1.11.4. Proof by Cases

This method utilizes the fact that the implication $h_1 \vee h_2 \vee \ldots \vee h_n \to c$ is equivalent to $(h_1 \to c) \wedge (h_2 \to c) \wedge \ldots \wedge (h_n \to c)$ where $h_1, h_2, \ldots, h_n$ are hypotheses and c is the conclusion. Thus if the cases $h_1 \to c, h_2 \to c, \ldots, h_n \to c$ are each proved to be true separately then $h_1 \vee h_2 \vee, \ldots, h_n \to c$ will be proved.

Illustration

Problem. Prove that for every positive integer n, $n^3 + n$ is always even.

Proof. Let h_1: "n is a positive even integer."

h_2: "n is a positive odd integer." and

c: "n^3+n is an even integer."

Case 1 : To prove that $h_1 \to c$ is true.

Let n be a positive even integer.

Then $n = 2k$ for same positve integer k.

$$\therefore n^3+n = 8k^3+2k = 2(4k^3+k) \text{ which is even.}$$

Thus $h_1 \to c$ is true.

Case 2 : To prove that $h_2 \to c$ is true.

Let n be a positive odd integer.

Then $n = 2k+1$ for some positive integer k.

$$\therefore n^3+n = (8k^3+12k^2+6k+1)+(2k+1)$$

$$= 2(4k^3+6k^2+4k+1) \text{ which is even.}$$

Thus $h_2 \to c$ is true.

Since $(h_1 \to c)$ and $(h_2 \to c)$ both are true, therefore, $(h_1 \to c) \wedge (h_2 \to c)$ is true, i.e., $(h_1 \vee h_2) \to c$ is true.

Since $h_1 \vee h_2$ means n is a positive integer (even or odd), the proof is complete.

1.11. 5. Proof by the Principles of Mathmatical Induction

This is an important method of proof of those theorems that are based on positive integers or natural numbers.

Let us give the formal statement of these principles.

First Principle of Mathematical Induction : Let $P(n)$ be a proposition involving positive integer n. Then $P(n)$ is true for all positive integral values of n provided that

(i) $P(1)$ is true,

and (ii) $P(m+1)$ is true whenever $P(m)$ is true for some positive integer m.

Thus there are three steps in the proof by Mathematical Induction.

- **Inductive base :** Verify that $P(1)$ is true.
- **Inductive hypothesis :** Assume that $P(m)$ is true for some arbitrary positive integer m.
- **Inductive step :** Verify that $P(m+1)$ is true on the basis of inductive hypothesis.

Note that in the inductive step we are to prove that $P(m) \rightarrow P(m+1)$ is a tautology for any choice of positive integer m.

Another form of First Principle of Mathematical Induction is as follows :

Let $P(m)$ is a proposition involving positive integer n. Then $P(n)$ is true for all positive integral values of n provided that

(i) $P(n_0)$ is true for some fixed positive integer n_0

and (ii) $P(m+1)$ is true whenever $P(m)$ is true for some positive integer $m \geq n_0$.

Second Principle of Mathematical Induction

Let $P(n)$ be a proposition for $n \in \mathbb{Z}^+$. Then $P(n)$ is true for all positive integral value of n provided

(i) $P(1)$ is true

and (ii) $P(m+1)$ true whenever $P(k)$ is true for all

$$k \in \{1, 2, 3, \ldots, m\} \in \mathbb{Z}^+$$

Illustration

Problem. Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Proof. Let $P(n)$ be the proposition

$$P(n): \text{“ } 1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2 \text{ ”}$$

where n is a positive integer.

Inductive base : $1^3=\left[\frac{1.(1+1)}{2}\right]^3$. $\therefore P(1)$ is true

Inductive hypothesis : Let $P(m)$ be true for some arbitrary positive integer m.

Then $1^3+2^3+\ldots+m^3=\left[\frac{m(m+1)}{2}\right]^2$

Inductive step :

$$1^3+2^3+\ldots+m^3+(m+1)^3=\left[\frac{m(m+1)}{2}\right]^2+(m+1)^3$$

$$=\frac{(m+1)^2}{4}\left[m^2+4(m+1)\right]=\frac{(m+1)^2(m+2)^2}{4}=\left[\frac{(m+1)\{(m+1)+1\}}{2}\right]^2.$$

$\therefore P(m+1)$ is true whenever $P(m)$ is true for arbitrary positive integer m. But $P(1)$ is true. Hence by the Principle of Mathematical Induction, $P(n)$ is true for all positive integral values of n.

$$\therefore 1^3+2^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2 \quad \textit{(Proved.)}$$

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