Assignment 3:
Eternal Elegance: Exploring Fourier Transform Applications in Image Processing

Homeworks Guidelines and Policies

- ***What you must hand in.*** It is expected that the students submit an assignment report (HW3_[student_id].pdf) as well as required source codes (.m or .py) into an archive file (HW3_[student_id].zip). Please combine all your reports just into a single .pdf file.
- ***Pay attention to problem types.*** Some problems are required to be solved *by hand* (shown by the icon), and some need to be implemented (shown by the icon). Please do not use implementation tools when it is asked to solve the problem by hand, otherwise you will be penalized and lose some points.
- ***Don't bother typing!*** You are free to solve by-hand problems on a paper and include their pictures in your report. Here, cleanness and readability are of high importance. Images should also have appropriate quality.
- ***Reports are critical.*** Your work will be evaluated mostly by the quality of your report. Do not forget to explain your answers clearly, and provide enough discussions when needed.
- ***Appearance matters!*** In each homework, 5 points (out of a possible 100) belong to compactness, expressiveness, and neatness of your report and codes.
- ***MATLAB is also allowable.*** By default, we assume you implement your codes in Python. If you are using MATLAB, you have to use the equivalent functions when it is asked to use specific Python functions.
- ***Be neat and tidy!*** Your codes must be separated for each question, and for each part. For example, you have to create a separate .m file for part b. of question 3, which must be named 'p3b.m'.
- ***Use bonus points to improve your score.*** Problems with bonus points are marked by the icon. These problems usually include uncovered related topics, or those that are only mentioned briefly in the class.
- ***Moodle access is essential.*** Make sure you have access to Moodle, because that is where all assignments as well as course announcements are posted. Homework submissions are also made through Moodle.

- ***Assignment Deadline.*** Please submit your work **before the end of June 27th**.
- ***Delay policy.*** During the semester, students are given only 10 free late days which they can use them in their own ways. Afterwards, there will be a 15% penalty for every late day, and no more than four late days will be accepted.
- ***Collaboration policy.*** We encourage students to work together, share their findings, and utilize all the resources available. However you are not allowed to share codes/answers or use works from the past semesters. Violators will receive a zero for that particular problem.
- ***Any questions?*** If there is any question, please do not hesitate to contact us through the Telegram group chat or following email addresses: m.ebadpour@aut.ac.ir , Peymanhashemi@aut.ac.ir , and atiyeh.moghadam@aut.ac.ir .

1. Practicing the Basics of 1-D Fourier Transform (10 Pts.)

Keywords: *Fourier Transform, Inverse Fourier Transform, Duality/Linearity/Time Shift/Convolution Property of Fourier Transform, Dirac Delta Function*

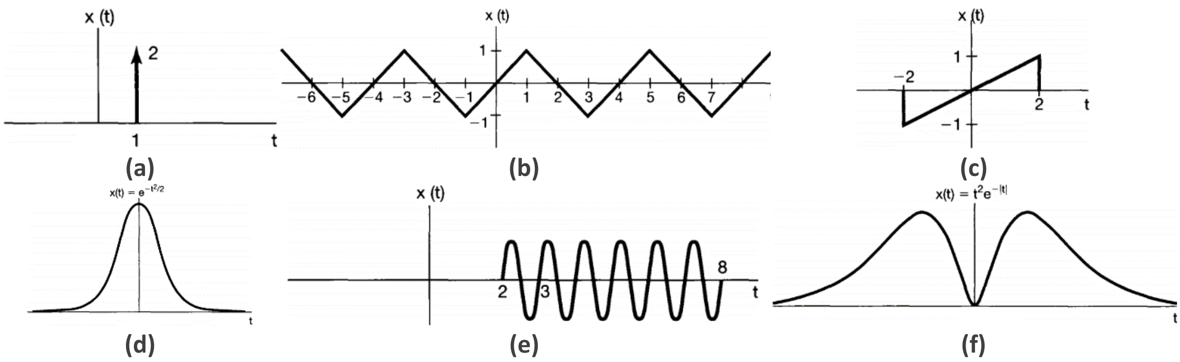
The **Fourier Transform** stands as an extraordinary milestone in the annals of scientific discovery. It offers us a profound understanding that virtually all phenomena in our world can be depicted as waveforms, encompassing functions of time, space, or other variables. By employing the Fourier transform, we gain an exceptional and formidable perspective on these waveforms, disassembling them into an alternative representation comprised of sine and cosine components.

The objective of this problem is to provide you with an opportunity to reinforce your understanding of the theoretical foundation of the Fourier transform, before encountering some image-related applications within this domain.

Let's determine which, if any, of the real signals depicted in the figure satisfy the following conditions for their Fourier transforms:

a. $\mathrm{Re}\{X(j\omega)\} = 0$ b. $\mathrm{Im}\{X(j\omega)\} = 0$ c. $\int_{-\infty}^{\infty} X(j\omega)d\omega = 0$

d. $\int_{-\infty}^{\infty} \omega X(j\omega)d\omega = 0$ e. $X(j\omega)$ is periodic. f. There is a real α such that $e^{j\alpha\omega}X(j\omega)$ is real.



(a) (b) (c) (d) (e) (f)

Finally, let f = [3, 1, -2, 3, 0, -1] and h = [-2, 2, -1].

g) Pad both f and h with zeros to a length of 10 and find the convolution $f * g$.

h) Find the convolution by calculating a product of discrete Fourier transforms.

i) Plot $F_p(u)$, $G_p(u)$ and $H_p(u)$ as points in the complex plane for $0 \le u \le 9$.

j) Calculate and interpret $\sum g_p^2(n)$ and $\sum \left|G_p(u)\right|^2$.

2. The Illusory Nature of Frequency Domain Image Analysis (14 Pts.)

Keywords: *Frequency Domain, Fourier Analysis, Magnitude, Fourier Transform, Frequency Mask*

As can be expected, **Fourier Transform** Could be considered as a useful tool in image processing. It decomposes an image into sine and cosine components, where the output represents the image in the **Fourier** or **Frequency Domain**. In Fourier domain image, each point represents a specific frequency contained in the spatial domain image.

In this problem, you are about to get familiar with basics of image analysis in the frequency domain. You will also perform some manipulations there and see the results back in spatial domain.

a. Based on your knowledge about 2D discrete Fourier transform, guess the properties and the general appearance of the corresponding spectrum of each of the following images. Include sufficient explanations about your answer on your report as well as implementation.
   - a-1. Horizontal black line in a white background
   - a-2. Horizontal white line in a black background
   - a-3. Vertical black line in a white background
   - a-4. Diagonal black line in a white background
   - a-5. Multiple horizontal black parallel lines in a white background
   - a-6. Multiple concentric black circles in a white background
   - a-7. A grid of black circles in a white background
   - a-8. A black and white chessboard
   - a-9. A collection of similar objects (like coffee beans)
   - a-10. Human fingerprint

   Note: Assume that the above shapes (lines, circles, etc.) are somewhat thick.

b. Read images “sketch.jpg”, “piano_keys.jpg”, “chessboard.jpg” and “trump_and_flag.jpg”. Compute and display the magnitude and phase of the input images, and explain which specific structure in each image yields the results you obtained. Specify the relations between the images in the spatial domain and the frequency domain.

Figure 1 Input images of part b., each with a specific structure

c. Read the image “parallel_man.jpg” and display its corresponding spectrums in frequency domain (you may need to apply histogram equalization), alongside your observations based on the image appearance in spatial domain. Now try to remove the background (vertical lines), by creating an appropriate mask and multiplying it with the Fourier transform of the image. Display the result. Then remove the man in the middle (horizontal lines) in a similar manner.

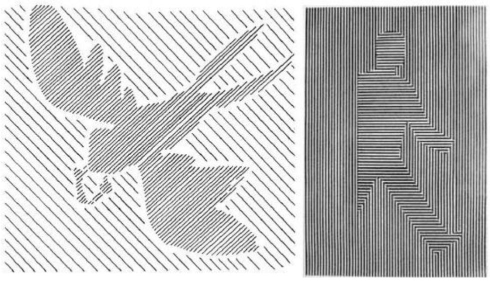


Figure 2 Input images of part c. (right) and part d. (left)

d. Do the same for the image “phoenix.jpg”. Note that the lines forming the image are now diagonal.

3. Exploring Matching in the Frequency Domain (16 Pts.)

Keywords: *Pattern Matching, Image Filtering, Mask (Frequency Domain), Image Thresholding*

Pattern Matching (not to be mistaken with "template matching") in the area of image processing, is the process of checking a given image for the presence of some pre-specified patterns. The theory of Fourier transform offers a powerful tool for doing pattern matching, especially when it comes to binary images, and letters and digits.

In this problem, we want to find all locations in an input image which are occupied by a certain letter. We need an image of that separated letter which can be used as a mask.

Read the image "math.png". By applying image filtering in frequency domain, you have to find the locations of letter 'w' inside the input image. Use "letter_w.png" as a filter mask.

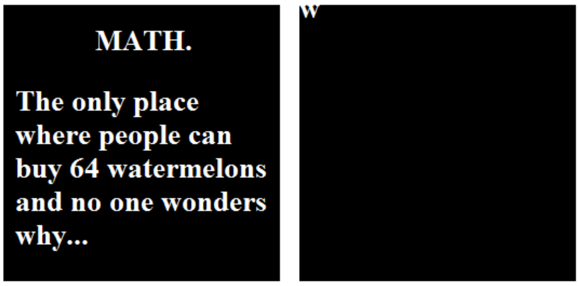


Figure 3 Input image (left) and the mask for letter 'w' (right)

a. Transform both the image and the mask into the frequency domain and multiply them. Then, apply the inverse Fourier transform to the resulting Fourier image and scale the output. Theoretically, the image you obtained so far is identical to the result of convolving the image and the mask in the spatial domain. Hence, it is expected that the image shows high values at locations where the expected pattern exists. To highlight those locations, image thresholding is probably needed.
   Note: It is not necessary to find all the letters. It is also possible to find some incorrect locations.
b. Now let's try a modified method. First, threshold the Fourier image of the mask to specify the most important frequencies which construct the desired letter in the spatial domain. Then, multiply the modified mask with the Fourier image of the text image, and apply inverse Fourier transform to the obtained image. After thresholding the resultant image, compare the performance of this method with the method used in part a.

Note: Display the results you obtain after each step in both parts. Your final result in each part must be an image where the locations of the letter 'w' you found is clearly visible by white dots.

4. 2D Discrete Fourier Transform and Geometric Transformations! (15 Pts.)

Keywords: *Fourier Transform (Properties), Geometric Transformation, Affine Group, Image Reflection, Image Translation, Image Rotation, Image Scaling, Image Shearing*

Image **Geometric Transformation** is the process of applying a function to an image, where the domain and range are sets of points of the image. Geometric transformation is mostly useful in the area of **Image Registration**. When it comes to images, translation, Rotation, Scaling and Shearing are among the most useful geometric transformations.

The goal of this problem is to examine the effects of some geometric transformations on the image Fourier transform result, and the possibility of applying them using the frequency domain.

a. Read the image “café_wall_illusion.png”, and display its Fourier spectrum. Apply the following affine transformations on it, and investigate how these transformations affect the image Fourier spectrum.
   a1. Reflection
   a2. Translation (arbitrary amount)
   a3. Rotation (by 45, 90 and 180 degrees)
   a4. Scaling (by the factor of 0.5 and 2)
   a5. Shearing (arbitrary amount)

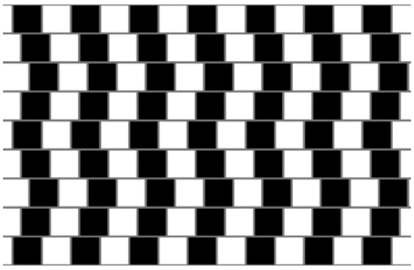


Figure 4 “Café wall illusion”. Can you believe that the horizontal lines are parallel?! (link)

b. Now, you have to implement simple geometric transformations using image representation in frequency domain. Please read the image “checker_shadow_illusion.png” and display the spectrum associated with it. Then Implement the following transformations upon it:
   b1. Translation (arbitrary amount)
   b2. Rotation (by 45, 90 and 180 degrees)
   b3. Scaling (by the factor of 0.5 and 2)

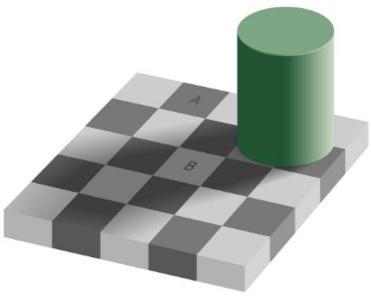


Figure 5 "Checker shadow illusion". The squares A and B are of identical brightness! (link)

5. A Deeper Look to the Image Filtering in Frequency Domain (17 Pts.)

Keywords: *Convolution Theorem*, *Image Filtering*, *Bandlimiting*

Convolution Theorem has provided a powerful tool for **Image Filtering** in frequency domain. According to this theorem, instead of performing a convolution in the spatial domain – which is not computationally efficient - filtering could be easily done in the frequency domain by multiplying two 2D Fourier transforms of the image and the filter.

Our goal in this problem is to get familiar with some well-known image filtering techniques in frequency domain. You are going to work with images "farewell.jpg" for low-pass filtering, and "old_friends.jpg" for band-pass filtering.

Figure 6 The input image for low-pass filtering is added with a Gaussian noise

a. Implement a function to perform ideal low-pass filtering on the input image. It must take the input image and two parameters `n` and `width` as its arguments, and return the frequency domain representation of an ideal low-pass filter of size $n \times n$. For spatial frequencies less than `width`, your filter should be one and zero otherwise. Apply it on "farewell.jpg" with three arbitrary settings, and display the results.
b. Implement a function to perform Gaussian low-pass filtering on the input image. It must take the input image and two parameters `n` and `variance` as its arguments, and return the frequency domain representation of a Gaussian low-pass filter of size $n \times n$. Your filter must be a Gaussian variance of parameter `variance` centred on the zero spatial frequency. Apply it on "farewell.jpg" with three arbitrary settings, and display the results.
c. Do you notice any ringing effect? In which method and for what frequency cut-offs? Explain the reason.
d. Implement a function to perform ideal band-pass filtering on the input image. It must take the input image and three parameters `n`, `centre` and `width` as its arguments, and return the frequency domain representation of an ideal band-pass filter of size $n \times n$. Your filter must be one inside a band of width `width` centred on spatial frequency `centre` and zero otherwise. Apply it on "old_friends.jpg" with three arbitrary settings, and display the results.

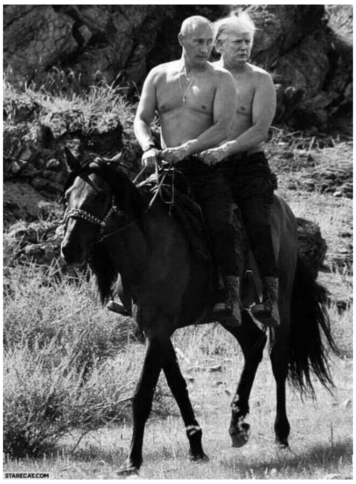


Figure 7 Input image of part d. and part e.

e. Implement a function to perform Gaussian band-pass filtering on the input image. It must take the input image and three parameters `n`, `centre` and `variance` as its arguments, and return the frequency domain representation of a Gaussian band-pass filter of size $n \times n$. Your filter must be an annulus with Gaussian cross-section with variance `variance` and mean `centre`. Apply it on "old_friends.jpg" with three arbitrary settings, and display the results.

6. Smoothing Halftone Image and Removal of Moiré Pattern using FFT (23 Pts.)

Keywords: *Halftone Image, Moiré Pattern, Image Filtering, Notch Filters*

Halftone Technique is the process of applying dots of different size and spacing in the image, so that the image tone looks continuous. As a traditional method of printing books and newspapers, it makes images extracted from old documents look dotted, as can be seen in Figure 9.

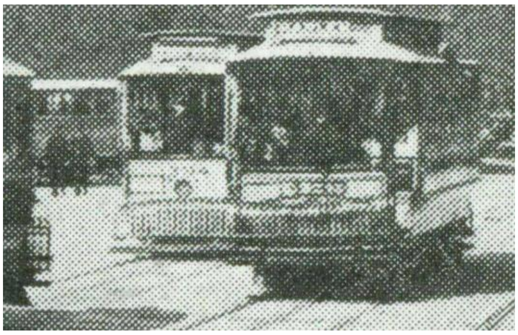


Figure 9 Example of a halftone image, taken from a newspaper. White dots are clearly visible.

Another close, yet not identical, concept in image processing is **Moiré Pattern**, which is a visual perception that happens when viewing a set of lines or dots that is superimposed on another set of lines or dots. These sets differ in relative size, angle or spacing, making strange-looking wavy pattern on the image. Figure 10 demonstrate the effect on some areas of the man's coat.

Figure 10 Moiré pattern can be seen in a large area of the man's coat

Both effects are usually undesirable, especially when it comes to digital imaging. Your goal in this problem is to deal with these artefacts in the frequency domain, which provides powerful capabilities for handling them.

a. Read the image "little_donald.png". As you can see, halftone effect is clearly visible, even without zooming in. Transform the image to the frequency domain, and comment on the result. Based on what you have learnt from image filtering in frequency domain, try to implement a method in order to reduce the artefacts in the input image.

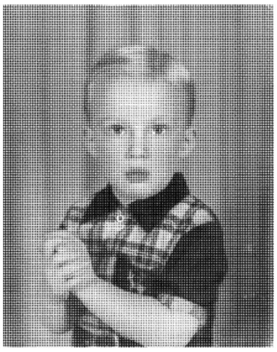


Figure 11 Trump as a child. Halftone effect is artificially

b. Moiré pattern is a very common artefact in medical imaging, making **Moiré Pattern Removal** an active research topic in the field of Radiography.
Read the image "trump_x-ray.png", and display the corresponding spectrum in the frequency domain, alongside your observations. Then try to reduce the effect of Moiré Pattern using an appropriate method.

c. Repeat part b. for the image "finger_x-ray.png". Note that the Moiré pattern in the input image in this case is angled.

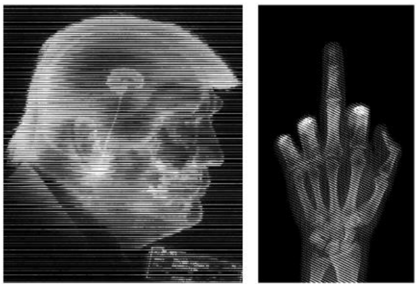


Figure 8 Input radiographic images with Moiré pattern artefacts, corresponding to part b. (left) and part c. (right).

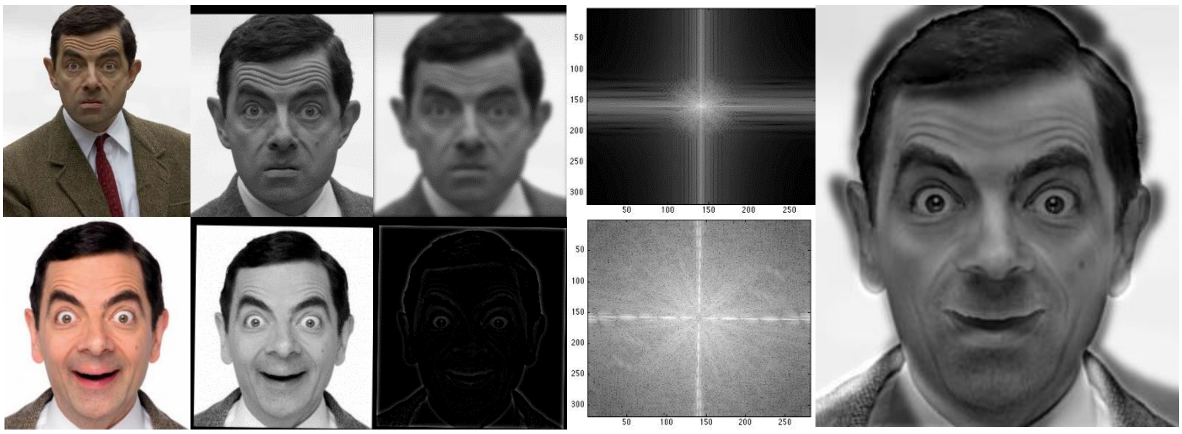
7. Smoothing Halftone Image and Removal of Moiré Pattern using FFT (+20 Pts.)

Keywords: *Image Filtering, Frequency Domain, Bandpass Filters, Hybrid Images, Image Alignment*

[Entire score of this problem assumed as bonus points] Have you ever felt happy and sad at the same time? This, in fact, is a normal feeling known as the *Bittersweet emotion*, which is experienced quite often in life. Just remember the last time you accidentally came across your childhood toys and started feeling nostalgic about your old days.

Figure 13 The woman appears to be sad in the large image, while her smile gradually appears as the size grows smaller. You can also blur your vision if you wish to see her mood changes.

We aim to bring this special feeling into images by using fundamentals of **Image Filtering** in **Fourier Domain**. As you know, high frequency details of an image tend to be more visible from close range, whereas low frequency signal is more noticeable from a distance. We intuitively make use of this fact by merging high frequency content of one image with the low frequency content of another to obtain a combined image whose interpretation depends on the distance where it is being seen. More details can be found in [this paper](#).



(a) (b) (c) (d) (e)

Figure 14 The process of creating a hybrid image (a) Initial face images (b) Images are aligned so that they match visually (c) The result of applying band-pass filters on the images (d) Fourier spectrum of the filtered images. As can be seen, only low frequencies are preserved in the upper spectrum, while the opposite is true in the lower one (e) Resultant hybrid image

You are given several frames taken from Donald Trump and Joe Biden in their final presidential debate. For each of these guys, choose two suitable images according to your preference. You are required to perform the following steps on each set of images.

a. **Image alignment.** A function (`align_imgs.m` for MATLAB and `align_imgs.py` for Python) is provided for you, which takes two images and two pairs of points, and align them so that the two pairs of points will have approximately equal coordinates. Use this function to align the input images. Display the resultant aligned images and their amplitudes of the Fourier transform.
Note: The Python function is not tested, hence it is not guaranteed to work flawlessly.

b. **Image filtering.** Apply low-pass filter on the first image using a standard Gaussian filter, and high-pass filter on the other by subtracting the image filtered with Gaussian filter from the original one. Choose proper values for cut-off frequencies. Display the results as well as their logarithmic amplitude of the Fourier transform.

c. **Merge Images.** Merge the images you obtained in the previous part, and display the final image and the corresponding amplitude of the Fourier transform.

d. **Visualisation.** Apply a Gaussian filter with five increasing cut-off values on the resultant images in order to illustrate the process of transformation of one expression into another.

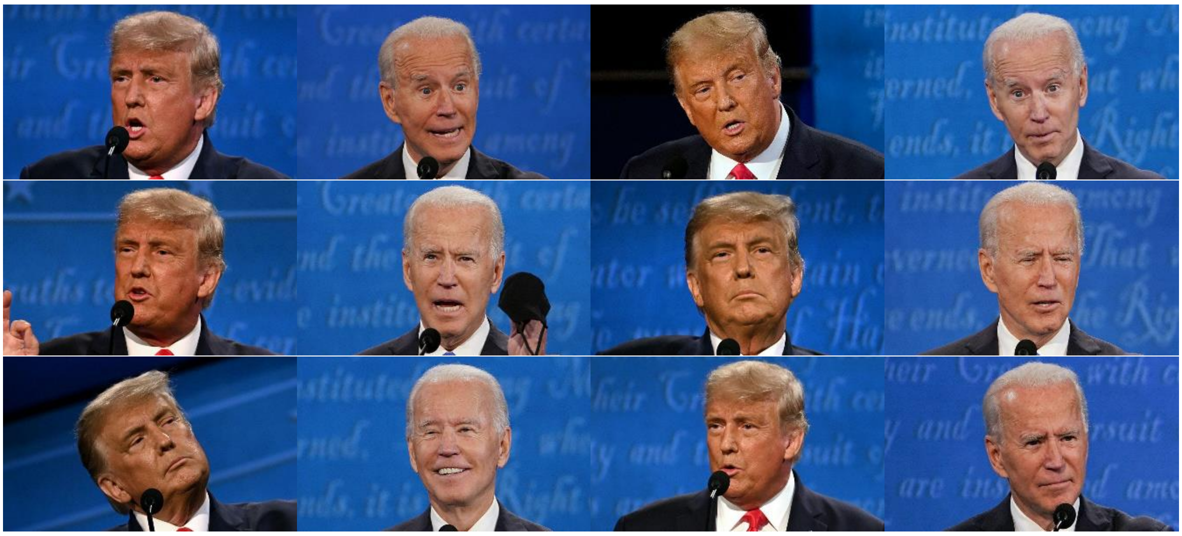


Figure 15 Different facial expressions were given by the candidates during the final debate of 2020 United States presidential election. Two images for each person must be selected as the input images of the algorithm.

Good Luck!

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