1.11.1 Direct Proof

It uses the implication $(h_1 \wedge h_2 \wedge h_3 \wedge ... \wedge h_n) \to c$ which is proved to be a tautology where $h_1, h_2, h_3, ..., h_n$ are hypotheses and c is the conclusion.

Illustrations

Problem 1. Prove that product of two odd integers is always odd.

Proof. Let m and n be two odd integers. Then there exists integers r and s such that

$m = 2r + 1$ and $n = 2s + 1$

Then $mn = (2r+1)(2s+1)$

$= 4rs + 2r + 2s + 1$ which is an odd integer as $4rs + 2r + 2s$ is an even integers

Thus (m is and odd integer) $\wedge$ (n is an odd integer) $\to$ (mn is an odd iteger) is universally valid. Hence the proof.

Problem 2. Prove that sum of two rational numbers is a rational number.

Proof. Let x and y be two rational numbers. Then there exists integers p, q, r, s such that $x = \frac{p}{q}$ and $y = \frac{r}{s}$

Now $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$

Since $(ps + rq)$ and qs are integers, therefore, $x + y$ is a rational number. Thus (x is a rational number) $\wedge$ (y is rational number) $\to$ ($x + y$ is a rational number) is univerally valid. Hence the proof.

1.11. 2. Indirect Proof

If uses the tautology $(p \to q) \leftrightarrow ((\sim q) \to (\sim p))$. This states that *an implication is equivalent to its contrapositive.* Thus to prove $p \to q$ indirectly, we assume that q is false (the statement $\sim q$) and show that p is then false (the statement $\sim p$).

Illustration

Problem. Let n be an integer. Prove that if n^2 is odd then n is also odd.

Proof. Let us consider the statements, p : " n^2 is odd" and q : " n is odd". Then it is required to prove that $p \to q$ is true whenever p and q both are true.

Instead, we prove the contrapositive $\sim q \to \sim p$. Suppose that n is not odd (the statement $\sim q$) i.e., n is even. Let $n = 2k$, where k is an integer.

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so n^2 is even (the statement $\sim p$).

Thus we show that " if n is even then n^2 is also even", which is the contrapositive of the given statement. Hence the implication "if n^2 is odd then n is odd" is universally true, Hence the proof.

1.11. 3. Proof by Contradiction

This method is based on the tautology $((p \to q) \wedge (\sim q)) \to (\sim p)$. In other words, the argument

is valid. $$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

Informally, it states that, *if a statement p implies a false statement q, then p must also be false.*

This method is applied to the case where q is a contradiction in the above argument. Usually in such a case, q is taken as the contradiction $r \wedge \sim r$. Thus any statement that implies a contradiction must be false. The use of this method is as follows.

Suppose we wish to show that a statement q is logically follows from the statements $p_1, p_2, \dots, p_n$. Assume that $\sim q$ is true (i.e. q is false). Introduce this statement as an additional hypothesis.

Now if it can be shown that the hypothesis $p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge (\sim q)$ implies a contradiction, then one of the statements $p_1, p_2, \ldots, p_n$, $\sim q$ must be false. It follows that if all p_i's are ture then $\sim q$ must be false, i.e., q must be true. Thus q follows from $p_1, p_2, \ldots, p_n$.

Illustration

Problem. Prove that $\sqrt{2}$ is not a rational number.

Proof. Let q : "$\sqrt{2}$ is not a rational number". We assume that $\sim q$ is true, i.e., $\sqrt{2}$ is a rational number.

Then there exist integers m and n that are mutually prime such that $\sqrt{2} = \dfrac{m}{n}$.

Then $m^2 = 2n^2$ ... (1)

$\therefore m^2$ is even. It implies that m is even.

Let $m = 2k$ where k is an integer.

Putting in (1) we get $2n^2 = 4k^2$ or $n^2 = 2k^2$ implying that n^2 is even and so n is even. Thus m, n both are even and as such they are not prime to each other as they have a common factor 2. Hence $\sim q$ is false. Thus our q is true. Thus $\sqrt{2}$ is not a rational number. Hence proved.