ASYMPTOTIC PROOFS

CS2009- Design and Analysis of Algorithms

Example: $n^2 + n = O(n^3)$

Proof:

- Here, we have $f(n) = n^2 + n$, and $g(n) = n^3$
- Notice that if $n \geq 1$, $n \leq n^3$ is clear.
- Also, notice that if $n \geq 1$, $n^2 \leq n^3$ is clear.
- **Side Note:** In general, if $a \leq b$, then $n^a \leq n^b$ whenever $n \geq 1$. This fact is used often in these types of proofs.

- Therefore,

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- We have just shown that

$$n^2 + n \leq 2n^3 \text{ for all } n \geq 1$$

- Thus, we have shown that $n^2 + n = O(n^3)$
(by definition of Big-O, with $n_0 = 1$, and $c = 2$.)

Example: $n^3 + 4n^2 = \Omega(n^2)$

Proof:

- Here, we have $f(n) = n^3 + 4n^2$, and $g(n) = n^2$
- It is not too hard to see that if $n \geq 0$,

$$n^3 \leq n^3 + 4n^2$$

- We have already seen that if $n \geq 1$,

$$n^2 \leq n^3$$

Thus when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 4n^2$$

Therefore,

$$1n^2 \leq n^3 + 4n^2 \text{ for all } n \geq 1$$

Thus, we have shown that $n^3 + 4n^2 = \Omega(n^2)$ (by definition of Big-Ω, with $n_0 = 1$, and $c = 1$.)

Example: $n^2 + 5n + 7 = \Theta(n^2)$

Proof:

- When $n \geq 1$,

$$n^2 + 5n + 7 \leq n^2 + 5n^2 + 7n^2 \leq 13n^2$$

- When $n \geq 0$,

$$n^2 \leq n^2 + 5n + 7$$

- Thus, when $n \geq 1$

$$1n^2 \leq n^2 + 5n + 7 \leq 13n^2$$

Thus, we have shown that $n^2 + 5n + 7 = \Theta(n^2)$ (by definition of Big-Θ, with $n_0 = 1$, $c_1 = 1$, and $c_2 = 13$.)

Show that $\frac{1}{2}n^2 + 3n = \Theta(n^2)$

Proof:

- Notice that if $n \geq 1$,

$$\frac{1}{2}n^2 + 3n \leq \frac{1}{2}n^2 + 3n^2 = \frac{7}{2}n^2$$

- Thus,

$$\frac{1}{2}n^2 + 3n = O(n^2)$$

- Also, when $n \geq 0$,

Also, when $n \geq 0$,

$$\frac{1}{2}n^2 \leq \frac{1}{2}n^2 + 3n$$

So

$$\frac{1}{2}n^2 + 3n = \Omega(n^2)$$

Since $\frac{1}{2}n^2 + 3n = O(n^2)$ and $\frac{1}{2}n^2 + 3n = \Omega(n^2)$,

$$\frac{1}{2}n^2 + 3n = \Theta(n^2)$$

Show that $(n \log n - 2n + 13) = \Omega(n \log n)$

Proof: We need to show that there exist positive constants c and n_0 such that

$$0 \le c\, n \log n \le n \log n - 2n + 13 \text{ for all } n \ge n_0.$$

Since $\quad n \log n - 2n \le n \log n - 2n + 13,$

we will instead show that

$$c\, n \log n \le n \log n - 2n,$$

which is equivalent to

$$c \le 1 - \frac{2}{\log n}, \text{ when } n > 1.$$

If $n \ge 8$, then $2/(\log n) \le 2/3$, and picking $c = 1/3$ suffices. Thus if $c = 1/3$ and $n_0 = 8$, then for all $n \ge n_0$, we have

$$0 \le c\, n \log n \le n \log n - 2n \le n \log n - 2n + 13.$$

Thus $(n \log n - 2n + 13) = \Omega(n \log n)$.

Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Proof:

- We need to find positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 n^2 \le \frac{1}{2}n^2 - 3\,n \le c_2 n^2 \text{ for all } n \ge n_0$$

- Dividing by n^2, we get

$$0 \le c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$$

- $c_1 \le \frac{1}{2} - \frac{3}{n}$ holds for $n \ge 10$ and $c_1 = 1/5$
- $\frac{1}{2} - \frac{3}{n} \le c_2$ holds for $n \ge 10$ and $c_2 = 1$.
- Thus, if $c_1 = 1/5$, $c_2 = 1$, and $n_0 = 10$, then for all $n \ge n_0$,

$$0 \le c_1 n^2 \le \frac{1}{2}n^2 - 3\,n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Thus we have shown that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.

- Prove $5n\log_2 n + 8n - 200 = O(n\log_2 n)$

$$\begin{aligned} 5n\log_2 n + 8n - 200 &\le 5n\log_2 n + 8n \\ &\le 5n\log_2 n + 8n\log_2 n \quad \text{for } n \ge 2 \ (\log_2 n \ge 1) \\ &\le 13n\log_2 n \end{aligned}$$

- $5n\log_2 n + 8n - 200 \le 13n\log_2 n$ for all $n \ge 2$
- $5n\log_2 n + 8n - 200 = O(n\log_2 n)$ [$c = 13,\ n_0 = 2$]

- Prove $n^2 + 42n + 7 = O(n^2)$
-
$$\begin{aligned} n^2 + 42n + 7 &\leq n^2 + 42n^2 + 7n^2 \quad \text{for } n \geq 1 \\ &= 50n^2 \end{aligned}$$
- So, $n^2 + 42n + 7 \leq 50n^2$ for all $n \geq 1$
- $n^2 + 42n^2 + 7n^2 = O(n^2)$ [$c = 50,\ n_0 = 1$]

Decide whether these statements are **True** or **False**. You must briefly justify all your answers to receive full credit.

1. If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $h(n) = \Theta(f(n))$

   Solution: True. Θ is transitive.

2. If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $h(n) = \Omega(f(n))$

   Solution: True. O is transitive, and $h(n) = \Omega(f(n))$ is the same as $f(n) = O(h(n))$

3. If $f(n) = O(g(n))$ and $g(n) = O(f(n))$ then $f(n) = g(n)$

   Solution: False: $f(n) = n$ and $g(n) = n + 1$.

4. $\dfrac{n}{100} = \Omega(n)$

   Solution: True. $\frac{n}{100} < c * n$ for $c = \frac{1}{200}$.

T F For all positive f(n), $f(n) + o(f(n)) = \Theta(f(n))$.

Let $f(n) = n^2$

Then, $n^2 + o(n^2) = \Theta(n^2)$, For small o, $f(n) < cg(n)$ i.e. $o(n^2)$ should be less than n^2. Thus, Equation is True

T F For all positive f(n), g(n) and h(n), if $f(n) = O(g(n))$ and $f(n) = \Omega(h(n))$, then $g(n) + h(n) = \Omega(f(n))$

$f(n) = n$, $g(n) = n\log n$, $h(n) = \log n$, Thus, $n\log n + \log n = \Omega(n)$, Thus Equation is True

T F If $f(n)= O(g(n))$ and $f(n)= \Omega(g(n))$, then we have $(f(n))^2 = \Theta((g(n))^2)$

If $f(n) = 2n$, g(n) can be n or (2n-1) or any equation with linear n in order satisfy both $f(n)= O(g(n))$ and $f(n)= \Omega(g(n))$ simultaneously. Thus True

T F If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then we have $f(n) = g(n)$

From above statement, it is clear that f(n) and g(n) can be different