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Beta & Gamma Functions

Beta Function or First Eulerian Integral: A function of the form,

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ;\ m,\ n>0$$

is called Beta function or first Eulerian integral and it is denoted by, $\beta(m,n)$.

$$i.e,\ \ \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ;\ m,\ n>0.$$

Gamma Function or Second Eulerian Integral: A function of the form,

$$\int_0^\infty e^{-x}x^{n-1}dx \quad ;\ n>0$$

is called Gamma function or second Eulerian integral and it is denoted by, $\Gamma(n)$.

$$i.e,\ \ \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \quad ;\ n>0.$$

Properties of Beta and Gamma functions: The properties are given below:

1. $\beta(m,n)=\beta(n,m)$
2. $\Gamma(1)=1$
3. $\Gamma(n+1)=n\Gamma(n) \quad ;\ n>0$
4. $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
5. $\displaystyle\int_0^\infty e^{-kx}x^{n-1}dx=\frac{\Gamma(n)}{k^n} \qquad ;\ k,n>0$
6. $\Gamma(m)\Gamma(1-m)=\dfrac{\pi}{\sin m\pi} \qquad ;0<m<1$
7. $\Gamma(1/2)=\sqrt{\pi}$
8. $\displaystyle\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$.
9. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^p x\cos^q x\,dx=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.

Theorem-01: Prove that $\beta(m,n)=\beta(n,m)$.

Proof: We know, the beta function is

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; m,\ n>0 \quad \cdots\cdots\cdots(1)$$

Let $x=1-t \qquad \therefore\ dx=-dt$

$$when\ x=0 \qquad then\quad t=1$$

$$when\ x=1 \qquad then\quad t=0$$

From (1) we get,

$$\beta(m,n)=\int_1^0 (1-t)^{m-1}\,t^{n-1}(-dt) \quad ; m,\ n>0$$

$$=\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

$$=\beta(n,m) \quad \textbf{(Proved)}$$

Theorem-02: Prove that $\Gamma(1/2)=\sqrt{\pi}$.

Proof: We know, the beta function is

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; m,\ n>0$$

If $m=n=\dfrac{1}{2}$ then,

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)}=\int_0^1 \frac{1}{\sqrt{x}.\sqrt{1-x}}\,dx$$

$$\Rightarrow \frac{\left\{\Gamma(1/2)\right\}^2}{\Gamma(1)}=\int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$\Rightarrow \frac{\left\{\Gamma(1/2)\right\}^2}{\Gamma(1)}=\int_0^1 \frac{dx}{\sqrt{-\left(x^2-x\right)}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{-\left[x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left[x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2\right]}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}\frac{\left(x - 1/2\right)}{1/2}\right]_0^1$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}\left(2x-1\right)\right]_0^1$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \left[\sin^{-1}\left(2.1-1\right) - \sin^{-1}\left(2.0-1\right)\right]$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \sin^{-1}\left(1\right) - \sin^{-1}\left(-1\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \sin^{-1}\left(1\right) + \sin^{-1}\left(1\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\sin^{-1}\left(1\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\sin^{-1}.\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \left\{\Gamma(1/2)\right\}^2 = \pi$$

$\therefore\ \Gamma(1/2) = \sqrt{\pi}$ **(Proved)**

Theorem-03: Prove that i). $\Gamma(1)=1$; ii). $\Gamma(n+1)=n\Gamma(n)$.

Proof: We know, the Gamma function is

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \quad ; \quad n>0 \quad \ldots\ldots\ldots (1)$$

If $n=1$ then, from (1) we get,

$$\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}dx$$

$$=\int_0^\infty e^{-x}dx$$

$$=\left[-e^{-x}\right]_0^\infty$$

$$=\left(-e^{-\infty}+e^{0}\right)$$

$$=(0+1)$$

$$=1$$

$\therefore\ \Gamma(1)=1$ **(Proved)**

Again, replacing n by $(n+1)$ in (1) we get,

$$\Gamma(n+1)=\int_0^\infty e^{-x}x^{n}dx$$

$$=\left[-x^n e^{-x}\right]_0^\infty+n\int_0^\infty e^{-x}x^{n-1}dx \qquad \text{[integrating by parts]}$$

$$=0+n\Gamma(n)$$

$$=n\Gamma(n)$$

$\therefore \Gamma(n+1)=n\Gamma(n)$ **(Proved)**

Theorem-04: Establish the relation between Gamma and Beta function.

Or, Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof: From the definition of Gamma function we can write

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \qquad ; n>0$$

Assume $x=\lambda u \;\therefore\; dx=\lambda du$.

Limit: when $x=0$, then $u=0$ and when $x=\infty$, then $u=\infty$.

From above relation we have

$$\Gamma(n)=\int_0^\infty e^{-\lambda u}(\lambda u)^{n-1}\lambda du$$

$$=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n-1}\lambda du$$

$$=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n}du \cdots\cdots\cdots\cdots(i)$$

Again,

$$\Gamma(m)=\int_0^\infty e^{-\lambda}\lambda^{m-1}d\lambda \quad \cdots\cdots\cdots\cdots(ii)$$

Multiplying (i) and (ii) we get

$$\Gamma(n)\Gamma(m)=\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n}du\int_0^\infty e^{-\lambda}\lambda^{m-1}d\lambda$$

$$\Rightarrow \Gamma(m)\Gamma(n)=\int_0^\infty\int_0^\infty e^{-\lambda u}u^{n-1}\lambda^{n}e^{-\lambda}\lambda^{m-1}d\lambda du$$

$$=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+u)}\lambda^{m+n-1}d\lambda\right]u^{n-1}du$$

$$=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+u)}\lambda^{m+n-1}d\lambda\right]u^{n-1}du$$

$$=\int_0^\infty\left[\frac{\overline{|m+n}}{(1+u)^{m+n}}\right]u^{n-1}du \qquad \left[\because \frac{\overline{|n}}{k^n}=\int_0^\infty e^{-kx}x^{n-1}dx\right]$$

$$=\overline{|m+n}\int_0^\infty \frac{u^{n-1}}{(1+u)^{m+n}}du$$

$$=\overline{|m+n}\times\beta(m,n) \qquad \left[\because \beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx\right]$$

$$\therefore \beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ **(Proved)**

Theorem-05: Prove that $\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx=\dfrac{\overline{\left|\frac{p+1}{2}\right.}\;\overline{\left|\frac{q+1}{2}\right.}}{2\overline{\left|\frac{p+q+2}{2}\right.}}$

Proof: We know that

$$\beta(m\,,\,n)=\int_0^\infty x^{m-1}(1-x)^{n-1}dx$$

Let $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta d\theta$.

Limit: $x=0 \Rightarrow \theta=0$ *and* $x=\infty \Rightarrow \theta=\frac{\pi}{2}$.

Now,

$$\beta(m\,,\,n)=\int_0^{\frac{\pi}{2}}(\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\times 2\sin\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2m-2}\theta(\cos^2\theta)^{n-1}\times 2\sin\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2m-2}\theta\cos^{2n-2}\theta\times 2\sin\theta\cos\theta d\theta$$

$$\therefore\ \beta(m\,,\,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Assume $2m-1=p$ *and* $2n-1=q \Rightarrow m=\frac{p+1}{2}$ *and* $n=\frac{q+1}{2}$.

Now from above equation we get

$$\beta\left(\tfrac{p+1}{2}\,,\,\tfrac{q+1}{2}\right)=2\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta$$

Using the relation between beta and gamma function $\beta(m\,,\,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$**,** we have

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}=2\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$ **(Proved)**

Problem-01: Evaluate $\int_0^{\frac{\pi}{2}}\cos^7 x\,dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}}\cos^7 x\,dx$

$$=\int_0^{\frac{\pi}{2}}\sin^0 x\cos^7 x\;dx$$

$$=\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{0+7+2}{2}\right)}$$

Exer.-01: $\int_0^{\frac{\pi}{2}}\cos^4 x\,dx$

Ans: $\frac{3\pi}{16}$

Exer.-02: $\int_0^{\frac{\pi}{2}}\sin^5 x\,dx$

Ans: $\frac{8}{15}$

$$=\frac{\left\lceil\frac{1}{2}\right.\;\left\lceil\frac{8}{2}\right.}{2\;\left\lceil\frac{9}{2}\right.}$$

$$=\frac{\left\lceil\frac{1}{2}\right.\;\left\lceil 4\right.}{2\;\left\lceil\frac{9}{2}\right.}$$

$$=\frac{\left\lceil\frac{1}{2}\right.\;\left\lceil 3+1\right.}{2\;\left\lceil\frac{7}{2}+1\right.}$$

$$=\frac{\left\lceil\frac{1}{2}\right.\;3!}{2\;\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\left\lceil\frac{1}{2}\right.}$$

$$=\frac{3.2.1}{2\;\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}}$$

$$=\frac{16}{35}$$ **(Ans.)**

Exer.-03: $\int_0^{\frac{\pi}{2}} \sin^8 x\ dx$

Ans: $\frac{35\pi}{256}$

Problem-02: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\ dx$

Solution: Solution: Let, $I=\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$

$$=\int_0^{\frac{\pi}{2}} \sin^6 x\cos^0 x\ \ dx$$

$$=\frac{\left\lceil\frac{6+1}{2}\right.\left\lceil\frac{0+1}{2}\right.}{2\left\lceil\frac{6+0+2}{2}\right.}$$

$$=\frac{\left\lceil\frac{7}{2}\right.\;\left\lceil\frac{1}{2}\right.}{2\;\left\lceil\frac{8}{2}\right.}$$

$$=\frac{\left\lceil\frac{7}{2}\right.\;\left\lceil\frac{1}{2}\right.}{2\;\left\lceil 4\right.}$$

$$=\frac{\Gamma\left(\frac{5}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}{2\;\Gamma(3+1)}$$

$$=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{2.\,3!}$$

$$=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2.\,3.2.1}$$

$$=\frac{5.\pi}{32} \qquad \textbf{(Ans.)}$$

Problem-03: Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x \, dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x \, dx$

$$=\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{4+3+2}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{4}{2}\right)}{2\;\Gamma\left(\frac{9}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma(2)}{2\;\Gamma\left(\frac{7}{2}+1\right)}$$

$$=\frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2.\,\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{2}{35} \qquad \textbf{(Ans.)}$$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx$

Ans: $\frac{8}{693}$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x \, dx$

Ans: $\frac{7\pi}{2048}$

Exer.-06: $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x \, dx$

Ans: $\frac{2}{63}$

Problem-04: Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \, dx$

Solution: Let, $I=\int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \, dx$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin 2x \cos^4 x \, dx$

Ans: $\frac{1}{3}$

$$=\int_0^{\frac{\pi}{2}} \cos^3 x\left(\cos^2 x-\sin^2 x\right)\,dx$$

$$=\int_0^{\frac{\pi}{2}} \left(\cos^5 x-\cos^3 x\sin^2 x\right)\,dx$$

$$=\int_0^{\frac{\pi}{2}} \cos^5 x dx-\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\;dx$$

$$=\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{0+5+2}{2}\right)}-\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{2+3+2}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma 3}{2\Gamma\left(\frac{7}{2}\right)}-\frac{\Gamma\left(\frac{3}{2}\right)\Gamma 2}{2\Gamma\left(\frac{7}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2+1)}{2\Gamma\left(\frac{5}{2}+1\right)}-\frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma(1+1)}{2\Gamma\left(\frac{5}{2}+1\right)}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right).2\Gamma 1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}-\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right).1\Gamma 1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$=\frac{8}{15}-\frac{2}{15}$$

$$=\frac{8-2}{15}$$

$$=\frac{6}{15}$$

$$=\frac{2}{5}\qquad \textbf{(Ans.)}$$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin 2x\sin^2 x\cos^5 x\,dx$

Ans: $\frac{4}{63}$

Problem-05: Evaluate $\int_0^{2\pi} \sin^4 x\cos^6 x\;dx$

Solution: Let, $I=\int_0^{2\pi} \sin^4 x\cos^6 x\;dx$

$$=2\int_0^{\pi} \sin^4 x\cos^6 x\;dx$$

$$=4\int_0^{\frac{\pi}{2}} \sin^4 x\cos^6 x\;dx$$

Exer.-06: $\int_0^{\pi} \sin^2 x\cos^4 x\,dx$

Ans: $\frac{\pi}{16}$

$$=4.\frac{\left\lceil\frac{4+1}{2}\right.\left\lceil\frac{6+1}{2}\right.}{2\left\lceil\frac{4+6+2}{2}\right.}$$

$$=2.\frac{\left\lceil\frac{5}{2}\right.\left\lceil\frac{7}{2}\right.}{\lceil 6}$$

$$=2.\frac{\left\lceil\frac{3}{2}+1\right.\left\lceil\frac{5}{2}+1\right.}{\lceil 5+1}$$

$$=2.\frac{\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right..\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right.}{5.4.3.2.1}$$

$$=\frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{5.4.3}$$

$$=\frac{1}{60}\times\frac{45\pi}{32}$$

$$=\frac{3\pi}{128}$$ **(Ans.)**

Problem-06: Evaluate $\int_0^{\pi} x\sin^6 x\cos^4 x\ dx$

Exer.-07: $\int_0^{\pi} x\sin^2 x\cos^4 x\,dx$

Ans: $\frac{\pi^2}{32}$

Exer.-08: $\int_0^{\pi} x\sin x\cos^2 x\,dx$

Ans: $\frac{\pi}{3}$

Solution: Let, $I=\int_0^{\pi} x\sin^6 x\cos^4 x\ dx$

$$=\int_0^{\pi}(\pi-x)\sin^6(\pi-x)\cos^4(\pi-x)\ dx$$

$$=\int_0^{\pi}(\pi-x)\sin^6 x\cos^4 x\ dx$$

$$=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx-\int_0^{\pi}x\sin^6 x\cos^4 x\ dx$$

$$=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx-I$$

$$\therefore\ 2I=\pi\int_0^{\pi}\sin^6 x\cos^4 x\ dx$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^6 x\cos^4 x\ dx$$

$$=2\pi.\frac{\left\lceil\frac{6+1}{2}\right.\left\lceil\frac{4+1}{2}\right.}{2\left\lceil\frac{6+4+2}{2}\right.}$$

$$=\pi.\frac{\left\lceil\frac{7}{2}\right.\left\lceil\frac{5}{2}\right.}{\lceil 6}$$

$$=\pi.\frac{\left\lceil\frac{5}{2}+1\right.\left\lceil\frac{3}{2}+1\right.}{\lceil 5+1}$$

$$=\pi.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right. .\frac{3}{2}.\frac{1}{2}\left\lceil\frac{1}{2}\right.}{5.4.3.2.1}$$

$$=\pi.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}\ .\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{120}$$

$$=\frac{1}{120}\times\frac{45\pi^2}{32}$$

$$=\frac{3\pi^2}{256}$$

$$\therefore\ I=\frac{3\pi^2}{512}\ \textbf{(Ans.)}$$

Problem-08: Evaluate $\int_0^1 x^3\left(1-x^2\right)^{\frac{5}{2}}dx$

Solution: Let, $I=\int_0^1 x^3\left(1-x^2\right)^{\frac{5}{2}}dx$

Put $x=\sin\theta \quad \therefore\ dx=\cos\theta d\theta$

$Limit: when\ x=0 \quad then\ \theta=0$

$when\ x=1 \quad then\ \theta=\frac{\pi}{2}$

Now, $I=\int_0^{\frac{\pi}{2}}\sin^3\theta\left(1-\sin^2\theta\right)^{\frac{5}{2}}\cos\theta d\theta$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\left(\cos^2\theta\right)^{\frac{5}{2}}\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^5\theta\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^6\theta d\theta$$

$$=\frac{\left\lceil\frac{3+1}{2}\right.\left\lceil\frac{6+1}{2}\right.}{2\left\lceil\frac{3+6+2}{2}\right.}$$

Exer.-09: $\int_0^1 x^2\left(1-x^2\right)^{\frac{1}{2}}dx$

Ans: $\frac{\pi}{16}$

Exer.-10: $\int_0^1 x^4\left(1-x^2\right)^{\frac{3}{2}}dx$

Ans: $\frac{3\pi}{256}$

Exer.-11: $\int_0^1 x^6\left(1-x^2\right)^{\frac{1}{2}}dx$

Ans: $\frac{5\pi}{256}$

$$= \frac{\Gamma(2)\,\Gamma\left(\frac{7}{2}\right)}{2\,\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\Gamma(1+1)\,\Gamma\left(\frac{5}{2}+1\right)}{2\,\Gamma\left(\frac{9}{2}+1\right)}$$

$$= \frac{1\Gamma(1)\,.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{2}{63} \qquad \textbf{(Ans.)}$$

Problem-09: Evaluate $\int_0^1 \frac{x^5}{\sqrt{(1-x^2)}}dx$

Exer.-12: $\int_0^1 \frac{x^6}{\sqrt{(1-x^2)}}dx$

Ans: $\frac{5\pi}{32}$

Solution: Let, $I = \int_0^1 \frac{x^3}{\sqrt{(1-x^2)}}dx$

Put $x = \sin\theta \quad \therefore\ dx = \cos\theta d\theta$

$Limit: when\ x = 0 \quad then\ \theta = 0$

$when\ x = 1 \quad then\ \theta = \frac{\pi}{2}$

Now, $I = \int_0^{\frac{\pi}{2}} \frac{\sin^5\theta}{\sqrt{(1-\sin^2\theta)}}.\cos\theta d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^5\theta}{\sqrt{\cos^2\theta}}.\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^5\theta}{\cos\theta}.\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^5\theta d\theta$$

$$= \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{5+0+2}{2}\right)}$$

$$= \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{\Gamma(2+1)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{2}+1\right)}$$

$$= \frac{2.1\,\Gamma(1)\,.\,\Gamma\left(\frac{1}{2}\right)}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{8}{15} \quad \textbf{(Ans.)}$$

Problem-10: Evaluate $\int_0^\infty \frac{x^3}{\left(1+x^2\right)^{\frac{9}{2}}}dx$

Solution: Let, $I = \int_0^\infty \frac{x^3}{\left(1+x^2\right)^{\frac{9}{2}}}dx$

Put $x = \tan\theta \quad \therefore\ dx = \sec^2\theta d\theta$

$Limit: when\ x = 0 \quad then\ \theta = 0$

$when\ x = \infty \quad then\ \theta = \frac{\pi}{2}$

Now, $I = \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\left(1+\tan^2\theta\right)^{\frac{9}{2}}}.\sec^2\theta d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\left(\sec^2\theta\right)^{\frac{9}{2}}}.\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\sec^9\theta}.\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3\theta}{\sec^7\theta}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3\theta}{\cos^3\theta}.\cos^7\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta d\theta$$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{3+4+2}{2}\right)}$$

$$= \frac{\Gamma 2 \, \Gamma\frac{5}{2}}{2\,\Gamma\frac{9}{2}}$$

$$= \frac{\Gamma(1+1)\,\Gamma\left(\frac{3}{2}+1\right)}{2\,\Gamma\left(\frac{7}{2}+1\right)}$$

$$= \frac{1\,\Gamma 1 \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\frac{1}{2}}{2 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\frac{1}{2}}$$

$$= \frac{2}{35} \quad \textbf{(Ans.)}$$

Problem-11: Evaluate $\int_0^1 x^4 (1-x)^{\frac{3}{2}}\, dx$

Solution: Let, $I = \int_0^1 x^4 (1-x)^{\frac{3}{2}}\, dx$

$$= \int_0^1 x^{5-1} (1-x)^{\frac{5}{2}-1}\, dx$$

$$= \beta\left(5, \frac{5}{2}\right)$$

$$= \frac{\Gamma 5\, \Gamma\frac{5}{2}}{\Gamma\left(5+\frac{5}{2}\right)}$$

$$= \frac{\Gamma(4+1)\, \Gamma\frac{5}{2}}{\Gamma\left(\frac{13}{2}+1\right)}$$

$$= \frac{4.3.2.1\,\Gamma\frac{5}{2}}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2}\,\Gamma\frac{5}{2}}$$

$$= \frac{256}{15015} \quad \textbf{(Ans.)}$$

Exer.-13: $\int_0^1 x^3 (1-x)^3\, dx$

Ans: $\frac{1}{140}$

Problem-12: Show that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

Solution: Let, $I = \int_0^\infty e^{-x^2}\, dx$

Put, $x^2 = z$

Exer.-14: Show that $\int_0^\infty e^{-3x^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{3}}$

Exer.-15: Show that $\int_0^\infty e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}$

$$\therefore \quad 2xdx = dz$$

$$\Rightarrow \; dx = \frac{1}{2\sqrt{z}} dz$$

$Limit: when\ x = 0\ then\ z = 0$

$when\ x = \infty\ then\ z = \infty$

Now, $I = \int_0^\infty e^{-z} . \frac{1}{2\sqrt{z}} dz$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2}$$

$$= \frac{\sqrt{\pi}}{2}$$

$\therefore \quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ **(Showed.)**

Problem-13: Show that $\int_0^\infty e^{-x} x^{\frac{3}{2}} dx = \frac{3\sqrt{\pi}}{4}$ **Exer.-16: Show that** $\int_0^\infty \sqrt{x} e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$

Solution: Let, $I = \int_0^\infty e^{-x} x^{\frac{3}{2}} dx$

$$= \int_0^\infty e^{-x} x^{\frac{5}{2}-1} dx$$

$$= \Gamma\left(\frac{5}{2}\right)$$

$$= \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{3}{2} . \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3\sqrt{\pi}}{4}$$

$\therefore \quad \int_0^\infty e^{-x} x^{\frac{3}{2}} dx = \frac{3\sqrt{\pi}}{4}$ **(Showed)**

Problem-14: Show that $\int_0^\infty e^{-3x} x^{\frac{3}{2}} dx = \frac{\sqrt{3\pi}}{36}$ **Exer.-17: Show that** $\int_0^\infty e^{-4x^2} dx = \frac{3\sqrt{\pi}}{128}$

Solution: Let, $I = \int_0^\infty e^{-3x} x^{\frac{3}{2}} dx$

$$= \int_0^\infty e^{-3x} x^{\frac{5}{2}-1} dx$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{3^{\frac{5}{2}}}$$

$$= \frac{\Gamma\left(\frac{3}{2}+1\right)}{\sqrt{3^5}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{9\sqrt{3}}$$

$$= \frac{\sqrt{3\pi}}{36}$$

$$\therefore \int_0^\infty e^{-3x} x^{\frac{3}{2}} dx = \frac{\sqrt{3\pi}}{36}$$ **(Showed.)**