Stability Theory

Stability: The solution $x(t) = x(t; t_0, x_0)$ of the differential equation $x'(t) = f(t, x)$ is called stable if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any solution $\bar{x}(t) = x(t; t_0, \bar{x}_0)$ of $x'(t) = f(t, x)$.

$$\|\bar{x}_0 - x_0\| \le \delta \Longrightarrow \|\bar{x}(t) - x(t)\| < \varepsilon \quad \text{for all } t \ge t_0.$$

Unstable: The solution $x(t) = x(t; t_0, x_0)$ of the differential equation $x'(t) = f(t, x)$ is called unstable if it is not stable.

Asymptotic stability: The solution $x(t) = x(t; t_0, x_0)$ of the differential equation $x'(t) = f(t, x)$ is called asymptotically stable if it is stable and if there exists a $\delta_0 > 0$ such that for any solution $\bar{x}(t) = x(t; t_0, \bar{x}_0)$ of $x'(t) = f(t, x)$

$$\|\bar{x}_0 - x_0\| \le \delta_0 \Longrightarrow \|\bar{x}(t) - x(t)\| \longrightarrow 0 \;\; as\; t \longrightarrow \infty\,.$$

Uniform stability: The solution $x(t) = x(t; t_0, x_0)$ of the differential equation $x'(t) = f(t, x)$ is called uniformly stable if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for any solution $\bar{x}(t) = x(t; t_0, \bar{x}_0)$ of $x'(t) = f(t, x)$, the inequalities

$$t \ge t_0 \text{ and } \|\bar{x}(t_1) - x(t_1)\| \le \delta \text{ imply } \|\bar{x}(t) - x(t)\| < \varepsilon, \text{for all } t \ge t_1\,.$$

Question-01: Prove that all solutions of $\bar{x}'(t) = A(t)\bar{x}$ where $A(t)$ is an $n \times n$ continuous matrix in $[0, \infty]$ and $\bar{x}$ is an n vector, are stable if and only if they are bounded.

<u>Answer</u>: First we suppose that all solutions of

$$\bar{x}'(t) = A(t)\bar{x} \qquad \cdots (1)$$

are bounded. Let $\phi(t)$ be a solution of (1) passing through the point (t_0, x_0). We shall show that it is stable, that is, for any solution $\psi(t; t_0, \xi)$ of (1), we have to show that for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequality

$$\|\xi - x_0\| \le \delta \Longrightarrow \|\psi(t) - \phi(t)\| \le \varepsilon, \quad for\ all\ \ t \ge t_0 \text{ holds.}$$

Let $\Phi(t)$ be a fundamental matrix of (1). Then we have

$$\phi(t) = \Phi(t)x_0$$

and $\psi(t) = \Phi(t)\xi$

since all solutions of (1) are bounded, so there exists a constant $M > 0$ such that

$$\|\Phi(t)\| \leq M,\ for\ all\ t \geq t_0 \qquad \cdots (2)$$

Now $\|\psi(t) - \phi(t)\| = \|\Phi(t)\xi - \Phi(t)x_0\|$

$$\Rightarrow \|\psi(t) - \phi(t)\| = \|\Phi(t)\xi\|\|\xi - x_0\|$$

$$\Rightarrow \|\psi(t) - \phi(t)\| \leq M\|\xi - x_0\|$$

$$\Rightarrow \|\psi(t) - \phi(t)\| \leq M\delta \qquad where\ \|\xi - x_0\| \leq \delta$$

If we choose $M\delta < \varepsilon \Rightarrow \delta < \frac{\varepsilon}{M}$, then

$$\|\psi(t) - \phi(t)\| < \varepsilon$$

Here M and δ are both positive, so $\varepsilon > 0$.

Thus, for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for any solution $\psi(t; t_0, \xi)$ of (1) we have

$$\|\xi - x_0\| \leq \delta \Rightarrow \|\psi(t) - \phi(t)\| < \varepsilon$$

Therefore all solutions of (1) are stable.

Conversely, suppose that all solutions of (1) are stable. We have to show that they are bounded.

Since all solutions of (1) are stable, so its zero solution $\phi(t; t_0, x_0) = 0$ is also stable, that is for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|\xi - 0\| \leq \delta \Rightarrow \|\psi(t; t_0, \xi) - 0\| < \varepsilon,\ \ \ni\ t \geq t_0$$

that is, $\|\xi\| \leq \delta \Rightarrow \|\psi(t; t_0, \xi)\| < \varepsilon,\ \ for\ all\ \ t \geq t_0 \qquad \cdots (3)$

We know that $\psi(t; t_0, \xi) = \Phi(t)\xi$ and hence from (3) we have

$$\|\psi(t; t_0, \xi)\| = \|\Phi(t)\xi\| < \varepsilon \qquad \cdots (4)$$

Let ξ be a vector with $\frac{\delta}{2}$ in the ith place and zero elsewhere.

Then from (4) we have

$$\left\|\Phi(t)\frac{\delta}{2}\right\| < \varepsilon$$

$\Rightarrow \left\|\Phi(t)\frac{\delta}{2}\right\| < \varepsilon$, where $\varphi_i(t)$ is the ith column of $\Phi(t)$.

$$\Rightarrow \|\phi_i(t)\| < \frac{2\varepsilon}{\delta}$$

Therefore, $\|\phi_i(t)\| < \frac{2\varepsilon}{\delta}$, for all $t \geq t_0;\ i = 1,2,\cdots,n$

$$\Rightarrow \|\Phi(t)\| < \frac{2n\varepsilon}{\delta} = M, \qquad say$$

Thus, $\|\psi(t; t_0, \xi)\| = \|\Phi(t)\xi\| = \|\Phi(t)\|\|\xi\| \leq M\|\xi\|$

Hence all solutions of (1) are bounded. (**Proved**)

Question-02: If all the characteristic roots of A have negative real parts, then every solution of $\bar{x}'(t) = A\bar{x}(t)$ where $A = (a_{ij})$ is a constant matrix, is asymptotically stable.

Solution: Given that $\bar{x}'(t) = A\bar{x}(t)$ $\cdots(1)$

where $A = (a_{ij})$ is a constant matrix.

Since all the characteristic roots of A have negative real parts, so there exists positive constants α and M such that

$$\|\Phi(t)\| \leq Me^{-\alpha(t-t_0)}, \text{ for all } t \geq t_0 \qquad \cdots(2)$$

where $\Phi(t)$ is a fundamental matrix of (1) satisfying $\Phi(t_0) = I$. If $\phi(t; t_0, x_0)$ and $\psi(t; t_0, \xi)$ be two solutions of (1) with initial values x_0 and ξ at $t = t_0$, then we have

$$\phi(t) = \Phi(t)x_0$$

and $$\psi(t) = \Phi(t)\,\xi$$

First we show that the solutions of (1) are stable and next we show that they are asymptotically stable.

For stability we need to show that $\forall\, \varepsilon > 0\; \exists\; \delta = \delta(\varepsilon) > 0$ such that

$$\|\xi - x_0\| \leq \delta \Longrightarrow \|\psi(t) - \phi(t)\| < \varepsilon \text{ , for all } t \geq t_0 \text{ holds.}$$

Now $\|\psi(t; t_0, \xi) - \phi(t; t_0, x_0)\| = \|\Phi(t)\xi - \Phi(t)x_0\|$

$$\Longrightarrow \|\psi(t) - \phi(t)\| = \|\Phi(t)\|\|\xi - x_0\|$$

$$\Longrightarrow \|\psi(t) - \phi(t)\| \leq M\delta e^{-\alpha(t-t_0)} \qquad \cdots(3)$$

$$\Longrightarrow \|\psi(t) - \phi(t)\| < M\delta \qquad \because\; e^{-\alpha(t-t_0)}\; is\; decreasing\; for\; \alpha > 0\,.$$

If we select $M\delta = \varepsilon \Longrightarrow \delta = \frac{\varepsilon}{M}$, then

$$\|\psi(t) - \phi(t)\| < \varepsilon$$

Therefore all solutions of (1) are stable.

Also from (3) we have

$$\|\psi(t) - \phi(t)\| \leq M\delta e^{-\alpha(t-t_0)}$$

$$\Longrightarrow \|\psi(t; t_0, \xi) - \phi(t; t_0, x_0)\| \to 0\; as\; t \to \infty, since\; \alpha > 0.$$

Hence every solution of (1) is asymptotically stable. (**Proved**)

Question-03: Let $\Phi(t)$ be a fundamental matrix of the system $\bar{x}'(t) = A(t)\bar{x}(t)$ such that $\Phi(t_0) = I$. Then prove that the system is asymptotically stable iff $\|\Phi(t)\| \to 0$ as $t \to \infty$.

Solution: Given that $\bar{x}'(t) = A(t)\bar{x}(t)$ $\cdots(1)$

Let $\Phi(t)$ be a fundamental matrix of the system (1) such that $\Phi(t_0) = I$.

First suppose that all solutions of (1) are asymptotically stable.

We have to show that $\|\Phi(t)\| \to 0$ as $t \to \infty$.

Since all the solutions of (1) are asymptotically stable, its zero solution $\phi(t; t_0, x_0) \equiv 0$ is asymptotically stable.

That is, $\|\phi(t)\| \to 0$ as $t \to \infty$

$\Rightarrow \|\Phi(t)x_0\| \to 0$ as $t \to \infty$ $\quad \because \phi(t) = \Phi(t)x_0$

$\Rightarrow \|\Phi(t)\|\|x_0\| \to 0$ as $t \to \infty$

$\Rightarrow \|\Phi(t)\| \to 0$ as $t \to \infty$ $\quad$ since $\|x_0\| \neq 0$

Conversely, suppose that $\|\Phi(t)\| \to 0$ as $t \to \infty$

We have to show that all the solutions of (1) are asymptotically stable.

Let $\phi(t)$ be a solution of (1) with $\phi(t_0) = x_0$.

Then $\quad \phi(t) = \Phi(t)x_0$ $\cdots(2)$

Since $\Phi(t)$ is continuous, it follows that every solution of (1) is bounded and hence it is stable.

From (2) we have

$$\|\phi(t)\| = \|\Phi(t)x_0\|$$

$$\Rightarrow \|\phi(t)\| = \|\Phi(t)\|\|x_0\| \to 0 \text{ as } t \to \infty$$

Since $\|\Phi(t)\| \to 0$ as $t \to \infty$

$$\Rightarrow \|\phi(t)\| \to 0 \text{ as } t \to \infty.$$

Hence all solutions of (1) are asymptotically stable. (**Proved**)

Question-04: Show that zero solution of $\bar{x}'(t) = a(t)\bar{x}(t)$ is asymptotically stable iff $\int_0^t a(s)ds \to -\infty \; as \; t \to \infty$ and is uniformly stable iff $\int_0^t a(s)ds$ is bounded above for $t > t_1 \geq 0$.

Solution Given that $\bar{x}'(t) = a(t)\bar{x}(t)$ $\cdots(1)$

$$or, \frac{\bar{x}'(t)}{\bar{x}(t)} = a(t)$$

$$or, \int_0^t \frac{d\bar{x}(t)}{\bar{x}(t)} = \int_0^t a(s)\, ds$$

$$or, [ln\bar{x}(t)]_0^t = \int_0^t a(s)\, ds$$

$$or, ln\bar{x}(t) - ln\bar{x}(0) = \int_0^t a(s)\, ds$$

$$or, ln\frac{\bar{x}(t)}{\bar{x}(0)} = \int_0^t a(s)\, ds$$

$$or, \frac{\bar{x}(t)}{\bar{x}(0)} = e^{\int_0^t a(s)ds}$$

$$or, \bar{x}(t) = \bar{x}(0)e^{\int_0^t a(s)ds}$$

(i) First we suppose that the solution of (1) is asymptotically stable. We have to show that $\int_0^t a(s)ds \to -\infty \ as\ t \to \infty$.

Now from the definition of asymptotically stable we have, there exists $\delta_0 > 0$ such that

$$|\bar{x}(0) - 0| \le \delta_0 \Longrightarrow \left|\bar{x}(0)e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$or, |\bar{x}(0)| \le \delta_0 \Longrightarrow |\bar{x}(0)|\left|e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$\Longrightarrow \left|e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$\Longrightarrow e^{\int_0^t a(s)ds} \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$\Longrightarrow \int_0^t a(s)\, ds \longrightarrow -\infty\ \ as\ t \longrightarrow \infty$$

Conversely, suppose that $\int_0^t a(s)\, ds \longrightarrow -\infty\ \ as\ t \longrightarrow \infty$

We have to show that the solution of (1) is asymptotically stable.

That is, there exists $\delta_0 > 0$ such that

$$|\bar{x}(0)| \le \delta_0 \Longrightarrow \left|\bar{x}(0)e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$or, |\bar{x}(0)| \le \delta_0 \Longrightarrow |\bar{x}(0)|\left|e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty.$$

Since $\int_0^t a(s)\, ds \longrightarrow -\infty\ \ as\ t \longrightarrow \infty$

$$\therefore\ |\bar{x}(0)|\left|e^{\int_0^t a(s)ds}\right| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

which implies that the zero solution is asymptotically stable.

Hence solution of (1) is asymptotically stable if and only if

$$\int_0^t a(s)\,ds \longrightarrow -\infty\ \ as\ t \longrightarrow \infty.$$

(ii) We assume that the zero solution of (1) is uniformly stable.

We shall show that $\int_0^t a(s)ds$ is bounded above for $t > t_1 \geq 0$. From the definition of uniform stability we have, for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for $t_1 \geq t_0$ and $|\bar{x}(0)| \leq \delta$

$$\Longrightarrow \left|\bar{x}(0)e^{\int_{t_1}^t a(s)ds}\right| < \varepsilon \ \text{ for all } t \geq t_1$$

$$\Longrightarrow |\bar{x}(0)|\left|e^{\int_{t_1}^t a(s)ds}\right| < \varepsilon \ \text{ for all } t \geq t_1$$

$$\Longrightarrow \left|e^{\int_{t_1}^t a(s)ds}\right| < \frac{\varepsilon}{\delta} \qquad \because |\bar{x}(0)| \leq \delta$$

$$\Longrightarrow e^{\int_{t_1}^t a(s)ds} < e^M, \qquad \text{where } \frac{\varepsilon}{\delta} = e^M, for\ all\ M > 0.$$

$$\Longrightarrow \int_{t_1}^t a(s)\,ds < M, \qquad \text{for all } t \geq t_1$$

$$\Longrightarrow \int_0^t a(s)\,ds < M, \qquad \text{where } t_1 = 0$$

$$\Longrightarrow \int_0^t a(s)\,ds \text{ is bounded above.}$$

Conversely, suppose that $\int_0^t a(s)ds$ is bounded above. We have to show that the solution of (1) is uniformly stable. For this we prove that, for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for $t_1 \geq t_0$ we have

$$|\bar{x}(0)| \leq\ \delta \Longrightarrow \left|\bar{x}(0)e^{\int_{t_1}^t a(s)ds}\right| < \varepsilon.$$

Since $\int_0^t a(s)ds$ is bounded above, so there exists $M > 0$ such that

$$\left|\int_0^t a(s)\,ds\right| \leq M, \qquad \forall\, t \geq 0$$

$$\Longrightarrow \left|\int_{t_1}^t a(s)\,ds\right| \leq M, \qquad \forall\ \ t > t_1 \geq 0$$

$$\Longrightarrow \left|e^{\int_{t_1}^t a(s)ds}\right| \leq e^M = M_1, \text{ say}$$

Now, $\left|\bar{x}(0)e^{\int_{t_1}^t a(s)ds}\right| = |\bar{x}(0)|\left|e^{\int_{t_1}^t a(s)ds}\right|$

$$\Rightarrow \left|\bar{x}(0)e^{\int_{t_1}^{t} a(s)ds}\right| \leq \delta M_1$$

$$\Rightarrow \left|\bar{x}(0)e^{\int_{t_1}^{t} a(s)ds}\right| \leq \frac{\varepsilon}{2}, \text{ where } \frac{\varepsilon}{2} = \delta M_1 \Rightarrow \delta = \frac{\varepsilon}{2M_1}$$

Thus, for $\varepsilon > 0$ we have a $\delta = \delta(\varepsilon) > 0$ such that for $t > t_1 \geq 0$,

$$|\bar{x}(0)| \leq \delta \Rightarrow \left|\bar{x}(0)e^{\int_{t_1}^{t} a(s)ds}\right| < \varepsilon$$

Hence the solution of (1) is uniformly stable.

Problem

Problem-01: Determine if system $\bar{x}' = A\bar{x}$, where $A = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is stable, asymptotically stable.

<u>Solution</u>: Given that $\bar{x}' = A\bar{x}$ $\cdots (1)$

where $A = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Let λ be the eigen value. Then the characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} -1 & 0 & 0 \\ -3 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ -3 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & 0 & 0 \\ -3 & -2-\lambda & 0 \\ 2 & 1 & -3-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} -1-\lambda & 0 & 0 \\ -3 & -2-\lambda & 0 \\ 2 & 1 & -3-\lambda \end{vmatrix}$$

$$=(-1-\lambda)\{(-2-\lambda)(-3-\lambda) - 0\} - 0 + 0$$

$$= (-1-\lambda)(-2-\lambda)(-3-\lambda)$$

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$or, (-1-\lambda)(-2-\lambda)(-3-\lambda)=0$$

$$or, (\lambda+1)(\lambda+2)(\lambda+3)=0$$

$$\therefore\ (\lambda+1)=0\ ,\ or\ (\lambda+2)=0,\ \ or\ (\lambda+3)=0$$

$$\therefore\ \lambda=-1,-2,-3$$

Since all values of λ are negative, so the solutions are asymptotically stable and as such they are also stable. Thus the solutions of the given system are not unstable.

Problem-02: Test the asymptotically stability of the system

$$\bar{x}'(t)=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}\bar{x}(\text{t})$$

Solution: Given that $\bar{x}'=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}\bar{x}$ $\cdots(1)$

where $A=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}$

Let λ be the eigen value. Then the characteristic matrix of A is

$$A-\lambda I=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & 1-\lambda\end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta=|A-\lambda I|$$

$$=\begin{vmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & 1-\lambda\end{vmatrix}$$

$$=(1-\lambda)\{(2-\lambda)(1-\lambda)+1\}+1\{3(1-\lambda)+2\}+4\{3-2(2-\lambda)\}$$

$$=(1-\lambda)(\lambda^2-3\lambda+3)+(5-3\lambda)+4(-1+2\lambda)$$

$$=\lambda^2-3\lambda+3-\lambda^3+3\lambda^2-3\lambda+5-3\lambda-4+8\lambda$$

$$=-\lambda^3+4\lambda^2-\lambda+4$$

The characteristic equation of A is

$$|A-\lambda I|=0$$

$$or, -\lambda^3+4\lambda^2-\lambda+4=0$$

$$or, \lambda^3-4\lambda^2+\lambda-4=0$$

$$or, \lambda^2\,(\lambda-4)+1(\lambda-4)=0$$

$$or, (\lambda-4)(\lambda^2+1)=0$$

$$\therefore\ \lambda-4=0\,,\ or\ \lambda^2+1=0$$

$$\therefore\ \lambda=4\,,\ or\ \lambda=\pm i$$

$$\therefore\ \lambda=4,\pm i$$

Since all real parts of λ are not negative, so the solutions are not asymptotically stable.

Problem-03: Prove that the zero solution of

(i) $\bar{x}'' + \bar{x} = 0$ is uniformly stable but not asymptotically stable and

(ii) $\bar{x}'(t) = (sinlnt + coslnt - 1.25)\bar{x}(t)$ is asymptotically stable, but not uniformly stable.

Solution: (i) Given that $\bar{x}'' + \bar{x} = 0$

$$or,\ \bar{x}'' = -\bar{x} \qquad \cdots(1)$$

Let $\bar{x}_1 = \bar{x}$ and $\bar{x}_2 = \bar{x}'$

Then $\quad \bar{x}_1{}' = \bar{x}' = \bar{x}_2$

$$\Rightarrow \bar{x}_1{}' = 0.\bar{x}_1 + \bar{x}_2$$

and $\quad \bar{x}_2{}' = \bar{x}'' = -\bar{x}$

$$\Rightarrow \bar{x}_2{}' = \bar{x}'' = -\bar{x}_1 + 0.\bar{x}_2$$

$$\therefore \begin{pmatrix}\bar{x}_1{}' \\ \bar{x}_2{}'\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

The characteristic equation is

$$\left|\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix}-\lambda & 1\\ -1 & -\lambda\end{vmatrix} = 0$$

$$\Rightarrow \lambda^2+1=0$$

$$\Rightarrow \lambda^2=-1$$

$$\Rightarrow \lambda^2=i^2$$

$$\Longrightarrow \lambda = i\,,-i$$

Therefore the eigen functions (solutions) are $\phi_1(t) = cost$ and $\phi_2(t) = sint$.

The wronskian is

$$W(\phi_1\,,\ \phi_2) = \begin{vmatrix} cost & sint \\ -sint & cost \end{vmatrix}$$

$$= cos^2 t + sin^2 t$$

$$= 1 \neq 0$$

Therefore the solutions are linearly independent.

The fundamental matrix is

$$\psi(\mathrm{t}) = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix}$$

$$\therefore \psi^{-1}(\mathrm{t}) = \frac{Adjoint\ of\ \psi(\mathrm{t})}{determinant\ of\ \psi(\mathrm{t})}$$

$$= \frac{1}{1}\begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$$

$$= \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$$

$$\therefore \psi(\mathrm{t})\psi^{-1}(\mathrm{s}) = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix}\begin{pmatrix} cos\,s & -sin\,s \\ sin\,s & cos\,s \end{pmatrix}$$

$$= \begin{pmatrix} cost\,cos\,s + sint\,sin\,s & sint\,cos\,s - cost\,sin\,s \\ -sint\,cos\,s + cost\,\sin s & \sin t \sin s + \cos t \cos s \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$$

$$\therefore \|\psi(\mathrm{t})\psi^{-1}(\mathrm{s})\| = \sqrt{cos^2(t-s) + sin^2(t-s) + cos^2(t-s) + sin^2(t-s)}$$

$$= \sqrt{1+1} = \sqrt{2} < M \qquad ,M > \sqrt{2}, s \leq t < \infty$$

Therefore the zero solutions are uniformly stable.

Also $\|\psi(\mathrm{t})\| = \sqrt{cos^2 t + sin^2 t + cos^2 t + sin^2 t} = \sqrt{1+1} = \sqrt{2} \nrightarrow 0, \quad as\ t \to \infty$.

Hence the zero solutions are not asymptotically stable.

(ii) Given that $\bar{x}'(t) = (sinlnt + coslnt - 1.25)\bar{x}(t)$ $\cdots(1)$

Let (t) be a solution of (1). Then we have

$$\phi'(t) = (sinlnt + coslnt - 1.25)\phi(t)$$

$$or,\ \frac{\phi'(t)}{\phi(t)} = sinlnt + coslnt - 1.25$$

$$or,\ \int \frac{d\phi(t)}{\phi(t)} = \int (sinlnt + coslnt - 1.25)dt$$

$$or, ln\phi(t) = tsinlnt - 1.25t + lnc$$

$$or, \phi(t) = c\, e^{(tsinlnt - 1.25t)} \quad \cdots (2)$$

At $t = 0$ we have $\phi(0) = c\, e^0 = c\,.\,1 = c$

Putting this value in (2) we get the general solution of (1) is

$$\phi(t) = \phi(0)\, e^{(tsinlnt - 1.25t)} \quad \cdots (3)$$

Also the zero solution of (1) is $\phi(t) \equiv 0$.

To show that the zero solution $\phi(t) = 0$ is asymptotically stable, we first show that it is stable and then show that it is asymptotically stable.

To show stable, we have to show that

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ such that } |\phi(0)| \le \delta \Longrightarrow |\phi(t)| < \varepsilon$$

Now $|\phi(t)| = \left|\phi(0)\, e^{(tsinlnt - 1.25t)}\right|$

$\Longrightarrow |\phi(t)| = |\phi(0)|\left| e^{(tsinlnt - 1.25t)}\right|$

$\Longrightarrow |\phi(t)| \le |\phi(0)|\,,\ \forall\, \mathrm{t} \ge 0$

$\Longrightarrow |\phi(t)| \le \delta$

Taking $\delta < \varepsilon$ we get, $|\phi(t)| \le \varepsilon$

Therefore any solution of (1) is stable.

For asymptotically stable we have

$$|\phi(t)| = \left|\phi(0)\, e^{(tsinlnt - 1.25t)}\right|$$

$$or, |\phi(t)| = |\phi(0)|\left| e^{(tsinlnt - 1.25t)}\right|$$

$$or, |\phi(t)| = |\phi(0)| e^{-(1.25t - tsinlnt)}$$

$$or, |\phi(t)| = |\phi(0)| e^{-(1.25t - tsinlnt)} \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

$$or, |\phi(t)| \longrightarrow 0\ \ as\ t \longrightarrow \infty$$

Therefore any solution of (1) is asymptotically stable.

<u>2nd part</u>: In this case we have to show that the zero solution of (1) is not uniformly stable.

Let $= e^{2n\pi + \frac{\pi}{3}}$, where $n = 0$ or positive integer.

$$\Longrightarrow lnt = 2n\pi + \frac{\pi}{3}$$

$$\Longrightarrow sinlnt = sin\left(2n\pi + \frac{\pi}{3}\right) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\ ; n = 0\,, 1\,, 2\,, \cdots$$

Also let $t_1 = e^{2n\pi + \frac{\pi}{6}}$, where $n = 0$ or positive integer.

$$\Longrightarrow lnt_1 = 2n\pi + \frac{\pi}{6}$$

$$\Longrightarrow sinlnt_1 = sin\left(2n\pi + \frac{\pi}{6}\right) = sin\frac{\pi}{6} = \frac{1}{2} \; ; n = 0\,,1\,,2\,,\cdots$$

From (3) we have

$$\phi(t) = \phi(0)\; e^{(tsinlnt - 1.25t)}$$

$$= \phi(0)\; e^{t(sinlnt - 1.25)}$$

$$= \phi(0)\; e^{e^{2n\pi+\frac{\pi}{3}}\left(\frac{\sqrt{3}}{2} - 1.25\right)}$$

$$= \phi(0)\; e^{e^{2n\pi+\frac{\pi}{3}}(0.86 - 1.25)}$$

$$= \phi(0)\; e^{-0.39\, e^{2n\pi+\frac{\pi}{3}}}$$

and $\quad \phi(t_1) = \phi(0)\; e^{(t_1 sinlnt_1 - 1.25t_1)}$

$$= \phi(0)\; e^{t_1(sinlnt_1 - 1.25)}$$

$$= \phi(0)\; e^{e^{2n\pi+\frac{\pi}{6}}\left(\frac{1}{2} - 1.25\right)}$$

$$= \phi(0)\; e^{e^{2n\pi+\frac{\pi}{6}}(0.5 - 1.25)}$$

$$= \phi(0)\; e^{-0.75\, e^{2n\pi+\frac{\pi}{6}}}$$

Now $\frac{\phi(t)}{\phi(t_1)} = \frac{\phi(0)\; e^{-0.39\, e^{2n\pi+\frac{\pi}{3}}}}{\phi(0)\; e^{-0.75\, e^{2n\pi+\frac{\pi}{6}}}}$

$$= e^{-0.39\, e^{2n\pi+\frac{\pi}{3}} \; + \; 0.75\, e^{2n\pi+\frac{\pi}{6}}}$$

$$= e^{\left(0.75 - 0.39 e^{\frac{\pi}{6}}\right) e^{2n\pi+\frac{\pi}{6}}}$$

Since $0.75 > 0.39e^{\frac{\pi}{6}}$, so we have

$$\frac{\phi(t)}{\phi(t_1)} \longrightarrow 0 \;\; as\; n \longrightarrow \infty$$

Therefore the solution is not uniformly stable.

Problem-04: Determine the stability, the asymptotic stability or the instability of the system

$$\bar{X}' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix} \bar{X}$$

Solution: Given that $\bar{X}' = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix} \bar{X}$

$$or,\ \bar{X}' = A\bar{X} \qquad \cdots (1)$$

Where $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix}$

The characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & 1 & -1 \\ -3 & 3-\lambda & -5 & 4 \\ 8 & -4 & 4-\lambda & -4 \\ 15 & -10 & 11 & -11-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & -1 & 1 & -1 \\ -3 & 3-\lambda & -5 & 4 \\ 8 & -4 & 4-\lambda & -4 \\ 15 & -10 & 11 & -11-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & -1 & 0 & 0 \\ -3 & 3-\lambda & -2-\lambda & -1 \\ 8 & -4 & -\lambda & -\lambda \\ 15 & -10 & 1 & -\lambda \end{vmatrix} \qquad \begin{matrix} c_3' = c_2 + c_3 \\ c_4' = c_3 + c_4 \end{matrix}$$

$$= \begin{vmatrix} 1-\lambda & -1 & 0 & 0 \\ -3 & 3-\lambda & -2-\lambda & 1+\lambda \\ 8 & -4 & -\lambda & 0 \\ 15 & -10 & 1 & -1-\lambda \end{vmatrix} \qquad c_4' = c_4 - c_3$$

$$= (1+\lambda) \begin{vmatrix} 1-\lambda & -1 & 0 & 0 \\ -3 & 3-\lambda & -2-\lambda & 1 \\ 8 & -4 & -\lambda & 0 \\ 15 & -10 & 1 & -1 \end{vmatrix}$$

$$= (1+\lambda)\left[(1-\lambda)\begin{vmatrix} 3-\lambda & -2-\lambda & 1 \\ -4 & -\lambda & 0 \\ -10 & 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -3 & -2-\lambda & 1 \\ 8 & -\lambda & 0 \\ 15 & 1 & -1 \end{vmatrix}\right]$$

$$= (1+\lambda)[(1-\lambda)(-\lambda^2-3\lambda+4)+4\lambda-8]$$

$$= (1+\lambda)(-\lambda^2-3\lambda+4+\lambda^3+3\lambda^2-4\lambda+4\lambda-8)$$

$$= (1+\lambda)(\lambda^3+2\lambda^2-3\lambda-4)$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$or,\ (1+\lambda)(\lambda^3+2\lambda^2-3\lambda-4) = 0$$

$$or, (1+\lambda)(\lambda^3+\lambda^2+\lambda^2+\lambda-4\lambda-4) = 0$$

$$or, (1+\lambda)\{\lambda^2(\lambda+1)+\lambda(\lambda+1)-4(\lambda+1)\} = 0$$

$$or, (\lambda+1)(\lambda+1)(\lambda^2+\lambda-4) = 0$$

$$\therefore\ (\lambda+1) = 0\,,\ or\ (\lambda+1) = 0,\ \ or\ \lambda^2+\lambda-4 = 0$$

$$or\ \lambda = \frac{-1\pm\sqrt{1-4.1.(-4)}}{2.1}$$

$$= \frac{-1\pm\sqrt{17}}{2}$$

$$\therefore\ \lambda = -1\,,-1\,,\frac{-1-\sqrt{17}}{2}\,,\frac{-1+\sqrt{17}}{2}$$

$$\therefore \lambda_1 = -1 < 0,\qquad \lambda_2 = -1 < 0,\qquad \lambda_3 = \frac{-1-\sqrt{17}}{2} < 0,\qquad \lambda_4 = \frac{-1+\sqrt{17}}{2} > 0$$

Since all values of λ are not negative, so the solutions are not asymptotically stable and hence not stable. Thus the solutions of the given system are unstable.

Problem-05: Show that the zero solution of $\bar{x}'' + \bar{x} = 0$ is uniformly stable but not asymptotically stable.

Solution: Given that $\bar{x}'' + \bar{x} = 0$

$$or,\ \bar{x}'' = -\bar{x} \qquad \cdots (1)$$

Let $\bar{x}_1 = \bar{x}$ and $\bar{x}_2 = \bar{x}'$

Then $\quad \bar{x}_1{}' = \bar{x}' = \bar{x}_2$

$$\Rightarrow \bar{x}_1{}' = 0.\bar{x}_1 + \bar{x}_2$$

and $\quad \bar{x}_2{}' = \bar{x}'' = -\bar{x}$

$$\Rightarrow \bar{x}_2{}' = \bar{x}'' = -\bar{x}_1 + 0.\bar{x}_2$$

$$\therefore \begin{pmatrix} \bar{x}_1{}' \\ \bar{x}_2{}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The characteristic equation is

$$\left| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Longrightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^2 + 1 = 0$$

$$\Longrightarrow \lambda^2 = -1$$

$$\Longrightarrow \lambda^2 = i^2$$

$$\Longrightarrow \lambda = i\,, -i$$

Therefore the eigen functions (solutions) are $\phi_1(t) = cost$ and $\phi_2(t) = sint$.

The wronskian is

$$W(\phi_1\,,\ \phi_2) = \begin{vmatrix} cost & sint \\ -sint & cost \end{vmatrix}$$

$$= cos^2t + sin^2t$$

$$= 1 \neq 0$$

Therefore the solutions are linearly independent.

The fundamental matrix is

$$\psi(\mathrm{t}) = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix}$$

$$\therefore \psi^{-1}(\mathrm{t}) = \frac{Adjoint\ of\ \psi(\mathrm{t})}{determinant\ of\ \psi(\mathrm{t})}$$

$$= \frac{1}{1} \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$$

$$= \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$$

$$\therefore \psi(\mathrm{t})\psi^{-1}(\mathrm{s}) = \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix} \begin{pmatrix} cos\ s & -sin\ s \\ sin\ s & cos\ s \end{pmatrix}$$

$$= \begin{pmatrix} cost\ cos\ s + sint\ sin\ s & sint\ cos\ s - cost\ sin\ s \\ -sint\ cos\ s + cost\ \sin s & \sin t \sin s + \cos t \cos s \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$$

$$\therefore \|\psi(\mathrm{t})\psi^{-1}(\mathrm{s})\| = \sqrt{cos^2(t-s) + sin^2(t-s) + cos^2(t-s) + sin^2(t-s)}$$

$$= \sqrt{1+1} = \sqrt{2} < M \qquad , M > \sqrt{2}, s \leq t < \infty$$

Therefore the zero solutions are uniformly stable.

Also $\|\psi(t)\| = \sqrt{cos^2t + sin^2t + cos^2t + sin^2t} = \sqrt{1+1} = \sqrt{2} \nrightarrow 0, \quad as\ t \to \infty$.

Hence the zero solutions are not asymptotically stable. (**Showed**)

Problem-06: Determine whether each solution $\bar{x}(t)$ of the differential equation $\bar{x}' = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix}\bar{x}$ is stable, asymptotically stable or unstable.

<u>Solution</u>: Given that $\bar{x}' = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix}\bar{x}$

$$\Rightarrow \bar{x}' = A\bar{x} \qquad \cdots (1)$$

where $A = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Let λ be the eigen value. Then the characteristic matrix of A is

$$A - \lambda I = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{pmatrix}$$

The characteristic polynomial of A is,

$$\Delta = |A - \lambda I|$$

$$= \begin{vmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\{(-1-\lambda)(-1-\lambda) + 4\} - 0 + 0$$

$$= (-1-\lambda)(\lambda^2 + 2\lambda + 5)$$

$$= -\lambda^2 - 2\lambda - 5 - \lambda^3 - 2\lambda^2 - 5\lambda$$

$$= -\lambda^3 - 3\lambda^2 - 7\lambda - 5$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$or, -\lambda^3 - 3\lambda^2 - 7\lambda - 5 = 0$$

$$or, \lambda^3 + 3\lambda^2 + 7\lambda + 5 = 0$$

$$or, \lambda^3 + \lambda^2 + 2\lambda^2 + 2\lambda + 5\lambda + 5 = 0$$

$$or, \lambda^2(\lambda + 1) + 2\lambda(\lambda + 1) + 5(\lambda + 1) = 0$$

$$or, (\lambda + 1)(\lambda^2 + 2\lambda + 5) = 0$$

$$\therefore \ \lambda + 1 = 0\ , \ or\ \lambda^2 + 2\lambda + 5 = 0$$

$$or, \lambda = -1\ , \ or\ \lambda = \frac{-2 \pm \sqrt{2^2 - 4.1.5}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$$\therefore \ \lambda = -1, -1 \pm 2i$$

Since the real characteristic root and the real parts of complex characteristic roots of λ are negative, so the solutions are asymptotically stable and as such they are also stable. Thus the solutions of the given system are not unstable.