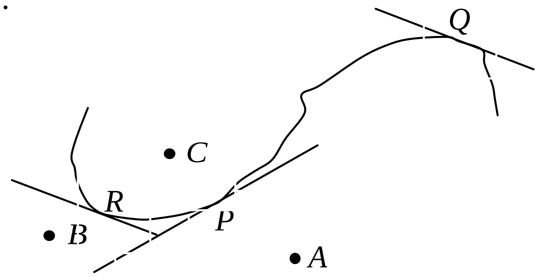


Fig. (iii)

Thus, in Fig.(iii) the curve at P is convex with respect to A, and concave with respect to B or C. The curve at Q is concave with respect to A. Again, the curve at R is convex to B and concave to C.

Point of Inflection:

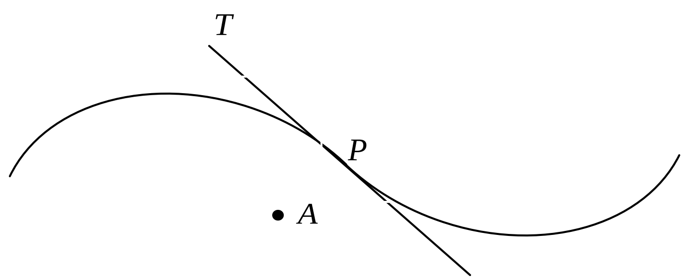
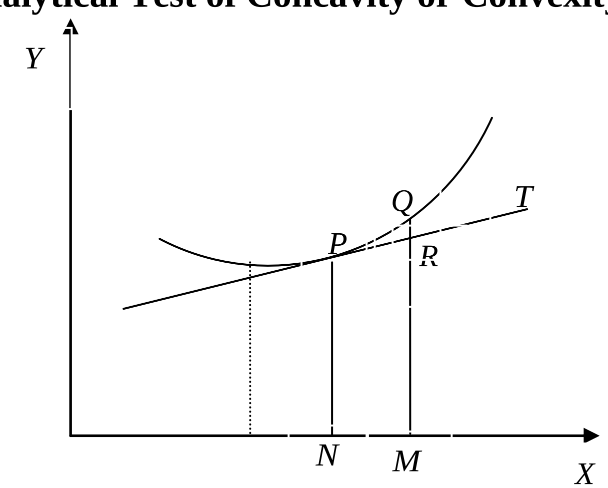


Fig. (i)

In some curves, at a particular point P on it, the tangent line crosses the curve, as in Fig. (i). At this point, clearly the curve, on one side of P, is convex, and on the other side it is concave with respect to any point A (not lying on the tangent line). Such a point on a curve is defined to be a point of inflection (or a point of contrary flexure).

Analytical Test of Concavity or Convexity (with respect to the x-axis):



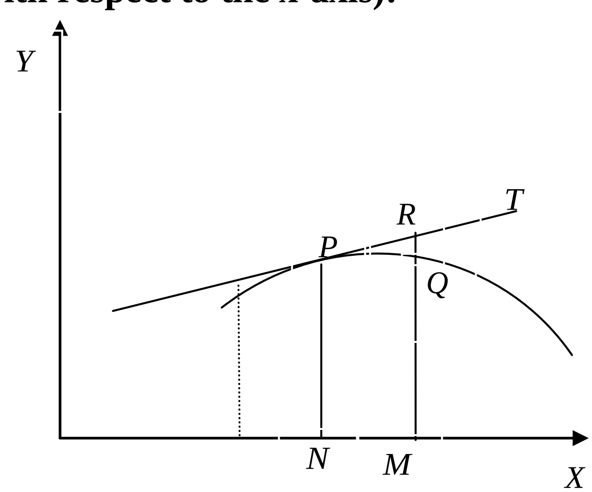


Fig. (i) **Fig. (ii)**

Let $P(x,y)$ be a point on the curve $y=f(x)$, Q a neighbouring point whose abscissa is $x+h$ (h being small, positive or negative). Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R.

The equation to PT is

$$Y-y=f'(x)(X-x)$$

and abscissa X of R being $x+h$, its ordinate

$$RM=Y=y+hf'(x).$$

Also the ordinate of Q is

$$QM=f(x+h)$$

$$=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h),\ 0<\theta<1$$

$$\therefore QM-RM=\frac{h^2}{2!}f''(x+\theta h). \qquad \cdots(1)$$

Now assuming $f''(x)$ to be continuous at P and $\neq 0$ there, $f''(x+\theta h)$ has the same sign as that of $f''(x)$ when $|h|$ is sufficiently small.

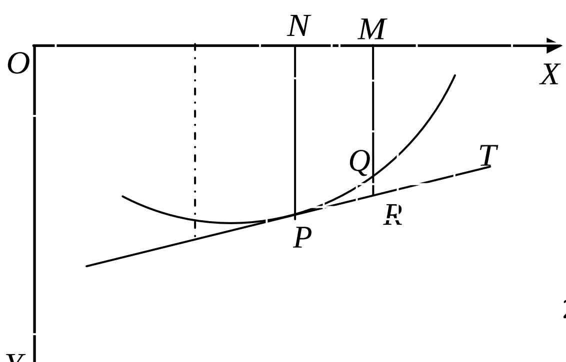
Hence from (1), $QM-RM$ has the same sign as that of $f''(x)$, for positive as well as negative values of h, provided it is sufficiently small in magnitude.

Firstly, let the ordinate PN or y be positive.

If $f''(x)$ is positive, from (1) $QM>RM$ for Q on either side of P in its neighborhood, and so the curve in the neighborhood of P (on either side of it) is entirely above the tangent i.e. on the side opposite to the foot N on the x-axis of the ordinate PN, as in Fig.(i). Hence, the curve at P is convex with respect to the x-axis.

Again if $f''(x)$ is negative, from (1) $QM<RM$ for Q on either side of P, and so the curve in the near P is entirely below the tangent i.e. on the same side of N, as in Fig.(ii). Hence, the curve at P is concave with respect to the x-axis.

Secondly, let the ordinate PN or y be negative.



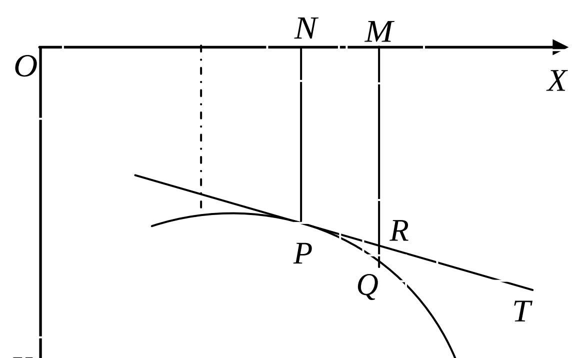


Fig. (i) **Fig. (ii)**

If $f''(x)$ is positive, from (1) $QM > RM$ for Q on either side of P, and as both are negative, QM is numerically less than RM, as in Fig.(i). The curve, therefore, at P lies on the same side as N is with respect to the tangent PT. Hence, the curve at P is concave with respect to the x-axis.

Again if $f''(x)$ is negative, we similarly get the curve at P convex with respect to the x-axis, as in Fig. (ii).

Combining the two cases, we get the following criterion for convexity or concavity of a curve at a point with respect to the x- axis:

If $y\dfrac{d^2y}{dx^2}$ is positive at P, the curve at P is convex to the x-axis.

If $y\dfrac{d^2y}{dx^2}$ is negative at P, the curve at P is concave to the x-axis.

Analytical condition for point of inflection: Let $P(x,y)$ be a point on the curve $y = f(x)$, Q a neighboring point whose abscissa is $x+h$ (h being small, positive or negative). Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R.

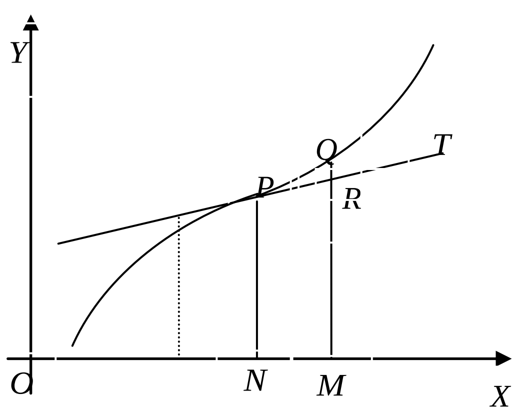


Fig.(i)

The equation to PT is

$$Y - y = f'(x)(X - x)$$

and abscissa X of R being $x+h$,its ordinate

$$RM = Y = y + hf'(x).$$

Let $f''(x) = 0$ at P, and $f'''('x) \neq 0$.

Then the ordinate of Q is

$$QM = f(x+h)$$
$$= f(x) + hf'(x) + \frac{h^3}{3!} f'''(x+\theta h),\ 0<\theta<1$$
$$\therefore QM - RM = \frac{h^3}{3!} f'''(x+\theta h),$$

and the sign of this for sufficiently small $|h|$ is the same as that of $\frac{h^3}{3!} f'''(x)$, which has got opposite signs for positive and negative values of h, whatever be the sign of $f'''(x)$ at P. Thus, near P the curve is above the tangent on one side of P, and below the tangent on the other side, as in Fig. (i). Hence, P is a point of inflection.

Thus, the condition that P is a point of inflection on the curve $y = f(x)$ is that, at P,

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0.$$

Problem-01: Examine the curve $y = \sin x$ regarding its convexity or concavity to the x- axis, and determine its point of inflection, if any.

Solution: The given curve is, $y = \sin x$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \cos x$$
$$\therefore \frac{d^2y}{dx^2} = -\sin x$$
$$\therefore \frac{d^3y}{dx^3} = -\cos x$$
$$\therefore y\frac{d^2y}{dx^2} = -\sin^2 x \qquad \cdots(2)$$

Here equation (2) is negative for all values of x excepting those which make $\sin x = 0$, i.e. $x = n\pi$, n being any integer , positive or negative.

Thus, the curve is concave to the x-axis at every point, excepting at points where it crosses the x-axis.

At these points, given by $x = n\pi$, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0.$$

Hence, those points where the curve crosses the x-axis are points of inflection.

Problem-02: Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin.

Solution: The given curve is, $y^3 = 8x^2$ $\cdots(1)$

The curve can be written as, $y = 2x^{\frac{2}{3}}$ $\cdots(2)$

Differentiating (2) with respect to x, we get

$$\frac{dy}{dx} = \frac{4}{3}x^{-\frac{1}{3}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{4}{9}x^{-\frac{4}{3}}$$

$$\therefore y\frac{d^2y}{dx^2} = -\frac{8}{9}x^{-\frac{2}{3}} \qquad \cdots(3)$$

Here equation (3) is negative for all values of x excepting at the origin. Thus, the curve is concave to the foot of the ordinate everywhere except at the origin.

Problem-03: Prove that $\left(a-2, -2/e^2\right)$ is a point of inflection of the curve $y = (x-a)e^{x-a}$.

Solution: The given curve is, $y = (x-a)e^{x-a}$ $\cdots(1)$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = e^{x-a} + (x-a)e^{x-a} = (1+x-a)e^{x-a}$$

$$\therefore \frac{d^2y}{dx^2} = e^{x-a} + (1+x-a)e^{x-a} = (2+x-a)e^{x-a}$$

$$\therefore \frac{d^3y}{dx^3} = e^{x-a} + (2+x-a)e^{x-a} = (3+x-a)e^{x-a} \qquad \cdots(2)$$

At $\left(a-2, -2/e^2\right)$, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = e^{-2} \neq 0.$$

Hence the point $\left(a-2, -2/e^2\right)$ is a point of inflection. **(Showed)**

Homework:

Problem-01: Prove that the curve $y = e^x$ is convex to the x-axis at every point.

Problem-02: Prove that the curve $y = \cos^{-1} x$ is everywhere concave to the y-axis excepting where it crosses the y-axis.

Problem-03: Show that the curve $y = \ln x$ is convex to the foot of the ordinate in the range $0 < x < 1$, and concave where $x > 1$. Show also that the curve is convex everywhere to the y-axis.

Problem-04: Prove that the origin is a point of inflection of the curves $y = x^2 \ln(1-x)$ and $y = x\cos 2x$.

Maximum (relative maximum or local maximum) value of a function: A function $f(x)$ is said to have a maximum value at $x = a$ if $f(a) > f(x)$ for all values of x in the open interval $(a-h, a+h)$, where h is a small positive number.

Minimum (relative minimum or local minimum) value of a function: A function $f(x)$ is said to have a minimum value at $x = a$ if $f(a) < f(x)$ for all values of x in the open interval $(a-h, a+h)$, where h is a small positive number.

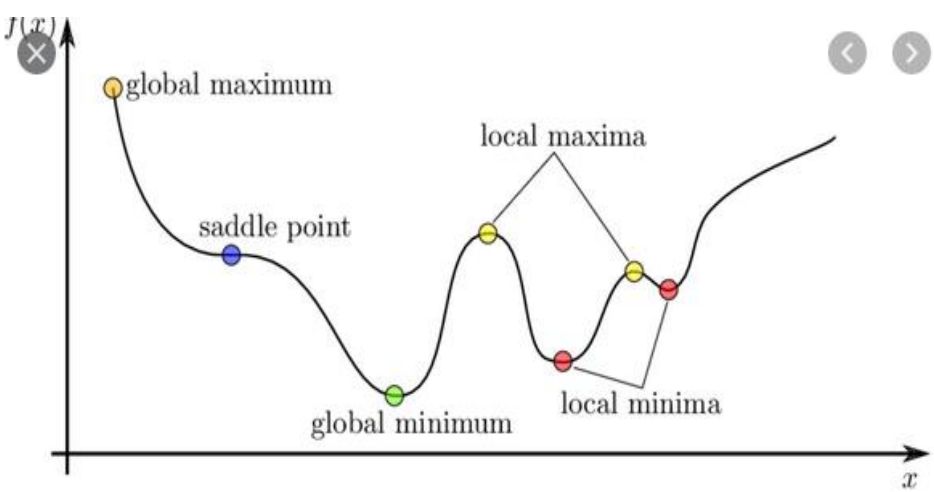
Global (absolute) maximum value of a function: A function $f(x)$ is said to have a global maximum value at $x = a$ if $f(a) > f(x)$ for all values of x in the domain of the function.

Global (absolute) minimum value of a function: A function $f(x)$ is said to have a global minimum value at $x = a$ if $f(a) < f(x)$ for all values of x in the domain of the function.

Critical point: A point on a curve in which derivative is zero or function is not differentiable.

Stationary point: A stationary point is a point on the curve where gradient of a function is zero. If gradient of the curve changes sign at stationary point then it called turning point otherwise horizontal Inflection.

Saddle point: A saddle point is a point in the domain of a function which is a stationary point but not a local extremum.



Theorem-07: State and prove Fermat's Theorem for finding maxima and minima.

OR

State and prove interior extremum theorem.

Statement: If $f(x)$ has maximum or minimum value at $x=a$ and $f'(a)$ exists, then $f'(a)=0$.

Proof: Let us consider that $f(x)$ is a function of x with the assumption that $f(x)$ is continuous and derivable and finite for all values of x in the neighborhood of $x=a$.

At $x=a$ the value of $f(x)$ is $f(a)$.

Consider two values of x, namely $a+h$ and $a-h$ in the neighborhood and on either side of $x=a$, h being very small.

By Taylor's theorem we get,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots+R_n$$

where R_n is the remainder after n terms.

Here h is very small so that by neglecting 2nd and higher degree term we get,

$$f(a+h)=f(a)+hf'(a)$$

$$or,\ f(a+h)-f(a)=hf'(a) \qquad \cdots(1)$$

Again replacing h by $-h$ in (1), we get

$$f(a-h)-f(a)=-hf'(a) \qquad \cdots(2)$$

We know that for maximum or minimum the sign of $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be the same. So from (1) and (2), we conclude that if $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same, then $f'(a)=0$ otherwise they have different signs.

Hence, if $f(x)$ has maximum or minimum value at $x=a$ and $f'(a)$ exists, then $f'(a)=0$. **(Proved)**

Theorem-08: State and prove the sufficient conditions for the existence of extreme values of a function.

Statement: If $f(x)$ is continuous at $x=a$, $f'(a)=0$ and $f''(a)\neq 0$ then

(1) $f(x)$ has maximum value $f(a)$ at $x=a$, if $f''(a)<0$

(2) $f(x)$ has minimum value $f(a)$ at $x=a$, if $f''(a)>0$.

Again if $f'(a)=0, f''(a)=0,\cdots, f^n(a)\neq 0$ then

(3) for maximum and minimum, n must be an even number and $f^n(a)$ should be negative for maximum and positive for minimum.

Proof: Let us consider that $f(x)$ is a function of x with the assumption that $f(x)$ is continuous and derivable and finite for all values of x in the neighborhood of $x=a$.

At $x=a$ the value of $f(x)$ is $f(a)$.

Consider two values of x, namely $a+h$ and $a-h$ in the neighborhood and on either side of $x=a$, h being very small.

By Taylor's theorem we get,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\frac{h^4}{4!}f^{iv}(a)+\cdots+R_n \qquad \cdots(1)$$

where R_n is the remainder after n terms.

Here h is very small so that by neglecting 3rd and higher degree terms we get,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)$$

$$or,\ f(a+h)-f(a)=hf'(a)+\frac{h^2}{2!}f''(a)$$

$$\therefore\ f(a+h)-f(a)=\frac{h^2}{2!}f''(a)\qquad [\because f'(a)=0]\qquad \cdots(2)$$

Again replacing h by $-h$ in (2), we get

$$f(a-h)-f(a)=\frac{h^2}{2!}f''(a)\qquad \cdots(3)$$

We know that for maximum or minimum the sign of $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be the same. From (2) and (3), since $h^2>0$, so we conclude that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same.

When $f(a)$ is maximum value then from (1) and (2), by the definition we have

$$f(a+h)-f(a)<0 \Rightarrow \frac{h^2}{2!}f''(a)<0\ \ \therefore f''(a)<0$$

and $$f(a-h)-f(a)<0 \Rightarrow \frac{h^2}{2!}f''(a)<0\ \ \therefore f''(a)<0$$

Hence $f(x)$ has maximum value $f(a)$ at $x=a$, if $f''(a)<0$.

Again when $f(a)$ is minimum value then from (1) and (2), by the definition we have

$$f(a+h)-f(a)>0 \Rightarrow \frac{h^2}{2!}f''(a)>0\ \ \therefore f''(a)>0$$

and $$f(a-h)-f(a)>0 \Rightarrow \frac{h^2}{2!}f''(a)>0\ \ \therefore f''(a)>0$$

Hence $f(x)$ has minimum value $f(a)$ at $x=a$, if $f''(a)>0$.

Again if $f'(a)=0, f''(a)=0$ then neglecting 4^{th} and higher degree terms we get from (1),

$$f(a+h)-f(a)=\frac{h^3}{3!}f'''(a)\qquad \cdots(4)$$

Again replacing h by $-h$ in (4), we get

$$f(a-h)-f(a)=-\frac{h^3}{3!}f'''(a)\qquad \cdots(5)$$

For maximum or minimum the sign of $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be the same. So from (4) and (5) if we conclude that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same sign then $f'''(a)=0$ otherwise they have different signs.

Again neglecting 5^{th} and higher degree terms we get from (1),

$$f(a+h)-f(a)=\frac{h^4}{4!}f^{iv}(a)\qquad \cdots(6)$$

Again replacing h by $-h$ in (6), we get

$$f(a-h)-f(a)=\frac{h^4}{4!}f^{iv}(a)\qquad \cdots(7)$$

For maximum or minimum the sign of $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be the same. So from (6) and (7), since $h^4>0$, so we conclude that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same sign.

When $f(a)$ is maximum value then from (6) and (7), by the definition we have

$$f(a+h)-f(a)<0 \Rightarrow \frac{h^4}{4!} f^{iv}(a)<0 \quad \therefore f^{iv}(a)<0$$

and $$f(a-h)-f(a)<0 \Rightarrow \frac{h^4}{4!} f^{iv}(a)<0 \quad \therefore f^{iv}(a)<0$$

Hence $f(x)$ has maximum value $f(a)$ at $x=a$, if $f^{iv}(a)<0$.

Again when $f(a)$ is minimum value then from (6) and (7), by the definition we have

$$f(a+h)-f(a)>0 \Rightarrow \frac{h^4}{4!} f^{iv}(a)>0 \quad \therefore f^{iv}(a)>0$$

and $$f(a-h)-f(a)>0 \Rightarrow \frac{h^4}{4!} f^{iv}(a)>0 \quad \therefore f^{iv}(a)>0$$

Hence $f(x)$ has minimum value $f(a)$ at $x=a$, if $f^{iv}(a)>0$.

Proceeding in this way we find that in general:

If $f'(a)=0, f''(a)=0, \cdots, f^{n-1}(a)=0$ *and* $f^n(a)\neq 0$ then for maximum and minimum, n must be an even number and $f^n(a)$ should be negative for maximum and positive for minimum. **(Proved)**

Working rule for finding maxima and minima:

If function $f(x)$ be given, find $f'(x)$ and equate it to zero. Solve this equation for x.

Let its roots be $a_1, a_1, a_3, \ldots \quad \ldots \ldots$

Find $f''(x)$ and hence find $f''(a_1), f''(a_2), \ldots \ldots \ldots \ldots$

- ✓ If $f''(a_1)$ is negative we have a maximum at $x=a_1$.
- ✓ If $f''(a_1)$ is positive we have a minimum at $x=a_1$.
- ✓ If $f''(a_1)=0$ find $f'''(x)$ and then $f'''(a_1)$.
- ✓ If $f'''(a_1)\neq 0$, then there is neither maxima nor minima at $x=a_{1.}$

If $f'''(a_1)=0$; find $f^{iv}(x)$ and then $f^{iv}(a_1)$.

If $f^{iv}(a_1)$ is negative, then $f(x)$ is maximum and if $f^{iv}(a_1)$ is positive, then $f(x)$ is minimum at $x=a_1$.

If $f^{iv}(a_1)=0$ Then find $f^{v}(a_1)$ and so on.

- ✓ $f'(a_1)=f''(a_1)=\ldots\ldots\ldots\ldots\ldots = f^{n-1}(a_1)=0$ *and* $f^n(a_1)\neq 0$
- ✓ if n be odd, then there is neither maxima nor minima at $x=a_1$
- ✓ if n be even and if $f^n(a_1)$ is negative then $f(x)$ is maximum at $x=a_1$ and
- ✓ if $f^n(a_1)$ is positive then $f(x)$ is minimum at $x=a_1$.

Problem-01: Find the maximum and minimum values of $y=x^5-5x^4+5x^3-10$.

Solution: The given function is,

$$y=f(x)=y=x^5-5x^4+5x^3-10 \qquad \cdots(1)$$

Differentiating with respect to x we get,

$$f'(x)=5x^4-20x^3+15x^2 \qquad \cdots(2)$$

We know that for maximum and minimum values,

$$f'(x)=0$$
$$\Rightarrow 5x^4-20x^3+15x^2=0$$

$$\Rightarrow x^4 - 4x^3 + 3x^2 = 0$$
$$\Rightarrow x^2\left(x^2 - 4x + 3\right) = 0$$
$$\Rightarrow x^2\left(x-3\right)\left(x-1\right) = 0$$
$$\therefore x = 0,\, 1,\, 3$$

Again, differentiating eq.(2) with respect to x we get,

$$f''(x) = 20x^3 - 60x^2 + 30x$$

For $x = 1$ we get,

$$f''(1) = 20 - 60 + 30 = -10 < 0$$

Therefore, the given function is maximum at $x = 1$.

The maximum value is,

$$f\left(1\right) = 1 - 5 + 5 - 10 = -9 \qquad (Ans.)$$

For $x = 3$ we get,

$$f''(3) = 540 - 540 + 90 = 90 > 0$$

Therefore, the given function is minimum at $x = 3$.

The minimum value is,

$$f\left(3\right) = 243 - 405 + 135 - 10 = -37 \qquad (Ans.)$$

For $x = 0$ we get,

$$f''(0) = 0$$

Therefore the test fails.

$$\therefore\ f'''(x) = 60x^2 - 120x + 30$$
$$\therefore\ f'''(0) = 30 \neq 0$$

Therefore, the given function is neither maximum nor minimum at $x = 0$ $\qquad (Ans.)$

Problem-02: Find the extremmum values of $y = \dfrac{x^3}{3} + ax^2 - 3a^2x$.

Solution: The given function is,

$$y = f(x) = \frac{x^3}{3} + ax^2 - 3a^2x \qquad \cdots(1)$$

Differentiating with respect to x we get,

$$f'(x) = \frac{3x^2}{3} + 2ax - 3a^2$$
$$= x^2 + 2ax - 3a^2 \qquad \cdots(2)$$

We know that for maximum and minimum values,

$$f'(x) = 0$$
$$\Rightarrow x^2 + 2ax - 3a^2 = 0$$
$$\Rightarrow x^2 + 3ax - ax - 3a^2 = 0$$
$$\Rightarrow x(x+3a) - a(x+3a) = 0$$
$$\Rightarrow (x+3a)(x-a) = 0$$
$$\therefore x = a,\ -3a$$

Again, differentiating eq.(2) with respect to x we get,

$$f''(x) = 2x + 2a$$

For $x = -3a$ we get,

$$f''(-3a) = -6a + 2a = -4a < 0$$

Therefore, the given function is maximum at $x = -3a$.

The maximum value is,

$$f\left(-3a\right)=9a^3 \qquad (Ans.)$$

For $x=a$ we get,

$$f''(a)=2a+2a=4a>0$$

Therefore, the given function is minimum at $x=a$

The minimum value is,

$$f\left(a\right)=\frac{5}{3}a^3 \qquad (Ans).$$

Problem-03: Investigate for what values of x the function $y=5x^3-3x^2-2x+5$ is a maximum or minimum. Find also the maximum and minimum values.

Solution: The given function is,

$$y=f(x)=5x^3-3x^2-2x+5 \qquad \cdots(1)$$

Differentiating with respect to x we get,

$$f'(x)=15x^2-6x-2 \qquad \cdots(2)$$

We know that for maximum and minimum values,

$$f'(x)=0$$

$$\Rightarrow 15x^2-6x-2=0$$

$$\Rightarrow x=\frac{-(-6)\pm\sqrt{(-6)^2-4.15.(-2)}}{2.15}$$

$$=\frac{6\pm\sqrt{36+120}}{30}$$

$$=\frac{6\pm\sqrt{156}}{30}$$

$$\therefore x=0.616,\ -0.216$$

Again, differentiating eq.(2) with respect to x we get,

$$f''(x)=30x-6$$

For $x=0.616$ we get,

$$f''(0.616)=30\times 0.616-6=12.48>0$$

Therefore, the given function is minimum at $x=0.616$.

The minimum value is,

$$f\left(0.616\right)=5\left(0.616\right)^3-3\left(0.616\right)^2-2\left(0.616\right)+5=3.8 \qquad (Ans.)$$

For $x=-0.216$ we get,

$$f''(-0.216)=30\left(-0.216\right)-6=-12.48<0$$

Therefore, the given function is maximum at $x=-0.216$

The maximum value is,

$$f\left(a\right)=5\left(-0.216\right)^3-3\left(-0.216\right)^2-2\left(-0.216\right)+5=5.24 \qquad (Ans.)$$

Problem-04: Show that $y=x^{\frac{1}{x}}\ \left(x>0\right)$ is a maximum at $x=e$. Deduce that $e^{\pi}>\pi^{e}$.

Solution: The given function is,

$$y=f(x)=x^{\frac{1}{x}}\quad\left(x>0\right) \qquad \cdots(1)$$

Differentiating with respect to x we get,

$$f'(x) = x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right) \qquad \cdots(2)$$

We know that for maximum and minimum values,

$$f'(x)=0$$

$$\Rightarrow x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)=0$$

$$\Rightarrow \frac{1}{x^2} - \frac{\ln x}{x^2} = 0$$

$$\Rightarrow 1-\ln x = 0$$

$$\Rightarrow \ln x = 1$$

$$\therefore\ x = e$$

Again, differentiating eq.(2) with respect to x we get,

$$f''(x)=x^{\frac{1}{x}}\left\{-\frac{2}{x^3} - \frac{x-2x\ln x}{x^4} + \frac{1}{x^4} - \frac{2\ln x}{x^4} + \frac{(\ln x)^2}{x^4}\right\}$$

For $x=e$ we get,

$$f''(e)=e^{\frac{1}{e}}\left\{-\frac{2}{e^3} - \frac{e-2e\ln e}{e^4} + \frac{1}{e^4} - \frac{2\ln e}{e^4} + \frac{(\ln e)^2}{e^4}\right\}$$

$$=e^{\frac{1}{e}}\left\{-\frac{2}{e^3} - \frac{e-2e}{e^4} + \frac{1}{e^4} - \frac{2}{e^4} + \frac{1}{e^4}\right\}$$

$$=-\frac{1}{e^3}\cdot e^{\frac{1}{e}} < 0$$

Since $f''(x)$ is negative at $x=e$, so the given function is maximum at $x=e$. **(Showed)**

The maximum value is,

$$f\ (e)=e^{\frac{1}{e}}\ .$$

2nd part: Since $f(x)$ is maximum for $x=e$ so

$$f(e) > f(\pi)$$

$$or,\ e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$$

$$\therefore\ e^{\pi} > \pi^{e}$$ **(Deduced)**

Problem-05: Find the maximum and minimum values for $f(x)=1+2\sin x+3\cos^2 x$.

Solution: The given function is,

$$y = f\ (x)=1+2\sin x+3\cos^2 x \qquad \cdots(1)$$

Differentiating with respect to x we get,

$$f'(x) = 2\cos x - 6\cos x\sin x \qquad \cdots(2)$$

We know that for maximum and minimum values,

$$f'(x)=0$$

$$\Rightarrow 2\cos x - 6\cos x\sin x = 0$$

$$\Rightarrow 2\cos x(1-3\sin x) = 0$$

$$\therefore\ \cos x = 0 \quad or\ \ 1-3\sin x = 0$$

$$\Rightarrow \quad x=\frac{\pi}{2} \quad or \ x=\sin^{-1}\left(\frac{1}{3}\right)$$

Again, differentiating eq.(2) with respect to x we get,

$$f''(x) = -2\sin x - 6\left(\cos^2 x - \sin^2 x\right) = -2\sin x - 6\cos 2x$$

For $x=\frac{\pi}{2}$ we get,

$$f''\left(\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2} - 6\cos 2\cdot\frac{\pi}{2} = -2\cdot 1 - 6\cdot(-1) = -2+6 = 4 > 0$$

Since $f''(x)$ is positive at $x=\frac{\pi}{2}$, so the given function is minimum at $x=\frac{\pi}{2}$.

The maximum value is,

$$f\left(\frac{\pi}{2}\right) = 1+2\sin\left(\frac{\pi}{2}\right)+3\cos^2\left(\frac{\pi}{2}\right) = 1+2+0 = 3.$$

For $x=\sin^{-1}\left(\frac{1}{3}\right)$ we get,

$$f''\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\} = -2\sin\sin^{-1}\left(\frac{1}{3}\right) - 6\cos 2\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}$$

$$= -\frac{2}{3} - 6\left[1-2\sin^2\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= -\frac{2}{3} - 6\left[1-2\cdot\left(\frac{1}{3}\right)^2\right]$$

$$= -\frac{2}{3} - 6\left(1-\frac{2}{9}\right)$$

$$= -\frac{2}{3} - \frac{42}{9} = -\frac{48}{9} < 0.$$

Since $f''(x)$ is negative at $x=\sin^{-1}\left(\frac{1}{3}\right)$, so the given function is minimum at $x=\sin^{-1}\left(\frac{1}{3}\right)$.

The maximum value is,

$$f\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\} = 1+2\sin\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}+3\cos^2\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}$$

$$= 1+\frac{2}{3}+3\left[1-\sin^2\left\{\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= 1+\frac{2}{3}+3\left(1-\frac{1}{9}\right)$$

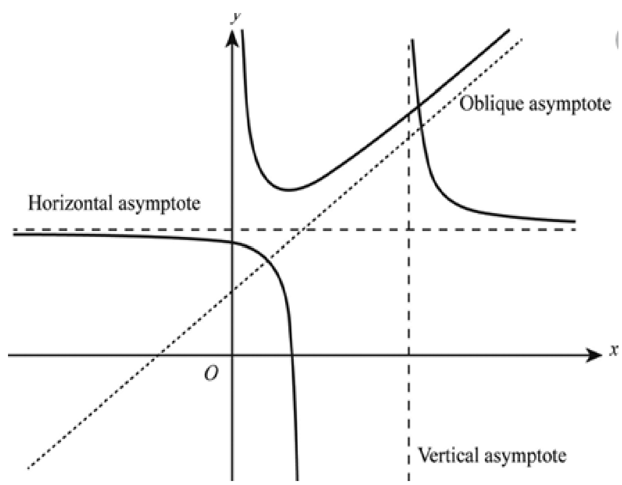
$$= 1+\frac{2}{3}+\frac{24}{9} = \frac{39}{9} = \frac{13}{3}.$$

Homework:

1. For what values of x the expression $f(x) = 3x^4 - 25x^2 + 60x$ is maximum or minimum? Find also the maximum and minimum values.
2. Examine $f(x) = x^6 - 12x^5 + 36x^4 + 4$ for maximum or minimum values.

3. Find maximum or minimum value of the function $y = 5x^3 - 3x^2 + 2x + 5$.
4. Find maximum or minimum value of the function $f(x) = 2x^4 + 3x^3 + 4x + 3$.
5. Show that $f(x) = x^3 - 6x^2 + 24x + 4$ has neither a maximum nor a minimum.
6. Show that the maximum value of $f(x) = x^2 \ln\left(\frac{1}{x}\right)$ is $\frac{1}{2e}$.
7. Show that the maximum value of $f(x) = x + \frac{1}{x}$ is less than its minimum value.
8. Show that the maximum value of $f(x) = \left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.
9. Show that the minimum value of $f(x) = \frac{x}{\ln x}$ is e.
10. Show that the maximum value of $f(x) = 4e^{2x} + 9e^{-2x}$ is 12.
11. Examine whether $f(x) = x^{\frac{1}{x}}$ possesses a maximum or a minimum and determine the same.
12. Investigate for what values of x the function $y = 5x^6 - 18x^5 + 15x^4 - 10$ is a maximum or minimum. Find also the maximum and minimum values.

Asymptotes: An asymptote of a curve $y = f(x)$ is a straight line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity.



There are three types of asymptotes:

(1) Vertical asymptotes,
(2) Horizontal asymptotes and
(3) Oblique asymptotes.

Question: If asymptotes not parallel to y-axis are given by $y = mx + c$, where m is any of the real finite roots of $\varphi_n(m) = 0$, then show that for each value of m,

$$c = -\frac{\varphi_{n-1}(m)}{\varphi_n'(m)}.$$

Answer: the general form of the equation of an algebraic curve of the nth degree (arranging in groups of homogeneous terms) is,

$$\left(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\cdots+a_ny^n\right)+\left(b_0x^{n-1}+b_1x^{n-2}y+b_2x^{n-3}y^2+\cdots+b_{n-1}y^{n-1}\right)+$$

$$\left(c_0x^{n-2}+c_1x^{n-3}y+c_2x^{n-4}y^2+\cdots+c_{n-2}y^{n-2}\right)+\cdots=0 \qquad \cdots(1)$$

The equation (1) also can be written as,

$$x^n\varphi_n\left(\frac{y}{x}\right)+x^{n-1}\varphi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\varphi_{n-2}\left(\frac{y}{x}\right)+\cdots=0 \qquad \cdots(2)$$

where φ_r is an algebraic polynomial of degree r.

Dividing equation (2) by x^n, we get

$$\varphi_n\left(\frac{y}{x}\right)+\frac{1}{x}\varphi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\varphi_{n-2}\left(\frac{y}{x}\right)+\cdots=0 \qquad \cdots(3)$$

Now if $y=mx+c$ ne an asymptote, where m and c are finite, then

$$\lim_{x\to\infty}\frac{y}{x}=m$$

Hence from (3), making $x\to\infty$, since m is finite, and the functions $\varphi_n(m), \varphi_{n-1}(m)$, $\cdots$,etc. which are algebraic polynomials in m are accordingly finite, we get

$$\varphi_n(m)=0.$$

Again, since in this case,

$$\lim_{x\to\infty}(y-mx)=c.$$

We can write

$$y-mx=c+u$$

where u is a function of x such that $u\to 0$ when $x\to\infty$.

Thus, $$\frac{y}{x}=m+\frac{c+u}{x}$$

From (3), we get

$$\varphi_n\left(m+\frac{c+u}{x}\right)+\frac{1}{x}\varphi_{n-1}\left(m+\frac{c+u}{x}\right)+\frac{1}{x^2}\varphi_{n-2}\left(m+\frac{c+u}{x}\right)+\cdots=0 \qquad \cdots(4)$$

Expanding each term by Taylor's theorem and using $\varphi_n(m)=0$, we get

$$\left\{\frac{c+u}{x}\varphi_n'(m)+\frac{(c+u)^2}{2!x^2}\varphi_n''(m)+\frac{(c+u)^3}{3!x^3}\varphi_n'''(m)+\cdots\right\}+\frac{1}{x}\left\{\frac{c+u}{x}\varphi_{n-1}'(m)+\frac{(c+u)^2}{2!x^2}\varphi_{n-1}''(m)+\right.$$

$$\left.\frac{(c+u)^3}{3!x^3}\varphi'''_{n-1}(m)+\cdots\right\}+\frac{1}{x^2}\left\{\frac{c+u}{x}\varphi'_{n-2}(m)+\frac{(c+u)^2}{2!x^2}\varphi''_{n-2}(m)+\frac{(c+u)^3}{3!x^3}\varphi'''_{n-2}(m)+\cdots\right\}+\cdots=0$$

$$\cdots(5)$$

Now multiplying throughout by x and making $x\to\infty$, we get

$$c\varphi'_n(m)+\varphi_{n-1}(m)=0 \qquad [\because\ u\to 0]$$

$$\therefore\ c=-\frac{\varphi_{n-1}(m)}{\varphi'_n(m)}$$

This is the required result. **(Showed)**

Working rule for asymptotes of algebraic curves: For an algebraic curve of the nth degree, first of all see if the term involving y^n is absent, in which case, the coefficient of the highest power of y involved in the equation (unless it is merely a constant independent of x) equated to zero will give asymptotes parallel to the y-axis.

Similarly, if the term involving x^n is absent, the coefficient of the highest available power of x equated to zero will in general give asymptotes parallel to the x-axis.

Next, replacing x by 1 and y by m in the homogeneous nth degree terms, get $\varphi_n(m)$. Similarly, get $\varphi_{n-1}(m)$ from the $(n-1)th$ degree terms, and if necessary $\varphi_{n-2}(m)$ from the $(n-2)th$ degree terms and so on. Now equating $\varphi_n(m)$ to zero, obtain the real finite roots m_1, m_2, etc. which will indicate the directions of the corresponding asymptotes (repeated roots giving in general a set of parallel asymptotes).

For each non-repeated root (m_1, say), a definite value c_1 of

$$c=-\frac{\varphi_{n-1}(m)}{\varphi'_n(m)}$$

is obtained, and the corresponding asymptote $y=m_1x+c_1$ determined.

Special cases:

(1). If for any $m=m_1$ we get $\varphi'_n(m)=0$, then there is no asymptote corresponding to this value.

(2). If for any $m=m_1$ we get $\varphi_n(m)=0,\ \varphi'_n(m)=0\ \textit{and}\ \varphi_{n-1}(m)=0$, then $c^2\varphi''_n(m)+2c\varphi'_{n-1}(m)+2\varphi_{n-2}(m)=0$ will give two values (say c_1, c_2) of c in general, and thereby giving two parallel asymptotes of the type $y=m_1x+c_1$ and $y=m_1x+c_2$.

Asymptotes of Polar curves: Let $r=f(\theta)$ be the polar equation to a curve. This may be written as,

$u=\frac{1}{r}=\frac{1}{f(\theta)}=F(\theta)\ say$. According to this assumption the formula for asymptotes is $r\sin(\theta-\alpha)=\frac{1}{F'(\alpha)}$, where α is the solution of $F(\theta)=0$.

Problem-01: Find the asymptotes of the cubic $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$.

Solution: The given curve is,

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0 \qquad \cdots(1)$$

This is an algebraic curve of third degree. Since the terms x^3 and y^3 are both present, so there are no asymptotes parallel to either the x-axis or the y-axis. It has oblique asymptotes of the type,

$$y=mx+c \qquad \cdots(2)$$

Putting 1 for x and m for y in all 3rd order terms of (1), we get

$$\varphi_n(m)=1-2m^3+m(2-m)=-2m^3-m^2+2m+1$$

$$\therefore\ \varphi_n'(m)=-6m^2-2m+2$$

Again putting 1 for x and m for y in all 2nd order terms of (1), we get

$$\varphi_{n-1}(m)=m(1-m)$$

Now $\varphi_n(m)=0$ gives,

$$2m^3+m^2-2m-1=0$$

$$or,\ 2m^3-2m^2+3m^2-3m+m-1=0$$

$$or,\ 2m^2(m-1)+3m(m-1)+1(m-1)=0$$

$$or,(m-1)\left(2m^2+3m+1\right)=0$$

$$\therefore\ m-1=0 \quad or\ \ 2m^2+3m+1=0$$

$$\Rightarrow\ m=1 \qquad or\quad m=\frac{-3\pm\sqrt{3^2-4\cdot 2\cdot 1}}{2\cdot 2}=\frac{-3\pm 1}{4}=-1,-\frac{1}{2}.$$

Therefore, the values are, $m=1,-1,-\frac{1}{2}$.

We know $\qquad c=-\frac{\varphi_{n-1}(m)}{\varphi_n'(m)}$

$$=\frac{m(1-m)}{6m^2+2m-2} \qquad \cdots(3)$$

For $m=1$, form (3) we get, $c=0$.

For $m=-1$, form (3) we get, $c=-1$.

For $m=-\frac{1}{2}$, form (3) we get, $c=\frac{1}{2}$.

Using $m=1$ and $c=0$ in (2) we have

$$y=x \Rightarrow x-y=0.$$

Using $m=-1$ and $c=-1$ in (2) we have

$$y=-x-1 \Rightarrow x+y+1=0.$$

Using $m=-\frac{1}{2}$ and $c=\frac{1}{2}$ in (2) we have

$$y=-\frac{1}{2}x+\frac{1}{2} \Rightarrow x+2y-1=0.$$

The required asymptotes are, $x-y=0$, $x+y+1=0$ and $x+2y-1=0$.

Problem-02: Find the asymptotes of $2x(y-5)^2=3(y-2)(x-1)^2$.

Solution: The given curve is,

$$2x(y-5)^2=3(y-2)(x-1)^2$$

$$or,\ 2xy^2-3x^2y-14xy+6x^2+38x-3y+6=0 \qquad \cdots(1)$$

This is an algebraic curve of third degree. Since the terms x^3 and y^3 are both absent, so there are asymptotes parallel to the x-axis and the y-axis.

The coefficient of the term y^2 is $2x$.

The asymptote parallel to the y-axis is

$$2x=0 \Rightarrow x=0.$$

The coefficient of the term x^2 is $-3y+6$.

The asymptote parallel to the x-axis is

$$-3y+6=0 \Rightarrow y=2.$$

It has oblique asymptotes of the type,

$$y=mx+c \qquad \cdots(2)$$

Putting 1 for x and m for y in all 3rd order terms and in all 2nd order terms of (1), we get

$$\varphi_n(m)=2m^2-3m$$

$$\therefore \varphi_n'(m) = 4m - 3$$

Again putting 1 for x and m for y in all 2nd order terms of (1), we get

$$\varphi_{n-1}(m) = -14m + 6$$

Now $\varphi_n(m) = 0$ gives,

$$2m^2 - 3m = 0$$

$$or,\ m(2m-3) = 0$$

$$\therefore m = 0 \quad or\ 2m - 3 = 0 \Rightarrow m = \frac{3}{2}.$$

Therefore, the values are, $m = 0, \frac{3}{2}$.

We know $\qquad c = -\dfrac{\varphi_{n-1}(m)}{\varphi_n'(m)}$

$$= \frac{14m-6}{4m-3} \qquad \cdots(3)$$

For $m = 0$, form (3) we get, $c = 2$.

For $m = \frac{3}{2}$, form (3) we get, $c = 5$.

Using $m = 0$ and $c = 2$ in (2) we have

$$y = 2.$$

Using $m = \frac{3}{2}$ and $c = 5$ in (2) we have

$$y = \frac{3}{2}x + 5 \Rightarrow 3x - 2y + 10 = 0.$$

The required asymptotes are, $x = 0$, $y = 2$ and $3x - 2y + 10 = 0$.

Problem-03: Find the asymptotes of the Folium of Descartes $x^3 + y^3 = 3axy$.

Solution: The given curve is,

$$x^3 + y^3 = 3axy$$

$$or,\ x^3 + y^3 - 3axy = 0 \qquad \cdots(1)$$

This is an algebraic curve of third degree. Since the terms x^3 and y^3 are both present, so there are no asymptotes parallel to either the x-axis or the y-axis. It has oblique asymptotes of the type,

$$y = mx + c \qquad \cdots(2)$$

Putting 1 for x and m for y in all 3rd order terms of (1), we get

$$\varphi_n(m) = 1 + m^3$$

$$\therefore \varphi_n'(m) = 3m^2$$

Again putting 1 for x and m for y in all 2nd order terms of (1), we get

$$\varphi_{n-1}(m) = -3am$$

Now $\varphi_n(m) = 0$ gives,

$$1 + m^3 = 0$$

$$or, (1+m)(m^2 - m + 1) = 0$$

$$\therefore m + 1 = 0 \quad or \ m^2 - m + 1 = 0$$

$$\Rightarrow m = -1 \quad or \ m^2 - m + 1 = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Therefore, the only real value of m is, $m = -1$.

We know $\quad c = -\dfrac{\varphi_{n-1}(m)}{\varphi_n'(m)}$

$$= \frac{a}{m} \qquad \cdots(3)$$

For $m = -1$, form (3) we get, $c = -a$.

Using $m = -1$ and $c = -a$ in (2) we have

$$y = -x - a \Rightarrow x + y + a = 0.$$

The required asymptote is, $x + y + a = 0$.

Problem-04: Find the asymptotes of $x^3 + 2x^2y + xy^2 - x + 1 = 0$.

Solution: The given curve is,

$$x^3 + 2x^2y + xy^2 - x + 1 = 0 \qquad \cdots(1)$$

This is an algebraic curve of third degree. Since the term x^3 is present so there is no asymptote parallel to the x-axis and since y^3 is absent, so there is an asymptote parallel to the y-axis.

The coefficient of the term y^2 is x.

The asymptote parallel to the y-axis is

$$x=0.$$

It has oblique asymptotes of the type,

$$y=mx+c \qquad \cdots(2)$$

Putting 1 for x and m for y in all 3rd order terms of (1), we get

$$\varphi_n(m)=1+2m+m^2$$

$$\therefore\ \varphi_n'(m)=2+2m$$

$$\therefore\ \varphi_n''(m)=2$$

Since 2nd order term is absent so

$$\varphi_{n-1}(m)=0$$

$$\therefore\ \varphi_{n-1}'(m)=0$$

Again putting 1 for x and m for y in all 1st order terms of (1), we get

$$\varphi_{n-2}(m)=-1$$

Now $\varphi_n(m)=0$ gives,

$$1+2m+m^2=0\Rightarrow m=-1,-1.$$

Therefore, the values are, $m=-1,-1$.

Since the roots are repeated so in the case of repeated roots we have

$$c^2\varphi_n''(m)+2c\varphi_{n-1}'(m)+2\varphi_{n-2}(m)=0 \qquad \cdots(3)$$

For $m=-1$, substituting the values of $\varphi_n''(m)$, $\varphi_{n-1}'(m)$ and $\varphi_{n-2}(m)$ in (3), we get,

$$2c^2+2c\cdot 0+2\cdot(-1)=0$$

$$or,\ 2c^2-2=0$$

$$or,\ c^2=1$$

$$\therefore\ \ c=\pm 1$$

Using $m=-1$ and $c=1$ in (2) we have

$$y=-x+1\Rightarrow x+y-1=0.$$

Using $m=-1$ and $c=-1$ in (2) we have

$$y=-x-1\Rightarrow x+y+1=0.$$

The required asymptotes are, $x=0$, $x+y-1=0$ and $x+y+1=0$.

Problem-05: Find the asymptotes, if any, of the curve $(r-a)\sin\theta=b$.

Solution: The given curve is,

$$(r-a)\sin\theta=b$$

$$or,\ r=\frac{b+a\sin\theta}{\sin\theta} \qquad \cdots(1)$$

Let $$u=\frac{1}{r}$$

$$\Rightarrow u=\frac{\sin\theta}{b+a\sin\theta}$$

$$\therefore\ F(\theta)=u=\frac{\sin\theta}{b+a\sin\theta}\quad (say) \qquad \cdots(2)$$

The directions in which $r\to\infty$ are given by

$$u=0$$

$$or,\ \frac{\sin\theta}{b+a\sin\theta}=0$$

$$or,\ \sin\theta=0$$

$$\therefore\ \theta=n\pi$$

Now differentiating (2) with respect to θ, we get

$$F'(\theta)=\frac{(b+a\sin\theta)\cos\theta-\sin\theta(a\cos\theta)}{(b+a\sin\theta)^2}$$

$$\therefore F'(\theta)=\frac{b\cos\theta}{(b+a\sin\theta)^2} \qquad \cdots(3)$$

Using $\theta=n\pi$ in (3), we get

$$F'(n\pi)=\frac{b\cos n\pi}{(b+a\sin n\pi)^2}=\frac{b\cos n\pi}{(b+a\cdot 0)^2}=\frac{bconn\pi}{b^2}=\frac{conn\pi}{b}$$

The required asymptote is given by,

$$r\sin(\theta-n\pi)=\frac{1}{F'(n\pi)}$$

$$or,\ r\sin(\theta-n\pi)=b\sec n\pi$$

$\therefore r\sin\theta = b$.

Homework:

Problem-01: Find the asymptotes of $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$.

Problem-02: Find the asymptotes of $y^2 - x^2 - 2x - 2y - 3 = 0$.

Problem-03: Find the asymptotes of $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$.

Problem-04: Find the asymptotes of $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$.

Problem-05: Find the asymptotes of $y^3 + x^2y + 2xy^2 - y + 1 = 0$.

Problem-06: Find the asymptotes, if any, of the curve $r = a(\cos\theta + \sec\theta)$.

Problem-07: Find the asymptotes, if any, of the curve $r = a\sec\theta + b\tan\theta$.

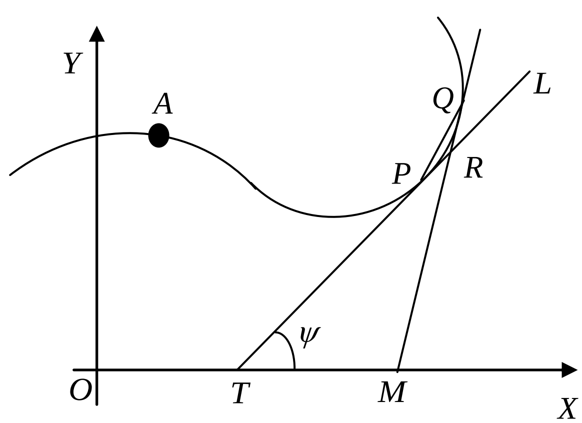
Problem-08: Find the asymptotes, if any, of the curve $r = 2a\tan\theta$.

Answers: P-1: $x - y = 0,\ 2x + y = 0$ *and* $2x + y + 1 = 0$; **P-2:** $x - y + 2 = 0$ *and* $x + y = 0$.

P-3: $x + 2y = 0,\ x + y - 1 = 0$ *and* $x - y + 1 = 0$; **P-4:** $4x + 12y + 9 = 0,\ 2x + 2y - 3 = 0$ *and* $4x - 4y + 1 = 0$.

P-5: $y = 0,\ x + y + 1 = 0$ *and* $x + y - 1 = 0$; **P-6:** $r\cos\theta = a$; **P-7:** $r\cos\theta = a \pm b$; **P-8:** $r\cos\theta = \pm 2a$.

Curvature: Let P be a given point on a curve, and Q be a point on the curve near P. Let the arc AP measured from some fixed point A on the curve be s, and the arc AQ be $s + \Delta s$; then arc $PQ = \Delta s$. Let TPL, MRQ be the tangents to the curve at P and Q, and let $\angle PTM = \psi$ and $\angle RMX = \psi + \Delta\psi$; then $\angle QRL = \Delta\psi$. Thus, $\Delta\psi$ is the change in the inclination of the tangent line as the point of contact of the tangent line describes the arc PQ.



The quotient $\dfrac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ.

The curvature at P is the limiting value, when it exists, of the average curvature when $Q \to P$ (from either side) along the curve, i.e. $\kappa = \lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s}$

$$\therefore \kappa = \frac{d\psi}{ds}.$$

Thus, the curvature is the rate of change of direction of the curve with respect to the arc, or roughly speaking, the curvature is the "rate at which the curve curves".

The reciprocal of the curvature at any point P is called the radius of curvature at P. It is denoted by ρ and defined as,

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}.$$

Envelopes: If each of the members of the family of curves $C \equiv F(x, y, \alpha) = 0$ touches a fixed curve E, then E is called the envelope of the family of curves C. The curve E also, at each point, is touched by some member of the family C.