Contd.

Ex1. Let (S,o) be a semigroup with the identity element e. If for each aϵS there exists an element a‘ in S such that a o a‘ = e. Show that (S,o) is a group.

Sol^n: It is given that e o a = a o e = a $\forall$ aϵS.

By the cond^n each element in S has a right inverse in S.

Let a'‘ be a right inverse of a‘

i.e. a‘o a'‘ = e.

Now a‘o a = (a‘o a) o e = (a‘o a) o (a‘o a'‘)

= a‘o (a o a‘) o a'‘, since o is associative

=(a‘o e) o a'‘ = a‘o a'‘ = e

Thus a‘o a = a o a‘ = e & this shows that a‘ is the inverse of a.

Thus each element in S has an inverse & hence (S,o) is a group.

Contd.

Ex2. Let (S,o) be a semigroup with a right identity element e. If for every two distinct elements a,b ϵ S there exists a unique x in S such that a o x = b. Show that (S,o) is a group.

Sol^n: By the cond^n, a o e = a ∀ a ϵ S.

Let, a≠e. Then there exists a unique element a' in S such that a o a' = e. This shows that a has a right e-inverse.

By the cond^n e o e = e. This shows that e has a right e-inverse.

Consequently, each element in S has a right e-inverse.

Thus (S. o) is a semigroup with a right identity element e and each element in S has a right e-inverse. Therefore (S, o) is a group.

Contd.

Ex3. Let (G,o) be a group & $a \in G$. Show that $aG = G$ where $aG=\{a \circ g : g \in G\}$

Sol^n: Let, $p \in aG$. Then $p = a \circ g$ for some $g \in G$

$a \circ g \in G$ since $a \in G$ & $g \in G$.

Therefore $p \in G$.

Thus $p \in aG \Rightarrow p \in G$. Therefore $aG \subset G$ --> (i)

Let, $q \in G$. There exists a unique element x in G such that $a \circ x = q$.

As $q = a \circ x$ & $x \in G$, $q \in aG$

Thus $q \in G \Rightarrow q \in aG$.

Therefore $G \subset aG$ --> (ii)

From (i) & (ii) $G=aG$.

Contd.

Subgroups

Let, (G,o) be a group & H be a non empty subset of G. H is said to be stable(closed) under o if $a \epsilon H$, $b \epsilon H \Rightarrow aob \, \epsilon H$. If H is stable under o then the restriction of o to $H \times H$ is a mapping from $H \times H \rightarrow H$. This restriction say * is a composition on H & is defined by $a*b=aob \ \forall \ a,b \epsilon H$. * is called the induced composition on H.

Def^n: Let (G,o) be a group & H be a non empty subset of G. If (H,o) is a group where o is the induced composition then (H,o) is said to be a subgroup of (G,o).

Ex1. Let (G,o) be a group & e be the identity element. G being a subset of G, (G,o) is a subgroup of (G,o). This subgroup (G,o) is said to be the improper subgroup of (G,o).

The singleton set {e} forms a group under the induced composition o. The subgroup ({e},o) is said to be the trivial subgroup of (G,o).

Contd.

The subgroup other than (G,o) and ({e},o) are said to be nontrivial proper subgroups of (G,o).

Ex2. (Q,+) is a group. Z is a non empty subset of Q and (Z,+) is a group. Therefore (Z,+) is a subgroup of (Q,+).

Ex3. (Q,+) is group. Q*=Q-{0} is a subset of Q and (Q*,.) is a group. But (Q*,.) is not a subgroup of (Q,+).

Contd.

Theorem: Let, (H,o) be a subgroup of (G,o). Then

i) the identity element of (H,o) is the identity element of (G,o)

ii) if $a \epsilon H$ then the inverse of a in (H,o) is same as the inverse of a in (G,o).

Proof: i) Let, e_H be the identity element in (H,o) & e_G be the identity element in (G,o).

Then e_H o h = h o e_H = h $\forall$ h in H.

Also e_G o h = h o e_G = h, considering h as an element of G.

If follows that h o e_H = h o e_G in G.

Therefore $e_H = e_G$ by left cancellation law in (G,o).

ii) Let, a' be the inverse of a in (H,o) & a^{-1} be the inverse of a in (G,o).

Then a'o a = a o a' = e_H, since (H,o) is a group.

Contd.

Also a^{-1} o a = a o a^{-1} = e_G

It follows that a’ o a = a^{-1} o a in (G,o) by (i)

Therefore a' = a^{-1} , by right cancellation law in (G,o).

Theorem: Let, (G,o) be a group. A non-empty subset H of G form a subgroup of (G,o) iff

i) a϶H, b϶H => aob ϵ H, and

ii) a϶H => a^{-1} ϵ H

Proof: Let (H,o) be a subgroup of (G,o).

Since (H,o) is a group, (i) and (ii) are satisfied.

Conversely, let H be a non-empty subset of G satisfying (i) and (ii).

Since (i) holds, H is closed under o.

Since H is a subset of G and o is associative on G, o is then associative on H.

Since (ii) holds, the inverse of each element in H exists in H.

Contd.

Let $a \epsilon H$. Then by (ii) $a^{-1} \epsilon H$. And since $a, a^{-1} \epsilon H$, (i) implies $a \circ a^{-1} = e \epsilon H$.

Therefore (H,o) is a group and hence (H,o) is a subgroup of (G,o).