Contd.

Tautology, Contradiction & Contingency

A statement which is true for all possible values of its constituent propositional variables is called a tautology.

A statement which is false for all possible values of its constituent propositional variables is called a contradiction.

A statement which is neither a tautology nor a contradiction is called a contingency.

Contd.

The statements $p \vee \sim p$, $p \wedge \sim p$, $\sim p \vee q$ is a tautology, contradiction & a contingency respectively.

p	q	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$	$\sim p \vee q$
T	**T**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**F**	**T**

Contd.

Logical Equivalence

Two propositions P and Q are said to be logically equivalent if $P \leftrightarrow Q$ is a tautology. If they are logically equivalent we write $P \equiv Q$.

Ex. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Check $p \rightarrow q \equiv \sim p \vee q$?

Contd.

Algebra of Proposition

The operations ~, ∧, ∨ for any proposition p,q,r satisfy the following proposition

1. Commutative properties

a) $p \vee q \equiv q \vee p$ b) $p \wedge q \equiv q \wedge p$

2. Associative properties

a) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

b) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

3. Distributive properties

a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

4. Idempotent properties

a) $p \vee p \equiv p$ b) $p \wedge p \equiv p$

5. Properties of components

a) $p \vee \sim p \equiv T$ b) $p \wedge \sim p \equiv F$

c) $\sim T \equiv F$ d) $\sim F \equiv T$

Contd.

De Morgan's Laws

e) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

f) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Involution property

g) $\sim(\sim p) \equiv p$

6. Identity properties

a) $p \wedge T \equiv p$

b) $p \wedge F \equiv F$

c) $p \vee T \equiv T$

d) $p \vee F \equiv p$

Contd.

Well Formed Formulas(wff)

A statement formula is a logical expression consisting of variables, parentheses & connective symbols. A statement formula is not a statement & has no truth value. If we substitute definite statements in place of variable in the given formula we get a statement.

A statement formula is called a wff if it can be generated by the following rules:

<W1> a statement variable p standing alone is a wff.

Contd.

<W2> if p is a wff then ~p is a wff

<W3> if p and q are wff then $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(p \leftrightarrow q)$ are wff.

<W4> a string of symbols is a wff iff it is obtained by finitely many applications of the rules W1, W2 and W3.

Ex. $\sim(p \vee q)$, $p \wedge \sim q$, $p \rightarrow (\sim p \wedge \sim q)$, $(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$

Contd.

Normal Forms:

The problem of determining in a finite no. of steps whether a given statement formula is a tautology or a contradiction or at least a contingency is known as a decision problem in proposition calculus.

We can check using truth table. The construction of truth table becomes tedious when the number of variables increases. If there are n variables in P or Q then 2^n rows in truth table.

Contd.

A better approach is to transform the expressions P and Q to some standard forms of expressions say, P’ and Q’ such that a simple comparison of P’ and Q’ shows whether $P \equiv Q$. These standard forms are called **normal forms or canonical forms.** There are 2 types of normal forms namely **disjunctive normal form(DNF) & conjunctive normal form(CNF).**

Contd.

Show how to find a specific formula from a given truth table.

Let, the truth table for the formula P(p,q,r)

p	q	r	P(p,q,r)
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

For true value

$P(p,q,r) \equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \rightarrow (1)$

For false value

$\sim T \equiv F$

$P(p,q,r) \equiv (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r) \rightarrow (2)$

$Eq^n(1)$ called sum of product(SOP)

$Eq^n(2)$ called product of sum(POS)

Contd.

Disjunctive Normal forms(DNF)

A logical expression which is equivalent to a given logical expression & which consists of a sum of elementary products is called a DNF of the given logical expression.

Ex. $p \vee (q \wedge r)$, $(p \wedge q) \vee (q \wedge \sim p)$,
$(p \wedge q) \vee (\sim p \wedge r) \vee (r \wedge \sim q)$

Procedure to obtain DNF of a given logical expression:

Step1. Remove conditional ($\rightarrow$) & biconditional ($\leftrightarrow$) from the given expression & obtain an equivalent expression containing the connectives $\sim$, $\wedge$, $\vee$ only.

Contd.

[replace $p \rightarrow q$ by $\sim p \vee q$

$p \leftrightarrow q$ by $(p \wedge q) \vee (\sim p \wedge \sim q)$ or by

$(\sim p \vee q) \wedge (p \vee \sim q)$]

Step2. Use De Morgan's law to eliminate ~ before sum & products by using double negation.

Step3. Apply distributive properties until a sum of elementary products is obtained.

Contd.

Ex.

Find the DNF of the expression $\sim(p \vee q) \leftrightarrow (p \wedge q)$

We know if $r \leftrightarrow s$

Then $r \leftrightarrow s \equiv (r \wedge s) \vee (\sim r \wedge \sim s)$

$\therefore \sim(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv (\sim(p \vee q) \wedge (p \wedge q)) \vee (\sim(\sim(p \vee q)) \wedge \sim(p \wedge q))$

$\equiv ((\sim p \wedge \sim q) \wedge (p \wedge q)) \vee (p \vee q) \wedge (\sim p \vee \sim q))$

[using De Morgan's Law]

$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$

[by distributive property]

$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$ [by distributive property]

Which is required DNF.

Contd.

Conjunctive Normal Forms(CNF)

A logical expression which is equivalent to a given logical expression & which consists of a product of elementary sums is called a CNF of the given expression.

The procedure to convert in CNF is similar to DNF.

Find the CNF of the expression $\sim(p \vee q) \leftrightarrow (p \wedge q)$

We know $r \leftrightarrow s \equiv (r \rightarrow s) \wedge (s \rightarrow r)$

Thus $\sim(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv (\sim(p \vee q) \rightarrow (p \wedge q) \wedge ((p \wedge q) \rightarrow \sim(p \vee q))$

$\equiv (\sim(\sim(p \vee q)) \vee (p \wedge q)) \wedge (\sim(p \wedge q) \vee (\sim(p \vee q)))$

[as $r \rightarrow s \equiv \sim r \vee s$]

Contd.

$\equiv ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$

[by De Morgan’s law]

$\equiv ((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$

[by distributive property]

$\equiv ((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$

[by distributive property]

$\equiv (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$

Which is required CNF.

Contd.

Without forming truth table prove that

$$p \vee (p \wedge q) \equiv p$$

Sol^{n}: $p \vee (p \wedge q)$

$\equiv (p \wedge T) \vee (p \wedge q)$ [as $p \wedge T \equiv p$]

$\equiv p \wedge (T \vee q)$ [by distributive property]

$\equiv p \wedge T$ [as $T \vee q \equiv T$]

$\equiv p$

$\therefore p \vee (p \wedge q) \equiv p$ (proved)