System of Differential Equations

System of Homogeneous Linear Differential Equations: Consider the following system of differential equations,

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\cdots\ \cdots\ \cdots\ &\cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

where the coefficients $a_{ij}\left(i=1,2,\cdots,n;\ j=1,2,\cdots,n\right)$ are real constants.

This is called system of homogeneous linear differential equations.

In matrix form the above system can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$or,\quad \frac{d\bar{x}}{dt} = A\bar{x}$$

where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

The solution of this system is

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

where $\phi_1, \phi_2, \ldots, \phi_n$ have a continuous derivative on real interval.

In fact,
$$\begin{aligned} x_1 &= \phi_1(t) \\ x_2 &= \phi_2(t) \\ &\vdots \\ x_n &= \phi_n(t) \end{aligned}$$

Fundamental set: If $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$, $a < t < b$, are linearly independent solutions of the homogeneous vector differential equation $\bar{x}'(t) = A(t)\bar{x}$, then the set$\{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$ is called the fundamental set of this equation on (a, b).

Fundamental Matrix: A matrix whose individual columns consist of a fundamental set of solutions of $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ is called a fundamental matrix of it.

Consider α_1, α_2,⋯, α_n form a fundamental set of solutions of $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ defined as

$$\alpha_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix}, \alpha_2(t) = \begin{pmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \vdots \\ \phi_{n2}(t) \end{pmatrix}, \cdots, \alpha_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}$$

Then the $n \times n$ square matrix

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}$$

is called the fundamental matrix.

Question-01: Prove that there exists fundamental set of solutions of the homogeneous system $\bar{x}'(t) = A(t)\bar{x}$, where $A(t)$ is a continuous matrix function.

Solution: Given that $\bar{x}'(t) = A(t)\bar{x}$

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \qquad \cdots (1)$$

we define a special set of constant vectors $u_1, u_2, \cdots, u_n$ as

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \cdots, u_n = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix}$$

That is, for each $i = 1, 2, \cdots, n$, u_1 has ith component one and all other components zero.

Now let $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ be the n solutions of (1) that satisfy the condition

$$\phi_i(t_0) = u_i \ , i = 1,2,\cdots,n$$

That is, $\phi_1(t_0) = u_1$, $\phi_2(t_0) = u_2$, $\cdots$, $\phi_n(t_0) = u_n$

where t_0 is an arbitrary point of (a, b).

Note that these solutions exist and are unique.

Now $W(\phi_1, \phi_2, \cdots, \phi_n)(t) = W(u_1, u_2, \cdots, u_n)$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= 1 \neq 0$$

$$\therefore W(\phi_1, \phi_2, \cdots, \phi_n)(t) \neq 0, \text{ for all } t \in [a, b].$$

This implies that the solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on $[a, b]$. Thus, $\phi_1, \phi_2, \cdots, \phi_n$ form a fundamental set of (1).

Hence there exists a fundamental set of solutions of the homogeneous linear differential equation $\bar{x}'(t) = A(t)\bar{x}$, where $A(t)$ is a continuous matrix function. (**Proved**)

Question-02: Prove that the solutions of $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ form an n-dimensional linear space, where $\bar{x}\,(t)$ is an n-dimensional vector and $A(t)$ is an $n \times n$ matrix.

<u>Solution</u>: Given that $\frac{d\bar{x}}{dt} = A(t)\bar{x}$

$$\bar{x}' = A(t)\bar{x} \qquad \cdots (1)$$

Since A is an $n \times n$ matrix, so all solutions of (1) contain n components. Therefore, every solution vector $\bar{x} = \bar{x}(t)$ belongs to R^n, where R^n is a linear space of dimension n.

Thus, the solution set $V \subseteq R^n$.

Since $\bar{x} = 0$ is the trivial solution of (1), so $0 \in V$.

Let $\bar{x}, \bar{y} \in V$. Then $\bar{x}' = A\bar{x}$ and $\bar{y}' = A\bar{y}$. For any scalar a , $b \in R$ we have

$$(a\bar{x} + b\bar{y})' = (a\bar{x})' + (b\bar{y})'$$
$$= a\bar{x}' + a\bar{y}'$$
$$= aA\bar{x} + aA\bar{y}$$
$$= A(a\bar{x}) + A(a\bar{y})$$
$$= A(a\bar{x} + a\bar{y})$$
$$\therefore a\bar{x} + a\bar{y} \in V.$$

Therefore, V is a subspace of R^n and hence V itself is a linear (vector) space.

Now we shall prove that V has the dimension n. To prove this we show that V has a basis with n vectors.

Let $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ be the n solutions of (1) with initial conditions

$$\phi_1(t_0) = e_1 , \phi_2(t_0) = e_2 , \cdots , \ \phi_n(t_0) = e_n , \ t_0 \in (t_1, t_2).$$

where $e_1 , e_2 , \cdots, e_n$ are usual basis of R^n.

Since A is an $n \times n$ matrix, so the above n solutions exist and are unique with

$$\phi_i(t_0) = e_i , i = 1,2, \cdots, n.$$

Now we prove that the solution vectors $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ linearly independent and they generate (span) V.

Independent part: Let $c_1, c_2, \cdots, c_n$ are scalars such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0 , \ \forall t \ \in (t_1, t_2)$$

Replacing t by t_0 we get

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0) = 0$$

$$or, c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0$$

$$or, c_1(1, 0, \cdots ,0) + c_2(0, 1, \cdots ,0) + \cdots + c_n(0, 0, \cdots ,1) = 0$$

$$or, (c_1, c_2, \cdots, c_n) = (0, 0, \cdots ,0)$$

$$or, c_1 = 0, c_2 = 0, \cdots, c_n = 0$$

$$or, c_1 = c_2 = \cdots = c_n = 0$$

Hence the solutions are linearly independent.

Generator part:

Since the solutions $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ are linearly independent , so any solution $\phi(t)$ of (1) can be written as

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) \qquad \cdots (2)$$

We shall prove that, this representation is unique.

If possible let,

$$\phi(t) = d_1\phi_1(t) + d_2\phi_2(t) + \cdots + d_n\phi_n(t) \qquad \cdots (3)$$

where $d_1, d_2, \cdots, d_n$ are scalars.

From (2) and (3)

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = d_1\phi_1(t) + d_2\phi_2(t) + \cdots + d_n\phi_n(t)$$

$$or, (c_1 - d_1)\phi_1(t) + (c_2 - d_2)\phi_2(t) + \cdots + (c_n - d_n)\phi_n(t) = 0$$

Since $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ are linearly independent so,

$$(c_1 - d_1) = 0, (c_2 - d_2) = 0, \cdots, (c_n - d_n) = 0$$

$or, c_1 = d_1, c_2 = d_2, \cdots, c_n = d_n$

Using these values in (3) we get

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t)$$

which is same as (2).

Hence the representation (2) is unique.

Thus, any solution $\phi(t)$ can be expressed as a unique linear combination of n solutions $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$. That is $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ generates V.

Therefore the solutions set $\{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$ is a basis of V and such dim $V = n$.

Hence the set of all solutions forms a linear space of dimension n. (**Proved**)

System of Non-homogeneous Linear Differential Equations: Consider the following system of differential equations,

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + F_1(t) \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + F_2(t) \\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + F_n(t)
\end{aligned}$$

where the coefficients $a_{ij}\left(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n\right)$ are real constants.

This is called system of non- homogeneous linear differential equations.

In matrix form the above system can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

$$or,\ \frac{d\bar{x}}{dt} = A\bar{x} + F(t)$$

where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$.

Question-03: State and prove the variation of constant formula.

Solution: Statement: If φ is a fundamental matrix of $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ on $[a, b]$, then

$$\bar{x}(t) = \varphi(t)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(u)F(u)du$$

is the unique solution of

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \ ,$$

where $\varphi(t)$ is a fundamental matrix satisfying $\varphi(t_0) = I, \bar{x}$ is an n vector, $A(t)$ is an $n \times n$ matrix, $F(t)$ is an n vector on $(t_0\, , t)$ and $t_0, t\epsilon[a, b]$.

Proof: Here the homogeneous and non-homogeneous systems are

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \qquad \cdots (1)$$

and $$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \qquad \cdots (2)$$

If $\varphi(t)$ is a fundamental matrix of (1), then the general solution of (1) is

$$\bar{x}(t) = \varphi(t)c \qquad \cdots (3)$$

where c is an arbitrary n rowed constant vector.

For variation of constants (parameters), we replace c of (3) by $v(t)$, and so we have

$$\bar{x}(t) = \varphi(t)v(t) \qquad \cdots (4)$$

We now determine $v(t)$ so that (4) is a solution of (2) with the condition $v(t_0) = \bar{x}_0$.

Differentiating (4) with respect to t we get,

$$\frac{d}{dt}[\bar{x}(t)] = \frac{d}{dt}\left[\varphi(t)v(t)\right]$$

$$\Rightarrow \frac{d\bar{x}}{dt} = \varphi'(t)v(t) + \varphi(t)v'(t) \qquad \cdots (5)$$

By putting the values of (4) and (5) in (2) we get,

$$\varphi'(t)v(t) + \varphi(t)v'(t) = A(t)\varphi(t)v(t) + F(t) \qquad \cdots (6)$$

Since $\varphi(t)$ is a fundamental matrix of the homogeneous system (1), so $\bar{x} = \bar{x}(t) = \varphi(t)$ and $|\varphi(t)| \neq 0$ on $[a, b]$ and hence $\varphi^{-1}(t)$ exists and unique.

Therefore (1) gives,

$$\frac{d}{dt}[\varphi(t)] = A(t)\varphi(t)$$

$$\Rightarrow \varphi'(t) = A(t)\varphi(t) \qquad \cdots (7)$$

Putting this value in (6) we get,

$$A(t)\varphi(t)v(t) + \varphi(t)v'(t) = A(t)\varphi(t)v(t) + F(t)$$

$$\Rightarrow \varphi(t)v'(t) = F(t)$$

$$\Rightarrow \varphi^{-1}(t)\varphi(t)v'(t) = \varphi^{-1}(t)F(t)$$

$$\Rightarrow Iv'(t) = \varphi^{-1}(t)F(t), \quad \text{where } I \text{ is an identity matrix}$$

$$\Rightarrow \frac{dv(t)}{dt} = \varphi^{-1}(t)F(t)$$

By integrating from t_0 to t we get,

$$v(t) = \bar{x}_0 + \int_{t_0}^{t} \varphi^{-1}(u)F(u)du \qquad \because v(t_0) = \bar{x}_0$$

Putting this in (4) we get,

$$\bar{x}(t) = \varphi(t)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(u)F(u)du \qquad \cdots (8)$$

If $x_0 = 0$ then (8) reduces to

$$\bar{x}(t) = \varphi(t)\int_{t_0}^{t} \varphi^{-1}(u)F(u)du. \qquad \textbf{(Proved)}$$

<u>Question-04</u>: Prove that the solution of the nonhomogeneous system $\bar{x}'(t) = A(t)\bar{x} + b(t)$, $\bar{x}(t_0) = \bar{x}_0$ is $\bar{x}(t) = \varphi(t)\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds$, where $A(t)$ is an $n \times n$ continuous matrix and $\varphi(t)$ is a fundamental matrix of the corresponding homogeneous system and $\varphi(t_0) \neq I$.

<u>Solution</u>: The given non-homogeneous equation is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + b(t) \qquad \cdots (1)$$

The corresponding homogeneous equation of (1) is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \quad \cdots (2)$$

Given $\varphi(t)$ is fundamental matrix of (2).

We shall prove that solution $\bar{x}(t)$ of (1) can be expressed as

$$\bar{x}(t) = \varphi(t)\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds \quad \cdots (3)$$

with initial condition $\bar{x}(t_0) = \bar{x}_0$ $\cdots (4)$

Let $\psi(t_0) = \bar{x}(t_0) = \bar{x}_0$

we know that if

(i) ψ_0 is any solution of (1)

(ii) $\psi_1, \psi_2, \cdots, \psi_n$ are fundamental sets of (2)

(iii) φ is a fundamental matrix of (2) having ψ_k as its individual columns

then $\psi_0(t) = \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds$ $\cdots (5)$

$$\psi(t) = \sum_{k=1}^{n} c_k\psi_k(t) + \psi_0(t) \quad \cdots (6)$$

$$\varphi(t)c = \sum_{k=1}^{n} c_k\psi_k(t) \quad \cdots (7)$$

where c_k are suitably chosen constants and c is an arbitrary n −rowed constant vector.

Using (5) and (7) we get from (6)

$$\psi(t) = \varphi(t)c + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds \quad \cdots (8)$$

Putting $t = t_0$ in (8) we get

$$\psi(t_0) = \varphi(t_0)c + \varphi(t_0)\int_{t_0}^{t_0} \varphi^{-1}(s)b(s)\,ds$$

$$\Rightarrow \bar{x}_0 = \varphi(t_0)c + 0$$

$$\Rightarrow \bar{x}_0 = \varphi(t_0)c$$

$$\Rightarrow \varphi^{-1}(t_0)\bar{x}_0 = \varphi^{-1}(t_0)\varphi(t_0)c$$

$$\Rightarrow \varphi^{-1}(t_0)\bar{x}_0 = Ic$$

$$\Rightarrow c = \varphi^{-1}(t_0)\bar{x}_0 \quad \cdots (9)$$

Putting the value of c in (8) we get

$$\psi(t) = \varphi(t)\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds$$

$$or, \bar{x}(t) = \varphi(t)\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds \qquad \cdots (10)$$

Equations (3) and (10) are same.

Thus, with the initial condition $\bar{x}(t_0) = \bar{x}_0$, the solution of (1) is

$$\bar{x}(t) = \varphi(t)\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(s)b(s)\,ds \qquad \textbf{(Proved)}$$

Problem

Problem-01: Find a fundamental set of solutions of the system of equations $\bar{x}' = A(t)\bar{x}$, where

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \text{ and } \bar{x} = \bar{x}(t).$$

Solution: Given that $\bar{x}' = A(t)\bar{x}$ $\cdots (1)$

where $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$

We shall find a solution of the form

$$\bar{x}(t) = e^{\lambda t}u \qquad \cdots (2)$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix}$$

$$=(4-\lambda)(-1-\lambda) + 6$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$or,\ \lambda^2 - 3\lambda + 2 = 0$$

$$or,\ \lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$or, \lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$or, (\lambda-2)(\lambda-1)=0$$

$$\therefore\ (\lambda-1)=0\ ,\ or\ (\lambda-2)=0$$

$$\therefore\ \lambda=1\ ,2$$

Now $(A-\lambda I)u=0$ gives

$$\begin{pmatrix}4-\lambda & -3\\ 2 & -1-\lambda\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=0 \qquad \cdots(3)$$

For $\lambda=1$ we get from (3)

$$\begin{pmatrix}3 & -3\\ 2 & -2\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=0$$

$$or, \begin{pmatrix}3u_1-3u_2\\ 2u_1-2u_2\end{pmatrix}=0$$

$$or, \left.\begin{matrix}3u_1-3u_2=0\\ 2u_1-2u_2=0\end{matrix}\right\}$$

$$or, \quad \left.\begin{matrix}3u_1-3u_2=0\\ 0=0\end{matrix}\right\} \qquad L_2'=3L_2-2L_1$$

$$or, \quad u_1-u_2=0 \qquad L_1'=\frac{1}{3}L_1$$

Putting $u_2=1$ we get $u_1=1$.

$$\therefore\ u=\begin{pmatrix}1\\ 1\end{pmatrix}$$

For $\lambda=2$ we get from (3)

$$\begin{pmatrix}2 & -3\\ 2 & -3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=0$$

$$or, \begin{pmatrix}2u_1-3u_2\\ 2u_1-3u_2\end{pmatrix}=0$$

$$or, \left.\begin{matrix}2u_1-3u_2=0\\ 2u_1-3u_2=0\end{matrix}\right\}$$

$or,\quad 2u_1-3u_2=0$, since both equations are same.

Putting $u_2=2$ we get $u_1=3$.

$$\therefore\ u=\begin{pmatrix}3\\ 2\end{pmatrix}$$

The solutions are

$$\varphi_1(t)=e^t\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}e^t\\ e^t\end{pmatrix}$$

$$\varphi_2(t)=e^{2t}\begin{pmatrix}3\\ 2\end{pmatrix}=\begin{pmatrix}3e^{2t}\\ 2e^{2t}\end{pmatrix}$$

The Wronskian of the solutions is

$$W\left(\varphi_1(t), \varphi_2(t)\right) = \begin{vmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}$$
$$= 2e^{3t} - 3e^{3t}$$
$$= -e^{3t} \neq 0.$$

Thus, $\varphi_1(t), \varphi_2(t)$ are linearly independent solutions of the given system.

The fundamental set is

$$\left\{\varphi_1(t), \varphi_2(t)\right\} = \left\{\begin{pmatrix} e^t \\ e^t \end{pmatrix}, \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}\right\} \qquad \textbf{(Ans)}$$

Problem-02: Solve the system: $x_1' = 3x_1 - x_2$, $x_2' = 4x_1 - x_2$

<u>Solution</u>: Given that $\begin{aligned} x_1' &= 3x_1 - x_2 \\ x_2' &= 4x_1 - x_2 \end{aligned}$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \frac{d\bar{x}}{dt} = A\bar{x}(t) \qquad \cdots (1)$$

where $\bar{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

The solution of (1) is

$$\bar{x}(t) = e^{\lambda t} v \qquad \cdots (2)$$

where v is an eigen vector.

The characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-1-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1) = 0$$

$$\therefore \lambda = 1\,, 1$$

Now $(A - \lambda I)v = 0$ gives

$$\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \qquad \text{where } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \cdots (3)$$

For $\lambda = 1$ we get from (3)

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} 2v_1 - v_2 &= 0 \\ 4v_1 - 2v_2 &= 0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} 2v_1 - v_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad L_2' = 2L_1 - L_2$$

$$\Rightarrow 2v_1 - v_2 = 0 \qquad \cdots (4)$$

Putting $v_2 = 2$ in (4) we get $v_1 = 1$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The solution is

$$\bar{x}_1(t) = \varphi_1(t) = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}.$$

Let $\bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$ be another solution of (1).

Then (1) must be satisfied by $\varphi_2(t)$.

$$\varphi_2'(t) = A\varphi_2(t)$$

$$\Rightarrow \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} e^t + \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3c_1 + 3c_2 t - c_3 - c_4 t \\ 4c_1 + 4c_2 t - c_3 - c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} = \begin{pmatrix} 3c_1 - c_3 + 3c_2 t - c_4 t \\ 4c_1 - c_3 + 4c_2 t - c_4 t \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_2 t = 3c_1 - c_3 + 3c_2 t - c_4 t \\ c_3 + c_4 + c_4 t = 4c_1 - c_3 + 4c_2 t - c_4 t \end{cases}$$

$$\Rightarrow \begin{cases} -2c_1 + c_2 + c_3 + (-2c_2 + c_4)t = 0 \\ -4c_1 + 2c_3 + +c_4 + (-4c_2 + 2c_4)t = 0 \end{cases}$$

This will be true if

$$\begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -4c_1 + 2c_3 + +c_4 = 0 \\ -4c_2 + 2c_4 = 0 \end{cases}$$

$$or,\begin{cases}-2c_1+c_2+c_3=0\\ -2c_2+c_4=0\\ -2c_2++c_4=0\\ -2c_2+c_4=0\end{cases}\qquad \begin{matrix}L_3'=L_3-2L_1\\ L_4'=\frac{1}{2}L_4\end{matrix}$$

$$or,\begin{cases}-2c_1+c_2+c_3=0\\ -2c_2+c_4=0\end{cases}$$

Putting $c_4=2$ and $c_3=1$ we get $c_2=1$ and $c_1=1$

The solution is

$$\bar{x}_2(t)=\varphi_2(t)=\begin{pmatrix}1+t\\1+2t\end{pmatrix}e^t.$$

Therefore the solutions of the given system are

$$\bar{x}_1(t)=\begin{pmatrix}e^t\\2e^t\end{pmatrix} \text{ and } \bar{x}_2(t)=\begin{pmatrix}1+t\\1+2t\end{pmatrix}e^t\ .$$

Problem-03: Compute a fundamental matrix for the system:

$$\frac{d\bar{x}}{dt}=\begin{bmatrix}7&-1&6\\-10&4&-12\\-2&1&-1\end{bmatrix}\bar{x}\,,\ \ \bar{x}\ =\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ and solve it.}$$

Solution: Given that $\frac{d\bar{x}}{dt}=\begin{bmatrix}7&-1&6\\-10&4&-12\\-2&1&-1\end{bmatrix}\bar{x}$ $\cdots(1)$

where $A=\begin{bmatrix}7&-1&6\\-10&4&-12\\-2&1&-1\end{bmatrix}$ and $\bar{x}\ =\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

We shall find a solution of the form

$$\bar{x}\,(t)=e^{\lambda t}u \qquad \cdots(2)$$

where $u=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of A is

$$A-\lambda I=\begin{pmatrix}7&-1&6\\-10&4&-12\\-2&1&-1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}7&-1&6\\-10&4&-12\\-2&1&-1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}$$

$$=\begin{pmatrix}7-\lambda&-1&6\\-10&4-\lambda&-12\\-2&1&-1-\lambda\end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix}$$

$$=(7-\lambda)\{(4-\lambda)(-1-\lambda)+12\}+1\{-10(-1-\lambda)-24\}+6\{-10+2(4-\lambda)\}$$

$$=(7-\lambda)(\lambda^2-3\lambda+8)+(10\lambda-14)+(-12-12\lambda)$$

$$=-\lambda^3+10\lambda^2-29\lambda+56+10\lambda-14-12-12\lambda$$

$$=-\lambda^3+10\lambda^2-31\lambda+30$$

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$or, -\lambda^3+10\lambda^2-31\lambda+30=0$$

$$or, \lambda^3-10\lambda^2+31\lambda-30=0$$

$$or, \lambda^2(\lambda-2)-8\lambda(\lambda-2)+15(\lambda-2)=0$$

$$or, (\lambda-2)(\lambda^2-8\lambda+15)=0$$

$$or, (\lambda-2)\{\lambda(\lambda-3)-5(\lambda-3)\}=0$$

$$or, (\lambda-2)(\lambda-3)(\lambda-5)=0$$

$$\therefore\ (\lambda-2)=0\ ,\ or\ (\lambda-3)=0,\ \ or\ (\lambda-5)=0$$

$$\therefore\ \lambda=2\ ,3\ ,5$$

Now $(A-\lambda I)u=0$ gives

$$\begin{pmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=0 \qquad \cdots(3)$$

For $\lambda=2$ we get from (3)

$$\begin{pmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=0$$

$$or, \begin{pmatrix} 5u_1-u_2+6u_3 \\ -10u_1+2u_2-12u_3 \\ -2u_1+u_2-3u_3 \end{pmatrix}=0$$

$$or, \left.\begin{aligned} 5u_1-u_2+6u_3&=0 \\ -10u_1+2u_2-12u_3&=0 \\ -2u_1+u_2-3u_3&=0 \end{aligned}\right\}$$

$$or, \left.\begin{aligned} 5u_1-u_2+6u_3&=0 \\ 0&=0 \\ 3u_2-3u_3&=0 \end{aligned}\right\} \qquad \begin{aligned} L_2'&=2L_1+L_2 \\ L_3'&=2L_1+5L_3 \end{aligned}$$

$$or,\quad \left.\begin{aligned} 5u_1 - u_2 + 6u_3 &= 0 \\ u_2 - u_3 &= 0 \end{aligned}\right\} \qquad L_2' = \tfrac{1}{3}L_2$$

Here u_3 is a free variable.

Putting $u_3 = -1$ we get $u_2 = -1$ and $u_1 = 1$.

$$\therefore\ u = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Again for $\lambda = 3$ we get from (3)

$$\begin{pmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$or,\begin{pmatrix} 4u_1 - u_2 + 6u_3 \\ -10u_1 + u_2 - 12u_3 \\ -2u_1 + u_2 - 4u_3 \end{pmatrix} = 0$$

$$or,\left.\begin{aligned} 4u_1 - u_2 + 6u_3 &= 0 \\ -10u_1 + u_2 - 12u_3 &= 0 \\ -2u_1 + u_2 - 4u_3 &= 0 \end{aligned}\right\}$$

$$or,\quad \left.\begin{aligned} 4u_1 - u_2 + 6u_3 &= 0 \\ -3u_2 + 6u_3 &= 0 \\ u_2 - 2u_3 &= 0 \end{aligned}\right\} \qquad \begin{aligned} L_2' &= 5L_1 + 2L_2 \\ L_3' &= L_1 + 2L_3 \end{aligned}$$

$$or,\quad \left.\begin{aligned} 4u_1 - u_2 + 6u_3 &= 0 \\ -3u_2 + 6u_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad L_3' = L_2 + 3L_3$$

$$or,\quad \left.\begin{aligned} 4u_1 - u_2 + 6u_3 &= 0 \\ u_2 - 2u_3 &= 0 \end{aligned}\right\} \qquad L_2' = -\tfrac{1}{2}L_2$$

Here u_3 is a free variable.

Putting $u_3 = -1$ we get $u_2 = -2$ and $u_1 = 1$.

$$\therefore\ u = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Again for $\lambda = 5$ we get from (3)

$$\begin{pmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$or,\begin{pmatrix} 2u_1 - u_2 + 6u_3 \\ -10u_1 - u_2 - 12u_3 \\ -2u_1 + u_2 - 6u_3 \end{pmatrix} = 0$$

$$or,\left.\begin{aligned} 2u_1 - u_2 + 6u_3 &= 0 \\ -10u_1 - u_2 - 12u_3 &= 0 \\ -2u_1 + u_2 - 6u_3 &= 0 \end{aligned}\right\}$$

$$or, \quad \left.\begin{aligned} 2u_1 - u_2 + 6u_3 &= 0 \\ -6u_2 + 18u_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad \begin{aligned} L_2' &= 5L_1 + L_2 \\ L_3' &= L_1 + L_3 \end{aligned}$$

$$or, \quad \left.\begin{aligned} 2u_1 - u_2 + 6u_3 &= 0 \\ u_2 - 3u_3 &= 0 \end{aligned}\right\} \qquad L_3' = -\tfrac{1}{6}L_2$$

Here u_3 is a free variable.

Putting $u_3 = -2$ we get $u_2 = -6$ and $u_1 = 3$.

$$\therefore\ u = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$$

The solutions are

$$\varphi_1(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix}$$

$$\varphi_2(t) = e^{3t}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$$

$$\varphi_3(t) = e^{5t}\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} = \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

The wronskian of the solutions is

$$W\left(\varphi_1(t), \varphi_2(t), \varphi_3(t)\right) = \begin{vmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{vmatrix}$$

$$= e^{2t}(4e^{8t} - 6e^{8t}) - e^{3t}(2e^{7t} - 6e^{7t}) + 3e^{5t}(e^{5t} - 2e^{5t})$$

$$= -2e^{10t} + 4e^{10t} - 3e^{10t}$$

$$= -e^{10t} \neq 0.$$

Thus, $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

$$\phi(t) = \begin{pmatrix} e^{2t} & e^{3t} & 3e^{5t} \\ -e^{2t} & -2e^{3t} & -6e^{5t} \\ -e^{2t} & -e^{3t} & -2e^{5t} \end{pmatrix}$$

The general solution of the given system is

$$\overline{x}(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t)$$

$$= c_1e^{2t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + c_3e^{5t}\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}.$$

Problem-04: Find the fundamental matrix for $\frac{d\bar{x}}{dt} = \begin{bmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{bmatrix} \bar{x}$ where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

__Solution__: Given that $\frac{d\bar{x}}{dt} = \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix} \bar{x}$ $\cdots (1)$

where $A = \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

We shall find a solution of the form

$$\bar{x}(t) = e^{\lambda t} u \qquad \cdots (2)$$

where $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 5-\lambda & 2 & -2 \\ 7 & -\lambda & -2 \\ 11 & 1 & -3-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 5-\lambda & 2 & -2 \\ 7 & -\lambda & -2 \\ 11 & 1 & -3-\lambda \end{vmatrix}$$

$$=(5-\lambda)\{-\lambda(-3-\lambda)+2\} - 2\{7(-3-\lambda)+22\} - 2\{7+11\lambda\}$$

$$= (5-\lambda)(\lambda^2+3\lambda+2) - 2(1-7\lambda) - 14 - 22\lambda$$

$$= (5\lambda^2 + 15\lambda + 10 - \lambda^3 - 3\lambda^2 - 2\lambda) - 2 + 14\lambda - 14 - 22\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 13\lambda + 10 - 16 - 8\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$or,\ -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$or, \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$or, \lambda^3 - \lambda^2 - \lambda^2 + \lambda - 6\lambda + 6 = 0$$

$$or, \lambda^2(\lambda - 1) - \lambda(\lambda - 1) - 6(\lambda - 1) = 0$$

$$or, (\lambda - 1)(\lambda^2 - \lambda - 6) = 0$$

$$or, (\lambda - 1)(\lambda^2 - 3\lambda + 2\lambda - 6) = 0$$

$$or, (\lambda - 1)\{\lambda(\lambda - 3) + 2(\lambda - 3)\} = 0$$

$$or, (\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

$$\therefore\ (\lambda - 1) = 0\ ,\ or\ (\lambda + 2) = 0,\ \ or\ (\lambda - 3) = 0$$

$$\therefore\ \lambda = 1\ ,-2\ ,3$$

Now $(A - \lambda I)u = 0$ gives

$$\begin{pmatrix} 5-\lambda & 2 & -2 \\ 7 & -\lambda & -2 \\ 11 & 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \qquad \cdots (3)$$

For $\lambda = -2$ we get from (3)

$$\begin{pmatrix} 7 & 2 & -2 \\ 7 & 2 & -2 \\ 11 & 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$or, \begin{pmatrix} 7u_1 + 2u_2 - 2u_3 \\ 7u_1 + 2u_2 - 2u_3 \\ 11u_1 + u_2 - u_3 \end{pmatrix} = 0$$

$$or, \left.\begin{matrix} 7u_1 + 2u_2 - 2u_3 = 0 \\ 7u_1 + 2u_2 - 2u_3 = 0 \\ 11u_1 + u_2 - u_3 = 0 \end{matrix}\right\}$$

$$or, \left.\begin{matrix} 7u_1 + 2u_2 - 2u_3 = 0 \\ 0 = 0 \\ -15u_2 + 15u_3 = 0 \end{matrix}\right\} \qquad \begin{matrix} L_2' = L_2 - L_1 \\ L_3' = 7L_3 - 112L_1 \end{matrix}$$

$$or, \left.\begin{matrix} 7u_1 + 2u_2 - 2u_3 = 0 \\ u_2 - u_3 = 0 \end{matrix}\right\} \qquad L_2' = -\frac{1}{15}L_2$$

Here u_3 is a free variable.

Putting $u_3 = 1$ we get $u_2 = 1$ and $u_1 = 0$.

$$\therefore\ u = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Again for $\lambda = 1$ we get from (3)

$$\begin{pmatrix} 4 & 2 & -2 \\ 7 & -1 & -2 \\ 11 & 1 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$or,\begin{pmatrix} 4u_1+2u_2-2u_3 \\ 7u_1-u_2-2u_3 \\ 11u_1+u_2-4u_3 \end{pmatrix}=0$$

$$or,\left.\begin{aligned} 4u_1+2u_2-2u_3&=0 \\ 7u_1-u_2-2u_3&=0 \\ 11u_1+u_2-4u_3&=0 \end{aligned}\right\}$$

$$or,\left.\begin{aligned} 4u_1+2u_2-2u_3&=0 \\ -18u_2+6u_3&=0 \\ -18u_2+6u_3&=0 \end{aligned}\right\} \qquad \begin{aligned} L_2'&=4L_2-7L_1 \\ L_3'&=4L_3-11L_1 \end{aligned}$$

$$or,\left.\begin{aligned} 4u_1+2u_2-2u_3&=0 \\ -18u_2+6u_3&=0 \\ 0&=0 \end{aligned}\right\} \qquad L_3'=L_3-L_2$$

$$or,\left.\begin{aligned} 2u_1+u_2-u_3&=0 \\ 3u_2-u_3&=0 \end{aligned}\right\} \qquad \begin{aligned} L_1'&=\tfrac{1}{2}L_1 \\ L_2'&=-\tfrac{1}{6}L_2 \end{aligned}$$

Here u_3 is a free variable.

Putting $u_3=3$ we get $u_2=1$ and $u_1=1$.

$$\therefore\ u=\begin{pmatrix}1\\1\\3\end{pmatrix}$$

Again for $\lambda=3$ we get from (3)

$$\begin{pmatrix} 2 & 2 & -2 \\ 7 & -3 & -2 \\ 11 & 1 & -6 \end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0$$

$$or,\begin{pmatrix} 2u_1+2u_2-2u_3 \\ 7u_1-3u_2-2u_3 \\ 11u_1+u_2-6u_3 \end{pmatrix}=0$$

$$or,\left.\begin{aligned} 2u_1+2u_2-2u_3&=0 \\ 7u_1-3u_2-2u_3&=0 \\ 11u_1+u_2-6u_3&=0 \end{aligned}\right\}$$

$$or,\left.\begin{aligned} 2u_1+2u_2-2u_3&=0 \\ -20u_2+10u_3&=0 \\ -20u_2+10u_3&=0 \end{aligned}\right\} \qquad \begin{aligned} L_2'&=2L_2-7L_1 \\ L_3'&=2L_3-11L_1 \end{aligned}$$

$$or,\left.\begin{aligned} 2u_1+2u_2-2u_3&=0 \\ -20u_2+10u_3&=0 \\ 0&=0 \end{aligned}\right\} \qquad L_3'=L_3-L_2$$

$$or,\left.\begin{aligned} u_1+u_2-u_3&=0 \\ 2u_2-u_3&=0 \end{aligned}\right\} \qquad \begin{aligned} L_1'&=\tfrac{1}{2}L_1 \\ L_3'&=-\tfrac{1}{10}L_2 \end{aligned}$$

Here u_3 is a free variable.

Putting $u_3 = 2$ we get $u_2 = 1$ and $u_1 = 1$.

$$\therefore\ u = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

The solutions are

$$\varphi_1(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$\varphi_2(t) = e^{t}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{pmatrix}$$

$$\varphi_3(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix}$$

The wronskian of the solutions is

$$W\left(\varphi_1(t), \varphi_2(t), \varphi_3(t)\right) = \begin{vmatrix} 0 & e^{t} & e^{3t} \\ e^{-2t} & e^{t} & e^{3t} \\ e^{-2t} & 3e^{t} & 2e^{3t} \end{vmatrix}$$

$$= 0 - e^{t}(2e^{t} - e^{t}) + e^{3t}(3e^{-t} - e^{-t})$$

$$= -e^{2t} + 2e^{2t}$$

$$= e^{2t} \neq 0.$$

Thus, $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

$$\varphi(t) = \begin{pmatrix} 0 & e^{t} & e^{3t} \\ e^{-2t} & e^{t} & e^{3t} \\ e^{-2t} & 3e^{t} & 2e^{3t} \end{pmatrix} \qquad \textbf{(Ans)}$$

Problem-05: Compute a fundamental matrix for the system of linear differential equation

$x_1' = -x_1 + x_2 - x_3, \quad x_2' = -2x_2 - 9x_3, \quad x_3' = x_2 - 2x_3$. Hence solve the system.

Solution: Given that

$$x_1' = -x_1 + x_2 - x_3$$

$$x_2' = -2x_2 - 9x_3$$

$$x_3' = x_2 - 2x_3$$

In matrix form, the given system can be written as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & -9 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \frac{d\bar{x}}{dt} = A\bar{x} \qquad \cdots (1)$$

where $A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & -9 \\ 0 & 1 & -2 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

We shall find a solution of the form

$$\bar{x}(t) = e^{\lambda t} u \qquad \cdots (2)$$

where $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is an eigen vector.

And also we shall find fundamental matrix.

The characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & -9 \\ 0 & 1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & -9 \\ 0 & 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & 1 & -1 \\ 0 & -2-\lambda & -9 \\ 0 & 1 & -2-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} -1-\lambda & 1 & -1 \\ 0 & -2-\lambda & -9 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$=(-1-\lambda)\{(-2-\lambda)(-2-\lambda)+9\} - 1\{0-0\} - 1\{0-0\}$$

$$= (-1-\lambda)(\lambda^2 + 4\lambda + 13)$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$or, (-1-\lambda)(\lambda^2 + 4\lambda + 13) = 0$$

$$or, (1+\lambda)(\lambda^2 + 4\lambda + 13) = 0$$

$$\therefore\ (1+\lambda) = 0\,, \quad or\ \lambda^2 + 4\lambda + 13 = 0$$

$$\Rightarrow\ \lambda = -1 \qquad or,\ \lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.13}}{2.1}$$

$$= \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$=\frac{-4\pm\sqrt{-36}}{2}$$

$$=\frac{-4\pm 6i}{2}$$

$$=-2+3i,-2-3i$$

$$\therefore\ \lambda=-1,-2+3i,-2-3i$$

Now $(A-\lambda I)u=0$ gives

$$\begin{pmatrix}-1-\lambda & 1 & -1\\ 0 & -2-\lambda & -9\\ 0 & 1 & -2-\lambda\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0 \qquad \cdots(3)$$

For $\lambda=-1$ we get from (3)

$$\begin{pmatrix}0 & 1 & -1\\ 0 & -1 & -9\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0$$

$$or,\begin{pmatrix}u_2-u_3\\ -u_2-9u_3\\ u_2-u_3\end{pmatrix}=0$$

$$\left.\begin{aligned}u_2-u_3&=0\\ or,-u_2-9u_3&=0\\ u_2-u_3&=0\end{aligned}\right\}$$

$$or,\quad\left.\begin{aligned}u_2-u_3&=0\\ -10u_3&=0\\ 0&=0\end{aligned}\right\}\qquad \begin{aligned}L_2'&=L_1+L_2\\ L_3'&=L_3-L_1\end{aligned}$$

$$or,\quad\left.\begin{aligned}u_2-u_3&=0\\ -10u_3&=0\end{aligned}\right\}$$

Solving these equations we get $u_3=0$ and $u_2=0$

Putting $u_1=1$ we get

$$\therefore\ u=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Again for $\lambda=-2+3i$ we get from (3)

$$\begin{pmatrix}1-3i & 1 & -1\\ 0 & -3i & -9\\ 0 & 1 & -3i\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0$$

$$or,\begin{pmatrix}(1-3i)u_1+u_2-u_3\\ -3iu_2-9u_3\\ u_2-3iu_3\end{pmatrix}=0$$

$$or,\quad\left.\begin{aligned}(1-3i)u_1+u_2-u_3&=0\\ -3iu_2-9u_3&=0\\ u_2-3iu_3&=0\end{aligned}\right\}$$

$$or, \quad \left.\begin{aligned} (1-3i)u_1 + u_2 - u_3 &= 0 \\ -3iu_2 - 9u_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad L_3' = 3iL_3 + L_2$$

$$or, \quad \left.\begin{aligned} (1-3i)u_1 + u_2 - u_3 &= 0 \\ -3iu_2 - 9u_3 &= 0 \end{aligned}\right\}$$

Here u_3 is a free variable.

Putting $u_3 = 1$ we get $u_2 = 3i$ and $u_1 = 1$.

$$\therefore \; u = \begin{pmatrix} 1 \\ 3i \\ 1 \end{pmatrix}$$

Again for $\lambda = -2 - 3i$ we get from (3)

$$\begin{pmatrix} 1+3i & 1 & -1 \\ 0 & 3i & -9 \\ 0 & 1 & 3i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$$or, \begin{pmatrix} (1+3i)u_1 + u_2 - u_3 \\ 3iu_2 - 9u_3 \\ u_2 + 3iu_3 \end{pmatrix} = 0$$

$$or, \quad \left.\begin{aligned} (1+3i)u_1 + u_2 - u_3 &= 0 \\ 3iu_2 - 9u_3 &= 0 \\ u_2 + 3iu_3 &= 0 \end{aligned}\right\}$$

$$or, \quad \left.\begin{aligned} (1+3i)u_1 + u_2 - u_3 &= 0 \\ 3iu_2 - 9u_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \qquad L_3' = -3iL_3 + L_2$$

$$or, \quad \left.\begin{aligned} (1+3i)u_1 + u_2 - u_3 &= 0 \\ 3iu_2 - 9u_3 &= 0 \end{aligned}\right\}$$

Here u_3 is a free variable.

Putting $u_3 = 1$ we get $u_2 = -3i$ and $u_1 = 1$.

$$\therefore \; u = \begin{pmatrix} 1 \\ -3i \\ 1 \end{pmatrix}$$

The solutions are

$$\varphi_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix}$$

$$\varphi_2(t) = e^{(-2+3i)t}\begin{pmatrix} 1 \\ 3i \\ 1 \end{pmatrix} = \begin{pmatrix} e^{(-2+3i)t} \\ 3ie^{(-2+3i)t} \\ e^{(-2+3i)t} \end{pmatrix}$$

$$\varphi_3(t) = e^{(-2-3i)t}\begin{pmatrix}1\\-3i\\1\end{pmatrix} = \begin{pmatrix}e^{(-2-3i)t}\\-3ie^{(-2-3i)t}\\e^{(-2-3i)t}\end{pmatrix}$$

The Wronskian of the solutions is

$$W\left(\varphi_1(t),\varphi_2(t),\varphi_3(t)\right) = \begin{vmatrix}e^{-t} & e^{(-2+3i)t} & e^{(-2-3i)t}\\0 & 3ie^{(-2+3i)t} & -3ie^{(-2-3i)t}\\0 & e^{(-2+3i)t} & e^{(-2-3i)t}\end{vmatrix}$$

$$= e^{-t}(3i+3i)e^{(-2+3i)t}e^{(-2-3i)t}$$

$$= 6ie^{-t}e^{-4t}$$

$$= 6ie^{-5t} \neq 0.$$

Thus, $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are linearly independent solutions of the given system and hence form a fundamental matrix.

The fundamental matrix is

$$\varphi(t) = \begin{pmatrix}e^{-t} & e^{(-2+3i)t} & e^{(-2-3i)t}\\0 & 3ie^{(-2+3i)t} & -3ie^{(-2-3i)t}\\0 & e^{(-2+3i)t} & e^{(-2-3i)t}\end{pmatrix}$$

The general solution of the given system is

$$\bar{x}(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t)$$

$$= c_1e^{-t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2e^{(-2+3i)t}\begin{pmatrix}1\\3i\\1\end{pmatrix} + c_3e^{(-2-3i)t}\begin{pmatrix}1\\-3i\\1\end{pmatrix}.$$

where c_1, c_2 and c_3 are arbitrary constants.

Problem-06: Solve $\bar{x}'(t) = \begin{bmatrix}3 & -1\\4 & -1\end{bmatrix}\bar{x}(t) + \begin{bmatrix}1\\t\end{bmatrix}, \bar{x}(0) = \begin{bmatrix}1\\-1\end{bmatrix}.$

Solution: Given that: $\frac{d\bar{x}}{dt} = \begin{bmatrix}3 & -1\\4 & -1\end{bmatrix}\bar{x}(t) + \begin{bmatrix}1\\t\end{bmatrix}$ $\cdots(1)$

and $\bar{x}(0) = \begin{bmatrix}1\\-1\end{bmatrix}.$ $\cdots(2)$

The general form of non-homogeneous system is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \quad \cdots(3)$$

with condition $\bar{x}(t_0) = \bar{x}_0$ $\cdots(4)$

If φ is a fundamental matrix of

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \quad \cdots(5)$$

then the solution of (3) can be expressed as

$$\bar{x}(t) = \varphi(\mathrm{t})\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t}\varphi^{-1}(u)F(u)du \qquad \cdots(6)$$

Comparing (1) with (3) and (2) with (4) we have

$$\left.\begin{array}{l} A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix},\ F(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \Rightarrow F(u) = \begin{bmatrix} 1 \\ u \end{bmatrix} \\ \bar{x}(t_0) = \bar{x}(0) \Rightarrow t_0 = 0 \ and\ \bar{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}\right\} \qquad \cdots(7)$$

For fundamental matrix of (5), let $\bar{x}(t) = e^{\lambda t}v$, where v is an eigen vector.

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-1-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1) = 0$$

$$\therefore \lambda = 1\,,1$$

Now $(A - \lambda I)v = 0$ gives

$$\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \quad \text{where } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \cdots(8)$$

For $\lambda = 1$ we get from (8)

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{r} 2v_1 - v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} 2v_1 - v_2 = 0 \\ 0 = 0 \end{array}\right\} \quad L_2' = 2L_1 - L_2$$

$$\Rightarrow 2v_1 - v_2 = 0 \qquad \cdots(9)$$

Putting $v_2 = 2$ in (9) we get $v_1 = 1$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \bar{x}_1(t) = \varphi_1(t) = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$$

Let $\bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$ be another solution of (5).

Then (5) must be satisfied by $\varphi_2(t)$.

$$\varphi_2'(t) = A\varphi_2(t)$$

$$\Rightarrow \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} e^t + \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3c_1 + 3c_2 t - c_3 - c_4 t \\ 4c_1 + 4c_2 t - c_3 - c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} = \begin{pmatrix} 3c_1 - c_3 + 3c_2 t - c_4 t \\ 4c_1 - c_3 + 4c_2 t - c_4 t \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_2 t = 3c_1 - c_3 + 3c_2 t - c_4 t \\ c_3 + c_4 + c_4 t = 4c_1 - c_3 + 4c_2 t - c_4 t \end{cases}$$

$$\Rightarrow \begin{cases} -2c_1 + c_2 + c_3 + (-2c_2 + c_4)t = 0 \\ -4c_1 + 2c_3 + +c_4 + (-4c_2 + 2c_4)t = 0 \end{cases}$$

This will be true if

$$\begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -4c_1 + 2c_3 + +c_4 = 0 \\ -4c_2 + 2c_4 = 0 \end{cases}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -2c_2 + +c_4 = 0 \\ -2c_2 + c_4 = 0 \end{cases} \qquad \begin{matrix} L_3' = L_3 - 2L_1 \\ L_4' = \frac{1}{2}L_4 \end{matrix}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \end{cases}$$

Putting $c_4 = 2$ and $c_3 = 1$ we get $c_2 = 1$ and $c_1 = 1$

$$\therefore \bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} 1 + t \\ 1 + 2t \end{pmatrix} e^t$$

$$W(\varphi_1, \varphi_2)(t) = \begin{vmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{vmatrix}$$

$$= (1+2t)e^{2t} - (2+2t)e^{2t}$$

$$= -e^{2t} \neq 0, \ for\ all\ t \epsilon\ R$$

The solutions $\varphi_1(t)$, $\varphi_2(t)$ are linearly independent.

The fundamental matrix for (5) is,

$$\varphi(\mathrm{t}) = \begin{pmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{pmatrix}$$

Here $\varphi(\mathrm{t}_0) = \varphi(0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq I$

$$\therefore \varphi^{-1}(\mathrm{t}) = \frac{1}{-e^{2t}} \begin{pmatrix} (1+2t)e^t & -(1+t)e^t \\ -2e^t & e^t \end{pmatrix}$$

$$= -e^t \begin{pmatrix} 1+2t & -1-t \\ -2 & 1 \end{pmatrix}$$

$$\varphi^{-1}(\mathrm{t}_0) = \varphi^{-1}(0) = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

and $\quad \varphi^{-1}(\mathrm{u}) = e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix}$

Now $\varphi(\mathrm{t})\varphi^{-1}(\mathrm{t}_0)\bar{x}_0 = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= e^t \begin{pmatrix} -2+3+3t \\ -4+3+6t \end{pmatrix}$$

$$= e^t \begin{pmatrix} 1+3t \\ -1+6t \end{pmatrix} \qquad \cdots (10)$$

and $\varphi(t) \int_{t_0}^{t} \varphi^{-1}(u)F(u)du = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} du$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u+u+u^2 \\ 2-u \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t \begin{pmatrix} u^2e^{-u} - ue^{-u} - e^{-u} \\ -ue^{-u} + 2e^{-u} \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - 2ue^{-u} - 2e^{-u} + ue^{-u} + e^{-u} + e^{-u} \\ ue^{-u} + e^{-u} - 2e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - ue^{-u} \\ ue^{-u} - e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2e^{-t} - te^{-t} \\ te^{-t} - e^{-t} + 1 \end{pmatrix}$$

$$= e^te^{-t} \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2 - t \\ t - 1 + e^t \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 - t + t - 1 + e^t + t^2 - t + te^t \\ -2t^2 - 2t + t - 1 + e^t + 2t^2 - 2t + 2te^t \end{pmatrix}$$

$$= \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix} \qquad \cdots (11)$$

By putting the values of (9) and (10) in (6) we get

$$\varphi(t) = e^t \begin{pmatrix} 1+3t \\ -1+6t \end{pmatrix} + \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t + 3te^t + te^t - t + e^t - 1 \\ -e^t + 6te^t + 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4te^t + 2e^t - t - 1 \\ 8te^t - 3t - 1 \end{pmatrix} \qquad \textbf{(Ans)}$$

Problem-07: Solve: $\begin{matrix} x_1' = 3x_1 - x_2 + 1 \\ x_2' = 4x_1 - x_2 + t \end{matrix} \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution: Given that $\begin{matrix} x_1' = 3x_1 - x_2 + 1 \\ x_2' = 4x_1 - x_2 + t \end{matrix}$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$\Rightarrow \frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} 1 \\ t \end{pmatrix} \qquad \cdots (1)$$

where $\bar{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

and $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\cdots (2)$

The general form of non-homogeneous system is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \qquad \cdots (3)$$

with condition $\bar{x}(t_0) = \bar{x}_0$. $\cdots (4)$

If φ is a fundamental matrix of

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \qquad \cdots (5)$$

then the solution of (3) can be expressed as

$$\bar{x}\,(t) = \varphi(\mathrm{t})\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(u)F(u)du \qquad \cdots (6)$$

Comparing (1) with (3) and (2) with (4) we have

$$\left.\begin{matrix} A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}, \; F(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \Rightarrow F(u) = \begin{bmatrix} 1 \\ u \end{bmatrix} \\ \bar{x}\,(t_0) = \bar{x}\,(0) \Rightarrow t_0 = 0 \; and \; \bar{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{matrix}\right\} \qquad \cdots (7)$$

For fundamental matrix of (5), let $\bar{x}(t) = e^{\lambda t}v$, where v is an eigen vector.

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-1-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1) = 0$$

$$\therefore \lambda = 1\,, 1$$

Now $(A - \lambda I)v = 0$ gives

$$\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \quad \text{where } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \cdots (8)$$

For $\lambda = 1$ we get from (8)

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} 2v_1 - v_2 &= 0 \\ 4v_1 - 2v_2 &= 0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} 2v_1 - v_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad L_2' = 2L_1 - L_2$$

$$\Rightarrow 2v_1 - v_2 = 0 \qquad \cdots (9)$$

Putting $v_2 = 2$ in (9) we get $v_1 = 1$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \bar{x}_1(t) = \varphi_1(t) = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$$

Let $\bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$ be another solution of (5).

Then (5) must be satisfied by $\varphi_2(t)$.

$$\varphi_2'(t) = A\varphi_2(t)$$

$$\Rightarrow \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} e^t + \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3c_1 + 3c_2 t - c_3 - c_4 t \\ 4c_1 + 4c_2 t - c_3 - c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} = \begin{pmatrix} 3c_1 - c_3 + 3c_2 t - c_4 t \\ 4c_1 - c_3 + 4c_2 t - c_4 t \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_2 t = 3c_1 - c_3 + 3c_2 t - c_4 t \\ c_3 + c_4 + c_4 t = 4c_1 - c_3 + 4c_2 t - c_4 t \end{cases}$$

$$\Rightarrow \begin{cases} -2c_1 + c_2 + c_3 + (-2c_2 + c_4)t = 0 \\ -4c_1 + 2c_3 + +c_4 + (-4c_2 + 2c_4)t = 0 \end{cases}$$

This will be true if

$$\begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -4c_1 + 2c_3 + +c_4 = 0 \\ -4c_2 + 2c_4 = 0 \end{cases}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -2c_2 + +c_4 = 0 \\ -2c_2 + c_4 = 0 \end{cases} \qquad \begin{matrix} L_3' = L_3 - 2L_1 \\ L_4' = \frac{1}{2} L_4 \end{matrix}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \end{cases}$$

Putting $c_4 = 2$ and $c_3 = 1$ we get $c_2 = 1$ and $c_1 = 1$

$$\therefore \bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} 1+t \\ 1+2t \end{pmatrix} e^t$$

$$W(\varphi_1, \varphi_2)(t) = \begin{vmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{vmatrix}$$

$$= (1+2t)e^{2t} - (2+2t)e^{2t}$$

$$= -e^{2t} \neq 0, \ for\ all\ t \epsilon\ R$$

The solutions $\varphi_1(t)$, $\varphi_2(t)$ are linearly independent.

The fundamental matrix for (5) is,

$$\varphi(\mathrm{t}) = \begin{pmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{pmatrix}$$

Here $\varphi(\mathrm{t}_0) = \varphi(0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq I$

$$\therefore \varphi^{-1}(\mathrm{t}) = \frac{1}{-e^{2t}} \begin{pmatrix} (1+2t)e^t & -(1+t)e^t \\ -2e^t & e^t \end{pmatrix}$$

$$= -e^t \begin{pmatrix} 1+2t & -1-t \\ -2 & 1 \end{pmatrix}$$

$$\varphi^{-1}(\mathrm{t}_0) = \varphi^{-1}(0) = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

and $\quad \varphi^{-1}(\mathrm{u}) = e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix}$

Now $\varphi(\mathrm{t})\varphi^{-1}(\mathrm{t}_0)\bar{x}_0 = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= e^t \begin{pmatrix} -1+2+2t \\ -2+2+4t \end{pmatrix}$$

$$= e^t \begin{pmatrix} 1+2t \\ 4t \end{pmatrix} \qquad \cdots (10)$$

and $\varphi(t) \int_{t_0}^{t} \varphi^{-1}(u) F(u) du = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} du$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u+u+u^2 \\ 2-u \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t \begin{pmatrix} u^2e^{-u} - ue^{-u} - e^{-u} \\ -ue^{-u} + 2e^{-u} \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - 2ue^{-u} - 2e^{-u} + ue^{-u} + e^{-u} + e^{-u} \\ ue^{-u} + e^{-u} - 2e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - ue^{-u} \\ ue^{-u} - e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2e^{-t} - te^{-t} \\ te^{-t} - e^{-t} + 1 \end{pmatrix}$$

$$= e^t e^{-t} \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2 - t \\ t - 1 + e^t \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 - t + t - 1 + e^t + t^2 - t + te^t \\ -2t^2 - 2t + t - 1 + e^t + 2t^2 - 2t + 2te^t \end{pmatrix}$$

$$= \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix} \qquad \cdots (11)$$

By putting the values of (9) and (10) in (6) we get

$$\bar{x}(t) = e^t \begin{pmatrix} 1+2t \\ 4t \end{pmatrix} + \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t + 2te^t + te^t - t + e^t - 1 \\ 4te^t + 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3te^t + 2e^t - t - 1 \\ 6te^t + e^t - 3t - 1 \end{pmatrix} \qquad \textbf{(Ans)}$$

Question-08: Solve: $\overline{x'}(t) = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ t \end{bmatrix}, \bar{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Solution</u>: Given that: $\frac{d\bar{x}}{dt} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ t \end{bmatrix} \qquad \cdots (1)$

and $\qquad \bar{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad \cdots (2)$

The general form of non-homogeneous system is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \qquad \cdots (3)$$

with condition $\bar{x}(t_0) = \bar{x}_0$. $\cdots(4)$

If φ is a fundamental matrix of

$$\frac{d\bar{x}}{dt} = A(t)\bar{x}. \quad \cdots(5)$$

then the solution of (3) can be expressed as

$$\bar{x}(t) = \varphi(\mathrm{t})\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t}\varphi^{-1}(u)F(u)du \quad \cdots(6)$$

Comparing (1) with (3) and (2) with (4) we have

$$\left.\begin{array}{c} A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix},\ F(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \Rightarrow F(u) = \begin{bmatrix} 1 \\ u \end{bmatrix} \\ \bar{x}(t_0) = \bar{x}(0) \Rightarrow t_0 = 0 \ and\ \bar{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}\right\} \quad \cdots(7)$$

For fundamental matrix of (5), let $\bar{x}(t) = e^{\lambda t}v$, where v is an eigen vector.

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-1-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1) = 0$$

$$\therefore \lambda = 1\,,1$$

Now $(A - \lambda I)v = 0$ gives

$$\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \quad \text{where } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \cdots(8)$$

For $\lambda = 1$ we get from (8)

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{r} 2v_1 - v_2 = 0 \\ 4v_1 - 2v_2 = 0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{aligned} 2v_1 - v_2 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad L_2' = 2L_1 - L_2$$

$$\Rightarrow 2v_1 - v_2 = 0 \qquad \cdots (9)$$

Putting $v_2 = 2$ in (9) we get $v_1 = 1$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \bar{x}_1(t) = \varphi_1(t) = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$$

Let $\bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$ be another solution of (5).

Then (5) must be satisfied by $\varphi_2(t)$.

$$\varphi_2'(t) = A\varphi_2(t)$$

$$\Rightarrow \begin{pmatrix} c_2 \\ c_4 \end{pmatrix} e^t + \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} c_1 + c_2 t \\ c_3 + c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} e^t = \begin{pmatrix} 3c_1 + 3c_2 t - c_3 - c_4 t \\ 4c_1 + 4c_2 t - c_3 - c_4 t \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_3 + c_4 + c_4 t \end{pmatrix} = \begin{pmatrix} 3c_1 - c_3 + 3c_2 t - c_4 t \\ 4c_1 - c_3 + 4c_2 t - c_4 t \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_2 t = 3c_1 - c_3 + 3c_2 t - c_4 t \\ c_3 + c_4 + c_4 t = 4c_1 - c_3 + 4c_2 t - c_4 t \end{cases}$$

$$\Rightarrow \begin{cases} -2c_1 + c_2 + c_3 + (-2c_2 + c_4)t = 0 \\ -4c_1 + 2c_3 + +c_4 + (-4c_2 + 2c_4)t = 0 \end{cases}$$

This will be true if

$$\begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -4c_1 + 2c_3 + +c_4 = 0 \\ -4c_2 + 2c_4 = 0 \end{cases}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \\ -2c_2 + +c_4 = 0 \\ -2c_2 + c_4 = 0 \end{cases} \qquad \begin{aligned} L_3' &= L_3 - 2L_1 \\ L_4' &= \tfrac{1}{2} L_4 \end{aligned}$$

$$or, \begin{cases} -2c_1 + c_2 + c_3 = 0 \\ -2c_2 + c_4 = 0 \end{cases}$$

Putting $c_4 = 2$ and $c_3 = 1$ we get $c_2 = 1$ and $c_1 = 1$

$$\therefore \bar{x}_2(t) = \varphi_2(t) = \begin{pmatrix} 1 + t \\ 1 + 2t \end{pmatrix} e^t$$

$$W(\varphi_1, \varphi_2)(t) = \begin{vmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{vmatrix}$$

$$= (1+2t)e^{2t} - (2+2t)e^{2t}$$

$$= -e^{2t} \neq 0, \; for\ all\ t\epsilon\ R$$

The solutions $\varphi_1(t)$, $\varphi_2(t)$ are linearly independent.

The fundamental matrix for (5) is,

$$\varphi(\mathrm{t}) = \begin{pmatrix} e^t & (1+t)e^t \\ 2e^t & (1+2t)e^t \end{pmatrix}$$

Here $\varphi(\mathrm{t}_0) = \varphi(0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \neq I$

$$\therefore \varphi^{-1}(\mathrm{t}) = \frac{1}{-e^{2t}} \begin{pmatrix} (1+2t)e^t & -(1+t)e^t \\ -2e^t & e^t \end{pmatrix}$$

$$= -e^t \begin{pmatrix} 1+2t & -1-t \\ -2 & 1 \end{pmatrix}$$

$$\varphi^{-1}(\mathrm{t}_0) = \varphi^{-1}(0) = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

and $\quad \varphi^{-1}(\mathrm{u}) = e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix}$

Now $\varphi(\mathrm{t})\varphi^{-1}(\mathrm{t}_0)\bar{x}_0 = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 1-1-t \\ 2-1-2t \end{pmatrix}$$

$$= e^t \begin{pmatrix} -t \\ 1-2t \end{pmatrix} \cdots (9)$$

and $\varphi(t) \int_{t_0}^{t} \varphi^{-1}(u) F(u) du = e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u & 1+u \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} du$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t e^{-u} \begin{pmatrix} -1-2u+u+u^2 \\ 2-u \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \int_0^t \begin{pmatrix} u^2e^{-u} - ue^{-u} - e^{-u} \\ -ue^{-u} + 2e^{-u} \end{pmatrix} du$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - 2ue^{-u} - 2e^{-u} + ue^{-u} + e^{-u} + e^{-u} \\ ue^{-u} + e^{-u} - 2e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \left(\begin{bmatrix} -u^2e^{-u} - ue^{-u} \\ ue^{-u} - e^{-u} \end{bmatrix}_0^t \right)$$

$$= e^t \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2e^{-t} - te^{-t} \\ te^{-t} - e^{-t} + 1 \end{pmatrix}$$

$$= e^t e^{-t} \begin{pmatrix} 1 & 1+t \\ 2 & 1+2t \end{pmatrix} \begin{pmatrix} -t^2 - t \\ t-1+e^t \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 - t + t - 1 + e^t + t^2 - t + te^t \\ -2t^2 - 2t + t - 1 + e^t + 2t^2 - 2t + 2te^t \end{pmatrix}$$

$$= \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix} \qquad \cdots (10)$$

By putting the values of (9) and (10) in (6) we get

$$\bar{x}(t) = e^t \begin{pmatrix} -t \\ 1 - 2t \end{pmatrix} + \begin{pmatrix} te^t - t + e^t - 1 \\ 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -te^t + te^t - t + e^t - 1 \\ e^t - 2te^t + 2te^t - 3t + e^t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t - t - 1 \\ 2e^t - 3t - 1 \end{pmatrix}$$

Question-09: Solve $\bar{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \bar{x}(\pi) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

<u>Solution</u>: Given that: $\frac{d\bar{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad \cdots (1)$

and $\bar{x}(\pi) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \cdots (2)$

The general form of non-homogeneous system is

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} + F(t) \qquad \cdots (3)$$

with condition $\bar{x}(t_0) = \bar{x}_0$. $\qquad \cdots (4)$

If φ is a fundamental matrix of

$$\frac{d\bar{x}}{dt} = A(t)\bar{x} \qquad \cdots (5)$$

then the solution of (3) can be expressed as

$$\bar{x}(t) = \varphi(\mathrm{t})\varphi^{-1}(t_0)\bar{x}_0 + \varphi(t)\int_{t_0}^{t} \varphi^{-1}(u)F(u)du \qquad \cdots (6)$$

Comparing (1) with (3) and (2) with (4) we have

$$\left.\begin{array}{c} A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ F(t) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow F(u) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ \bar{x}(t_0) = \bar{x}(\pi) \Rightarrow t_0 = \pi \ and \ \bar{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array}\right\} \qquad \cdots (7)$$

For fundamental matrix of (5), let $\bar{x}(t) = e^{\lambda t}v$, where v is an eigen vector.

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\Rightarrow \lambda^2 = i^2$$

$$\therefore \lambda = \pm i$$

Now $(A - \lambda I)v = 0$ gives

$$\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0\,, \quad \text{where } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \cdots (8)$$

For $\lambda = i$ we get from (8)

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{matrix} -iv_1 + v_2 = 0 \\ -v_1 - iv_2 = 0 \end{matrix}\right\}$$

$$\Rightarrow \left.\begin{matrix} -iv_1 + v_2 = 0 \\ 0 = 0 \end{matrix}\right\} \quad L_2' = iL_2 - L_1$$

$$\Rightarrow -iv_1 + v_2 = 0 \qquad \cdots (9)$$

Putting $v_2 = 1$ in (9) we get $v_1 = -i$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\therefore \bar{x}_1(t) = \varphi_1(t) = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^{it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -ie^{it} \\ e^{it} \end{pmatrix}$$

For $\lambda = -i$ we get from (8)

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left.\begin{matrix} iv_1 + v_2 = 0 \\ -v_1 + iv_2 = 0 \end{matrix}\right\}$$

$$\Rightarrow \left.\begin{matrix} iv_1 + v_2 = 0 \\ 0 = 0 \end{matrix}\right\} \quad L_2' = iL_2 + L_1$$

$$\Rightarrow iv_1 + v_2 = 0 \qquad \cdots (10)$$

Putting $v_2 = 1$ in (9) we get $v_1 = i$.

$$\therefore v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\therefore \bar{x}_2(t) = \varphi_2(t) = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^{-it}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} ie^{-it} \\ e^{-it} \end{pmatrix}$$

The wronskian is

$$W(\varphi_1, \varphi_2)(t) = \begin{vmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{vmatrix}$$

$$= -ie^0 - ie^0$$

$$= -i - i$$

$$= -2i \neq 0$$

The solutions $\varphi_1(t)$, $\varphi_2(t)$ are linearly independent.

The fundamental matrix for (5) is,

$$\varphi(\mathrm{t}) = \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix}$$

Here $\varphi(\mathrm{t}_0) = \varphi(\pi) = \begin{pmatrix} -ie^{i\pi} & ie^{-i\pi} \\ e^{i\pi} & e^{-i\pi} \end{pmatrix} = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} \neq I$

$$\therefore \varphi^{-1}(\mathrm{t}) = \frac{Adjoint\ of\ \varphi(\mathrm{t})}{determinent\ of\ \varphi(\mathrm{t})}$$

$$= -\frac{1}{2i}\begin{pmatrix} e^{-it} & -ie^{-it} \\ -e^{it} & -ie^{it} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} ie^{-it} & e^{-it} \\ -ie^{it} & e^{it} \end{pmatrix}$$

$$\varphi^{-1}(\mathrm{t}_0) = \varphi^{-1}(\pi) = \frac{1}{2}\begin{pmatrix} ie^{-i\pi} & e^{-i\pi} \\ -ie^{i\pi} & e^{i\pi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & -1 \\ i & -1 \end{pmatrix}$$

and $\varphi^{-1}(\mathrm{u}) = \frac{1}{2}\begin{pmatrix} ie^{-iu} & e^{-iu} \\ -ie^{iu} & e^{iu} \end{pmatrix}$

Now $\varphi(\mathrm{t})\varphi^{-1}(\mathrm{t}_0)\bar{x}_0 = \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix}.\frac{1}{2}\begin{pmatrix} -i & -1 \\ i & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix}\begin{pmatrix} -i & -2 \\ i & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{it} + 2ie^{it} - e^{-it} - 2ie^{-it} \\ -ie^{it} - 2e^{it} + ie^{-it} - 2e^{-it} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2i\left(e^{it} - e^{-it}\right) - \left(e^{it} + e^{-it}\right) \\ -i\left(e^{it} - e^{-it}\right) - 2\left(e^{it} + e^{-it}\right) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2i.2isint - 2cost \\ -i.2isint - 2.2cost \end{pmatrix}$$

$$= \begin{pmatrix} -2sint - cost \\ sint - 2cost \end{pmatrix} \qquad \cdots (11)$$

and $\varphi(t)\int_{t_0}^{t}\varphi^{-1}(u)F(u)du = \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix}\int_{\pi}^{t}\frac{1}{2}\begin{pmatrix} ie^{-iu} & e^{-iu} \\ -ie^{iu} & e^{iu} \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix}du$

$$= \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \int_{\pi}^{t} \frac{1}{2} \begin{pmatrix} 3e^{-iu} \\ 3e^{iu} \end{pmatrix} du$$

$$= \frac{3}{2} \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \int_{\pi}^{t} \begin{pmatrix} e^{-iu} \\ e^{iu} \end{pmatrix} du$$

$$= \frac{3}{2} \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \left(\begin{bmatrix} \frac{e^{-iu}}{-i} \\ \frac{e^{iu}}{i} \end{bmatrix}_{\pi}^{t} \right)$$

$$= \frac{3}{2i} \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \begin{pmatrix} -e^{-it} + e^{-i\pi} \\ e^{it} - e^{i\pi} \end{pmatrix}$$

$$= -\frac{3i}{2} \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \begin{pmatrix} -e^{-it} - 1 \\ e^{it} + 1 \end{pmatrix}$$

$$= -\frac{3i}{2} \begin{pmatrix} i + ie^{it} + i + ie^{-it} \\ -1 - e^{it} + 1 + e^{-it} \end{pmatrix}$$

$$= -\frac{3i}{2} \begin{pmatrix} 2i + i\left(e^{it} + e^{-it}\right) \\ -\left(e^{it} - e^{-it}\right) \end{pmatrix}$$

$$= -\frac{3i}{2} \begin{pmatrix} 2i + 2icost \\ -2isint \end{pmatrix}$$

$$= 3 \begin{pmatrix} 1 + cost \\ -sint \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 3cost \\ -3sint \end{pmatrix} \qquad \cdots (12)$$

By putting the values of (9) and (10) in (6) we get

$$\bar{x}(t) = \begin{pmatrix} -2sint - cost \\ sint - 2cost \end{pmatrix} + \begin{pmatrix} 3 + 3cost \\ -3sint \end{pmatrix}$$

$$= \begin{pmatrix} -2sint - cost + 3 + 3cost \\ sint - 2cost - 3sint \end{pmatrix}$$

$$= \begin{pmatrix} 2cost - 2sint + 3 \\ -2cost - 2sint \end{pmatrix}$$ **(Ans)**