1 INTRODUCTION

1-1. WHAT IS A GRAPH?

A *linear*† *graph* (or simply a *graph*) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called *vertices*, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices. Often this diagram itself is referred to as the graph. The object shown in Fig. 1-1, for instance, is a graph.

Observe that this definition permits an edge to be associated with a vertex pair (v_i, v_i). Such an edge having the same vertex as both its end vertices is called a *self-loop* (or simply a *loop*. The word loop, however, has a different meaning in electrical network theory; we shall therefore use the term self-loop to avoid confusion). Edge e_1 in Fig. 1-1 is a self-loop. Also note that

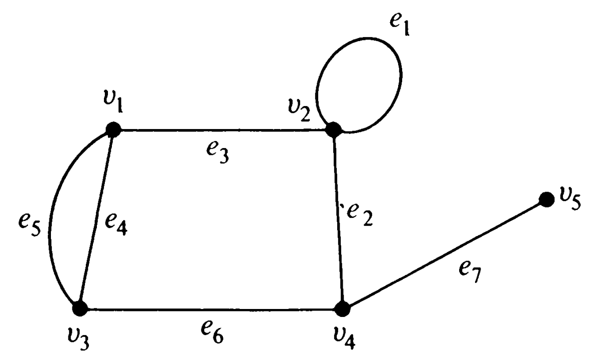

Fig. 1-1 Graph with five vertices and seven edges.

†The adjective "linear" is dropped as redundant in our discussions, because by a graph we always mean a linear graph. There is no such thing as a nonlinear graph.

the definition allows more than one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in Fig. 1-1. Such edges are referred to as *parallel edges*.

A graph that has neither self-loops nor parallel edges is called a *simple graph*. In some graph-theory literature, a graph is defined to be only a simple graph, but in most engineering applications it is necessary that parallel edges and self-loops be allowed; this is why our definition includes graphs with self-loops and/or parallel edges. Some authors use the term *general graph* to emphasize that parallel edges and self-loops are allowed.

It should also be noted that, in drawing a graph, it is immaterial whether the lines are drawn straight or curved, long or short: what is important is the incidence between the edges and vertices. For example, the two graphs drawn in Figs. 1-2(a) and (b) are the same, because incidence between edges and vertices is the same in both cases.

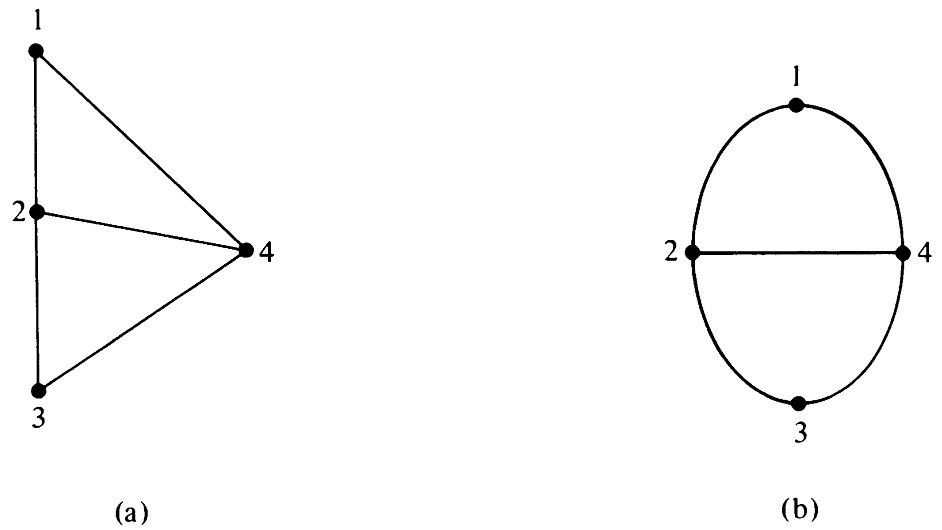

Fig. 1-2 Same graph drawn differently.

In a diagram of a graph, sometimes two edges may seem to intersect at a point that does not represent a vertex, for example, edges e and f in Fig. 1-3. Such edges should be thought of as being in different planes and thus having no common point. (Some authors break one of the two edges at such a crossing to emphasize this fact.)

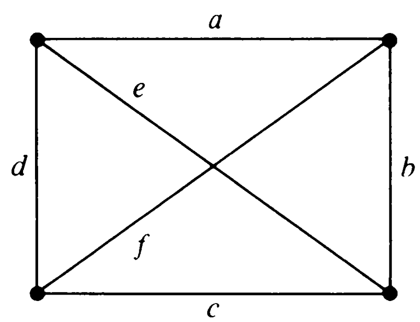

Fig. 1-3 Edges e and f have no common point.

A graph is also called a *linear complex*, a *1-complex*, or a *one-dimensional complex*. A vertex is also referred to as a *node*, a *junction*, a *point*, *0-cell*, or an *0-simplex*. Other terms used for an edge are a *branch*, a *line*, an *element*, a *1-cell*, an *arc*, and a *1-simplex*. In this book we shall generally use the terms graph, vertex, and edge.

1-2. APPLICATIONS OF GRAPHS

Because of its inherent simplicity, graph theory has a very wide range of applications in engineering, in physical, social, and biological sciences, in linguistics, and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The following are four examples from among hundreds of such applications.

Königsberg Bridge Problem: The Königsberg bridge problem is perhaps the best-known example in graph theory. It was a long-standing problem until solved by Leonhard Euler (1707–1783) in 1736, by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs as well as of the rest of topology. The problem is depicted in Fig. 1-4.

Two islands, C and D, formed by the Pregel River in Königsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia) were connected to each other and to the banks A and B with seven bridges, as shown in Fig. 1-4. The problem was to start at any of the four land areas of the city, A, B, C, or D, walk over each of the seven bridges *exactly* once, and return to the starting point (without swimming across the river, of course).

Euler represented this situation by means of a graph, as shown in Fig. 1-5. The vertices represent the land areas and the edges represent the bridges.

As we shall see in Chapter 2, Euler proved that a solution for this problem does not exist.

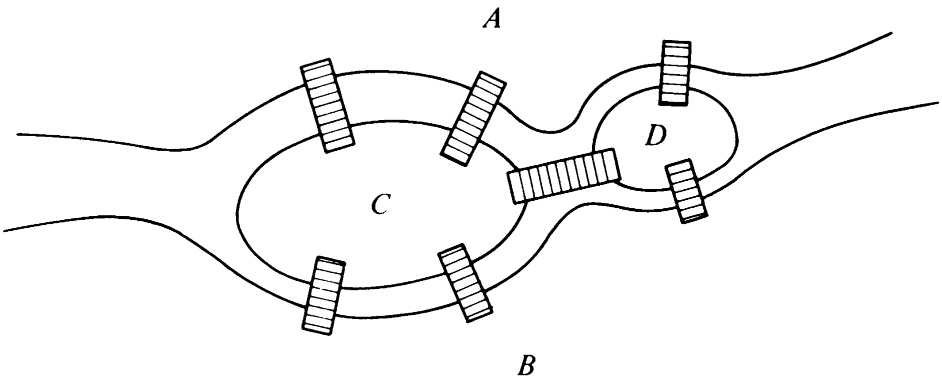

Fig. 1-4 Königsberg bridge problem.

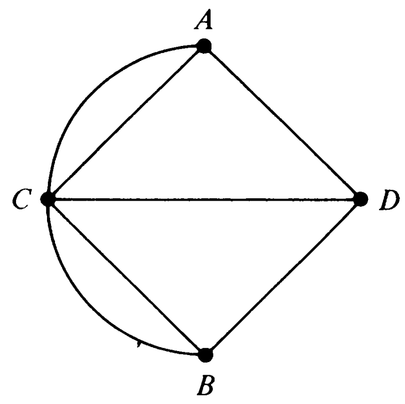

Fig. 1-5 Graph of Königsberg bridge problem.

The Königsberg bridge problem is the same as the problem of drawing figures without lifting the pen from the paper and without retracing a line (Problems 2-1 and 2-2). We all have been confronted with such problems at one time or another.

Utilities Problem: There are three houses (Fig. 1-6) H_1, H_2, and H_3, each to be connected to each of the three utilities—water (W), gas (G), and elec-

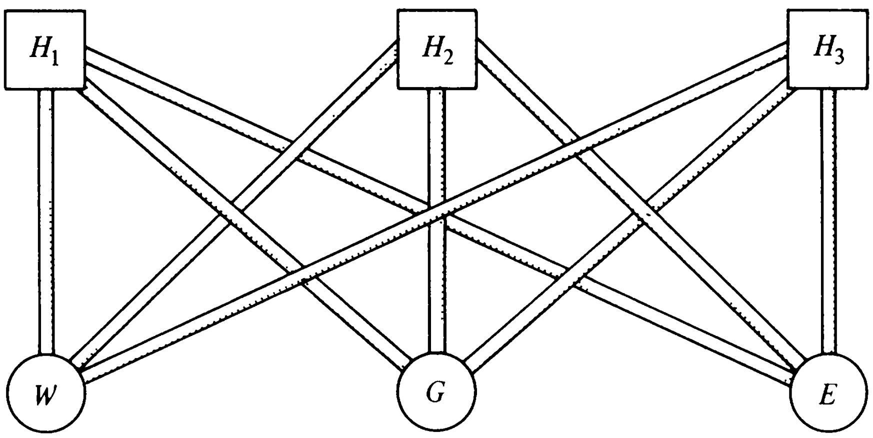

Fig. 1-6 Three-utilities problem.

tricity (E)—by means of conduits. Is it possible to make such connections without any crossovers of the conduits?

Figure 1-7 shows how this problem can be represented by a graph—the conduits are shown as edges while the houses and utility supply centers are vertices. As we shall see in Chapter 5, the graph in Fig. 1-7 cannot be drawn in the plane without edges crossing over. Thus the answer to the problem is no.

Electrical Network Problems: Properties (such as transfer function and input impedance) of an electrical network are functions of only two factors:

1. The nature and value of the elements forming the network, such as resistors, inductors, transistors, and so forth.
2. The way these elements are connected together, that is, the topology of the network.

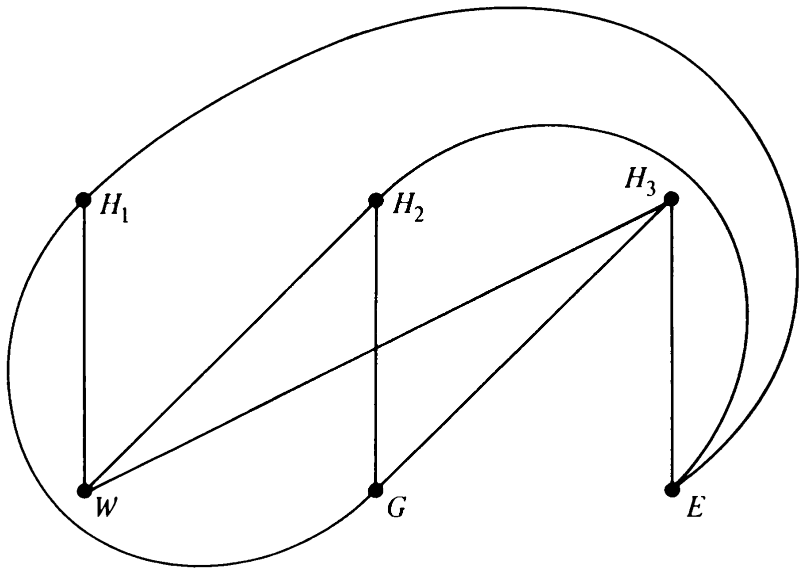

Fig. 1-7 Graph of three-utilities problem.

Since there are only a few different types of electrical elements, the variations in networks are chiefly due to the variations in topology. Thus electrical network analysis and synthesis are mainly the study of network topology. In the topological study of electrical networks, factor 2 is separated from 1 and is studied independently. The advantage of this approach will be clearer in Chapter 13, a chapter devoted solely to applying graph theory to electrical networks.

The topology of a network is studied by means of its graph. In drawing a graph of an electrical network the junctions are represented by vertices, and branches (which consist of electrical elements) are represented by edges, regardless of the nature and size of the electrical elements. An electrical network and its graph are shown in Fig. 1-8.

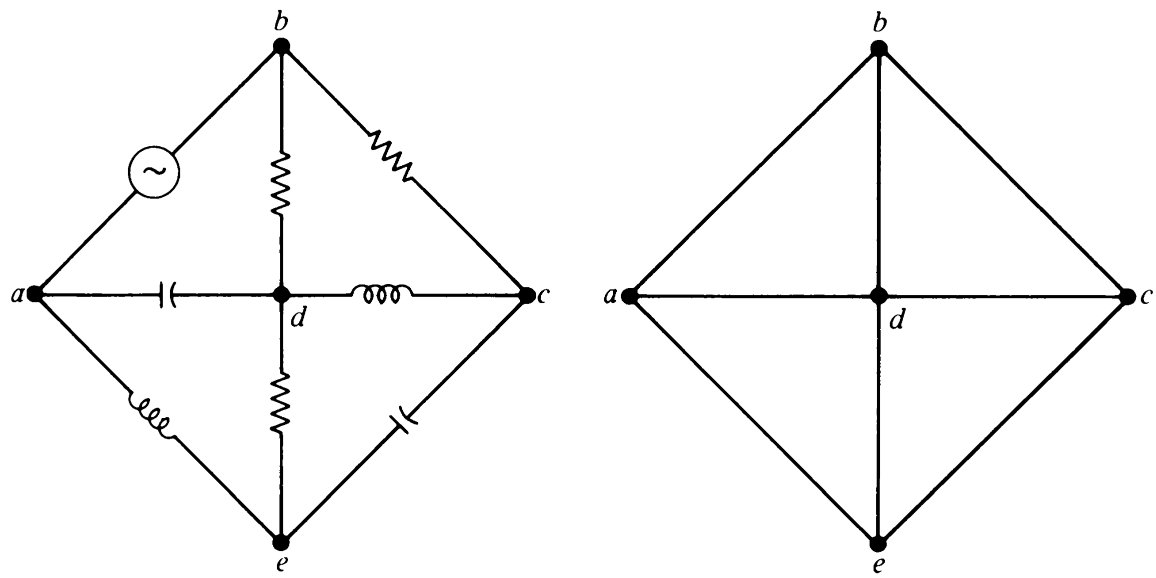

Fig. 1-8 Electrical network and its graph.

Seating Problem: Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch. How many days can this arrangement last?

This situation can be represented by a graph with nine vertices such that each vertex represents a member, and an edge joining two vertices represents the relationship of sitting next to each other. Figure 1-9 shows two possible

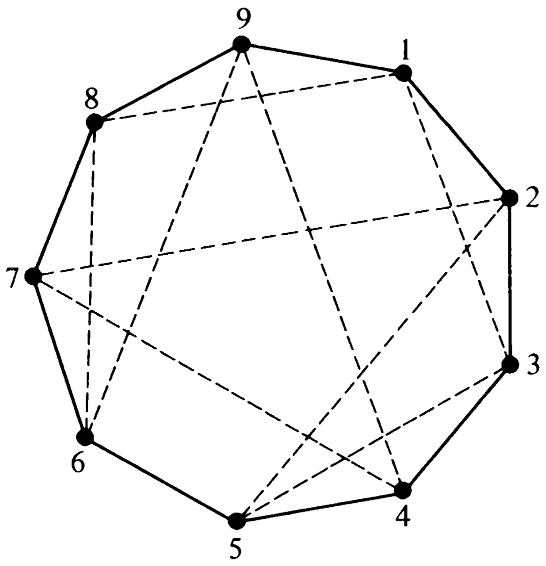

Fig. 1-9 Arrangements at a dinner table.

seating arrangements—these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines). It can be shown by graph-theoretic considerations that there are only two more arrangements possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is

$$\frac{n-1}{2}, \quad \text{if } n \text{ is odd,}$$

and

$$\frac{n-2}{2}, \quad \text{if } n \text{ is even.}$$

The reader has probably noticed that three of the four examples of applications above are puzzles and not engineering problems. This was done to avoid introducing at this stage background material not pertinent to graph theory. More substantive applications will follow, particularly in the last four chapters.

1-3. FINITE AND INFINITE GRAPHS

Although in our definition of a graph neither the vertex set V nor the edge set E need be finite, in most of the theory and almost all applications these sets are finite. A graph with a finite number of vertices as well as a finite number of edges is called a *finite graph;* otherwise, it is an *infinite graph.* The graphs in Figs. 1-1, 1-2, 1-5, 1-7, and 1-8 are all examples of finite graphs. Portions of two infinite graphs are shown in Fig. 1-10.

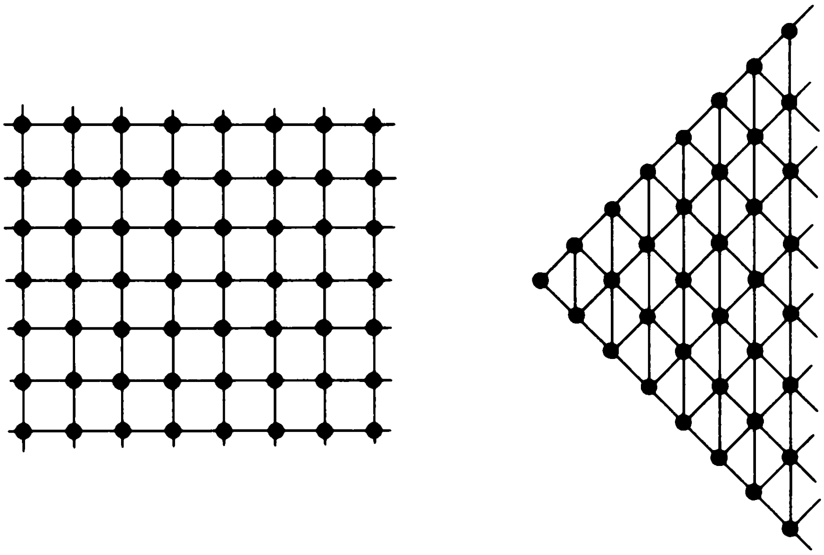

Fig. 1-10 Portions of two infinite graphs.

In this book we shall confine ourselves to the study of finite graphs, and unless otherwise stated the term "graph" will always mean a finite graph.

1-4. INCIDENCE AND DEGREE

When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be *incident with* (*on* or *to*) each other. In Fig. 1-1, for example, edges e_2, e_6, and e_7 are incident with vertex v_4. Two nonparallel edges are said to be *adjacent* if they are incident on a common vertex. For example, e_2 and e_7 in Fig. 1-1 are adjacent. Similarly, two vertices are said to be adjacent if they are the end vertices of the same edge. In Fig. 1-1, v_4 and v_5 are adjacent, but v_1 and v_4 are not.

The number of edges incident on a vertex v_i, with self-loops counted twice, is called the *degree*, $d(v_i)$, of vertex v_i. In Fig. 1-1, for example, $d(v_1) =$

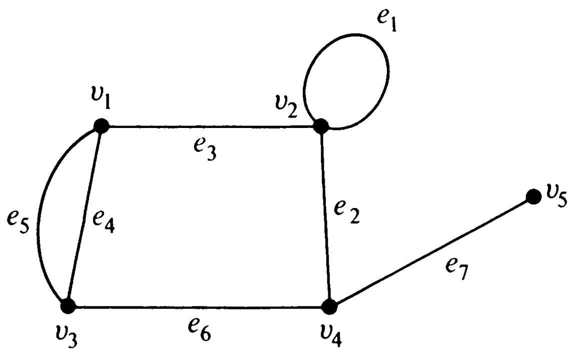

Fig. 1-1 A graph with five vertices and seven edges.

$d(v_3) = d(v_4) = 3$, $d(v_2) = 4$, and $d(v_5) = 1$. The degree of a vertex is sometimes also referred to as its *valency*.

Let us now consider a graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

$$\sum_{i=1}^{n} d(v_i) = 2e. \tag{1-1}$$

Taking Fig. 1-1 as an example, once more,

$$\begin{aligned} &d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) \\ &= 3 + 4 + 3 + 3 + 1 = 14 = \text{twice the number of edges.} \end{aligned}$$

From Eq. (1-1) we shall derive the following interesting result.

THEOREM 1-1

The number of vertices of odd degree in a graph is always even.

Proof: If we consider the vertices with odd and even degrees separately, the quantity in the left side of Eq. (1-1) can be expressed as the sum of two sums, each taken over vertices of even and odd degrees, respectively, as follows:

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k). \tag{1-2}$$

Since the left-hand side in Eq. (1-2) is even, and the first expression on the right-hand side is even (being a sum of even numbers), the second expression must also be even:

$$\sum_{\text{odd}} d(v_k) = \text{an even number.} \tag{1-3}$$

Because in Eq. (1-3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem. ■

A graph in which all vertices are of equal degree is called a *regular graph* (or simply a *regular*). The graph of three utilities shown in Fig. 1-7 is a regular of degree three.

1-5. ISOLATED VERTEX, PENDANT VERTEX, AND NULL GRAPH

A vertex having no incident edge is called an *isolated vertex*. In other words, isolated vertices are vertices with zero degree. Vertices v_4 and v_7 in Fig. 1-11, for example, are isolated vertices. A vertex of degree one is called

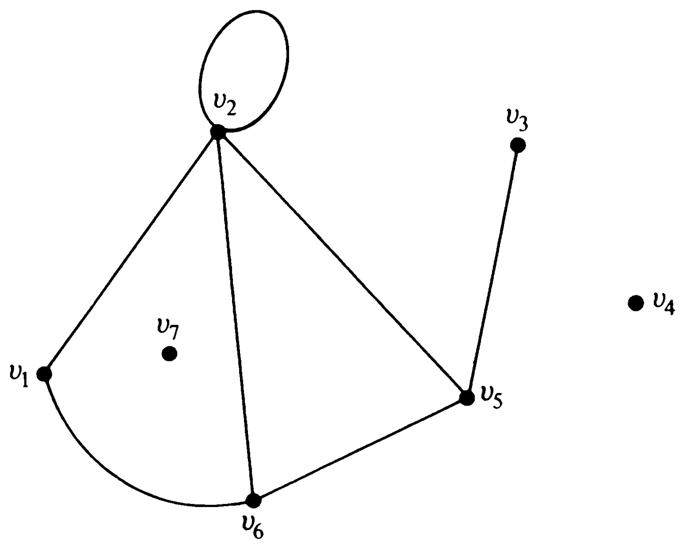

Fig. 1-11 Graph containing isolated vertices, series edges, and a pendant vertex.

a *pendant vertex* or an *end vertex*. Vertex v_3 in Fig. 1-11 is a pendant vertex. Two adjacent edges are said to be in *series* if their common vertex is of degree two. In Fig. 1-11, the two edges incident on v_1 are in series.

In the definition of a graph $G = (V, E)$, it is possible for the edge set E to be empty. Such a graph, without any edges, is called a *null graph*. In other words, every vertex in a null graph is an isolated vertex. A null graph of six vertices is shown in Fig. 1-12. Although the edge set E may be empty, the

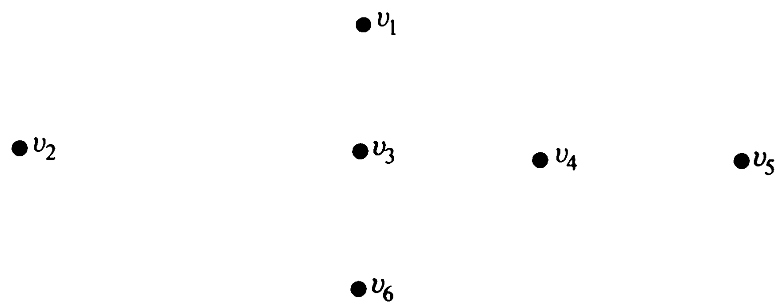

Fig. 1-12 Null graph of six vertices.

vertex set V must not be empty; otherwise, there is no graph. In other words, by definition, a graph must have at least one vertex.†

†Some authors (see, for example, [2-9], p. 1, or [15-58], p. 17) do allow the case in which the vertex set V is also empty. This they call the null graph, and they call a graph with $E = \emptyset$ and $V \neq \emptyset$ a *vertex graph*. For our purposes this distinction is of no consequence. For a lively discussion on paradoxes arising out of different definitions of the null graph, see pp. 40–41 in *Theory of Graphs: a Basis for Network Theory*, by L. M. Maxwell and M. B. Reed (Pergamon Press, N. Y. 1971).

1-6. A BRIEF HISTORY OF GRAPH THEORY

As mentioned before, graph theory was born in 1736 with Euler's paper in which he solved the Königsberg bridge problem [1-4].† For the next 100 years nothing more was done in the field.

In 1847, G. R. Kirchhoff (1824–1887) developed the theory of trees for their applications in electrical networks [1-6]. Ten years later, A. Cayley (1821–1895) discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons C_nH_{2n+2} [1-3].

About the time of Kirchhoff and Cayley, two other milestones in graph theory were laid. One was the *four-color conjecture*, which states that four colors are sufficient for coloring any atlas (a map on a plane) such that the countries with common boundaries have different colors.

It is believed that A. F. Möbius (1790–1868) first presented the four-color problem in one of his lectures in 1840. About 10 years later, A. De Morgan (1806–1871) discussed this problem with his fellow mathematicians in London. De Morgan's letter is the first authenticated reference to the four-color problem. The problem became well known after Cayley published it in 1879 in the first volume of the *Proceedings of the Royal Geographic Society*. To this day, the four-color conjecture is by far the most famous unsolved problem in graph theory; it has stimulated an enormous amount of research in the field [1-11].

The other milestone is due to Sir W. R. Hamilton (1805–1865). In the year 1859 he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden, regular dodecahedron (a polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner; see Fig. 2-21). The corners were marked with the names of 20 important cities: London, NewYork, Delhi, Paris, and so on. The object in the puzzle was to find a route along the edges of the dodecahedron, passing through each of the 20 cities exactly once [1-12].

Although the solution of this specific problem is easy to obtain (as we shall see in Chapter 2), to date no one has found a necessary and sufficient condition for the existence of such a route (called Hamiltonian circuit) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920s. One of the pioneers in this period was D. König. He organized the work of other mathematicians and his own and wrote the first book on the subject, which was published in 1936 [1-7].

The past 30 years has been a period of intense activity in graph theory—both pure and applied. A great deal of research has been done and is being

†Bracketed numbers refer to references at the end of chapters.

done in this area. Thousands of papers have been published and more than a dozen books written during the past decade. Among the current leaders in the field are Claude Berge, Oystein Ore (recently deceased), Paul Erdös, William Tutte, and Frank Harary.

SUMMARY

In this chapter some basic concepts of graph theory have been introduced, and some elementary results have been obtained. An attempt has also been made to show that graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationships among them.

REFERENCES

As an elementary text on graph theory, Ore's book [1-10] is recommended. Busacker and Saaty [1-2] is a good intermediate-level book. Seshu and Reed [1-13] is specially suited for electrical engineers. Berge [1-1] and Ore [1-9] are good general texts, but are somewhat advanced. Harary's book [1-5] contains an excellent treatment of the subject. It is compact and clear, but it contains no applications and is written for an advanced student of graph theory. For relating graph theory to the rest of topology one should read [1-8], a well-written elementary book on important aspects of topology. The entertaining book of Rouse Ball [1-12] contains a variety of puzzles and games to which graphs have been applied.

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1-2. Busacker, R. G., and T. L. Saaty, *Finite Graphs and Networks: An Introduction with Applications*, McGraw-Hill Book Company, New York, 1965.

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PROBLEMS

1-1. Draw all simple graphs of one, two, three, and four vertices.

1-2. Draw graphs representing problems of (a) two houses and three utilities; (b) four houses and four utilities, say, water, gas, electricity, and telephone.

1-3. Name 10 situations (games, activities, real-life problems, etc.) that can be represented by means of graphs. Explain what the vertices and the edges denote.

1-4. Draw the graph of the Wheatstone bridge circuit.

1-5. Draw graphs of the following chemical compounds: (a) CH_4, (b) C_2H_6, (c) C_6H_6, (d) N_2O_3. (*Hint:* Represent atoms by vertices and chemical bonds between them by edges.)

1-6. Draw a graph with 64 vertices representing the squares of a chessboard. Join these vertices appropriately by edges, each representing a move of the knight. You will see that in this graph every vertex is of degree two, three, four, six, or eight. How many vertices are of each type?

1-7. Given a maze as shown in Fig. 1-13, represent this maze by means of a graph such that a vertex denotes either a corridor or a dead end (as numbered). An edge represents a possible path between two vertices. (This is one of numerous mazes that were drawn or built by the Hindus, Greeks, Romans, Vikings, Arabs, etc.)

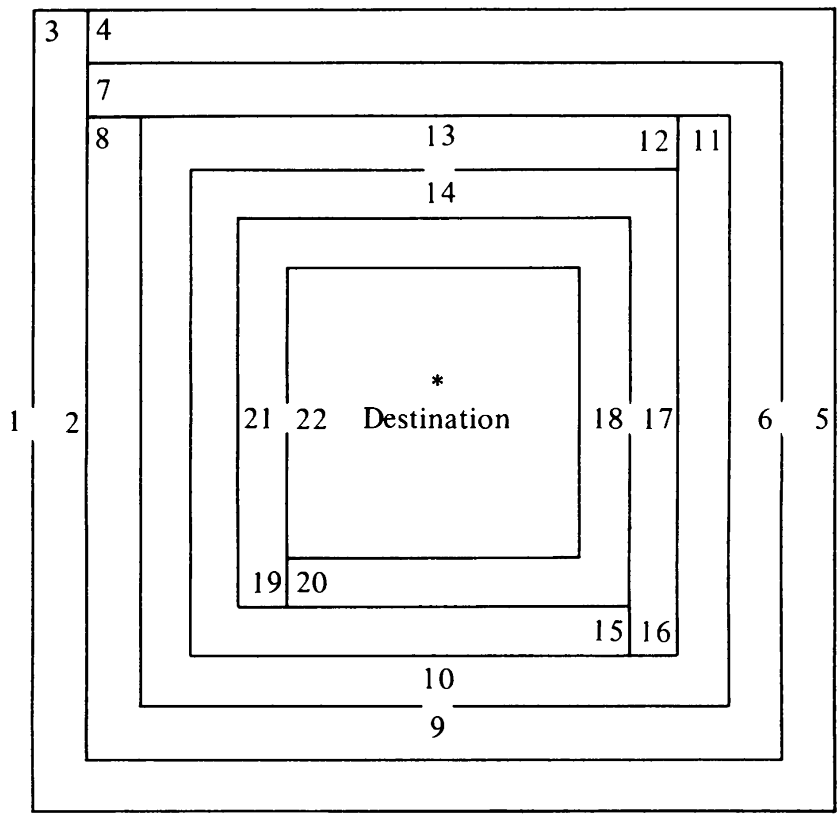


Fig. 1-13 A maze.

1-8. *Decanting problem.* You are given three vessels A, B, and C of capacities 8, 5, and 3 gallons, respectively. A is filled, while B and C are empty. Divide the liquid in A into two equal quantities. [*Hint:* Let a, b, and c be the amounts of liquid in A, B, and C, respectively. We have $a + b + c = 8$ at all times. Since at least one of the vessels is always empty or full, at least one of the following equations must always be satisfied: $a = 0$, $a = 8$; $b = 0$, $b = 5$; $c = 0$, $c = 3$. You will find that with these constraints there are 16 possible states (situations) in this process. Represent this problem by means of a 16-vertex graph. Each vertex stands for a state and each edge for a permissible change of states between its two end vertices. Now when you look at this graph it will be clear to you how to go from state (8, 0, 0) to state (4, 4, 0).] This is the classical decanting problem.

1-9. Convince yourself that an infinite graph with a finite number of edges (i.e., a graph with a finite number of edges and an infinite number of vertices) must have an infinite number of isolated vertices.

1-10. Show that an infinite graph with a finite number of vertices (i.e., a graph with a finite number of vertices and an infinite number of edges) will have at least one pair of vertices (or one vertex in case of parallel self-loops) joined by an infinite number of parallel edges.

1-11. Convince yourself that the maximum degree of any vertex in a simple graph with n vertices is $n - 1$.

1-12. Show that the maximum number of edges in a simple graph with n vertices is $n(n - 1)/2$.

2 PATHS AND CIRCUITS

This chapter serves two purposes. The first is to introduce additional concepts and terms in graph theory. These concepts, such as paths, circuits, and Euler graphs, deal mainly with the nature of connectivity in graphs. The degree of vertices, which is a local property of each vertex, will be shown to be related to the more global properties of the graph.

The second purpose is to illustrate with examples how to solve actual problems using graph theory. The celebrated Königsberg bridge problem, which was introduced in Chapter 1, will be solved. The solution of the seating arrangement problem, also introduced in Chapter 1, will be obtained by means of Hamiltonian circuits. A third problem, which involves stacking four multicolored cubes, will also be solved. These three unrelated problems will demonstrate the problem-solving power of graph theory. The reader may attempt to solve these problems without using graphs; the difficulty of such an approach will quickly convince him of the value of graph theory.

2-1. ISOMORPHISM

In geometry two figures are thought of as equivalent (and called congruent) if they have identical behavior in terms of geometric properties. Likewise, two graphs are thought of as equivalent (and called *isomorphic*) if they have identical behavior in terms of graph-theoretic properties. More precisely: Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e is incident on vertices v_1 and v_2 in G; then the corresponding edge e' in G' must be incident on the vertices v'_1 and v'_2 that correspond to

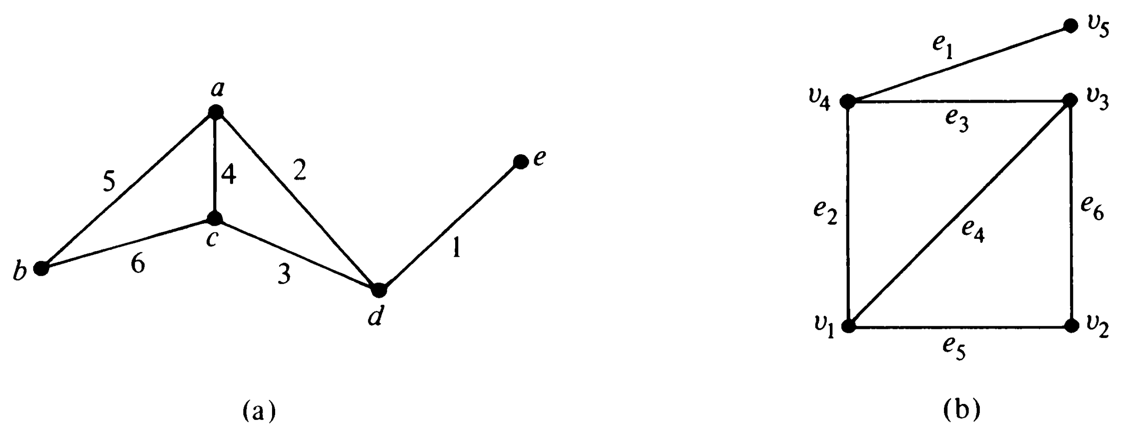


Fig. 2-1 Isomorphic graphs.

v_1 and v_2, respectively. For example, one can verify that the two graphs in Fig. 2-1 are isomorphic. The correspondence between the two graphs is as follows: The vertices a, b, c, d, and e correspond to v_1, v_2, v_3, v_4, and v_5, respectively. The edges 1, 2, 3, 4, 5, and 6 correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively.

Except for the labels (i.e., names) of their vertices and edges, isomorphic graphs are the same graph, perhaps drawn differently. As indicated in Chapter 1, a given graph can be drawn in many different ways. For example, Fig. 2-2 shows two different ways of drawing the same graph.

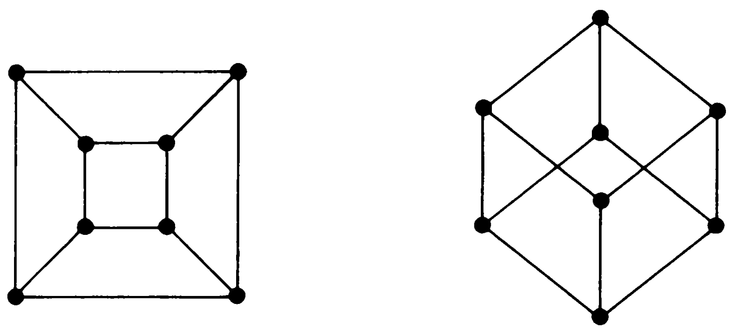
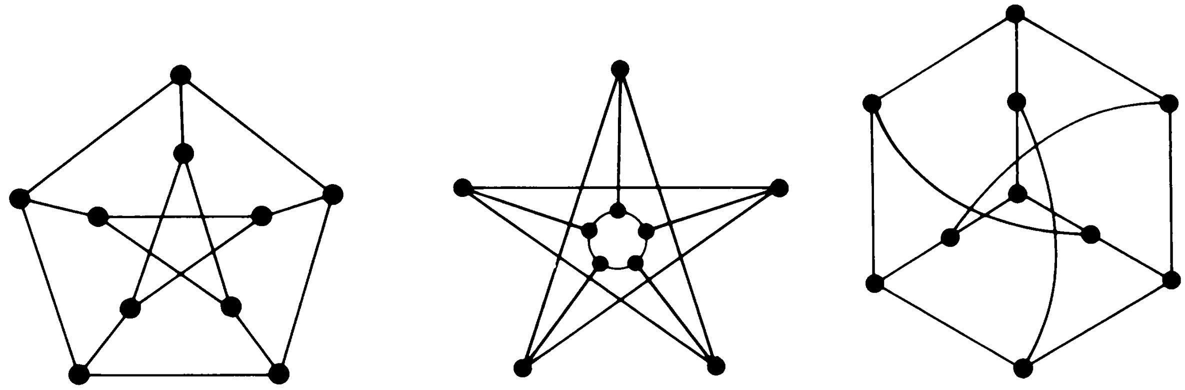


Fig. 2-2 Isomorphic graphs.

It is not always an easy task to determine whether or not two given graphs are isomorphic. For instance, the three graphs shown in Fig. 2-3 are all isomorphic, but just by looking at them you cannot tell that. It is left as an exercise for the reader to show, by redrawing and labeling the vertices and edges, that the three graphs in Fig. 2-3 are isomorphic (see Problem 2-3).

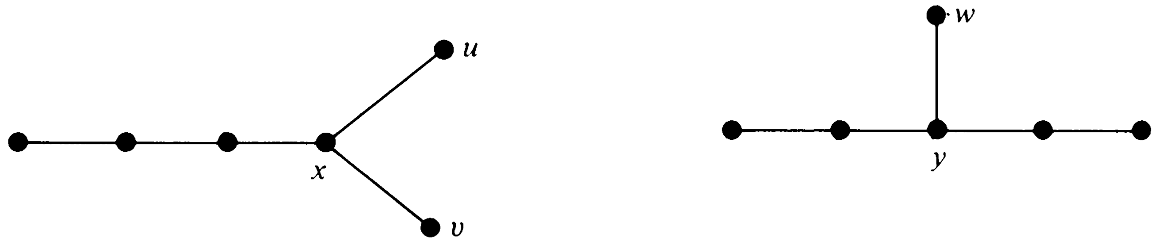
It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.



(a) (b) (c)

Fig. 2-3 Isomorphic graphs.



(a) (b)

Fig. 2-4 Two graphs that are not isomorphic.

However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 2-4 satisfy all three conditions, yet they are not isomorphic. That the graphs in Figs. 2-4(a) and (b) are not isomorphic can be shown as follows: If the graph in Fig. 2-4(a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y, while in (a) there are two pendant vertices, u and v, adjacent to x.

Finding a simple and efficient criterion for detection of isomorphism is still actively pursued and is an important unsolved problem in graph theory. In Chapter 11 we shall discuss various proposed algorithms and their programs for automatic detection of isomorphism by means of a computer. For now, we move to a different topic.

2-2. SUBGRAPHS

A graph g is said to be a *subgraph* of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G. For instance, the graph in Fig. 2-5(b) is a subgraph of the one in Fig. 2-5(a). (Obviously, when considering a subgraph, the original graph must

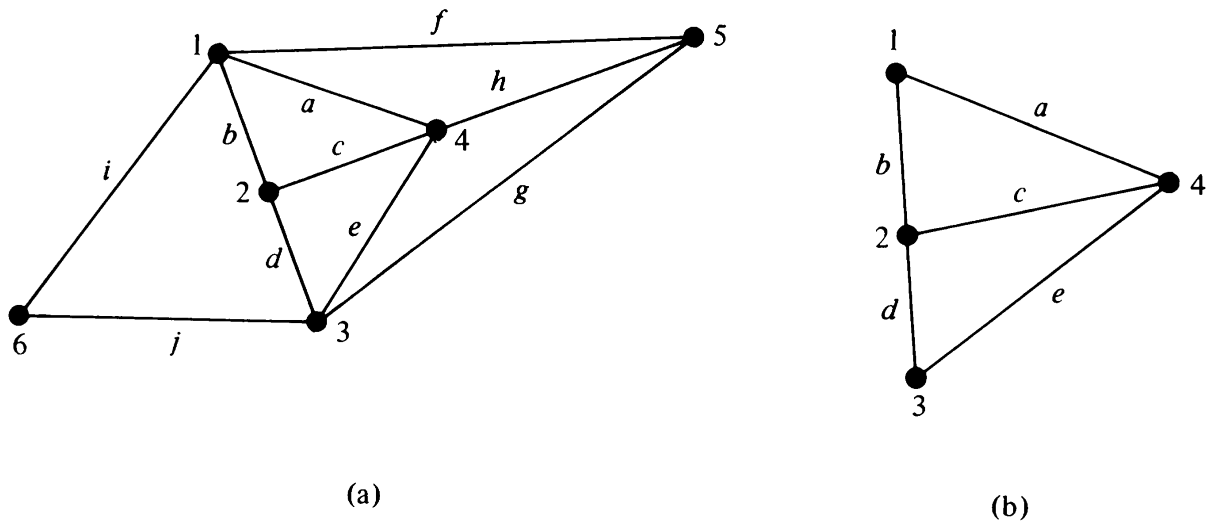


Fig. 2-5 Graph (a) and one of its subgraphs (b).

not be altered by identifying two distinct vertices, or by adding new edges or vertices.) The concept of subgraph is akin to the concept of subset in set theory. A subgraph can be thought of as being contained in (or a part of) another graph. The symbol from set theory, $g \subset G$, is used in stating "g is a subgraph of G."

The following observations can be made immediately:

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

Edge-Disjoint Subgraphs: Two (or more) subgraphs g_1 and g_2 of a graph G are said to be *edge disjoint* if g_1 and g_2 do not have any edges in common. For example, the two graphs in Figs. 2-7(a) and (b) are edge-disjoint subgraphs of the graph in Fig. 2-6. Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common. Subgraphs that do not even have vertices in common are said to be *vertex disjoint.* (Obviously, graphs that have no vertices in common cannot possibly have edges in common.)

2-3. A PUZZLE WITH MULTICOLORED CUBES

Now we shall take a brief pause to illustrate, with an example, how a problem can be solved by using graphs. Two steps are involved here: First, the physical problem is converted into a problem of graph theory. Second, the graph-theory problem is then solved. Let us consider the following problem, a well-known puzzle available in toy stores (under the name *Instant Insanity*).

Problem: We are given four cubes. The six faces of every cube are variously colored blue, green, red, or white. Is it possible to stack the cubes one on top of another to form a column such that no color appears twice on any of the four sides of this column? (Clearly, a trial-and-error method is unsatisfactory, because we may have to try all 41,472 ($= 3 \times 24 \times 24 \times 24$) possibilities.)

Solution: Step 1: Draw a graph with four vertices B, G, R, and W—one for each color (Fig. 2-6). Pick a cube and call it cube 1; then represent its

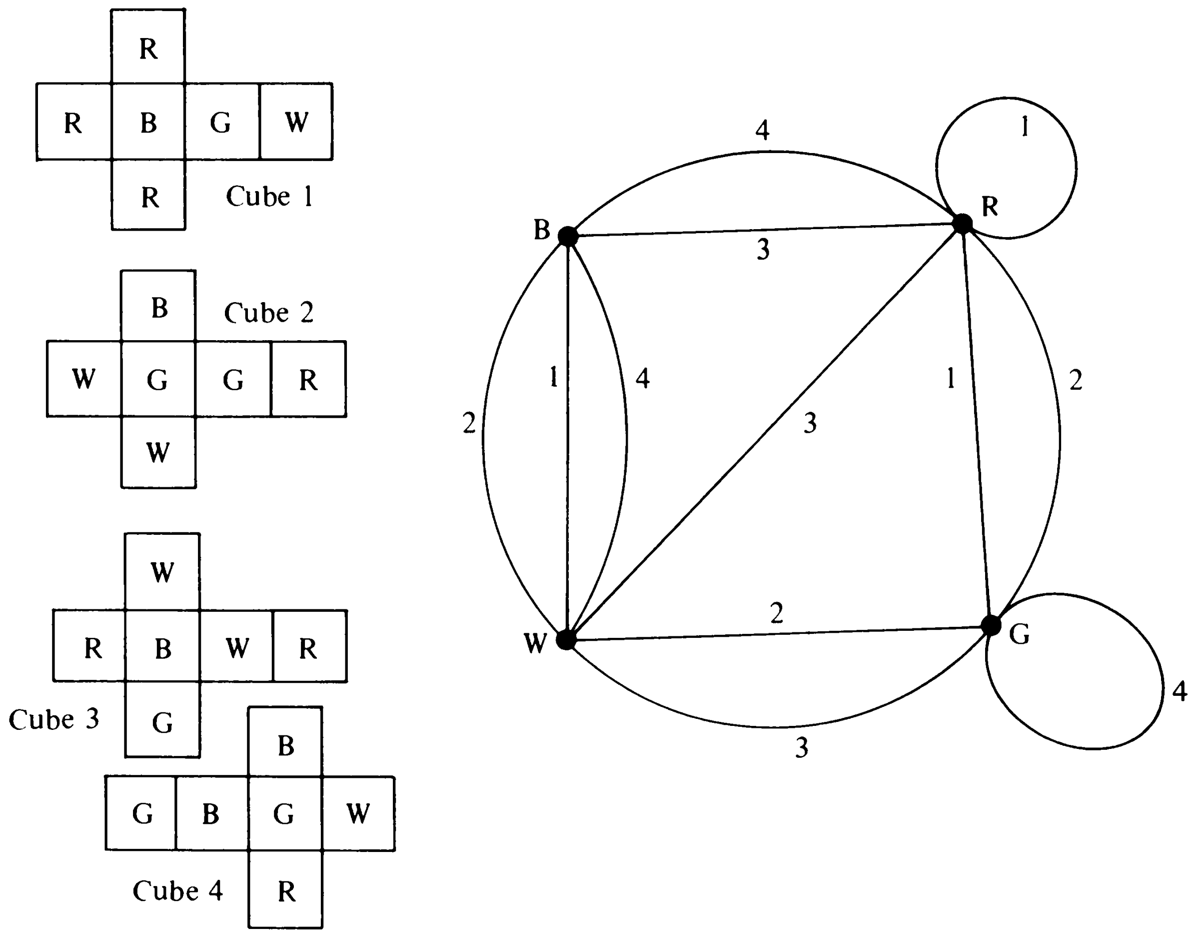


Fig. 2-6 Four cubes unfolded and the graph representing their colors.

three pairs of opposite faces by three edges, drawn between the vertices with appropriate colors. In other words, if a blue face in cube 1 has a white face opposite to it, draw an edge between vertices B and W in the graph. Do the same for the remaining two pairs of faces in cube 1. Put label 1 on all three edges resulting from cube 1. A self-loop with the edge labeled 1 at vertex R, for instance, would result if cube 1 had a pair of opposite faces both colored

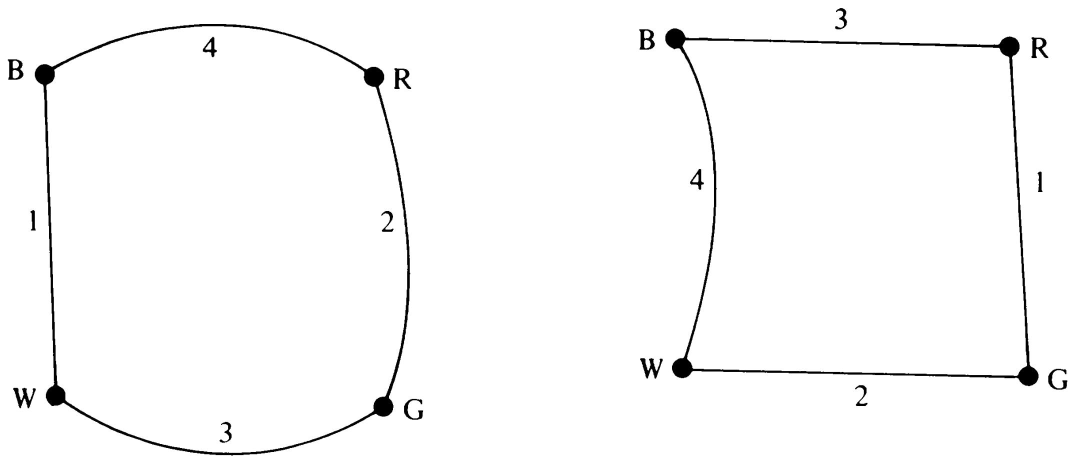
red. Repeat the procedure for the other three cubes one by one on the same graph until we have a graph with four vertices and 12 edges. A particular set of four colored cubes and their graph are shown in Fig. 2-6.

Step 2: Consider the graph resulting from this representation. The degree of each vertex is the total number of faces with the corresponding color. For the cubes of Fig. 2-6, we have five blue faces, six green, seven red, and six white.

Consider two opposite vertical sides of the desired column of four cubes, say facing north and south. A subgraph (with four edges) will represent these eight faces—four facing south and four north. Each of the four edges in this subgraph will have a different label—1, 2, 3, and 4. Moreover, no color occurs twice on either the north side or south side of the column if and only if every vertex in this subgraph is of degree two.

Exactly the same argument applies to the other two sides, east and west, of the column.

Thus the four cubes can be arranged (to form a column such that no color appears more than once on any side) if and only if there exist two edge-disjoint subgraphs, each with four edges, each of the edges labeled differently, and such that each vertex is of degree two. For the set of cubes shown in Fig. 2-6, this condition is satisfied, and the two subgraphs are shown in Fig. 2-7.

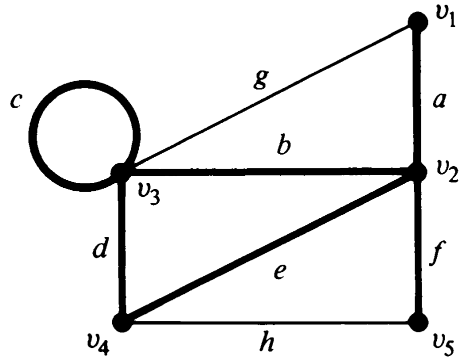


(a) North-South Subgraph (b) East-West Subgraph

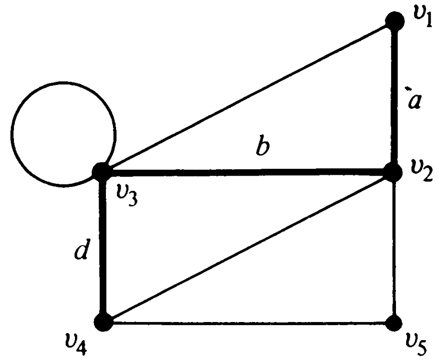
Fig. 2-7 Two edge-disjoint subgraphs of the graph in Fig. 2-6.

2-4. WALKS, PATHS, AND CIRCUITS

A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears (is covered or traversed) more than once in a walk. A vertex, however, may appear more than once. In Fig. 2-8(a), for instance, $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a walk shown with



(a) An Open Walk



(b) A Path of Length Three

Fig. 2-8 A walk and a path.

heavy lines. A walk is also referred to as an *edge train* or a *chain*. The set of vertices and edges constituting a given walk in a graph G is clearly a subgraph of G.

Vertices with which a walk begins and ends are called its *terminal vertices*. Vertices v_1 and v_5 are the terminal vertices of the walk shown in Fig. 2-8(a). It is possible for a walk to begin and end at the same vertex. Such a walk is called a *closed walk*. A walk that is not closed (i.e., the terminal vertices are distinct) is called an *open walk* [Fig. 2-8(a)].

An open walk in which no vertex appears more than once is called a *path* (or a *simple path* or an *elementary path*). In Fig. 2-8, $v_1\ a\ v_2\ b\ v_3\ d\ v_4$ is a path, whereas $v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is not a path. In other words, a path does not intersect itself. The number of edges in a path is called the *length of a path*. It immediately follows, then, that an edge which is not a self-loop is a path of length one. It should also be noted that a self-loop can be included in a walk but not in a path (Fig. 2-8).

The terminal vertices of a path are of degree one, and the rest of the vertices (called *intermediate vertices*) are of degree two. This degree, of course, is counted only with respect to the edges included in the path and not the entire graph in which the path may be contained.

A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*. That is, a circuit is a closed, non-

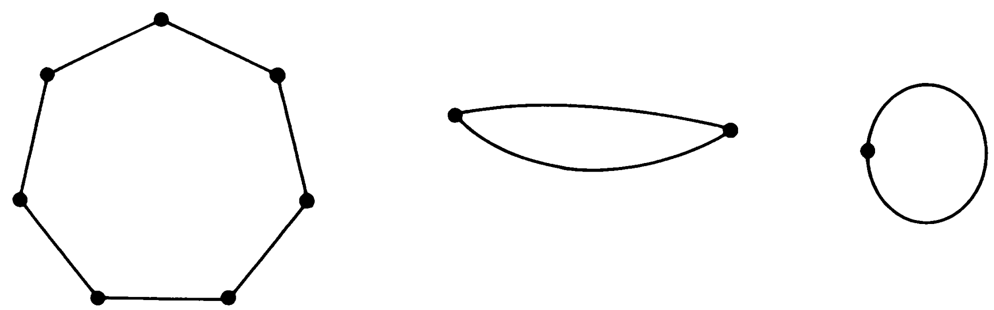


Fig. 2-9 Three different circuits.

intersecting walk. In Fig. 2-8(a), $v_2\ b\ v_3\ d\ v_4\ e\ v_2$ is, for example, a circuit. Three different circuits are shown in Fig. 2-9. Clearly, every vertex in a circuit is of degree two; again, if the circuit is a subgraph of another graph, one must count degrees contributed by the edges in the circuit only.

A circuit is also called a *cycle*, *elementary cycle*, *circular path*, and *polygon*. In electrical engineering a circuit is sometimes referred to as a *loop*—not to be confused with self-loop. (Every self-loop is a circuit, but not every circuit is a self-loop.)

The definitions in this section are summarized in Fig. 2-10. The arrows are in the direction of increasing restriction.

You may have observed that although the concepts of a path and a circuit are very simple, the formal definition becomes involved.

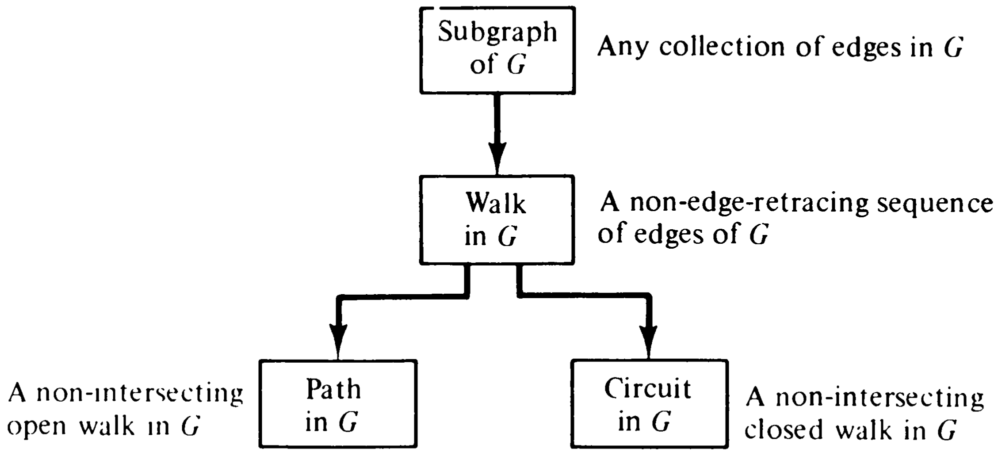


Fig. 2-10 Walks, paths, and circuits as subgraphs.

2-5. CONNECTED GRAPHS, DISCONNECTED GRAPHS, AND COMPONENTS

Intuitively, the concept of *connectedness* is obvious. A graph is connected if we can reach any vertex from any other vertex by traveling along the edges. More formally:

A graph G is said to be *connected* if there is at least one path between every pair of vertices in G. Otherwise, G is *disconnected*. For instance, the graph in Fig. 2-8(a) is connected, but the one in Fig. 2-11 is disconnected. A null graph of more than one vertex is disconnected (Fig. 1-12).

It is easy to see that a disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a *component*. The graph in Fig. 2-11 consists of two components. Another way of looking at a component is as follows: Consider a vertex v_i in a disconnected graph G. By definition, not all vertices of G are joined by paths to v_i. Vertex v_i and all the vertices of G that have paths to v_i, together with all the edges incident on them, form a component. Obviously, a component itself is a graph.

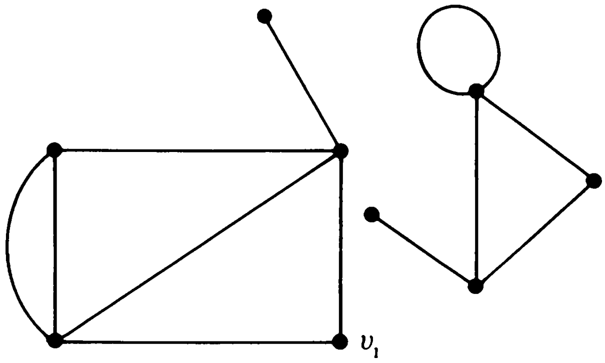


Fig. 2-11 A disconnected graph with two components.

THEOREM 2-1

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof: Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2. Hence, if a partition exists, G is not connected.

Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence the partition. ■

Two interesting and useful results involving connectedness are:

THEOREM 2-2

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: Let G be a graph with all even vertices† except vertices v_1 and v_2, which are odd. From Theorem 1-1, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component, and hence must have a path between them. ■

THEOREM 2-3

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n - k)(n - k + 1)/2$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$. Thus we have

$$n_1 + n_2 + \cdots + n_k = n,$$
$$n_i \geq 1.$$

†For brevity, a vertex with odd degree is called an *odd vertex*, and a vertex with even degree an *even vertex*.

The proof of the theorem depends on an algebraic inequality†

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k). \tag{2-1}$$

Now the maximum number of edges in the ith component of G (which is a simple connected graph) is $\frac{1}{2}n_i(n_i - 1)$. (See Problem 1-12.) Therefore, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i - 1)n_i = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2} \tag{2-2}$$

$$\leq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2}, \qquad \text{from (2-1)}$$

$$= \frac{1}{2}\cdot(n-k)(n-k+1). \quad \blacksquare \tag{2-3}$$

It may be noted that Theorem 2-3 is a generalization of the result in Problem 1-12. The solution to Problem 1-12 is given by (2-3), where $k = 1$.

Now we are equipped to handle the Königsberg bridge problem introduced in Chapter 1.

2-6. EULER GRAPHS

As mentioned in Chapter 1, graph theory was born in 1736 with Euler's famous paper in which he solved the Königsberg bridge problem. In the same paper, Euler posed (and then solved) a more general problem: In what type of graph G is it possible to find a closed walk running through every edge of G exactly once? Such a walk is now called an *Euler line*, and a graph that consists of an Euler line is called an *Euler graph*. More formally:

If some closed walk in a graph contains all the edges of the graph, then the walk is called an *Euler line* and the graph an *Euler graph*.

By its very definition a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an *Euler graph* is always connected, except for any isolated vertices the graph may have. Since isolated vertices do not contribute anything to the understanding of an Euler graph, it is hereafter assumed that Euler graphs do not have any isolated vertices and are therefore connected.

Now we shall state and prove an important theorem, which will enable us to tell immediately whether or not a given graph is an Euler graph.

†*Proof:* $\sum_{i=1}^{k}(n_i - 1) = n - k$. Squaring both sides,

$$\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

or $\sum_{i=1}^{k}(n_i^2 - 2n_i) + k +$ nonnegative cross terms $= n^2 + k^2 - 2nk$ because $(n_i - 1) \geq 0$, for all i. Therefore, $\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n = n^2 - (k-1)(2n-k)$. $\blacksquare$

THEOREM 2-4

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

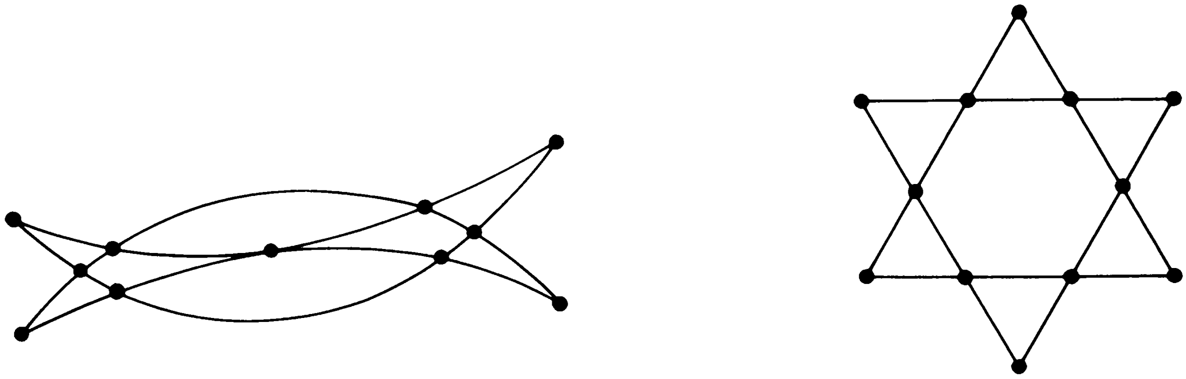
Proof: Suppose that G is an Euler graph. It therefore contains an Euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v—with one we "entered" v and with the other "exited." This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph. ■

Königsberg Bridge Problem: Now looking at the graph of the Königsberg bridges (Fig. 1-5), we find that not all its vertices are of even degree. Hence, it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

One often encounters Euler lines in various puzzles. The problem common to these puzzles is to find how a given picture can be drawn in one continuous line without retracing and without lifting the pencil from the paper. Two such pictures are shown in Fig. 2-12. The drawing in Fig. 2-12(a) is called *Mohammed's scimitars* and is believed to have come from the Arabs. The one in Fig. 2-12(b) is, of course, the *star of David.* (Equal time!)

In defining an Euler line some authors drop the requirement that the walk be closed. For example, the walk a 1 c 2 d 3 a 4 b 5 d 6 e 7 b in Fig. 2-13, which includes all the edges of the graph and does not retrace any edge, is not closed. The initial vertex is a and the final vertex is b. We shall call such an open walk that includes (or traces or covers) all edges of a graph without retracing any edge a *unicursal line* or an *open Euler line*. A (connected) graph that has a unicursal line will be called a *unicursal graph*.



(a) (b)

Fig. 2-12 Two Euler graphs.

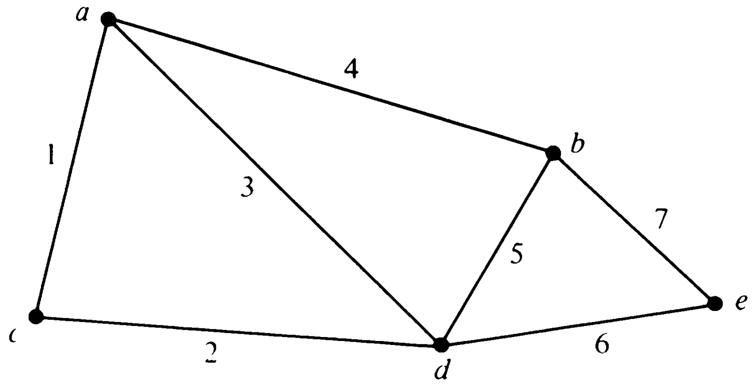


Fig. 2-13 Unicursal graph.

It is clear that by adding an edge between the initial and final vertices of a unicursal line we shall get an Euler line. Thus a connected graph is unicursal if and only if it has exactly two vertices of odd degree. This observation can be generalized as follows:

THEOREM 2-5

In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Proof: Let the odd vertices of the given graph G be named $v_1, v_2, \ldots, v_k$; $w_1, w_2, \ldots, w_k$ in any arbitrary order. Add k edges to G between the vertex pairs (v_1, w_1), (v_2, w_2), $\ldots$, (v_k, w_k) to form a new graph G'.

Since every vertex of G' is of even degree, G' consists of an Euler line p. Now if we remove from p the k edges we just added (no two of these edges are incident on the same vertex), p will be split into k walks, each of which is a unicursal line: The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem. ■

We shall interrupt our study of Euler graphs to define some commonly used graph-theoretic operations. One of these operations is required immediately in the next section; others will be needed later.

2-7. OPERATIONS ON GRAPHS

As is the case with most mathematical entities, it is convenient to consider a large graph as a combination of small ones and to derive its properties from those of the small ones. Since graphs are defined in terms of the sets of vertices and edges, it is natural to employ the set-theoretical terminology to define operations between graphs. In particular:

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \cup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$. Likewise, the *intersection* $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2. The *ring sum* of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but *not* in both. Two graphs and their union, intersection, and ring sum are shown in Fig. 2-14.†

It is obvious from their definitions that the three operations just mentioned are commutative. That is,

$$G_1 \cup G_2 = G_2 \cup G_1, \qquad G_1 \cap G_2 = G_2 \cap G_1,$$
$$G_1 \oplus G_2 = G_2 \oplus G_1.$$

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph G,

$$G \cup G = G \cap G = G,$$

and

$$G \oplus G = \text{a null graph.}$$

If g is a subgraph of G, then $G \oplus g$ is, by definition, that subgraph of G which remains after all the edges in g have been removed from G. Therefore, $G \oplus g$ is written as $G - g$, whenever $g \subseteq G$. Because of this complementary nature, $G \oplus g = G - g$ is often called the complement of g in G.

Decomposition: A graph G is said to have been *decomposed* into two subgraphs g_1 and g_2 if

$$g_1 \cup g_2 = G,$$

and

$$g_1 \cap g_2 = \text{a null graph.}$$

†If an edge e_i is in two graphs G_1 and G_2, its end vertices in G_1 must have the same labels as in G_2.

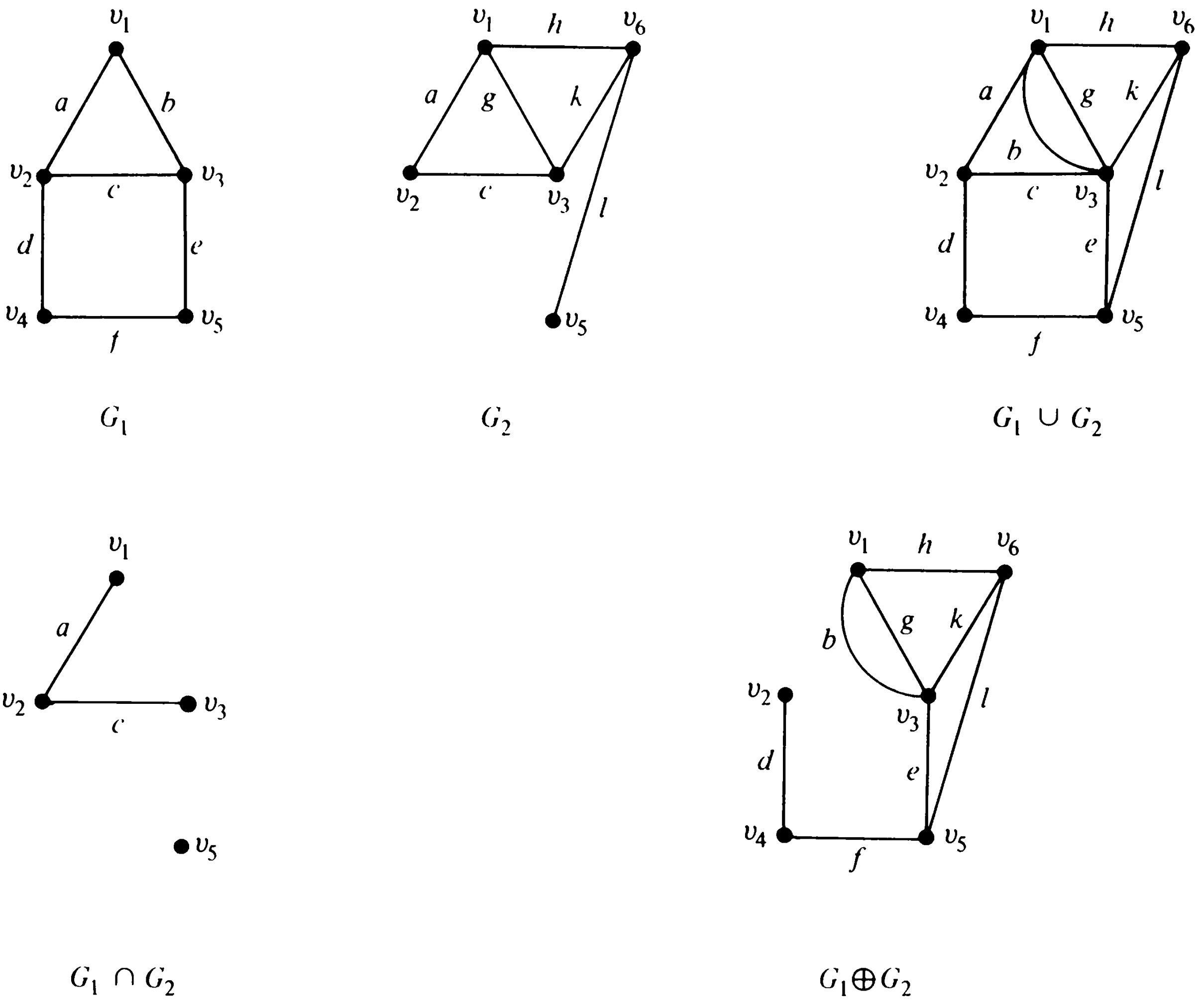


Fig. 2-14 Union, intersection, and ring sum of two graphs.

In other words, every edge of G occurs either in g_1 or in g_2, but not in both. Some of the vertices, however, may occur in both g_1 and g_2. In decomposition, isolated vertices are disregarded. A graph containing m edges $\{e_1, e_2, \ldots, e_m\}$ can be decomposed in $2^{m-1} - 1$ different ways into pairs of subgraphs g_1, g_2 (why?).

Although union, intersection, and ring sum have been defined for a pair of graphs, these definitions can be extended in an obvious way to include any finite number of graphs. Similarly, a graph G can be decomposed into more than two subgraphs—subgraphs that are (pairwise) edge disjoint and collectively include every edge in G.

Deletion: If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting (i.e., removing) v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex. (See Fig. 2-15.) If e_j is an edge in G, then $G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices. Therefore $G - e_j = G \oplus e_j$.

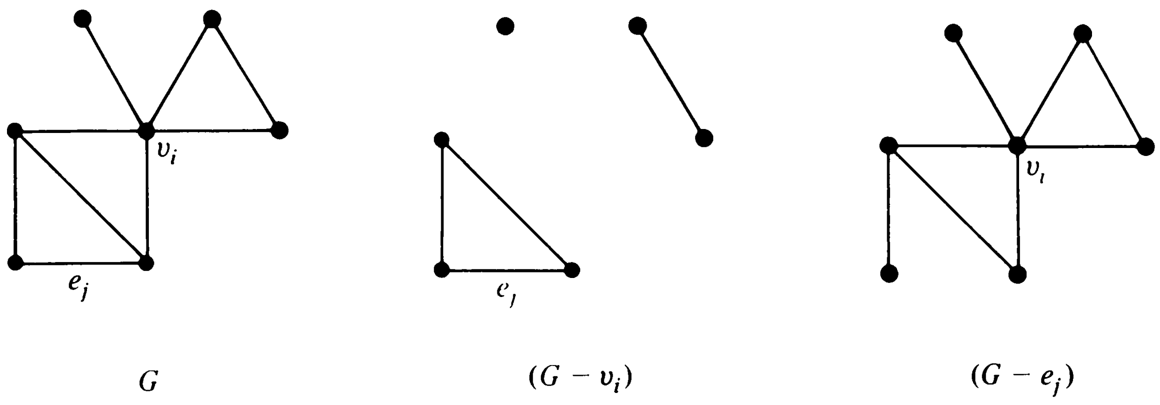


Fig. 2-15 Vertex deletion and edge deletion.

Fusion: A pair of vertices a, b in a graph are said to be *fused* (*merged* or *identified*) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one. See Fig. 2-16 for an example.

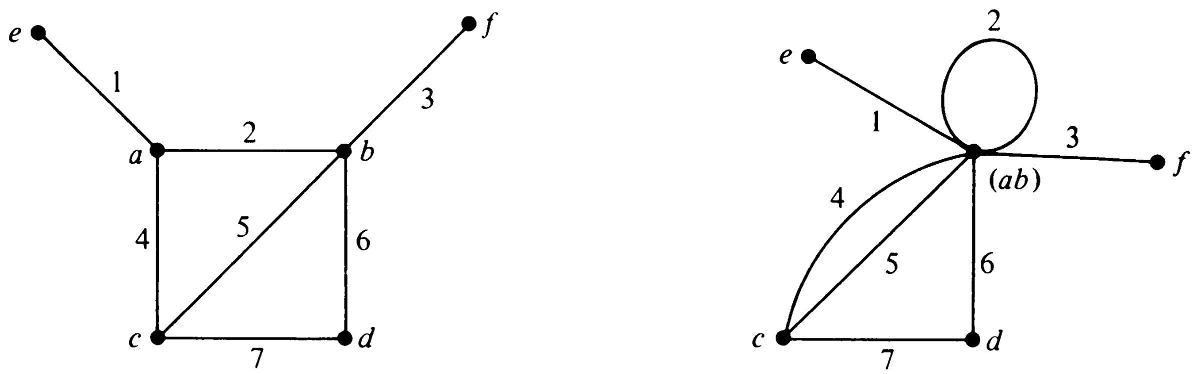


Fig. 2-16 Fusion of vertices a and b.

These are some of the elementary operations on graphs. More complex operations have been defined and are used in graph-theory literature. For a survey of such operations see the paper by Harary and Wilcox [2-10].

2-8. MORE ON EULER GRAPHS

The following are some more results on the important topic of Euler graphs.

THEOREM 2-6

A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Proof: Suppose graph G can be decomposed into circuits; that is, G is a union of edge-disjoint circuits. Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even. Hence G is an Euler graph.

Conversely, let G be an Euler graph. Consider a vertex v_1. There are at least two edges incident at v_1. Let one of these edges be between v_1 and v_2. Since vertex v_2 is also of even degree, it must have at least another edge, say between v_2 and v_3. Proceeding in this fashion, we eventually arrive at a vertex that has previously been traversed, thus forming a circuit Γ. Let us remove Γ from G. All vertices in the remaining graph (not necessarily connected) must also be of even degree. From the remaining graph remove another circuit in exactly the same way as we removed Γ from G. Continue this process until no edges are left. Hence the theorem. ■

Arbitrarily Traceable Graphs: Consider the graph in Fig. 2-17, which is an Euler graph. Suppose that we start from vertex a and trace the path $a\ b\ c$.

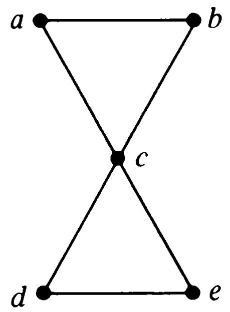


Fig. 2-17 Arbitrarily traceable graph from c.

Now at c we have the choice of going to a, d, or e. If we took the first choice, we would only trace the circuit $a\ b\ c\ a$, which is not an Euler line. Thus, starting from a, we cannot trace the entire Euler line simply by moving along any edge that has not already been traversed. This raises the following interesting question:

What property must a vertex v in an Euler graph have such that an Euler line is always obtained when one follows any walk from vertex v according to the single rule that whenever one arrives at a vertex one shall select *any* edge (which has not been previously traversed)?

Such a graph is called an *arbitrarily traceable graph from vertex v*. For instance, the Euler graph in Fig. 2-17 is an arbitrarily traceable graph from vertex c, but not from any other vertex. The Euler graph in Fig. 2-18 is not arbitrarily traceable from any vertex; the graph in Fig. 2-19 is arbitrarily

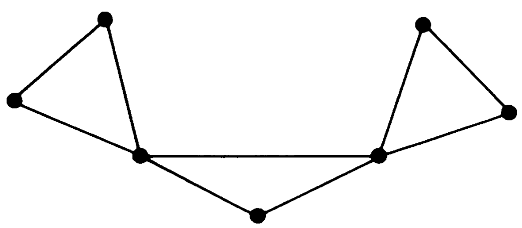


Fig. 2-18 Euler graph; not arbitrarily traceable.

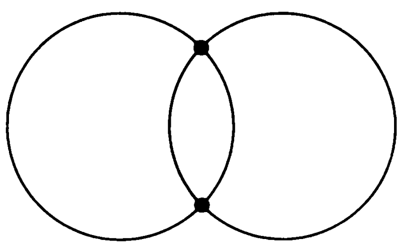


Fig. 2-19 Arbitrarily traceable graph from all vertices.