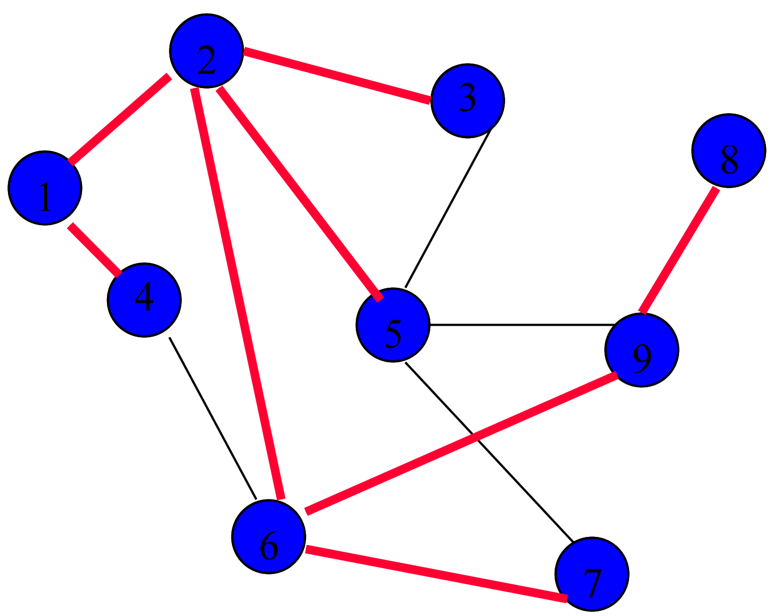
Spanning Tree



Breadth-first search from vertex 1.

Breadth-first spanning tree.

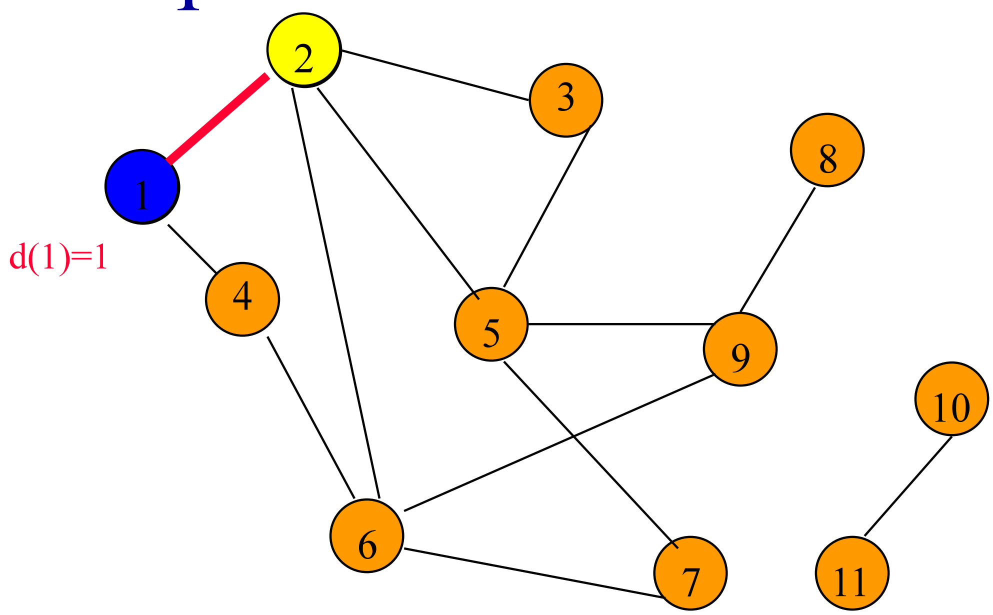
Spanning Tree

- Start a breadth-first search at any vertex of the graph.
- If graph is connected, the n-1 edges used to get to unvisited vertices define a spanning tree (breadth-first spanning tree).
- Time
  - $O(n^2)$ when adjacency matrix used
  - $O(n+e)$ when adjacency lists used (e is number of edges)

Depth-First Search

```
DFS(G)
for each vertex u ∈ V [G]
      do color[u] ← WHITE
         π[u] ← NIL
time ← 0
for each vertex u ∈ V [G]
      do if color[u] = WHITE
            then DFS-VISIT(u)
DFS-VISIT(u)
color[u] ← GRAY       ▹White vertex u has just been discovered
time ← time +1
d[u]←time
for each v ∈ Adj[u]   ▹Explore edge(u, v).
      do if color[v] = WHITE
            then π[v] ← u
                 DFS-VISIT(v)
color[u]←BLACK        ▹ Blacken u; it is finished.
f [u] ▹ time ← time +1
```

Depth-First Search Example

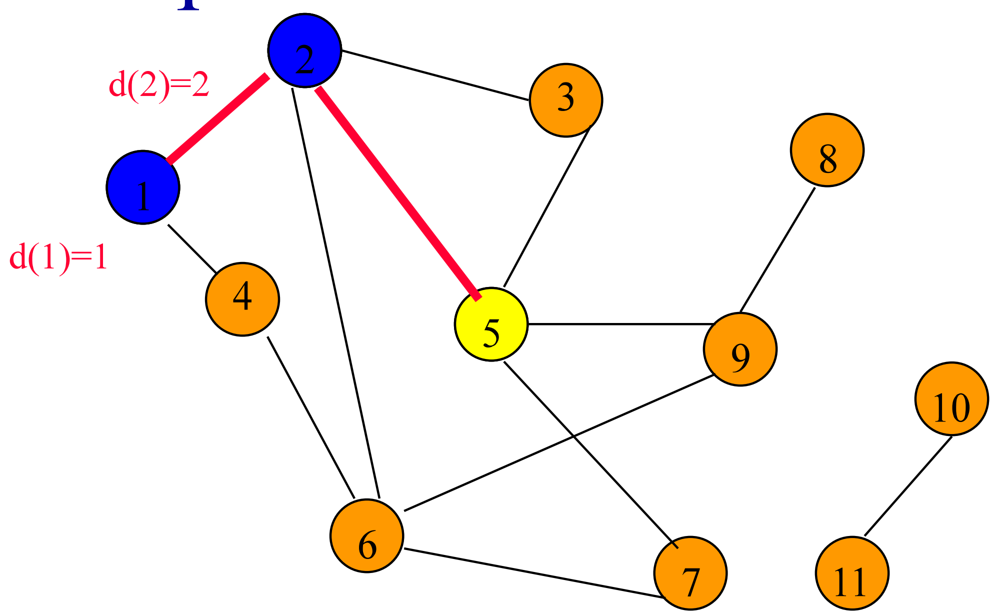


Start search at vertex 1.

Label vertex 1 and do a depth first search from either 2 or 4.

Suppose that vertex 2 is selected.

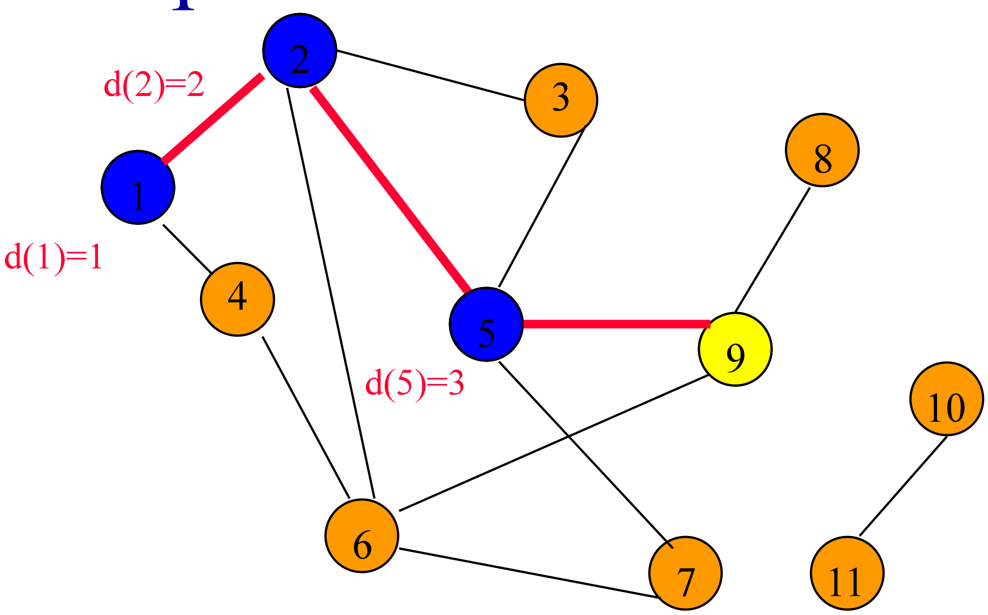
Depth-First Search Example



Label vertex 2 and do a depth first search from either 3, 5, or 6.

Suppose that vertex 5 is selected.

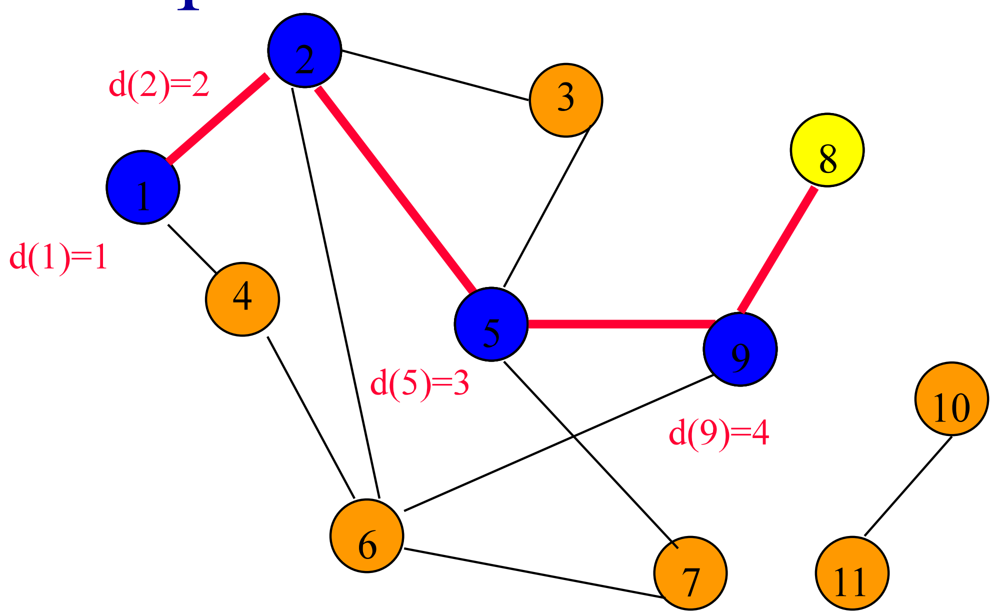
Depth-First Search Example



Label vertex 5 and do a depth first search from either 3, 7, or 9.

Suppose that vertex 9 is selected.

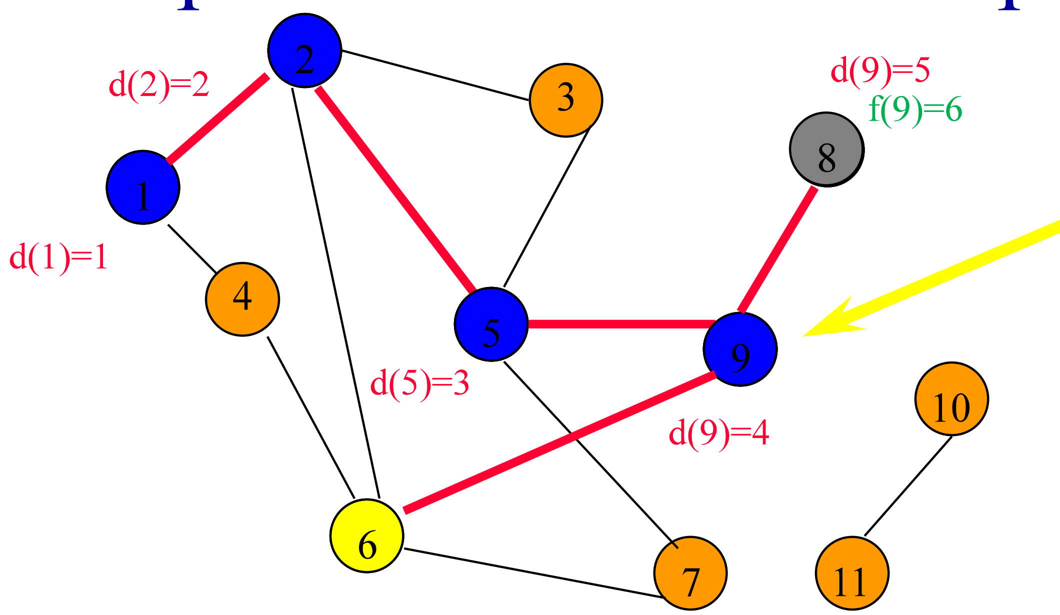
Depth-First Search Example



Label vertex 9 and do a depth first search from either 6 or 8.

Suppose that vertex 8 is selected.

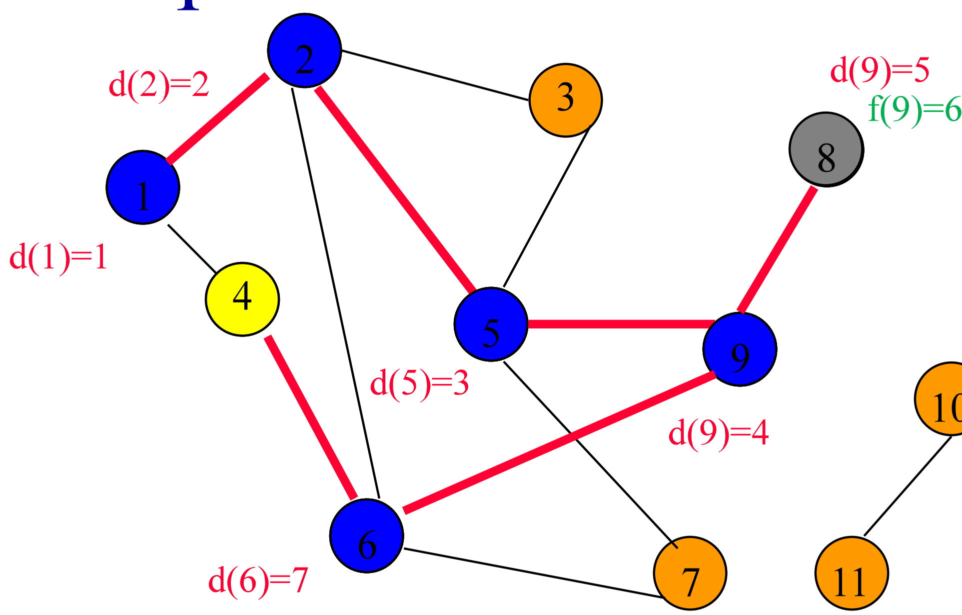
Depth-First Search Example



Label vertex 8 and return to vertex 9.

From vertex 9 do a dfs(6).

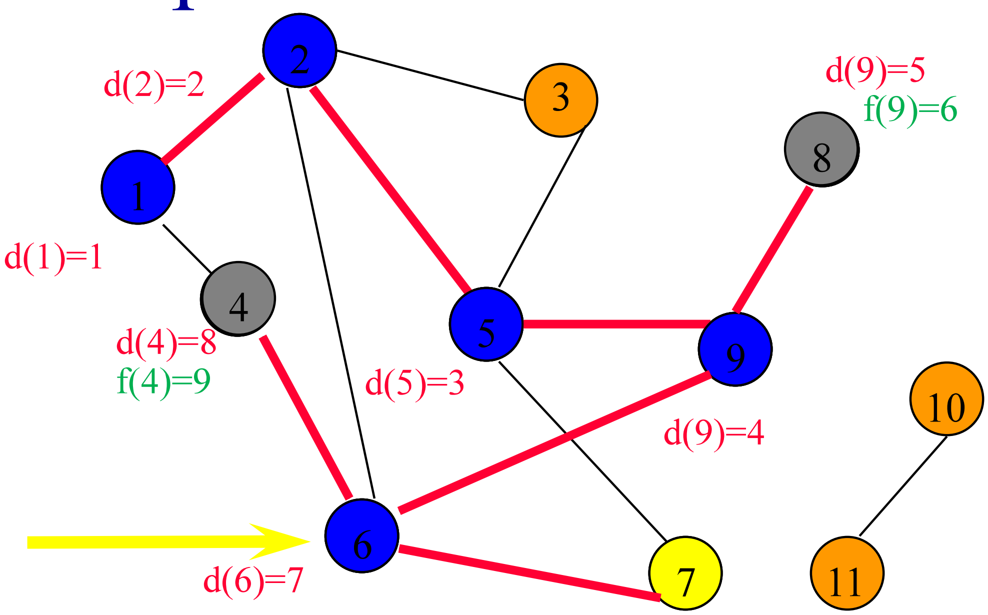
Depth-First Search Example



Label vertex 6 and do a depth first search from either 4 or 7.

Suppose that vertex 4 is selected.

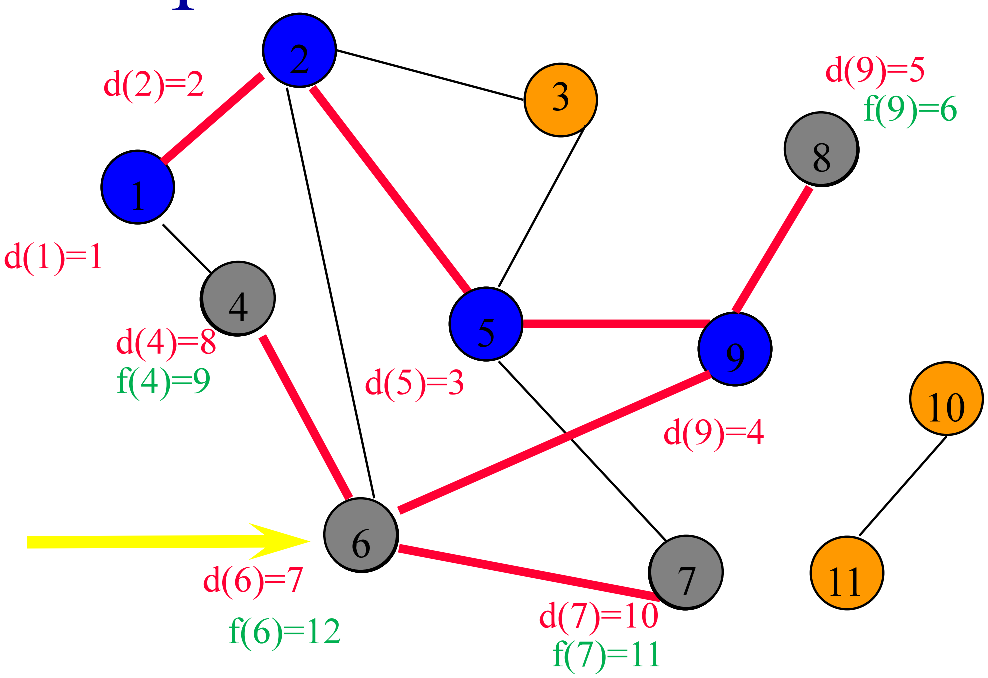
Depth-First Search Example



Label vertex 4 and return to 6.

From vertex 6 do a dfs(7).

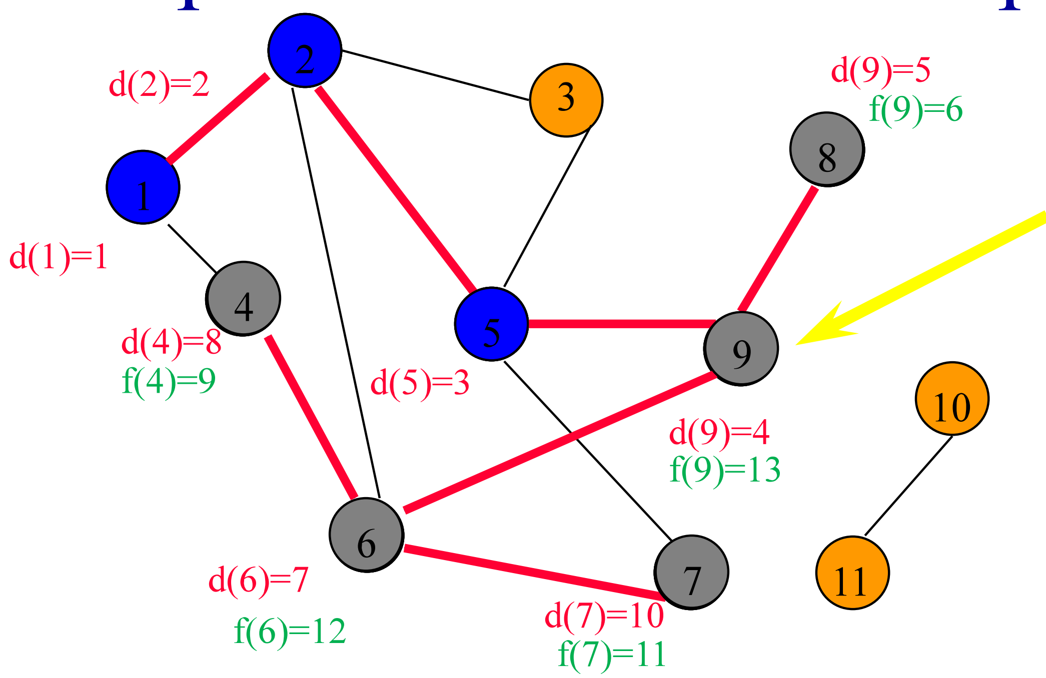
Depth-First Search Example



Label vertex 7 and return to 6.

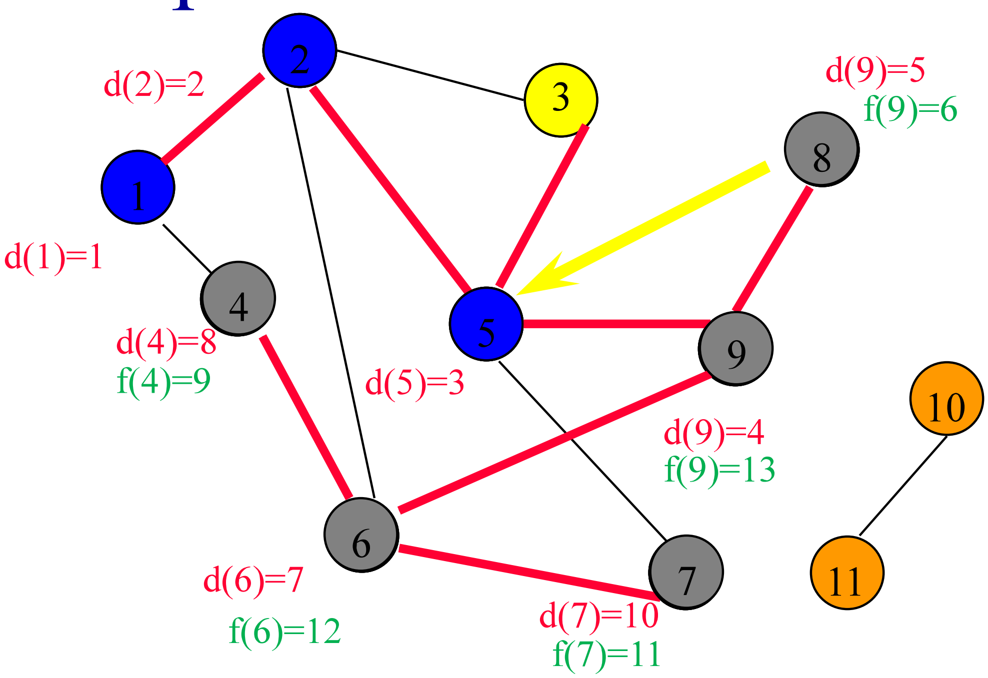
Return to 9.

Depth-First Search Example



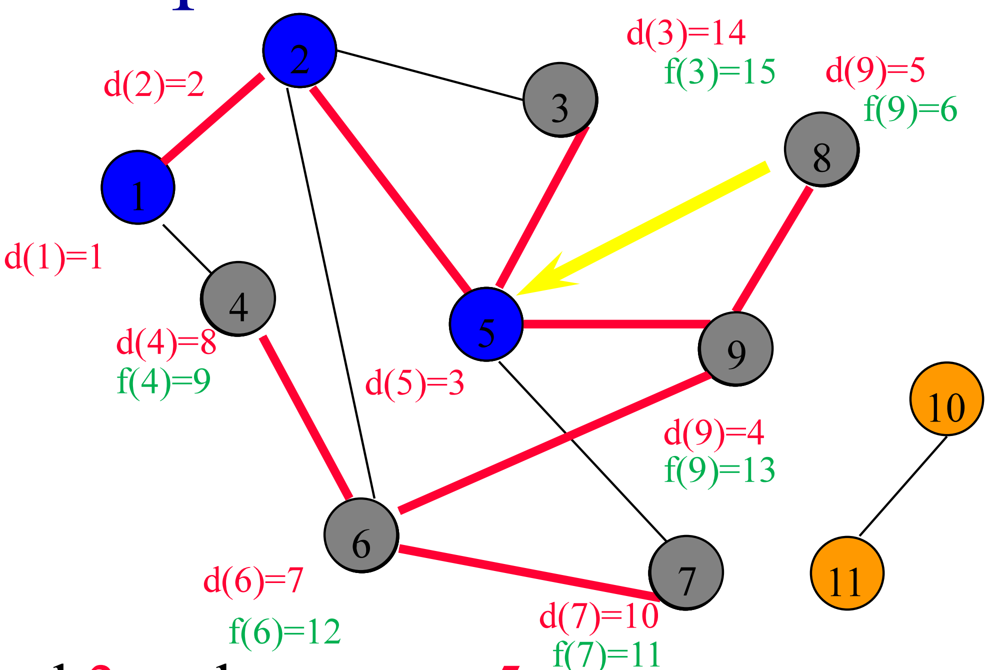
Return to 5.

Depth-First Search Example



Do a dfs(3).

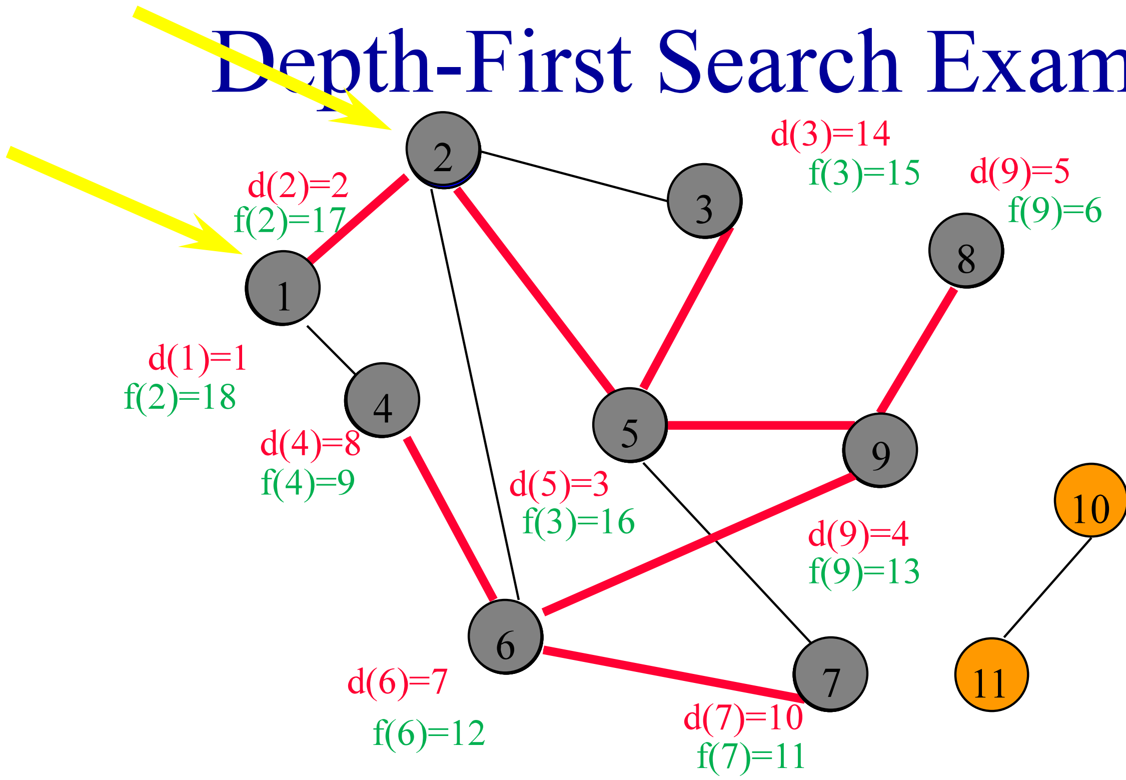
Depth-First Search Example



Label 3 and return to 5.

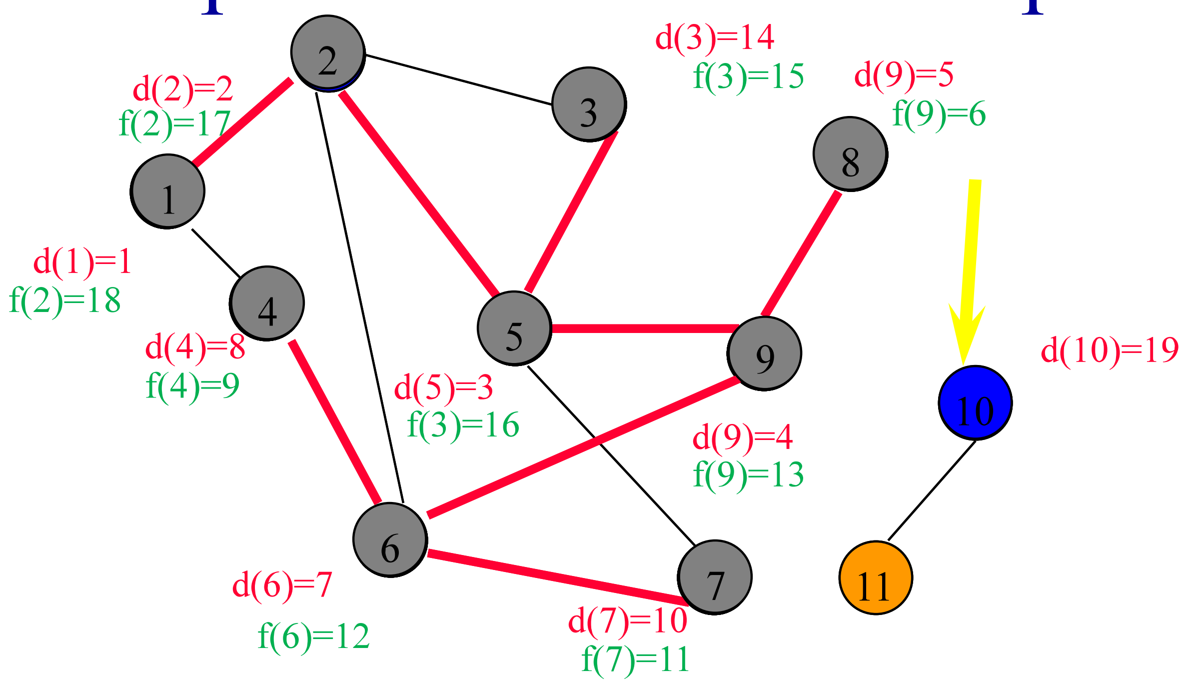
Return to 2.

Depth-First Search Example



Return to 1.

Depth-First Search Example



Return to invoking method.

Depth-First Search Properties

- Same complexity as BFS.
- Same properties with respect to path finding, connected components, and spanning trees.
- Edges used to reach unlabeled vertices define a depth-first spanning tree when the graph is connected.
- There are problems for which BFS is better than DFS and vice versa.

Depth-First Search Properties

Theorem : (Parenthesis theorem)

In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

- the intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest,
- the interval $[d[u], f[u]]$ is contained entirely within the interval $[d[v], f[v]]$, and u is a descendant of v in a depth-first tree, or
- the interval $[d[v], f[v]]$ is contained entirely within the interval $[d[u], f[u]]$, and v is a descendant of u in a depth-first tree.

Depth-First Search Properties

Corollary : (Nesting of Descendants' Intervals)

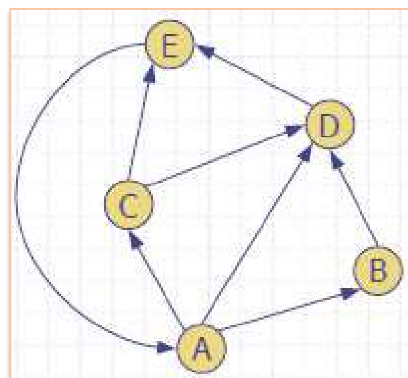
Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if $d[u] < d[v] < f[v] < f[u]$.

Theorem : (White-path theorem)

In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $d[u]$ that the search discovers u, vertex v can be reached from u along a path consisting entirely of white vertices.

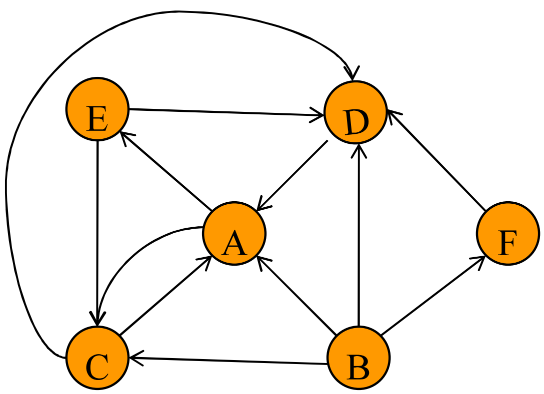
Directed Graphs (Digraphs)

- A **digraph is a graph**
  – whose edges are all directed.
- Short for "directed graph"
- Application
  – Scheduling: edge (A,B) means task A must be completed before B can be started.



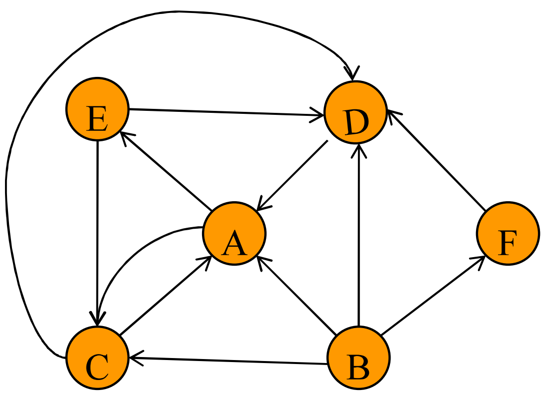
Example: DFS on Directed Graphs

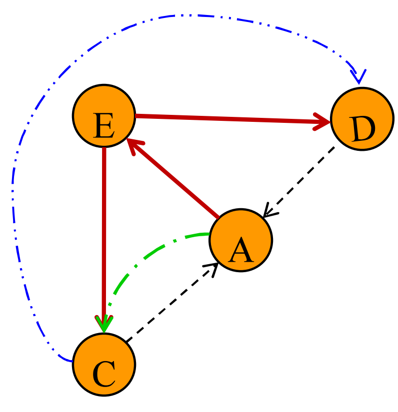
- DFS starts at A, then B,…



Example: DFS on Directed Graphs

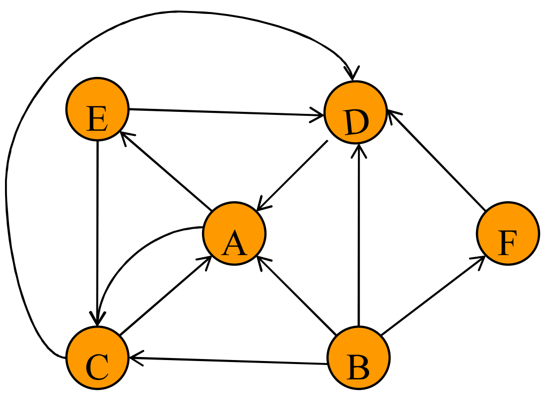
- DFS starts at A, then B,…

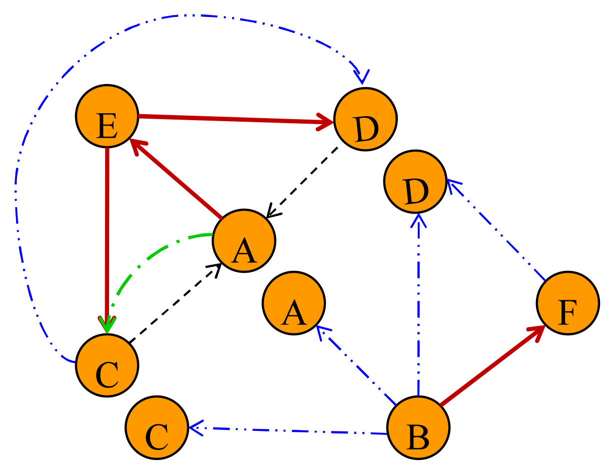




Example: DFS on Directed Graphs

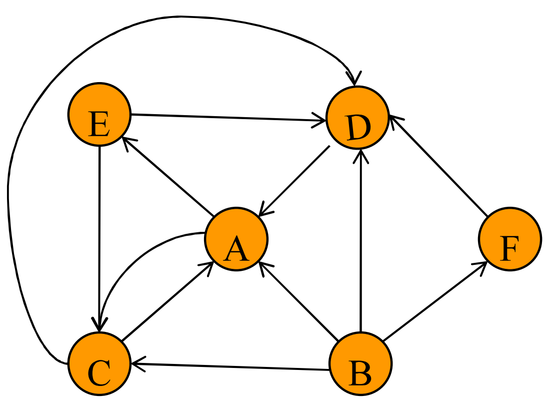
- DFS starts at A, then B,…

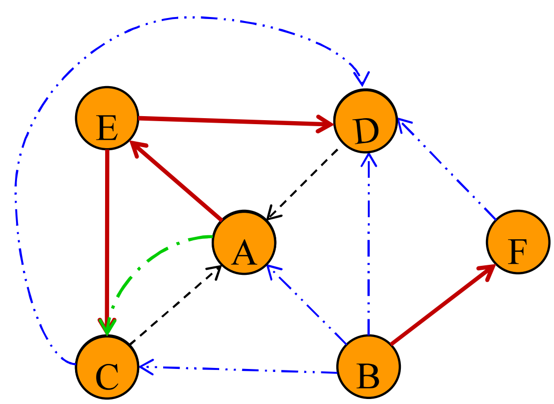




Example: DFS on Directed Graphs

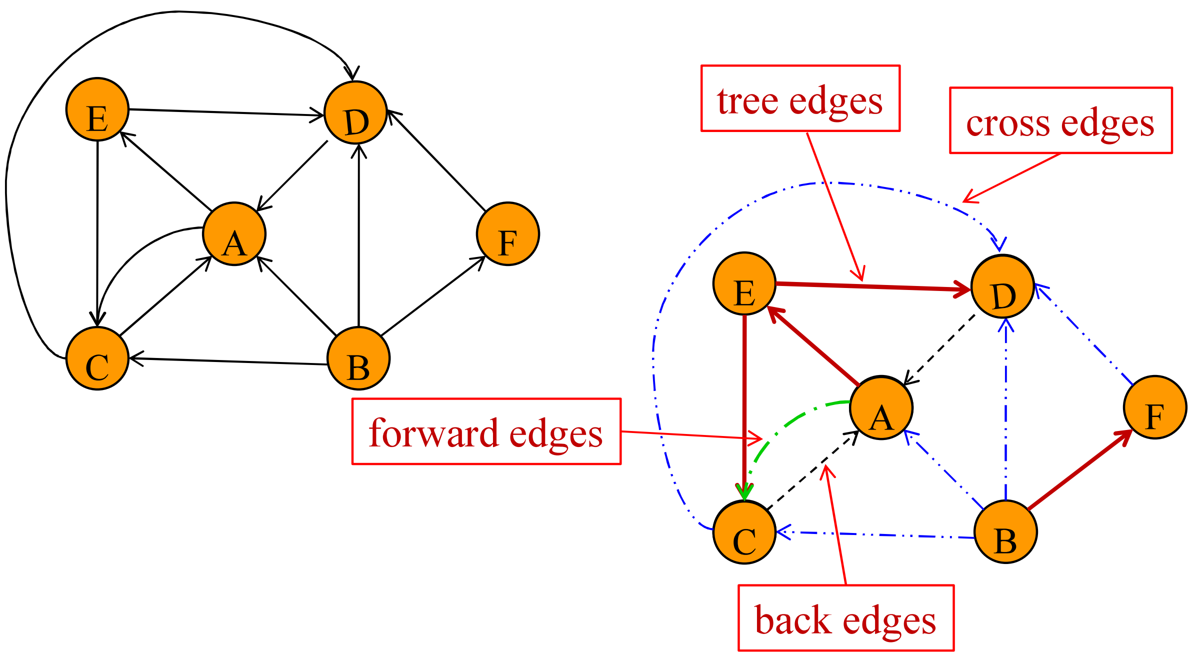
- DFS starts at A, then B,…





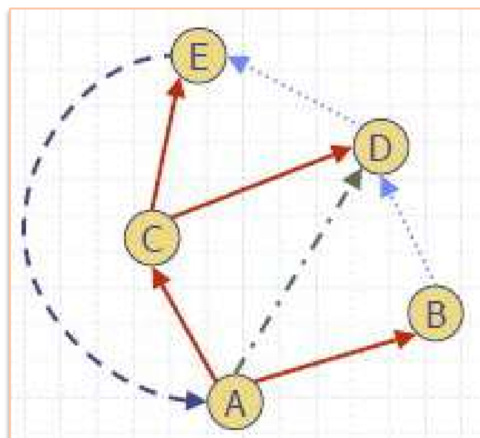
Example: DFS on Directed Graphs

- DFS starts at A, then B,…

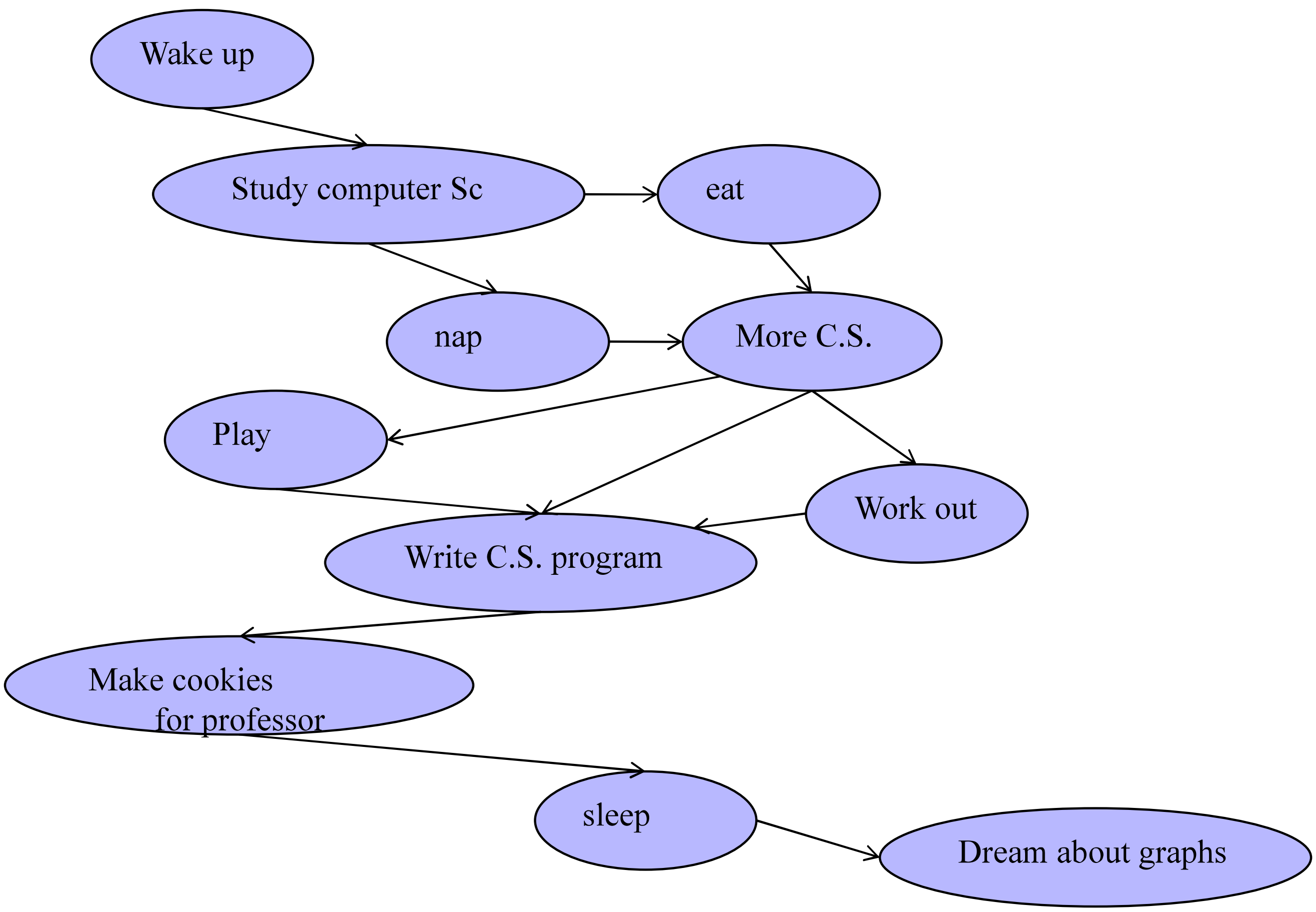


Classification of edges

- ***Tree edges*** are edges in the depth-first forest Gπ. Edge (*u, v*) is a tree edge if v was first discovered by exploring edge (*u, v*).
- ***Back edges*** are those edges (*u, v*) connecting a vertex u to an ancestor v in a depthfirst tree. Self-loops, which may occur in directed graphs, are considered to be back edges.
- ***Forward edges*** are those nontree edges (*u, v*) connecting a vertex u to a descendant v in a depth-first tree.
- ***Cross edges*** are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

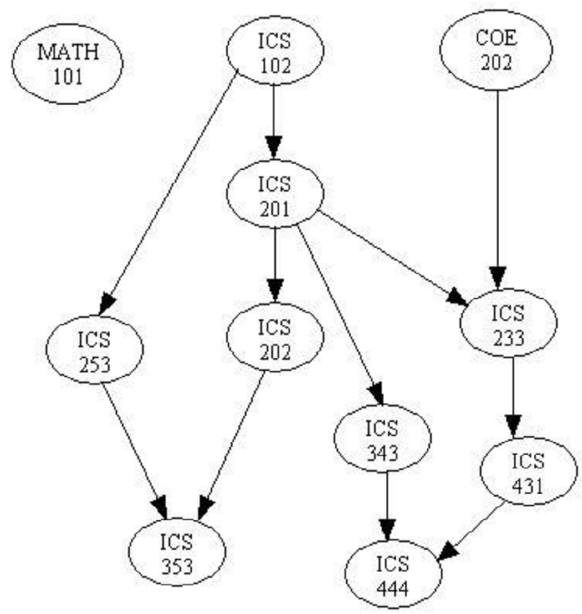


Example of digraph: A typical student day



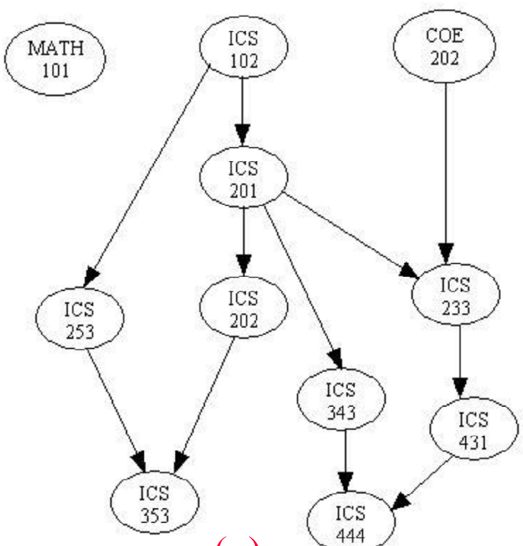
Linear Ordering

- There are many problems involving a set of tasks in which some of the tasks must be done before others.
- For example, consider the problem of taking a course only after taking its prerequisites.
- Is there any systematic way of linearly arranging the courses in the order that they should be taken ?
- Yes ! - Topological sort.

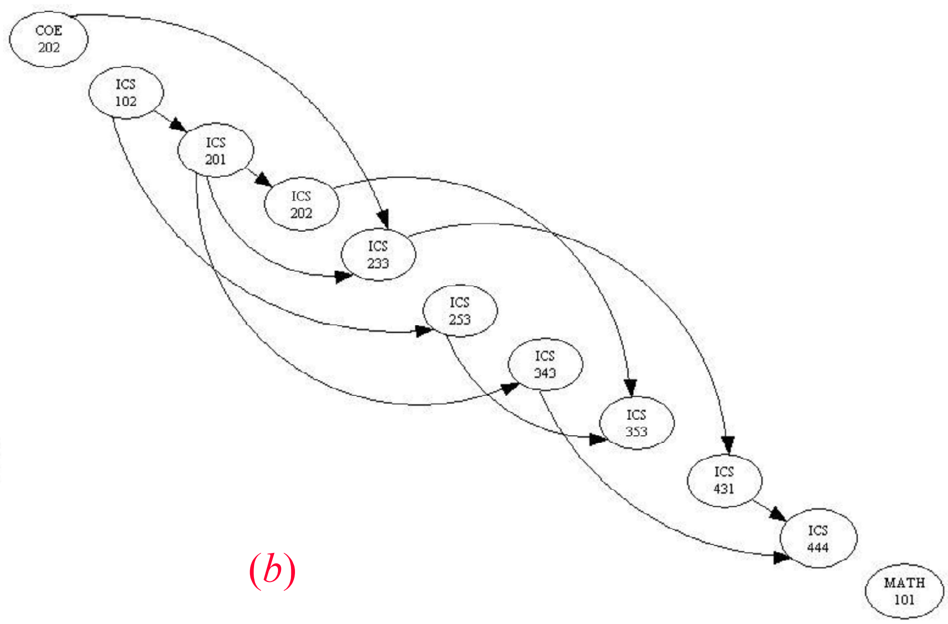


Definition of Topological Sort

- Topological sort is a method of arranging the vertices in a directed acyclic graph (DAG), as a sequence, such that no vertex appear in the sequence before its predecessor.
- The graph in (*a*) can be topologically sorted as in (*b*)



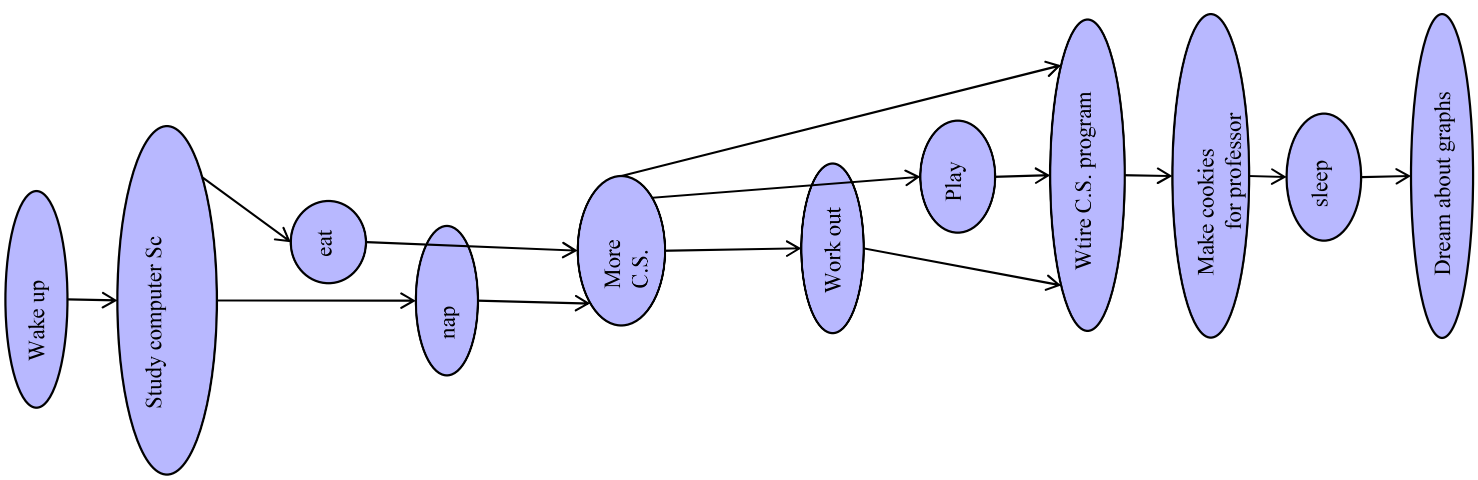
(*a*)



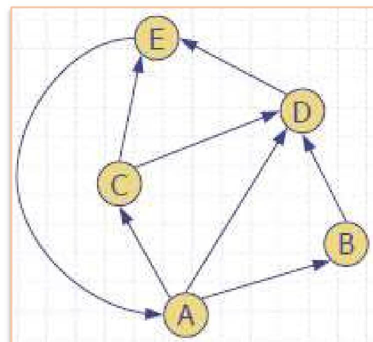
(*b*)

Topological Sorting

Number vertices, so that (u,v) in E implies $u < v$



Topological Sorting



- A directed acyclic graph (DAG) is a digraph that has no directed cycles.
- A topological ordering of a digraph is a numbering $v_1, \ldots, v_n$ of the vertices such that for every edge (v_i, v_j), we have $i < j$.

Theorem: A digraph admits a topological ordering if and only if it is a DAG.

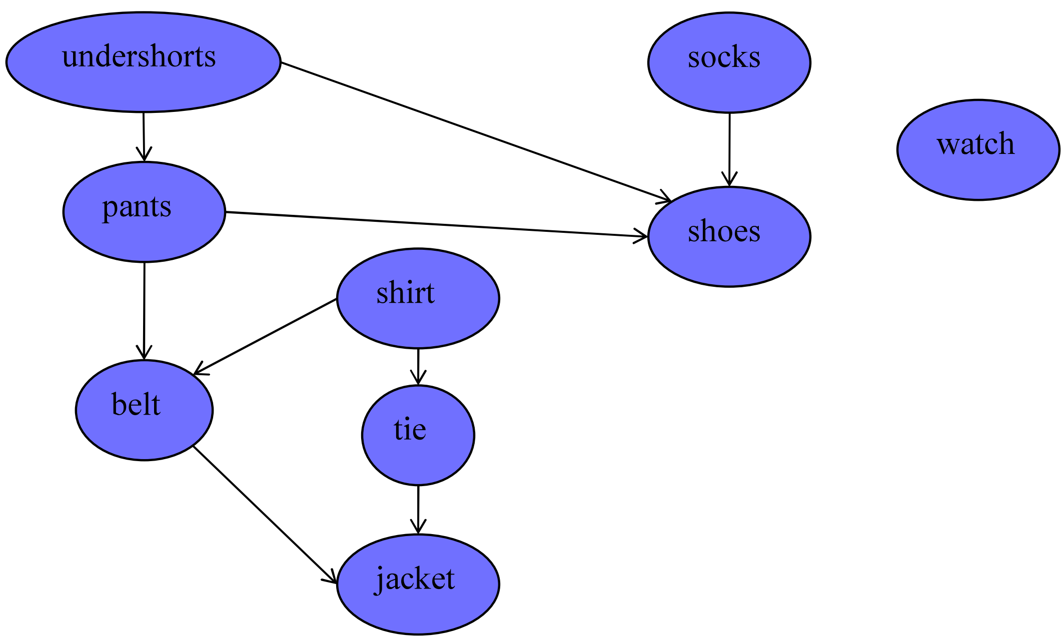
Lemma: A digraph is DAG if and only if a DFS of G yields no back edges.

Algorithm for Topological Sorting

Topological Sort(G)

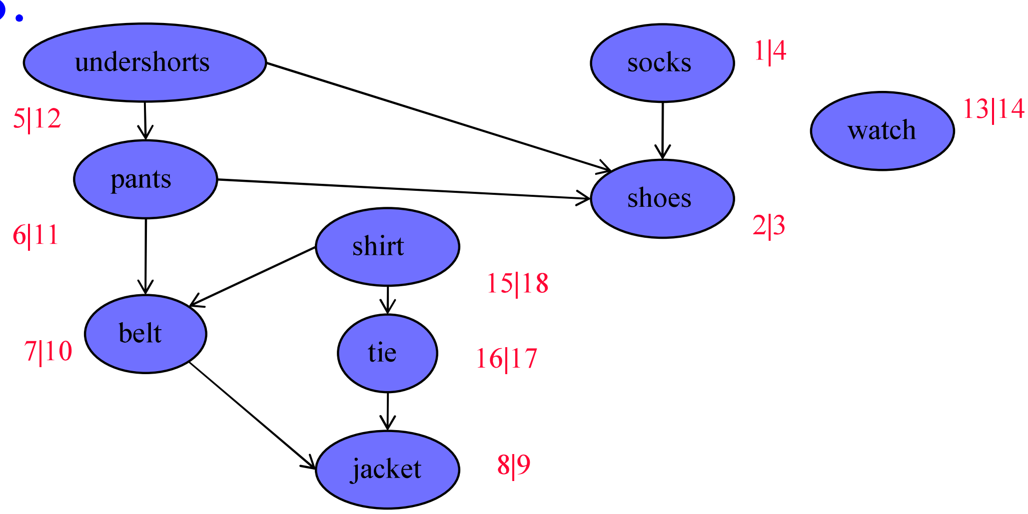
1. Call **DFS(G)** to compute finishing time *f*[*v*] for each vertex *v*.
2. As each vertex is finished, insert it onto the front of a linked list.
3. **Return** the linked list of vertices.

Example: Digraph

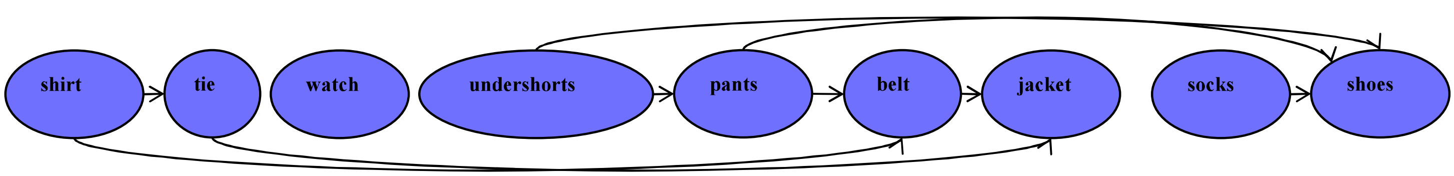


Example: Topological Sorting

DFS:



Events ordering after Topological Sort:



Time complexity for topological sort

- Runtime for a topological sort is O(V+E)
- Since DFS takes O(V+E) time
- and O(1) time to insert each of the |V| vertices onto the front of the linked list.