Shape Statistics via Skeletal Structures

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Papers

The compilation thesis consists of three papers

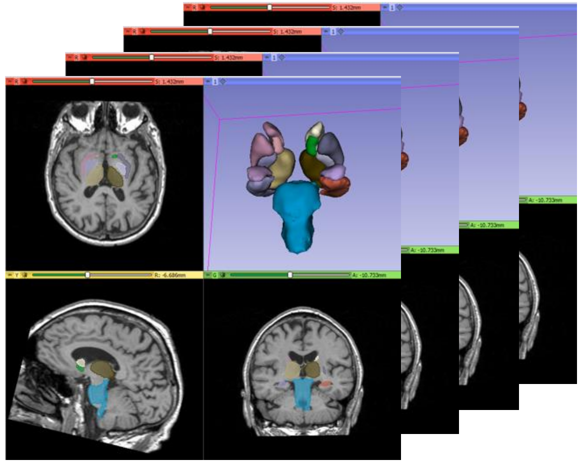
Paper I
“**Statistical Analysis of Locally Parameterized Shapes.**” (2022)
Taheri, Mohsen, and Jörn Schulz.

Paper II
“**Fitting the Discrete Swept Skeletal Representation to Slabular Objects.**” (2023)
Taheri, Mohsen, Stephen M. Pizer, and Jörn Schulz.

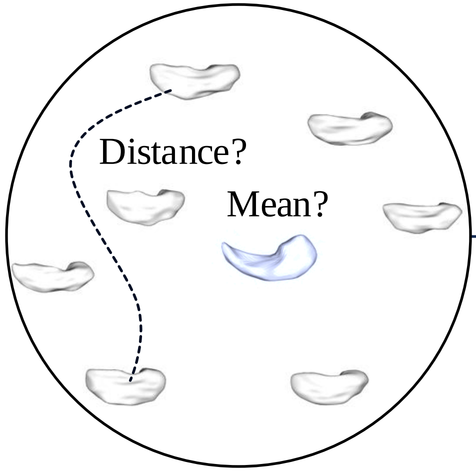
Paper III
“**The Mean Shape under the Relative Curvature Condition.**” (2024)
Taheri, Mohsen, Stephen M. Pizer, and Jörn Schulz.

Introduction

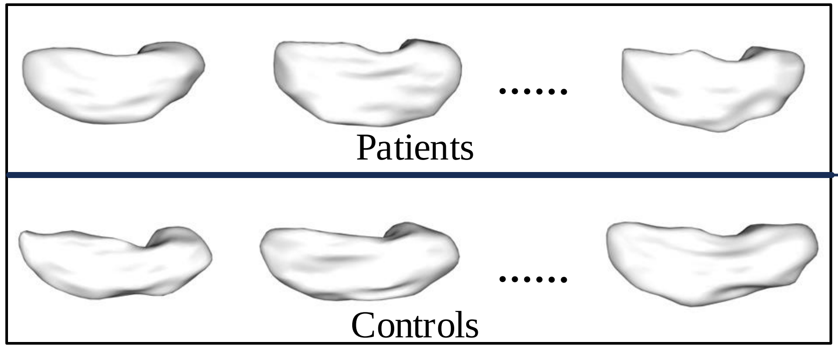
Statistical shape analysis is crucial for medical researchers because it enables them to study **shape and formation** of groups of objects by addressing critical **key questions**.



MRI scans with 3D segmentations



A set of left hippocampi



Key questions

What is the **mean**?
What is the **distribution**?
What is the **shape space**?
What are the **distances**?
What are the **modes of variation**?

Does any dissimilarity **exist**?
Where is the **location** of the dissimilarity?
What is the **intensity** of the dissimilarity?
What is the **type** of the dissimilarity
(e.g., bending, protrusion, shrinkage, etc.)?

Answering the key questions can lead to **early diagnosis** and **better treatment**, ultimately **improving patients' quality of life**.

General Idea

“***Shape** is all geometrical information that remains when location, scale and rotation are removed from an object*”.

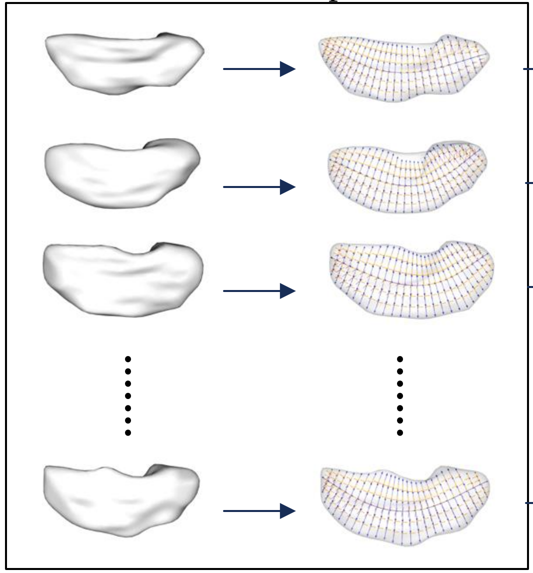
David George Kendall

Dryden, Ian L., and Kanti V. Mardia. *Statistical shape analysis* (1998).

General idea

- Representing the shapes of the objects as a finite set of **corresponding geometric object properties (GOPs)**, such as local width and curvature across the sample.
- Using the corresponding GOPs to answer the key questions.

Object | Shape representation | Corresponding GOPs



GOPs 1= $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, \upsilon_1, \ldots, \upsilon_{n_c})_1$

GOPs 2= $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, \upsilon_1, \ldots, \upsilon_{n_c})_2$

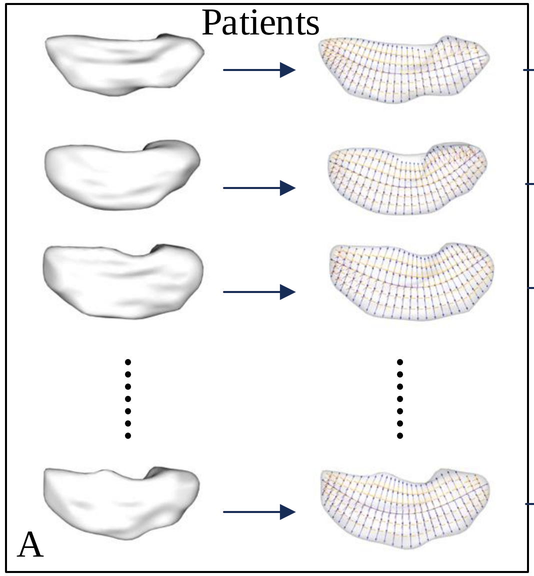
GOPs 3= $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, \upsilon_1, \ldots, \upsilon_{n_c})_3$

⋮

GOPs N= $(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, \upsilon_1, \ldots, \upsilon_{n_c})_N$

Answering the key questions

Example

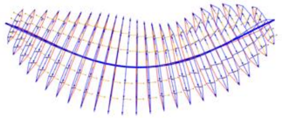


$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_1$$

$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_2$$

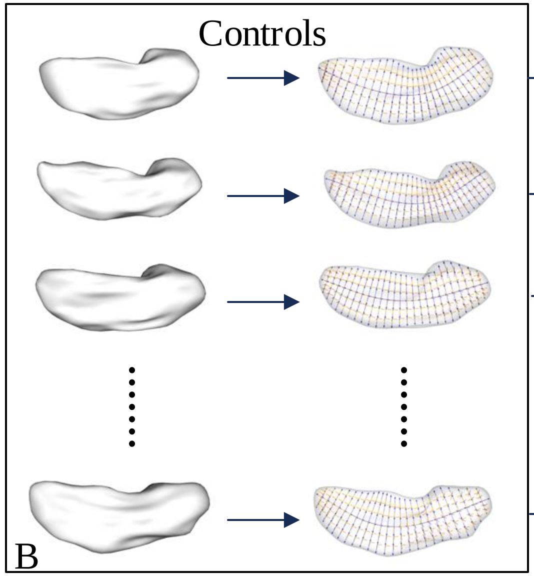
$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_3$$

$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_{N_1}$$



$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_{\mu_A}$$

Mean shape group A

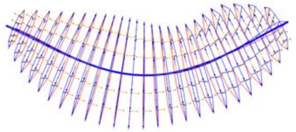


$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_1$$

$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_2$$

$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_3$$

$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_{N_2}$$



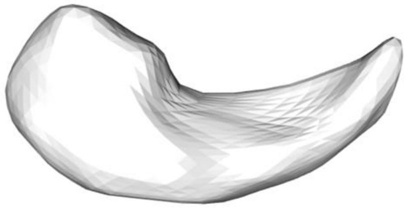
$$(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_{n_s}^*, r_1, \ldots, r_{n_s}, F_1^*, \ldots, F_{n_p}^*, \boldsymbol{v}_1^*, \ldots, \boldsymbol{v}_{n_c}^*, v_1, \ldots, v_{n_c})_{\mu_B}$$

Mean shape group B

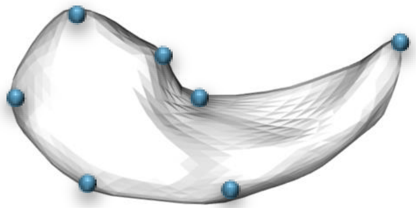
Element-wise comparison to explain local dissimilarities

Shape representation

There are various shape representations such as landmark models, boundary point distribution models (PDMs), normal vector field representations, skeletal representations, etc.

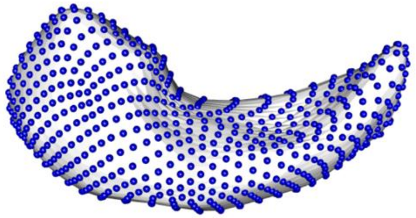


A hippocampus



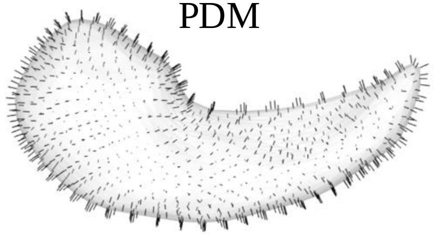
Landmark model

GOPs based on interior or boundary landmarks



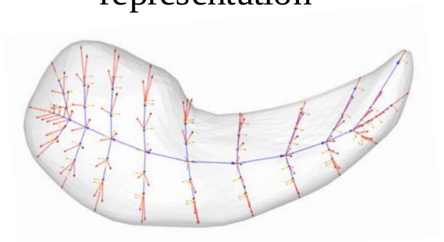
PDM

GOPs based on boundary points



Boundary normal representation

GOPs based on normal vectors



skeletal representation

GOPs based on interior vectors

⋮

Etc.

Which one and why?

Objective

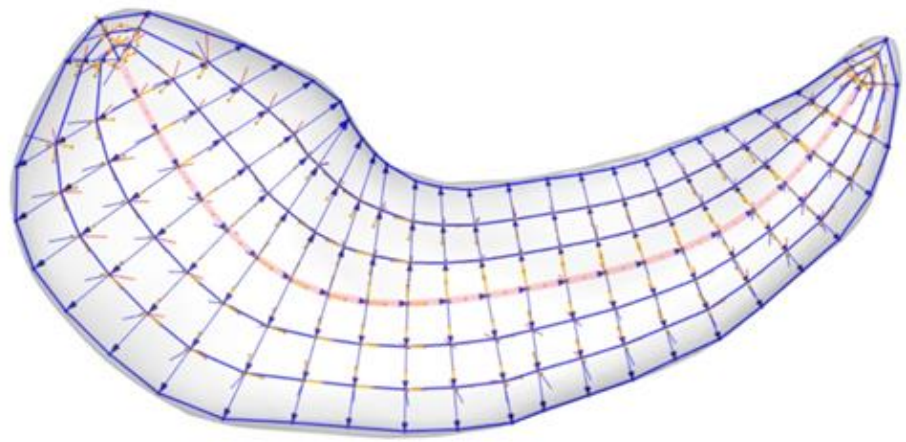
A proper shape representation must

- provide a **good correspondence** across the population.
- be able to answer the questions regarding local dissimilarities (i.e., **existence, location, intensity** and **type**).
- be **invariant** to the act of Euclidean similarity transformations of **translation, rotation** and **scaling**.
- Etc.

A robust and invariant shape representation can be established based on the **skeletal structure** of the objects.

The focus of the thesis is on developing skeletally-based shape representations, along with the **necessary mathematical tools**, to facilitate the analysis of **slabular objects (SlOs)**.

What is the skeletal structure?
What is an SlO?
Why SlOs?



Skeletal representation

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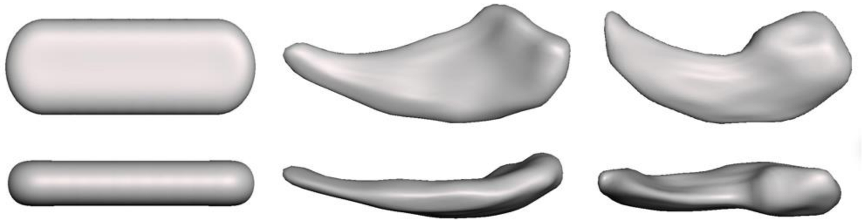
Slabular objects

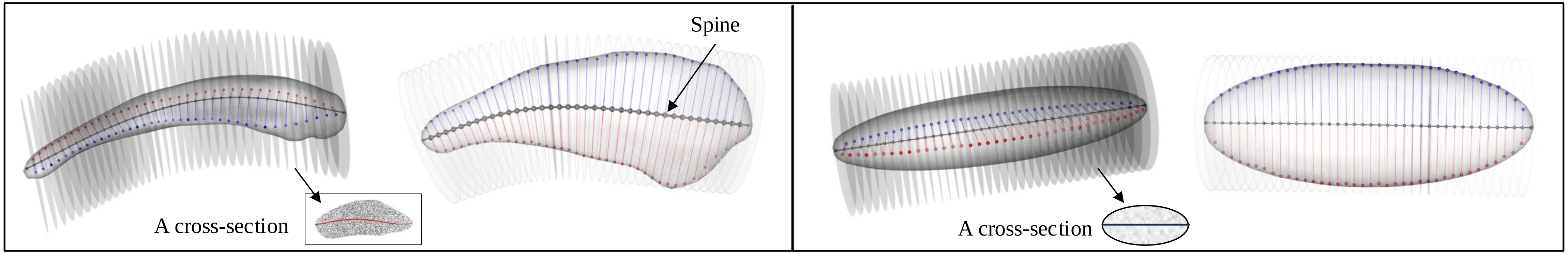
A variety of human body parts, like the kidney, and subcortical structures, such as the caudate nucleus and hippocampus, are elliptical slab-shaped objects with elliptical topologies.



A slab · A caudate · A hippocampus

A **slabular object (SlO)** is a 3D **swept region** based on a smooth sequence of **slicing planes** along a **central curve** called **spine** such that the slicing planes do not intersect within the object and **each cross-section** is a 2D swept region.

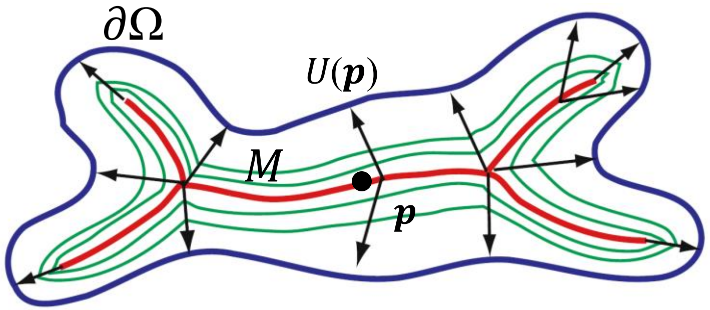
An SlO is analogous to an eccentric ellipsoid such that it has a **crest** (i.e., a closed boundary curve with convex and maximal principal curvature at each point**)** corresponding to the ellipsoid's crest and **two vertices** corresponding to the ellipsoid's vertices.



Left: A caudate nucleus as an SlO. Right: An eccentric ellipsoid.

SlO Skeletal Structures

Any object has a **skeletal structure** as a field of radial vectors U called **spokes** on the object's skeleton M as (M, U) satisfiying Damon's three conditions of radial curvature condition, edge curvature condition, and compatibility condition.

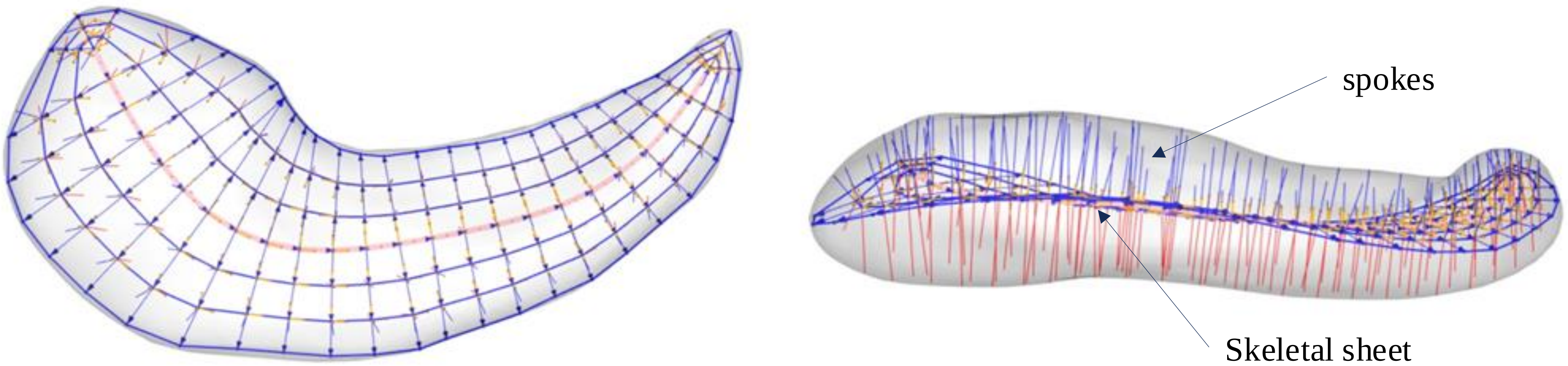


Thus, $\partial\Omega = \{\boldsymbol{p} + U(\boldsymbol{p}) \mid \boldsymbol{p} \in M\}$.

Damon, James. "Smoothness and geometry of boundaries associated to skeletal structures I: Sufficient conditions for smoothness." (2003).

Blum's medial skeletal structure is a form of skeletal structure in which the spokes with a common tail position have equal lengths.

The skeleton of an SlO can be seen as a two-dimensional topological disk called the **skeletal sheet**.

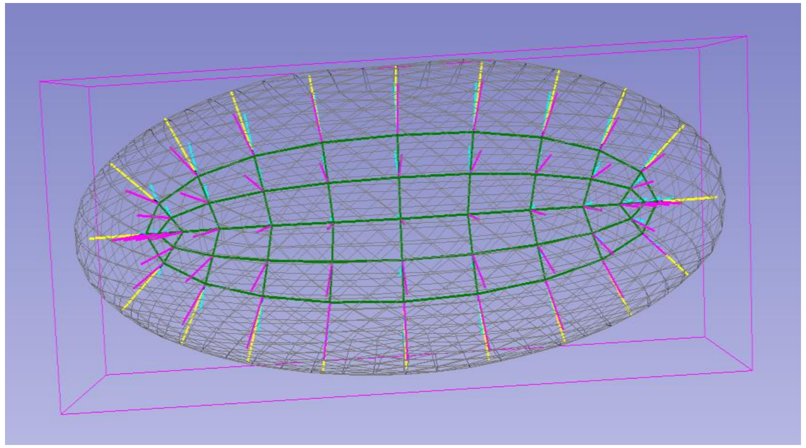


Skeletal structure of a hippocampus

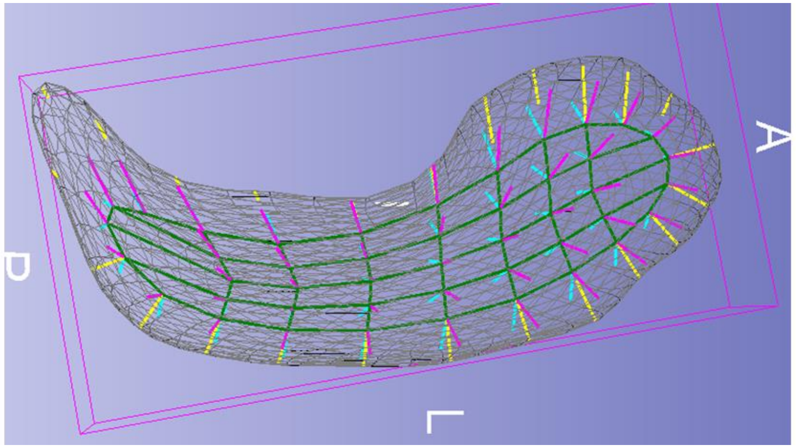
DSRep

The **discrete skeletal representation (DSRep)** can be defined in correspondence with the discrete medial skeletal structure of an eccentric ellipsoid.

Such correspondence can be established via the act of deformation.



DSRep of an eccentric ellipsoid



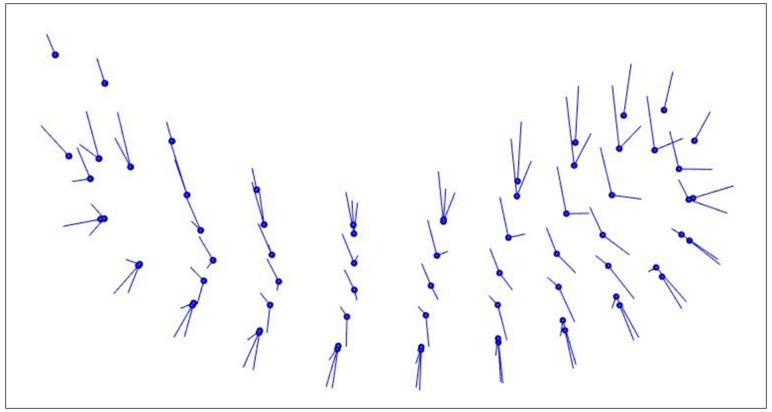
DSRep of a hippocampus

Liu, Z., et al., *Fitting unbranching skeletal structures to objects.* (2021).

A DSRep can be represented as a set of spokes with positional, directional, and length components.

$$s = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n, r_1, \dots, r_n)$$

$$\boldsymbol{p}_i \in \mathbb{R}^3, \boldsymbol{u}_i \in S^2, r_i \in \mathbb{R}^+, i = 1, \dots, n$$



The GOPs are not invariant so **alignment** is necessary to remove the act of similarity transformations.

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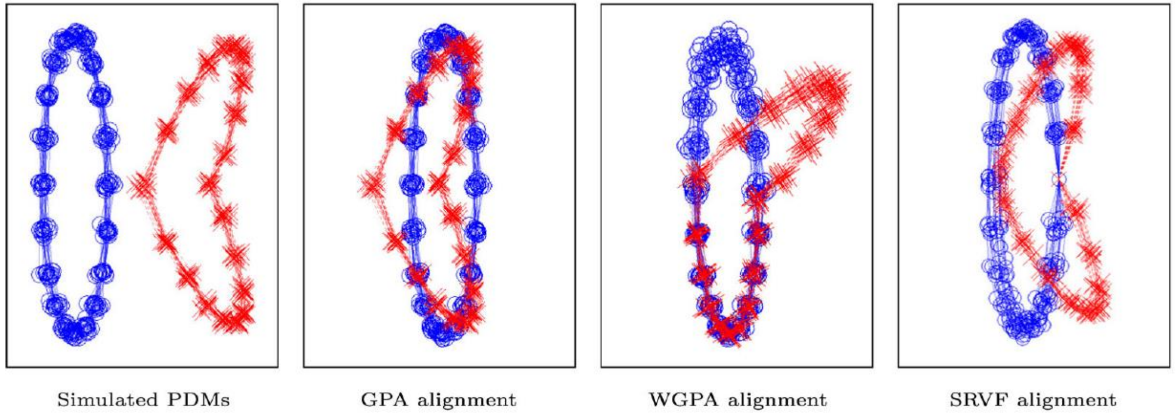
The problem of alignment

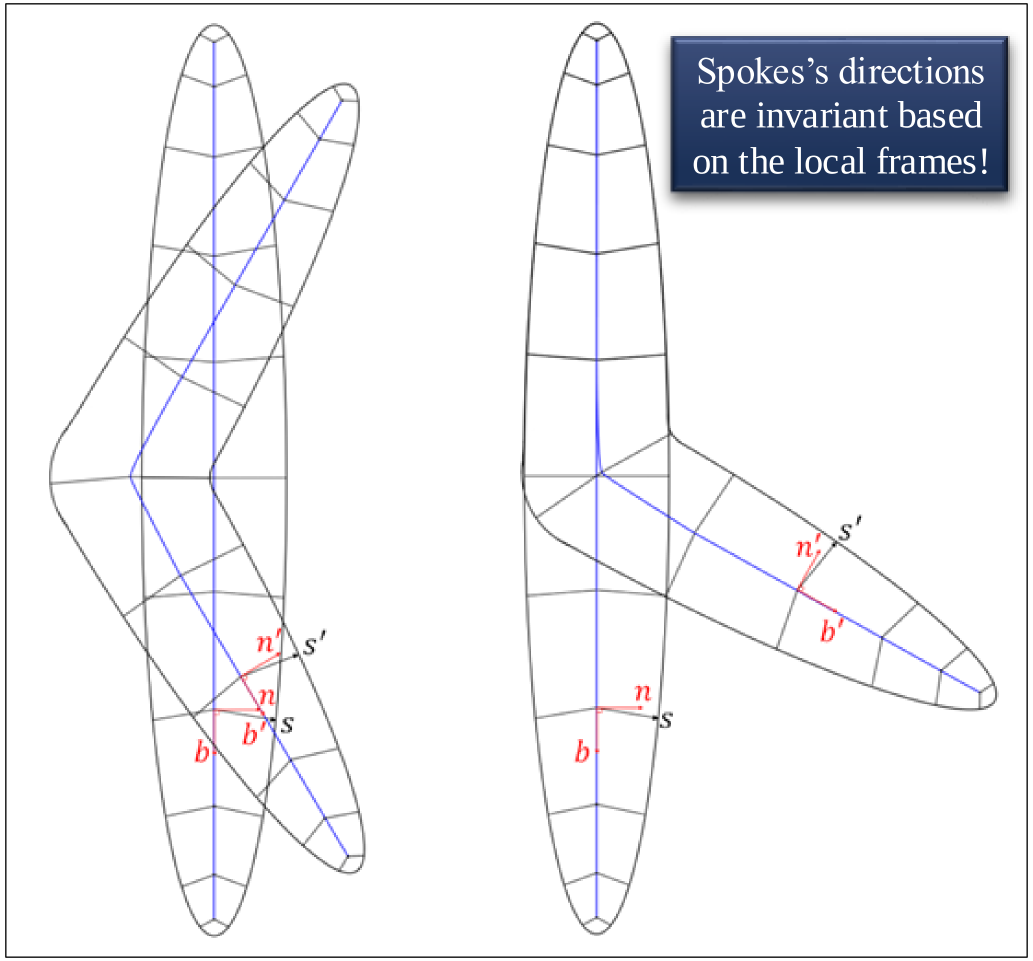
The process of alignment introduces bias and misleads the analysis, making it nearly impossible to accurately describe local dissimilarities, even for simple objects.

Lele, Subhash R., and Joan T. Richtsmeier.
An invariant approach to statistical analysis of shapes (2001).



(Generalized Procrustes Analysis) (Weighted GPA) (Square Root Velocity Function)

PDM analysis based on different alignment methods



DSRep alignment based on GPA (left) and WGPA (right)

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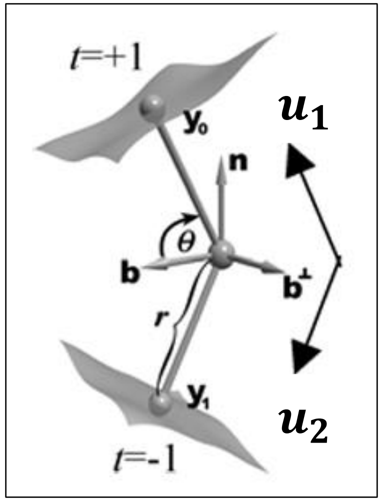
Local frames

In medial skeletal structures, we can define local frames based on the symmetric spokes tangent to the skeletal sheet.

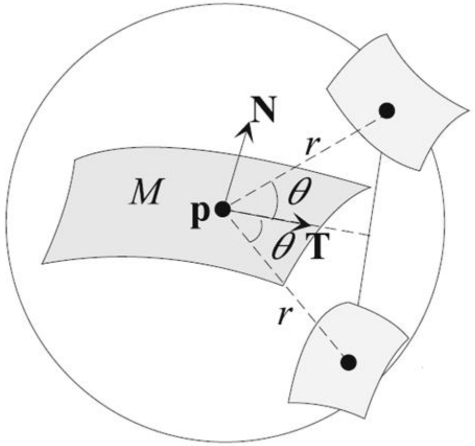
Based on the local frames some features like **spokes' directions** become **invariant.**

In m-rep, the **frames' orientations are not invariant** as they are based on the global coordinate system, so alignment is necessary.



Medial atom

Pizer, S. M., Fletcher, P. T., Joshi, S., Thall, A., Chen, J. Z., Fridman, Y., & Chaney, E. L. "Deformable m-reps for 3D medical image segmentation" (2003).



Medial frame

Giblin, P. J. and B. B. Kimia: 'A Formal Classification of 3D Medial Axis Points and their Local Geometry'. (2004)

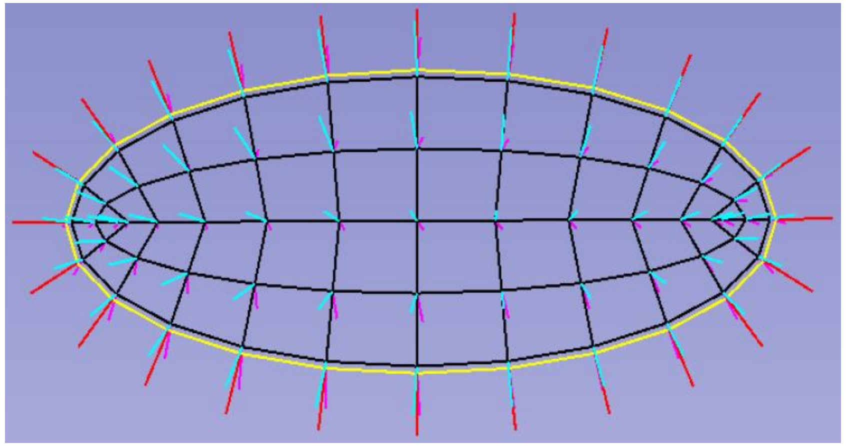
$$\mathrm{M} = (\mathrm{m}_1, \ldots, \mathrm{m}_n)$$
$$\mathrm{m}_i = (\boldsymbol{p}_i, r, \boldsymbol{u}_{i1}, \boldsymbol{u}_{i2})$$
$$i = 1, \ldots, n$$

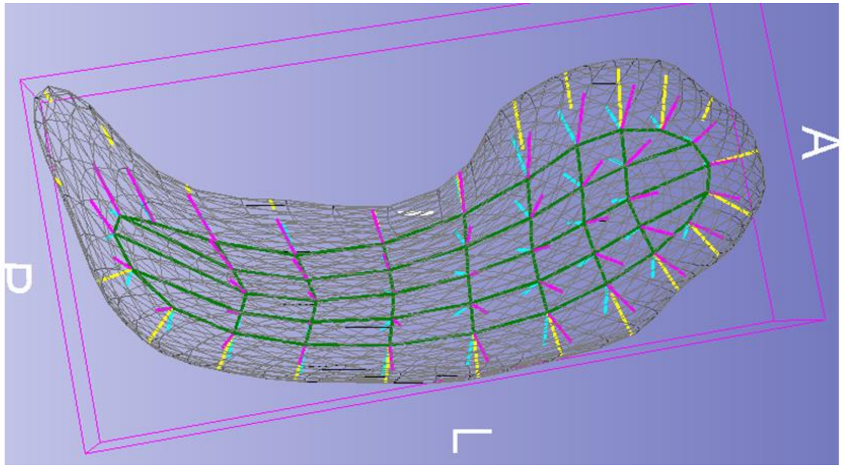
Medial representation (m-rep) as a tupel of medial atoms

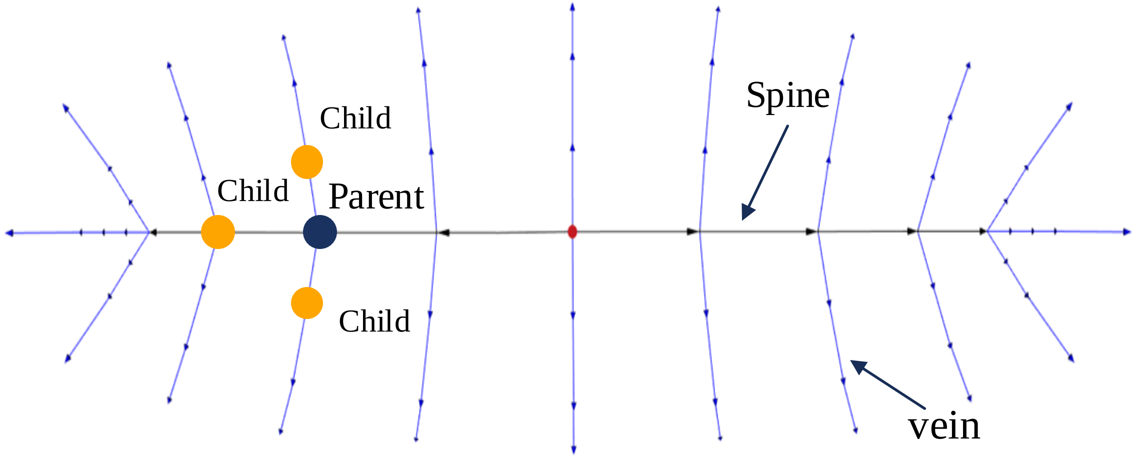
M-rep alignment

Fletcher, P. T., Lu, C., Pizer, S. M., & Joshi, S. " Principal geodesic analysis for the study of nonlinear statistics of shape " (2004).

Hierarchical structure

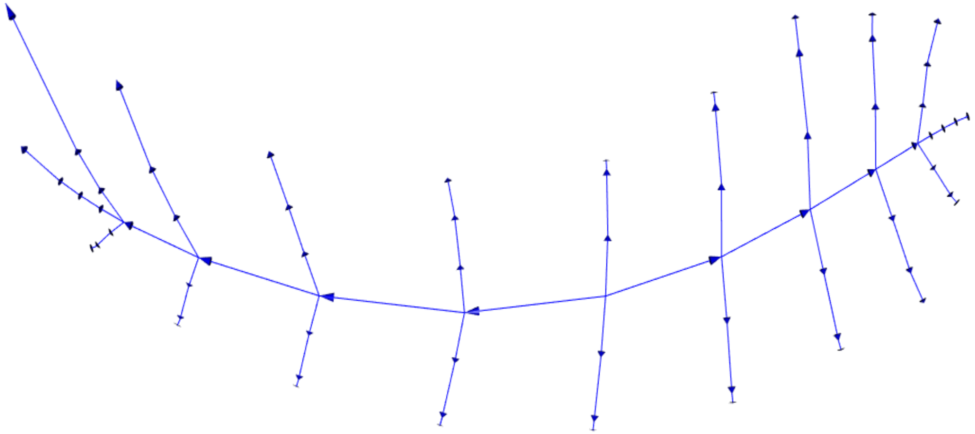




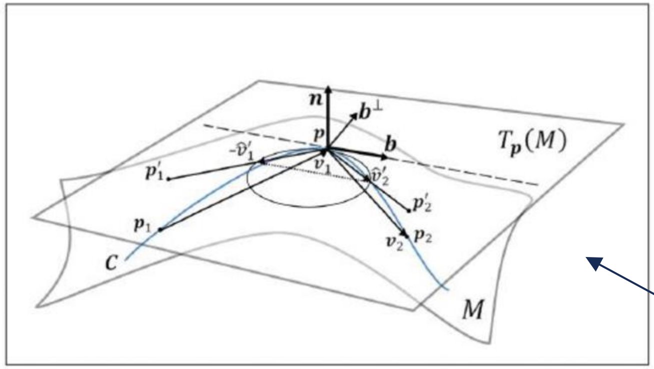


The hierarchical structure of the ellipsoid's skeletal sheet is based on Blum's grassfire flow, resembling a spanning tree.

Based on the hierarchical structure each frame has a parent frame.

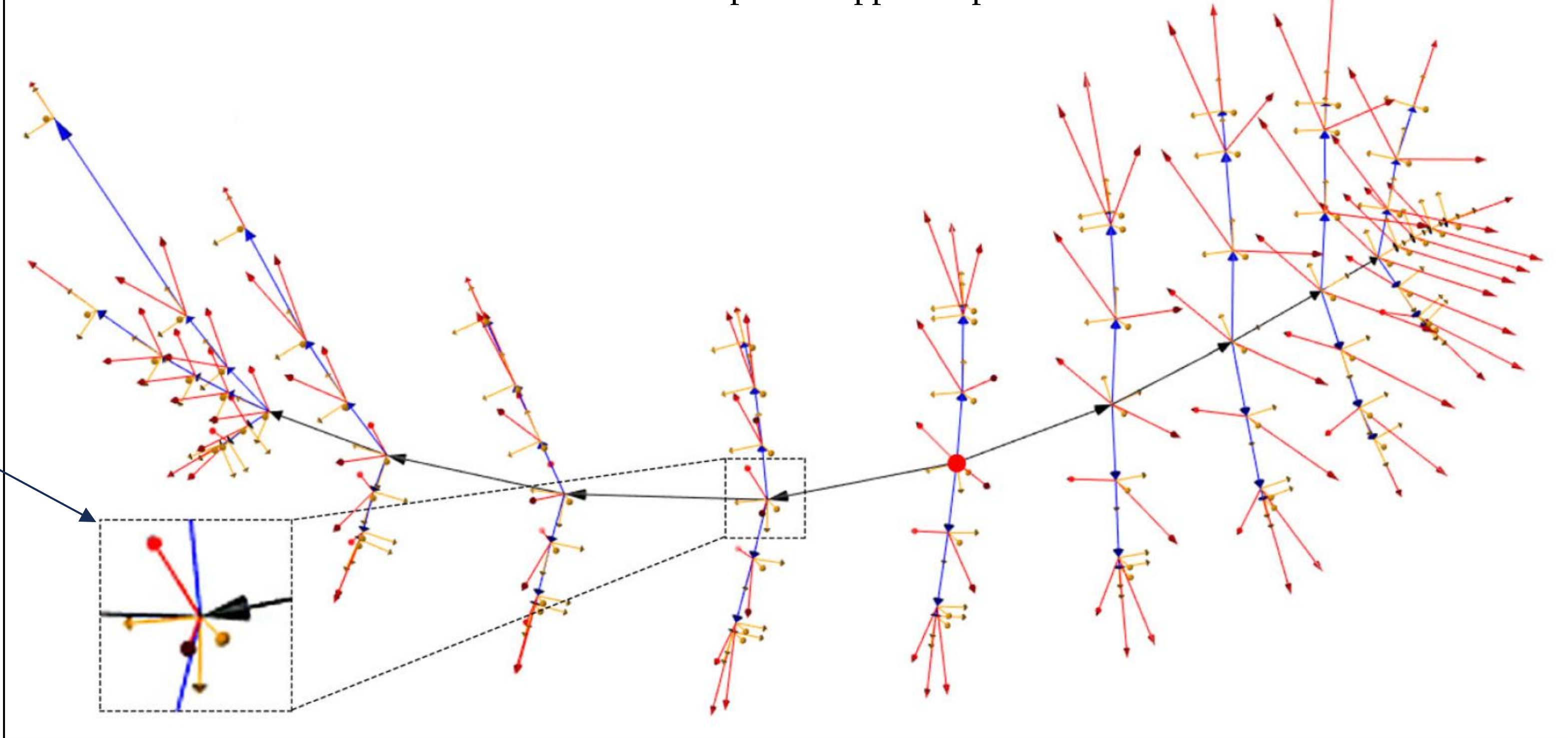


The hierarchical structure of the hippocampus's skeletal sheet.



Material frame with one element normal to the sheet and one element tangent to the vein.

An LPDSRep of a hippocampus.



$$s^{\mathrm{LP}} = (\underbrace{\boldsymbol{u}_1^*, \dots, \boldsymbol{u}_{n_s}^*}, \underbrace{r_1, \dots, r_{n_s}}, \underbrace{F_1^*, \dots, F_{n_p}^*}, \underbrace{\boldsymbol{v}_1^*, \dots, \boldsymbol{v}_{n_c}^*}, \underbrace{v_1, \dots, v_{n_c}})$$

Locally Parameterized DSRep (LPDSRep)

Spokes' directions on S^2 based on their local frame

Spokes' lengths in $\mathbb{R}^+$

Frames' orientations in *SO(3)* based on their **parent frames**

Connections' directions on S^2 based on their local frames

Connections' lengths in $\mathbb{R}^+$

LPDSRep is an invariant shape representation.

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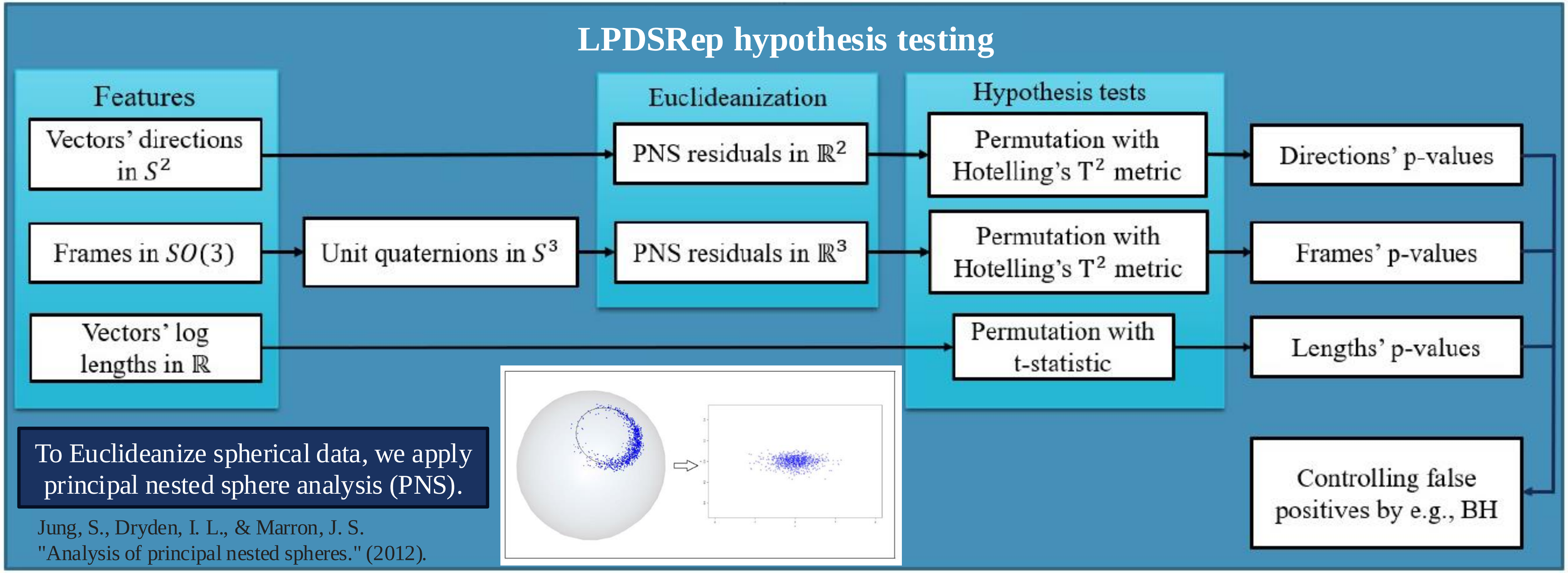
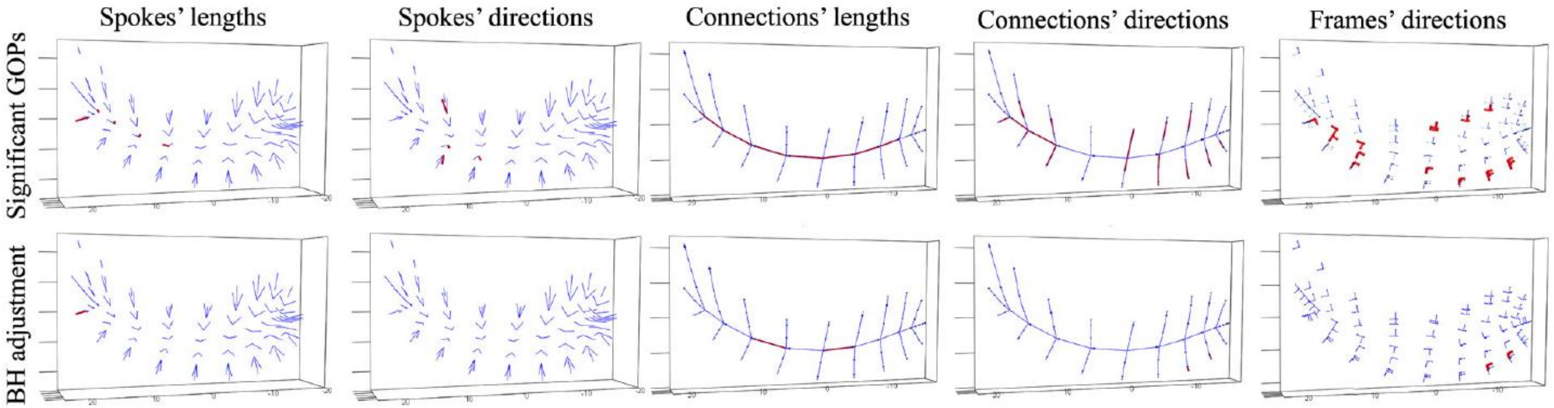
Hypothesis pipeline





PD vs. CG. Red indicate significant GOPs. FDR=0.05 for BH adjustment.

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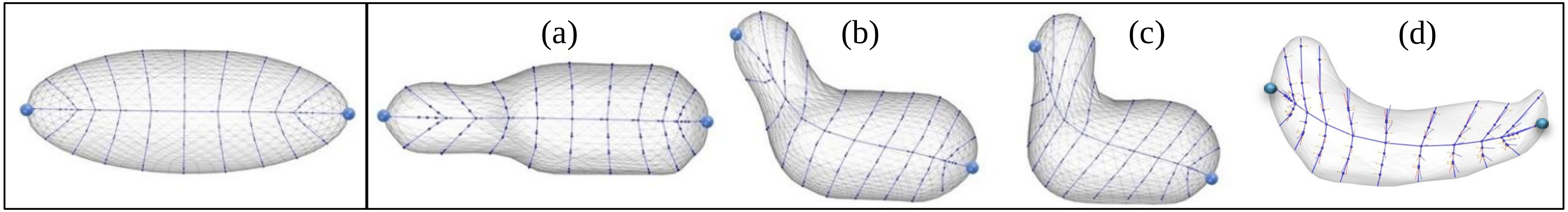
Correspondence issue

There are concerns regarding LPDSRep model fitting via deformation which **requires precise boundary registration.**

Achieving such boundary registration, which involves **crest-to-crest** and **vertex-to-vertex** correspondence, is challenging and is the subject of several ongoing studies.



- Correspondence between LPDSRep of the ellipsoid and objects (b), (c), and (d) is not desirable.
- The skeletal structure of the objects (c) and (d) seems chaotic.

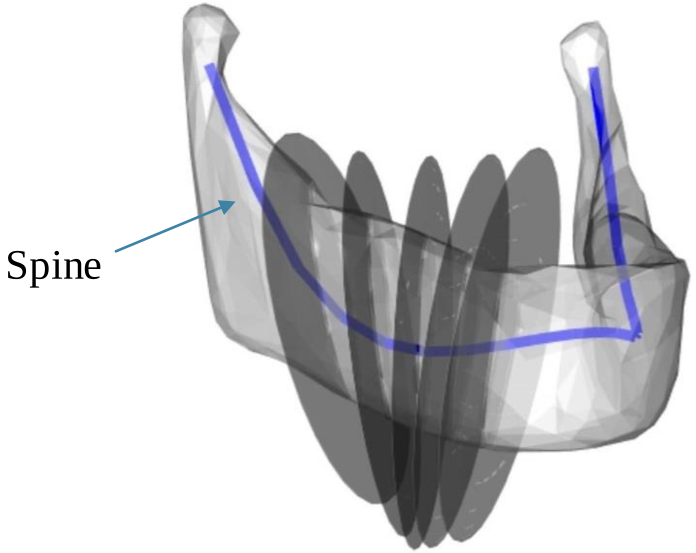
We realize an SlO as a swept region. Thus, the skeletal representation should effectively capture and **reflect key properties of a swept region**, including the spine and the slicing planes.

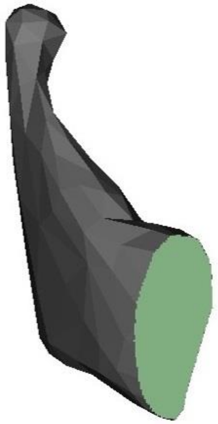
Such skeletal structure can be established based on **Damon's swept skeletal structures.**

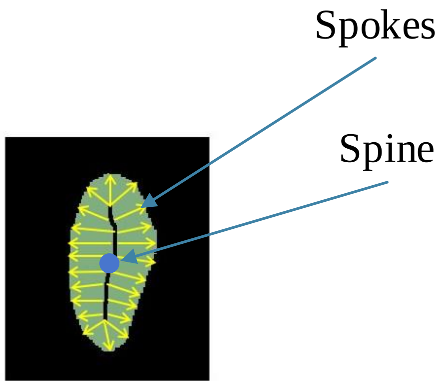
Swept skeletal structure

We say that (M, U) is a *swept skeletal structure* if for each $x \in M$ with say $x \in \Pi_t$, and for each value $U(x)$ of U at x, $U(x) \in \Pi_t$.

Damon, Swept regions and surfaces (2008)

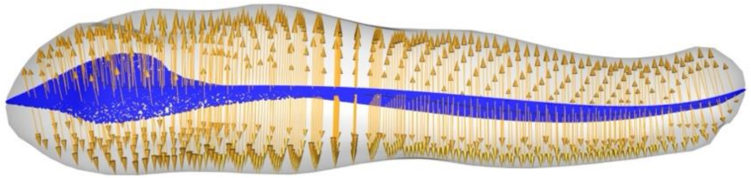


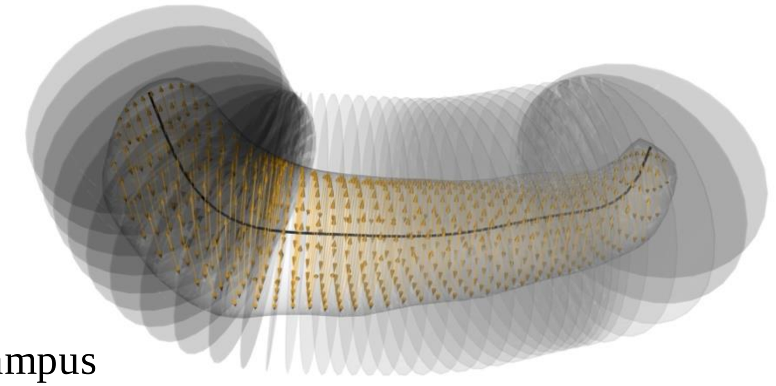




The skeletal structure of a cross-section

The union of the cross-sections' skeletal structures forms the object's swept skeletal structure.



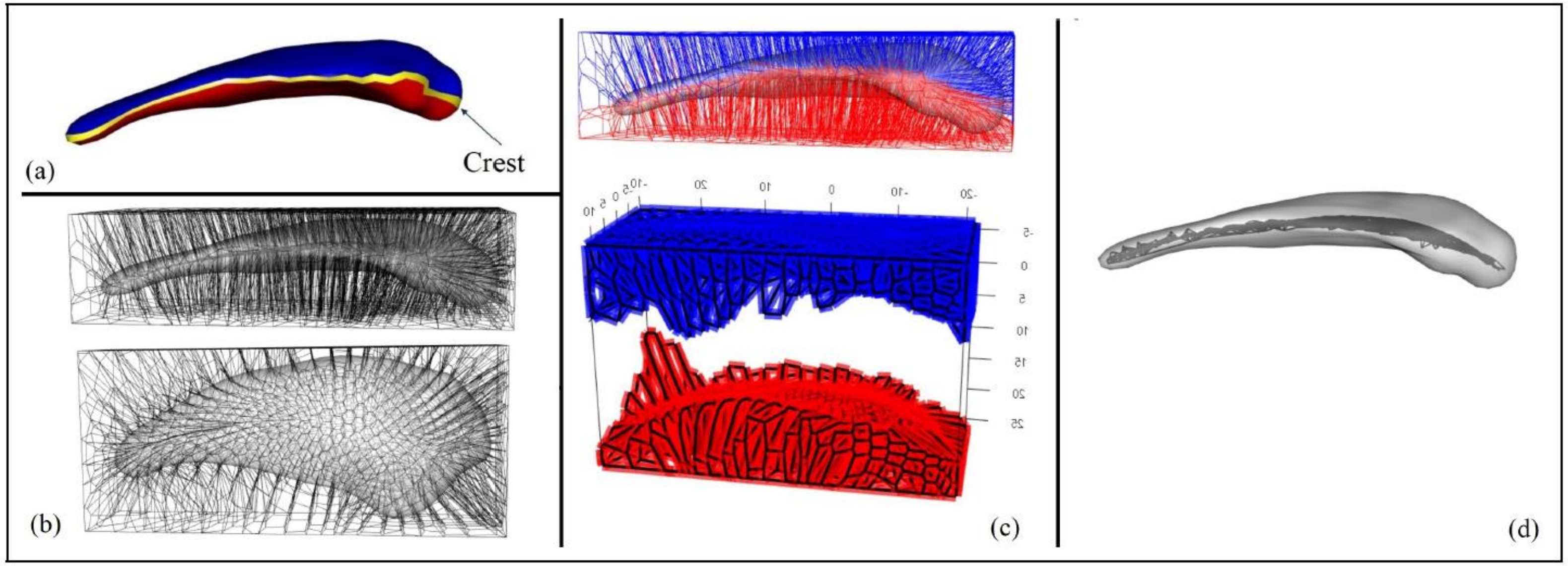


Swept skeletal structure of a hippocampus

Model fitting can be effectively achieved using the central medial skeleton as a reference.

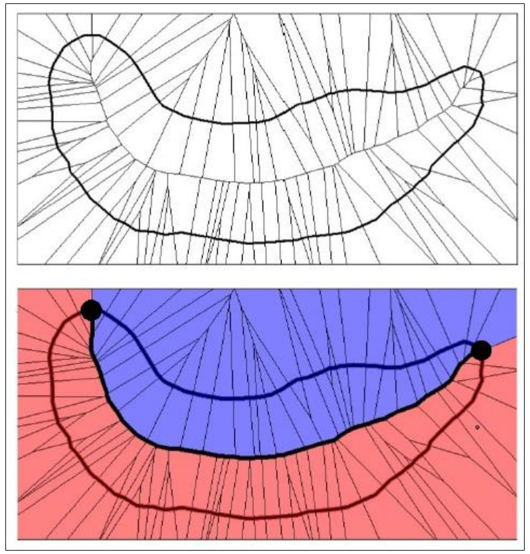
The centeral medial skeleton (CMS)

The CMS of an SlO can be calculated by dividing the Voronoi diagram of the SlO based on the crest of the object.



The CMS of an SlO

(a) The top part, the bottom part, and the crest of a caudate are depicted in blue, red, and yellow, respectively. (b) Voronoi diagram of the caudate in two views. (c) Top: Illustration of the two sub-regions in blue and red. Bottom: The sub-regions are separated to provide a better intuition. (d) The CMS is depicted as a black triangle surface.



CMS of a 2D tube

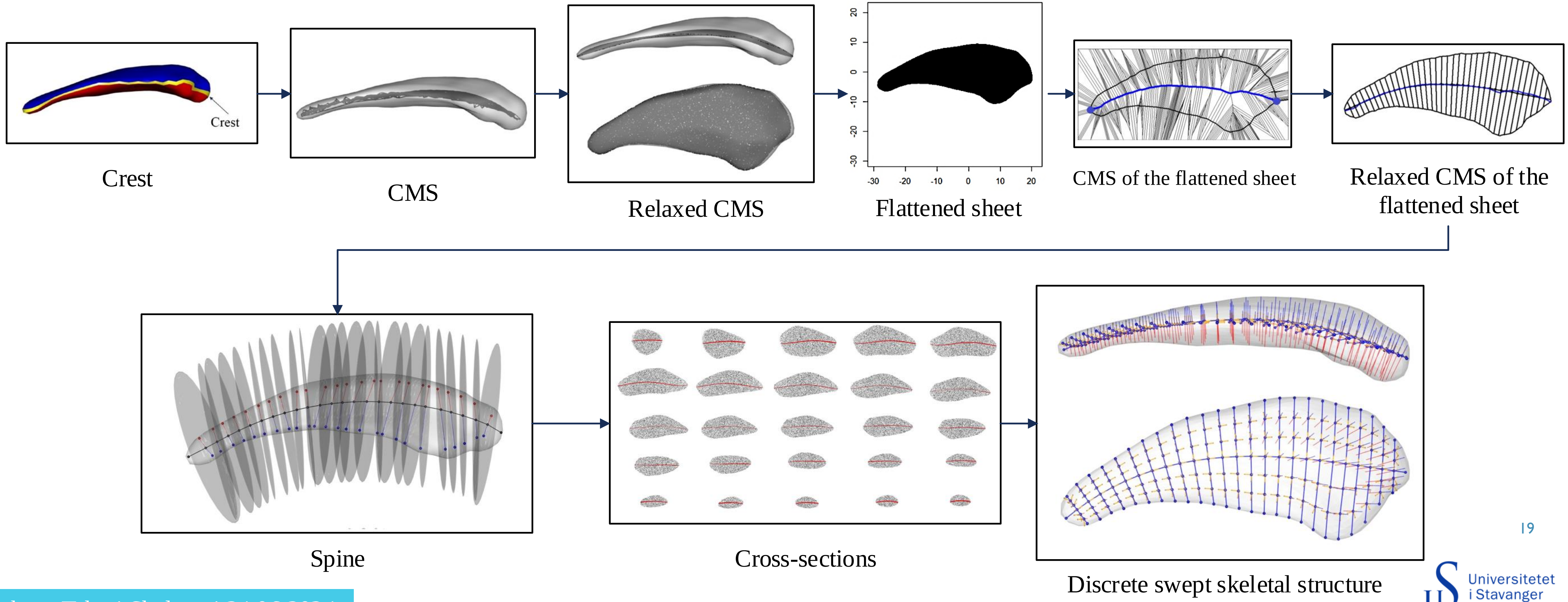
Top: The Voronoi diagram of a 2D GC.
Bottom: The CMS as the black curve.

The CMS can be seen as a unique subset of the medial skeleton that has no holes, discontinuity, or branches and divides the object into two parts, namely the top and bottom parts.

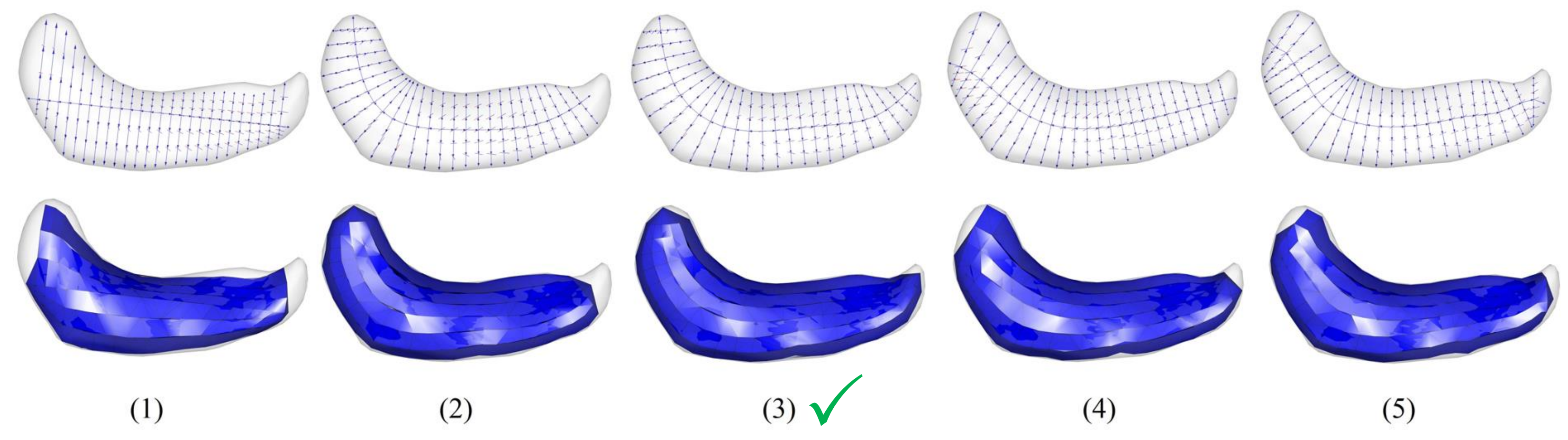
Model fitting procedure

Calculating the crest → Calculating the CMS → Defining the skeletal sheet as the relaxed CMS → Flattening the relaxed CMS → Defining the spine based on the center curve of the flattened sheet → Calculating the cross-sections → Finding the skeletal structure of the cross-sections → The union of the cross-sections' skeletal structures forms the swept skeletal structure of the SlO

Model fitting procedure



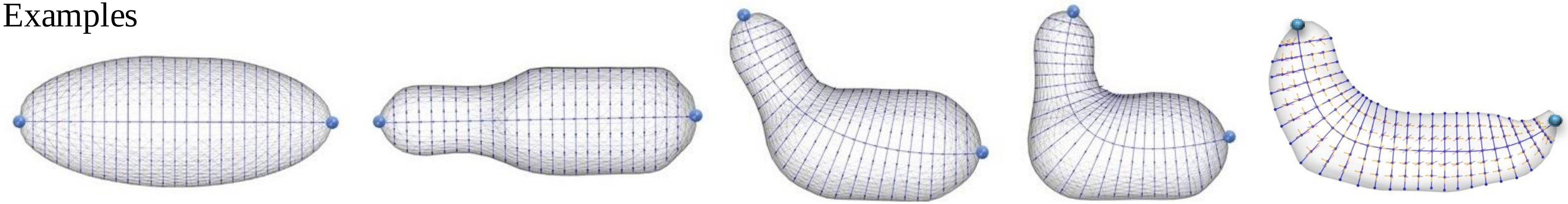
Volume-coverage, skeletal-symmetry and tidiness



Goodness of fit

Fit	PR degrees $\tilde{M}_{\Omega_3}, \Gamma'_2$	Volume-coverage	skeletal-symmetry	Average tidiness $\zeta^{\dagger}$	Strict tidiness $\zeta^{\ddagger}$	Score 1	Score 2
#1	1, 1	0.812	0.724	1	1	0.588	0.588
#2	3, 4	0.877	0.883	0.969	0.861	0.750	0.667
#3	4, 4	0.890	0.873	0.973	0.890	0.756	0.692
#4	4, 5	0.876	0.861	0.972	0.870	0.733	0.656
#5	7, 7	0.878	0.894	0.971	0.735	0.762	0.577

Examples



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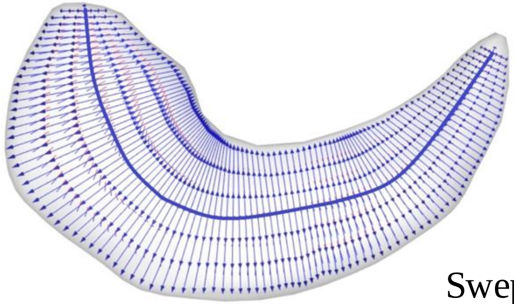
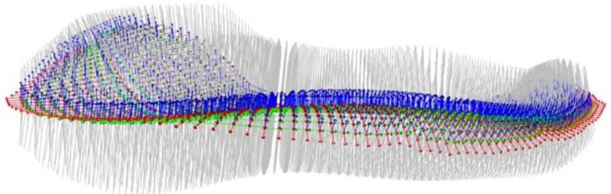
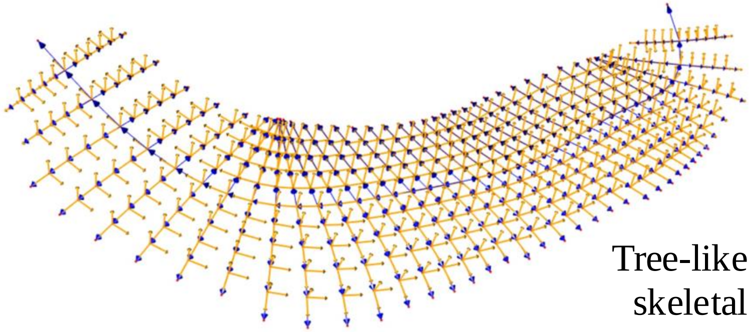
LPDSSRep

Analogous to LPDSRep we can define local frames on the skeletal sheet and define an invariant shape representation called **locally parameterized discrete swept skeletal representation (LPDSSRep).**

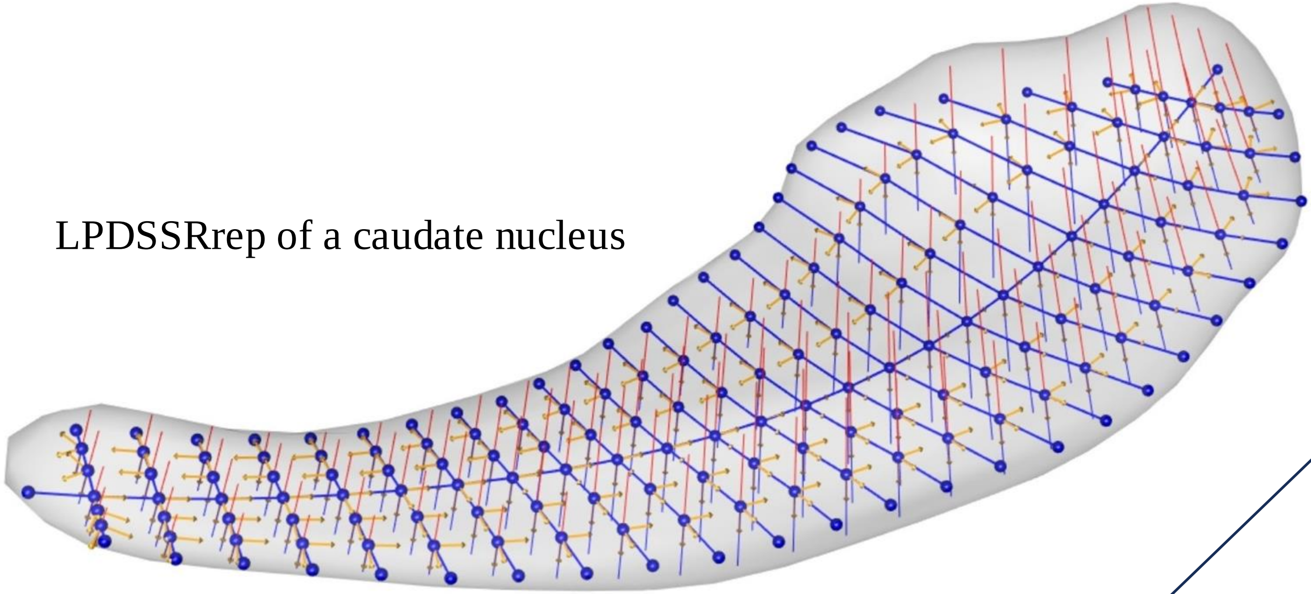




Swept skeletal structure of a hippocampus

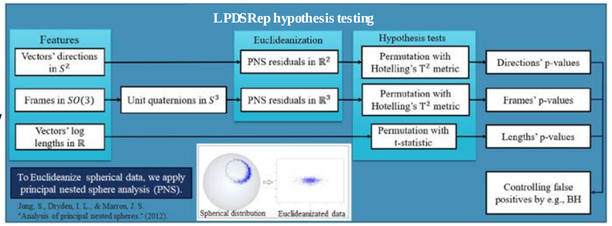


Tree-like structure of the skeletal sheet equiped with the local frames



LPDSSRrep of a caudate nucleus

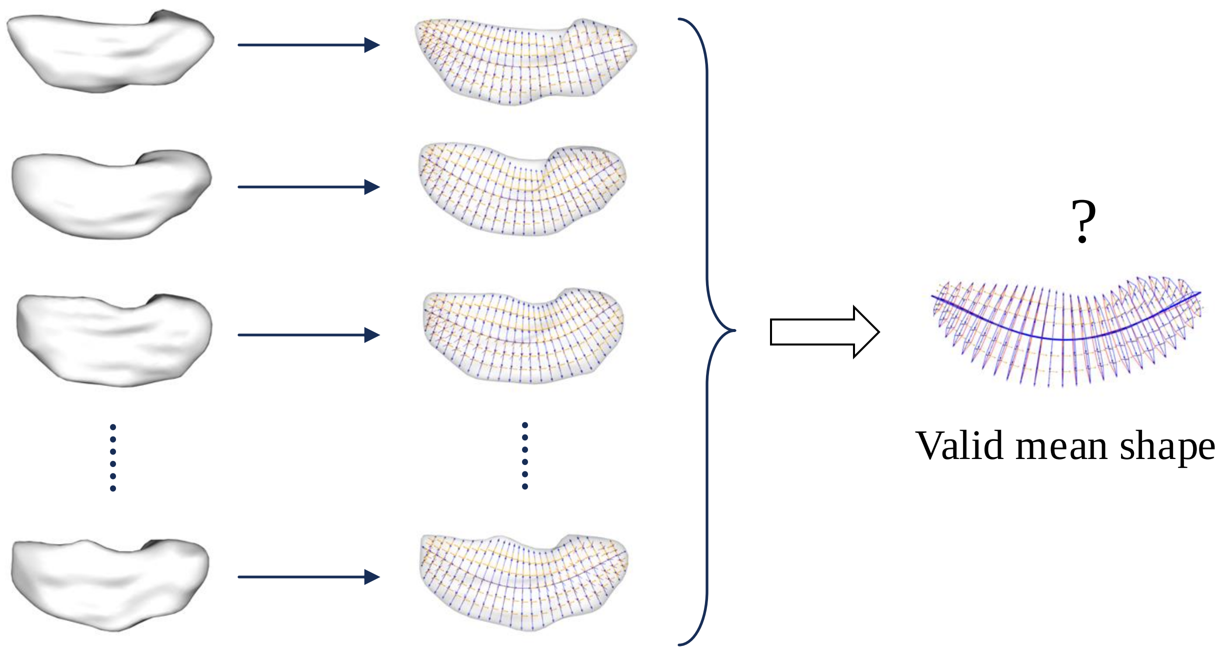
LPDSRep hypothesis framework is applicable for LPDSSRep analysis

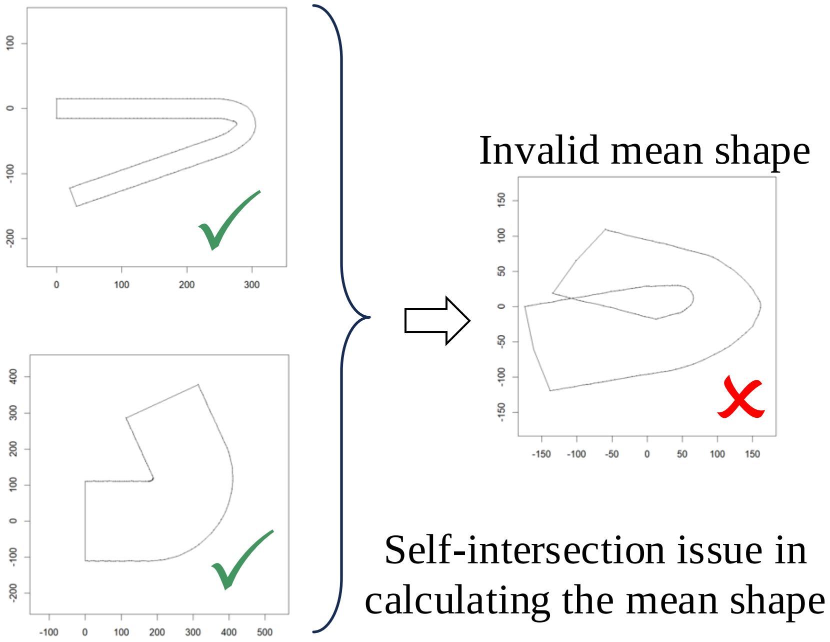


LPDSSRep $s = (F_1^*, \dots, F_{n_f}^*, \boldsymbol{v}_1^*, \dots, \boldsymbol{v}_{n_c}^*, \boldsymbol{u}_1^{\pm *}, \dots, \boldsymbol{u}_{n_f}^{\pm *}, v_1, \dots, v_{n_c}, r_1^{\pm}, \dots, r_{n_f}^{\pm})$

What is the mean?

Calculating a **valid mean shape** is essential as it represents the central tendency of the objects and serves as a fundamental component in statistical analyses like hypothesis testing and classification.





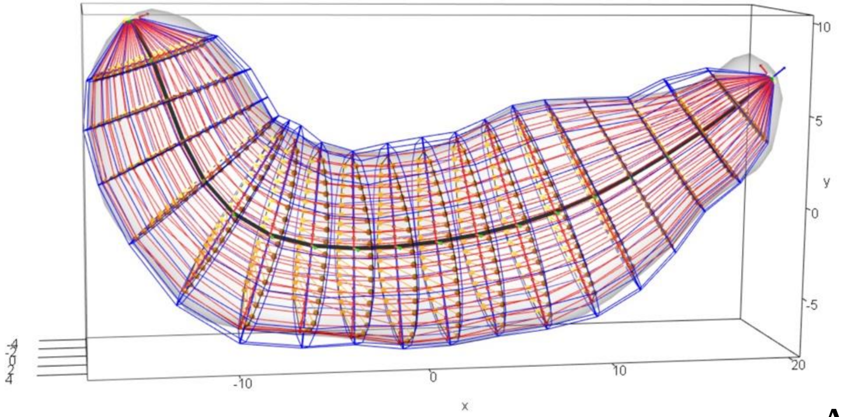
To calculate the SlO mean shape, we need to define the **shape space**, and **distance**. The definition of shape space and distance for SlOs as swept regions is a complex problem because of the problem of **self-intersection**.

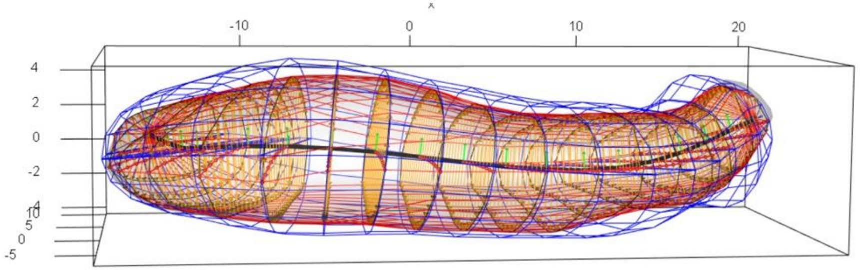
It is possible to calculate the mean for a class of SlOs called **Elliptical SlOs (E-SlOs).**

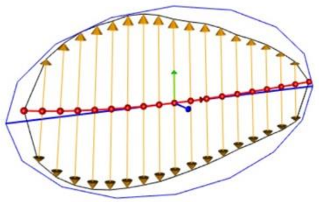
E-SlO & RCC

An **elliptical slabular object (E-SlO)** is a slabular object whose cross-sections can be represented by elliptical disks.

An E-SlO can be seen as an **elliptical tube**.





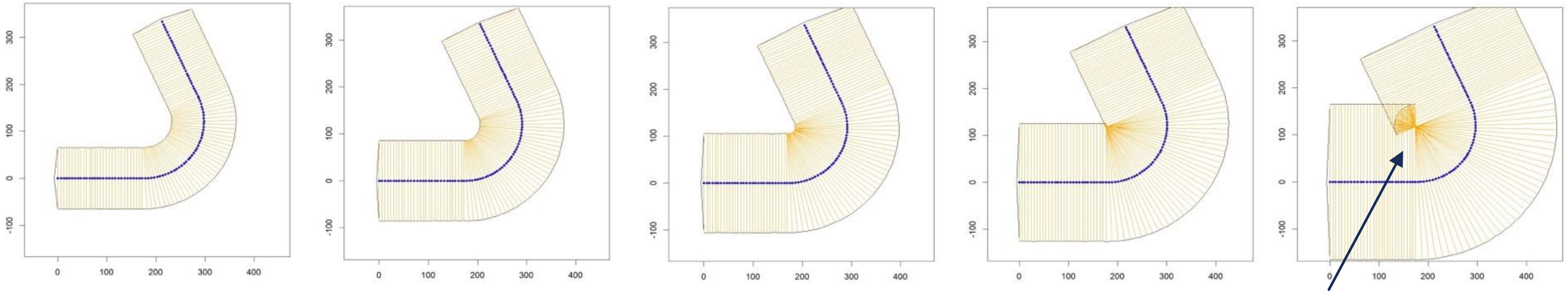


An elliptical cross-section

A hippocampus as an E-SlO.

An **E-SlOs** as a swept region must satisfy **Damon’s relative curvature condition (RCC).**

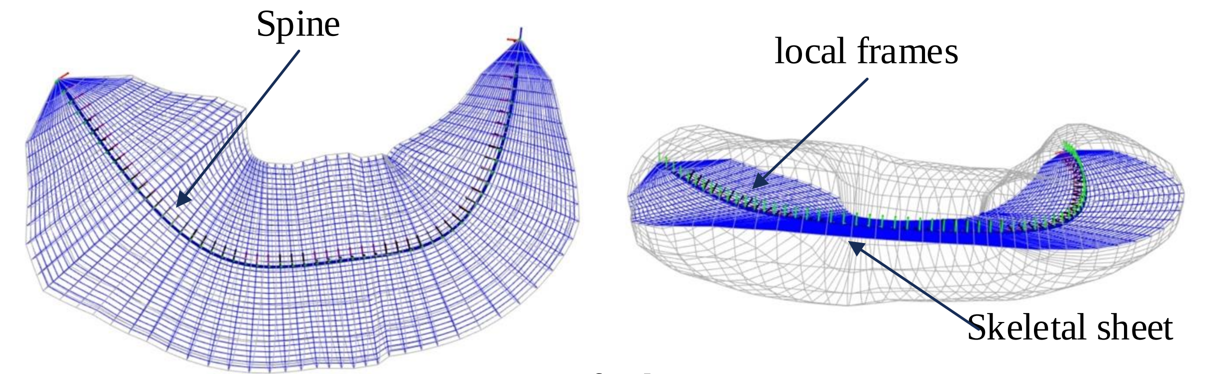
The RCC ($r < \frac{1}{\kappa}$) defines a curvature tolerance for the center curve to ensure the cross-sections do not intersect within the object.



The RCC is violated

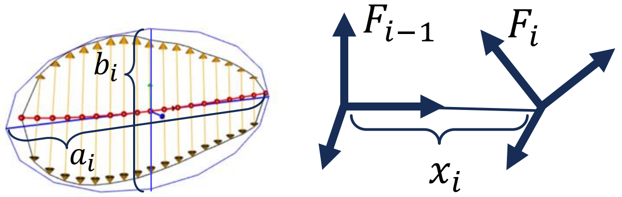
ETRep

An **elliptical tube representation (ETRep)** is a shape representation based on a sequence of elliptical cross-sections.



ETRep of a hippocampus

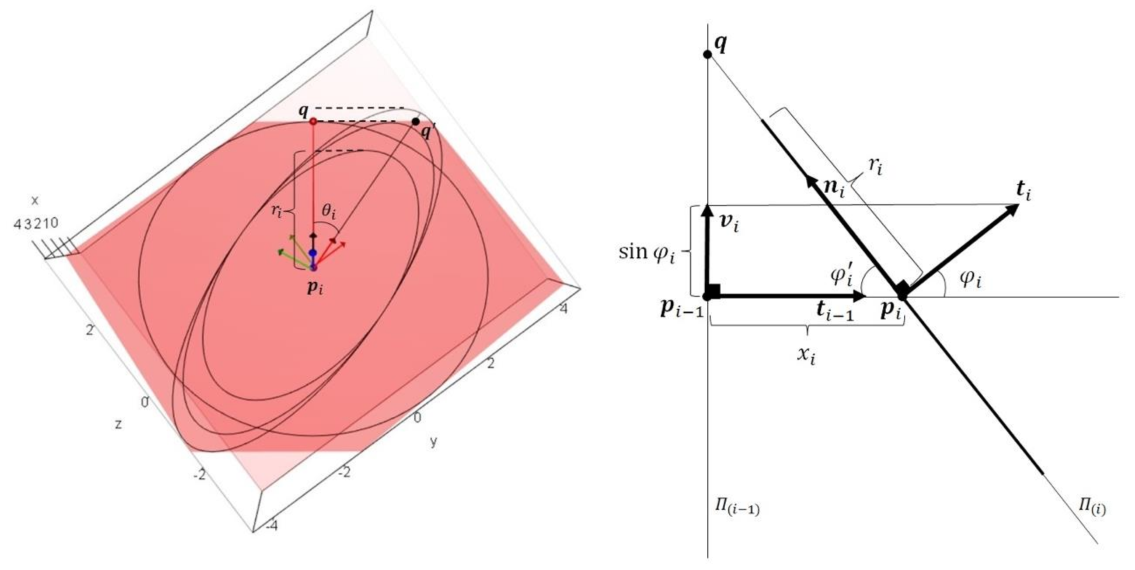
ETRep: $s = (\omega_i)_{i=0}^{n} = ((F_i^*, x_i, a_i, b_i)_i)_{i=0}^{n}$



$(SO(3) \times (\mathbb{R}^+)^3)^{n+1}$

Space

An ETRep is an invarianet shape representation.



Two consequative cross-sections

RCC

$$\forall i;\ r_i \le \frac{x_i}{\sin\varphi_i}$$

r_i: width along the normal

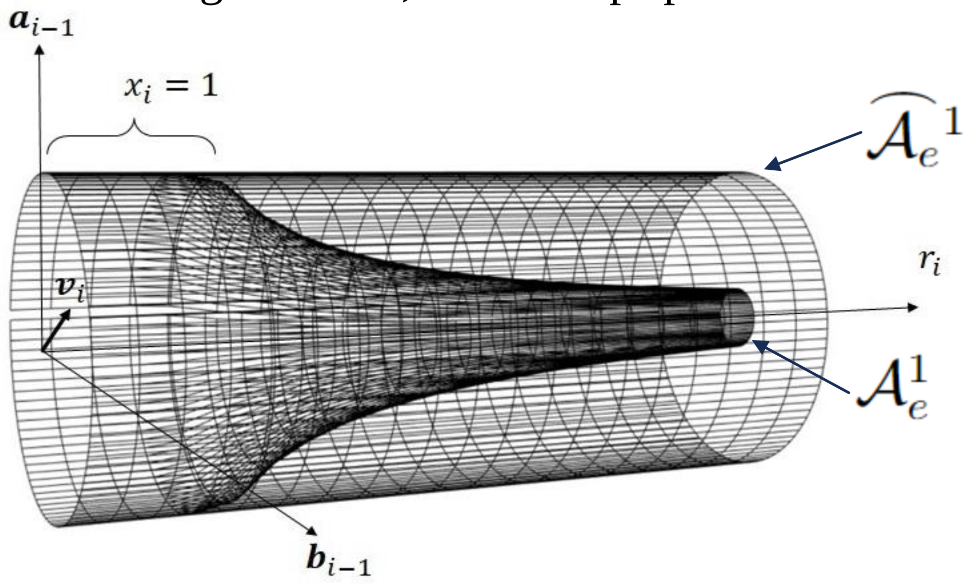
φ_i: bending angle

$$r_i = \left|a_i \cos\left(\tan^{-1}\left(\frac{-b_i}{a_i}\tan\theta_i\right)\right)\cos\theta_i - b_i \sin\left(\tan^{-1}\left(\frac{-b_i}{a_i}\tan\theta_i\right)\right)\sin\theta_i\right|$$

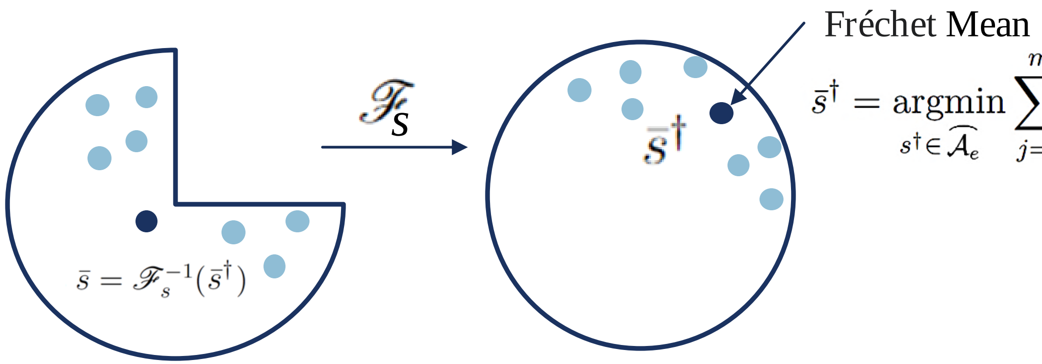
θ_i: degree of twist

ETRep space

By considering the RCC, the ETRep space is a **non-convex high-dimentional space** with a **hyperbolic boundary**.



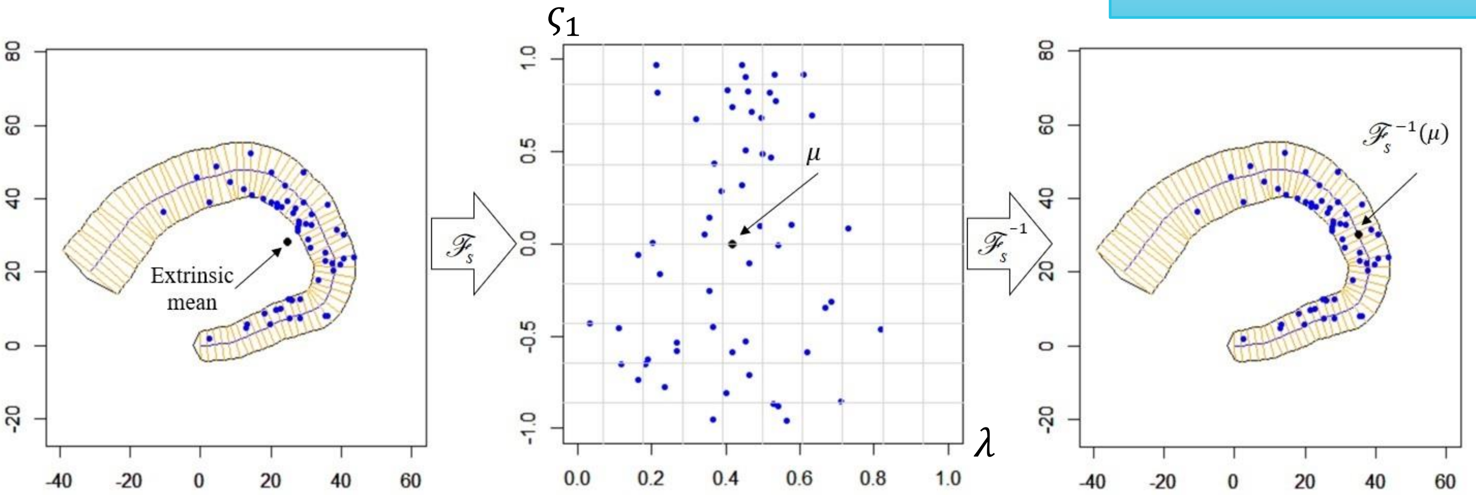
The space of a cross-section.



$$\bar{s}^\dagger = \underset{s^\dagger\in\widehat{\mathcal{A}}_e}{\operatorname{argmin}} \sum_{j=1}^{m} d^2_{s^\dagger}(s^\dagger, s_j^\dagger)$$

$$\mathscr{F}_s(\boldsymbol{w}) : \mathcal{A}_e \to \widehat{\mathcal{A}}_e$$

The mapping can be defined based on the intrinsic skeletal coordinate system of the non-convex space.



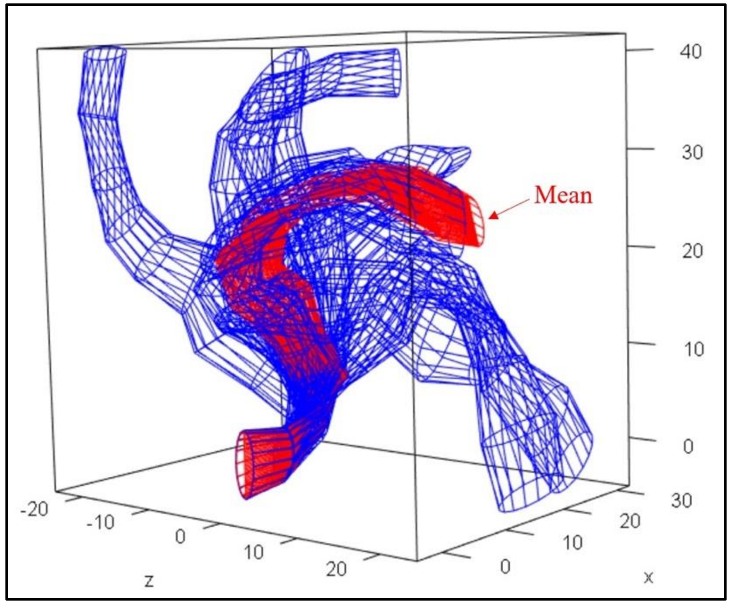
Intrinsic skeletal coordinate system of a 2D swept region

$$\boldsymbol{p} = \boldsymbol{p}_\lambda + \varsigma_1 \overrightarrow{\boldsymbol{p}_\lambda \boldsymbol{q}} \in \Omega_E$$

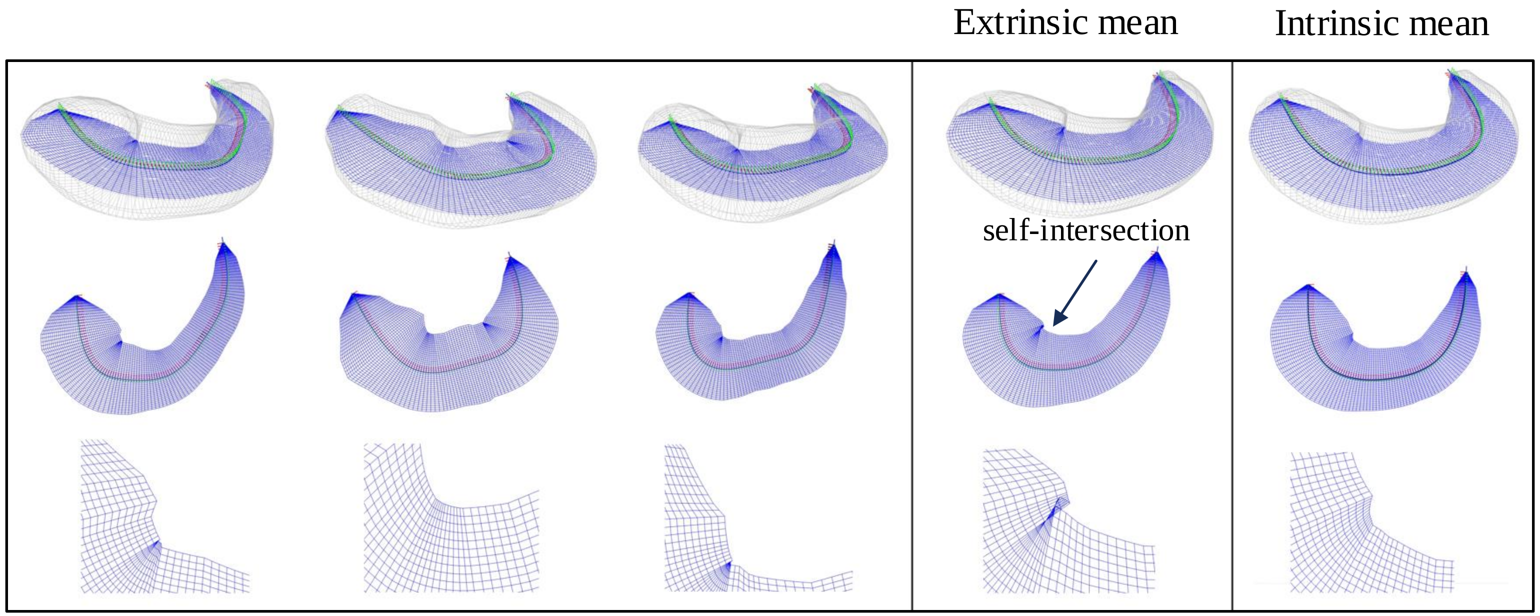
$\lambda \in [0,1]$ is the ratio of the central curve

$$\varsigma_1 = \frac{\|\overrightarrow{p_\lambda p}\|}{\|\overrightarrow{p_\lambda q}\|} \in [0, 1]$$

Example

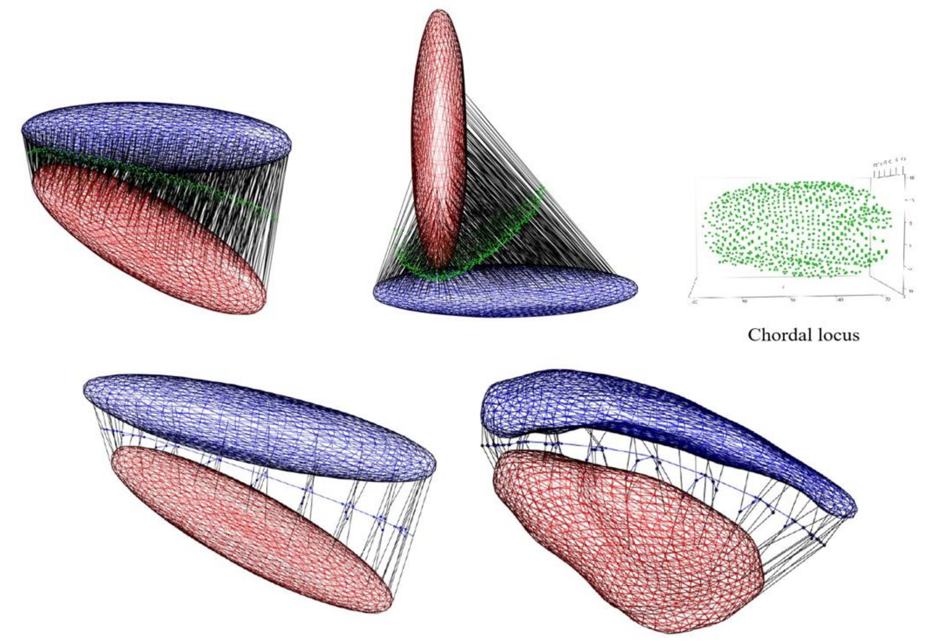


Intrinsic mean of a set of ETReps

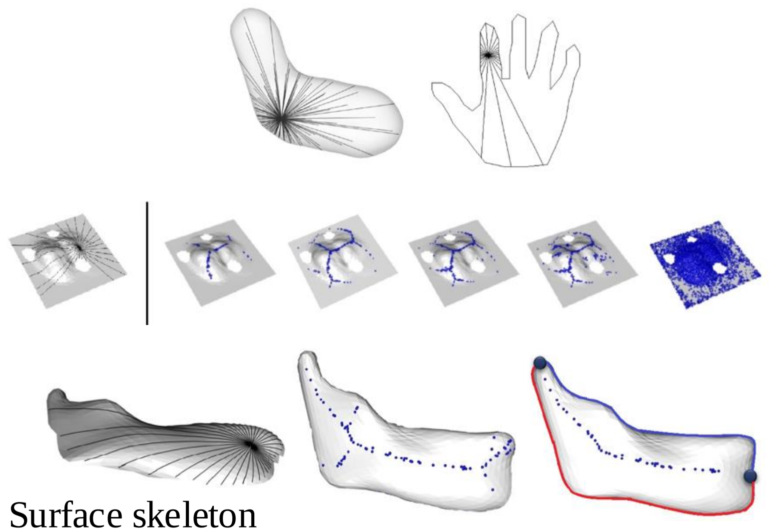


Left column: Three RTReps of three hippocampi. Middle column: The extrinsic mean exhibiting a self-intersection problem. Right column: The intrinsic mean without self-intersection.

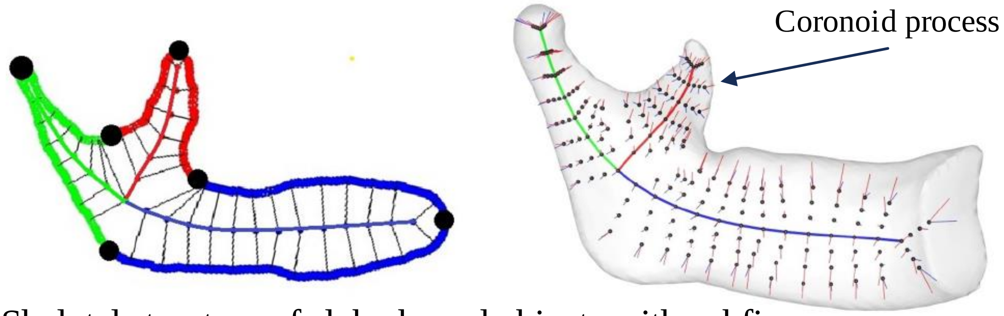
Future works



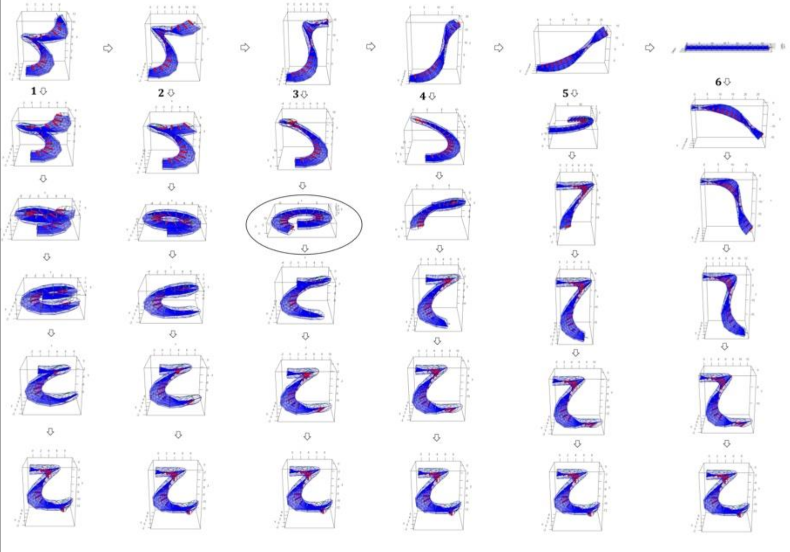
Intermediate LPDSRep



Surface skeleton



Skeletal structure of slab-shaped objects with subfigures



Problem of non-local self-intersection

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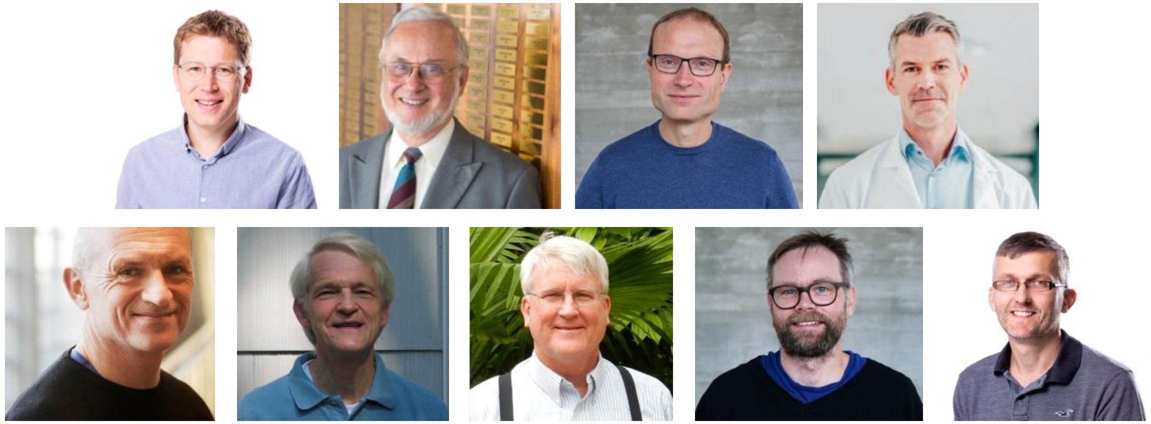

Thank you

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