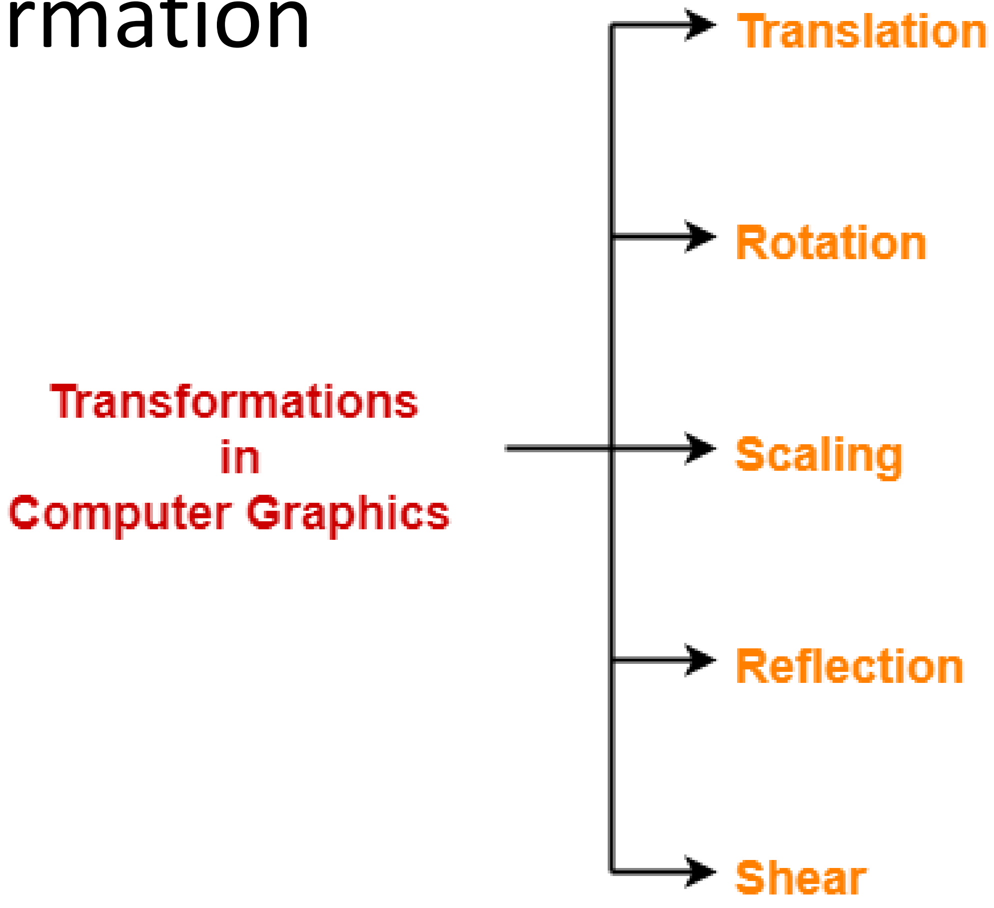
3D Transformation

Dr. Mousumi Dutt

3D Transformation

- Add z-coordinates part in 2D transformations
- Matrix formation of transformation
- Homogeneous Coordinates



3D Translation

Consider a point object O has to be moved from one position to another in a 3D plane.

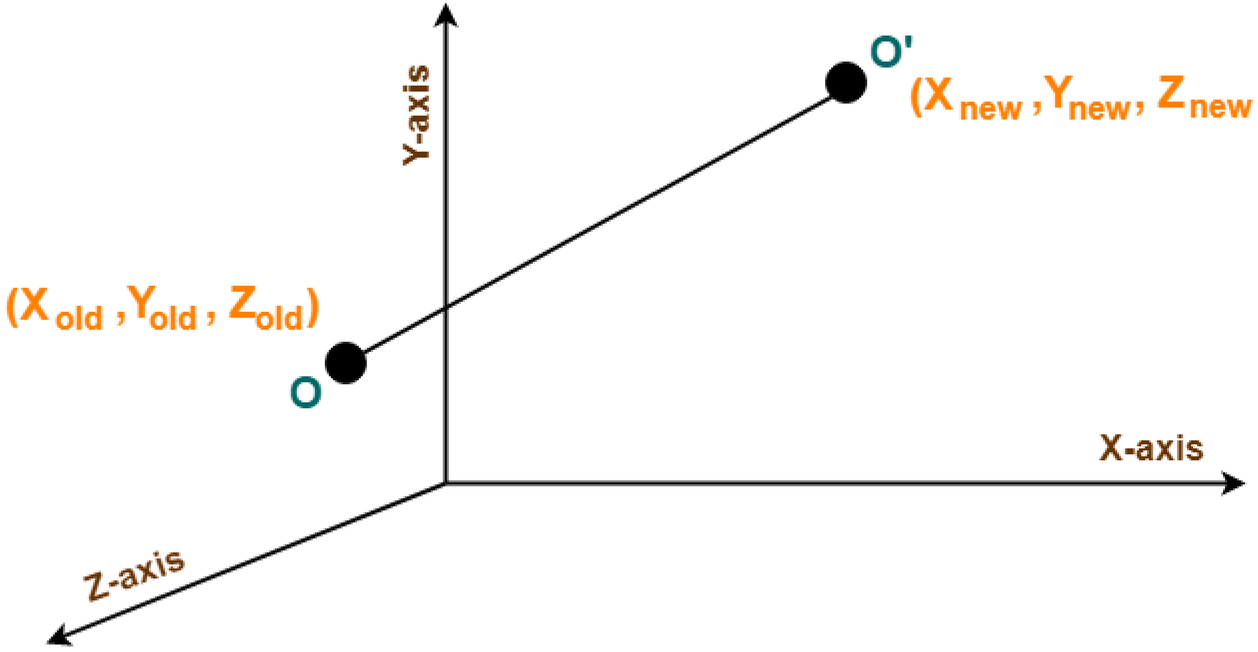
Let-

- Initial coordinates of the object O = $(X_{old}, Y_{old}, Z_{old})$
- New coordinates of the object O after translation = $(X_{new}, Y_{new}, Z_{old})$
- Translation vector or Shift vector = (T_x, T_y, T_z)

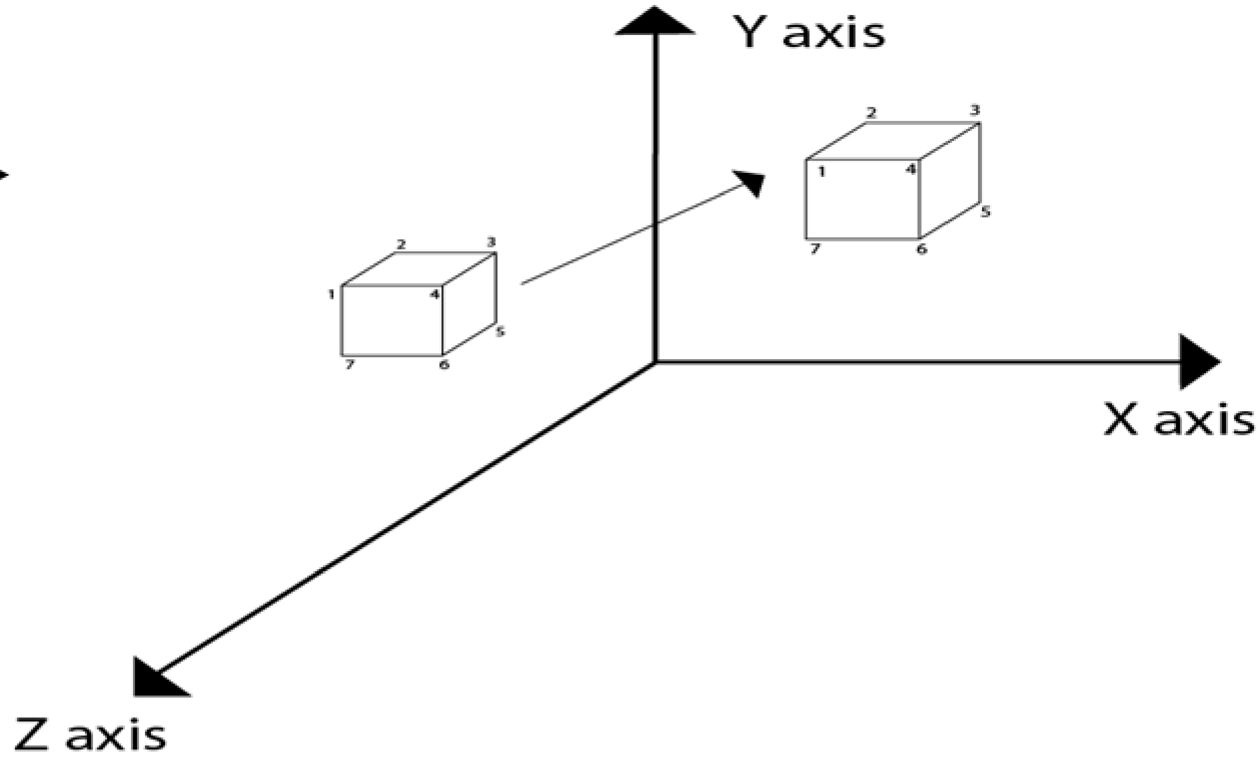
Given a Translation vector (T_x, T_y, T_z)-

- T_x defines the distance the X_{old} coordinate has to be moved.
- T_y defines the distance the Y_{old} coordinate has to be moved.
- T_z defines the distance the Z_{old} coordinate has to be moved.

3D Translation



3D Translation in Computer Graphics



3D Translation

This translation is achieved by adding the translation coordinates to the old coordinates of the object as-

- $X_{new} = X_{old} + T_x$ (This denotes translation towards X axis)
- $Y_{new} = Y_{old} + T_y$ (This denotes translation towards Y axis)
- $Z_{new} = Z_{old} + T_z$ (This denotes translation towards Z axis)

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Translation Matrix

3D Translation

Given a 3D object with coordinate points A(0, 3, 1), B(3, 3, 2), C(3, 0, 0), D(0, 0, 0). Apply the translation with the distance 1 towards X axis, 1 towards Y axis and 2 towards Z axis and obtain the new coordinates of the object.

Given-

- Old coordinates of the object = A (0, 3, 1), B(3, 3, 2), C(3, 0, 0), D(0, 0, 0)
- Translation vector = $(T_x, T_y, T_z) = (1, 1, 2)$

<u>For Coordinates A(0, 3, 1)</u>

Let the new coordinates of A = $(X_{new}, Y_{new}, Z_{new})$.

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 0 + 1 = 1$
- $Y_{new} = Y_{old} + T_y = 3 + 1 = 4$
- $Z_{new} = Z_{old} + T_z = 1 + 2 = 3$

Thus, New coordinates of A = (1, 4, 3).

3D Translation

Given a 3D object with coordinate points A(0, 3, 1), B(3, 3, 2), C(3, 0, 0), D(0, 0, 0). Apply the translation with the distance 1 towards X axis, 1 towards Y axis and 2 towards Z axis and obtain the new coordinates of the object.

For Coordinates B(3, 3, 2)

Let the new coordinates of B = (X_{new}, Y_{new}, Z_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 3 + 1 = 4$
- $Y_{new} = Y_{old} + T_y = 3 + 1 = 4$
- $Z_{new} = Z_{old} + T_z = 2 + 2 = 4$

Thus, New coordinates of B = (4, 4, 4).

3D Translation

Given a 3D object with coordinate points A(0, 3, 1), B(3, 3, 2), C(3, 0, 0), D(0, 0, 0). Apply the translation with the distance 1 towards X axis, 1 towards Y axis and 2 towards Z axis and obtain the new coordinates of the object.

For Coordinates C(3, 0, 0)

Let the new coordinates of C = $(X_{new}, Y_{new}, Z_{new})$.

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 3 + 1 = 4$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$
- $Z_{new} = Z_{old} + T_z = 0 + 2 = 2$

Thus, New coordinates of C = (4, 1, 2).

3D Translation

Given a 3D object with coordinate points A(0, 3, 1), B(3, 3, 2), C(3, 0, 0), D(0, 0, 0). Apply the translation with the distance 1 towards X axis, 1 towards Y axis and 2 towards Z axis and obtain the new coordinates of the object.

For Coordinates D(0, 0, 0)

Let the new coordinates of D = (X_{new}, Y_{new}, Z_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 0 + 1 = 1$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$
- $Z_{new} = Z_{old} + T_z = 0 + 2 = 2$

Thus, New coordinates of D = (1, 1, 2).

Thus, New coordinates of the object = A (1, 4, 3), B(4, 4, 4), C(4, 1, 2), D(1, 1, 2).

3D Translation

Example: A point has coordinates in the x, y, z direction i.e., (5, 6, 7). The translation is done in the x-direction by 3 coordinate and y direction. Three coordinates and in the z- direction by two coordinates. Shift the object. Find coordinates of the new position.

Solution: Co-ordinate of the point are (5, 6, 7)

Translation vector in x direction = 3

Translation vector in y direction = 3

Translation vector in z direction = 2

Translation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_x & T_y & T_z & 1 \end{pmatrix}$$

3D Translation

Example: A point has coordinates in the x, y, z direction i.e., (5, 6, 7). The translation is done in the x-direction by 3 coordinate and y direction. Three coordinates and in the z- direction by two coordinates. Shift the object. Find coordinates of the new position.

Multiply co-ordinates of point with translation matrix

$$(x^1 y^1 z^1) = (5,6,7,1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 3 & 2 & 1 \end{pmatrix}$$

x becomes x^1=8
y becomes y^1=9
z becomes z^1=9

= [5+0+0+30+6+0+30+0+7+20+0+0+1] = [8991]

3D Scaling

Scaling is used to change the size of an object. The size can be increased or decreased. The scaling three factors are required S_x S_y and S_z.

S_x=Scaling factor in x- direction

S_y=Scaling factor in y-direction

S_z=Scaling factor in z-direction

- Scaling may be used to increase or reduce the size of object.
- Scaling subjects the coordinate points of the original object to change.
- Scaling factor determines whether the object size is to be increased or reduced.
- If scaling factor > 1, then the object size is increased.
- If scaling factor < 1, then the object size is reduced.

3D Scaling

Consider a point object O has to be scaled in a 3D plane.

Let-

- Initial coordinates of the object O = (X_{old}, Y_{old},Z_{old})
- Scaling factor for X-axis = S_x
- Scaling factor for Y-axis = S_y
- Scaling factor for Z-axis = S_z
- New coordinates of the object O after scaling = (X_{new}, Y_{new}, Z_{new})

This scaling is achieved by using the following scaling equations-

- $X_{new} = X_{old} \times S_x$
- $Y_{new} = Y_{old} \times S_y$
- $Z_{new} = Z_{old} \times S_z$

3D Scaling

In Matrix form, the above scaling equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Scaling Matrix

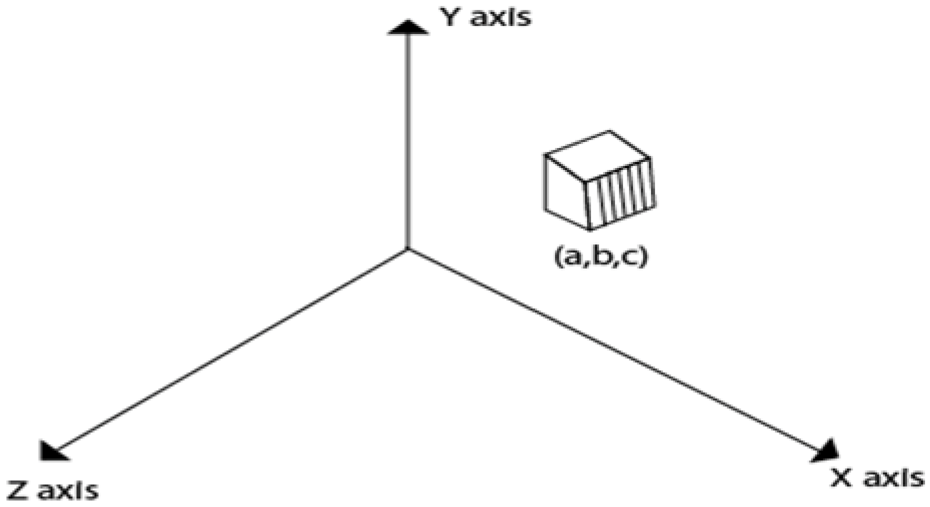
Scaling of the object relative to a fixed point

Following are steps performed when scaling of objects with fixed point (a, b, c). It can be represented as below:

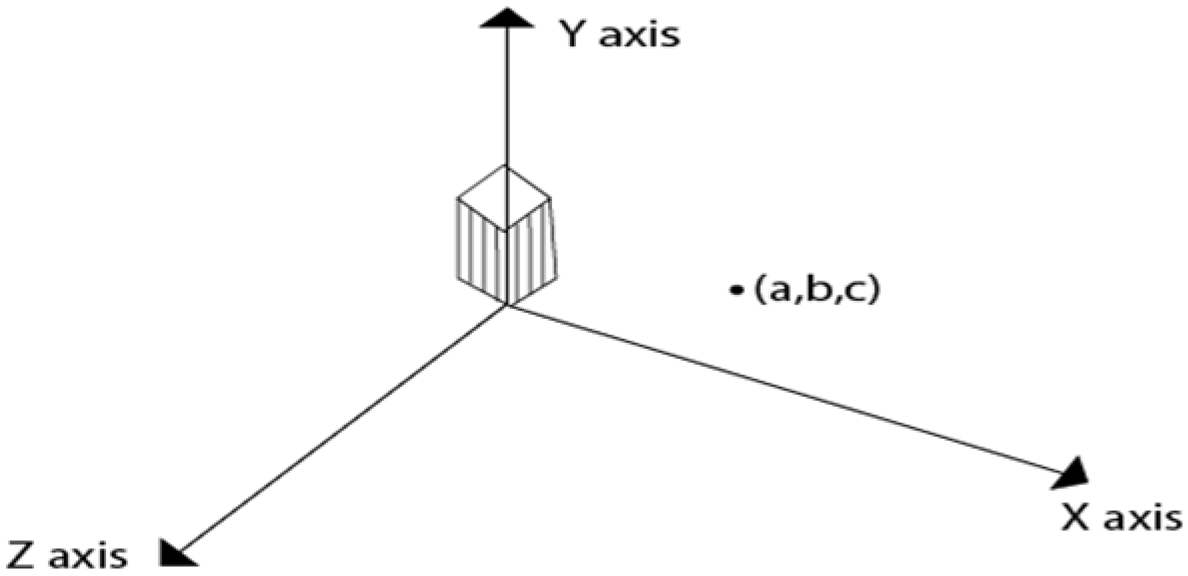
1. Translate fixed point to the origin
2. Scale the object relative to the origin
3. Translate object back to its original position.

If all scaling factors $S_x=S_y=S_z$.Then scaling is called as uniform. If scaling is done with different scaling vectors, it is called a differential scaling.

Scaling of the object relative to a fixed point

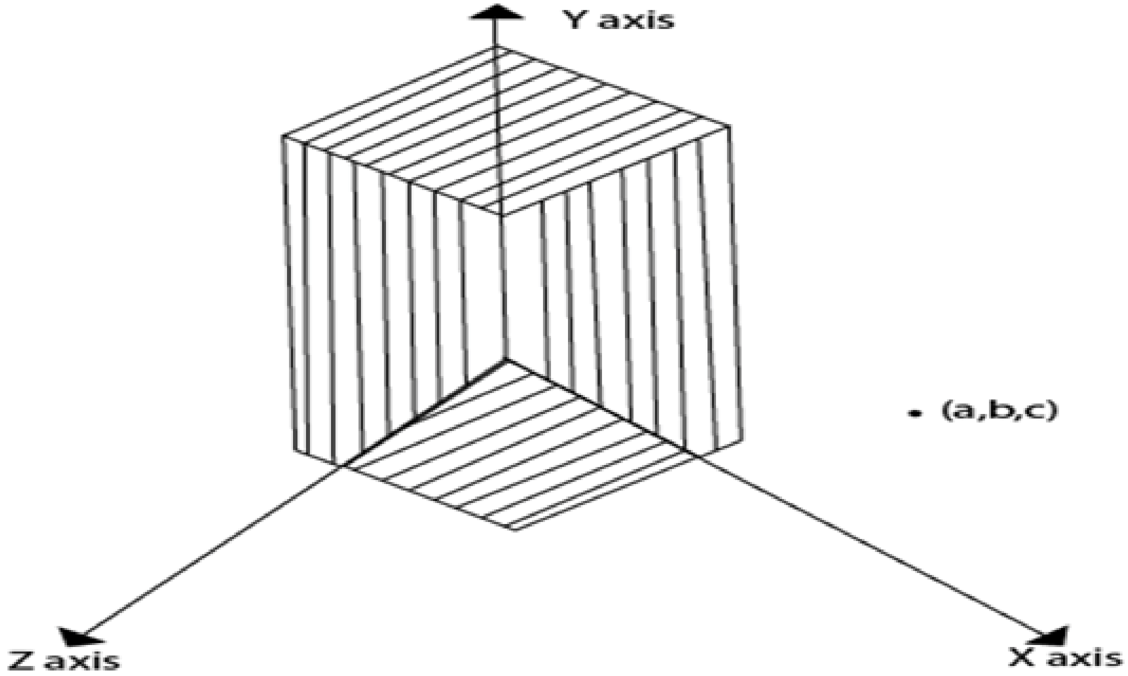


Object & point for scaling
(a)

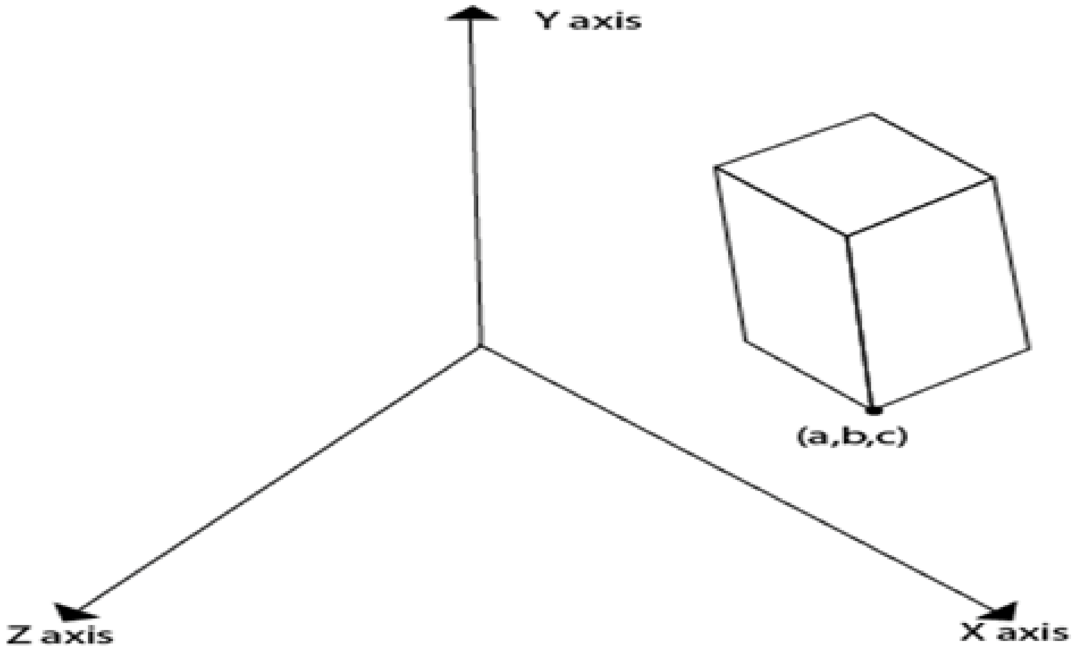


Object transfer at origin
(b)

Scaling of the object relative to a fixed point



Scaling of object
(c)



Object shifted to original position after scaling
(d)

Scaling of the object relative to a fixed point

$$\mathbf{T}(x_f, y_f, z_f) \cdot \mathbf{S}(s_x, s_y, s_z) \cdot \mathbf{T}(-x_f, -y_f, -z_f) = \begin{bmatrix} s_x & 0 & 0 & (1 - s_x)x_f \\ 0 & s_y & 0 & (1 - s_y)y_f \\ 0 & 0 & s_z & (1 - s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Scaling

Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X axis, 3 towards Y axis and 3 towards Z axis and obtain the new coordinates of the object.

- Old coordinates of the object = A (0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0)
- Scaling factor along X axis = 2
- Scaling factor along Y axis = 3
- Scaling factor along Z axis = 3

3D Scaling

Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X axis, 3 towards Y axis and 3 towards Z axis and obtain the new coordinates of the object.

<u>For Coordinates A(0, 3, 3)</u>

Let the new coordinates of A after scaling = (X_{new}, Y_{new}, Z_{new}).

Applying the scaling equations, we have-

- $X_{new} = X_{old} \times S_x = 0 \times 2 = 0$
- $Y_{new} = Y_{old} \times S_y = 3 \times 3 = 9$
- $Z_{new} = Z_{old} \times S_z = 3 \times 3 = 9$

Thus, New coordinates of corner A after scaling = (0, 9, 9).

3D Scaling

Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X axis, 3 towards Y axis and 3 towards Z axis and obtain the new coordinates of the object.

For Coordinates B(3, 3, 6)

Let the new coordinates of B after scaling = (X_{new}, Y_{new}, Z_{new}).

Applying the scaling equations, we have-

- $X_{new} = X_{old} \times S_x = 3 \times 2 = 6$
- $Y_{new} = Y_{old} \times S_y = 3 \times 3 = 9$
- $Z_{new} = Z_{old} \times S_z = 6 \times 3 = 18$

Thus, New coordinates of corner B after scaling = (6, 9, 18).

3D Scaling

Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X axis, 3 towards Y axis and 3 towards Z axis and obtain the new coordinates of the object.

For Coordinates C(3, 0, 1)

Let the new coordinates of C after scaling = (X_{new}, Y_{new}, Z_{new}).

Applying the scaling equations, we have-

- $X_{new} = X_{old} \times S_x = 3 \times 2 = 6$
- $Y_{new} = Y_{old} \times S_y = 0 \times 3 = 0$
- $Z_{new} = Z_{old} \times S_z = 1 \times 3 = 3$

Thus, New coordinates of corner C after scaling = (6, 0, 3).

3D Scaling

Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X axis, 3 towards Y axis and 3 towards Z axis and obtain the new coordinates of the object.

For Coordinates D(0, 0, 0)

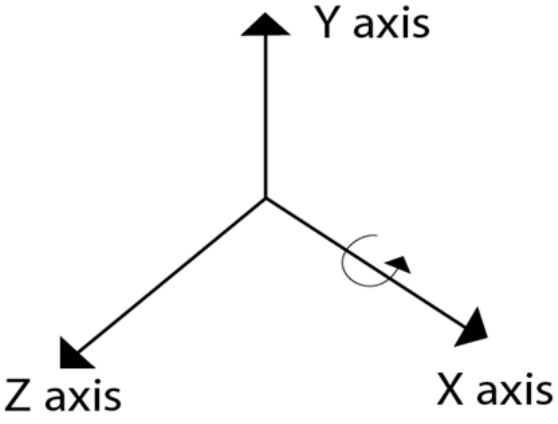
Let the new coordinates of D after scaling = (X_{new}, Y_{new}, Z_{new}).

Applying the scaling equations, we have-

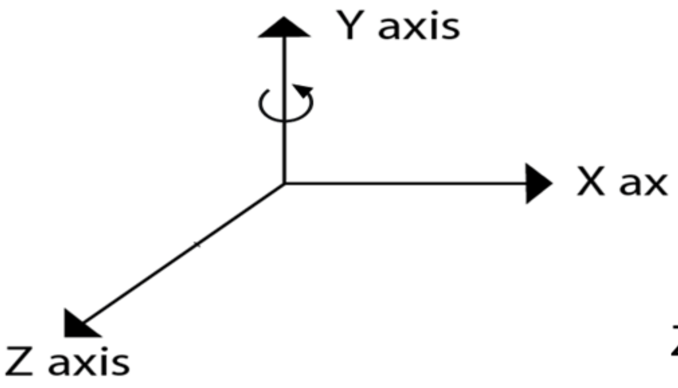
- $X_{new} = X_{old} \times S_x = 0 \times 2 = 0$
- $Y_{new} = Y_{old} \times S_y = 0 \times 3 = 0$
- $Z_{new} = Z_{old} \times S_z = 0 \times 3 = 0$

Thus, New coordinates of corner D after scaling = (0, 0, 0).

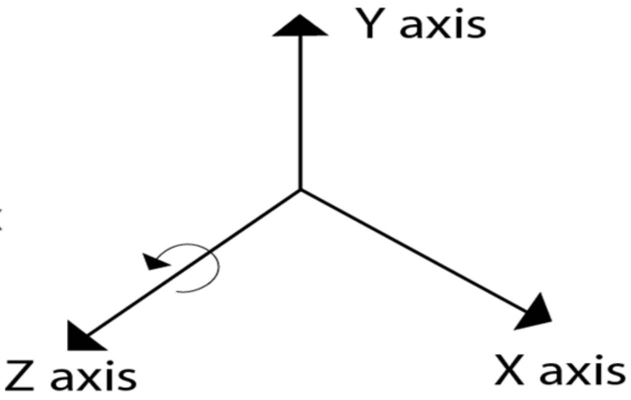
3D Rotation



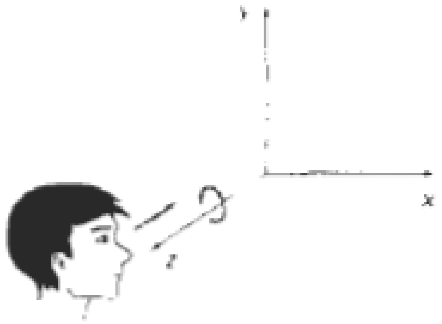
rotation object X axis anticlockwise

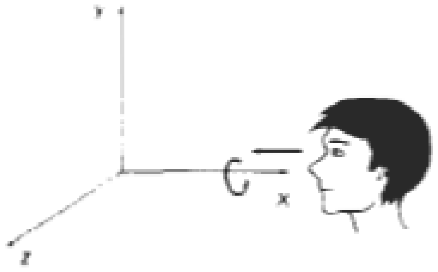


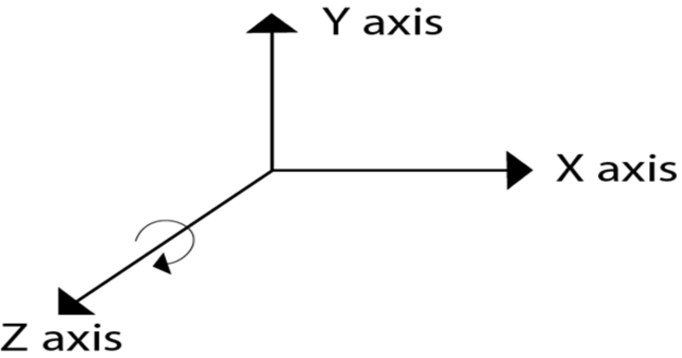
rotation object Y axis anticlockwise



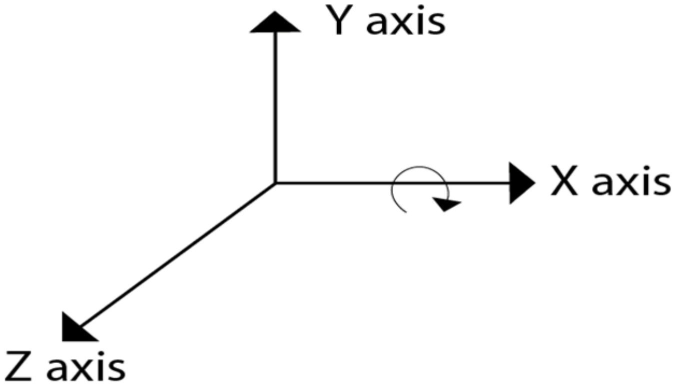
rotation object Z axis anticlockwise


(a)

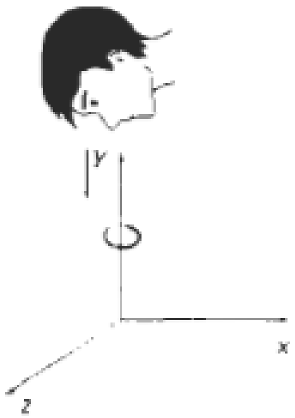

(b)



rotation about Z axis clockwise



rotation about X axis clockwise



3D Rotation

Consider a point object O has to be rotated from one angle to another in a 3D plane.

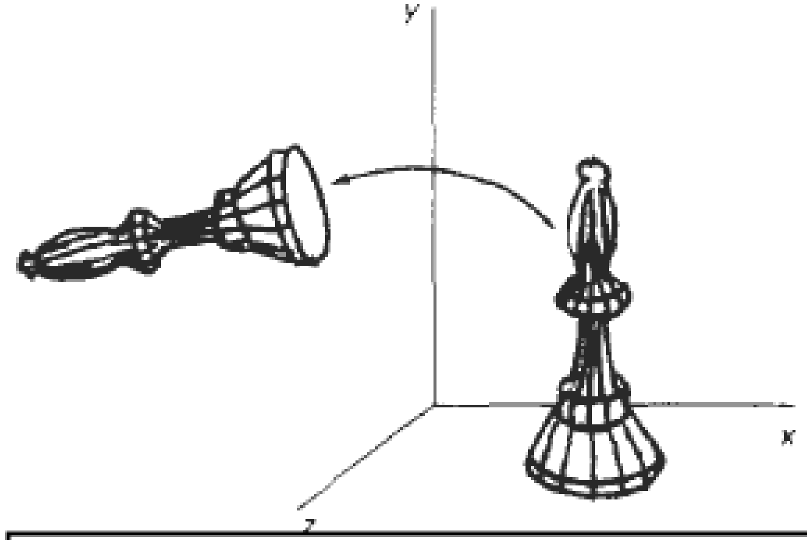
Let-

- Initial coordinates of the object O = (X_{old}, Y_{old}, Z_{old})
- Initial angle of the object O with respect to origin = Φ
- Rotation angle = θ
- New coordinates of the object O after rotation = (X_{new}, Y_{new}, Z_{new})

In 3 dimensions, there are 3 possible types of rotation-

- X-axis Rotation
- Y-axis Rotation
- Z-axis Rotation

3D Rotation about z-axis



This rotation is achieved by using the following rotation equations-

- $X_{new} = X_{old} \times \cos\theta - Y_{old} \times \sin\theta$
- $Y_{new} = X_{old} \times \sin\theta + Y_{old} \times \cos\theta$
- $Z_{new} = Z_{old}$

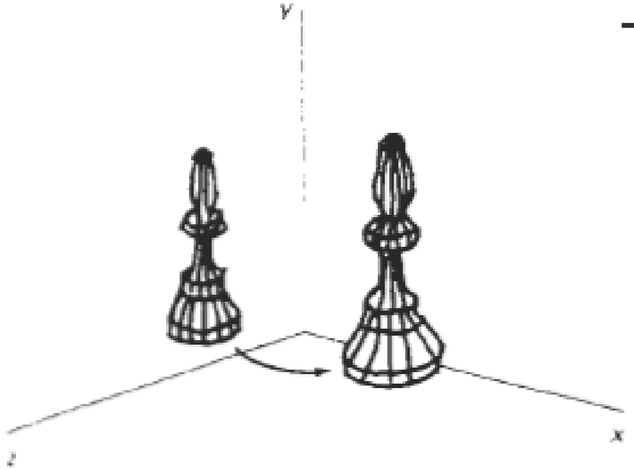
In Matrix form, the above rotation equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix

(For Z-Axis Rotation)

3D Rotation about y-axis



This rotation is achieved by using the following rotation equations-

- $X_{new} = Z_{old} \times \sin\theta + X_{old} \times \cos\theta$
- $Y_{new} = Y_{old}$
- $Z_{new} = Y_{old} \times \cos\theta - X_{old} \times \sin\theta$

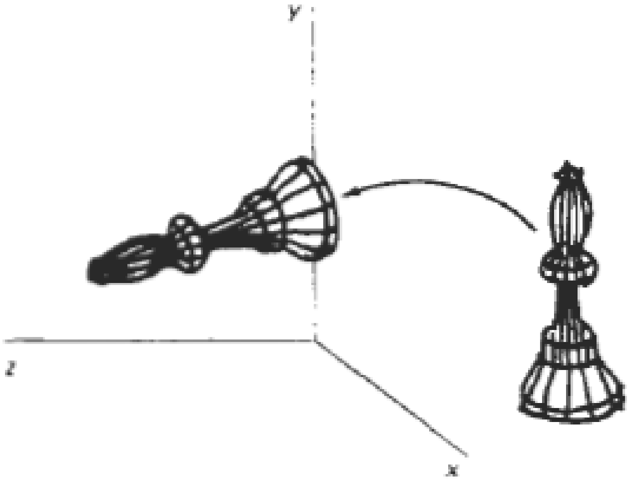
In Matrix form, the above rotation equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix

(For Y-Axis Rotation)

3D Rotation about x-axis



This rotation is achieved by using the following rotation equations-

- $X_{new} = X_{old}$
- $Y_{new} = Y_{old} \times \cos\theta - Z_{old} \times \sin\theta$
- $Z_{new} = Y_{old} \times \sin\theta + Z_{old} \times \cos\theta$

In Matrix form, the above rotation equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Rotation Matrix

(For X-Axis Rotation)

3D Rotation about arbitrary axis

1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
2. Perform the specified rotation about that axis.
3. Translate the object so that the rotation axis is moved back to its original position.

The steps in this sequence are illustrated in Fig. 11-8. Any coordinate position **P** on the object in this figure is transformed with the sequence shown as

$$\mathbf{P}' = \mathbf{T}^{-1} \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T} \cdot \mathbf{P}$$

where the composite matrix for the transformation is

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T}$$

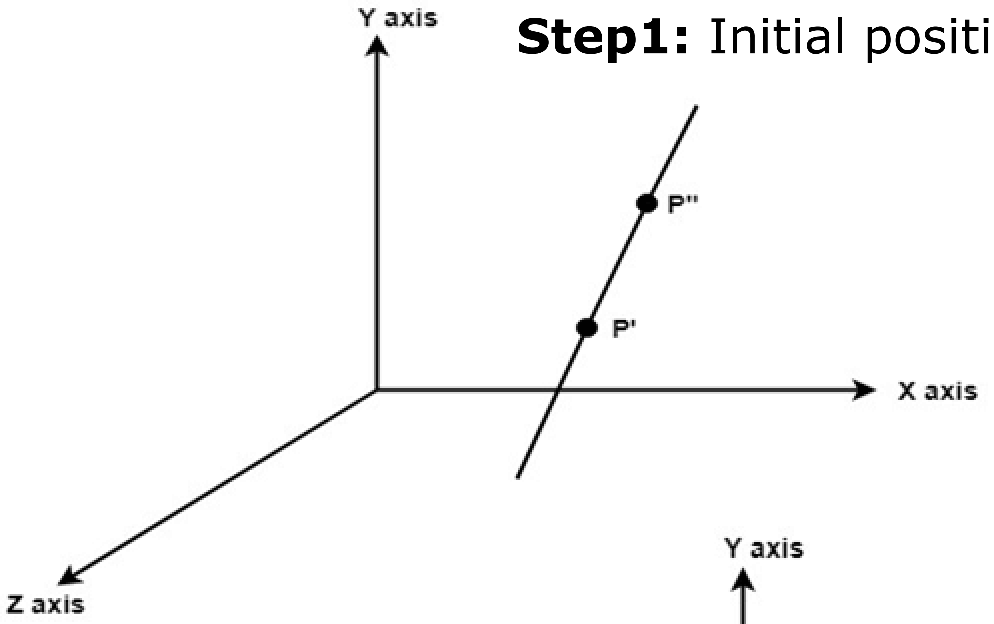
which is of the same form as the two-dimensional transformation sequence for rotation about an arbitrary pivot point.

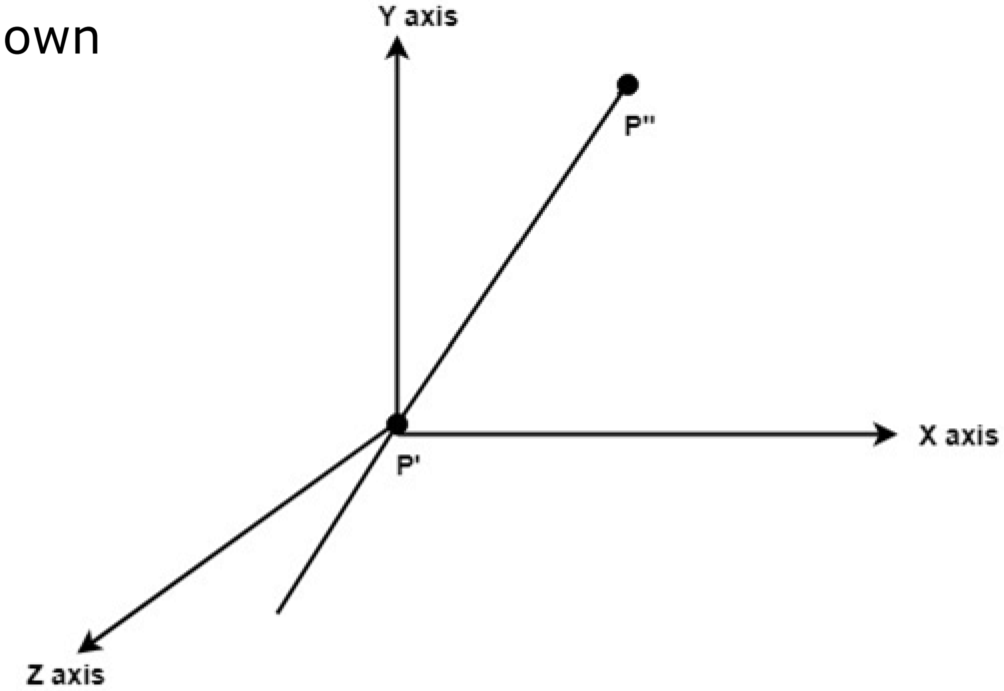
3D Rotation about arbitrary axis

1. Translate the object so that the rotation axis passes through the coordinate origin.
2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
3. Perform the specified rotation about that coordinate axis.
4. Apply inverse rotations to bring the rotation axis back to its original orientation.
5. Apply the inverse translation to bring the rotation axis back to its original position.

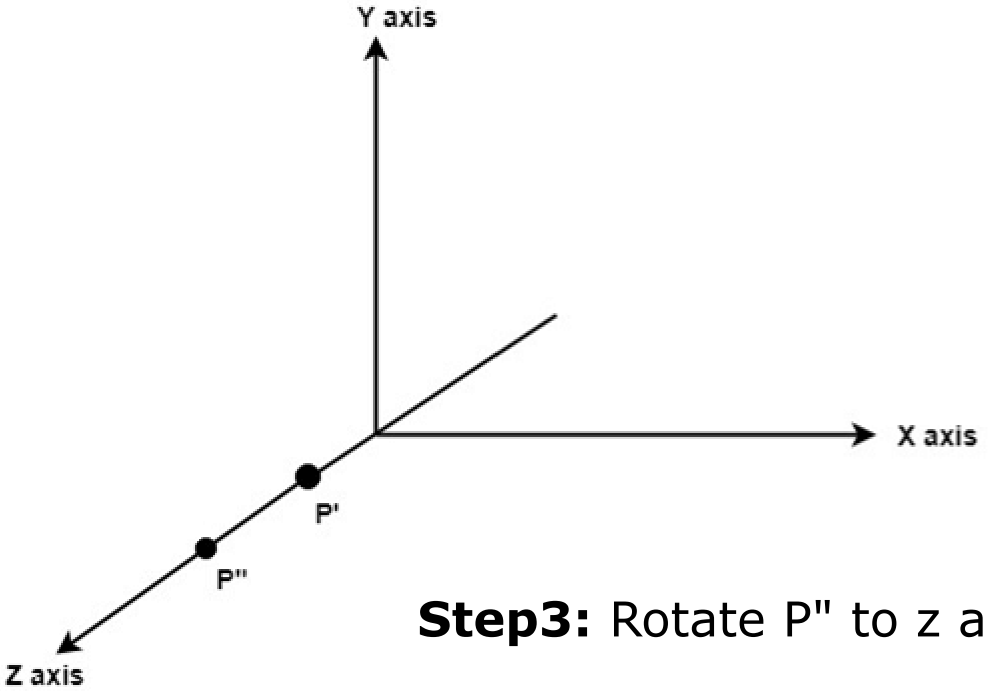
3D Rotation about arbitrary axis

Step1: Initial position of P' and P" is shown





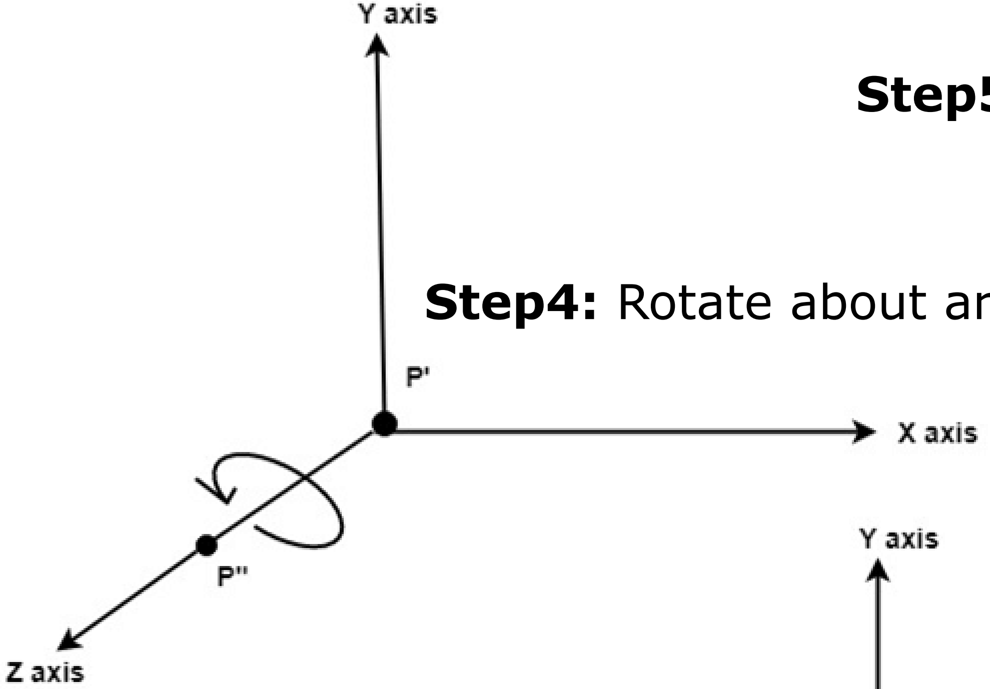
Step2: Translate object P' to origin



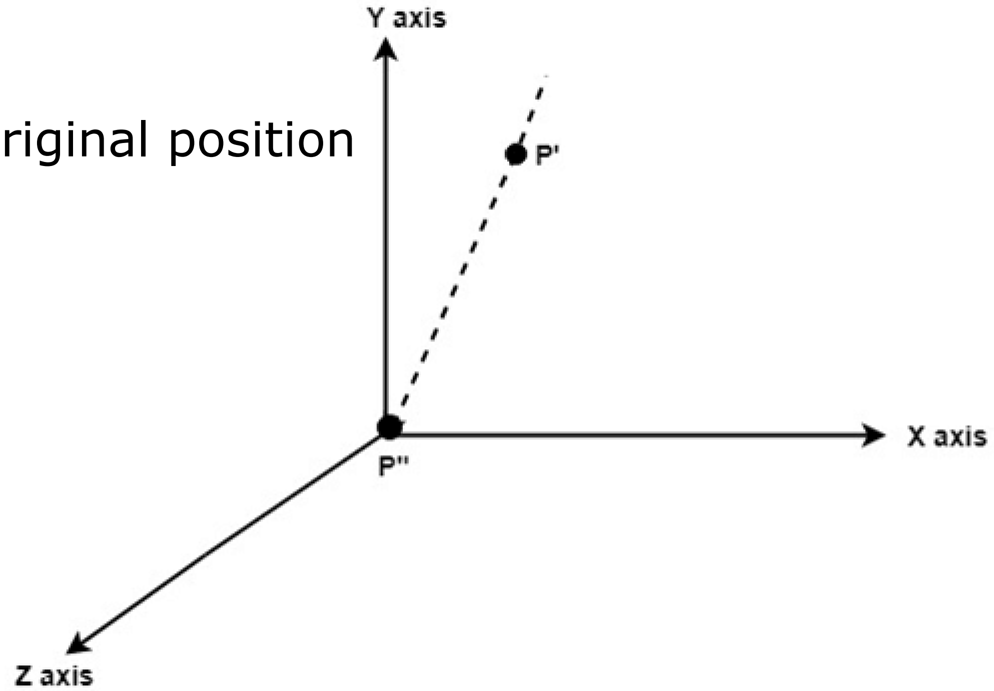
Step3: Rotate P" to z axis so that it aligns along the z-axis

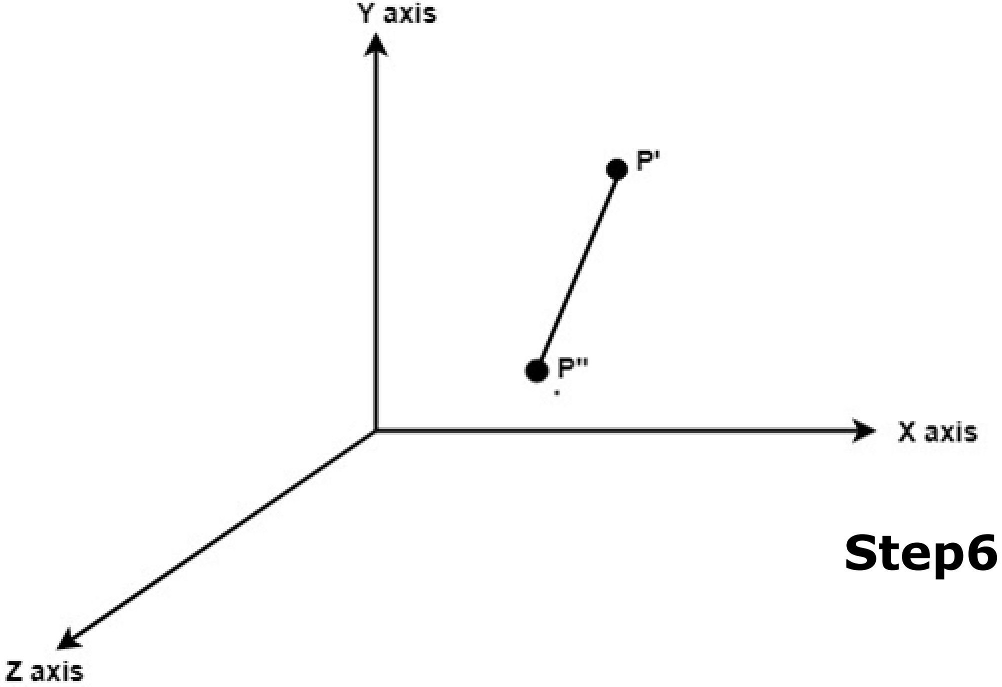
3D Rotation about arbitrary axis

Step4: Rotate about around z- axis



Step5: Rotate axis to the original position





Step6: Translate axis to the original position.

3D Rotation about arbitrary axis

We will arbitrarily choose the Z axis to map the rotation axis onto. The rotation axis is defined by 2 points: P1(x1,y1,z1) and P2(x2,y2,z2.). These 2 points define a vector:

V = (x2 - x1 , y2 - y1 , z2 - z1) = (dx, dy, dz)

which has a unit vector

U = V ÷ |V| where |V| is the length of V = (VxV) = (dx^2 + dy^2 + dz^2)^1/2

Now **U** = (a,b,c) where

a = dx/|V|, b = dy/|V|, c = dz/|V| (these are called the direction cosines of x, y, and z)

(Note: the direction cosine of x = cos A where A = angle of **V** with respect to x axis)

3D Rotation about arbitrary axis

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\alpha = \frac{\mathbf{u}' \cdot \mathbf{u}_z}{|\mathbf{u}'|\,|\mathbf{u}_z|} = \frac{c}{d}$$

where d is the magnitude of $\mathbf{u}'$:

$$d = \sqrt{b^2 + c^2}$$

$$\mathbf{u}' \times \mathbf{u}_z = \mathbf{u}_x\,|\mathbf{u}'|\,|\mathbf{u}_z|\sin\alpha$$

the Cartesian form for the cross product gives us

$$\mathbf{u}' \times \mathbf{u}_z = \mathbf{u}_x \cdot b$$

Equating the right sides

$$|\mathbf{u}_z| = 1 \text{ and } |\mathbf{u}'| = d$$

$$d\sin\alpha = b$$

$$\sin\alpha = \frac{b}{d}$$

3D Rotation about arbitrary axis

Now we can perform the first translation (of the rotation axis to pass through the origin) by using the matrix **T** (-x1, -y1, -z 1), i.e., move the point P1 to the origin. Next we want to rotate to align **V** with Z axis. We do this in 2 steps:

1. Rotate **V** about X axis to put **V** in XZ plane.
2. Rotate **V** about Y to align with Z.

For rotation about X axis we need to find cos A, sin A where A = angle between projection of **U** (in YZ plane) and Z axis. Note: **U'** is no longer a unit vector, i.e. |U'| =/ 1

Uz = unit vector along z axis = (0,0,1)

now (U)' * (Uz) = |U'| * |Uz| cos A = d * cos A

|Uz | = (1)1/2 = 1

(U')(Uz) = 0 * 0 + b * 0 + c * 1 = c

therefore c = d * (cos A) => cos A = c/d

Now the cross product of 2 vectors (V1) X (V2) = W |V| * |V2|sin q where W is perpendicular to plane of V1, V2

so (U') X (Uz) = Ux|U' |* |Uz|sin A = Ux d * sin A

3D Rotation about arbitrary axis

$$\text{So } R_x(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c/d & b/a & 0 \\ 0 & -b/a & c/a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y(B) = \begin{pmatrix} d & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ -a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rx(a) --> Rotates **U** into XZ plane

Now compute Ry(B) for rotation to z-axis.

After rotation about x-axis the vector is as below:

Uy" = 0 since in XZ plane

Uz" = d = |U'| since just rotated U' into XZ plane

again from dot product:

cos B = U" *(Uz) / |U"||Uz| = 0 * a + 0 * 0 + 1 * d = d

Note: |U"|*|Uz| = 1 since both are unit vectors

from the cross product U" x Uz = Uz|U"||Uz|sin B

(from matrix) = Uy x (-a)

therefore sin B = - a

3D Rotation about arbitrary axis

$$\cos\beta = \frac{\mathbf{u}'' \cdot \mathbf{u}_z}{|\mathbf{u}''|\ |\mathbf{u}_z|} = d$$

since $|\mathbf{u}_z| = |\mathbf{u}''| = 1$

$$\mathbf{u}'' \times \mathbf{u}_z = \mathbf{u}_y |\mathbf{u}''|\ |\mathbf{u}_z| \sin\beta$$

with the Cartesian form

$$\mathbf{u}'' \times \mathbf{u}_z = \mathbf{u}_y \cdot (-a)$$

$$\sin\beta = -a$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x^{-1}(\alpha) \cdot \mathbf{R}_y^{-1}(\beta) \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\beta) \cdot \mathbf{R}_x(\alpha) \cdot \mathbf{T}$$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u}'_z = \mathbf{u}$$

$$\mathbf{u}'_y = \frac{\mathbf{u} \times \mathbf{u}_x}{|\mathbf{u} \times \mathbf{u}_x|}$$

$$\mathbf{u}'_x = \mathbf{u}'_y \times \mathbf{u}'_z$$

$$\mathbf{u}'_x = (u'_{x1}, u'_{x2}, u'_{x3})$$

$$\mathbf{u}'_y = (u'_{y1}, u'_{y2}, u'_{y3})$$

$$\mathbf{u}'_z = (u'_{z1}, u'_{z2}, u'_{z3})$$

$$\mathbf{R} = \begin{bmatrix} u'_{x1} & u'_{x2} & u'_{x3} & 0 \\ u'_{y1} & u'_{y2} & u'_{y3} & 0 \\ u'_{z1} & u'_{z2} & u'_{z3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotation about arbitrary axis

The result of this transformation is that V (= Rotation axis) is coincident with z axis.
Then apply

Rz(q) =	(cos q sin q 0 0)
	(-sin q cos q 0 0)
	(0 0 1 0)
	(0 0 0 1)

Then we must apply inverse transformations to get R.A. back to original position.
Therefore, the complete composite transformation matrix is as follows:.

R(q) = T * Rx(A) * Ry(B) * Rz(q) * RY-1(B) * Rx-1(A) * T-1

n1 = a(x), n2 = b(y), n3 = c(z)

3D Rotation about arbitrary axis

[R] =

Row1	n12 + (1 - n12)cosq	n1n2(1 - cosq) + n3sinq	n1n3(1 - cosq) - n2sinq	0
Row2	n1n2(1 - cosq) - n3sinq	n22 + (1 - n22)cosq	n2n3(1 - cosq) + n1sinq	0
Row3	n1n3(1 - cosq) + n2sinq	n2n3(1 - cosq) - n1sinq	n32 + (1 - n32)cosq	0
Row 4	0	0	0	1

3D Rotation

Given a homogeneous point (1, 2, 3). Apply rotation 90 degree towards X, Y and Z axis and find out the new coordinate points.

<u>Solution-</u>

Given-

- Old coordinates = (X_{old}, Y_{old}, Z_{old}) = (1, 2, 3)
- Rotation angle = θ = 90º

<u>For X-Axis Rotation-</u>

Let the new coordinates after rotation = (X_{new}, Y_{new}, Z_{new}).

Applying the rotation equations, we have-

- $X_{new} = X_{old} = 1$
- $Y_{new} = Y_{old} \times \cos\theta - Z_{old} \times \sin\theta = 2 \times \cos 90° - 3 \times \sin 90° = 2 \times 0 - 3 \times 1 = -3$
- $Z_{new} = Y_{old} \times \sin\theta + Z_{old} \times \cos\theta = 2 \times \sin 90° + 3 \times \cos 90° = 2 \times 1 + 3 \times 0 = 2$

Thus, New coordinates after rotation = (1, -3, 2).

3D Rotation

Given a homogeneous point (1, 2, 3). Apply rotation 90 degree towards X, Y and Z axis and find out the new coordinate points.

For Y-Axis Rotation-

Let the new coordinates after rotation = $(X_{new}, Y_{new}, Z_{new})$.

Applying the rotation equations, we have-

- $X_{new} = Z_{old} \times \sin\theta + X_{old} \times \cos\theta = 3 \times \sin 90° + 1 \times \cos 90° = 3 \times 1 + 1 \times 0 = 3$
- $Y_{new} = Y_{old} = 2$
- $Z_{new} = Y_{old} \times \cos\theta - X_{old} \times \sin\theta = 2 \times \cos 90° - 1 \times \sin 90° = 2 \times 0 - 1 \times 1 = -1$

Thus, New coordinates after rotation = (3, 2, -1).

3D Rotation

Given a homogeneous point (1, 2, 3). Apply rotation 90 degree towards X, Y and Z axis and find out the new coordinate points.

For Z-Axis Rotation-

Let the new coordinates after rotation = $(X_{new}, Y_{new}, Z_{new})$.

Applying the rotation equations, we have-

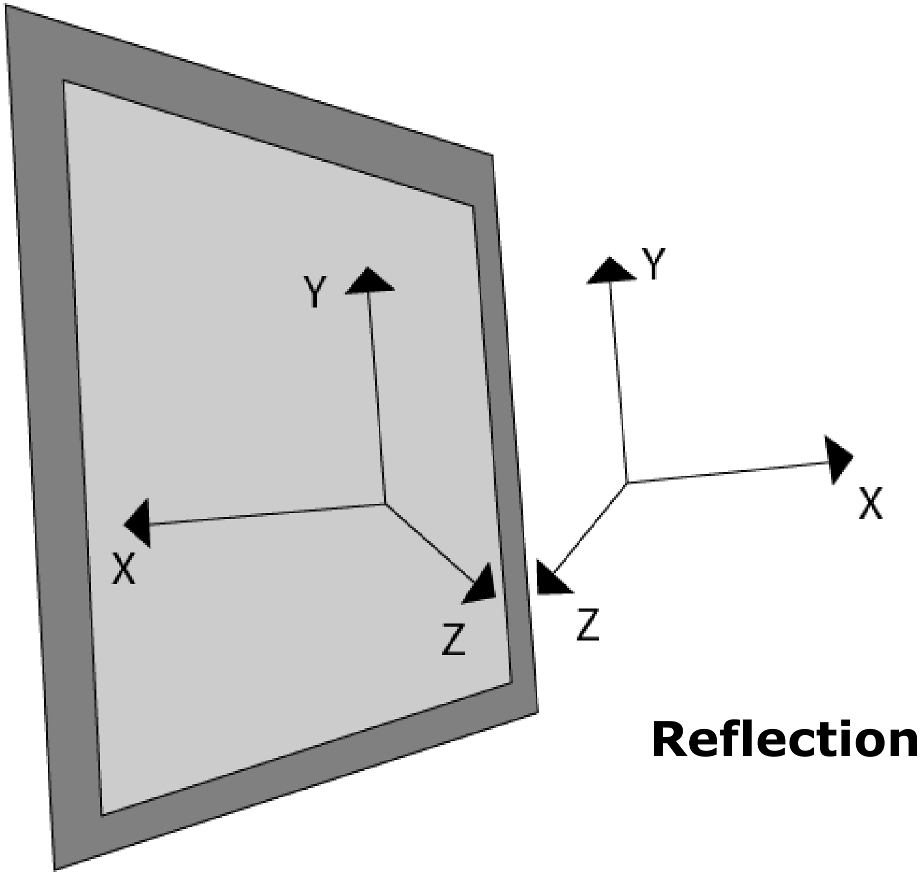
- $X_{new} = X_{old} \times \cos\theta - Y_{old} \times \sin\theta = 1 \times \cos 90° - 2 \times \sin 90° = 1 \times 0 - 2 \times 1 = -2$
- $Y_{new} = X_{old} \times \sin\theta + Y_{old} \times \cos\theta = 1 \times \sin 90° + 2 \times \cos 90° = 1 \times 1 + 2 \times 0 = 1$
- $Z_{new} = Z_{old} = 3$

Thus, New coordinates after rotation = (-2, 1, 3)

3D Reflection

It is also called a mirror image of an object. For this reflection axis and reflection of plane is selected. Three-dimensional reflections are similar to two dimensions. Reflection is 180° about the given axis. For reflection, plane is selected (xy,xz or yz). Following matrices show reflection respect to all these three planes.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Reflection relative to XY plane

3D Reflection

Reflection relative to YZ plane

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflection relative to ZX plane

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3D Reflection

- Reflection is a kind of rotation where the angle of rotation is 180 degree.
- The reflected object is always formed on the other side of mirror.
- The size of reflected object is same as the size of original object.

Consider a point object O has to be reflected in a 3D plane.

Let-

- Initial coordinates of the object O = (X_{old}, Y_{old}, Z_{old})
- New coordinates of the reflected object O after reflection = (X_{new}, Y_{new},Z_{new})

In 3 dimensions, there are 3 possible types of reflection-

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XY plane and find out the new coordinates of the object.

Solution-

Given-

- Old corner coordinates of the triangle = A (3, 4, 1), B(6, 4, 2), C(5, 6, 3)
- Reflection has to be taken on the XY plane

For Coordinates A(3, 4, 1)

Let the new coordinates of corner A after reflection = (X_{new}, Y_{new}, Z_{new}).

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 3$
- $Y_{new} = Y_{old} = 4$
- $Z_{new} = -Z_{old} = -1$

Thus, New coordinates of corner A after reflection = (3, 4, -1).

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XY plane and find out the new coordinates of the object.

For Coordinates B(6, 4, 2)

Let the new coordinates of corner B after reflection = (X_{new}, Y_{new}, Z_{new}).

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 6$
- $Y_{new} = Y_{old} = 4$
- $Z_{new} = -Z_{old} = -2$

Thus, New coordinates of corner B after reflection = (6, 4, -2).

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XY plane and find out the new coordinates of the object.

For Coordinates C(5, 6, 3)

Let the new coordinates of corner C after reflection = (X_{new}, Y_{new}, Z_{new}).

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 5$
- $Y_{new} = Y_{old} = 6$
- $Z_{new} = -Z_{old} = -3$

Thus, New coordinates of corner C after reflection = (5, 6, -3).

Thus, New coordinates of the triangle after reflection = A (3, 4, -1), B(6, 4, -2), C(5, 6, -3).

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XZ plane and find out the new coordinates of the object.

Solution-

Given-

- Old corner coordinates of the triangle = A (3, 4, 1), B(6, 4, 2), C(5, 6, 3)
- Reflection has to be taken on the XZ plane

For Coordinates A(3, 4, 1)

Let the new coordinates of corner A after reflection = $(X_{new}, Y_{new}, Z_{new})$.

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 3$
- $Y_{new} = -Y_{old} = -4$
- $Z_{new} = Z_{old} = 1$

Thus, New coordinates of corner A after reflection = (3, -4, 1).

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XZ plane and find out the new coordinates of the object.

For Coordinates B(6, 4, 2)

Let the new coordinates of corner B after reflection = (X_{new}, Y_{new}, Z_{new}).

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 6$
- $Y_{new} = -Y_{old} = -4$
- $Z_{new} = Z_{old} = 2$

Thus, New coordinates of corner B after reflection = (6, -4, 2).

3D Reflection

Given a 3D triangle with coordinate points A(3, 4, 1), B(6, 4, 2), C(5, 6, 3). Apply the reflection on the XZ plane and find out the new coordinates of the object.

<u>For Coordinates C(5, 6, 3)</u>

Let the new coordinates of corner C after reflection = (X_{new}, Y_{new}, Z_{new}).

Applying the reflection equations, we have-

- $X_{new} = X_{old} = 5$
- $Y_{new} = -Y_{old} = -6$
- $Z_{new} = Z_{old} = 3$

Thus, New coordinates of corner C after reflection = (5, -6, 3).

Thus, New coordinates of the triangle after reflection = A (3, -4, 1), B(6, -4, 2), C(5, -6, 3).

3D Shear

It is change in the shape of the object. It is also called as deformation. Change can be in the x -direction or y -direction or both directions in case of 2D. If shear occurs in both directions, the object will be distorted. But in 3D shear can occur in three directions.

Consider a point object O has to be sheared in a 3D plane.

Let-

- Initial coordinates of the object O = (X_{old}, Y_{old}, Z_{old})
- Shearing parameter towards X direction = Sh_x
- Shearing parameter towards Y direction = Sh_y
- Shearing parameter towards Z direction = Sh_z
- New coordinates of the object O after shearing = (X_{new}, Y_{new}, Z_{new})

3D Shear

Shearing in X axis is achieved by using the following shearing equations-

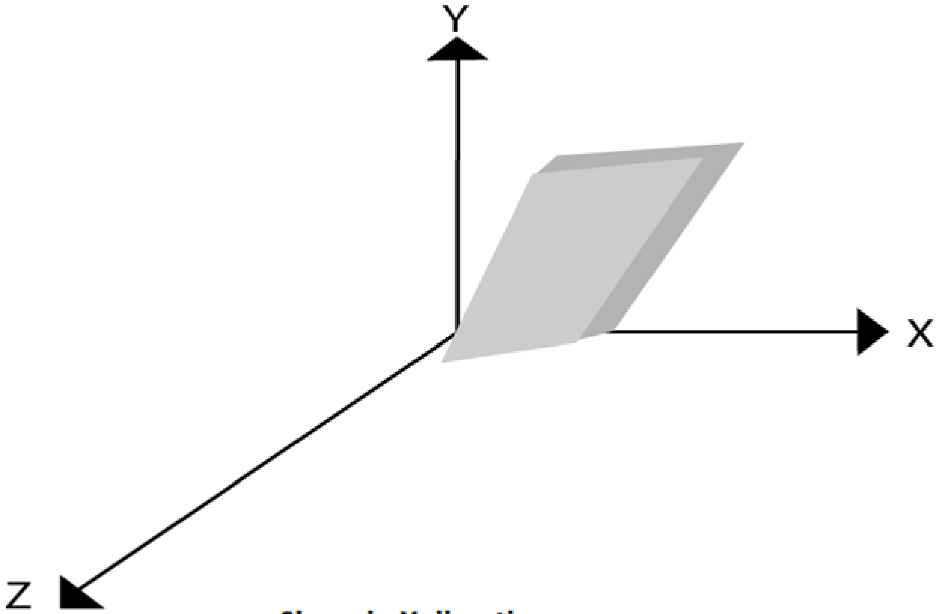
- $X_{new} = X_{old}$
- $Y_{new} = Y_{old} + Sh_y \times X_{old}$
- $Z_{new} = Z_{old} + Sh_z \times X_{old}$

In Matrix form, the above shearing equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ Sh_y & 1 & 0 & 0 \\ Sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In X axis)



Shear in X direction
(a)

3D Shear

Shearing in Y axis is achieved by using the following shearing equations-

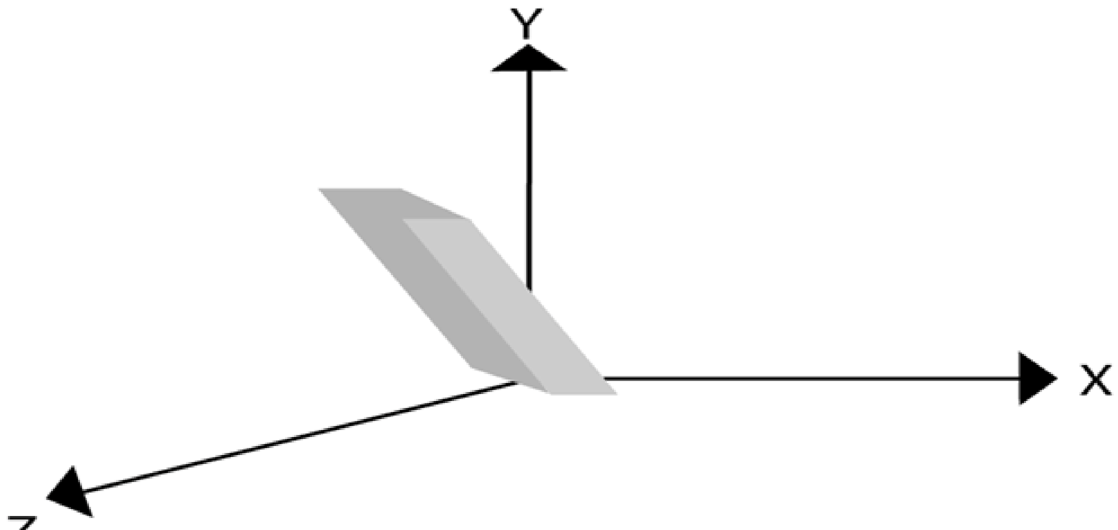
- $X_{new} = X_{old} + Sh_x \times Y_{old}$
- $Y_{new} = Y_{old}$
- $Z_{new} = Z_{old} + Sh_z \times Y_{old}$

In Matrix form, the above shearing equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In Y axis)



Shear in Y direction
(c)

3D Shear

Shearing in Z axis is achieved by using the following shearing equations-

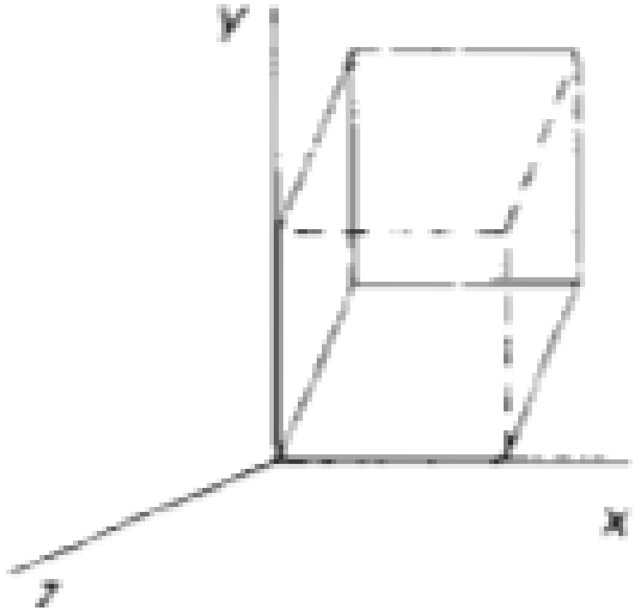
- $X_{new} = X_{old} + Sh_x \times Z_{old}$
- $Y_{new} = Y_{old} + Sh_y \times Z_{old}$
- $Z_{new} = Z_{old}$

In Matrix form, the above shearing equations may be represented as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & Sh_x & 0 \\ 0 & 1 & Sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In Z axis)



3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

Solution-

Given-

- Old corner coordinates of the triangle = A (0, 0, 0), B(1, 1, 2), C(1, 1, 3)
- Shearing parameter towards X direction (Sh_x) = 2
- Shearing parameter towards Y direction (Sh_y) = 2
- Shearing parameter towards Y direction (Sh_z) = 3

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

<u>Shearing in X Axis-</u>

<u>For Coordinates A(0, 0, 0)</u>

Let the new coordinates of corner A after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} = 0$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 0 + 2 \times 0 = 0$
- $Z_{new} = Z_{old} + Sh_z \times X_{old} = 0 + 3 \times 0 = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates B(1, 1, 2)

Let the new coordinates of corner B after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} = 1$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 1 + 2 \times 1 = 3$
- $Z_{new} = Z_{old} + Sh_z \times X_{old} = 2 + 3 \times 1 = 5$

Thus, New coordinates of corner B after shearing = (1, 3, 5).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates C(1, 1, 3)

Let the new coordinates of corner C after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} = 1$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 1 + 2 \times 1 = 3$
- $Z_{new} = Z_{old} + Sh_z \times X_{old} = 3 + 3 \times 1 = 6$

Thus, New coordinates of corner C after shearing = (1, 3, 6).
Thus, New coordinates of the triangle after shearing in X axis = A (0, 0, 0), B(1, 3, 5), C(1, 3, 6).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

Shearing in Y Axis-

For Coordinates A(0, 0, 0)

Let the new coordinates of corner A after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 0 + 2 \times 0 = 0$
- $Y_{new} = Y_{old} = 0$
- $Z_{new} = Z_{old} + Sh_z \times Y_{old} = 0 + 3 \times 0 = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates B(1, 1, 2)

Let the new coordinates of corner B after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 1 + 2 \times 1 = 3$
- $Y_{new} = Y_{old} = 1$
- $Z_{new} = Z_{old} + Sh_z \times Y_{old} = 2 + 3 \times 1 = 5$

Thus, New coordinates of corner B after shearing = (3, 1, 5).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates C(1, 1, 3)

Let the new coordinates of corner C after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 1 + 2 \times 1 = 3$
- $Y_{new} = Y_{old} = 1$
- $Z_{new} = Z_{old} + Sh_z \times Y_{old} = 3 + 3 \times 1 = 6$

Thus, New coordinates of corner C after shearing = (3, 1, 6).
Thus, New coordinates of the triangle after shearing in Y axis = A (0, 0, 0), B(3, 1, 5), C(3, 1, 6).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

Shearing in Z Axis-

For Coordinates A(0, 0, 0)

Let the new coordinates of corner A after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Z_{old} = 0 + 2 \times 0 = 0$
- $Y_{new} = Y_{old} + Sh_y \times Z_{old} = 0 + 2 \times 0 = 0$
- $Z_{new} = Z_{old} = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates B(1, 1, 2)

Let the new coordinates of corner B after shearing = $(X_{new}, Y_{new}, Z_{new})$.

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Z_{old} = 1 + 2 \times 2 = 5$
- $Y_{new} = Y_{old} + Sh_y \times Z_{old} = 1 + 2 \times 2 = 5$
- $Z_{new} = Z_{old} = 2$

Thus, New coordinates of corner B after shearing = (5, 5, 2).

3D Shear

Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

For Coordinates C(1, 1, 3)

Let the new coordinates of corner C after shearing = (X_{new}, Y_{new}, Z_{new}).

Applying the shearing equations, we have-

- $X_{new} = X_{old} + Sh_x \times Z_{old} = 1 + 2 \times 3 = 7$
- $Y_{new} = Y_{old} + Sh_y \times Z_{old} = 1 + 2 \times 3 = 7$
- $Z_{new} = Z_{old} = 3$

Thus, New coordinates of corner C after shearing = (7, 7, 3).
Thus, New coordinates of the triangle after shearing in Z axis = A (0, 0, 0), B(5, 5, 2), C(7, 7, 3).