Negation of a Quantified Statement

Parts (3) & (4) of the following theorem show the equivalences of negated quantified statements.

Theorem:

Let the universe of discourse be a finite set $S = \{a_1, a_2, \dots, a_n\}$ & $P(x)$ be a propositional fn defined on S. Then

1) $(\forall x)\, P(x) \equiv P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n)$
2) $(\exists x)\, P(x) \equiv P(a_1) \vee P(a_2) \vee \dots \vee P(a_n)$
3) $\sim(\forall x\, P(x)) \equiv (\exists x) \sim P(x)$
4) $\sim((\exists x)\, P(x)) \equiv (\forall x) \sim P(x)$

Proof: (1) & (2) follows from the def^n of quantifier.

(3) By (1), $\sim(\forall x\, P(x)) \equiv \sim(P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n))$

$$\equiv \sim P(a_1) \vee \sim P(a_2) \vee \dots \vee \sim P(a_n)$$

[by De Morgan's Law]

$$\equiv (\exists x) \sim P(x) \quad [\text{by } (2)]$$

Hence proved.

(4) By (2), $\sim(\exists x\, P(x)) \equiv \sim(P(a_1) \vee P(a_2) \vee \cdots \vee P(a_n))$

$\equiv \sim P(a_1) \wedge \sim P(a_2) \wedge \cdots \wedge \sim P(a_n)$

[by De Morgan's Law]

$\equiv (\forall x\, P(x))$ [by (1)]

Hence proved.

Illustration

Consider the statement: "All students of B.Tech have taken the course of Discrete Mathematics". It can be written as

"$\forall x\, P(x)$" where

domain of $x \rightarrow$ set of all students of B.Tech

$P \rightarrow$ is the predicate "have taken the course of Discrete Mathematics"

The negation of the above statement is "It is not the case that all students of B.Tech. have taken the course of Discrete Mathematics".

or equivalently,

"There exists a student of B.Tech. who has not taken the course of Discrete Mathematics."

Hence the negation of "$\forall x\, P(x)$" is $\exists x \sim P(x)$ i.e., $\sim((\forall x)\, P(x)) \equiv ((\exists x) \sim P(x))$

which verifies part(3) theorem.

Again consider the statement:

"There is a student of B.Tech. who has taken the course of ~~dis~~ Discrete Mathematics".

This can be written as $\exists x\, P(x)$ where

domain of $x \rightarrow$ set of all students of B.Tech

$P \rightarrow$ predicate "has the course of Discrete Mathematics".

The negation of the above statement is

"It is not the case that there is a student of B.Tech who has taken the course of Discrete Mathematics".

or equivalently,

"All students of B.Tech have not taken the course of Discrete Mathematics".

Hence the negation of
$(\exists x)\, P(x)$ is $(\forall x) \sim P(x)$ i.e.,
$\sim((\exists x)\, P(x)) \equiv (\forall x) \sim P(x)$ which verifies part (4) ~~of~~ theorem.

Quantified statements	Negated Quantified statements
1). $(\forall x)\, P(x)$ (all are true)	i) $(\exists x) \sim P(x)$ (at least 1 is false)
2). $(\forall x) \sim P(x)$ (all are false)	2) $(\exists x)\, P(x)$ (at least 1 is true)
3). $(\exists x)\, P(x)$ (at least 1 is true)	3) $(\forall x) \sim P(x)$ (all are false)
4) $(\exists x) \sim P(x)$ (at least 1 is false)	4) $(\forall x)\, P(x)$ (all are true)

Table: Equivalences of quantified statements & their negations