transitivity of the relation $\preceq$. In other words, Q is $\mathcal{NP}$-complete. In symbols:

$$P \preceq Q \text{ and } P \in \mathcal{NPC} \Rightarrow Q \in \mathcal{NPC}.$$

Cook and Levin made a fundamental breakthrough by showing that there do indeed exist $\mathcal{NP}$-complete problems. More precisely, they proved that the satisfiability problem for boolean formulae is $\mathcal{NP}$-complete. We now describe this problem, and examine the theoretical and practical implications of their discovery.

3.3.2 Boolean formulae and satisfiability

A *boolean variable* is a variable which takes on one of two values, $false$ or $true$. Boolean variables may be combined into *boolean formulae*, which may be defined recursively as follows.

- Every boolean variable is a boolean formula.
- If f is a boolean formula, then so too is $(\neg f)$, the *negation* of f.
- If f_1 and f_2 are boolean formulae, then so too are:
  - $(f_1 \vee f_2)$, the *disjunction* of f_1 and f_2,
  - $(f_1 \wedge f_2)$, the *conjunction* of f_1 and f_2.

These three operations may be thought of informally as 'not f', 'f_1 or f_2', and 'f_1 and f_2', respectively.

An assignment of values to the variables is called a *truth assignment*. Given a truth assignment, the value of the formula may be computed according to the following rules:

- if $f = false$, then $(\neg f) = true$, else $(\neg f) = false$;
- if $f_1 = true$ or $f_2 = true$, then $(f_1 \vee f_2) = true$, else $(f_1 \vee f_2) = false$;
- if $f_1 = true$ and $f_2 = true$, then $(f_1 \wedge f_2) = true$, else $(f_1 \vee f_2) = false$.

Two boolean formulae are *equivalent* (written $\equiv$) if they take the same value for each assignment of the variable involved. It follows easily from the above rules that disjunction and conjunction are *commutative* and *associative*. Hence, all the formulae obtained from k subformulae $f_1, f_2, \ldots, f_k$ by means of disjunction are all equivalent. Any of these is denoted by $(f_1 \vee f_2 \vee \cdots \vee f_k)$.

A boolean formula is *satisfiable* if there is a truth assignment of its variables for which the value of the formula is $true$. Clearly, some boolean formulae are satisfiable and some are not. This poses the general problem:

Problem 3.5 (SAT ; Boolean Satisfiability)**.**
Instance: a boolean formula f.
Decide: Is f satisfiable?