5.1 Let

$$s_x = \frac{vx_{max} - vx_{min}}{wx_{max} - wx_{min}} \quad \text{and} \quad s_y = \frac{vy_{max} - vy_{min}}{wy_{max} - wy_{min}}$$

Express window-to-viewport mapping in the form of a composite transformation matrix.

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -wx_{min} & -wy_{min} & 1 \end{pmatrix} \cdot \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ vx_{min} & vy_{min} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ -s_x\, wx_{min} + vx_{min} & -s_y\, wy_{min} + vy_{min} & 1 \end{pmatrix}$$

5.2 Find the normalization transformation that maps a window whose lower left corner is at (1, 1) and upper right corner is at (3, 5) onto

(a) a viewport that is the entire normalized device screen, and

(b) a viewport that has lower left corner at (0, 0) and upper right corner $\left(\frac{1}{2}, \frac{1}{2}\right)$.

From Solved Problem 5.1, we need only identify the appropriate parameters.

(a) The window parameters are $wx_{min} = 1$, $wx_{max} = 3$, $wy_{min} = 1$, and $wy_{max} = 5$. The viewport parameters are $vx_{min} = 0$, $vx_{max} = 1$, $vy_{min} = 0$, and $vy_{max} = 1$. Then $s_x = \frac{1}{2}$, $s_y = \frac{1}{4}$, and

$$N = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ -1/2 & -1/4 & 1 \end{pmatrix}$$

(b) The window parameters are the same as in (a). The viewport parameters are now $vx_{min} = 0$, $vx_{max} = \frac{1}{2}$, $vy_{min} = 0$, $vy_{max} = \frac{1}{2}$. Then $s_x = \frac{1}{4}$, $s_y = \frac{1}{8}$, and

$$N = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/8 & 0 \\ -1/4 & -1/8 & 1 \end{pmatrix}$$

5.3 Find the complete viewing transformation that maps a window in world coordinates with x extent 1 to 10 and y extent 1 to 10 onto a viewport with x extent $\frac{1}{4}$ to $\frac{3}{4}$ and y extent 0 to $\frac{1}{2}$ in normalized device space, and then maps a workstation window with x extent $\frac{1}{4}$ to $\frac{1}{2}$ and y extent $\frac{1}{4}$ to $\frac{1}{2}$ in the normalized device space into a workstation viewport with x extent 1 to 10 and y extent 1 to 10 on the physical display device.

From Solved Problem 5.1, the parameters for the normalization transformation are $wx_{\min} = 1$, $wx_{\max} = 10$, $wy_{\min} = 1$, $wy_{\max} = 10$, and $vx_{\min} = \frac{1}{4}$, $vx_{\max} = \frac{3}{4}$, $vy_{\min} = 0$, and $vy_{\max} = \frac{1}{2}$. Then

$$s_x = \frac{1/2}{9} = \frac{1}{18} \qquad s_y = \frac{1/2}{9} = \frac{1}{18}$$

and

$$N = \begin{pmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{18} & 0 \\ \frac{7}{36} & \frac{-1}{18} & 1 \end{pmatrix}$$

The parameters for the workstation transformation are $wx_{\min} = \frac{1}{4}$, $wx_{\max} = \frac{1}{2}$, $wy_{\min} = \frac{1}{4}$, $wy_{\max} = \frac{1}{2}$ and $vx_{\min} = 1$, $vx_{\max} = 10$, $vy_{\min} = 1$, and $vy_{\max} = 10$. Then

$$s_x = \frac{9}{1/4} = 36 \qquad s_y = \frac{9}{1/4} = 36$$

and

$$W = \begin{pmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ -8 & -8 & 1 \end{pmatrix}$$

The complete viewing transformation V is

$$V = N \cdot W = \begin{pmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{18} & 0 \\ \frac{7}{36} & \frac{-1}{18} & 1 \end{pmatrix} \cdot \begin{pmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ -8 & -8 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & -10 & 1 \end{pmatrix}$$

5.4 Find a normalization transformation from the window whose lower left corner is at (0, 0) and upper right corner is at (4, 3) onto the normalized device screen so that aspect ratios are preserved.

The window aspect ratio is $a_w = \frac{4}{3}$. Unless otherwise indicated, we shall choose a viewport that is as large as possible with respect to the normalized device screen. To this end, we choose the x extent from 0 to 1 and the y extent from 0 to $\frac{3}{4}$. So

$$a_v = \frac{1}{3/4} = \frac{4}{3}$$

As in Solved Problem 5.2, with parameters $wx_{\min} = 0$, $wx_{\max} = 4$, $wy_{\min} = 0$, $wy_{\max} = 3$ and $vx_{\min} = 0$, $vx_{\max} = 1$, $vy_{\min} = 0$, $vy_{\max} = \frac{3}{4}$,

$$N = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5.5 Find the normalization transformation N which uses the rectangle $A(1, 1)$, $B(5, 3)$, $C(4, 5)$, $D(0, 3)$ as a window [Fig. 5.16(a)] and the normalized device screen as a viewport [Fig. 5.16(b)].

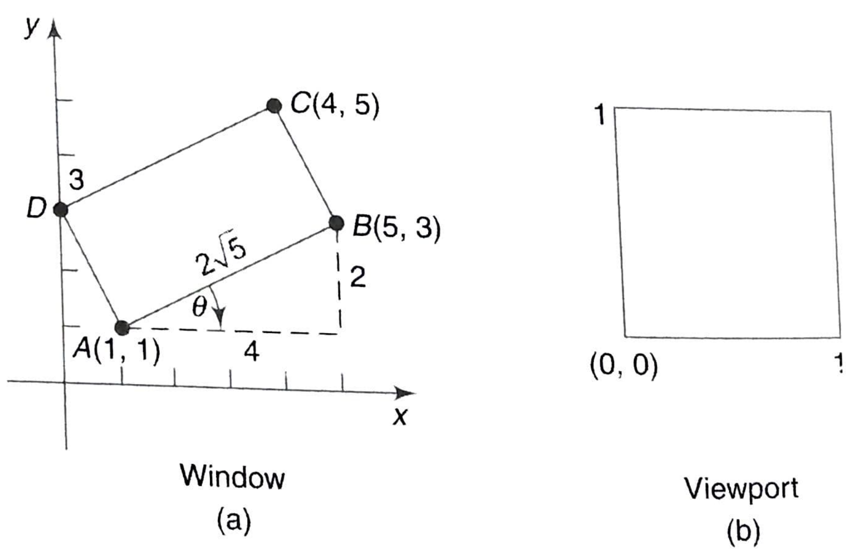

Fig. 5.16

We will first rotate the rectangle about A so that it is aligned with the coordinate axes. Next, as in Solved Problem 5.1, we calculate s_x and s_y and finally we compose the rotation and the transformation N (from Problem 5.1) to find the required normalization transformation N_R.

The slope of the line segment $\overline{AB}$ is

$$m = \frac{3-1}{5-1} = \frac{1}{2}$$

Looking at Fig. 5.11, we see that $-\theta$ will be the direction of the rotation. The angle θ is determined from the slope of a line (Appendix 1) by the equation $\tan\theta = \frac{1}{2}$. Then

$$\sin\theta = \frac{1}{\sqrt{5}} \text{ and so } \sin(-\theta) = -\frac{1}{\sqrt{5}},\ \cos\theta = \frac{2}{\sqrt{5}},\ \cos(-\theta) = -\frac{2}{\sqrt{5}}$$

The rotation matrix about $A(1, 1)$ is then (Chapter 4, Problem 4.4):

$$R_{-\theta,A} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \left(1-\frac{3}{\sqrt{5}}\right) & \left(1-\frac{1}{\sqrt{5}}\right) & 1 \end{pmatrix}$$

The x extent of the rotated window is the length of $\overline{AB}$. Similarly, the y extent is the length of $\overline{AD}$. Using the distance formula (Appendix 1) to calculate these lengths yields

$$d(A, B) = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5} \quad d(A,D) = \sqrt{1^2+2^2} = \sqrt{5}$$

Also, the x extent of the normalized device screen is 1, as is the y extent. Calculating s_x and s_y,

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{1}{2\sqrt{5}} \qquad s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } y \text{ extent}} = \frac{1}{\sqrt{5}}$$

so

$$N = \begin{pmatrix} \frac{1}{2\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ \frac{-1}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} & 1 \end{pmatrix}$$

The normalization transformation is then

$$N_R = R_{-\theta,A} \cdot N = \begin{pmatrix} \frac{1}{5} & \frac{-1}{5} & 0 \\ \frac{1}{10} & \frac{2}{5} & 0 \\ \frac{-3}{10} & \frac{-1}{5} & 1 \end{pmatrix}$$

5.6 Let R be the rectangular window whose lower left-hand corner is at $L(-3, 1)$ and upper right-hand corner is at $R(2, 6)$. Find the region codes for the endpoints in Fig. 5.17.

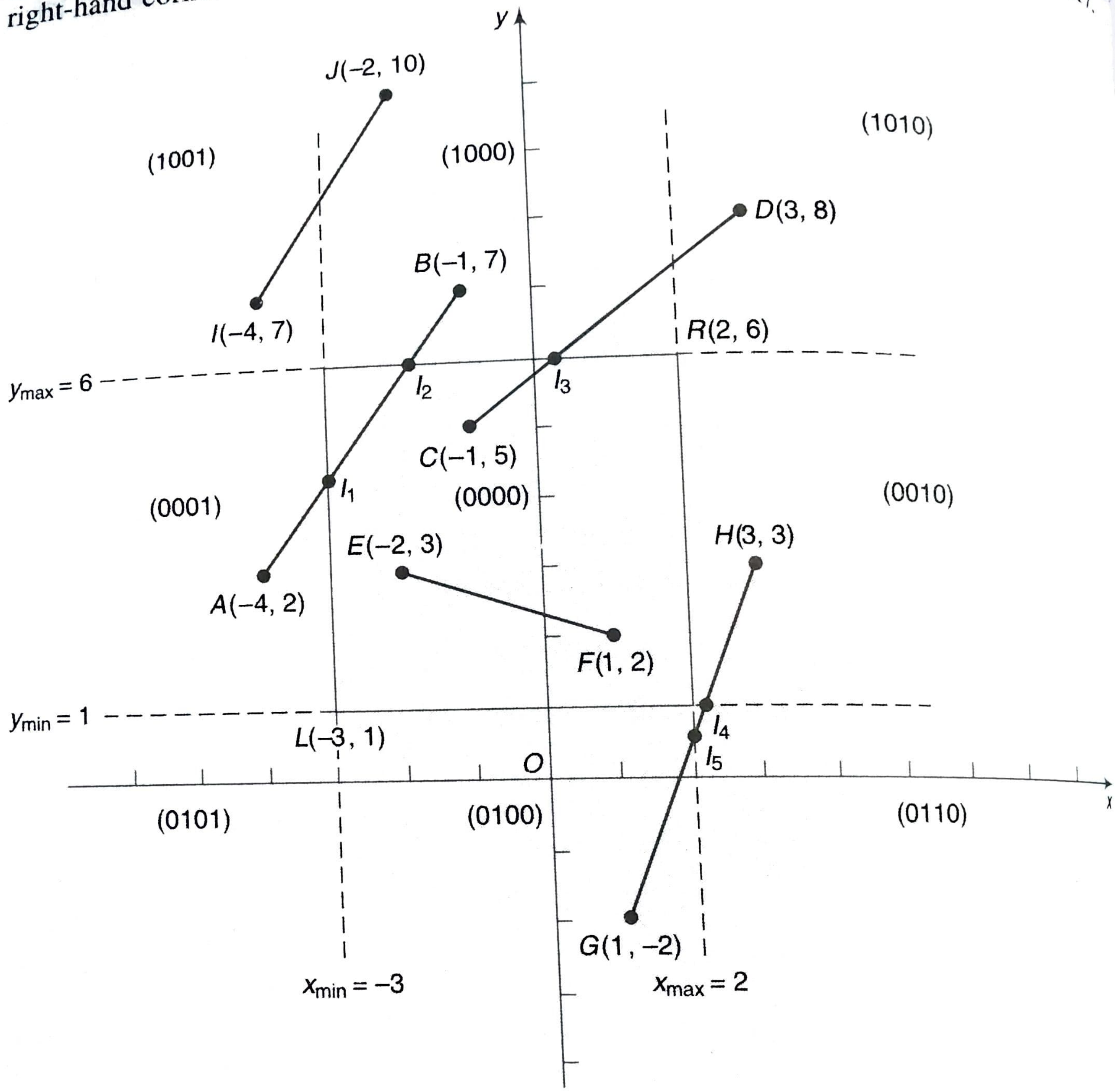

Fig. 5.17

The region code for point (x, y) is set according to the scheme

$$\text{Bit 1} = \text{sign}(y - y_{max}) = \text{sign}(y - 6) \qquad \text{Bit 3} = \text{sign}(x - x_{max}) = \text{sign}(x - 2)$$
$$\text{Bit 2} = \text{sign}(y_{min} - y) = \text{sign}(1 - y) \qquad \text{Bit 4} = \text{sign}(x_{min} - x) = \text{sign}(-3 - x)$$

Here

$$\text{sign}(a) = \begin{cases} 1 & \text{if } a \text{ is positive} \\ 0 & \text{otherwise} \end{cases}$$

So

$$A(-4, 2) \rightarrow 0001 \qquad F(1, 2) \rightarrow 0000$$
$$B(-1, 7) \rightarrow 1000 \qquad G(1, -2) \rightarrow 0100$$
$$C(-1, 5) \rightarrow 0000 \qquad H(3,3) \rightarrow 0010$$

$$D(3, 8) \rightarrow 1010 \qquad I(-4, 7) \rightarrow 1001$$
$$E(-2, 3) \rightarrow 0000 \qquad J(-2, 10) \rightarrow 1000$$

5.7 Clipping against rectangular windows whose sides are aligned with the x and y axes involves computing intersections with vertical and horizontal lines. Find the intersection of a line segment $\overline{P_1P_2}$ [joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$] with

(a) the vertical line $x = a$ and
(b) the horizontal line $y = b$.

We write the equation of $\overline{P_1P_2}$ in parametric form (Appendix 1, Problem A1.23):

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1), \end{cases} \qquad 0 \leq t \leq 1 \qquad \begin{matrix}(5.1)\\(5.2)\end{matrix}$$

(a) Since $x = a$, we substitute this into equation (5.1) and find $t = (a - x_1)/(x_2 - x_1)$. Then, substituting this value into equation (5.2), we find that the intersection point is $x_I = a$ and

$$y_I = y_1 + \left(\frac{a - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$

(b) Substituting $y = b$ into equation (5.2), we find $t = (b - y_1)/(y_2 - y_1)$. When this is placed into equation (5.1), the intersection point is $y_1 = b$ and

$$x_I = x_1 + \left(\frac{b - y_1}{y_2 - y_1}\right)(x_2 - x_1)$$

5.8 Find the clipping categories for the line segments in Solved Problem 5.6 (see Fig. 5.17).

We place the line segments in their appropriate categories by testing the region codes found in Problem 5.6.

Category 1 (visible): $\overline{EF}$ since the region code for both endpoints is 0000.
Category 2 (not visible): $\overline{IJ}$ since (1001) AND (1000) = 1000 (which is not 0000).
Category 3 (candidates for clipping): $\overline{AB}$ since (0001) AND (1000) = 0000, $\overline{CD}$ since (0000) AND(1010) = 0000, and $\overline{GH}$ since (0100) AND (0010) = 0000.

5.9 Use the Cohen-Sutherland algorithm to clip the line segments in Problem 5.6 (see Fig. 5.17).

From Solved Problem 5.8, the candidates for clipping are $\overline{AB}$, $\overline{CD}$ and $\overline{GH}$. In clipping $\overline{AB}$, the code for A is 0001. To push the 1 to 0, we clip against the boundary line $x_{\min} = -3$. The resulting intersection point is $I_1\left(-3, 3\frac{2}{3}\right)$. We clip (do not display) $\overline{AI_1}$ and work on $\overline{I_1B}$. The code for I_1 is 0000. The clipping category for $\overline{I_1B}$ is 3 since

(0000) AND (1000) is (0000). Now B is outside the window (i.e. its code is 1000), so we push the 1 to a 0 by clipping against the line $y_{max} = 6$. The resulting intersection is $I_2\left(-1\frac{3}{5}, 6\right)$. Thus $\overline{I_2 B}$ is clipped. The code for I_2 is 0000. The remaining segment $\overline{I_1 I_2}$ is displayed since both endpoints lie in the window (i.e. their codes are 0000).

For *clipping* $\overline{CD}$, we start with D since it is outside the window. Its code is 1010. We push the first 1 to a 0 by clipping against the line $y_{max} = 6$. The resulting intersection I_3 is $\left(\frac{1}{3}, 6\right)$ and its code is 0000. Thus $\overline{I_3 D}$ is clipped and the remaining segment $\overline{CI_3}$ has both endpoints coded 0000 and so it is displayed.

For *clipping* $\overline{GH}$, we can start with either G or H since both are outside the window. The code for G is 0100, and we push the 1 to a 0 by clipping against the line $y_{min} = 1$. The resulting intersection point is $I_4\left(2\frac{1}{5}, 1\right)$, and its code is 0010. We clip $\overline{GI_4}$ and work on $\overline{I_4 H}$. Segment $\overline{I_4 H}$ is not displayed since (0010) AND (0010) = 0010.

5.10 Clip line segment $\overline{CD}$ of Solved Problem 5.6 by using the midpoint subdivision process.

The midpoint subdivision process is based on repeated bisections. To avoid continuing indefinitely, we agree to say that a point (x_1, y_1) lies on any of the boundary lines of the rectangle, say, boundary line $x = x_{max}$, for example, if $-\text{TOL} \le x_1 - x_{max} \le \text{TOL}$. Here TOL is a prescribed tolerance, some small number that is set before the process begins.

To clip $\overline{CD}$, we determine that it is in category 3. For this problem we arbitrarily choose TOL = 0.1. We find the midpoint of $\overline{CD}$ to be $M_1(1, 6.5)$. Its code is 1000.

So $\overline{M_1 D}$ is not displayed since (1000) AND (1010) = 1000. We further subdivide $\overline{CM_1}$ since (0000) AND (1000) = 0000. The midpoint of $\overline{CM_1}$ is $M_2(0, 5.75)$; the code for M_2 is 0000. Thus $\overline{CM_2}$ is displayed since both endpoints are 0000 and $\overline{M_2 M_1}$ is a candidate for clipping. The midpoint of $\overline{M_2 M_1}$ is $M_3(0.5, 6.125)$, and its code is 1000. Thus $\overline{M_3 M_1}$ is chipped and $\overline{M_2 M_3}$ is subdivided. The midpoint of $\overline{M_2 M_3}$ is $M_4(0.25, 5.9375)$, whose code is 0000. However, since $y_1 = 5.9375$ lies within the tolerance 0.1 of the boundary line $y_{max} = 6$ – that is, $6 - 5.9375 = 0.0625 < 0.1$, we agree that M_4 lies on the boundary line $y_{max} = 6$. Thus $\overline{M_2 M_4}$ is displayed and $\overline{M_4 M_3}$ is not displayed. So the original line segment $\overline{CD}$ is clipped at M_4 and the process stops.

5.11 Suppose that in an implementation of the Cohen–Sutherland algorithm we choose boundary lines in the top–bottom–right–left order to clip a line in category 3, draw a picture to show a worst-case scenario, i.e. one that involves the highest number of iterations.

See Fig. 5.18.

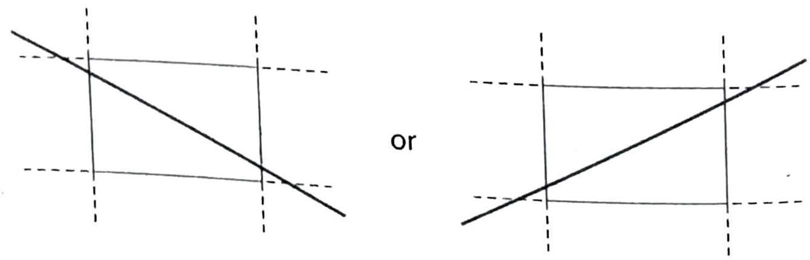

Fig. 5.18

5.12 Use the Liang–Barsky algorithm to clip the lines in Fig. 5.19.

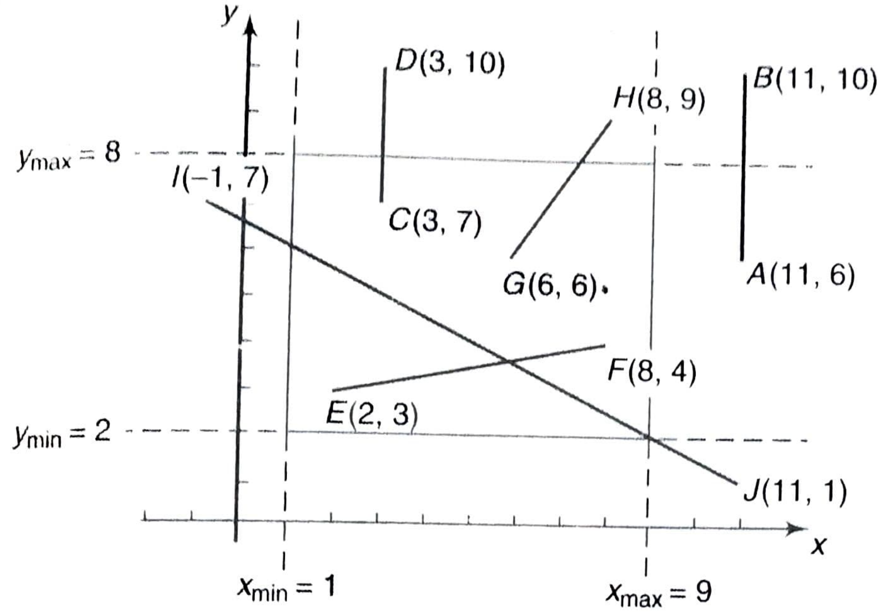

Fig. 5.19

For line AB, we have

$$\begin{aligned} p_1 &= 0 & q_1 &= 10 \\ p_2 &= 0 & q_2 &= -2 \\ p_3 &= -4 & q_3 &= 4 \\ p_4 &= 4 & q_4 &= 2 \end{aligned}$$

Since $p_2 = 0$ and $q_2 < -2$, AB is completely outside the right boundary.

For line CD, we have

$$\begin{aligned} p_1 &= 0 & q_1 &= 2 & & \\ p_2 &= 0 & q_2 &= 6 & & \\ p_3 &= -3 & q_3 &= 5 & r_3 &= -\frac{5}{3} \\ p_4 &= 3 & q_4 &= 1 & r_4 &= \frac{1}{3} \end{aligned}$$

Thus $u_1 = \max\left(0, -\frac{5}{3}\right) = 0$ and $u_2 = \min\left(1, \frac{1}{3}\right) = \frac{1}{3}$. Since $u_1 < u_2$ the two endpoints of the clipped line are (3, 7) and $\left(3, 7 + 3\left(\frac{1}{3}\right)\right) = 3, 8$.

For line EF, we have

$$\begin{aligned} p_1 &= -6 & q_1 &= 1 & r_1 &= -\frac{1}{6} \\ p_2 &= 6 & q_2 &= 7 & r_2 &= \frac{7}{6} \\ p_3 &= -1 & q_3 &= 1 & r_3 &= -\frac{1}{1} \\ p_4 &= 1 & q_4 &= 5 & r_4 &= \frac{5}{1} \end{aligned}$$

Thus $u_1 = \max\left(0, -\frac{1}{6}, -1\right) = 0$ and $u_2 = \min\left(1, \frac{7}{6}, 5\right) = 1$. Since $u_1 = 0$ and $u_2 = 1$, line EF is completely inside the clipping window.

For line GH, we have

$$p_1 = -2 \qquad q_1 = 5 \qquad r_1 = -\frac{5}{2}$$

$$p_2 = 2 \qquad q_2 = 3 \qquad r_2 = \frac{3}{2}$$

$$p_3 = -3 \qquad q_3 = 4 \qquad r_3 = -\frac{4}{3}$$

$$p_4 = 3 \qquad q_4 = 2 \qquad r_4 = \frac{2}{3}$$

Thus $u_1 = \max\left(0, -\frac{5}{2}, -\frac{4}{3}\right) = 0$ and $u_2 = \min\left(1, \frac{3}{2}, \frac{2}{3}\right) = \frac{2}{3}$. Since $u_1 < u_2$, the two end-points of the clipped line are (6, 6) and $\left(6 + 2\left(\frac{2}{3}\right), 6 + 3\left(\frac{2}{3}\right)\right) = \left(7\frac{1}{3}, 8\right)$

For line IJ, we have

$$p_1 = -12 \qquad q_1 = -2 \qquad r_1 = \frac{1}{6}$$

$$p_2 = 12 \qquad q_2 = 10 \qquad r_2 = \frac{5}{6}$$

$$p_3 = 6 \qquad q_3 = 5 \qquad r_3 = \frac{5}{6}$$

$$p_4 = -6 \qquad q_4 = 1 \qquad r_4 = -\frac{1}{6}$$

Thus $u_1 = \max\left(0, \frac{1}{6}, -\frac{1}{6}\right) = \frac{1}{6}$ and $u_2 = \min\left(1, \frac{5}{6}, \frac{5}{6}\right) = \frac{5}{6}$. Since $u_1 < u_2$, the two end-points of the clipped line are $\left(-1 + 12\left(\frac{1}{6}\right), 7 + (-6)\left(\frac{1}{6}\right)\right)$ and $\left(-1 + 12\left(\frac{5}{6}\right), 7 + (-6)\left(\frac{5}{6}\right)\right)$ = (9, 2)

5.13 How can we determine whether a point $P(x, y)$ lies to the left or to the right of a line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$?

Refer to Fig. 5.20. Form the vectors **AB** and **AP**. If the point P is to the left of **AB**, then by the definition of the cross product of two vectors (Appendix 2) the vector **AB** × **AP** points in the direction of the vector **K** perpendicular to the xy plane (see Fig. 5.20). If it lies to the right, the cross product points in the direction $-\mathbf{K}$. Now

$$\mathbf{AB} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J}$$

$$\mathbf{AP} = (x - x_1)\mathbf{I} + (y - y_1)\mathbf{J}$$

So

$$\mathbf{AB} \times \mathbf{AP} = [(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)]\mathbf{K}$$

Then the direction of this cross product is determined by the number

$$\bar{C} = (x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)$$

If $\bar{C}$ is positive, P lies to the left of **AB**. If $\bar{C}$ is negative, then P lies to the right of **AB**.

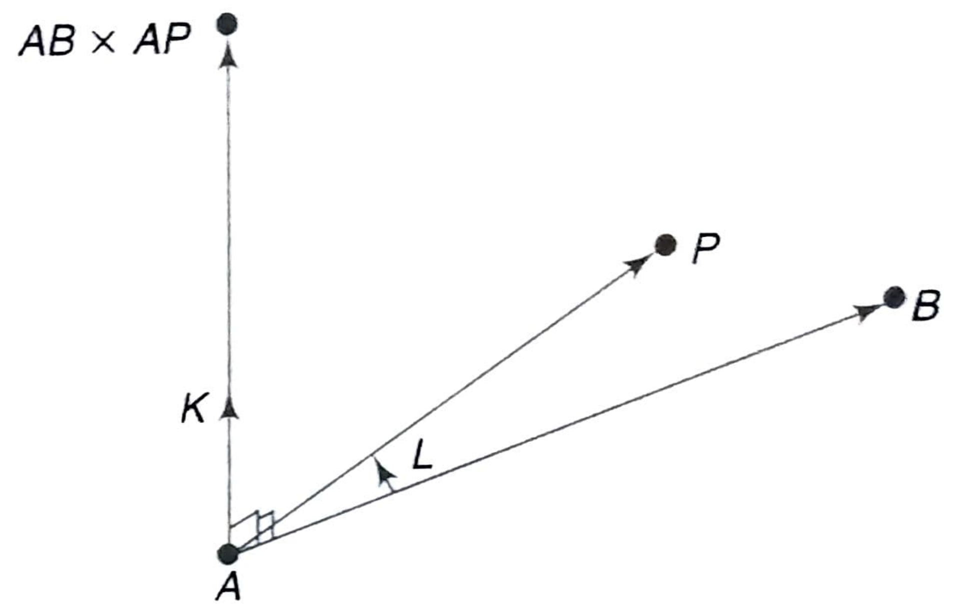

Fig. 5.20