Divide-and-Conquer

- **Divide** the problem into a number of sub-problems
  - Similar sub-problems of smaller size
- **Conquer** the sub-problems
  - Solve the sub-problems recursively
  - Sub-problem size small enough $\Rightarrow$ solve the problems in straightforward manner
- **Combine** the solutions of the sub-problems
  - Obtain the solution for the original problem

Merge Sort Approach

- To sort an array $A[p \,..\, r]$:
- **Divide**
  - Divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each
- **Conquer**
  - Sort the subsequences recursively using merge sort
  - When the size of the sequences is 1 there is nothing more to do
- **Combine**
  - Merge the two sorted subsequences

Merge Sort

p			q				r
1	2	3	4	5	6	7	8
5	2	4	7	1	3	2	6

Alg.: MERGE-SORT(A, p, r)

if p < r ▷ Check for base case

then $q \leftarrow \lfloor (p + r)/2 \rfloor$ ▷ Divide

MERGE-SORT(A, p, q) ▷ Conquer

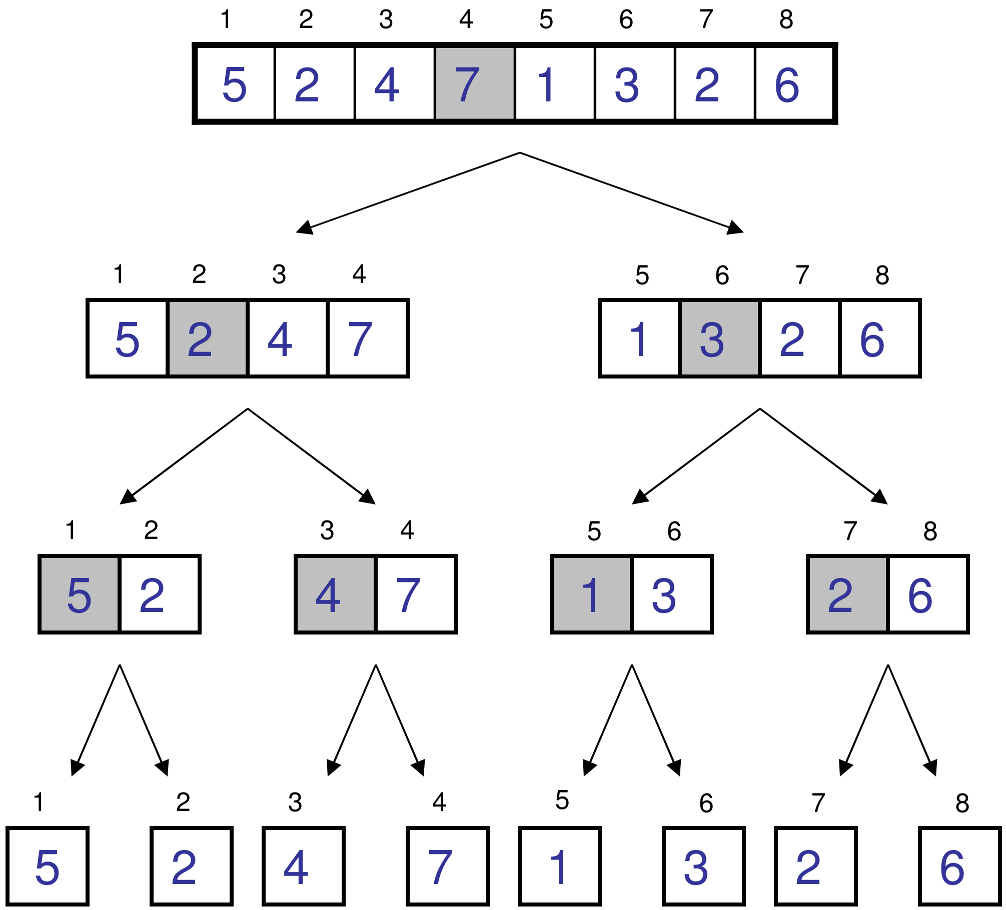
MERGE-SORT(A, q + 1, r) ▷ Conquer

MERGE(A, p, q, r) ▷ Combine

- Initial call: MERGE-SORT(A, 1, n)

Example – n Power of 2

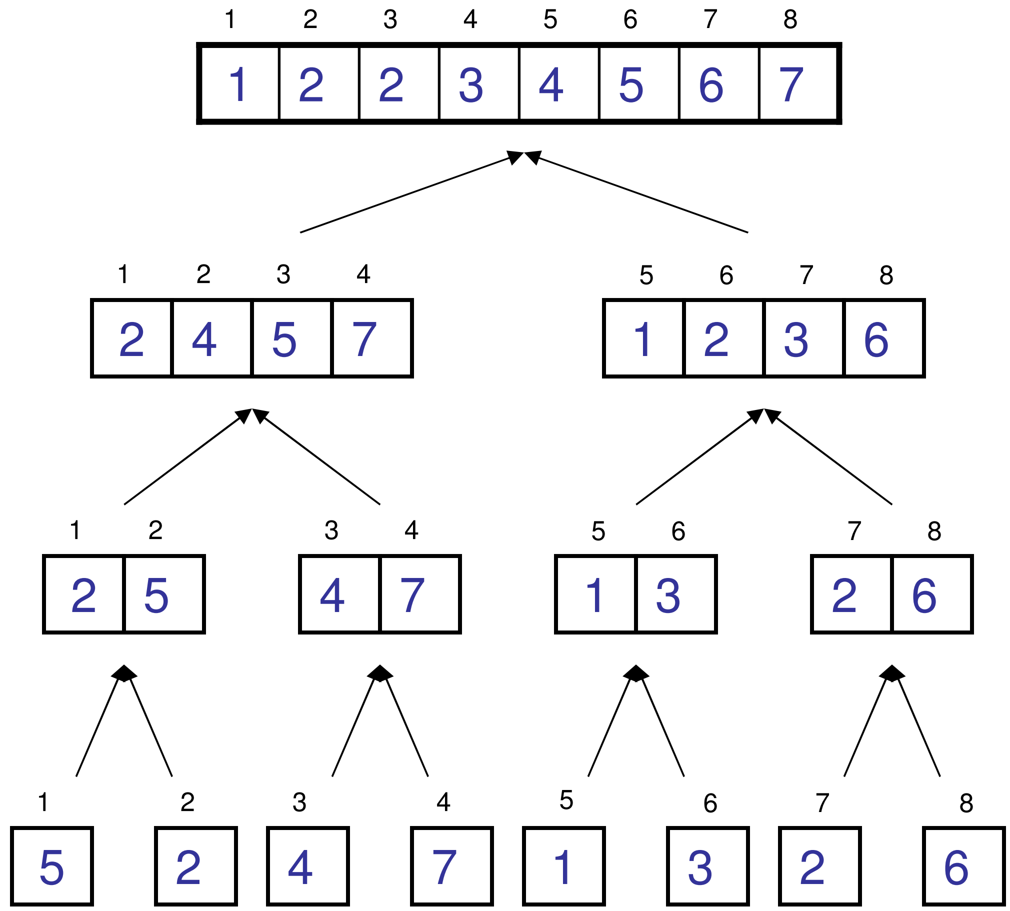
Divide



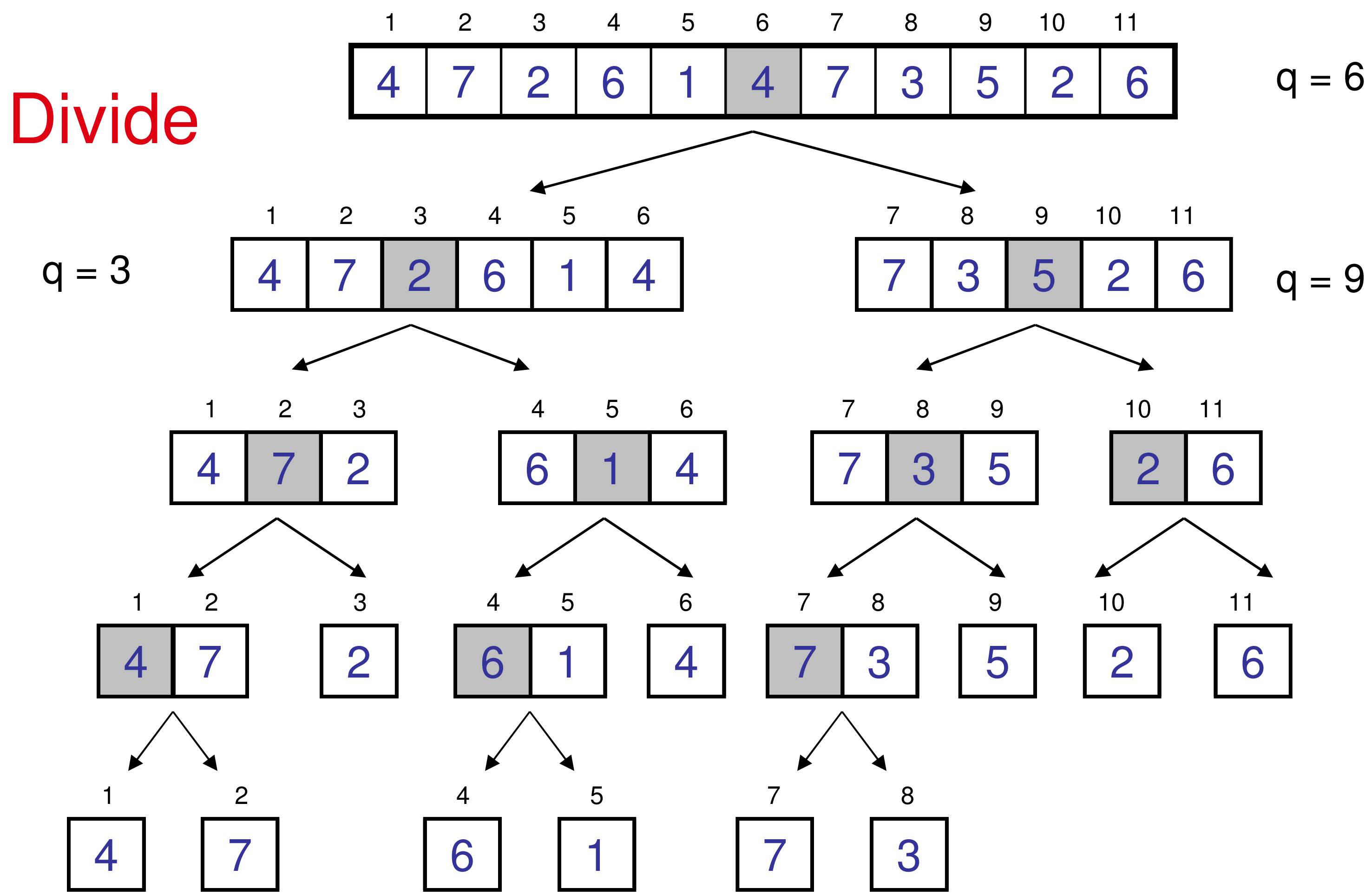
q = 4

Example – n Power of 2

Conquer and Merge

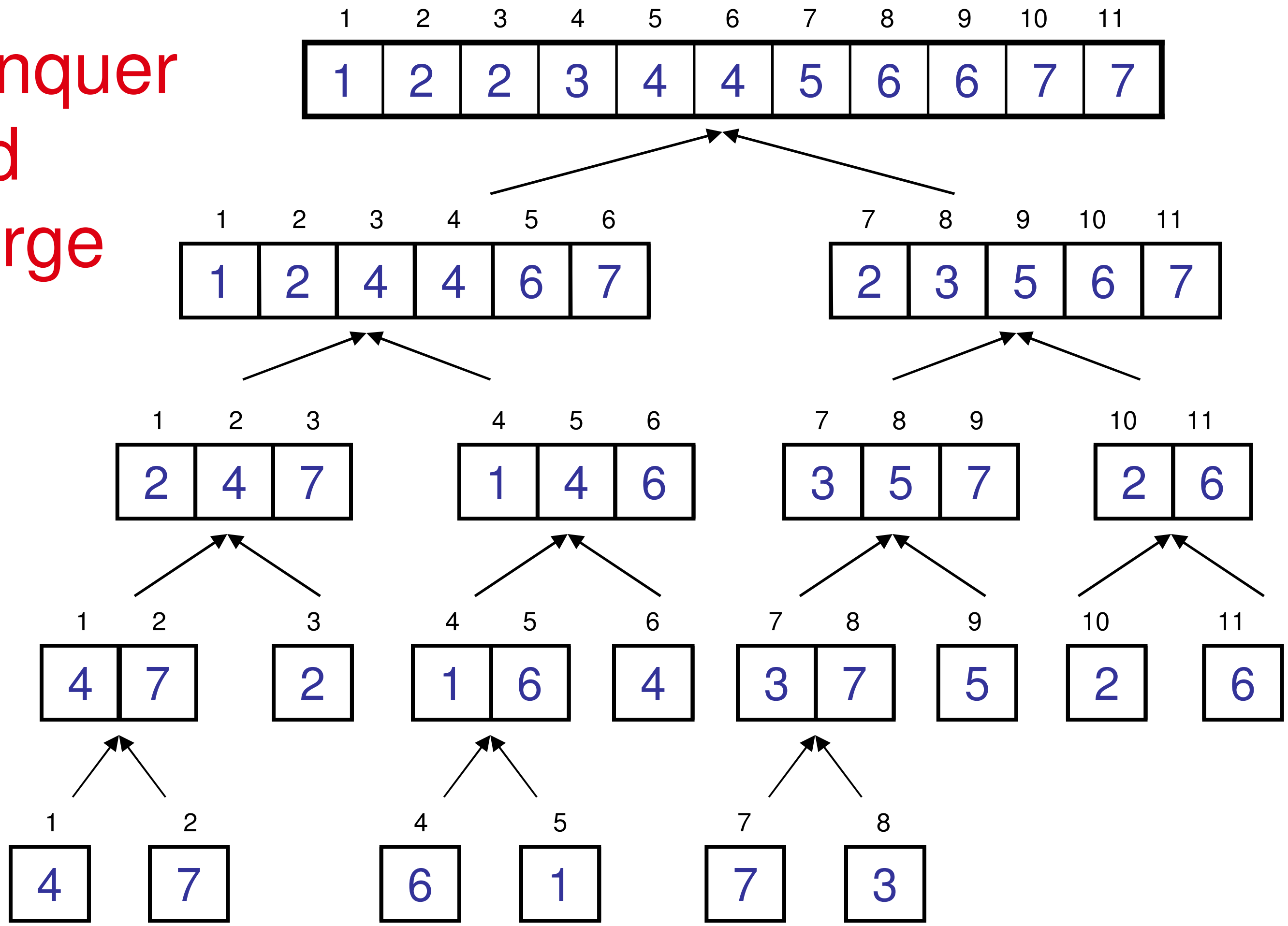


Example – n Not a Power of 2



Example – n Not a Power of 2

Conquer and Merge

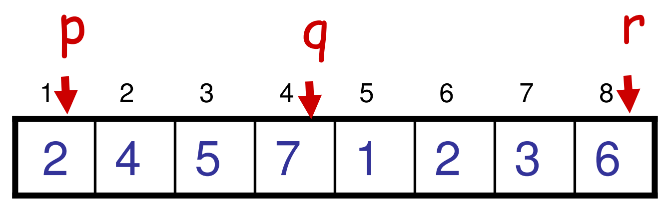


Merging

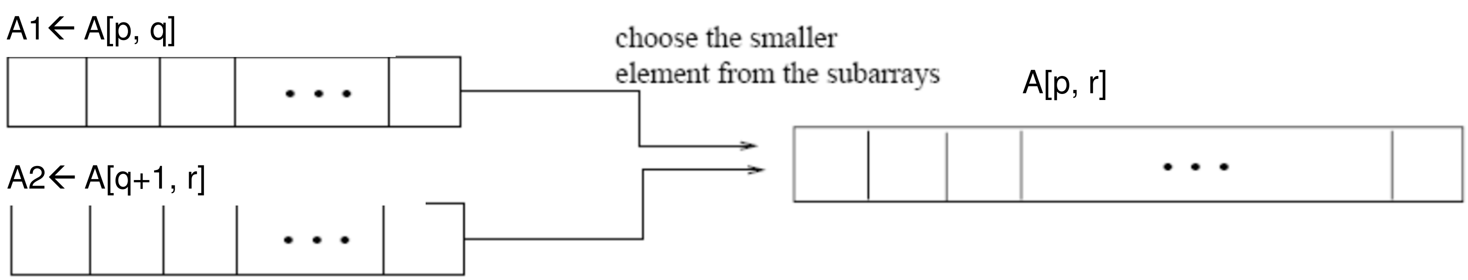
p			q				r
1	2	3	4	5	6	7	8
2	4	5	7	1	2	3	6

- **Input:** Array A and indices p, q, r such that $p \le q < r$
  - Subarrays $A[p \,..\, q]$ and $A[q + 1 \,..\, r]$ are sorted
- **Output:** One single sorted subarray $A[p \,..\, r]$

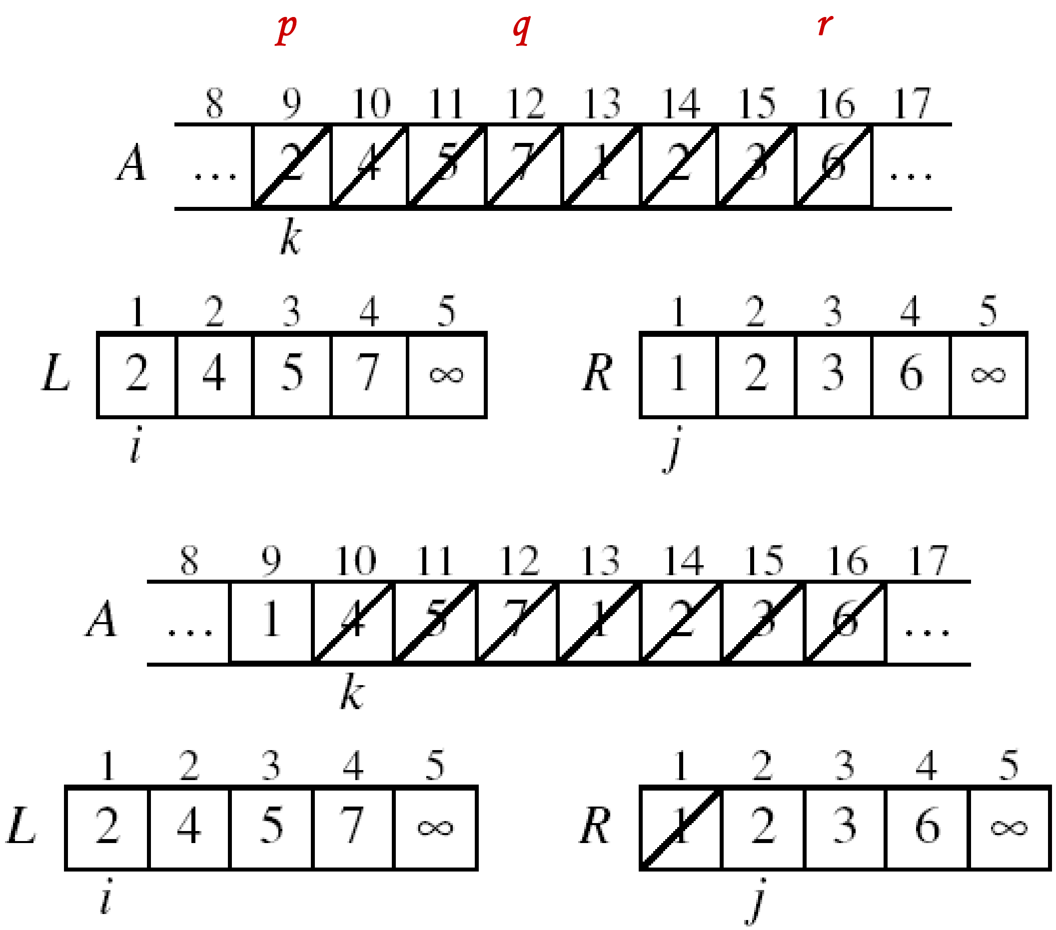
Merging



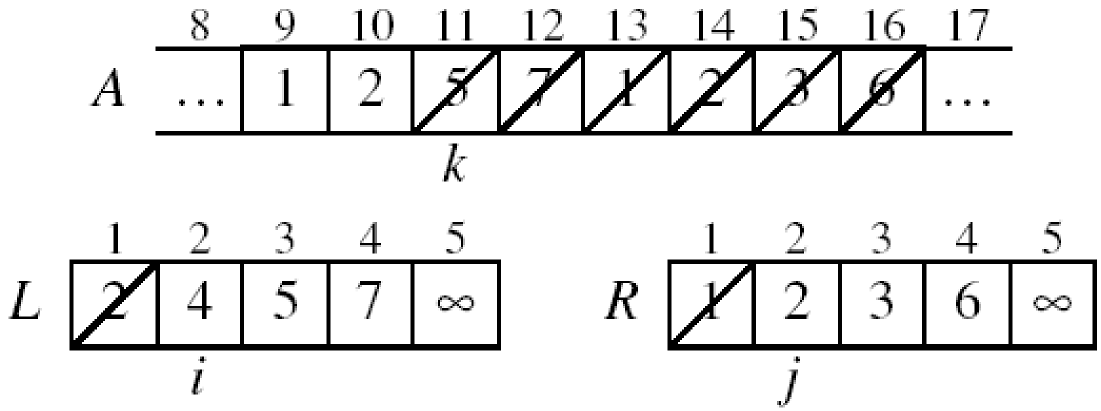
- Idea for merging:
  - Two piles of sorted cards
    - Choose the smaller of the two top cards
    - Remove it and place it in the output pile
  - Repeat the process until one pile is empty
  - Take the remaining input pile and place it face-down onto the output pile

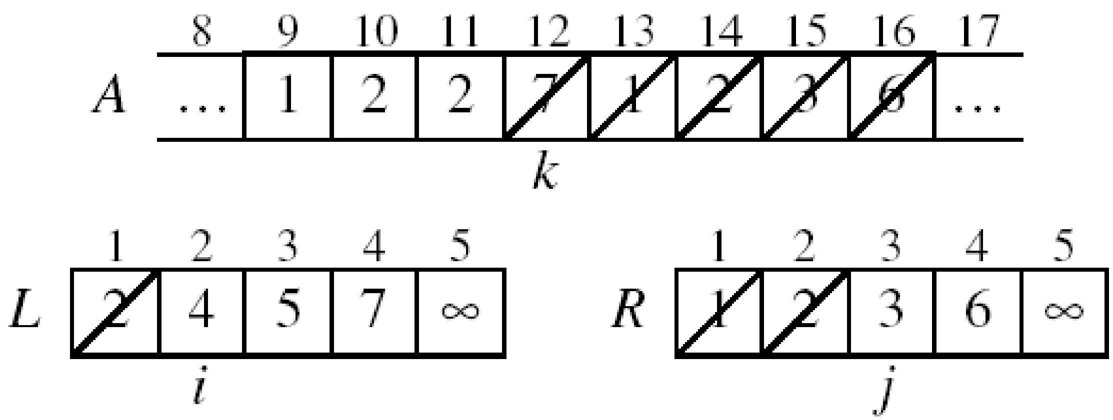


Example: MERGE(A, 9, 12, 16)

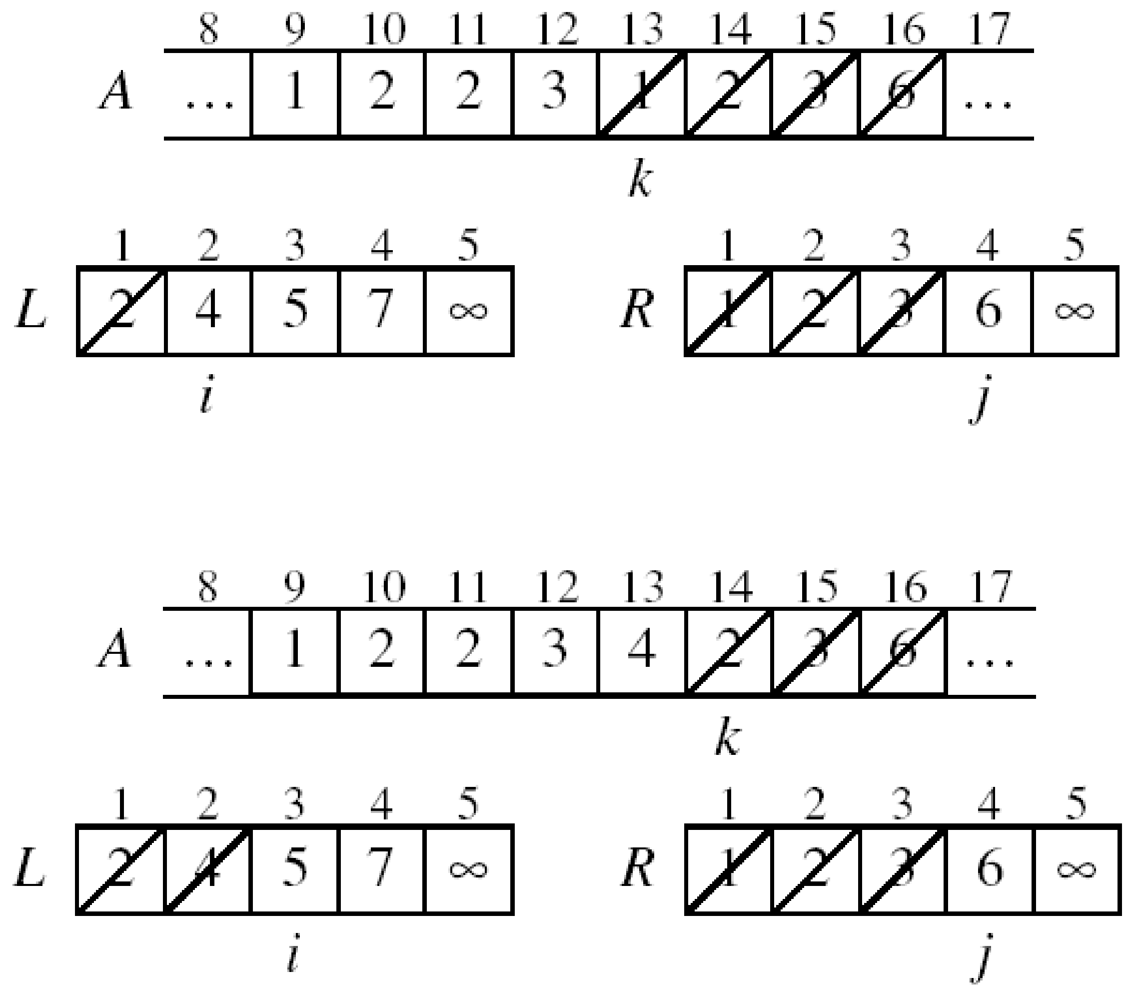


Example: MERGE(A, 9, 12, 16)





Example (cont.)



Example (cont.)

	8	9	10	11	12	13	14	15	16	17
A	…	1	2	2	3	4	5	~~3~~	~~6~~	…
								k		

	1	2	3	4	5
L	~~2~~	~~4~~	~~5~~	7	∞
				i	

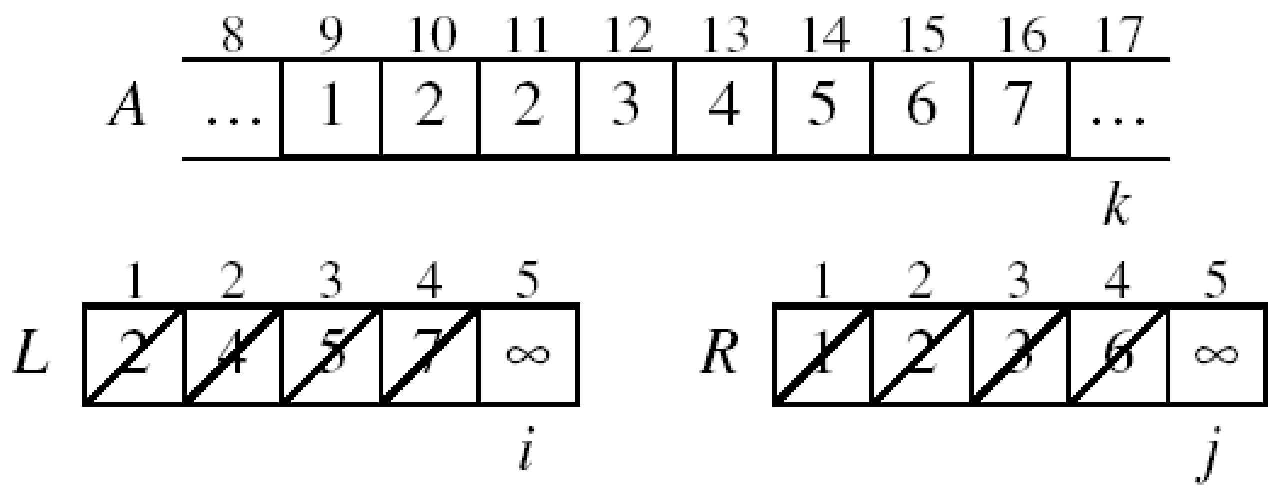
	1	2	3	4	5
R	~~1~~	~~2~~	~~3~~	6	∞
				j	

	8	9	10	11	12	13	14	15	16	17
A	…	1	2	2	3	4	5	6	~~6~~	…
									k	

	1	2	3	4	5
L	~~2~~	~~4~~	~~5~~	7	∞
				i	

	1	2	3	4	5
R	~~1~~	~~2~~	~~3~~	~~6~~	∞
					j

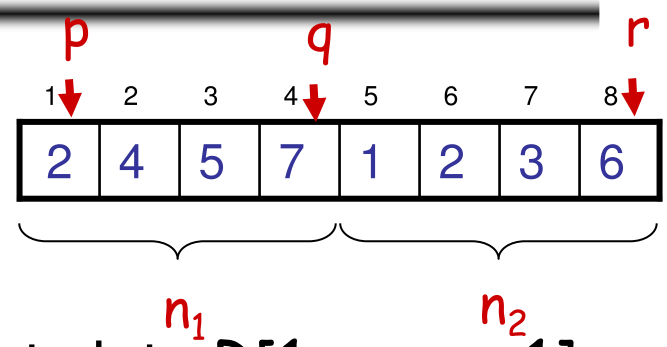
Example (cont.)



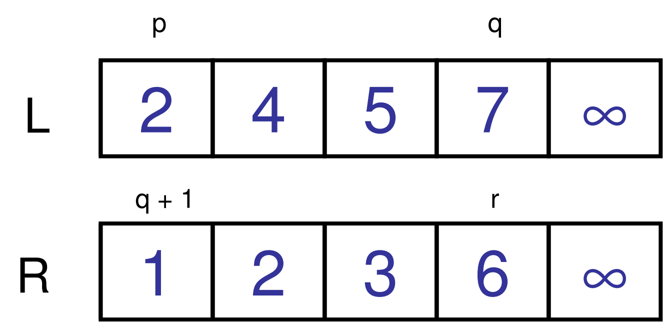
Done!

Merge - Pseudocode

Alg.: MERGE(A, p, q, r)

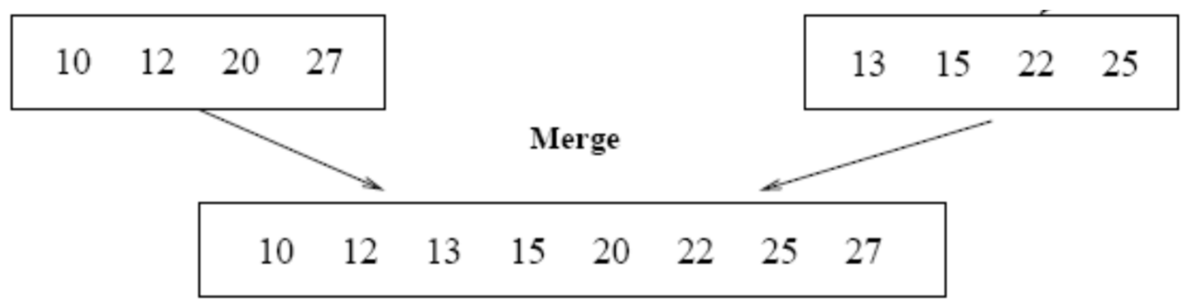


1. Compute n_1 and n_2
2. Copy the first n_1 elements into $L[1 \,..\, n_1 + 1]$ and the next n_2 elements into $R[1 \,..\, n_2 + 1]$
3. $L[n_1 + 1] \leftarrow \infty$; $R[n_2 + 1] \leftarrow \infty$
4. $i \leftarrow 1$; $j \leftarrow 1$
5. **for** $k \leftarrow p$ **to** r
6. **do if** $L[\, i \,] \le R[\, j \,]$
7. **then** $A[k] \leftarrow L[\, i \,]$
8. $i \leftarrow i + 1$
9. **else** $A[k] \leftarrow R[\, j \,]$
10. $j \leftarrow j + 1$



Running Time of Merge (assume last **for** loop)

- Initialization (copying into temporary arrays):
  - $\Theta(n_1 + n_2) = \Theta(n)$
- Adding the elements to the final array:
  - n iterations, each taking constant time $\Rightarrow \Theta(n)$
- Total time for Merge:
  - $\Theta(n)$



Analyzing Divide-and Conquer Algorithms

- The recurrence is based on the three steps of the paradigm:
  - $T(n)$ – running time on a problem of size n
  - **Divide** the problem into a subproblems, each of size n/b: takes $D(n)$
  - **Conquer** (solve) the subproblems $aT(n/b)$
  - **Combine** the solutions $C(n)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

MERGE-SORT Running Time

- **Divide:**
  - compute q as the average of p and r: $D(n) = \Theta(1)$
- **Conquer:**
  - recursively solve 2 subproblems, each of size $n/2$
    $\Rightarrow 2T(n/2)$
- **Combine:**
  - MERGE on an n-element subarray takes $\Theta(n)$ time
    $\Rightarrow C(n) = \Theta(n)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Solve the Recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

Use Master's Theorem:

Compare n with $f(n) = cn$

Case 2: $T(n) = \Theta(n\lg n)$

Merge Sort - Discussion

- Running time insensitive of the input
- Advantages:
  - Guaranteed to run in $\Theta(nlgn)$
- Disadvantage
  - Requires extra space $\approx$N