Functions

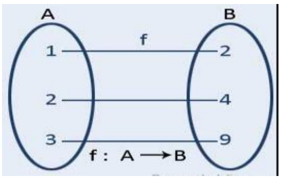
Function: If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then y is called a function of x and it is denoted by the following symbol,

$$y = f(x)$$

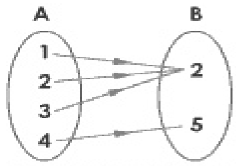
where x is independent variable and y is dependent variable. The inverse of this function is denoted by $f^{-1}(y) = x$.

Example: $y = x^2 + x + 1$; $y = sinx$; $y = e^x$; $y = lnx$ etc.

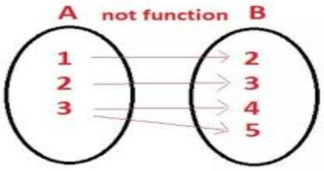
Alternatively, let A and B be two non empty sets. A mapping $f : A \to B$ is called function if each element of A is assigned by unique element of B.



Function



Function



It is not function

Types of functions: There are many types of functions. These have been discussed as:

Even function: A function $y = f(x)$ is called an even function if it satisfies the condition

$f(-x) = f(x)$.

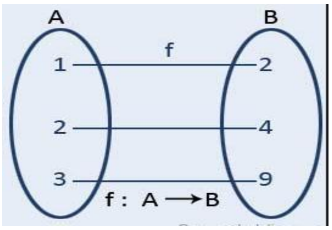
Example: $y = cosx,\ y = x^4, etc.$ are even functions.

Odd function: A function $y = f(x)$ is called an odd function if it satisfies the condition

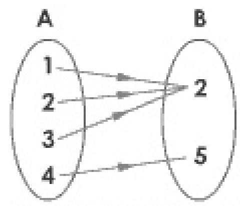
$f(-x) = -f(x)$.

Example: $y = sinx,\ y = x^3, etc.$ are odd functions.

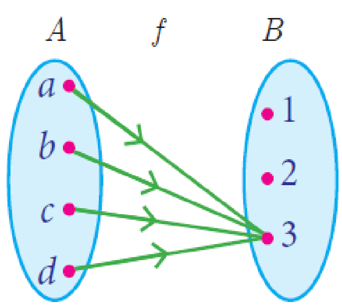
One–one Function: Let f map A into B, i.e, $f : A \to B$. Then f is called a one-one function if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, $f : A \to B$ is one-one if $f(a) = f(b)$ implies $a = b$ or, equivalently, $a \neq b$ implies $f(a) \neq f(b)$.



Onto Function: Let f be a function of A into B. Then f is called a onto function if every element of B appears as the image of at least one element of A. More briefly, $f: A \to B$ is onto function if $f(A) = B$.



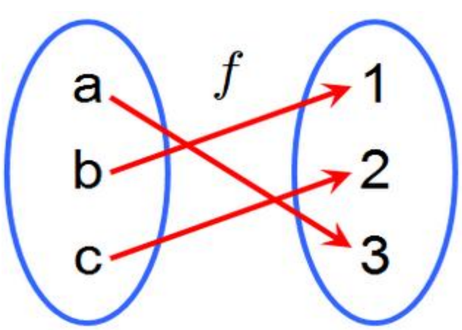
Constant Function: A function f of A into B is called a constant function if the same element in B is assigned to every element in A. More briefly, $f: A \to B$ is a constant function if the range of f consists of only one element.



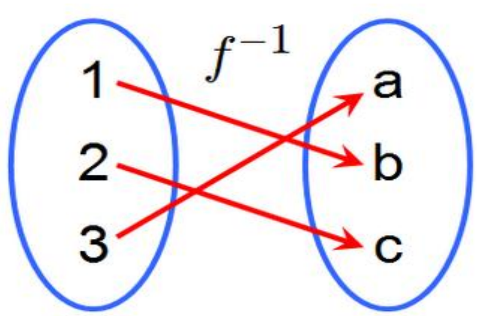
Inverse Function: Let f be a function of A into B. In general, $f^{-1}(b)$ could consist of more than one element or might even be the empty set $\varnothing$. Now if $f: A \to B$ is a one-one function and an onto function, then for each $b \in B$ the inverse $f^{-1}(b)$ will consist of a single element in A. We therefore have a rule that assigns to each $b \in B$ a unique element $f^{-1}(b)$ in A. Accordingly, f^{-1} is a function of B into A and we can write

$$f^{-1}: B \to A$$

In this situation, when $f: A \to B$ is one-one and onto, we call f^{-1} the inverse function of f.

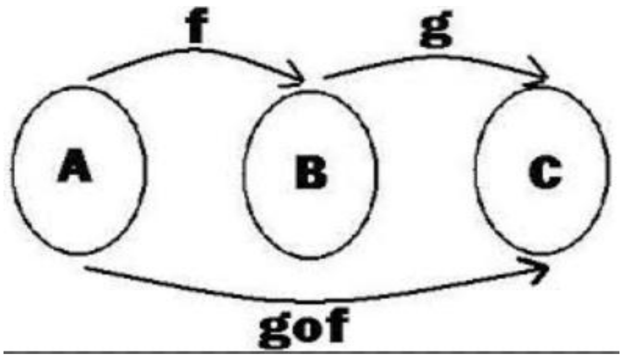


Function, $f: A \to B$



Inverse function, $f^{-1}: B \to A$

Composition of Functions: Let f be a function of A into B and g be a function of B into C. We illustrate the functions below,



Let $a \in A$; then its image $f\left(a\right)$ is in B which is the domain of g. Accordingly, we can find the image of $f\left(a\right)$ under the mapping g, that is, we can find $g\left(f\left(a\right)\right)$. Thus we have a rule which assigns to each element $a \in A$ a corresponding element $g\left(f\left(a\right)\right) \in C$. In other words, we have a function of A into C. This new function is called product function or composition function of f and g and it is denoted by

$$\left(g \circ f\right) \quad \text{or} \quad \left(gf\right)$$

More briefly, $f : A \to B$ and $g : B \to C$ then we define a function $\left(g \circ f\right) : A \to C$ by

$$\left(g \circ f\right)\left(a\right) = g\left(f\left(a\right)\right).$$

<u>Domain</u>: The set of all values of x for which the function $y = f(x)$ is defined, is called domain of the function. Simply domain is the set of all allowable x-values.

Mathematically, $D_f = \left\{x : y = f(\mathrm{x}) \in \mathrm{R} \text{ and } \mathrm{x} \in \mathrm{R}\right\}$.

<u>Range</u>: The set of all values of y corresponding to the x values for which the function $y = f(x)$ is defined, is called range of the function. Simply range is the set of all possible y-values.

Mathematically, $R_f = \left\{\mathrm{y} : y = f(\mathrm{x}) \in \mathrm{R} \text{ and } \mathrm{x} \in \mathrm{R}\right\}$.

<u>Interval</u>: If the set of all real numbers lie between two real numbers a and b, where $a < b$ then the set of all real numbers is called an interval.

Intervals are four kinds:

a) The set $\{x \epsilon \mathcal{R}: a \leq x \leq b\}$ is called a closed interval, denoted by $[a, b]$.
b) The set $\{x \epsilon \mathcal{R}: a < x < b\}$ is called an open interval, denoted by (a, b).
c) The set $\{x \epsilon \mathcal{R}: a < x \leq b\}$ is called a left half open interval, denoted by $(a, b]$.
d) The set $\{x \epsilon \mathcal{R}: a \leq x < b\}$ is called a right half open interval, denoted by $[a, b)$.

Problem 01: Find the domain and range of the function $y = 2x + 5$.

Solution: Given function is,

$y = 2x + 5$

Here, y gives real values for all real values of x.
So, the domain of the given function is,

$$D_f = R$$

Again, we have,

$$y = 2x + 5$$

$or,\ 2x = y - 5$

$or,\ x = \dfrac{y-5}{2}$

Here, x gives real values for all real values of y.
So, the range of the given function is,

$$R_f = R \text{ (Ans)}$$

H.W:
Find the domain and range of the following functions

1. $y = 3x + 5$ Ans: $D_f = R$ and $R_f = R$
2. $y = 4x - 3$ Ans: $D_f = R$ and $R_f = R$
3. $y = ax + b$ Ans: $D_f = R$ and $R_f = R$

Problem 02: Find the domain and range of the function $y = x^2 + 3x + 2$.

Solution: Given function is,

$$y = x^2 + 3x + 2$$

Here, y gives real values for all real values of x.
So, the domain of the given function is,

$$D_f = R$$

Again, we have

$y = x^2 + 3x + 2$

$or,\ x^2 + 3x + (2 - y) = 0$

In the above equation the values of x will be real if and only if its $Discriminant \geq 0$.

$i.e,\ 3^2 - 4.1.(2-y) \geq 0 \qquad ;\ [b^2 - 4ac \geq 0]$

$or,\ 9 - 4(2-y) \geq 0$

$or,\ 9 - 8 + 4y \geq 0$

$or,\ 1 + 4y \geq 0$

$or,\ 4y \geq -1$

$or,\ y \geq -\dfrac{1}{4}$

Therefore the range of the given function is,

$$R_f = [-\frac{1}{4}, \infty) \text{ (Ans)}$$

Alternative way, For range we have

$y = x^2 + 3x + 2$

$$or,\ x^2+3x+2=y$$

$$or,\ x^2+2.x.\frac{3}{2}+\left(\frac{3}{2}\right)^2+2-\frac{9}{4}=y$$

$$or,\ \left(x+\frac{3}{2}\right)^2-\frac{1}{4}=y$$

$$or,\ \left(x+\frac{3}{2}\right)^2=y+\frac{1}{4}$$

$$or,\ x+\frac{3}{2}=\pm\sqrt{y+\frac{1}{4}}$$

$$or,\ x=\pm\sqrt{y+\frac{1}{4}}-\frac{3}{2}$$

Here, x is defined if

$$y+\frac{1}{4}\geq 0$$

$$or,\ \ y\geq -\frac{1}{4}$$

Therefore the range of the given function is,

$$R_f=[-\frac{1}{4},\infty)\ (\text{Ans})$$

H.W:
Find the domain and range of the following quadratic functions

1. $y=x^2+5x+6$ Ans: $D_f=R$ and $R_f=[-\frac{1}{4},\infty)$
2. $y=-x^2+5x-6$ Ans: $D_f=R$ and $R_f=(-\infty,\frac{1}{4}]$
4. $y=-x^2+1$ Ans: $D_f=R$ and $R_f=(-\infty,1]$
5. $y=x^2+4x+7$ Ans: $D_f=R$ and $R_f=[3,\infty)$
6. $y=x^2-4x+3$ Ans: $D_f=R$ and $R_f=[-1,\infty)$
7. $y=(x+2)^2+3$ Ans: $D_f=R$ and $R_f=[3,\infty)$

Problem 03: Find the domain and range of the function $y=\dfrac{x-3}{2x+1}$.

Solution: Given function is,

$$y=\frac{x-3}{2x+1}$$

Here, y is undefined if

$$2x+1\ =\ 0$$

$$or,\ x=-\frac{1}{2}$$

So, y gives real values for all real values of x except $x = -\frac{1}{2}$.

Therefore, the domain of the given function is

$$D_f = R - \left\{-\frac{1}{2}\right\}.$$

Again we have,

$$y = \frac{x-3}{2x+1}$$

$$or,\ 2xy + y = x - 3$$

$$or,\ x - 2xy = y + 3$$

$$or,\ x(1-2y) = y + 3$$

$$or,\ x = \frac{y+3}{1-2y}$$

Here, x is undefined if

$$1 - 2y = 0$$

$$or,\ y = \frac{1}{2}$$

So, x gives real values for all real values of y except $y = \frac{1}{2}$.

Therefore, the range of the given function is

$$R_f = R - \left\{\frac{1}{2}\right\} \qquad \text{(Ans)}$$

Problem 04: Find the domain and range of the function $y = \frac{x^2-4}{x-2}$.

Solution: Given function is,

$$y = \frac{x^2-4}{x-2}$$

Here, y is undefined if

$$x - 2 = 0$$

$$or,\ x = 2$$

So, y gives real values for all real values of x except $x = 2$.
Therefore, the domain of the given function is

$$D_f = R - \{2\}.$$

Again we have,

$$y = \frac{x^2-4}{x-2}$$

$$or,\ y = \frac{(x+2)(x-2)}{x-2} \quad ;\ x \neq 2$$

$$or,\ y = x + 2 \quad ;\ x \neq 2$$

$$or,\ x = y - 2 \quad ;\ x \neq 2$$

Here, x is defined for all real values of y except $y = 4$

Therefore, the range of the given function is

$$R_f = R - \{4\} \qquad \text{(Ans)}$$

H.W:
Find the domain and range of the following quadratic functions

1. $y = \dfrac{x}{x+1}$ Ans: $D_f = R - \{-1\}$ and $R_f = R - \{1\}$
2. $y = \dfrac{1+x}{5-x}$ Ans: $D_f = R - \{5\}$ and $R_f = R - \{-1\}$
3. $y = \dfrac{2}{x+3}$ Ans: $D_f = R - \{-3\}$ and $R_f = R - \{0\}$
4. $y = \dfrac{x-3}{x^2-9}$ Ans: $D_f = R - \{-3, 3\}$ and $R_f = R - \left\{0, \dfrac{1}{6}\right\}$
5. $y = \frac{4x+3}{x^2+1}$ Ans: $D_f = R \text{ and } R_f = [-1,4]$

Problem 05: Find the domain and range of the function $y = \sqrt{2x+5}$.
Solution: Given function is,

$y = \sqrt{2x+5}$

Here, y gives real values iff

$$2x+5 \geq 0$$

$or,\ 2x \geq -5$

$or,\ x \geq -\dfrac{5}{2}$

Therefore, the domain of the given function is

$$D_f = \left[-\frac{5}{2}, \infty\right).$$

Again,

$$y = \sqrt{2x+5} \quad \cdots\cdots(1)$$

The values of y in (1) are positive or zero, *i.e,* $y \not< 0$.

Now $y^2 = 2x+5$; $y \not< 0$. [Squaring both sides]

$2x+5 = y^2 \qquad ; y \not< 0.$

$2x = y^2 - 5 \qquad ; y \not< 0.$

$x = \dfrac{y^2-5}{2} \qquad ; y \not< 0.$

Here, x is defined for $y \geq 0$.

Therefore, the range of the given function is

$$R_f = \{y : y \geq 0\}$$
$$= [0, \infty) \ \textbf{\textit{(Ans).}}$$

Problem 06: Find the domain and range of the function $y=-\sqrt{1-2x}$.

Solution: Given function is,

$$y=-\sqrt{1-2x}$$

Here, y gives real values iff

$$1-2x \geq 0$$
$$or,\ -2x \geq -1$$
$$or,\ 2x \leq 1$$
$$or,\ x \leq \frac{1}{2}$$

Therefore, the domain of the given function is

$$D_f=\left(-\infty,\ \frac{1}{2}\right].$$

Again, we have,

$$y=-\sqrt{1-2x}\ \cdots\cdots(1)$$

The values of y in (1) are negative or zero ,$i.e,\ y \ngtr 0$.

Now $y^2=1-2x;\ y \ngtr 0$ [Squaring both sides]

$$1-2x=y^2;\ y \ngtr 0$$
$$2x=1-y^2;\ y \ngtr 0$$
$$x=\frac{1-y^2}{2};\ y \ngtr 0$$

Here, x is defined for $y \leq 0$.

Therefore, the range of the given function is

$$R_f=\{y : y \leq 0\}$$
$$=(-\infty, 0]$$ **(Ans).**

H.W:

Find the domain and range of the following functions

1. $y=\sqrt{2x-1}$ Ans: $D_f=\left[\frac{1}{2},\ \infty\right)$ and $R_f=[0,\ \infty)$
2. $y=\sqrt{1-5x}$ Ans: $D_f=\left(-\infty,\ \frac{1}{5}\right]$ and $R_f=[0,\ \infty)$
3. $y=\sqrt{2x-1}+5$ Ans: $D_f=\left[\frac{1}{2},\ \infty\right)$ and $R_f=[5,\ \infty)$
4. $y=\sqrt{x+6}-3$ Ans: $D_f=[-6,\ \infty)$ and $R_f=[-3,\ \infty)$
5. $y=5-\sqrt{8-2x}$ Ans: $D_f=(-\infty,\ 4]$ and $R_f=[5,\ -\infty)$
6. $y=-\sqrt{x-1}$ Ans: $D_f=[1,\ \infty)$ and $R_f=(-\infty, 0]$

7. $y = -\sqrt{1-4x}$ Ans: $D_f = \left(-\infty, \frac{1}{4}\right]$ and $R_f = \left(-\infty, 0\right]$

Problem 07: Find the domain and range of the function $y = \sqrt{x^2 - 4x + 3}$.

Solution: Given function is,

$y = \sqrt{x^2 - 4x + 3}$

Here, y gives real values iff,

$$x^2 - 4x + 3 \geq 0$$

$or,\ x^2 - 3x - x + 3 \geq 0$

$or,\ x(x-3) - 1(x-3) \geq 0$

$or,\ (x-3)(x-1) \geq 0$

This inequality is satisfied if

$x \leq 1\ or\ \ x \geq 3$

Therefore, the domain of the given function is,

$D_f = \{x : x \leq 1\} \cup \{x : x \geq 3\}$

$= (-\infty, 1] \cup [3, \infty)$

$= R - (1, 3)$

Again, we have,

$$y = \sqrt{x^2 - 4x + 3}\ \cdots\cdots (1)$$

The values of y in (1) are positive or zero *i.e*, ,$y \not< 0$.

Now, $y^2 = x^2 - 4x + 3 ; y \not< 0$ [Squaring both sides]

$x^2 - 4x + 3 - y^2 = 0 ; y \not< 0$

$x^2 - 4x + (3 - y^2) = 0 ; y \not< 0$

In the above equation the values of x will be real if and only if it's $Discriminant \geq 0$.

$i.e, (-4)^2 - 4 \times 1.(3 - y^2) \geq 0 ; y \not< 0 [\, b^2 - 4ac \geq 0\,]$

$or,\ 16 - 4(3 - y^2) \geq 0 ; y \not< 0$

$or,\ 16 - 12 + 4y^2 \geq 0 ; y \not< 0$

$or,\ 4 + 4y^2 \geq 0 ; y \not< 0$

$or,\ 1 + y^2 \geq 0 ; y \not< 0$

Here, x is defined for $y \geq 0$.

So the range of the given function is

$$R_f = \{y : y \geq 0\}$$

$=[0,\infty)$ **(Ans).**

Problem 08: Find the domain and range of the function $y=\sqrt{x^2+1}$.
Solution: Given function is,

$y=\sqrt{x^2+1}$

Here, y gives real values iff,

$$x^2+1\geq 0$$

This inequality is satisfied for all real values of *x*.

Therefore the domain of the given function is,

$$D_f=R.$$

Again, we have,

$$y=\sqrt{x^2+1}\ \ldots\ \ldots (1)$$

The values of *y* in (1) are positive and lowest value is 1,*i.e,y* $\not< 1$.

Now $y^2=x^2+1 \quad ;y\not< 1$ [Squaring both sides]

$\Rightarrow x^2+1-y^2=0 \quad ;y\not< 1$

$\Rightarrow x^2+0.x+(1-y^2)=0 \ ; y\not< 1$

In the above equation the values of *x* will be real if and only if its *Discriminant* ≥ 0.

i.e, $0^2-4.1.(1-y^2)\geq 0 ;y\not< 1 [b^2-4ac\geq 0]$

$or,\ -4(1-y^2)\geq 0 ; y\not< 1$

$or,\ 4y^2-4\geq 0 ; y\not< 1$

$or,\ y^2-1\geq 0 ; y\not< 1$

Here, *x* is defined for all $y\geq 1$.

$$R_f=\{y : y\geq 1\}$$

$=[1,\infty)$ **(Ans).**

Problem 09: Find the domain and range of the function $y=\sqrt{4-x^2}$.
Solution: Given function is,

$y=\sqrt{4-x^2}$

Here, y gives real values iff,

$4-x^2\geq 0$

$or,\ (2+x)(2-x)\geq 0$

This inequality is satisfied if,

$-2\leq x\leq 2$

Therefore, the domain of the given function is,

$D_f = \{x : -2 \le x \le 2\}$

$= [-2, 2]$

Again, we have,

$$y = \sqrt{4 - x^2} \quad \dots \dots (1)$$

The values of y in (1) are positive and lowest value is zero ,*i.e,* $y \nless 0$.

Now $y^2 = 4 - x^2$; $y \nless 0$ [Squaring both sides]

$\Rightarrow x^2 + y^2 - 4 = 0$; $y \nless 0$

$\Rightarrow x^2 + 0.x + (y^2 - 4) = 0$; $y \nless 0$

In the above equation the values of x will be real if and only if it's $Discriminant \ge 0$.

i.e, $0^2 - 4.1.(y^2 - 4) \ge 0$; $y \nless 0$ [$b^2 - 4ac \ge 0$]

$or,\ -4y^2 + 16 \ge 0$; $y \nless 0$

$or,\ y^2 - 4 \le 0$; $y \nless 0$ [Dividing by -4]

Here, x is defined for all $0 \le y \le 2$.

Therefore the range of the given function is,

$R_f = \{y : 0 \le y \le 2\}$

$= [0, 2]$ (Ans.)

H.W:

Find the domain and range of the following functions

1. $y = \sqrt{x^2 - 3}$ Ans: $D_f = R - (-\sqrt{3}, \sqrt{3})$ and $R_f = [0, \propto)$
2. $y = \sqrt{x^2 - 25}$ Ans: $D_f = R - (-5, 5)$ and $R_f = [0, \propto)$
3. $y = \sqrt{x^2 + 3x}$ Ans: $D_f = R - (-3, 0)$ and $R_f = [0, \propto)$
4. $y = \sqrt{x^2 - 2x}$ Ans: $D_f = R - (0, 2)$ and $R_f = [0, \propto)$
5. $y = \sqrt{x^2 + 3}$ Ans: $D_f = R$ and $R_f = [3, \propto)$
6. $y = \sqrt{x^2 + 25}$ Ans: $D_f = R$ and $R_f = [25, \propto)$
7. $y = \sqrt{16 - x^2}$ Ans: $D_f = [-4, 4]$ and $R_f = [0, 4]$
8. $y = \sqrt{x^2 - 2x + 2}$ Ans: $D_f = R$ and $R_f = [1, \infty)$

Problem 10: Find the domain and range of the function $y = \dfrac{1}{\sqrt{2x + 3}}$.

Solution: Given function is,

$$y=\frac{1}{\sqrt{2x+3}}$$

Here, y gives real values iff,

$$2x+3>0$$

$$or,\ 2x>-3$$

$$or,\ x>-\frac{3}{2}$$

Therefore the domain of the given function is $D_f=\{x:x>-\frac{3}{2}\}$.

$$D_f=\left(-\frac{3}{2},\infty\right)$$

Again, we have,

$$y=\frac{1}{\sqrt{2x+3}}\ \cdots\cdots(1)$$

The values of y in (1) are positive and lowest value is near to 0,*i.e,* $y>0$.

Now, $y^2=\dfrac{1}{2x+3}$; $y>0$

$$or,\ 2x+3=\frac{1}{y^2} \quad ; y>0$$

$$or,\ 2x=\frac{1}{y^2}-3 \quad ; y>0$$

$$or,\ x=\frac{1}{2}\left(\frac{1}{y^2}-3\right) \quad ; y>0$$

Here, $x$is defined for all $y>0$.

Therefore the range of the given function is

$$R_f=\{y:y>0\}$$

$$=(0,\infty)$$ **(Ans)**