Streaming Motions

<u>Question-01</u>: State and prove circle theorem.

OR

State and prove Milne Thomson circle theorem.

<u>Statement</u>: Let $w = f(z)$ be the complex potential of a two dimensional irrotational motion of an incompressible inviscid fluid with no rigid boundaries. Further, let $f(z)$ have no singularities within the circle $|z| = a$. Then if a circular cylinder is inserted in the flow field, the new complex potential will be

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

where $\bar{f}$ is the complex conjugate of f.

<u>Proof:</u> Let C be the cross-section of the circular cylinder $|z| = a$, then on the circle $z\bar{z} = a^2$ we have

$$w = f(z) + \bar{f}(\bar{z})$$

which would be a real quantity and hence

$$\phi + i\psi = f(z) + \bar{f}(\bar{z})$$

i.e. $\phi + i\psi = real\ quantity \Rightarrow \psi = 0$ **.**

Thus the circle is a streamline.

Further, if the point z lies outside the circle then the point $\bar{z} = \dfrac{a^2}{z}$ will lie inside the circle and vice-versa for all the singularities of $f(z)$ and $\bar{f}(\bar{z})$ lie in the domain $|z| > a$ and $|z| < a$ respectively. Hence $\bar{f}\left(\dfrac{a^2}{z}\right)$ is the complex potential of the image of the system $f(z)$ in the circle $|z| = a$.

Hence $w = f(z) + \bar{f}(\bar{z})$ **(Proved)**

<u>Question-02:</u> State and prove Blasius Theorem.

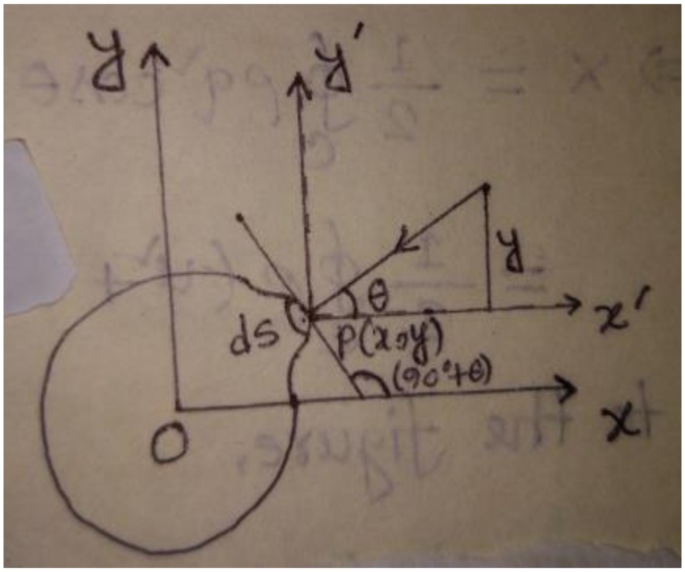
<u>Statement:</u> Let a cylinder of an arbitrary shape be placed in a liquid which is moving steadily and irrotationally. Let X, Y and M be the components along the axes and momentum about the origin of the pressure thrust on the cylinder.

If the external forces are absent, then

$$X - iY = \frac{1}{2}\rho i \oint_c \left(\frac{dw}{dz}\right)^2 dz$$

and $$M = real\ \ part\ of -\frac{1}{2}\rho\oint_c \left(\frac{dw}{dz}\right)^2 zdz$$

where the integrals are taken around the contour of the cylinder and w represents complex potential.



Proof: Let p be the pressure at $p(x, y)$ on the cross-section of the cylinder. Let θ be the angle which the normal at p make with the positive direction of the axis of x.

Then we have

$$X = -\oint_c p\cos\theta ds \qquad (1)$$

$$Y = -\oint_c p\sin\theta ds \qquad (2)$$

and $$M = -\oint_c xp\sin\theta ds + \oint_c yp\cos\theta ds \qquad (3)$$

since the motion is steady and external forces are absent, then we can write from Bernoullis theorem that

$$\frac{p}{\rho} + \frac{1}{2}q^2 = c \qquad \text{where } c \text{ is a constant}$$

From this we can get

$$p = c\rho - \frac{1}{2}\rho q^2$$

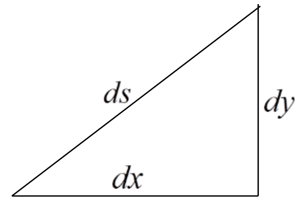
Hence from equation (1), we get

$$X = -\oint_c \left(c\rho - \frac{1}{2}\rho q^2\right)\cos\theta ds$$

$$= \frac{1}{2}\oint_c \rho q^2 \cos\theta ds$$

$$= \frac{1}{2}\oint_c \rho\left(u^2+v^2\right)\cos\theta ds$$

According to the figure,



$$\frac{dy}{ds} = \sin\left(\frac{\pi}{2}+\theta\right)$$

$$or,\ \cos\theta ds = dy$$

$$\therefore\ X = \frac{1}{2}\oint_c \rho\left(u^2+v^2\right)dy$$

Similarly, $Y = -\frac{1}{2}\oint_c \rho\left(u^2+v^2\right)dx$

and $X - iY = \frac{1}{2}\oint_c \rho\left(u^2+v^2\right)\left(dy + idx\right)$

$$= \frac{1}{2}\rho i\oint_c \left(u^2+v^2\right)\left(dx - idy\right) \qquad (4)$$

and $M = -\frac{1}{2}\oint_c \rho x\left(u^2+v^2\right)dx - \frac{1}{2}\oint_c \rho y\left(u^2+v^2\right)dy$

$$= -\frac{1}{2}\oint_c \rho x\left(u^2+v^2\right)\left(xdx + ydy\right) \qquad (5)$$

Clearly the curve c is a streamline.

Its equation is given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dx+idy}{u+iv} = \frac{dx-idy}{u-iv}$$

From the last two ratios

$$\frac{dx+idy}{u+iv} = \frac{dx-idy}{u-iv}$$

$$or,\ \frac{dx+idy}{dx-idy} = \frac{u+iv}{u-iv}$$

$$or,\ \frac{dx+idy}{dx-idy} = \frac{u^2+v^2}{\left(u-iv\right)^2}$$

$$\therefore\ \left(u^2+v^2\right)\left(dx-idy\right) = \left(u-iv\right)^2$$

From equation (4), we get

$$X - iY = \frac{1}{2}\oint_c \rho i\left(u-iv\right)^2\left(dx+idy\right)$$

$$=\frac{1}{2}\oint_c \rho i\left(\frac{dw}{dz}\right)^2 dz$$

From equation (5), we have

$$M=-\frac{1}{2}\oint_c \rho\left(u^2+v^2\right)(xdx+ydy)$$

$$=\operatorname{Re}al\ \ part\ of\ -\frac{1}{2}\oint_c \rho(u-iv)^2(x+iy)(dx+idy)$$

$$=\operatorname{Re}al\ \ part\ of\ -\frac{1}{2}\rho\oint_c\left(\frac{dw}{dz}\right)^2 zdz$$

(**Proved)**

Problem-01: A circular cylinder is placed in a uniform stream. Find the force acting on the cylinder. The complex potential for the undisturbed motion is given by $w=(u-iv)z$.

Solution: The complex potential for the undisturbed motion is given by

$$w=(u-iv)z$$

if we insert a circular cylinder $|z|=a$ in the flow field then by circle theorem the new complex potential will be

$$w=(u-iv)z+(u-iv)\frac{a^2}{z}$$

if the pressure thrust on the contour of the cylinder be represented by a force (X,Y) and a couple of momentum M then by Blasious theorem

$$X-iY=\frac{1}{2}\rho i\oint_c\left(\frac{dw}{dz}\right)^2 dz \qquad (1)$$

and $$M=Real\ \ part\ of\ -\frac{1}{2}\rho\oint_c\left(\frac{dw}{dz}\right)^2 zdz \qquad (2)$$

Now $$\frac{dw}{dz}=(u-iv)-(u+iv)\frac{a^2}{z^2}$$

From equation (1), we get

$$X-iY=\frac{1}{2}\rho i\oint_c\left[(u-iv)-(u+iv)\frac{a^2}{z^2}\right]^2 dz$$

$$=0$$

$$\therefore\ X=0,\ Y=0$$

From equation (2), we get

$$M = Real\ \ part\ \ of -\frac{1}{2}\rho\oint_c \left(\frac{dw}{dz}\right)^2 zdz$$

$$= Real\ \ part\ \ of -\frac{1}{2}\rho\oint_c \left[(u-iv)-(u+iv)\frac{a^2}{z^2}\right]^2 zdz$$

$$= Real\ \ part\ \ of -\frac{1}{2}\rho\oint_c \left[(u-iv)^2 - 2(u^2+v^2)\frac{a^2}{z^2}+(u+iv)^2\frac{a^4}{z^4}\right]zdz$$

$$= Real\ \ part\ \ of -\frac{1}{2}\rho\times 2\pi i\left[-2(u^2+v^2)a^2\right]$$

[By Cauchy Residue theorem]

$$=0.$$

Thus $X=0,\ Y=0,\ M=0$.

Problem-02: If a two dimensional motion of a liquid has complex potential $w=U\left(z+\frac{a^2}{z}\right)+ik\ln\left(\frac{z}{a}\right)$ where a, U and k are real and positive, then show that

1) the velocity at infinity is U in the negative sense of the real axis.
2) the circle $|z|=a$ is a streamline.
3) there are two stagnation points.
4) the circulation around the circle is $2\pi k$.
5) find (X,Y).

Solution: Given that

$$w=U\left(z+\frac{a^2}{z}\right)+ik\ln\left(\frac{z}{a}\right) \qquad (1)$$

$$\therefore\ \frac{dw}{dz}=U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z} \qquad (2)$$

if $z\to\infty$ then from (2), we have

$$-\frac{dw}{dz}=-U$$

which shows that the velocity at infinity is U in the negative sense.

2nd part: From (1), we have

$$w=U\left(z+\frac{a^2}{z}\right)+ik\ln\left(\frac{z}{a}\right)$$

$$or,\ \phi+i\psi=U\left(re^{i\theta}+\frac{a^2}{r}e^{-i\theta}\right)+ik\ln\left(\frac{re^{i\theta}}{a}\right)$$

$$or,\ \phi+i\psi=U\left(r\cos\theta+ir\sin\theta+\frac{a^2}{r}\cos\theta-i\frac{a^2}{r}\sin\theta\right)+ik\ln\left(\frac{r}{a}\right)-k\theta$$

Separating real and imaginary parts, we have the velocity potential and stream function respectively.

$$\phi=U\left(r\cos\theta+\frac{a^2}{r}\cos\theta\right)-k\theta$$

and $$\psi=U\left(r\sin\theta-\frac{a^2}{r}\sin\theta\right)+k\ln\left(\frac{r}{a}\right)$$

The lines of equipotential are given by

$$\phi=constant$$

$$or,\ U\left(r\cos\theta+\frac{a^2}{r}\cos\theta\right)-k\theta=constant$$

The streamlines are given by

$$\psi=constant$$

$$or,\ U\left(r\sin\theta-\frac{a^2}{r}\sin\theta\right)+k\ln\left(\frac{r}{a}\right)=constant$$

One of the streamlines is given by

$$U\left(r\sin\theta-\frac{a^2}{r}\sin\theta\right)+k\ln\left(\frac{r}{a}\right)=0$$

it is possible if $r=a$

$$or,\ r^2=a^2$$

$$or,\ x^2+y^2=a^2$$

$$i.e.\ |z|=a$$

which shows that the circle is a streamline.

3rd part: The stagnation points are given by

$$\frac{dw}{dz}=0$$

$$or,\ U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z}=0$$

$$or,\ U\left(z^2-a^2\right)+ikz=0$$

$$or,\ Uz^2+ikz-Ua^2=0$$

which is a quadratic equation in z, So there are two values of z and hence there are two stagnation points.

4th part: If the pressure thrust on the contour of the cylinder be represented by a force (X, Y) and a couple of momentum M then by Blasious theorem

$$X - iY = \frac{1}{2}\rho i \oint_c \left(\frac{dw}{dz}\right)^2 dz \qquad (3)$$

and

$$M = real\ \ part\ of\ -\frac{1}{2}\rho \oint_c \left(\frac{dw}{dz}\right)^2 zdz \qquad (4)$$

From equation (3), we get

$$X - iY = \frac{1}{2}\rho i \oint_c \left[U\left(1 - \frac{a^2}{z^2}\right) + \frac{ik}{z}\right]^2 dz$$

$$= \frac{1}{2}\rho i \times 2\pi i \times 2Uik \qquad \text{[By Cauchy Residue theorem]}$$

$$= -2\pi i \rho Uk$$

$$\therefore\ X = 0,\ Y = -2\pi\rho Uk$$

Now

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{k^2}{z^2} + \frac{2ikU}{z}\left(1 - \frac{a^2}{z^2}\right)$$

$$= U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} - \frac{k^2}{z^2} + \frac{2ikU}{z} - \frac{2ikUa^2}{z^3}$$

$$= \frac{U^2z^4 - 2U^2a^2z^2 + U^2a^4 - k^2z^2 + 2ikUz^3 - 2ikUa^2z}{z^4}$$

$$\therefore \left(\frac{dw}{dz}\right)^2 z = \frac{U^2z^4 - 2U^2a^2z^2 + U^2a^4 - k^2z^2 + 2ikUz^3 - 2ikUa^2z}{z^3}$$

The pole is at $z = 0$ of order 3.
The residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{(3-1)!}\frac{d^{3-1}}{dz^{3-1}}\left\{z^3 \cdot \frac{U^2z^4 - 2U^2a^2z^2 + U^2a^4 - k^2z^2 + 2ikUz^3 - 2ikUa^2z}{z^3}\right\}$$

$$= \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(U^2z^4 - 2U^2a^2z^2 + U^2a^4 - k^2z^2 + 2ikUz^3 - 2ikUa^2z\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\frac{d}{dz}\left(4U^2z^3 - 4U^2a^2z - 2k^2z + 6ikUz^2 - 2ikUa^2\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\left(12U^2z^2 - 4U^2a^2 - 2k^2 + 12ikUz\right)$$

$$= \frac{1}{2}\left(-4U^2a^2 - 2k^2\right)$$

$$= -2U^2a^2 - k^2$$

$$\therefore \oint_c \left(\frac{dw}{dz}\right)^2 zdz = 2\pi i\left(-2U^2a^2 - k^2\right)$$

$$=-2\pi i\left(2U^2a^2+k^2\right)$$

From equation (5), we have

$$M = Real\ \ part\ of -\frac{1}{2}\rho\oint_c\left(\frac{dw}{dz}\right)^2 zdz$$

$$= Real\ \ part\ of -\frac{1}{2}\rho\left\{-2\pi i\left(2U^2a^2+k^2\right)\right\}$$

$$= Real\ \ part\ of\ \rho\pi i\left(2U^2a^2+k^2\right)$$

$$=0.$$

5th part: On the cylinder $|z|=a$ we have

$$z=ae^{i\theta}$$

From (1), we have

$$w=U\left(ae^{i\theta}+\frac{a^2}{ae^{i\theta}}\right)+ik\ln\left(\frac{ae^{i\theta}}{a}\right)$$

$$or,\ \phi+i\psi=U\left(ae^{i\theta}+ae^{-i\theta}\right)-k\theta$$

$$or,\ \phi+i\psi=Ua\left(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta\right)-k\theta$$

$$or,\ \phi+i\psi=2Ua\cos\theta-k\theta$$

Equating real and imaginary parts, we have

$$\phi=2Ua\cos\theta-k\theta \quad \text{and} \quad \psi=0$$

From it is obvious that $\psi=0$ on $|z|=a$, hence it is a streamline. Again in going once round the cylinder $|z|=a$ in the direction of θ increasing we see that θ increases by 2π and hence

$$\phi=2Ua\cos(\theta+2\pi)-k(\theta+2\pi)$$

$$=2Ua\cos\theta-k(\theta+2\pi)$$

This shows that ϕ decreases by an amount $2\pi k$.

But $circulation = decrease\ in\ \phi\ in\ going\ once\ round\ the\ cylinder$

$$=2\pi k \quad (\textbf{Showed})$$

Problem-03: Show that for a liquid streaming past a fixed cylinder of radius a, the velocity potential and the stream function are given by

$$\phi=U\left(r+\frac{a^2}{r}\right)\cos\theta$$

$$\psi=U\left(r-\frac{a^2}{r}\right)\sin\theta$$

OR

Determine streaming motion past a fixed circular cylinder.

OR

Show that the complex potential $w = U\left(z + \frac{a^2}{z}\right)$ represents a streaming motion past a circular cylinder. Hence find the stagnation points.

Solution: We know that the uniform stream having velocity $-Ui$ gives rise to a complex potential Uz. We consider $f(z) = Uz$.

Now if a circular cylinder is inserted in the flow field, then for the region $|z| \ge a$, we have the complex potential

$$
\begin{aligned}
w &= Uz + \frac{Ua^2}{z} \\
&= U\left(z + \frac{a^2}{z}\right) \\
&= U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) \\
&= U\left(r\cos\theta + ir\sin\theta + \frac{a^2}{r}\cos\theta - i\frac{a^2}{r}\sin\theta\right)
\end{aligned}
$$

Equating real and imaginary parts we have,
The velocity potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta$$

the stream function

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

(Showed)

Now $$\frac{dw}{dz} = U - \frac{Ua^2}{z^2}$$

For stagnation point

$$
\begin{aligned}
&\frac{dw}{dz} = 0 \\
&or,\ U - \frac{Ua^2}{z^2} = 0 \\
&or,\ 1 - \frac{a^2}{z^2} = 0 \\
&\therefore\ z = \pm a
\end{aligned}
$$

Hence the stagnation poins are $z = a$ and $z = -a$.

The fluid speed is given by

$$q = \left|\frac{dw}{dz}\right|$$

$$= \left|U\left(1 - \frac{a^2}{z^2}\right)\right|$$

$$= U\left|1 - \frac{a^2}{z^2}\right|.$$