All-Pairs Shortest Paths Problem
The Floyd-Warshall Algorithm

Given a weighted digraph $G = (V, E)$ with weight function w: $E \rightarrow R$, (R, is the set of real numbers) determine the length of the shortest path (i.e., distance between all pairs of vertices in G). Here we assume that there are no cycles with zero or negative cost.

Decomposition

- **Definition:** The vertices $v_2, v_2, \ldots, v_{l-1}$ are called the intermediate vertices of the path $p = \{v_1, v_2, v_2, \ldots, v_{l-1}, v_l\}$.
- Let $c^{(k)}_{ij}$ be the length of the shortest path from i to j such that all intermediate vertices on the path (in any) are in set $\{1, 2, \ldots, k\}$.
- $c^{(0)}_{ij}$ is set to be w_{ij}, i.e., no intermediate vertex.
- Let $D^{(k)}$ be the $n \times n$ matrix $[c^{(k)}_{ij}]$

Decomposition

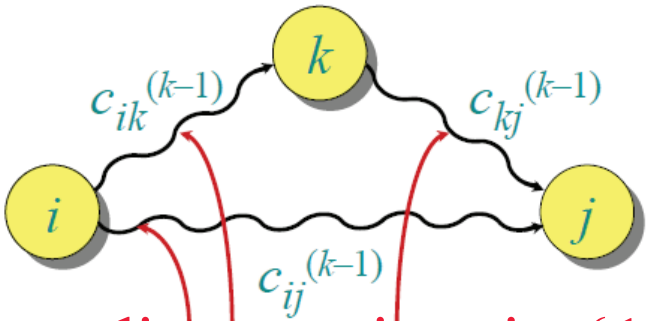
- $c^{(k)}_{ij}$ is the distance from i to j.
- our aim is to compute $D^{(n)}$.
- Subproblems:
  - compute $D^{(k)}$ for $k = 0, 1, \ldots, n$.

Structure of shortest paths

- **Observation 1:** A shortest path does not contain the same vertex twice.
- **Observation 2:** For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1,2,\ldots,k\}$ there are two possibilities:
  - k is not a vertex on the path, the shortest such path has length $c^{(k-1)}_{ij}$
  - k is a vertex on the path, the shortest such path has length $c^{(k-1)}_{ik} + c^{(k-1)}_{kj}$.

Floyd-Warshall recurrence

- Consider a shortest path from i to j containing the vertex k. It consists of a subpath from i to k and k to j.
- Each subpath can only contain intermediate vertices in $\{1,2,\ldots, k\text{-}1\}$ and must be as short as possible, namely they have lengths $c^{(k\text{-}1)}_{ik}$ and $c^{(k\text{-}1)}_{kj}$.
- Hence the path has length $c^{(k\text{-}1)}_{ik} + c^{(k\text{-}1)}_{kj}$.

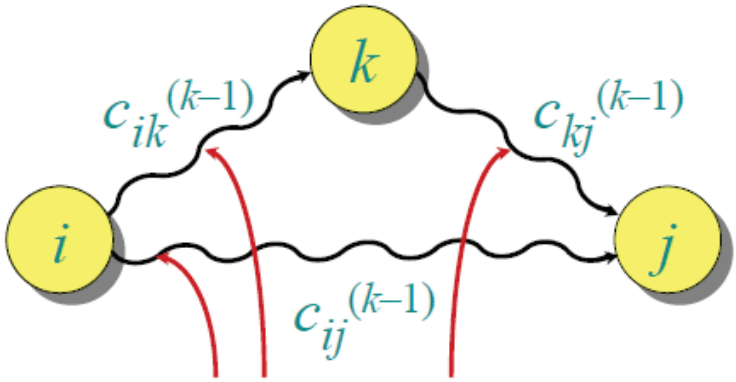


intermediate vertices in $\{1, 2, \ldots, k\}$

Floyd-Warshall recurrence

- Combining the two cases we get

$$c^{(k)}_{ij} = \min_k \{c^{(k-1)}_{ij}, c^{(k-1)}_{ik} + c^{(k-1)}_{kj}\}$$



intermediate vertices in $\{1, 2, \ldots, k\}$

Bottom-up Computation

- Bottom: $D^{(0)}$

  $= [w_{ij}]$ the weight matrix.
- Compute $D^{(k)}$ from $D^{(k\text{-}1)}$ using

  $c^{(k)}_{ij} = \min\{c^{(k\text{-}1)}_{ij},\ c^{(k\text{-}1)}_{ik} + c^{(k\text{-}1)}_{kj}\}$ for $k = 1,2,\ldots,n$

Pseudocode for Floyd-Warshall

Floyd-Warshall(W)

$n \leftarrow$ rows[W]

$D^{(0)} \leftarrow W$

for $k \leftarrow 1$ to n

    do for $i \leftarrow 1$ to n

        do for $j \leftarrow 1$ to n

            do $c^{(k)}_{ij} \leftarrow \min\{c^{(k-1)}_{ij}, c^{(k-1)}_{ik} + c^{(k-1)}_{kj}\}$

return $D^{(n)}$

Extracting the Shortest Paths

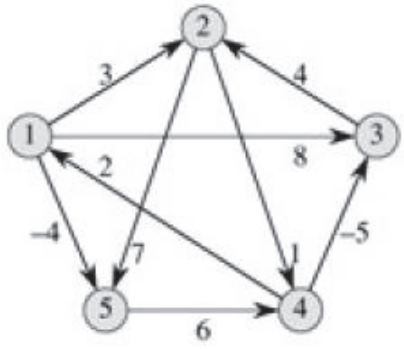
- Construct a predecessor matrix Π from the D matrix.
- Given the predecessor matrix Π, **print-all-pairs-shortest-path** procedure can use to print vertices on a given shortest path.
- The predecessor pointers π_{ij} can be used to extract the final path. The idea is as follows.
- Whenever we discover that the shortest path from i to j passes through an intermediate vertex k, we set $\pi_{ij} = k$
- If the shortest path does not pass through any intermediate vertex, then $\pi_{ij} = \text{NIL}$.

Pseudocode for Floyd-Warshall (Extracting the Shortest Paths)

Floyd-Warshall(W)
$n \leftarrow$rows[W]
$C^{(0)} \leftarrow W = [w_{ij}]$
$\Pi^{(0)} \leftarrow [\pi^{(0)}_{ij}]$
for $k \leftarrow 1$ to n
    do for $i \leftarrow 1$ to n
        do for $j \leftarrow 1$ to n
            do $c^{(k)}_{ij} \leftarrow \min\{c^{(k-1)}_{ij}, c^{(k-1)}_{ik} + c^{(k-1)}_{kj}\}$
            $\pi_{ij} \leftarrow k$
return $\Pi^{(n)}$

$\pi^{(0)}_{ij} = NIL$ if $i=j$ or $w_{ij} = \infty$
$= i$ if $i \neq j$ and $w_{ij} < \infty$

Example



Floyd-Warshall(W)
$n \leftarrow$rows[W]
$D^{(0)} \leftarrow W$
for $k \leftarrow 1$ to n
  for $i \leftarrow 1$ to n
    for $j \leftarrow 1$ to n
      $c^{(k)}_{ij} \leftarrow \min\{c^{(k-1)}_{ij}, c^{(k-1)}_{ik} + c^{(k-1)}_{kj}\}$
      $\pi_{ij} \leftarrow k$
return $D^{(n)}$

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

Run time comparison with Floyd-Warshall Algorithm

- $O(V^2E)$ Bellman-Ford (sparse graph)
- $O(V^4)$ Bellman-Ford (dense graph)
- $O(VE \lg V)$ Dijkstra (binary-heap) (sparse graph)
- $O(V^3 \lg V)$ Dijkstra (binary-heap) (dense graph)
- $O(V^2 \lg V + VE)$ Dijkstra (Fibonacci-heap) (sparse graph)
- $O(V^2 \lg V + V^3)$ Dijkstra (Fibonacci-heap) (dense graph)

  –ve weight edges are **not** allowed in case of Dijkstra's algorithm

- Dynamic Programming based algorithms:
  - $O(V^4)$ matrix multiplication methods
  - $O(V^3 \lg V)$ repeated squaring
  - $O(V^3)$ Floyd-Warshall Algorithm

Transitive closure of a directed graph

- Given a digraph G=(V,E) we may wish to find weather there is a path in G from i to j for all vertex pairs i,j.
- the transitive closure of G is defined as the graph G* =(V,E*) E* ={(i,j): there is a path from *i* to *j* in G}

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

Transitive closure of a directed graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with $(\vee, \wedge)$ instead of $(\min, +)$:

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Time $= \Theta(n^3)$.