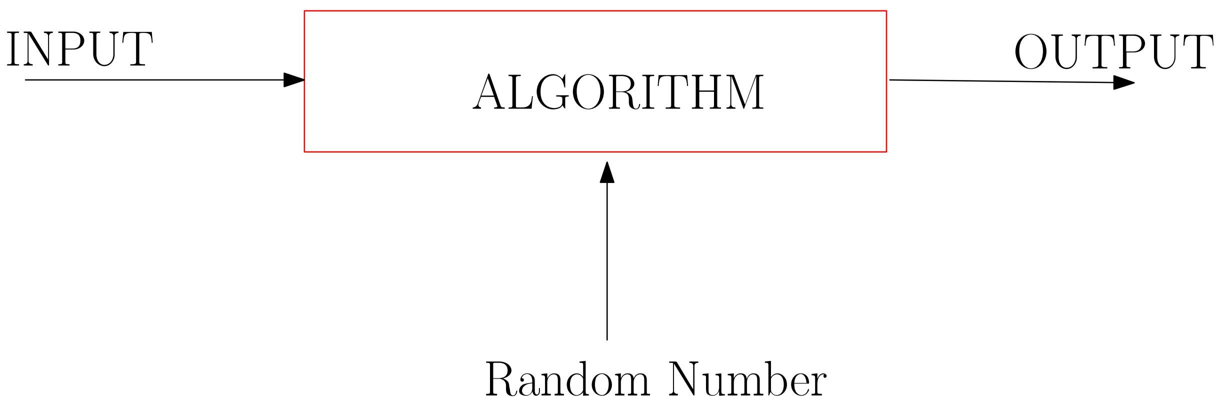
Introduction

Goal of a Deterministic Algorithm

- The solution produced by the algorithm is correct, and
- the number of computational steps is same for different runs of the algorithm with the same input.

Randomized Algorithm



Randomized Algorithm

- In addition to the input, the algorithm uses a source of pseudo random numbers. During execution, it takes random choices depending on those random numbers.
- The behavior (output) can vary if the algorithm is run multiple times on the same input.

Advantage of Randomized Algorithm

The Paradigm

Instead of making a guaranteed good choice, make a random choice and hope that it is good. This helps because guaranteeing a good choice becomes difficult sometimes.

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Average Case Analysis

analyzes the expected running time of deterministic algorithms assuming a suitable random distribution on the input.

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- Getting true random numbers is almost impossible.

Types of Randomized Algorithms

Definition

Las Vegas: a randomized algorithm that always returns a correct result. But the running time may vary between executions.

Example: Randomized QUICKSORT Algorithm

Definition

Monte Carlo: a randomized algorithm that terminates in polynomial time, but might produce erroneous result.

Example: Randomized MINCUT Algorithm

Some basic ideas from Probability

Expectation

Random variable

A function defined on a sample space is called a random variable. Given a random variable X, $Pr[X = j]$ means X's probability of taking the value j.

Expectation – "the average value"

The expectation of a random variable X is defined as:
$E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X = j]$

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- So, the expectation of X, $E[X] = \sum_{j=0}^{\infty} j \cdot Pr[X=j] = \frac{1}{p}$.

Conditional Probability and Independent Event

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The conditional probability of X given Y is

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Independent Events

Two events X and Y are independent, if $Pr[(X = x) \cap (Y = y)] = Pr[X = x] \cdot Pr[Y = y]$. In particular, if X and Y are independent, then

$$Pr[X = x \mid Y = y] = Pr[X = x]$$

A Result on Intersection of events

Let $\eta_1, \eta_2, \ldots, \eta_n$ be n events not necessarily independent. Then,

$Pr[\cap_{i=1}^{n} \eta_i] = Pr[\eta_1] \cdot Pr[\eta_2 \mid \eta_1] \cdot Pr[\eta_3 \mid \eta_1 \cap \eta_2] \cdots Pr[\eta_n \mid \eta_1 \cap \ldots \cap \eta_{n-1}]$.

The proof is by induction on n.

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- The coupon collection process in phase j when you have already collected j different coupons and are buying to get a new type.
- A new type of coupon ends phase j and you enter phase $j + 1$.

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Lemma

The expected number of jeans bought in phase j, $E[X_j] = \frac{n}{n-j}$.

- The success probability, p in the j-th phase is $\frac{n-j}{n}$.
- By the bound on waiting for success, the expected number of jeans bought $E[X_j]$ is $\frac{1}{p} = \frac{n}{n-j}$.

The expectation

Theorem

The expected number of jeans bought before all n types of coupons are collected is $E[X] = nH_n = \Theta(n \log n)$.

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Proof

$$E[X] = \sum_{j=0}^{n-1} E[X_j] = n \sum_{j=0}^{n-1} \frac{1}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = nH_n = \Theta(n \log n)$$

Randomized Quick Sort

Deterministic Quick Sort

The Problem:

Given an array $A[1 \ldots n]$ containing n (comparable) elements, sort them in increasing/decreasing order.

QSORT(A, p, q)

- If $p \geq q$, EXIT.
- Compute $s \leftarrow$ correct position of $A[p]$ in the sorted order of the elements of A from p-th location to q-th location.
- Move the pivot $A[p]$ into position $A[s]$.
- Move the remaining elements of $A[p - q]$ into appropriate sides.
- QSORT(A, p, $s - 1$);
- QSORT(A, $s + 1$, q).

Complexity Results of QSORT

- An INPLACE algorithm
- The worst case time complexity is $O(n^2)$.
- The average case time complexity is $O(n \log n)$.

Randomized Quick Sort

An Useful Concept - The Central Splitter

It is an index s such that the number of elements less (resp. greater) than $A[s]$ is at least $\frac{n}{4}$.

- The algorithm randomly chooses a key, and checks whether it is a central splitter or not.
- If it is a central splitter, then the array is split with that key as was done in the QSORT algorithm.
- It can be shown that the expected number of trials needed to get a central splitter is constant.

Randomized Quick Sort

RandQSORT(A, p, q)

1: If $p \geq q$, then EXIT.

2: While no central splitter has been found, execute the following steps:

  2.1: Choose uniformly at random a number $r \in \{p, p+1, \ldots, q\}$.

  2.2: Compute $s =$ number of elements in A that are less than $A[r]$, and
  $t =$ number of elements in A that are greater than $A[r]$.

  2.3: If $s \geq \frac{q-p}{4}$ and $t \geq \frac{q-p}{4}$, then $A[r]$ is a central splitter.

3: Position $A[r]$ in $A[s+1]$, put the members in A that are smaller than the central splitter in $A[p \ldots s]$ and the members in A that are larger than the central splitter in $A[s+2 \ldots q]$.

4: RandQSORT(A, p, s);

5: RandQSORT(A, $s+2$, q).

Analysis of RandQSORT

Fact: One execution of Step 2 needs $O(q-p)$ time.

Question: How many times Step 2 is executed for finding a central splitter ?

Result:

The probability that the randomly chosen element is a central splitter is $\frac{1}{2}$.

Recall "Waiting for success"

If p be the probability of success of a random experiment, and we continue the random experiment till we get success, the expected number of experiments we need to perform is $\frac{1}{p}$.

Implication in Our Case

- The expected number of times Step 2 needs to be repeated to get a central splitter (success) is 2 as the corresponding success probability is $\frac{1}{2}$.
- Thus, the expected time complexity of Step 2 is $O(n)$

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- By linearity of expectations, the expected time for all partitions of size $n \cdot (\frac{3}{4})^j$ is $O(n)$.
- Number of levels of recursion $= \log_{\frac{4}{3}} n = O(\log n)$.
- Thus, the expected running time is $O(n \log n)$.