1.12.3. Quantified Forms of Simple and Compound Sentences

As stated earlier a ***simple statement function*** of one variable is defined to be an expression consisting of predicate symbol and a variable. Thus $M(x)$ is a simple statement function expressing "*x is a man*" so that predicate M is "*is a man.*"

A ***compound statement function*** is obtained by combining one or more simple statement functions with logical connectives.

Thus if $M(x)$: "*x is a man*". and $H(x)$: "*x is mortal* " be two simple statement functions then we can form several compound statement functions from them such as

$$M(x) \wedge H(x),\ M(x) \rightarrow H(x),\ \sim H(x),\ M(x) \vee \sim H(x) \text{ etc.}$$

Now let us see how English sentences can be expressed as quantified statements.

A . Consider the sentence "*All men are mortal.*" Let us paraphrase this sentence as follows:

"*For all x, if x is a man, then x is mortal.* "

Symbolically, this can be represented as

$$(\forall x)(M(x) \to H(x)),\ M(x), H(x) \text{ being defined earlier.}$$

Similarly the sentence "*Any integer is either even or odd.*" can be paraphrased as

"*For all x, if x is an integer, then either x is even or x is odd.*"

Thus if $P(x)$: "*x is an integer.*", $Q(x)$: "*x is even.*" and $R(x)$: "*x is odd.*" be the statement functions then the above sentence can be represented as the following quantified statement:

$$(\forall x)\big(P(x) \to (Q(x) \vee R(x))\big)$$

B. Again consider the following sentences:

(a) "*Some men are clever.*"

(b) "*Some real numbers are either rational or irrational.*"

These sentences can be paraphrased as follows:

(a1) "*There exists an x such that x is a man and x is clever.*"

Or,

"*There exists at least one x such that x is a man and x is clever.*"

(b1) "*There exists an x such that x is a real number and either x is rational or x is irrational.*"

Or,

"*There exists at least one x such that x is a real number and either x is rational or x is irrational.*"

Thus if $M(x)$: "*x is a man*" and $C(x)$: "*x is clever.*" Then (a) can be represented as the following quantified statement:

$$(\exists x)(M(x) \wedge C(x))$$

Again if $R(x)$: "*x is a real number.*" $R_1(x)$: "*x is rational.*" and $R_2(x)$: "*x is irrational.*" then the sentence (b) can be represented as the following quantified statement:

$$(\exists x)\big(R(x) \wedge (R_1(x) \vee R_2(x))\big).$$

C. Consider the sentence

(c) *"If Samir is taller than Rabin then Rabin is not taller than Samir."*

Let T be the predicate denoting "*is taller than*" Then $T(x, y)$: "*x is taller than y*." The quantified statement with 2-place predicate T,

$$(\forall x)(\forall y)(T(x, y) \to \sim T(y, x))$$

represents the sentence

"For all x and y, if x is taller than y then y is not taller than x."

Thus if s represents '*Samir*' and r represents '*Rabin*' then the sentence (c) is represented as $T(s, r) \to \sim T(r, s)$

Illustrative Examples 5

Problem 1. Let the universe of discourse be the set $\mathbb{Z}$ of all integers. Find the truth values of each of the following statements:

(a) $(\forall x \in \mathbb{Z})\ x^2 = x$.

(b) $(\exists x \in \mathbb{Z})\ x^2 = x$.

Solution. Consider the proposition $P(x)$: "$x^2 = x$."

Then $(\forall x \in \mathbb{Z})\ P(x)$ is false because $4 \in \mathbb{Z}$ but $4^2 = 4$ is false.

$(\exists x \in \mathbb{Z})\ P(x)$ is true, because at least one substitution instance of $P(x)$ is true. In fact $P(0)$ and $P(1)$ are true.

Problem 2. Write quantified negated statement for each of the following sentences:

(a) For $x \in \mathbb{R}$, if $x > 5$ then $x^2 > 25$.

(b) For $x \in \mathbb{R}$, if $x^2 - 5x + 6 = 0$ then either $x = 3$ or $x = 5$.