Groupoid

Let G be a non empty set on which a BC o is defined. Some algebraic structure is imposed on G by the composition o and (G,o) become a algebraic system. The algebraic system (G,o) is said to be a groupoid. The groupoid (G,o) is comprised of two entities, the set G & the composition o on G.

Ex.1. (**Z**,+), (**Z**,-) are both groupoids. So same set make different groupoids.

Ex2. (**Q**,+), (**R**,+), (**Q**,.), (**R**,.) are groupoids.

Def^n: A groupoid (G,o) is said to be a commutative groupoid if the BC o is commutative.

An element e in G is said to be an identity element in the groupoid (G,o) if a o e = e o a = a ∀ a in G.

Contd.

Ex1. (**Z**,+) is a commutative groupoid but (**Z**,-) is not a commutative groupoid. Zero(0) is an identity element in (Z,+).

There is no identity element in (**Z**,-).

Def^n: An element e in G is said to be a right identity in the groupoid (G,o) if a o e =a ∀ a in G.

An element e in G is said to be a left identity in the groupoid (G,o) if e o a =a ∀ a in G.

Ex. In the groupoid (**Z**,+), zero(0) is a left identity as well as right identity. In groupoid (**Z**,.) 1 is the left identity as well as a right identity.

In the groupoid (**Z**,-) there is no left identity, but zero(0) is a right identity.

Contd.

Theorem: If a groupoid (G,o) contains an identity element then that element is unique.

Proof: If possible let there be 2 identity elements e and f in (G,o).

Then, $e \circ a = a \circ e = a$ & $f \circ a = a \circ f = a \ \forall \ a \in G$.

Now, $e \circ f = e$ by the property of f

& $e \circ f = f$ by the property of e

$\therefore e = f$.

Theorem: If a groupoid (G,o) contains a left identity as well as a right identity then are equal & the equal element is the identity element in the groupoid.

Proof: Let e be a left identity & f be a right identity in (G,o).

Then, $e \circ a = a \ \forall \ a \in G$

$a \circ f = a \ \forall \ a \in G$

Contd.

Now, e o f = f by the property of e

&   e o f = e by the property of f

∴ e = f.

This proves that e is an identity element in the groupoid & by the previous theorem e is the only identity element in the groupoid.

Def^{n}: Let, (G, o) be a groupoid containing the identity element e. An element a in G is said to be invertible if there exists an element a' in G such that a' o a = a o a' = e. a' is said to be an inverse of a in the groupoid.

An element a in G is said to be left invertible if there exists an element b in G such that b o a = e. b is said to be a left inverse of a in the groupoid.

An element a in G is said to be right invertible if there exists an element c in G such that a o c = e. c is said to be a right inverse of a in the groupoid.

Contd.

Ex. 1 is the identity element in the groupoid (Z,.). -1 in Z is invertible because x . (-1) = (-1) . x = 1 hold in Z for x=-1. 2 in Z has no left inverse in the groupoid as there is no element x in Z such that x . 2 = 1. Also 2 has no right inverse in the groupoid as there is no element y in Z such that 2 . y = 1.

Ex. 1 is the identity element in the groupoid (Q,.). 2 in Q is invertible because there exists an element ½ in Q such that ½ . 2 = 2 . ½ = 1. 0(zero) in Q is not invertible.

Defn. If e be just a left identity in the groupoid (G,o) then an element a in G is said to be left e-invertible if there exists an element b in G such that b . a = e & a is said to be right e-invertible if there exists an element c in G such that a o c = e. b is said to be a left e-inverse of a & c is said to be a right e-inverse of a.

Contd.

When e is just a right identity, then a left e-inverse & a right e-inverse of an element can be defined in a similar manner.

Ex. In the groupoid $(Z,-)$ 0(zero) is a right identity. An element a in Z has a left 0 - inverse as well as a right 0 - inverse in the groupoid.

Ex. In the groupoid $(Z,*)$ where $*$ is defined by $a*b= a+2b$, $a,b \in Z$, 0(zero) is a right identity. 3 in Z is left 0-invertible but not right 0-invertible. 4 in Z is left 0-invertible as well as right 0-invertible.