Definite Integration

Fundamental Theorem of Integral Calculus: If $f(x)$ be a bounded and continuous function defined in the interval $[a, b]$ where, $b > a$ and there exists a function $\varphi(x)$ such that $\varphi'(x) = f(x)$, then

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a)$$

This is called the fundamental theorem of integral calculus.

Integration as the limit of a sum: Let, $f(x)$ be a bounded and continuous function defined in the interval $[a, b]$ where a , b are finite quantities and $b > a$. If the interval $[a, b]$ be divided into n equal sub-intervals, each of length h, by the points $a+h,\ a+2h, \cdots\cdots a+(n-1)h$ so that $nh = b-a$ then the area enclosed by $f(x)$ is defined as

$$\lim_{h\to 0}\left[hf(a) + hf(a+h) + hf(a+2h) + \cdots\cdots + hf\{a+(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) \qquad where,\ nh = b-a$$

which is also defined as the definite integral of $f(x)$ with respect to x between the limits a and b , and is denoted by the symbol,

$$\int_a^b f(x)dx$$

where, a is called the lower limit and b is called the upper limit.

Therefore, $\displaystyle\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) \qquad where,\ nh = b-a$

NOTE:

1. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx\ ;OR, \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx\ ;OR, \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

2. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n-1} f\left(\frac{r}{n}\right) = \int_0^2 f(x)dx \quad OR,\ \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)dx$

3. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n-1} f\left(\frac{r}{n}\right) = \int_0^3 f(x)dx \quad OR, \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n} f\left(\frac{r}{n}\right) = \int_0^3 f(x)dx$

Problem-01: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{n+n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)}$$

$$=\int_0^1\frac{dx}{1+x}$$

$$=\left[\ln(1+x)\right]_0^1$$

$$=\ln(1+1)-\ln(1+0)$$

$$=\ln 2$$

Problem-02: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$=\left[\sin^{-1}x\right]_0^1$$

$$=\sin^{-1}.1-\sin^{-1}.0$$

$$=\sin^{-1}.\sin\frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

Problem-03: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

Solution: Given that, $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{1}{8n}\right]$

$$=\lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots\cdots+\frac{n^2}{(n+n)^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}$$

$$=\left[-\frac{1}{2}\frac{1}{(1+x)^2}\right]_0^1$$

$$=\left[-\frac{1}{2}\frac{1}{(1+1)^2}+\frac{1}{2}\frac{1}{(1+0)^2}\right]$$

$$=-\frac{1}{8}+\frac{1}{2}$$

$$=\frac{3}{8}$$

Problem-04: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots\cdots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\sqrt{1-x^2}\,dx$$

$$=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=\left[\frac{1.\sqrt{1-1^2}}{2}+\frac{1}{2}\sin^{-1}.1-\frac{0.\sqrt{1-0^2}}{2}-\frac{1}{2}\sin^{-1}.0\right]$$

$$=\frac{1}{2}\sin^{-1}.\sin\frac{\pi}{2}$$

$$=\frac{\pi}{4}$$

Problem-05: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots\cdots+\frac{n}{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\frac{dx}{1+x^2}$$

$$=\left[\tan^{-1}x\right]_0^1$$

$$=\tan^{-1}.1-\tan^{-1}.0$$

$$=\tan^{-1}.\tan\frac{\pi}{4}-\tan^{-1}.\tan 0$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$

Problem-06: Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

Solution: Given that, $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots\cdots+\frac{1}{n}\right]$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n.1-1^2}}+\frac{1}{\sqrt{2n.2-2^2}}+\cdots\cdots+\frac{1}{\sqrt{2n.n-n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2nr-r^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\left\{\left(\frac{r}{n}\right)-\left(\frac{r}{n}\right)^2\right\}}}$$

$$=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(1-x)^2}}$$

$$=-\left[\sin^{-1}(1-x)\right]_0^1$$

$$=-\left[\sin^{-1}(1-1)-\sin^{-1}(1-0)\right]$$

$$=-\sin^{-1}.0+\sin^{-1}.1$$

$$=-\sin^{-1}.\sin 0+\sin^{-1}.\sin\frac{\pi}{2}$$

$$=0+\frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

Problem-07: Evaluate $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{2n}\frac{1}{\sqrt{4n^2+r^2}}$

Solution: Given that, $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{2n}\frac{1}{\sqrt{4n^2+r^2}}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{\sqrt{4+\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^2\frac{dx}{\sqrt{4+x^2}}$$

$$=\int_0^2\frac{dx}{\sqrt{2^2+x^2}}$$

$$=\left[\ln\left(x+\sqrt{2^2+x^2}\right)\right]_0^2$$

$$=\left[\ln\left(2+\sqrt{2^2+2^2}\right)-\ln\left(0+\sqrt{2^2+0^2}\right)\right]$$

$$=\ln\left(2+\sqrt{8}\right)-\ln 2$$

$$=\ln\left(\frac{2+\sqrt{8}}{2}\right)$$

$$=\ln\left(1+\sqrt{2}\right)$$

Problem-08: Evaluate $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{3n}\frac{n}{3^2n^2+r^2}$

Solution: Given that, $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{3n}\frac{n}{3^2n^2+r^2}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{3n}\frac{1}{3^2+\left(\frac{r}{n}\right)^2}$$

$$=\int_0^3\frac{dx}{3^2+x^2}$$

$$=\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$=\left[\frac{1}{3}\tan^{-1}\left(\frac{3}{3}\right)-\frac{1}{3}\tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$=\frac{1}{3}\tan^{-1}.1$$

$$=\frac{1}{3}\tan^{-1}.\tan\frac{\pi}{4}$$

$$=\frac{1}{3}.\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

Some Definite integrations

Problem-01: Evaluate $\int_0^{\pi/2}\cos^2 x dx$

Solution: Let, $I=\int_0^{\pi/2}\cos^2 x dx$

$$=\frac{1}{2}\int_0^{\pi/2}2\cos^2 x dx$$

$$=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)dx$$

$$=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin 2.\frac{\pi}{2}}{2}\right)-\left(0+\frac{\sin 2.0}{2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)$$

$$=\frac{1}{2}\left(\frac{\pi}{2}+0\right)$$

$$=\frac{\pi}{4}$$

Problem-02: Evaluate $\int_0^{\pi/2}\frac{dx}{1+\cos x}$

Solution: Let, $I=\int_0^{\pi/2}\frac{dx}{1+\cos x}$

$$=\int_0^{\pi/2}\frac{dx}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}\int_0^{\pi/2}\sec^2\frac{x}{2}dx$$

$$=\frac{1}{2}\left[\frac{\tan\frac{x}{2}}{1/2}\right]_0^{\pi/2}$$

$$=\left[\tan\frac{x}{2}\right]_0^{\pi/2}$$

$$=\tan\frac{\pi}{4}-\tan\frac{0}{2}$$

$$=1$$

Problem-03: Evaluate $\int_0^{\ln 2}\frac{e^x}{1+e^x}dx$

Solution: Let, $I=\int_0^{\ln 2}\frac{e^x}{1+e^x}dx$

$$= \left[\ln\left(1+e^{x}\right)\right]_{0}^{\ln 2}$$

$$= \ln\left(1+e^{\ln 2}\right) - \ln\left(1+e^{0}\right)$$

$$= \ln(1+2) - \ln(1+1)$$

$$= \ln 3 - \ln 2$$

$$= \ln\frac{3}{2}$$

Problem-04: Evaluate $\int_{0}^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

Solution: Let, $I = \int_{0}^{\pi/3} \frac{\cos x dx}{3+4\sin x}$

$$= \frac{1}{4}\int_{0}^{\pi/3} \frac{4\cos x dx}{3+4\sin x}$$

$$= \frac{1}{4}\left[\ln(3+4\sin x)\right]_{0}^{\pi/3}$$

$$= \frac{1}{4}\left[\ln\left(3+4\sin\frac{\pi}{3}\right) - \ln(3+4\sin 0)\right]$$

$$= \frac{1}{4}\left[\ln\left(3+4.\frac{\sqrt{3}}{2}\right) - \ln 3\right]$$

$$= \frac{1}{4}\left[\ln\left(3+2\sqrt{3}\right) - \ln 3\right]$$

$$= \frac{1}{4}\ln\left(\frac{3+2\sqrt{3}}{3}\right)$$

Problem-05: Evaluate $\int_{0}^{\pi/2} (\sec\theta - \tan\theta) d\theta$

Solution: Let, $I = \int_{0}^{\pi/2} (\sec\theta - \tan\theta) d\theta$

$$= \int_{0}^{\pi/2} \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1-\sin\theta}{\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} \right) d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)} d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\cos\frac{\theta}{2} - s\,in\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} d\theta$$

$$= 2\int_0^{\pi/2} \frac{\frac{1}{2}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)} d\theta$$

$$= 2\left[\ln\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \right) \right]_0^{\pi/2}$$

$$= 2\left[\ln\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4} \right) - \ln\left(\cos\frac{0}{2} + \sin\frac{0}{2} \right) \right]$$

$$= 2\left[\ln\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \ln 1 \right]$$

$$= 2\left[\ln\left(\frac{2}{\sqrt{2}} \right) - 0 \right]$$

$$= 2\ln\sqrt{2}$$

$$= \ln 2$$

Problem-06: Evaluate $\int_0^{\pi/2} \cos 2x \cos 3x dx$

Solution: Let, $I = \int_0^{\pi/2} \cos 2x \cos 3x dx$

$$=\frac{1}{2}\int_0^{\pi/2} 2\cos 2x\cos 3x dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\cos(2x+3x)+\cos(2x-3x)\right]dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\cos 5x+\cos x\right]dx$$

$$=\frac{1}{2}\left[\frac{\sin 5x}{5}+\sin x\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\sin 5.\frac{\pi}{2}}{5}+\sin\frac{\pi}{2}\right)-\left(\frac{\sin 0}{5}+\sin 0\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{5}\sin\frac{5\pi}{2}+1\right)$$

$$=\frac{1}{10}\sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{2}$$

$$=\frac{1}{10}\sin\frac{\pi}{2}+\frac{1}{2}$$

$$=\frac{1}{10}+\frac{1}{2}$$

$$=\frac{3}{5}$$

Problem-07: Evaluate $\int_0^{\pi/2}\cos^7 x dx$

Solution: Let, $I=\int_0^{\pi/2}\cos^7 x dx$

$$=\int_0^{\pi/2}\cos^6 x\cos x dx$$

$$=\int_0^{\pi/2}\left(\cos^2 x\right)^3\cos x dx$$

$$= \int_0^{\pi/2} \left(1-\sin^2 x\right)^3 \cos x dx$$

put, $\sin x = t \qquad \therefore \cos x dx = dt$

when $x = 0$ then $t = 0$

when $x = \dfrac{\pi}{2}$ then $t = 1$

Now, $I = \int_0^1 \left(1-t^2\right)^3 dt$

$$= \int_0^1 \left(1-3t^2+3t^4-t^6\right) dt$$

$$= \left[t - t^3 + 3\frac{t^5}{5} - \frac{t^7}{7} \right]_0^1$$

$$= 1 - 1 + \frac{3}{5} - \frac{1}{7}$$

$$= \frac{16}{35}$$

Problem-08: Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Solution: Let, $\displaystyle I = \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

$$= \frac{1}{b^2} \int_0^{\pi/2} \frac{dx}{\cos^2 x \left\{ \left(\frac{a}{b}\right)^2 + \tan^2 x \right\}}$$

$$= \frac{1}{b^2} \int_0^{\pi/2} \frac{\sec^2 x dx}{\left(\frac{a}{b}\right)^2 + \tan^2 x}$$

put, $\tan x = t \qquad \therefore \sec^2 x dx = dt$

when $x = 0$ then $t = 0$

when $x = \dfrac{\pi}{2}$ then $t = \infty$

Now, $I=\frac{1}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}$

$$=\frac{1}{b^2}\left[\frac{1}{a/b}\tan^{-1}\frac{t}{a/b}\right]_0^{\infty}$$

$$=\frac{1}{b^2}\left[\frac{b}{a}\tan^{-1}\frac{bt}{a}\right]_0^{\infty}$$

$$=\frac{1}{ab}\left(\tan^{-1}\infty-\tan^{-1}0\right)$$

$$=\frac{1}{ab}\left(\tan^{-1}\tan\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2ab}$$

Problem-09: Evaluate $\int_0^{\pi/2}\frac{dx}{4+5\sin x}$

Solution: Let, $I=\int_0^{\pi/2}\frac{dx}{4+5\sin x}$

$$=\int_0^{\pi/2}\frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{dx}{\frac{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{1+\tan^2\frac{x}{2}}{4+4\tan^2\frac{x}{2}+10\tan\frac{x}{2}}dx$$

Exercise-01: $\int_0^{\pi/2}\frac{dx}{5+4\sin x}$

Ans: $\frac{2}{3}\tan^{-1}\frac{1}{3}$

Exercise-02: $\int_0^{\pi}\frac{dx}{2+\cos x}$

Ans: $\frac{\pi}{\sqrt{3}}$

$$= \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{4\tan^2 \frac{x}{2} + 10\tan \frac{x}{2} + 4} dx$$

put, $\tan \frac{x}{2} = t \qquad \therefore\ \sec^2 \frac{x}{2} dx = 2dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = 1$

Now, $I = \int_0^1 \frac{2dt}{4t^2 + 10t + 4}$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 + 5t/2 + 1}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 + 2.t.5/4 + \left(5/4\right)^2 + 1 - 25/16}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\left(t + 5/4\right)^2 - 9/16}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\left(t + 5/4\right)^2 - \left(3/4\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2 \times 3/4} \ln\left(\frac{t + 5/4 - 3/4}{t + 5/4 + 3/4}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} \ln\left(\frac{t + 1/2}{t + 2}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\ln\left(\frac{1 + 1/2}{1 + 2}\right) - \ln\left(\frac{1/2}{2}\right)\right]$$

$$= \frac{1}{3}\left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{4}\right)\right]$$

$$=\frac{1}{3}\ln\left(\frac{1/2}{1/4}\right)$$

$$=\frac{1}{3}\ln 2$$

Problem-10: Evaluate $\int_0^{\pi/2}\frac{dx}{5+3\cos x}$

Exercise-03: $\int_0^{\pi/2}\frac{dx}{3+5\cos x}$

Ans: $\frac{1}{4}\ln 3$

Exercise-04: $\int_0^{\pi/2}\frac{dx}{1+2\cos x}$

Ans: $\frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right)$

Solution: Let, $I=\int_0^{\pi/2}\frac{dx}{5+3\cos x}$

$$=\int_0^{\pi/2}\frac{dx}{5+3\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{dx}{\frac{5+5\tan^2\frac{x}{2}+3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{8+2\tan^2\frac{x}{2}}dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{4+\tan^2\frac{x}{2}}dx$$

put, $\tan\frac{x}{2}=t \qquad \therefore\ \sec^2\frac{x}{2}dx=2dt$

when $x=0$ then $t=0$

when $x=\frac{\pi}{2}$ then $t=1$

Now, $I=\frac{1}{2}\int_0^1\frac{2dt}{4+\mathrm{t}^2}$

$$=\int_0^1 \frac{dt}{2^2+t^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^1$$

$$=\frac{1}{2}\left(\tan^{-1}\frac{1}{2}-\tan^{-1}.0\right)$$

$$=\frac{1}{2}\tan^{-1}\frac{1}{2}$$

Problem-11: Evaluate $\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Solution: Let, $I=\int_0^1 \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

put, $1+x=\frac{1}{t}$ $\quad\therefore\ dx=-\frac{1}{t^2}dt$

when $x=0$ then $t=1$

when $x=1$ then $t={}^1\!/_2$

Now, $I=\int_1^{1/2} \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+2\left(\frac{1}{t}-1\right)-\left(\frac{1}{t}-1\right)^2}}$

$$=-\int_1^{1/2} \frac{dt}{t\sqrt{1+\frac{2}{t}-2-\left(\frac{1}{t^2}-\frac{2}{t}+1\right)}}$$

$$=-\int_1^{1/2} \frac{dt}{t\sqrt{\frac{2}{t}-1-\frac{1}{t^2}+\frac{2}{t}-1}}$$

$$=-\int_1^{1/2} \frac{dt}{t\sqrt{\frac{4}{t}-\frac{1}{t^2}-2}}$$

$$=-\int_{1}^{1/2}\frac{dt}{t\sqrt{\frac{4t-1-2t^2}{t^2}}}$$

$$=-\int_{1}^{1/2}\frac{dt}{\sqrt{-1-2t^2+4t}}$$

$$=-\int_{1}^{1/2}\frac{dt}{\sqrt{2}\sqrt{-1/2-t^2+2t}}$$

$$=-\frac{1}{\sqrt{2}}\int_{1}^{1/2}\frac{dt}{\sqrt{-1/2-\left(t^2-2t\right)}}$$

$$=-\frac{1}{\sqrt{2}}\int_{1}^{1/2}\frac{dt}{\sqrt{1-1/2-\left(t^2-2t+1\right)}}$$

$$=-\frac{1}{\sqrt{2}}\int_{1}^{1/2}\frac{dt}{\sqrt{1/2-\left(t-1\right)^2}}$$

$$=-\frac{1}{\sqrt{2}}\int_{1}^{1/2}\frac{dt}{\sqrt{\left(1/\sqrt{2}\right)^2-\left(t-1\right)^2}}$$

$$=-\frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{t-1}{1/\sqrt{2}}\right)\right]_{1}^{1/2}$$

$$=-\frac{1}{\sqrt{2}}\left[\sin^{-1}\sqrt{2}\left(t-1\right)\right]_{1}^{1/2}$$

$$=-\frac{1}{\sqrt{2}}\left[\sin^{-1}\sqrt{2}\left(1/2-1\right)-\sin^{-1}\sqrt{2}\left(1-1\right)\right]$$

$$=-\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(-1/2\right)$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{2}\left(1/2\right)$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}}$$

Problem-12: Evaluate $\int_0^1 \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}}$

Solution: Let, $I=\int_0^1 \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}}$

Put $x=\frac{1}{z}$ $\therefore dx=-\frac{1}{z^2}dz$

when $x=0$ then $z=\infty$

when $x=1$ then $z=1$

Now $I=\int_\infty^1 \frac{-\frac{1}{z^2}dz}{\left(1+\frac{1}{z^2}\right)\sqrt{1-\frac{1}{z^2}}}$

$$=\int_1^\infty \frac{zdz}{\left(z^2+1\right)\sqrt{z^2-1}}$$

Again let $z^2-1=t^2$ $or,$ $z^2=t^2+1$

$\therefore$ $zdz=tdt$

when $z=1$ then $t=0$

when $z=\infty$ then $t=\infty$

$$\therefore\ I=\int_0^\infty \frac{tdt}{\left(t^2+1+1\right)\sqrt{t^2}}$$

$$=\int_0^\infty \frac{dt}{2+t^2}$$

$$=\int_0^\infty \frac{dt}{\left(\sqrt{2}\right)^2+t^2}$$

$$=\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^\infty$$

$$=\frac{1}{\sqrt{2}}\left(\tan^{-1}.\infty-\tan^{-1}.0\right)$$

$$=\frac{1}{\sqrt{2}}\left(\tan^{-1}.\tan\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2\sqrt{2}}$$

General Properties of Definite Integrals: The general properties are,

1. $\int_a^b f(x)dx=\int_a^b f(z)dz$
2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$
3. $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$
4. $\int_0^a f(x)dx=\int_0^a f(a-x)dx$
5. $\int_0^{2a} f(x)dx=2\int_0^a f(x)dx \quad if \;\; f(2a-x)=f(x)$
6. $\int_{-a}^{a} f(x)dx=\begin{cases}2\int_0^a f(x)\,dx & if\;\; f(-x)=f(x)\\ 0 & if\;\; f(-x)=-f(x)\end{cases}$

Problem-01: Evaluate $\int_0^{\pi/2}\frac{dx}{1+\cot x}$

Solution: Let, $I=\int_0^{\pi/2}\frac{dx}{1+\cot x}$

$$=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$$

$$=\int_0^{\pi/2}\frac{\sin\left(\pi/2-x\right)}{\sin\left(\pi/2-x\right)+\cos\left(\pi/2-x\right)}dx$$

$$=\int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}dx$$

Now $2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2}$$

$$= \pi/2$$

$$\therefore \ I = \pi/4$$

Problem-02: Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad OR, \int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}} \quad OR, \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$

Solution: Let, $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\pi/2 - x\right)}}{\sqrt{\sin\left(\pi/2 - x\right)} + \sqrt{\cos\left(\pi/2 - x\right)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Now $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\pi/2} dx$$

$$=[x]_0^{\pi/2}$$

$$=\pi/2$$

$$\therefore\ I=\pi/4$$

Problem-03: Evaluate $\int_0^{\pi}\frac{xdx}{1+\sin x}$

Solution: Let, $I=\int_0^{\pi}\frac{xdx}{1+\sin x}$

$$=\int_0^{\pi}\frac{(\pi-x)}{1+\sin(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)}{1+\sin x}dx$$

Now $2I=\int_0^{\pi}\frac{x}{1+\sin x}dx+\int_0^{\pi}\frac{(\pi-x)}{1+\sin x}dx$

$$=\int_0^{\pi}\frac{x+\pi-x}{1+\sin x}dx$$

$$=\int_0^{\pi}\frac{\pi}{1+\sin x}dx$$

$$=\int_0^{\pi}\frac{\pi(1-\sin x)}{1-\sin^2 x}dx$$

$$=\pi\int_0^{\pi}\frac{(1-\sin x)}{\cos^2 x}dx$$

$$=\pi\int_0^{\pi}\sec^2 x(1-\sin x)dx$$

$$=\pi\int_0^{\pi}\left(\sec^2 x-\sec^2 x\sin x\right)dx$$

$$=\pi\int_0^{\pi}\left(\sec^2 x-\sec x\tan x\right)dx$$

$$=\pi\left[\tan x-\sec x\right]_0^{\pi}$$

$$=\pi\left[0+1-0+1\right]$$

$$=2\pi$$

$$\therefore\ I=\pi$$

Problem-04: Evaluate $\int_0^{\pi}\frac{x\sin xdx}{1+\cos^2 x}$

Solution: Let, $I=\int_0^{\pi}\frac{x\sin xdx}{1+\cos^2 x}$

$$=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

Now $2I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx+\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$$=\int_0^{\pi}\frac{(x+\pi-x)\sin x}{1+\cos^2 x}dx$$

$$=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

put $\cos x=t \quad \therefore\ -\sin xdx=dt$

when $x=0$ then $t=1$

when $x=\pi$ then $t=-1$

$$\therefore\ 2I=-\pi\int_1^{-1}\frac{dt}{1+t^2}$$

$$= \pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi \left[\tan^{-1} t \right]_{-1}^{1}$$

$$= \pi \left[\tan^{-1}.1 - \tan^{-1}(-1) \right]$$

$$= \pi \left[\tan^{-1}.1 + \tan^{-1}.1 \right]$$

$$= \pi \left[\tan^{-1}.\tan\frac{\pi}{4} + \tan^{-1}.\tan\frac{\pi}{4} \right]$$

$$= \pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$$

$$= \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}$$

Problem-05: Evaluate $\int_0^{\pi/2} \frac{xdx}{\sin x + \cos x}$

Solution: Let, $I = \int_0^{\pi/2} \frac{xdx}{\sin x + \cos x}$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin x + \cos x}$$

Now $2I = \int_0^{\pi/2} \frac{xdx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin x + \cos x}$

$$=\int_0^{\pi/2}\frac{\left(x+\frac{\pi}{2}-x\right)dx}{\sin x+\cos x}$$

$$=\int_0^{\pi/2}\frac{\frac{\pi}{2}dx}{\sin x+\cos x}$$

$$=\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}$$

$$=\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\sin\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}\cos x}$$

$$=\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$=\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx$$

$$=\frac{\pi}{2\sqrt{2}}\left[\ln\left\{\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right\}\right]_0^{\pi/2}$$

$$=\frac{\pi}{2\sqrt{2}}\left[\ln\left\{\sec\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\right\}-\ln\left\{\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right\}\right]$$

$$=\frac{\pi}{2\sqrt{2}}\left[\ln\left\{\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\}-\ln\left\{\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right\}\right]$$

$$=\frac{\pi}{2\sqrt{2}}\left[\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)\right]$$

$$=\frac{\pi}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$=\frac{\pi}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right)^2$$

$$= \frac{\pi}{\sqrt{2}} \ln\left(\sqrt{2}+1\right)$$

$$\therefore \ I = \frac{\pi}{2\sqrt{2}} \ln\left(\sqrt{2}+1\right)$$

Problem-06: Evaluate $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

Solution: Let, $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

put $x = \tan\theta \quad \therefore \ dx = \sec^2\theta d\theta$

when $x = 0$ then $\theta = 0$

when $x = 1$ then $\theta = \frac{\pi}{4}$

$$\therefore \ I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta} \sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \ln(1+\tan\theta) d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$$

Now $\quad 2I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta + \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$$= \int_0^{\pi/4} \ln\left\{(1+\tan\theta).\frac{2}{(1+\tan\theta)}\right\} d\theta$$

$$= \int_0^{\pi/4} \ln 2\,d\theta$$

$$= \ln 2\,[\theta]_0^{\pi/4}$$

$$= \frac{\pi}{4}\ln 2$$

$$\therefore\; I = \frac{\pi}{8}\ln 2$$

Problem-07: Evaluate $\int_0^{\pi/2} \ln\sin x\,dx \quad OR \quad \int_0^{\pi/2} \ln\cos x\,dx$

Solution: Let, $I = \int_0^{\pi/2} \ln\sin x\,dx$

$$= \int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2}-x\right) dx$$

$$= \int_0^{\pi/2} \ln\cos x\,dx$$

Now $\quad 2I = \int_0^{\pi/2} \ln\sin x\,dx + \int_0^{\pi/2} \ln\cos x\,dx$

$$= \int_0^{\pi/2} (\ln\sin x + \ln\cos x)\,dx$$

$$= \int_0^{\pi/2} \ln(\sin x \cos x)dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{1}{2}\sin 2x\right)dx$$

$$= \int_0^{\pi/2} \ln\sin 2x dx - \ln 2 \int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \ln\sin 2x dx - \ln 2 [x]_0^{\pi/2}$$

$$= I_1 - \frac{\pi}{2}\ln 2 \quad \dots\dots\dots (1)$$

where, $I_1 = \int_0^{\pi/2} \ln\sin 2x dx$

put $2x = t \quad \therefore\ dx = \frac{1}{2}dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = \pi$

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \ln\sin t dt$$

$$= \int_0^{\pi/2} \ln\sin t dt$$

$$= \int_0^{\pi/2} \ln\sin x dx$$

$$= I$$

From (1) we get

$$2I = I - \frac{\pi}{2}\ln 2$$

$$\Rightarrow I = -\frac{\pi}{2}\ln 2$$

$$\Rightarrow I = \frac{\pi}{2}\ln\frac{1}{2}$$

Problem-08: Evaluate $\int_0^{\pi/2} \ln \tan x dx$

Solution: Let, $I = \int_0^{\pi/2} \ln \tan x dx$

$$= \int_0^{\pi/2} \ln \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \ln \cot x dx$$

Now $\quad 2I = \int_0^{\pi/2} \ln \tan x dx + \int_0^{\pi/2} \ln \cot x dx$

$$= \int_0^{\pi/2} (\ln \tan x + \ln \cot x) dx$$

$$= \int_0^{\pi/2} \ln(\tan x \cot x) dx$$

$$= \int_0^{\pi/2} \ln 1 dx$$

$$= 0$$

$$\therefore \ I = 0$$