Basic Concept

Fluid Dynamics: Fluid dynamics is a branch of science which deals with the study of the motion of fluids.

Hydrodynamics: Hydrodynamics is a branch of science which deals with the study of the motion of incompressible fluids.

Solid and Fluid: Matter exists in two states such as,

a) Solid state,
b) Fluid state.

Solid can resist deformation of certain degree but the fluid cannot. The fluid is capable of changing shape and is capable of flowing.
The fluid is also divided in two states such as,

i. Liquid
ii. Gas

Liquid: The liquid has a definite volume but no definite shape such as water, milk, blood, oil, mercury, ethanol etc.

Gas: The gas has no definite volume and shape such as air, hydrogen, nitrogen oxygen etc.

Properties of Fluid: Some basic properties of fluid are as follows:

a) **Density**: The density of a fluid is the mass per unit volume and it is denoted by ρ. If δv is the volume around a point and δm is the mass within this volume, then the density is defined as,

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}.$$

The density depends on the space coordinates and the temperature i.e. $\rho = f(x, y, z, t)$. The density of water at $4^0 C$ is $1000\,kgm^{-3}$.

b) **Specific Weight**: The specific weight of a fluid is the weight per unit volume and it is denoted by γ. If ρ is the density and g is the gravitational acceleration of a fluid, then the specific weight is defined as,

$$\gamma = \rho g.$$

c) **Specific Volume**: The specific volume of a fluid is the volume per unit mass and it is denoted by v_s. It is clearly the reciprocal of the density, i.e.

$$v_s = \frac{1}{\rho}.$$

d) **Pressure**: The pressure of a fluid is the force per unit area and it is denoted by p.

If δA is the area around a point and δF is the applied force in this area, then the pressure is defined as,

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}.$$

e) **Viscosity**: The viscosity or internal friction of fluid particles is produced due to shearing stress and it is denoted by μ. It measures the resistance of flow of the fluid. Mathematically, it is defined as,

$$\tau = \mu \frac{du}{dy}$$

where, τ is shear stress, μ is viscosity and $\frac{du}{dy}$ is velocity gradient. This equation is called Newton's law of viscosity.

f) **Compressibility**: The compressibility of a fluid is defined as the variation of its density, with the variation of pressure. Mathematically,

$$dp \propto \frac{d\rho}{\rho}$$

$$or,\ dp = K \frac{d\rho}{\rho}$$

Here K is called the bulk modulus of the fluid.

g) **Temperature**: When two bodies of different heat content are brought into contact then some thermal energy will move from one body into other body. The body from where the thermal energy moves is said to be at a higher temperature while the body into which the energy flows is said to be at a lower temperature. When two bodies are in thermal equilibrium then they are said to have a common property, known as temperature T.

h) **Thermal Conductivity**: When a fluid in static equilibrium is heated non uniformly, heat may be transferred from regions of higher temperature to those of lower temperature. Consider a surface element situated at some point in the fluid. The heat flux (rate of flow per unit area) in the direction of the normal to the element is proportional to the rate of change of temperature at that points. The heat flow occurs in the direction of decreasing temperature. Let q_n denotes the heat flux and $\frac{\partial T}{\partial n}$ denotes the rate of change of temperature, then we have

$$q_n \propto -\frac{\partial T}{\partial n}$$

$$or,\ q_n = -K \frac{\partial T}{\partial n}$$

where, K is a positive proportionality factor known as the coefficient of thermal conductivity. The thermal conductivity is a function of temperature and pressure.

i) **Specific Heat**: The specific heat, C is defined as the amount of heat required to raise the temperature of a unit mass of medium by one degree, i.e.

$$C = \frac{\partial Q}{\partial t}$$

where, Q is the quantity of added per unit mass of the gas.

Types of fluids: Some types of fluids are as follows:

i. **Compressible Fluid**: A fluid is called compressible fluid if the volume changes when the pressure changes and it has variable density.

Example: Gases.

ii. **Incompressible Fluid**: A fluid is called incompressible fluid if the volume does not change when the pressure changes and the density is fixed.

Example: Liquids.

iii. **Viscous Fluid**: A fluid is said to be viscous when the normal as well as the shearing stresses exist.

Example: Paint, Coalter, Molases and heavy oil.

iv. **Inviscid Fluid**: A fluid is said to be non-viscous or inviscid when it does not exert any shearing stress whether at rest or in motion.

Example: Gases.

v. **Newtonian Fluid**: A fluid for which the viscosity does not change with the rate of deformation is said to be Newtonian fluid.

In other words, fluids which obey the Newton's law of viscosity are known as Newtonian fluids.

Example: Water, Air and Mercury.

vi. **Non-Newtonian Fluid**: A fluid for which the viscosity changes with the rate of deformation is said to be Non-Newtonian fluid.

In other words, fluids which do not obey the Newton's law of viscosity are known as Non-Newtonian fluids.

Example: Paint, Coalter and Polymer solutions.

Types of Flows: Some flows are as follows:

a) **Steady and Unsteady Flows:** A flow, in which the fluid properties (P, say) are independent of time so that the flow pattern remains unchanged with the time, is said to be steady. Mathematically we may write,

$$\frac{\partial P}{\partial t} = 0$$

where, P may be velocity, pressure , viscosity , temperature etc.

On the other hand, A flow, in which the fluid properties (P, say) are dependent on the time so that the flow pattern varies with the time, is said to be unsteady. Mathematically we may write,

$$\frac{\partial P}{\partial t} \neq 0$$

where, P may be velocity, pressure , viscosity , temperature etc.

Example: Water flowing through a tap at a constant rate is steady flow. Again, Water flowing through a tap at a changing rate is unsteady flow.

b) **Uniform and Non-uniform Flows**: A flow, in which the fluid particles possess equal velocities at each section of the channel or pipe is called uniform flow.

On the other hand, A flow, in which the fluid particles possess different velocities at each section of the channel or pipe is called non-uniform flow.

Example: Water flowing through a long straight pipe of uniform diameter at a constant rate is uniform flow. Again, Water flowing through a long straight pipe of non-uniform diameter at changing rate is non-uniform flow.

c) **Rotational and Irrotational Flows**: A flow, in which the fluid particles go on rotating about their own axes, while flowing, is said to be rotational.

On the other hand, A flow, in which the fluid particles do not rotate about their own axes, while flowing, is said to be irrotational.

d) **Laminar and Turbulent Flows**: A flow in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect is said to be laminar flow. In laminar flow, the flow velocity is low and viscosity is very high.

On the other hand, A flow in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles intersect is said to be turbulent flow. In turbulent flow, the flow velocity is high and viscosity is very low.

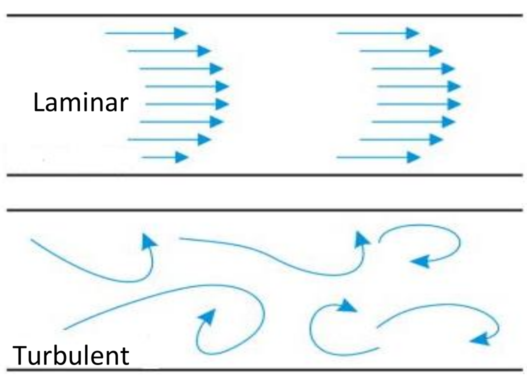


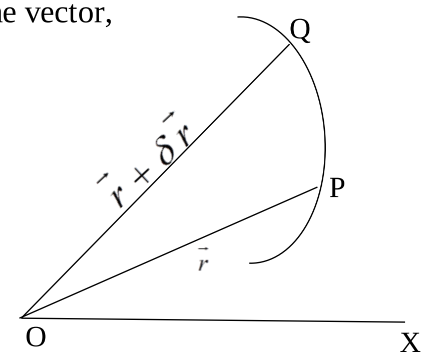
Figure 3.5: Laminar flow and Turbulent flow.

e) Barotropic flow: The flow is said to be barotropic when the pressure is a function of the density.

Velocity of a fluid particle: If a fluid particle reaches at P and Q from O in times t and $t+\delta t$ respectively such that $\overrightarrow{OP}=\vec{r}$ and $\overrightarrow{OQ}=\vec{r}+\delta\vec{r}$.

Then the velocity of the particle at P is defined by the vector,

$$\vec{q}=\lim_{\delta t\to 0}\frac{\vec{r}+\delta\vec{r}-\vec{r}}{\delta t}$$
$$=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}$$
$$=\frac{d\vec{r}}{dt}.$$



If u, v and w be the velocity components of the fluid particle in the direction of x, y and z axes respectively then

$$\vec{q}=u\,i+v\,j+w\,k \quad \cdots(1)$$

Also if (x, y, z) be the Cartesian coordinates of the point then,

$$\vec{r}=x\,i+y\,j+z\,k$$
$$\therefore \frac{d\vec{r}}{dt}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k$$
$$\Rightarrow \vec{q}=\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k \quad \cdots (2)$$

From (1) and (2) we have

$$u=\frac{dx}{dt}\ ,\ v=\frac{dy}{dt}\ \text{ and }\ w=\frac{dz}{dt}.$$

This is the velocity equation of fluid particle.

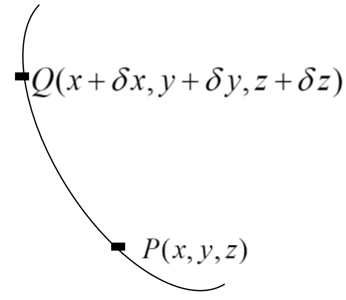
Question-01: Establish the relation between operators of individual and local rate changes.

OR

Obtain the material derivative in the form $\dfrac{d}{dt}=\dfrac{\partial}{\partial t}+\vec{q}\cdot\nabla$.

Answer: Let a fluid particle moves from $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Also we suppose that $f(x, y, z, t)$ be a scalar function associated with some property of fluid. Then the total or individual rate of change of f at the point P at time t is,

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$
$$=\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}$$
$$=\frac{\partial f}{\partial t}+\left(u\,i+v\,j+w\,k\right)\cdot\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)f$$



$$= \frac{\partial f}{\partial t} + \left(\vec{q} \cdot \nabla\right) f$$

Similarly for a vector point function ϕ , we have

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \left(\vec{q} \cdot \nabla\right)\phi$$

Thus for both the functions scalar or vector, we have found the operational equivalence

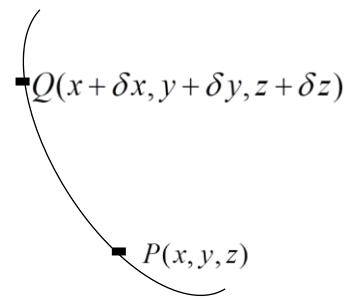
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \nabla$$

where $\frac{d}{dt}$ is individual or material rate of change , $\frac{\partial}{\partial t}$ is local rate of change and $\vec{q} \cdot \nabla$ is convective derivative term.

This is the relation between individual and local rate of changes.

Acceleration of a fluid particle: Let a fluid particle moves from $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Let $\vec{q} = u\,i + v\,j + wk$ be the velocity of the fluid particle at P and $\vec{q} + \delta\vec{q}$ be the velocity of the same fluid particle at Q.

Then the total rate of change of $\vec{q}$ at the point P at time t is,



$$\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{q}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{q}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{q}}{\partial z}\frac{dz}{dt}$$

$$\therefore \vec{a} = \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

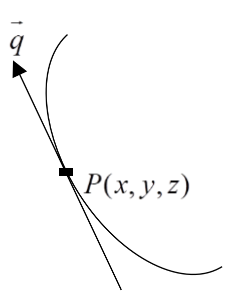
$$= \frac{\partial \vec{q}}{\partial t} + \left(u\,i + v\,j + wk\right) \cdot \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\vec{q}$$

$$= \frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla\right)\vec{q}$$

This is the acceleration of the fluid particle. This shows that the acceleration of a fluid particle can be expressed as the material derivative of the velocity.

Question-02: Define stream line. Establish a differential equation for stream line.

Answer: Streamline or Line of Flow: A streamline is a curve drawn in the fluid such that at any time, the direction of the tangent at any point gives the direction of the velocity of the fluid particle at that point.



Let $\vec{r} = xi + yj + zk$ be the position vector of a point $P(x, y, z)$ on a straight line and let $\vec{q} = ui + vj + wk$ be the fluid velocity at P. If $d\vec{r}$ be an element of the streamline passing through P, then since the direction of $d\vec{r}$ is the same as that of $\vec{q}$, we write

$$\vec{q} \times d\vec{r} = 0$$

$$or, \left(ui + vj + wk\right) \times \left(i\,dx + j\,dy + k\,dz\right) = 0$$

$$or, \begin{vmatrix} i & j & k \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$

$$or, \left(vdz - wdy\right)i + \left(wdx - udz\right)j + \left(udy - vdx\right)k = 0$$

Equating the coefficient of like vectors in both sides we get

$$vdz - wdy = 0 \Rightarrow \frac{dy}{v} = \frac{dz}{w} \qquad \cdots(1)$$

$$wdx - udz = 0 \Rightarrow \frac{dx}{u} = \frac{dz}{w} \qquad \cdots(2)$$

$$udy - vdx = 0 \Rightarrow \frac{dx}{u} = \frac{dy}{v} \qquad \cdots(3)$$

From (1), (2) and (3) we get

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

This is the differential equation for a stream line.

<u>Path line</u>: A curve which is traced out by the motion of a particular fluid particle is said to be a path

line. If $\vec{r} = xi + yj + zk$ be the position vector of a fluid particle at a point and $\vec{q} = ui + vj + wk$ be

the velocity of the particle at that point, then we can write,

$$\vec{q} = \frac{d\vec{r}}{dt}$$

$$or, \ ui + vj + wk = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}$$

$$\therefore \ u = \frac{dx}{dt}\ , \ v = \frac{dy}{dt}, \ w = \frac{dz}{dt}$$

These are the differential equation of the path lines.

Question-03: Write down the differences between stream line and path line.

Answer: It is important to note that stream lines are not, in general, the same as the path lines. Stream lines show how each particle is moving at a given instant of time while the path lines show how a given particle is moving at each instant. Except in the case of steady motion, u, v, w are always functions of the time and hence the stream lines go on changing with the time and the path line of any fluid particle will not in general coincide with a stream line. In the case of steady motion the stream lines remain unchanged as time progressed and hence they are also the path line.

Velocity Potential: When the expression $udx+vdy+wdz$ is an exact differential $-d\phi$, then ϕ is called the velocity potential or velocity function.

i.e. $udx+vdy+wdz=-d\phi$

$$=-\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)$$

$$\therefore u=-\frac{\partial\phi}{\partial x},\quad v=-\frac{\partial\phi}{\partial y}\quad,\ w=-\frac{\partial\phi}{\partial z}$$

In vector form we get, $\vec{q}=-\nabla\phi$

The negative sign in the equation ensures that flow take place from the higher to the lower potentials.

Rotational and Irrotational motion: Let, $\vec{q}$ be the velocity vector of fluid particle.

If $\nabla\times\vec{q}\neq 0$ then the motion is rotational.

If $\nabla\times\vec{q}=0$ then the motion is irrotational. This happens when $\vec{q}=-\nabla\phi$ i.e. the velocity potential exists. This field of $\vec{q}$ is called conservative.

Question-04: What is vorticity vector? Obtain the components of the vorticity in three dimensional Cartesian coordinates. Derive the differential equation of vortex line.

Answer: **Vorticity vector:** If $\vec{q}=ui+vj+wk$ be the velocity vector and the motion is rotational i.e. $\nabla\times\vec{q}\neq 0$, then the vector quantity $\vec{\xi}=\nabla\times\vec{q}$ is called vorticity vector.

2nd part: If ξ_x, ξ_y, ξ_z be the Cartesian components of $\vec{\xi}$ then,

$$\vec{\xi}=\nabla\times\vec{q}$$

$$or,\ \xi_x i+\xi_y j+\xi_z k=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)i+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)j+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)k$$

Equating the coefficients of like vector on both sides we get,

$$\xi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \;,\quad \xi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \;,\quad \xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} .$$

3rd part: A vortex line is a curve drawn in the fluid such that the tangent to it at each point is in the direction of the vorticity vector. Let $\vec{\xi} = \xi_x i + \xi_y j + \xi_z k$ be the vorticity vector and $\vec{r} = xi + yj + zk$ be the position vector of a fluid particle at a point P. If $d\vec{r}$ be an element of vortex line passing through P then since the direction of $d\vec{r}$ is the same as that of $\vec{\xi}$, we write

$$\vec{\xi} \times d\vec{r} = 0$$

$$or,\ \left(\xi_x i + \xi_y j + \xi_z k\right) \times \left(i\,dx + j\,dy + k\,dz\right) = 0$$

$$or,\ \begin{vmatrix} i & j & k \\ \xi_x & \xi_y & \xi_z \\ dx & dy & dz \end{vmatrix} = 0$$

$$or, \left(\xi_y dz - \xi_z dy\right) i + \left(\xi_z dx - \xi_x dz\right) j + \left(\xi_x dy - \xi_y dx\right) k = 0$$

Equating the coefficient of like vectors in both sides we get

$$\xi_y dz - \xi_z dy = 0 \Rightarrow \frac{dy}{\xi_y} = \frac{dz}{\xi_z} \qquad \cdots(1)$$

$$\xi_z dx - \xi_x dz = 0 \Rightarrow \frac{dx}{\xi_x} = \frac{dz}{\xi_z} \qquad \cdots(2)$$

$$\xi_x dy - \xi_y dx = 0 \Rightarrow \frac{dx}{\xi_x} = \frac{dy}{\xi_y} \qquad \cdots(3)$$

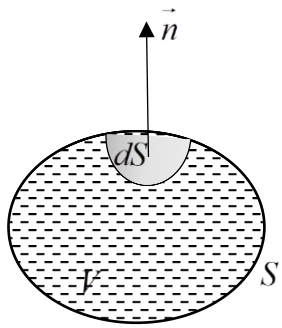
From (1), (2) and (3) we get

$$\frac{dx}{\xi_x} = \frac{dy}{\xi_y} = \frac{dz}{\xi_z}$$

This is the differential equation for a vortex line.

Question-05: Write down the significance of Equation of Continuity/Conservation of Mass. Derive the Equation of Continuity (vector form) by Euler's Method. Write it for steady motion of incompressible fluid.

Answer: **Significance of Equation of Continuity:** The law of conservation of mass states that fluid mass can be neither created nor destroyed. The equation of continuity expresses the law of conservation of mass in mathematical form. Thus, in continuous motion, this equation expresses that the rate of generation of mass within a given volume is equal to the net outflow of mass from the volume.



Consider a closed surface S in the moving fluid such that it encloses a volume V. Let $\vec{n}$ be the unit outward-drawn normal at any element dS, $\vec{q}$ be the fluid velocity and ρ be the density of the fluid.

Then the inward normal velocity is $=-\vec{q}\cdot\vec{n}$

Hence the mass of the fluid per unit time across dS is $=\rho\left(-\vec{q}\cdot\vec{n}\right)dS$

The mass of the fluid per unit time across the whole surface is $=-\int_S \vec{q}\cdot\vec{n}\rho dS$

$$=-\int_V \nabla\cdot\left(\rho\vec{q}\right)dV \quad \cdots(1)$$

[By Gauss Divergence Theorem]

Also the mass of fluid within volume V is $=\int_V \rho dV$

The rate of mass increase within volume V is $=\frac{\partial}{\partial t}\int_V \rho dV \quad \cdots(2)$

By conservation of fluid mass, From (1) and (2) we can write,

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \nabla\cdot\left(\rho\vec{q}\right)dV$$

$$or,\ \int_V\left[\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{q}\right)\right]dV=0$$

This is true for all volume if

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{q}\right)=0 \quad \cdots(3)$$

This is the equation of continuity.

The equation (3) can also be written as

$$\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\vec{q}+\vec{q}\cdot\nabla\rho=0$$

$$or,\ \left(\frac{\partial}{\partial t}+\vec{q}\cdot\nabla\right)\rho+\rho\nabla\cdot\vec{q}=0$$

$$or,\ \frac{d\rho}{dt}+\rho\nabla\cdot\vec{q}=0\ \cdots(4)$$

In case of steady flow, i.e. $\frac{\partial\rho}{\partial t}=0$, the equation gives,

$$\nabla\cdot\left(\rho\vec{q}\right)=0\ \cdots(5)$$

For a homogenous and incompressible fluid the density ρ is constant so $\frac{d\rho}{dt}=0$ and equation (4) reduces to,

$$\nabla\cdot\vec{q}=0\ \cdots(6)$$

Again if the homogeneous incompressible fluid is of the potential kind, then there exists a velocity potential ϕ such that

$$\vec{q}=-\nabla\phi$$

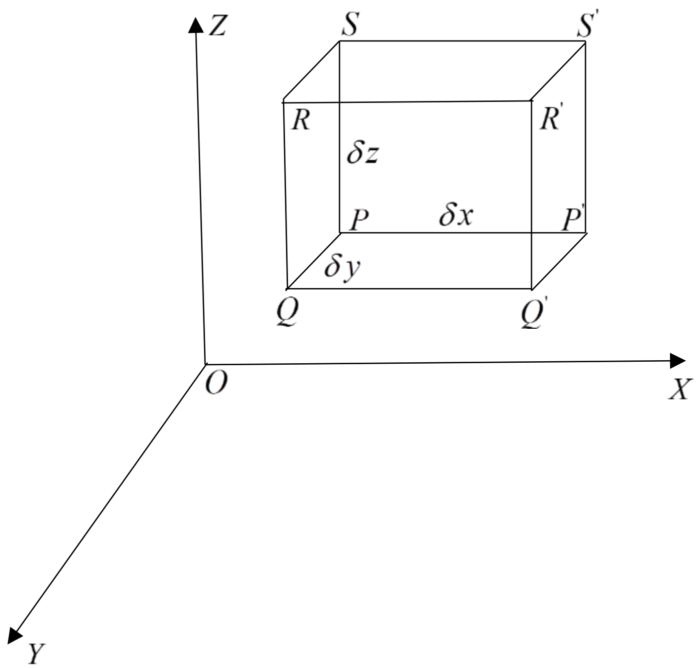
Then the equation (6) becomes,

$$\nabla^2\phi=0$$

which is called Laplace equation.

<u>Question-06</u>: Derive the Equation of Continuity in Cartesian coordinates.

<u>Answer</u>: Consider $\rho\left(x,y,z,t\right)$ be the density of the fluid at the point $P\left(x,y,z\right)$ and u , v , w be the velocity components parallel to the coordinate axes.



Construct a small parallelepiped with edges of length R , δy , δz parallel to their respective axes. Mass of the fluid passing through the face PQRS per unit time is $=\rho u\delta y\delta z$

$$= f(x,y,z) \quad \cdots(1)$$

Mass of the fluid passing out through the face $P'Q'R'S'$ per unit time is $= f(x+\delta x, y, z)$

$$= f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) \quad \cdots(2)$$

[By Taylor's theorem neglecting higher order terms]

The net gain in mass per unit time is

$$= \textit{mass that has come in through } PQRS - \textit{mass that has come out through } P'Q'R'S'$$

$$= f(x,y,z) - f(x,y,z) - \delta x \frac{\partial}{\partial x} f(x,y,z)$$

$$= -\delta x \frac{\partial}{\partial x} f(x,y,z)$$

$$= -\delta x \frac{\partial}{\partial x}(\rho u \delta y \delta z)$$

$$= -\delta x \delta y \delta z \frac{\partial}{\partial x}(\rho u) \quad \cdots(3)$$

Similarly, the net gain in mass due to the faces $PP'S'S$ and $QQ'R'S$ is

$$= -\delta x \delta y \delta z \frac{\partial}{\partial y}(\rho v) \quad \cdots(4)$$

The net gain in mass due to the faces $PP'Q'Q$ and $RR'S'S$ is

$$= -\delta x \delta y \delta z \frac{\partial}{\partial z}(\rho w) \quad \cdots(5)$$

The total mass flow into the elementary parallelepiped per unit time is,

$$= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x \delta y \delta z \quad \cdots(6)$$

But the mass of the fluid within the chosen element in time t is $= \rho \delta x \delta y \delta z$

The total mass gain in the element per unit time is $= \frac{\partial}{\partial t}(\rho \delta x \delta y \delta z)$

$$= \frac{\partial \rho}{\partial t} \delta x \delta y \delta z \quad \cdots(7)$$

By the law of conservation of fluid mass we can write,

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x \delta y \delta z$$

$$or, \ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$or, \ \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$or,\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\rho+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}=0$$

$$or,\ \frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \qquad \cdots(8)$$

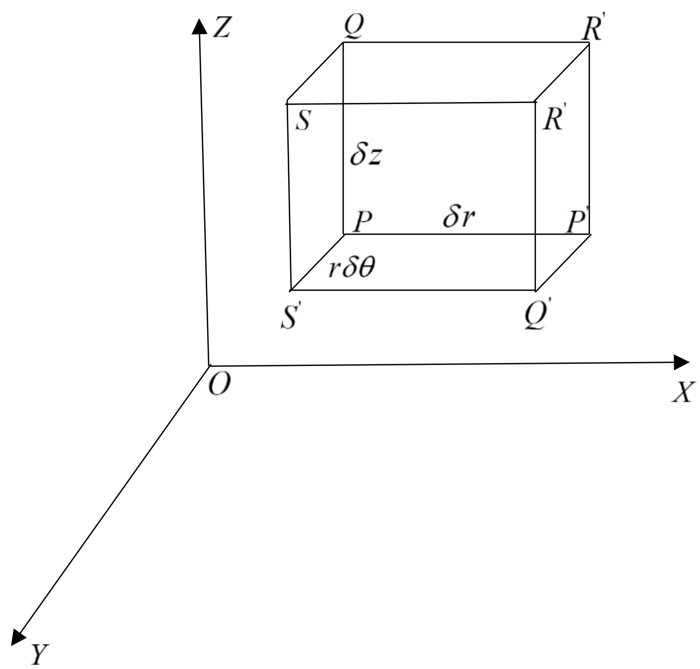
This is the equation of continuity in Cartesian coordinates.

If the is incompressible then $\frac{d\rho}{dt}=0$ the equation (8) reduces to,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,.$$

Question-07: Derive the Equation of Continuity in Cylindrical coordinates.

Answer: Consider $\rho(r,\theta,z,t)$ be the density of the fluid at the point $P(r,\theta,z)$ and q_r, q_θ, q_z be the velocity components parallel to the coordinate axes.



Construct a small parallelepiped with edges of length δr, $r\delta\theta$, δz parallel to their respective axes.

Mass of the fluid passing through the face PQRS per unit time is $=\rho r q_r \delta\theta\delta z$

$$=f(r,\theta,z) \cdots(1)$$

Mass of the fluid passing out through the face $P'Q'R'S'$ per unit time is $=f(r+\delta r,\theta,z)$

$$=f(r,\theta,z)+\delta r\frac{\partial}{\partial r}f(r,\theta,z) \qquad \cdots(2)$$

[By Taylor's theorem neglecting higher order terms]

The net gain in mass per unit time is

$$= mass\, that\, has\, come\, in\, through\, PQRS - mass\, that\, has\, come\, out\, through\, P'Q'R'S'$$

$$= f(r,\theta,z) - f(r,\theta,z) - \delta r \frac{\partial}{\partial r} f(r,\theta,z)$$

$$= -\delta r \frac{\partial}{\partial r} f(r,\theta,z)$$

$$= -\delta r \frac{\partial}{\partial r}(\rho r q_r \delta\theta\delta z)$$

$$= -\delta r\delta\theta\delta z \frac{\partial}{\partial r}(\rho r q_r) \quad \cdots (3)$$

Similarly, the net gain in mass due to the faces $PP'S'S \text{ and } QQ'R'S$ is

$$= -r\delta\theta \frac{\partial}{r\partial\theta}(\rho q_\theta \delta r\delta z)$$

$$= -\delta r\delta\theta\delta z \frac{\partial}{\partial\theta}(\rho q_\theta) \quad \cdots (4)$$

The net gain in mass due to the faces $PP'Q'Q \text{ and } RR'S'S$ is

$$= -r\delta r\delta\theta\delta z \frac{\partial}{\partial z}(\rho q_z) \quad \cdots (5)$$

The total mass flow into the elementary parallelepiped per unit time is,

$$= -\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial\theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]\delta r\delta\theta\delta z \quad \cdots (6)$$

But the mass of the fluid within the chosen element in time t is $= \rho r\delta r\delta\theta\delta z$

The total mass gain in the element per unit time is $= \frac{\partial}{\partial t}(\rho r\delta r\delta\theta\delta z)$

$$= \frac{\partial\rho}{\partial t} r\delta r\delta\theta\delta z \quad \cdots (7)$$

By the law of conservation of fluid mass we can write,

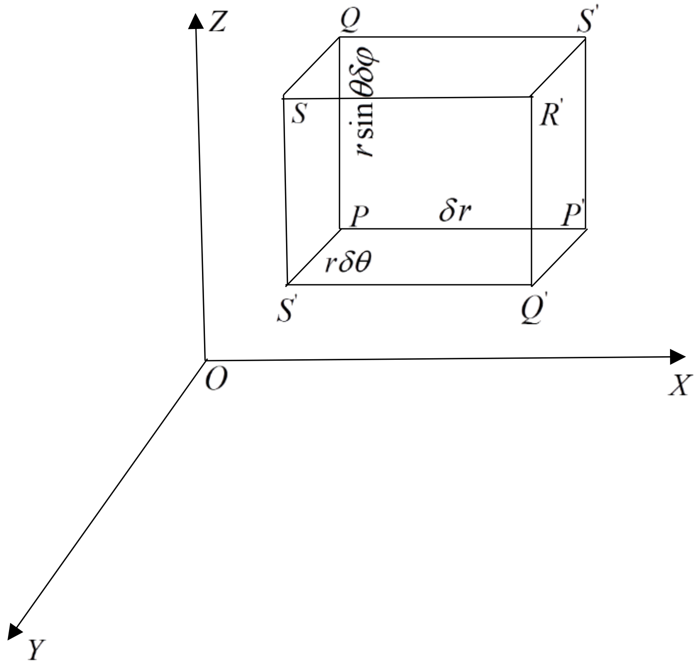
$$\frac{\partial\rho}{\partial t} r\delta r\delta\theta\delta z = -\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial\theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]\delta r\delta\theta\delta z$$

$$or,\ \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

This is the equation of continuity in Cylindrical Coordinates.

<u>Question-08</u>: Derive the Equation of Continuity in Spherical coordinates.

<u>Answer</u>: Consider $\rho(r,\theta,\varphi,t)$ be the density of the fluid at the point $P(r,\theta,\varphi)$ and q_r, q_θ, q_φ be the velocity components parallel to the coordinate axes.



Construct a small parallelepiped with edges of length δr, $r\delta\theta$, $r\sin\theta\delta\varphi$ parallel to their respective axes.

Mass of the fluid passing through the face PQRS per unit time is $=\rho r^2 q_r \sin\theta\delta\theta\delta\varphi$

$$= f(r,\theta,\varphi) \cdots (1)$$

Mass of the fluid passing out through the face $P'Q'R'S'$ per unit time is $= f(r+\delta r,\theta,\varphi)$

$$= f(r,\theta,\varphi)+\delta r\frac{\partial}{\partial r}f(r,\theta,\varphi) \quad \cdots (2)$$

[By Taylor's theorem neglecting higher order terms]

The net gain in mass per unit time is

$$= mass\, that\, has\, come\, in\, through\, PQRS - mass\, that\, has\, come\, out\, through\, P'Q'R'S'$$

$$= f(r,\theta,\varphi)- f(r,\theta,\varphi)-\delta r\frac{\partial}{\partial r}f(r,\theta,\varphi)$$

$$=-\delta r\frac{\partial}{\partial r}f(r,\theta,\varphi)$$

$$=-\delta r\frac{\partial}{\partial r}\left(\rho r^2 q_r \sin\theta\delta\theta\delta\varphi\right)$$

$$=-\delta r\delta\theta\delta\varphi\frac{\partial}{\partial r}\left(\rho r^2 q_r \sin\varphi\right) \quad \cdots (3)$$

Similarly, the net gain in mass due to the faces $PP'S'S\ and\ QQ'R'S$ is

$$= -r\delta\theta \frac{\partial}{r\partial\theta}\left(\rho q_\theta r \sin\theta \delta r \delta\varphi\right)$$

$$= -r\delta r\delta\theta\delta\varphi \frac{\partial}{\partial\theta}\left(\rho q_\theta \sin\theta\right) \ \cdots (4)$$

The net gain in mass due to the faces $PP'Q'Q$ and $RR'S'S$ is

$$= -r\sin\theta\delta\varphi \frac{\partial}{r\sin\theta\partial\varphi}\left(\rho q_\varphi r\delta r\delta\theta\right)$$

$$= -r\delta r\delta\theta\delta\varphi \frac{\partial}{\partial\varphi}\left(\rho q_\varphi\right) \ \cdots (5)$$

The total mass flow into the elementary parallelepiped per unit time is,

$$= -\left[\frac{\partial}{\partial r}\left(\rho r^2 q_r \sin\theta\right) + r\frac{\partial}{\partial\theta}\left(\rho q_\theta \sin\theta\right) + r\frac{\partial}{\partial\varphi}\left(\rho q_\varphi\right)\right]\delta r\delta\theta\delta\varphi \quad \cdots (6)$$

But the mass of the fluid within the chosen element in time t is $= \rho r^2 \sin\theta \delta r\delta\theta\delta\varphi$

The total mass gain in the element per unit time is $= \dfrac{\partial}{\partial t}\left(\rho r^2 \sin\theta\delta r\delta\theta\delta\varphi\right)$

$$= \frac{\partial\rho}{\partial t} r^2 \sin\theta\delta r\delta\theta\delta\varphi \ \cdots (7)$$

By the law of conservation of fluid mass we can write,

$$\frac{\partial\rho}{\partial t} r^2 \sin\theta\delta r\delta\theta\delta\varphi = -\left[\frac{\partial}{\partial r}\left(\rho r^2 q_r \sin\theta\right) + r\frac{\partial}{\partial\theta}\left(\rho q_\theta \sin\theta\right) + r\frac{\partial}{\partial\varphi}\left(\rho q_\varphi\right)\right]\delta r\delta\theta\delta\varphi$$

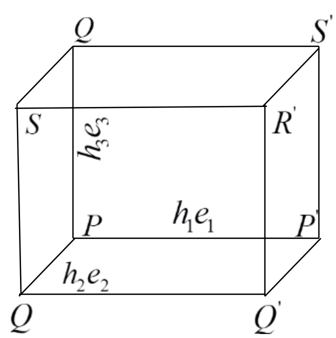
$$or, \ \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 q_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\rho q_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\rho q_\varphi\right) = 0$$

This is the equation of continuity in Spherical coordinates.

NOTE: In curvilinear coordinate system, if we have, $r = r\left(u_1, u_2, u_3\right)$ then

$$dr = \frac{\partial r}{\partial u_1}du_1 + \frac{\partial r}{\partial u_2}du_2 + \frac{\partial r}{\partial u_3}du_3$$

$$= h_1 e_1 du_1 + h_2 e_2 du_2 + h_3 e_3 du_3$$



Here $e_1 = \dfrac{\frac{\partial r}{\partial u_1}}{\left|\frac{\partial r}{\partial u_1}\right|}$, $e_2 = \dfrac{\frac{\partial r}{\partial u_2}}{\left|\frac{\partial r}{\partial u_2}\right|}$, $e_3 = \dfrac{\frac{\partial r}{\partial u_3}}{\left|\frac{\partial r}{\partial u_3}\right|}$ are unit vectors.

$$h_1 = \left|\frac{\partial r}{\partial u_1}\right|, h_2 = \left|\frac{\partial r}{\partial u_2}\right|, h_3 = \left|\frac{\partial r}{\partial u_3}\right|$$

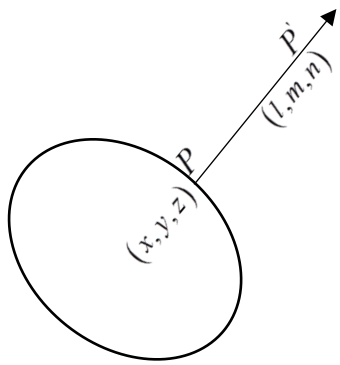
In Cylindrical system $\left(r,\theta,z\right)$**:** $x = r\cos\theta$, $y = r\sin\theta$ $z = z$, $h_r = 1, h_\theta = r$, $h_z = 1$

In Spherical system $\left(r,\theta,\varphi\right)$**:** $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$, $h_r = 1, h_\theta = r$, $h_\varphi = r\sin\theta$.

Question-09: Find the condition that the surface $F(x,y,z,t)=0$ may be a boundary surface of a fluid in motion.

Answer: Let the equation of the boundary be,

$$F(x,y,z,t)=0 \qquad \cdots(1)$$



Consider a point $P(x,y,z)$ on the boundary surface and let (l,m,n) be the direction cosines of the normal at P to the surface. After time δt, let P come to P' such that $PP' = \tau\delta t$ where τ is the normal velocity of the boundary at P. Evidently the projections of PP' on the coordinate axes are $l\tau\delta t$, $m\tau\delta t$, $n\tau\delta t$. Hence the coordinates of P' become $(x+l\tau\delta t, y+m\tau\delta t, z+n\tau\delta t)$. But the point P' lies on the boundary surface at time $t+\delta t$, so we have

$$F(x+l\tau\delta t, y+m\tau\delta t, z+n\tau\delta t, t+\delta t)=0 \quad \cdots(2)$$

Expanding by Taylor's theorem, we get

$$F(x,y,z,t)+l\tau\delta t\frac{\partial F}{\partial x}+m\tau\delta t\frac{\partial F}{\partial y}+n\tau\delta t\frac{\partial F}{\partial z}+\delta t\frac{\partial F}{\partial t}+\cdots=0$$

$$or,\ l\tau\delta t\frac{\partial F}{\partial x}+m\tau\delta t\frac{\partial F}{\partial y}+n\tau\delta t\frac{\partial F}{\partial z}+\delta t\frac{\partial F}{\partial t}=0 \qquad \left[\because F(x,y,z,t)=0\right]$$

$$or,\tau=-\frac{\frac{\partial F}{\partial t}}{l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}} \quad \cdots(3)$$

But $\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}$ be the direction ratios of the normal PP' to the surface $F(x,y,z,t)=0$, so we have

$$\frac{l}{\frac{\partial F}{\partial x}}=\frac{m}{\frac{\partial F}{\partial y}}=\frac{n}{\frac{\partial F}{\partial z}}=\frac{1}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}} \quad \cdots(4)$$

$$\therefore\ l=\frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}},\ m=\frac{\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}},\ n=\frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}}$$

Putting the values of l,m,n in (3), we get

$$\tau = -\frac{\dfrac{\partial F}{\partial t}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}} \quad \cdots (5)$$

But the normal component of the velocity of the fluid particle must be equal to the normal component of the velocity of the surface.

i.e. $\tau = lu + mv + nw$

$$= \frac{u\dfrac{\partial F}{\partial x} + v\dfrac{\partial F}{\partial y} + w\dfrac{\partial F}{\partial z}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}} \quad \cdots (6)$$

Equating (5) and (6), we get

$$-\frac{\partial F}{\partial t} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$

$$or,\ \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

This is the required condition.

Also when the boundary is at rest, $\dfrac{\partial F}{\partial t} = 0$ then the condition for representing the boundary surface is,

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0.$$

Problems

Problem-01: Determine the acceleration and components of acceleration of a fluid particle from the flow field

$$\vec{q} = i\,Ax^2y + j\,By^2zt + k\,Cz^2t$$

where A, B and C are constants. Also find the vorticity components.

Solution: The given flow field is,

$$\vec{q} = i\,Ax^2y + j\,By^2zt + k\,Cz^2t$$

Here, $u = Ax^2y$, $v = By^2zt$ and $w = Cz^2t$

$$\frac{\partial \vec{q}}{\partial t} = \frac{\partial}{\partial t}\left(i\,Ax^2y + j\,By^2zt + k\,Cz^2t\right) = j\,By^2z + k\,Cz^2$$

$$\frac{\partial \vec{q}}{\partial x} = \frac{\partial}{\partial x}\left(i\,Ax^2y + j\,By^2zt + k\,Cz^2t\right) = i\,2Axy$$

$$\frac{\partial \vec{q}}{\partial y} = \frac{\partial}{\partial y}\left(i\,Ax^2y + j\,By^2zt + k\,Cz^2t\right) = i\,Ax^2 + j\,2Byzt$$

$$\frac{\partial \vec{q}}{\partial z} = \frac{\partial}{\partial z}\left(i\,Ax^2y + j\,By^2zt + k\,Cz^2t\right) = j\,By^2t + k\,2Czt$$

Let $\vec{a}$ be the acceleration of a fluid particle, then

$$\vec{a} = \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

$$= j\,By^2z + k\,Cz^2 + Ax^2y\left(i\,2Axy\right) + By^2zt\left(i\,Ax^2 + j\,2Byzt\right) + Cz^2t\left(j\,By^2t + k\,2Czt\right)$$

$$= j\,By^2z + k\,Cz^2 + i2A^2x^3y^2 + i\,ABx^2y^2zt + j\,2B^2y^3z^2t^2 + j\,BCy^2z^2t^2 + k\,2C^2z^3t^2$$

$$= iA\left(2Ax^3y^2 + Bx^2y^2zt\right) + j\,B\left(y^2z + 2By^3z^2t^2 + Cy^2z^2t^2\right) + k\,C\left(z^2 + 2Cz^3t^2\right)$$

This is the required acceleration.

The components of acceleration are,

$$a_x = A\left(2Ax^3y^2 + Bx^2y^2zt\right)$$

$$a_y = B\left(y^2z + 2By^3z^2t^2 + Cy^2z^2t^2\right)$$

$$a_z = C\left(z^2 + 2Cz^3t^2\right)$$

2nd part: The vorticity components are,

$$\xi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 - By^2t = -By^2t$$

$$\xi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 - 0 = 0$$

$$\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - Ax^2 = -Ax^2$$

<u>Problem-02</u>: The velocity vector in the flow field is given by $\vec{q} = i\left(Az - By\right) + j\left(Bx - cz\right) + k\left(Cy - Ax\right)$ where A, B and C are constants. Determine the equation of the vortex lines.

<u>Solution</u>: The given flow field is,

$$\vec{q} = i\left(Az - By\right) + j\left(Bx - cz\right) + k\left(Cy - Ax\right)$$

Here, $u = Az - By$, $v = Bx - Cz$, $w = Cy - Ax$

The vortex components are,

$$\xi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = C + C = 2C$$

$$\xi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = A + A = 2A$$

$$\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = B + B = 2B$$

The equation of the vortex lines are,

$$\frac{dx}{\xi_x}=\frac{dy}{\xi_y}=\frac{dz}{\xi_z}$$

$$or,\ \frac{dx}{2C}=\frac{dy}{2A}=\frac{dz}{2B}$$

$$or,\ \frac{dx}{C}=\frac{dy}{A}=\frac{dz}{B}\quad\cdots(1)$$

From first two fractions we get

$$\frac{dx}{C}=\frac{dy}{A}$$

$$or,\ Adx-Cdy=0$$

Integrating, $Ax-Cy=c_1 \quad \cdots(2)$

From last two fractions we get,

$$\frac{dy}{A}=\frac{dz}{B}$$

$$or,\ Bdy-Adz=0$$

Integrating, $By-Az=c_2 \quad \cdots(3)$

Equations (2) and (3) constitute the required vortex lines.

Problem-03: Find vorticity of the fluid motion when velocity components are $u=x+y, v=-x-y$

Answer: The given velocity components are,

$$u=x+y,\ v=-x-y$$

The vortex components are,

$$\xi_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0-0=0$$

$$\xi_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0-0=0$$

$$\xi_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-1-1=-2$$

The vorticity of the fluid motion is,

$$\vec{\xi}=\xi_x i+\xi_y j+\xi_z k$$

$$or,\ \vec{\xi}=0i+0j-2k$$

$$or,\ \vec{\xi}=-2k$$

Problem-04: Find the streamlines and path lines when $u=\frac{x}{1+t},\ v=\frac{y}{1+t},\ w=\frac{z}{1+t}$.

Answer: Here we have,

$$u=\frac{x}{1+t},\ v=\frac{y}{1+t},\ w=\frac{z}{1+t}$$

The equation of streamlines are,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$or,\ \frac{dx}{\frac{x}{1+t}}=\frac{dy}{\frac{y}{1+t}}=\frac{dz}{\frac{z}{1+t}}$$

$$or,\ \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}\qquad\cdots(1)$$

Taking first two fractions we get,

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating, $\ln x=\ln y+\ln c_1$

$$\therefore x=c_1y\qquad\cdots(2)$$

Again, taking last two fractions we get,

$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating, $\ln y=\ln z+\ln c_2$

$$\therefore x=c_2y\qquad\cdots(3)$$

The desired streamlines are given by the intersection of (2) and (3).

2nd part: The path lines are,

$$u=\frac{dx}{dt}$$

$$or,\ \frac{x}{1+t}=\frac{dx}{dt}$$

$$or,\ \frac{dx}{x}=\frac{dt}{1+t}$$

Integrating $\ln x=\ln(1+t)+\ln c_1$

$$\therefore x=c_1(1+t)\qquad\cdots(4)$$

Again, $v=\frac{dy}{dt}$

$$or,\ \frac{y}{1+t}=\frac{dy}{dt}$$

$$or,\ \frac{dy}{y}=\frac{dt}{1+t}$$

Integrating $\ln x=\ln(1+t)+\ln c_2$

$$\therefore x=c_2(1+t)\qquad\cdots(5)$$

Again, $$w=\frac{dz}{dt}$$

$$or,\ \frac{z}{1+t}=\frac{dz}{dt}$$

$$or,\ \frac{dz}{z}=\frac{dt}{1+t}$$

Integrating $\ln z=\ln(1+t)+\ln c_3$

$$\therefore z=c_3(1+t) \quad \cdots(6).$$

These are the path lines.

Problem-05: The velocity field at a point in a fluid is given by $\vec{q}=\left(\frac{x}{t},y,0\right)$. Obtain path lines.

Answer: Here we have, $\vec{q}=\left(\frac{x}{t},y,0\right)$

$$\therefore\ u=\frac{x}{t},\ v=y,\ w=0$$

The path lines are,

$$u=\frac{dx}{dt}$$

$$or,\ \frac{x}{t}=\frac{dx}{dt}$$

$$or,\ \frac{dx}{x}=\frac{dt}{t}$$

Integrating, $\ln x=\ln t+\ln c_1$

$$\therefore x=c_1t$$

Again, $$v=\frac{dy}{dt}$$

$$or,\ y=\frac{dy}{dt}$$

$$or,\ \frac{dy}{y}=dt$$

Integrating $\ln y=t+\ln c_2$

$$\therefore y=c_2e^t$$

Again, $$w=\frac{dz}{dt}$$

$$or,\ 0=\frac{dz}{dt}$$

$$or,\ dz=0$$

Integrating, $z=c_3$.

Problem-06: Find the streamlines when the velocity field is $\vec{q}=i(3y-2z)+j(z-3x)+k(2x-y)$.

Answer: Here we have, $\vec{q}=i(3y-2z)+j(z-3x)+k(2x-y)$

$$u=3y-2z,\ v=z-3x,\ w=2x-y$$

The streamlines are,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$or,\ \frac{dx}{3y-2z}=\frac{dy}{z-3x}=\frac{dz}{2x-y}$$

$$or,\ \frac{dx}{3y-2z}=\frac{dy}{z-3x}=\frac{dz}{2x-y}=\frac{xdx+ydy+zdz}{0} \qquad \cdots(1)$$

Taking first fraction we get,

$$\frac{dx}{3y-2z}=\frac{xdx+ydy+zdz}{0}$$

$$or,\ xdx+ydy+zdz=0$$

Integrating, $\quad \dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}=\dfrac{c}{2}$

$$\therefore\ x^2+y^2+z^2=c$$

These are the required streamlines.

Problem-07: A velocity field is given by $\vec{q}=-xi+(y+t)j$. Find the equation of streamlines at time $t=2$ for this field.

Answer: Here we have, $\vec{q}=-xi+(y+t)j$

$$u=-x,\ v=y+t$$

Since the motion is two dimensional so $w=0$.

The streamlines are given by,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$or,\ \frac{dx}{-x}=\frac{dy}{y+t}=\frac{dz}{0}$$

Taking the first two fractions we get,

$$\frac{dx}{-x}=\frac{dy}{y+t}$$

$$or,\ \frac{dx}{x}+\frac{dy}{y+t}=0$$

Integrating, $\quad \ln x+\ln(y+t)=\ln c_1$

$$\therefore\ x(y+t)=c_1$$

Taking the last two fractions we get,

$$\frac{dy}{y+t}=\frac{dz}{0}$$
$$dz=0$$

Integrating, $z=c_2$

At $t=2$ we get,

$$x(y+2)=c_1 \quad \text{and} \quad z=c_2$$

The required streamlines are given by the curves of intersection of

$$x(y+2)=c_1 \quad \text{and} \quad z=c_2.$$

Problem-08: Show that $u=-\frac{2xyz}{(x^2+y^2)^2}, v=\frac{(x^2-y^2)z}{(x^2+y^2)^2}, w=\frac{y}{x^2+y^2}$ are the velocity components of possible liquid motion. Is this motion irrational?

Answer: Here we have, $u=-\frac{2xyz}{(x^2+y^2)^2},\ v=\frac{(x^2-y^2)z}{(x^2+y^2)^2},\ w=\frac{y}{x^2+y^2}$.

Now, $$\frac{\partial u}{\partial x}=-\frac{\partial}{\partial x}\left\{\frac{2xyz}{(x^2+y^2)^2}\right\}$$
$$=-\left\{\frac{(x^2+y^2)^2.2yz-2xyz.2(x^2+y^2).2x}{(x^2+y^2)^4}\right\}$$
$$=-\left\{\frac{2x^2yz+2y^3z-8x^2yz}{(x^2+y^2)^3}\right\}$$
$$=\frac{6x^2yz-2y^3z}{(x^2+y^2)^3} \quad \cdots(1)$$

and $$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left\{\frac{(x^2-y^2)z}{(x^2+y^2)^2}\right\}$$
$$=\left\{\frac{(x^2+y^2)^2(0-2y)z-(x^2-y^2)z.2(x^2+y^2).2y}{(x^2+y^2)^4}\right\}$$
$$=\left\{\frac{(x^2+y^2)(-2yz)-4yz(x^2-y^2)}{(x^2+y^2)^3}\right\}$$
$$=\left\{\frac{-2x^2yz-2y^3z-4x^2yz+4y^3z}{(x^2+y^2)^3}\right\}$$

$$= \frac{2y^3z - 6x^2yz}{\left(x^2+y^2\right)^3} \quad \cdots (2)$$

and $$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left\{\frac{y}{x^2+y^2}\right\}$$

$$= 0 \quad \cdots (3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{6x^2yz - 2y^3z}{\left(x^2+y^2\right)^3} + \frac{2y^3z - 6x^2yz}{\left(x^2+y^2\right)^3} + 0$$

$$= \frac{6x^2yz - 2y^3z + 2y^3z - 6x^2yz}{\left(x^2+y^2\right)^3}$$

$$\therefore \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since the equation of continuity is satisfied so the liquid motion is possible. (**Showed**)

2nd Part: We know, $\nabla \times \vec{q} = \xi_x i + \xi_y j + \xi_w k \qquad \cdots (1)$

Now, $$\xi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$= \frac{\partial}{\partial y}\left\{\frac{y}{x^2+y^2}\right\} - \frac{\partial}{\partial z}\left\{\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}\right\}$$

$$= \frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} - \frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= 0$$

$$\xi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$= -\frac{\partial}{\partial z}\left\{\frac{2xyz}{\left(x^2+y^2\right)^2}\right\} - \frac{\partial}{\partial x}\left\{\frac{y}{\left(x^2+y^2\right)}\right\}$$

$$= -\frac{2xy}{\left(x^2+y^2\right)^2} + \frac{2xy}{\left(x^2+y^2\right)^2}$$

$$= 0$$

$$\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{2xyz}{\left(x^2+y^2\right)^2}\right\}$$

$$=\frac{2xz\left(3y^2-x^2\right)}{\left(x^2+y^2\right)^3}-\frac{2xz\left(3y^2-x^2\right)}{\left(x^2+y^2\right)^3}$$

$$=0$$

Putting these values in (1) we get,

$$\nabla\times\vec{q}=0.$$

Hence the motion is irrotational.

Problem-09: If the velocity of an incompressible fluid at the point (x,y,z) is given by $\frac{3xz}{r^5}$, $\frac{3yz}{r^5}$, $\frac{3z^2-r^2}{r^5}$, where $r^2=x^2+y^2+z^2$, prove that the liquid motion is possible and that the velocity potential is $\frac{\cos\theta}{r^2}$.

Answer: Here we have, $u=\frac{3xz}{r^5}$, $v=\frac{3yz}{r^5}$, $w=\frac{3z^2-r^2}{r^5}$

where $r^2=x^2+y^2+z^2$.

$$\therefore\ 2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

Now, $\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{\frac{3xz}{r^5}\right\}$

$$=\frac{3zr^5-3xz.5r^4.\frac{\partial r}{\partial x}}{r^{10}}$$

$$=\frac{3zr^5-3xz.5r^4.\frac{x}{r}}{r^{10}}$$

$$=\frac{3zr^2-15x^2z}{r^7}\quad\cdots(1)$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left\{\frac{3yz}{r^5}\right\}$$

$$=\frac{3zr^5-3yz.5r^4.\frac{\partial r}{\partial y}}{r^{10}}$$

$$=\frac{3zr^5-3yz.5r^4.\frac{y}{r}}{r^{10}}$$

$$= \frac{3zr^2 - 15y^2z}{r^7} \quad \cdots(2)$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left\{\frac{3z^2 - r^2}{r^5}\right\}$$

$$= \frac{r^5\left(6z - 2r\dfrac{\partial r}{\partial z}\right) - \left(3z^2 - r^2\right)5r^4.\dfrac{\partial r}{\partial z}}{r^{10}}$$

$$= \frac{r^5\left(6z - 2r\dfrac{z}{r}\right) - \left(3z^2 - r^2\right)5r^4.\dfrac{z}{r}}{r^{10}}$$

$$= \frac{4zr^5 - 5zr^3\left(3z^2 - r^2\right)}{r^{10}}$$

$$= \frac{4zr^2 - 5z\left(3z^2 - r^2\right)}{r^7}$$

$$= \frac{9zr^2 - 15z^3}{r^7} \quad \cdots(3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3zr^2 - 15x^2z}{r^7} + \frac{3zr^2 - 15y^2z}{r^7} + \frac{9zr^2 - 15z^3}{r^7}$$

$$= \frac{15zr^2 - 15z\left(x^2 + y^2 + z^2\right)}{r^7}$$

$$= \frac{15zr^2 - 15zr^2}{r^7}$$

$$\therefore \ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since the equation of continuity is satisfied so the liquid motion is possible. (**Proved**)

2nd Part: For velocity potential ϕ we can write,

$$d\phi = -\left(udx + vdy + wdz\right)$$

$$= -\left(\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2 - r^2}{r^5}dz\right)$$

$$= -\left(\frac{3xzdx + 3yzdy + 3z^2dz - r^2dz}{r^5}\right)$$

$$= \frac{r^2dz - 3xzdx - 3yzdy - 3z^2dz}{r^5}$$

$$= \frac{r^2dz - 3z\left(xdx + ydy + zdz\right)}{r^5}$$

$$=\frac{r^2dz-3z.rdr}{r^5} \qquad \left[\because r^2=x^2+y^2+z^2 \Rightarrow 2rdr=2xdx+2ydy+2zdz\right]$$

$$=\frac{r^3d(z)-zd(r^3)}{(r^3)^2}$$

$$=d\left(\frac{z}{r^3}\right)$$

Integrating, $\phi=\dfrac{z}{r^3} \qquad \cdots(4)$

Since integrating constant has no effect on ϕ so it has been neglected.

In Spherical polar coordinates (r,θ,ϕ) we have $z=r\cos\theta$

The equation (4) becomes,

$$\phi=\frac{\cos\theta}{r^2}$$

This is the required velocity potential. (**Proved**)

<u>Problem-10</u>: Test whether the motion specified by $\vec{q}=\dfrac{k^2(xj-yi)}{x^2+y^2}$ $[k=cons\tan t]$, is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

<u>Answer</u>: Here we have, $\vec{q}=\dfrac{k^2(xj-yi)}{x^2+y^2}$ $\quad [k=cons\tan t]$

$$u=-\frac{k^2y}{x^2+y^2},\ v=\frac{k^2x}{x^2+y^2},\ w=0$$

Now, $\dfrac{\partial u}{\partial x}=-\dfrac{\partial}{\partial x}\left\{\dfrac{k^2y}{x^2+y^2}\right\}$

$$=\frac{2k^2xy}{(x^2+y^2)^2} \quad \cdots(1)$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left\{\frac{k^2x}{x^2+y^2}\right\}$$

$$=-\frac{2k^2xy}{(x^2+y^2)^2} \quad \cdots(2)$$

$$\frac{\partial w}{\partial z}=0 \quad \cdots(3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{2k^2xy}{\left(x^2+y^2\right)^2}-\frac{2k^2xy}{\left(x^2+y^2\right)^2}+0$$
$$=0$$
$$\therefore\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Since the equation of continuity is satisfied so the motion is possible. (**Showed**)

2nd Part: The equation of streamlines are,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$
$$or,\ \frac{dx}{-\dfrac{k^2y}{x^2+y^2}}=\frac{dy}{\dfrac{k^2x}{x^2+y^2}}=\frac{dz}{0}$$
$$or,\ \frac{dx}{-y}=\frac{dy}{x}=\frac{dz}{0}\qquad\cdots(4)$$

Taking the first two fractions we get,

$$\frac{dx}{-y}=\frac{dy}{x}$$
$$or,\ xdx+ydy=0$$

Integrating, $\dfrac{x^2}{2}+\dfrac{y^2}{2}=\dfrac{c_1}{2}$

$$or,\ x^2+y^2=c_1\quad\cdots(5)$$

Taking the last two fractions we get,

$$\frac{dy}{x}=\frac{dz}{0}$$
$$or,\ dz=0$$

Integrating, $z=c_2\quad\cdots(6)$

The required equation of streamlines are given by the intersection of the curves of equations (5) and (6).

3rd Part: We know, $\nabla\times\vec{q}=\xi_x i+\xi_y j+\xi_w k\qquad\cdots(7)$

Now, $\xi_x=\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}$

$$=\frac{\partial}{\partial y}\{0\}-\frac{\partial}{\partial z}\left\{\frac{k^2x}{x^2+y^2}\right\}$$
$$=0-0$$
$$=0$$
$$\xi_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}$$

$$= -\frac{\partial}{\partial z}\left\{\frac{k^2 y}{x^2+y^2}\right\} - \frac{\partial}{\partial x}\{0\}$$

$$= -\,0 - 0$$

$$= 0$$

$$\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial x}\left\{\frac{k^2 x}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{k^2 y}{x^2+y^2}\right\}$$

$$= \frac{\left(x^2+y^2\right)k^2 - k^2 x.2x}{\left(x^2+y^2\right)^2} + \frac{\left(x^2+y^2\right)k^2 - k^2 y.2y}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2k^2 - k^2x^2}{\left(x^2+y^2\right)^2} + \frac{x^2k^2 - k^2y^2}{\left(x^2+y^2\right)^2}$$

$$= 0$$

Putting these values in (7) we get,

$$\nabla \times \vec{q} = 0 .$$

Hence the motion is irrotational and the flow is of the potential kind.

2nd Part: For velocity potential ϕ we can write,

$$d\phi = -\left(udx + vdy + wdz\right)$$

$$or,\ \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = -\left(-\frac{k^2 y}{x^2+y^2}dx + \frac{k^2 x}{x^2+y^2}dy + 0.dz\right) \quad \cdots(8)$$

Equating the like terms of both sides of (8) we get,

$$\frac{\partial \phi}{\partial x} = \frac{k^2 y}{x^2+y^2} \qquad \cdots(9)$$

$$\frac{\partial \phi}{\partial y} = -\frac{k^2 x}{x^2+y^2} \qquad \cdots(10)$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \cdots(11)$$

By equation (9) we can say that the velocity potential ϕ is function of x, y only, so that $\phi = \phi\left(x, y\right)$.

Now integrating (9) we get,

$$\phi = k^2 \tan^{-1}\left(\frac{x}{y}\right) + f\left(y\right) \qquad \cdots(12)$$

where $f\left(y\right)$ is an arbitrary function of y.

From (12) we have,

$$\frac{\partial \phi}{\partial y} = -\frac{k^2 x}{x^2 + y^2} + f^{'}(y) \qquad \cdots (13)$$

By equations (10) and (13) we have

$$f^{'}(y) = 0$$

Integrating, $f(y) = c \;(cons\tan t)$

Since we can omit the constant while writing the velocity potential, the required velocity potential is,

$$\phi = k^2 \tan^{-1}\left(\frac{x}{y}\right).$$

Problem-11: Show that the function, $\phi = \frac{1}{2}\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$ gives the velocity potential of a possible motion.

Answer: The given velocity potential is,

$$\phi = \frac{1}{2}\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

$$or,\; \phi = \frac{1}{2}\left[\ln\left\{(x+a)^2 + y^2\right\} - \ln\left\{(x-a)^2 + y^2\right\}\right] \qquad \cdots (1)$$

We know, $u = -\frac{\partial \phi}{\partial x}$

$$= -\frac{1}{2}\frac{\partial}{\partial x}\left[\ln\left\{(x+a)^2 + y^2\right\} - \ln\left\{(x-a)^2 + y^2\right\}\right]$$

$$= -\frac{1}{2}\left[\frac{2(x+a)}{(x+a)^2 + y^2} - \frac{2(x-a)}{(x+a)^2 + y^2}\right]$$

$$= -\frac{x+a}{(x+a)^2 + y^2} + \frac{x-a}{(x-a)^2 + y^2}$$

$$\therefore \; \frac{\partial u}{\partial x} = \frac{(x+a)^2 - y^2}{\left[(x+a)^2 + y^2\right]^2} - \frac{(x-a)^2 - y^2}{\left[(x-a)^2 + y^2\right]^2} \qquad \cdots (2)$$

$$v = -\frac{\partial \phi}{\partial y}$$

$$= -\frac{1}{2}\frac{\partial}{\partial y}\left[\ln\left\{(x+a)^2 + y^2\right\} - \ln\left\{(x-a)^2 + y^2\right\}\right]$$

$$= -\frac{1}{2}\left[\frac{2y}{(x+a)^2 + y^2} - \frac{2y}{(x+a)^2 + y^2}\right]$$

$$= -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2}$$

$$\therefore \frac{\partial v}{\partial y} = -\frac{(x+a)^2-y^2}{\left[(x+a)^2+y^2\right]^2} + \frac{(x-a)^2-y^2}{\left[(x-a)^2+y^2\right]^2} \quad \cdots (3)$$

Adding (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since the equation of continuity is satisfied so the motion is possible. (**Showed**)

Problem-12: Show that the velocity potential $\phi = \frac{1}{2}a\left(x^2+y^2-2z^2\right)$ satisfies Laplace's equation. Also determine the streamlines.

Answer: The given velocity potential is,

$$\phi = \frac{1}{2}a\left(x^2+y^2-2z^2\right)$$

We know, $\vec{q} = -\nabla\phi$

$$= -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\frac{1}{2}a\left(x^2+y^2-2z^2\right)$$

$$= -\left(iax + jay - k2az\right)$$

Now, $\nabla \cdot \vec{q} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left\{-\left(iax + jay - k2az\right)\right\}$

$$or,\ -\nabla^2\phi = -\left(a+a-2a\right)$$

$$\therefore \nabla^2\phi = 0.$$

Hence the given velocity potential satisfies the Laplace equation. (**Showed**)

2nd Part: We know that,

$$u = -\frac{\partial\phi}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left\{\frac{1}{2}a\left(x^2+y^2-2z^2\right)\right\}$$

$$= -ax$$

$$v = -\frac{\partial\phi}{\partial y}$$

$$= -\frac{\partial}{\partial y}\left\{\frac{1}{2}a\left(x^2+y^2-2z^2\right)\right\}$$

$$= -ay$$

$$w = -\frac{\partial \phi}{\partial z}$$

$$= -\frac{\partial}{\partial z}\left\{\frac{1}{2}a\left(x^2 + y^2 - 2z^2\right)\right\}$$

$$= 2az$$

The equation of streamlines are,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$or,\ \frac{dx}{-ax} = \frac{dy}{ay} = \frac{dz}{2az}$$

$$or,\ \frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{2z} \qquad \cdots (1)$$

Taking the first two fractions we get,

$$\frac{dx}{-x} = \frac{dy}{-y}$$

$$or,\ \frac{dx}{x} = \frac{dy}{y}$$

Integrating, $\ln x = \ln y + \ln c_1$

$$or,\ x = yc_1 \quad \cdots (2)$$

Taking the last two fractions we get,

$$\frac{dy}{-y} = \frac{dz}{2z}$$

$$or,\ \frac{dy}{y} = -\frac{1}{2}\frac{dz}{z}$$

Integrating, $\ln y = -\frac{1}{2}\ln z + \ln c_2$

$$or,\ y = \frac{c_2}{\sqrt{z}} \qquad \cdots (3)$$

The required equation of streamlines are given by the intersection of the curves of equations (2) and (3).

Problem-13: Show that $\frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$ is a possible boundary surface of a liquid at time t.

Answer: Here we have,

$$F\left(x, y, z, t\right) = \frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - 1 = 0 \qquad \cdots (1)$$

which presents a possible boundary surface if it satisfies the boundary condition

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \quad \cdots(2)$$

and the values of u,v,w satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \cdots(3)$$

From (1), we have

$$\frac{\partial F}{\partial t}=-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^4}$$

$$\frac{\partial F}{\partial y}=\frac{2kt^2y}{b^2}$$

$$\frac{\partial F}{\partial z}=\frac{2kt^2z}{c^2}$$

Putting these values in (2) we get,

$$-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\frac{2xu}{a^2k^2t^4}+\frac{2kt^2yv}{b^2}+\frac{2kt^2zw}{c^2}=0$$

$$or,\ \frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2kyt}{b^2}(y+vt)+\frac{2ktz}{c^2}(z+wt)=0$$

which is true for

$$u-\frac{2x}{t}=0, \qquad y+vt=0, \qquad z+wt=0$$

$$or,\ u=\frac{2x}{t}, \qquad v=-\frac{y}{t}, \qquad w=-\frac{z}{t}$$

$$\therefore\ \frac{\partial u}{\partial x}=\frac{2}{t}, \qquad \frac{\partial v}{\partial y}=-\frac{1}{t}, \qquad \frac{\partial w}{\partial z}=-\frac{1}{t}$$

The equation (3) becomes,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{2}{t}-\frac{1}{t}-\frac{1}{t}=0$$

Since the equation of continuity is satisfied so equation (1) forms a boundary surface.

Exercise

Problem-01: Find the streamlines when $u = ax$, $v = ay$, $w = -2az$.

Problem-02: If the velocity is given by $\vec{q} = xi - yj$, determine the streamlines.

Problem-03: If the velocity is given by $\vec{q} = 3y^2 i - 6xj$, determine the streamlines.

Problem-04: Show that $u = U\left[1 - \frac{ay}{x^2+y^2} + \frac{b^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]$, $v = U\left[\frac{ax}{x^2+y^2} + \frac{2b^2xy}{\left(x^2+y^2\right)^2}\right]$, are the velocity components of possible liquid motion in two dimensions. Also show that the motion is irrotational.

Problem-05: Show that $u = \frac{3x^2 - r^2}{r^5}$, $v = \frac{3xy}{r^5}$, $w = \frac{3zx}{r^5}$, where $r^2 = x^2 + y^2 + z^2$ are the velocity components of possible liquid motion. Also show that the motion is irrotational and find the equation of streamlines.

Problem-06: Show that $u = 2cxy$, $v = c\left(a^2 + x^2 - y^2\right)$ are the velocity components of a possible fluid motion for an incompressible fluid. Show that the flow is of the potential kind. Find the velocity potential of this flow.

Problem-07: Show that the function, $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two dimensional fluid. Show that the streamlines at time are the curves $(x-t)^2 - (y-t)^2 = constant$.

Problem-08: Show that $\left(\frac{x^2}{a^2}\right)\tan^2 t + \left(\frac{y^2}{b^2}\right)\cot^2 t = 1$ is a possible boundary surface of a liquid at time.