**Q.1. Find the matrix that represents the rotation of an object by 30° about origin ? What are the new coordinates of the point P (2, – 4) after the rotation.**

**[MDU, BE(CSE)-5th Sem., Dec. 2006]**

**Ans.** Rotation – matrix of an object by 30° about origin is :

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

or

$$R_{30^\circ} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

The new coordinates of the point (2, – 4) after rotation can be found out by multiplying–

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} \sqrt{3}+2 \\ 1-2\sqrt{3} \end{bmatrix}$$

**Q.2. Show that the composition of two rotations is additive by concatenating the matrix representations for R ($\theta_1$) and R ($\theta_2$) to obtain :**

$$R(\theta_1) \times R(\theta_2) = R(\theta_1 + \theta_2)$$

**[UPTU, B.Tech (CSE/IT) Sem., 2003, 2004-05]**

**Ans.** As we know that the rotation matrix is :

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So,

$$R(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

and

$$R(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$\therefore \quad R(\theta_1) \times R(\theta_2) = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= R(\theta_1+\theta_2)$$

$$\therefore \quad R(\theta_1) \times R(\theta_2) = R(\theta_1+\theta_2)$$

proved

**Q.3. Show that the 2 × 2 matrix :**

$$[T] = \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$ **represents pure rotation.**

**Ans.** A pure rotation matrix means $|T| = 1$. Let us prove this.

Given : T

$$\therefore \quad |T| = \begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ \frac{-2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \left[\frac{1-t^2}{1+t^2}\right]^2 - \left[\frac{-2t}{1+t^2}\right]^2$$

$$= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2}$$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2}$$

$$= \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2}$$

$$= 1$$

**Hence proved.**

**Q.4. Show that a reflection about the line $y = -x$ is equivalent to a reflection relative to $y$-axis is followed by a counter clockwise rotation of 90°.**

**[UPTU, B.Tech (CSE/IT)-5th Sem., 2002, 2003, 2005-06]**

**Ans.** $\because$ **Reflection is about line $y = -x$**

$\therefore$ Transformation matrix for reflection about line $y = -x$ is :

$$T_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \ldots (1)$$

$\therefore$ Transformation matrix for reflection relative to $y$-axis is :

$$T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \dots (2)$$

∴ Transformation matrix for counter-clockwise rotation is :

$$T_3 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \dots (3)$$

Here,

$$\theta = 90^\circ$$

∴

$$T_3 = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

Here, we have to prove that

$$T_1 = T_2 * T_3$$

Now,

$$T_2 * T_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= T_1 \quad \text{(from equation – (1))}$$

**Hence proved.**

∴

$$T_1 = T_2 * T_3$$

**Q.5. Find the instance transformation which places a half-size copy of the square A (0, 0), B (2, 0), C (2, 2) and D (0, 2) defined in a master coordinate system into a world coordinate system in such a way that the centre of the square is at (– 3, – 3) in the world coordinate system.** **[UPTU, B.Tech (CSE/IT)-5th Sem., 2003 & 2004-05]**

**Ans.** Center of the square ABCD is at P (1, 1). We now translate P (1, 1) to the center of square P′ (–3, –3) in WCS–

$$t_x = -3 - 1 = -4$$

$$t_y = -3 - 1 = -4$$

Here,

$$S_x = S_y = \frac{1}{2}$$

∴

$$N_{\text{picture, square}} = T_V \cdot S_{\frac{1}{2}\cdot\frac{1}{2}} \cdot T_p$$

or

$$S_{\frac{1}{2}\cdot\frac{1}{2}} \cdot p = T_{-p} \times S_{\frac{1}{2}\cdot\frac{1}{2}} \cdot T_p$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{\text{picture, square}} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & -\frac{9}{2} \\ 0 & \frac{1}{2} & -\frac{9}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

**Q.6. Prove that the simultaneous shearing in both directions ($x$ and $y$) is not equal to the composition of pure shear along $x$-axis followed by pure shear along $y$-axis.**

**[UPTU, MCA-4th Sem., 2004]**

**Ans.** We know that simultaneous shearing,

$$S_h = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \quad \text{... (1)}$$

Shearing in $x$-direction, $S_{ha} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ & shearing in $y$-direction, $S_{hb} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$

∴ Shearing in $x$-direction followed by $y$-direction is–

$$= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+ab & a \\ b & 1 \end{bmatrix} \quad \text{... (2)}$$

∴ Equation (1) ≠ Equation (2)

**Hence, proved.**

**Q.7. Prove that 2D-rotation and scaling commute if $S_x = S_y$ or $\theta = n\pi$.**

**Ans.** Consider the transformation matrix for rotation about origin in anticlockwise direction as–

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and scaling matrix as–

$$S_{xy} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us now prove the commutative property for $S_x = S_y$.

*i.e.*, $\quad R.S = S.R$

Now, for

$$\theta = n\pi$$

$$R.S = \begin{bmatrix} \cos n\pi & \sin n\pi & 0 \\ -\sin n\pi & \cos n\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{... (1)}$$

Now,

$$S.R = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos n\pi & \sin n\pi & 0 \\ -\sin n\pi & \cos n\pi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{... (2)}$$

From equations (1) and (2) we get–

$$R \cdot S = S \cdot R \qquad (\text{if } S_x = S_y \text{ or } \theta = n\pi)$$ **Hence proved.**

**Q.8. Perform a 45° rotation of triangle A (0, 0), B (1, 1), C (5, 2) :**

**(*a*) About the origin.**

**(*b*) About point P (– 1, – 1).**

**[UPTU, MCA-4th Sem., 2003][MDU, BE(CSE)-5th Sem., May, 2009]**

**Ans. Given :** Triangle, ABC with coordinates as

A (0, 0), B (1, 1), C (5, 2)

∴ Triangle ABC in matrix form is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \qquad \text{(Row major)}$$

or

$$\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{(Column major form)}$$

**(*a*) Matrix of rotation is–**

$$R_{45^\circ} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, coordinates, A′, B′, C′ of the rotated triangle ABC can be found as

$$[A' \quad B' \quad C'] = R_{45°} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \qquad A' = (0, 0)$$

$$B' = (0, \sqrt{2})$$

$$C' = \left(\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$$

(*b*) **The rotation matrix is**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $$[A' \quad B' \quad C'] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & \left(\frac{3}{2}\sqrt{2}-1\right) \\ (\sqrt{2}-1) & (2\sqrt{2}-1) & \left(\frac{9}{2}\sqrt{2}-1\right) \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore$

$$A' = (-1, \sqrt{2}-1)$$

$$B' = (-1, 2\sqrt{2}-1)$$

and

$$C' = \left(\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1\right).$$

**Q.9. Compute the composite transformation matrix for the following transformation in the given order :**

**(*a*) Translate by (– 2, 1).** **(*b*) Rotate by 70°.**

**(*c*) Translate by 2, 3.** **[UPTU, B.Tech (CSE / IT)–5th Sem., 2005-06]**

**Ans.** The composite transformation matrix,

$$M = \overleftarrow{T(2,3) \cdot R(70^\circ) \cdot T(-2,1)}$$

Now, $\cos 70^\circ = 0.342$

and $\sin 70^\circ = 0.94$

$\therefore$

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.342 & -0.94 & 0 \\ 0.94 & 0.342 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.342 & -0.94 & 0.376 \\ 0.94 & 0.342 & 1.462 \\ 0 & 0 & 1 \end{bmatrix}$$

**Q.10. Show that coordinate rotation transformation and Geometric rotation transformation are related.**

**Ans.** Let $\bar{R}_{-\theta}$ = rotation transformation matrix when coordinate axis is rotated in clockwise direction

Then

$$\bar{R}_{-\theta} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= R_\theta$$

$\therefore$

$$[R_\theta]^T = R' = \bar{R}_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\because$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

or $$R_{-\theta} = \bar{R}_\theta = R_\theta^{-1}$$

Now, $$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad R_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \bar{R}_\theta$$

$$\therefore \quad R_{-\theta} = R_\theta^{-1} = \bar{R}_\theta \quad \text{and} \quad \bar{R}_{-\theta} = R_\theta .$$

**Q.11. Show that scaling followed by rotation is equivalent to shearing.**

**Ans.** Let the scaling matrix be–

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

and Rotation matrix be–

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad S * R = \begin{bmatrix} S_x \cos\theta & S_x \sin\theta \\ -S_y \sin\theta & S_y \cos\theta \end{bmatrix} \quad \text{(from (1) and (2))}$$

Now Shearing matrix is–

$$S_h = \begin{bmatrix} 1 & Sh_y \\ Sh_x & 1 \end{bmatrix}$$

From equations (3) and (4) we get

$$\left.\begin{array}{r} S_x \cos\theta = 1 \\ S_x \sin\theta = Sh_y \\ -S_y \sin\theta = Sh_x \\ S_y \cos\theta = 1 \end{array}\right]$$

$$\therefore \quad S_x = \frac{1}{\cos\theta} \quad \text{and} \quad S_y = \frac{1}{\cos\theta}$$

$$\therefore \quad Shx = -S_y \sin\theta = -\frac{1}{\cos\theta}\cdot\sin\theta = -\tan\theta$$

and $$Sh_y = S_x \sin\theta = \frac{1}{\cos\theta}\cdot\sin\theta$$

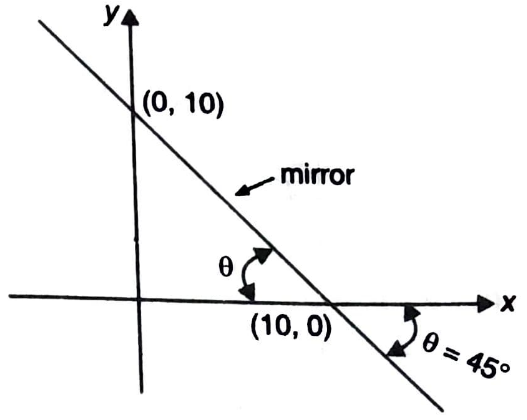
$$= \tan\theta$$

$$\therefore \quad \left.\begin{array}{r} Sh_x = -\tan\theta \\ Sh_y = \tan\theta \end{array}\right]$$

Hence proved

**Q.12.** A mirror is placed vertically such that it passes through the points (10, 0) and (0, 10). Find the reflected view of a triangle ABC with coordinates A (5, 50), B (20, 40), C (10, 70). [ UPTU, B.Tech (CSE / IT)–5th Sem., 2005-06]

**Ans.**




$$\therefore \quad \tan\theta = \frac{10}{10} = 1$$

$$\therefore \quad \theta = \tan^{-1} 1 = 45°.$$

Coordinates of triangle ABC in matrix form is

$$C = \begin{bmatrix} 5 & 50 & 1 \\ 20 & 40 & 1 \\ 10 & 70 & 1 \end{bmatrix}$$

Now, we translate mirror, so that it passes through the origin *i.e.*,

$$t_x = 0, \; t_y = -10$$

$\therefore$ It's transformation matrix is

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{bmatrix} \qquad \left[ \because T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \right]$$

$$\therefore \quad T_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$$

Now, we rotate the mirror by 45° anticlockwise so that it matches with the origin

$$\therefore \quad R_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad R_1^{-1} = \begin{bmatrix} \cos(-45^\circ) & \sin(-45^\circ) & 0 \\ -\sin 45^\circ & \cos(-45^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Transformation matrix for reflection about $x$-axis is–

$$R_{ref} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we rotate in reverse direction. Then, translate mirror to the original position. The ste followed here are :

1. Translate the mirror and object so that it passes through origin *i.e.*, $T_1$
2. Rotate mirror and object by 45° in anticlockwise *i.e.*, $R_1$
3. Now mirror matches with $x$-axis then reflect triangle ABC about $x$-axis *i.e.*, $R_{ref}$
4. Rotate mirror and object by 45° clockwise *i.e.*, $R_1^{-1}$
5. Then translate mirror and object back to its position matrix *i.e.*, $T_1^{-1}$

So resultant transformation matrix is

$$R = T_1 * R_1 * R_{ref} * R_1^{-1} * T_1^{-1}$$

*i.e.*,
$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{10}{\sqrt{2}} & -\frac{10}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{10}{\sqrt{2}} & -\frac{10}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 10 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$$

$$\therefore \quad R = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 10 & 10 & 1 \end{bmatrix}$$

$\therefore$ New coordinates for $\Delta$ ABC can be found by multiplying R with C *i.e.*,

$$R * C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 10 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 50 & 1 \\ 20 & 40 & 1 \\ 10 & 70 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -40 & 5 & 1 \\ -30 & -10 & 1 \\ -60 & 0 & 1 \end{bmatrix}$$

$\therefore$ After reflection the new coordinates are

$$A \rightarrow (-40, 5)$$
$$B \rightarrow (-30, -10)$$
$$C \rightarrow (-60, 0)$$

**Q.13. Find the transformation matrix that transforms a given square ABCD to half its size with center still remaining at the same position. The coordinates of the square are A (1, 1), B (3, 1), C (3, 3) and D (1, 3) center at (2, 2).**

**[MDU, BE (CSE)-5th Sem., May 2006, 2007 & UPTU, MCA-4th Sem., 2004]**

**Ans.** Given: $S_x = S_y = \frac{1}{2}$

and $\vec{v} = 2\hat{i} + 2\hat{j}$

Now, scaling w.r.t. any arbitrary point is given by

$$S_{Sx,\, Sy,\, p} = [T_v] \cdot S_{Sx,Sy} \cdot [T_{-v}]$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} = T$$

= Transformation which transforms the square ABCD to half of its size.

$\therefore$ Coordinates after calculation can be calculated by the following relation

$$[A' \quad B' \quad C' \quad D'] = S_{sx,sy,p}[A \quad B \quad C \quad D]$$

**Q.14. Reflect the triangle ABC about the line $3x - 4y + 8 = 0$. The position vector of the coordinate ABC is given as A (4, 1), B (5, 2) and C (4 , 3).**

**Ans.** Equation of line $3x - 4y + 8 = 0$

$$m = \frac{3}{4} = \tan\theta$$

$$\tan\theta = \frac{3}{4} \text{ so, } \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

The intersection of the line $3x - 4y + 8 = 0$ with

$$x = 0 \text{ is } y = 2 \Rightarrow (0, 2)$$
$$y = 0 \text{ is } x = -8/3 \Rightarrow (-8/3, 0)$$

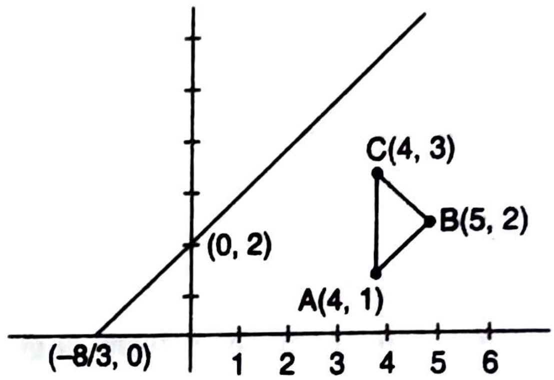



**Fig. 6.21.**

We know the composite transformation matrix [T] for reflection about the line which does not pass through origin is

$$[T] = [T_R][R_\theta][R_{ref}][R_\theta]^{-1}[T_R]^{-1}$$

$$[T_R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad [T_R]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [R_\theta]^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(R_\theta)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection about $x$-axis *i.e.*,

$$R_{ref} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Putting the values of $\cos\theta$ and $\sin\theta$, we get

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 2 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ -\frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 2 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} & 0 \\ -\frac{24}{25} & -\frac{7}{25} & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} & 0 \\ -\frac{24}{25} & -\frac{7}{25} & 0 \\ \frac{48}{25} & \frac{64}{25} & 1 \end{bmatrix}$$

Matrix for triangle ABC can be written as

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 4 & 1 & 1 \\ 5 & 2 & 1 \\ 4 & 3 & 1 \end{bmatrix}$$

The reflected co-ordinates can be calculated as follows.

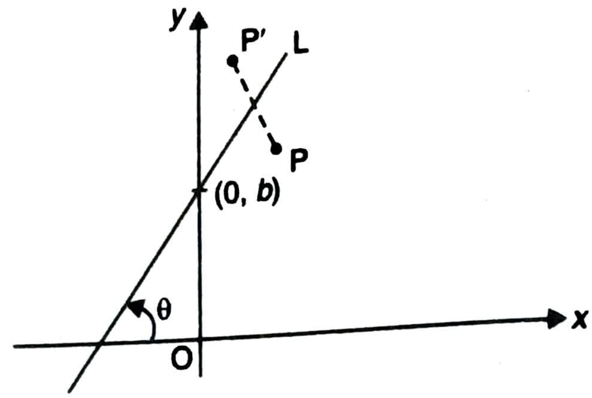
$$\begin{bmatrix} 4 & 1 & 1 \\ 5 & 2 & 1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} & 0 \\ -\frac{24}{25} & -\frac{7}{25} & 0 \\ \frac{48}{25} & \frac{64}{25} & 1 \end{bmatrix} = \begin{bmatrix} \frac{52}{25} & -\frac{39}{25} & 1 \\ \frac{35}{25} & -\frac{70}{25} & 1 \\ \frac{4}{25} & \frac{53}{25} & 1 \end{bmatrix}$$

Thus the reflected co-ordinates are

$$A \rightarrow \left(\frac{52}{25}, -\frac{39}{25}\right) \quad B \rightarrow \left(\frac{35}{25}, -\frac{70}{25}\right) \quad \text{and} \quad C \rightarrow \left(\frac{4}{25}, \frac{53}{25}\right)$$

**Q.15. Reflect a triangle whose vertices are A (– 1, 0), B (0, – 2) and C (1, 0) about the line $y = x + 2$.**

**Ans.** Firstly we draw its figure,




The required transformation is

$$M_L = T_{+V} \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_{-V} \qquad \text{...(1)}$$

where $\vec{V} = b\hat{j}$

Given : $y = x + 2$

$\therefore$ $b = 2$ and $m = \text{slope} = 1$

and $\Delta ABC = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\therefore \quad M_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$M_L = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\because \quad [A' \;\; B' \;\; C'] = M_L \cdot [A \;\; B \;\; C]$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & -2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore A' \rightarrow (-2, 1)$

$B' \rightarrow (-4, 2)$

$C' \rightarrow (-2, 3)$

**Note:** Similarly, we can find reflection about $y$-axis but we need $M_y$ now in equation (1)