13. A system of equations $AX = b$ where $A = (a_{ij})_{n\times n}$ is said to be diagonally dominant if

(a) $|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ for all i (b) $|a_{ii}| < \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ for all i

(c) $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$ for all i (d) $|a_{ii}| < \sum_{j=1}^{n} |a_{ij}|$ for all i

14. The iterative method is known as

(a) direct method (b) indirect method

(c) none of these

15. The solution of a system of equations is obtained by successive approximation method is known as

(a) direct method (b) indirect method

(c) both (a) and (b) (d) none of these

Answers

1.d	2.a	3.b	4.a	5.b	6.a	7.a	8.b	9.b	10.b
11.a	12.a	13.a	14.b	15.b					

6 NUMERICAL SOLUTIONS OF ALGEBRAIC EQUATIONS

6.1. Introduction.

In applied mathematics and engineering we frequently face the problem of finding one or more roots of the equation

$$f(x) = 0 \qquad \ldots \quad (1)$$

where $f(x)$ is, in general, a nonlinear function of the real variable x. But in most cases, it is very difficult to have explicit solutions of the equation (1) and, therefore, we proceed to look for a root of (1) numercially with any specified degree of accuracy. The numerical methods of finding these roots are called *iterative methods.*

The function $f(x)$ may have any one of the following forms:

(i) $f(x)$ is an algebraic or pclynomial function of degree n, say, so that

$$f(x) = a_\circ + a_1x + a_2x^2 + \cdots + a_{n-1}x^{x-1} + a_nx^n$$

where $a_i (i = 0,1,2,\cdots,n)$ are constants, real or complex, and $a_n \neq 0$. For example, $x^3 - 7x + 1$, $x^{12} + x^5 - 4x + 3$ etc. are algebraic functions. In such cases, the equation $f(x) = 0$ is called *algebraic equation.*

(ii) $f(x)$ is a transcendental function, i.e. $f(x)$ is a function of the form

$$f(x) = a_\circ + a_1x + a_2x^2 + \cdots to\ \infty,$$

where $a_i, (i = 0,1,2,\cdots)$, are constants, real or complex, and all $a_i \neq 0$. For example, $\sin x + 2x^2 - 1$, $e^x + \log x + 5$ etc. are transcendental functions, Here the equation $f(x) = 0$ is called *transcendental equation.*

Every value α of x for which the function $f(x)$ is zero, i.e., $f(\alpha) = 0$ is called a root or zero of the equation (1). In this chapter we shall discuss different numerical methods to compute the approximate real roots of an algebraic or transcendental equation $f(x) = 0$

To develop the methods we assume that

(i) The function $f(x)$ is continuous and continuously differentiable for a sufficient number of times.

(ii) $f(x)$ has no multiple root, i.e., if α is a real root of $f(x)=0$ then

$$f(\alpha)=0,\ f'(\alpha)\neq 0.$$

Determination of approximate (real) root of (1) by numerical methods to be discussed here, consists, in general, of the following two steps.

(i) Isolating the roots, i.e., finding the smallest possible interval $[a, b]$ containing one and only one root of (1).

(ii) Improving the values of the approximate roots, i.e refinding them to the desired degree of accuracy.

To implement the first step, we use the following theorem of a continuous function :

Theorem 1. If real valued function $f(x)$ is continuous in $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there is at least one real root of $f(x)=0$ in (a, b)

6.2. Iteration Processes.

Let the sequence $\{x_n\}$ of iterates of a root α of the equation

$$f(x)=0$$

is produced by a given method. Then the error ε_n involved at the n th iteration is given by

$$\varepsilon_n=\alpha-x_n. \qquad \dots \qquad (2)$$

If $\varepsilon_n\to 0$ as $n\to\infty$, we say that the iteration converges and the sequence $\{x_n\}$ converges to α. Otherwise, the iteration is divergent and the method of computation fails. Thus our primary task is to find the *condition of convergence of the iteration processes.* The error ε_{n+1} can be expressed in terms of ε_n, $\varepsilon_{n-1}, \varepsilon_{n-2}, \cdots$ which we call *error equation.*

If we define h_n by $h_n=x_{n+1}-x_n=\varepsilon_n-\varepsilon_{n+1}$, then h_n is an approximation of ε_n if x_{n+1} approximates α. If the iteration converges, then we can find an upper bound for $|\varepsilon_{n+1}|$ in terms of h_n. This is called *estimation of error.*

In case an iterative method converges, we can find two constants $p\geq 1$ and $q>0$ such that

$$\lim_{n\to\infty}\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^p}\right|=q \qquad \dots \qquad (3)$$

Here p is called the *order of convergence* and q is known as *asymptotic error constant.* The iterative method with $p>1$ generally converges rapidly.

The convergence of the iterative method also depends on the initial approximation x_0 of the root α. If this initial approximation is not satisfactory, the iterative method does not converge and then we look for the new computation.

The iterative method is self correct, i.e. if there is an accidental error in the calculations of iteration, the erroneous iterate acts as a new initial approximation leading to a correct result, provided that the error is not large enough for which the method fails.

6.3. Bisection Method.

A. *Basic principle and formula*

The method of bisection is the most simplest iterative method. It is also known as *half-interval* or *Bolzano method.* This method is based on Theorem 1 on the change of sign.

In this method, we first find out a sufficiently small interval $[a_0, b_0]$ containing the required root α of the equation (1). Then $f(a_0)f(b_0)<0$ and $f'(x)$ has the same sign in $[a_0, b_0]$ and so $f(x)$ is strictly monotonic in $[a_0, b_0]$

To generate the sequence $\{x_n\}$ of iterates, we put $x_0=a_0$ or b_0 and $x_1=\frac{1}{2}(a_0+b_0)$ and find $f(x_1)$. If $f(a_0)$ and $f(x_1)$ are of opposite signs, then set $a_1=a_0, b_1=x_1$ so that $[a_1, b_1]=[a_0, x_1]$. On the other hand, if $f(x_1)$ and $f(b_0)$ are of opposite signs then put $a_1=x_1, b_1=b_0$, i.e. $[a_1, b_1]=[x_1, b_0]$. Thus we see that $[a_1, b_1]$ contains the root α in either case.

Next set $x_2 = \frac{1}{2}(a_1 + b_1)$ and repeat the above process till we obtain

$$x_{n+1} = \frac{1}{2}(a_n + b_n) \quad \ldots \quad (4)$$

with desired accuracy with $x_n \to \alpha$ as $n \to \infty$.

B. *Convergence of bisection method*

Suppose the interval $[a_n, b_n]$ contains the root α and $f(a_n)f(b_n) < 0$.

Let $x_{n+1} = \frac{1}{2}(a_n + b_n)$.

If $f(a_n)f(x_{n+1}) < 0$, then set $a_{n+1} = a_n, b_{n+1} = x_{n+1}$.

On the other hand, if $f(x_{n+1})f(b_n) < 0$, then

$$a_{n+1} = x_{n+1}, b_{n+1} = b_n.$$

Thus in any case

$$\alpha \in [a_{n+1}, b_{n+1}], \ f(a_{n+1})f(b_{n+1}) < 0$$

and

$$b_{n+1} - a_{n+1} < \frac{1}{2}(b_n - a_n) < \ldots < \frac{b_0 - a_0}{2^n}.$$

If ε_{n+1} be the error in approximating α by x_{n+1}, then

$$\varepsilon_{n+1} = |\alpha - x_{n+1}| \le b_n - a_n < \frac{b_0 - a_0}{2^n} \to 0 \text{ as } n \to \infty.$$

Thus the iteration converges.

Since $\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{2}$, so the convergence in bisection method is linear.

C. *Advantage and disadvantage of bisection method*

Advantage. This method is very simple, as at any stage of iteration the approximate value of the desired root of the equation $f(x) = 0$ does not depend on the values $f(x_n)$ but on their signs only. Also the method is unconditionally and surely convergent.

Disadvantage. The method is very slow and requires large number of iteration to obtain moderately accurate results and hence it is laborious.

Example.1. Find the root of the equation $x \tan x = 1.28$, that lies in the interval $(0,1)$, correct to four places of decimal, using bisection method. *[W.B.U.T., CS-312,2005]*

Solution. Let $f(x) = x \tan x - 1 \cdot 28$.

$\therefore f(0) = -1 \cdot 28 < 0, f(1) = 0 \cdot 277408 > 0$.

So a root lies between 0 and 1.

Take $a_0 = 0$, $b_0 = 1$ so that $x_1 = \frac{1}{2}(0+1) = 0.5$. Since $f(0.5) = -1.006849 < 0$ and $f(1) > 0$, the root lies between 0.5 and 1. Thus we have $x_2 = \frac{1}{2}(0.5+1) = 0.75$.

Proceeding in this way, we obtain the following table :

No. of iteration (n)	a_n $f(a_n) < 0$	b_n $f(b_n) > 0$	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0	1	0.5	-1.006849
1	0.5	1	0.75	-0.581303
2	0.75	1	0.875	-0.232256
3	0.875	1	0.9375	-0.003058
4	0.9375	1	0.968750	0.129819
5	0.9375	0.96875	0.953125	0.061675
6	0.9375	0.961675	0.945312	0.028898
7	0.9375	0.945312	0.941406	0.012819
8	0.9375	0.941406	0.939453	0.004856
9	0.9375	0.939453	0.938477	0.000893
10	0.9375	0.938477	0.937988	-0.001084

11	0.937988	0.938477	0.938232	−0.000096
12	0.938232	0.938477	0.938354	0.000398
13	0.938232	0.938354	0.938293	0.000151
14	0.938232	0.938293	0.938263	0.000028

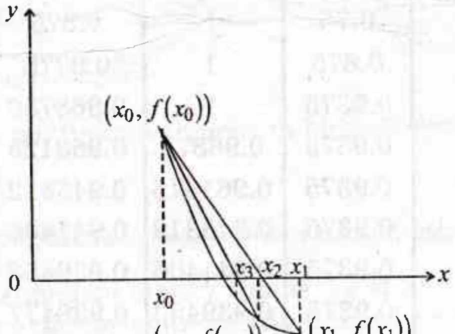
Thus a real root of the given equation is 0.9383 correct to four decimal places.

6.4 Regula-Falsi Method.

A. Basic principle and formula [W.B.U.T., CS-312,2003]

The regula-falsi method or false position method is also sometimes referred to as the method of *linear interpolation* and it is the oldest method for computing real roots of an equation $f(x)=0$.

To find a real root α of $f(x)=0$, we first choose a sufficiently small interval $[x_0,x_1]$ in which the root α lies. Then $f(x_0)$ and $f(x_1)$ must be of opposite signs so that $f(x_0)f(x_1)<0$ and the graph of $f(x)$ must cross the x-axis between $x=x_0$ and $x=x_1$. Since the interval $[x_0,x_1]$ is sufficiently small, the portion of the curve between $A[x_0,f(x_0)]$ and $B[x_1,f(x_1)]$ can be approximated by a secant line (straight line) and so the intersection of the secant AB with the x-axis gives an approximate value x_2 ,say, of the root.



The equation of the secant line AB is

$$y-f(x_1)=\frac{f(x_0)-f(x_1)}{x_0-x_1}(x-x_1)$$

Putting $y=0, x=x_2$, we derive

$$x_2=x_1-\frac{f(x_1)}{f(x_1)-f(x_0)}(x_1-x_0) \quad \ldots (5)$$

If $f(x_2)=0$, then x_2 is a root of $f(x)=0$; otherwise, $f(x_2)<0$ or $f(x_2)>0$ It $f(x_\circ)$ and $f(x_2)$ are of opposite signs the root lies between $x_\circ$ and x_2 and in this case we set $x_1=x_\circ$ and $x_2=x_1$. On the other hand if $f(x_1)$ and $f(x_2)$ are of opposite signs, the root lies between x_1 and x_2 and thus, in either case,

$$f(x_1)f(x_2)<0$$

Hence the next approximation of the root, say x_3 lies between x_1 and x_2 and get

$$x_3=x_2-\frac{f(x_2)}{f(x_2)-f(x_1)}(x_2-x_1) \quad \ldots (6)$$

The general formula based on the above process is

$$x_{n+1}=x_n-\frac{f(x_n)}{f(x_n)-f(x_{n-1})}(x_n-x_{n-1}),\ n=1,2,\cdots \quad \ldots (7)$$

This is *regula falsi iteration formula*. The process is repeated until the root is obtained to required degree of accuracy.

B. Convergence of regula falsi method.

Let α be a simple root of the equation $f(x)=0$. Then putting $x_n=\alpha+\varepsilon_n$ in (7), we get

$$\varepsilon_{n+1}=\varepsilon_n-\frac{(\varepsilon_n-\varepsilon_{n-1})f(\alpha+\varepsilon_n)}{f(\alpha+\varepsilon_n)-f(\alpha+\varepsilon_{n-1})}$$

$$= \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1})\left[\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots\right]}{\left[\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots\right] - \left[\varepsilon_{n-1} f'(\alpha) + \frac{1}{2}\varepsilon_{n-1}^2 f''(\alpha) + \cdots\right]}$$

(Expanding f in Taylor's series and noting $f(\alpha) = 0$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \frac{1}{2}(\varepsilon_n + \varepsilon_{n-1}) f''(\alpha) + \cdots}$$

$$= \varepsilon_n - \varepsilon_n \left[1 + \frac{1}{2}\varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right] \times \left[1 + \frac{1}{2}(\varepsilon_n + \varepsilon_{n-1}) \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \varepsilon_{n-1}\varepsilon_n + 0\left(\varepsilon_n^2 \varepsilon_{n-1} + \varepsilon_n \varepsilon_{n-1}^2\right), \quad \ldots (8)$$

so that

$$\varepsilon_{n+1} = C\, \varepsilon_{n-1}\varepsilon_n \quad \ldots (9)$$

where $C = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$ and we have neglect higher power of ε_n

The relation (9) is called the error equation.

To find the order of convergence, we set $\varepsilon_{n+1} = A\varepsilon_n^m$ and $\varepsilon_n = A\varepsilon_{n-1}^m$ where the constants A and m are to be determined. Then the equation (9) gives

$$A\varepsilon_n^m = C \cdot A^{-\frac{1}{m}} \varepsilon_n^{\frac{1}{m}} \varepsilon_n = CA^{-\frac{1}{m}} \varepsilon_n^{1+\frac{1}{m}} \quad \ldots (10)$$

$$\Rightarrow m = 1 + \frac{1}{m}$$

$$\Rightarrow m = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Neglecting the minus sign, we find that the order of the convergence of $\{x_n\}$ is $m = 1 \cdot 618$. Also from (10) we get

$$A = C^{m/m+1}$$

C. Advantage and disadvantage of regula falsi method. *[W.B.U.T., CS-312,2003]*

Advantage. The method is very simple and does no required to calculate the derivative of $f(x)$ which is difficult for some problems. Moreover, the method is evidently convergent.

Disadvantage. Sometimes the method is very slow and not suitable for practical computation. Also the initial interval in which the root lies is to be chosen very small.

Example.2. Find a root of the equation $x^3 - 2x - 5 = 0$ by Regula-Falsi method correct upto 4 places of decimal *[W.B.U.T. CS-312,2004]*

Solultion. Let $f(x) = x^3 - 2x - 5$.

$\therefore f(0) = -5, f(1) = -6, f(2) = -1, f(3) = 16$. So a real root lies between 2 and 3. We choose $x_0 = 2, x_1 = 3$ giving $f(x_0) = -1$ and $f(x_1) = 16$. Then the iteration (7) gives

$$x_2 = 3 - \frac{16}{16 - (-1)}(3 - 2) = 2 \cdot 05882 \text{ and } f(x_2) = -0 \cdot 39082$$

Proceeding in this way, the iteration (7) gives the following table :

No, of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)<0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	−1	16	2.05882	−0.39084
2	2.05882	3	−0.39084	16	2.08126	−0.147244
3	2.08126	3	−0.147244	16	2.08964	−0.054667
4	2.08964	3	−0.054667	16	2.09274	−0.020198
5	2.09274	3	−0.020198	16	2.09388	−0.007491
6	2.09388	3	−0.007491	16	2.09430	−0.002806
7	2.09430	3	−0.002806	16	2.09445	−0.001133
8	2.09445	3	−0.001133	16	2.094451	

Hence 2.0944 is a root of the given equation correct upto four decimal places

6.5. Newton-Raphson method.

A. *Basic principle and formula* [W.B.U.T., CS-312,2002]

Let x_0 be an initial approximation of the desired root α of the equation $f(x)=0$ and $x_1 = x_0 + h$ is the correct root

$$\therefore f(x_1)=0$$

$$\text{i.e., } f(x_0+h)=0, (\min(x_1,x_0)<h<\max(x_1,x_0))$$

$$\text{i.e., } f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\dots=0,$$

by Taylor's series, expansion

Neglecting the second and higher order terms, we obtain

$$f(x_0)+hf'(x_0)=0,$$

$$\text{i.e., } h=-\frac{f(x_0)}{f'(x_0)}$$

Thus a better approximation of the root α is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \quad \dots (11)$$

Repeating the above process and replacing x_0 by x_1 we obtain the second approximation of the root as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Proceeding in this way, we get the successive approximations $x_3, x_4, \dots, x_{n+1}$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \dots (12)$$

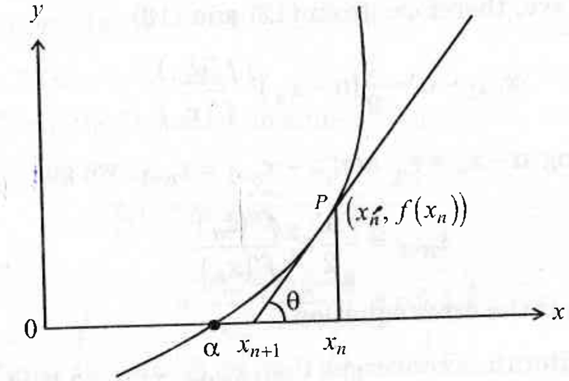
provided $f'(x_n)\neq 0, n=1,2,\dots$

The result (11) is known as *Newton-Raphson iteration formula.*

If $[a,b]$ be the initial interval in which the root α of the given equation $f(x)=0$ lies and $f'(x)\neq 0$, then the initial approximation may be started with $x_0=a$ or b.

B. Geometrical meaning of Newton-Raphson formula

Let the curve $y=f(x)$ cuts the x–axis at the point $x=\alpha$ so that α is a root of the equation $f(x)=0$.



If the tangent to the curve at the point $P(x_n, f(x_n))$ cuts the x–axis at the point $x=x_{n+1}$ and is inclined at an angle θ with the positive direction of the x–axis, then

$$f'(x_n)=\tan\theta=\frac{f(x_n)}{x_n-x_{n+1}}$$

so that
$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

Accordingly, the Newton-Raphson method may also be called the *method of tangents*.

C. *Convergence of Newton–Raphson method.* [W.B.U.T., CS-312,2003,2007]

Let α be a root of the equation $f(x)=0$

$$\therefore f(\alpha)=0$$

$$\text{i.e., } f(x_n+\alpha-x_n)=0$$

$$\text{i.e., } f(x_n)+(\alpha-x_n)f'(x_n)+\frac{(\alpha-x_n)^2}{2}f''(\xi_n)=0$$

$[\min|\alpha,x_n|<\xi_n<\max|\alpha,x_n|]$, by Taylor's theorem

$$\therefore -\frac{f(x_n)}{f'(x_n)} = (\alpha - x_n) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)} \qquad \dots \quad (13)$$

We have, therefore, from (12) and (13),

$$x_{n+1} - \alpha = \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}$$

Putting $\alpha - x_n = \varepsilon_n$ and $\alpha - x_{n+1} = \varepsilon_{n+1}$, we get

$$\varepsilon_{n+1} = -\frac{1}{2}\varepsilon_n^2 \frac{f''(\xi_n)}{f'(x_n)} \qquad \dots \quad (14)$$

which is the error equation.

If the iteration converges then $x_n, \xi_n \to \alpha$ as $n \to \infty$ so that

$$\lim_{n\to\infty}\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^2}\right| = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \qquad \dots \quad (15)$$

Hence Newton–Raphson method is a second order iteration process. So the convergence is quadratic and the constant asymptotic, error is equal to $\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$

D. *Advantage and disadvantage of Newton–Raphson method.*

Advantage. The rate of convergence of this method is quadratic. So the method converges more rapidly than other numerical method.

Disadvantage. In this method, the initial approximation must be chosen very close to the root; otherwise, the method will fail. Since the method depends on the derivative $f'(x)$, it may not be suitable for a function $f(x)$ whose derivative is difficult to compute. Also the method fails if $f'(x) = 0$ or small in the neighbourhood of the root.

Note. When $f'(x)$ is large in the neighbourhood of the real root, i.e., when the graph of function $y = f(x)$ is nearly vertical, then it crosses the x-axis. In this case, the method is very useful and the correct value of the root can be obtained more rapidly.

Example.3. Find the smallest positive root of the equation $3x^3 - 9x^2 + 8 = 0$ correct to four places decimal, using Newton–Raphson method. *[W.B.U.T., CS-312,2009]*

Solution. Let $f(x) = 3x^3 - 9x^2 + 8$

$\therefore f'(x) = 9x^2 - 18x$.

Then the iteration formula (12) gives

$$x_{n+1} = x_n - \frac{3x_n^3 - 9x_n^2 + 8}{9x_n^2 - 18x_n}$$

$$= \frac{6x_n^3 - 9x_n^2 - 8}{9x_n^2 - 18x_n}, n = 0, 1, 2, \dots \qquad \dots \quad (1)$$

Now $f(0) = 8, f(1) = 2, f(2) = -4$. So a positive root lies between 1 and 2. Choose $x_\circ = 1$

$\therefore$ From (1)

$$x_1 = \frac{6x_\circ^3 - 9x_\circ^2 - 8}{9x_\circ^2 - 18x_\circ} = \frac{6\times1^3 - 9\times1^2 - 8}{9\times1^2 - 18\times1} = 1\cdot22222$$

$$x_2 = \frac{6(1\cdot22222)^3 - 9(1\cdot22222)^2 - 8}{9(1\cdot22222)^2 - 18\times1\cdot222222} = 1\cdot22607$$

$$x_3 = \frac{6(1\cdot22607)^3 - 9(1\cdot22607)^2 - 8}{9(1\cdot22607)^2 - 18\times1\cdot22607} = 1\cdot22607$$

Hence positive real root of the given equation correct to four decimal places is 1.2261.

E. *Newton-Raphson method for finding an assigned root of a positive real number.* *[W.B.U.T., CS-312,2009]*

Suppose we are to find the m^{th} root of a real number R.

So let $x = \sqrt[m]{R}$

$\therefore x^m = R$

i.e., $x^m - R = 0$

Let $f(x) = x^m - R$

$\therefore f'(x) = mx^{m-1}$

The Newton–Raphson iteration formula (13) gives

$$x_{n+1} = x_n - \frac{x_n^m - R}{mx_n^{m-1}}$$

$$= \frac{(m-1)x_n^m + R}{mx_n^{m-1}}, n = 0, 1, 2, \cdots \quad \ldots \quad (16)$$

with $|x_{n+1} - x_n| < \varepsilon$, ε being the desired degree of accuracy.

It is obvious from (17) that for finding the square root of any positive number R (where $m = 2$), we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right), \ (n = 0,1,2,\cdots)$$

Example.4. Find the cube root of 10 upto 5 significant figures by Newton–Raphson method.

Solution. Let $x = \sqrt[3]{10}$

$\therefore x^3 - 10 = 0$

Let $f(x) = x^3 - 10$.

$\therefore f'(x) = 3x^2$

Now $f(0) = -10, f(1) = -9, f(2) = -2, f(3) = 17$ so that a real root of $f(x) = 0$ lies between 2 and 3. Hence using the iteration formula (16) with $m = 3$, we have

$$x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}, \ n = 0,1,2,\cdots$$

which gives with $x_0 = 2$

$$x_1 = \frac{2x_0^3 + 10}{3x_0^2} = 2.16666$$

Similarly, $x_2 = \dfrac{2(2.16666)^3 + 10}{3(2.16666)^2} = 2.15450$.

$x_3 = 2.15443$,

$x_4 = 2.15443$

Hence we have $x = \sqrt[3]{10} \simeq 2.1544$ correct to five significant figures.

ILLUSTRATIVE EXAMPLES

Ex 1. Find a real root of the transcendental equation $x^x + 2x - 2 = 0$, correct upto two decimal places using bisection method.

Solution. Let $f(x) = x^x + 2x - 2$.

Then $f(0\cdot5) = -0\cdot293, f(1) = 1$.

So a real root lies between 0.5 and 1.

Take $a_0 = 0\cdot5, b_0 = 1$ so that $x_1 = \dfrac{0\cdot5 + 1}{2} = 0\cdot75$.

Again, since $f(0\cdot75) = 0\cdot3059$, so the root lies between 0.75 and 1. Proceeding in this way, we construct the following table :

No. of iteration(n)	a_n ($f(a_n) < 0$)	b_n ($f(b_n) > 0$)	$x_{n+1} = \frac{1}{2}(a_n + b_n)$	$f(x_{n+1})$
0	0.5	1	0.75	0.3059
1	0.5	0.75	0.625	–0.0045
2	0.625	0.75	0.687	0.1466
3	0.625	0.687	0.656	0.0704
4	0.625	0.656	0.640	0.0325
5	0.625	0.640	0.632	0.0041
6	0.625	0.632	0.628	0.0036
7	0.625	0.628	0.626	

Then the root correct upto two decimal places is 0.63.

Ex 2. Find the smallest positive root of the equation $e^x = 4\sin x$ correct to four decimal places by bisection method.

Solution. The given equation is

$$4\sin x - e^x = 0 \cdot$$

Let $f(x) = 4\sin x - e^x$.

Since $f(0) = -1$, $f(1) = 0.64760$, so the smallest positive root lies between 0 and 1. Take $a_0 = 0, b_0 = 1$ so that

$$x_1 = \frac{1}{2}(a_0 + b_0) = 0.5.$$

Since $f(0.5) = 0.26898$, so the root lies between 0 and 0.5. Proceeding in this way, we construct the following table: (shown in the next page)

No. of iteration (n)	a_n ($f(a_n)<0$)	b_n ($f(b_n)>0$)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0	1	0.5	0.26898
1	0	0.5	0.25	–0.29440
2	0.25	0.5	0.375	0.01010
3	0.25	0.375	0.33333	–0.08690
4	0.33333	0.375	0.35416	–0.03770
5	0.35416	0.375	0.36458	–0.01360
6	0.36458	0.375	0.36979	–0.00168
7	0.36979	0.375	0.37240	0.00420
8	0.36979	0.37240	0.37109	0.00126
9	0.36979	0.37109	0.37044	–0.00240
10	0.37044	0.37109	0.37077	0.00047
11	0.37044	0.37077	0.37060	0.00011
12	0.37044	0.37060	0.37052	–0.00007
13	0.37052	0.37060	0.37056	

Thus the required smallest positive root is 0.3706 correct to four decimal places.

Ex. 3. Find a root of the equation $x\sin x + \cos x = 0$ using Newton-Raphson method correct upto 5 places of decimal. *[W.B.U.T.,CS-312, 2004]*

Solution. Let $f(x) = x\sin x + \cos x$

$\therefore f'(x) = x\cos x + \sin x - \sin x = x\cos x$

So the Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, ...$$

gives

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

$$= x_n - \frac{1}{x_n} - \tan x_n \quad \dots (1)$$

Now $f(0) = 1, f(1) = 1.38, f(2) = 1.4, f(3) = -0.56$

So a root lies between 2 and 3.

Choose $x_0 = 2.5$, for quick convergence.

Then we have, from (1),

$$x_1 = 2.5 - \frac{1}{2.5} - \tan 2.5$$

$$= 2.847022$$

$$x_2 = 2.847022 - \frac{1}{2.847022} - \tan 2.847022$$

$$= 2.799175$$

Similarly $x_3 = 2.798386$

$$x_4 = 2.798386$$

Hence a positive real root of the given equation correct to five decimal places is 2.79839.

Ex.4. Find out the root of the following equation using Regula falsi method $x^3 - 5x - 7 = 0$ that lies between 2 and 3, correct to 4 decimal places. *[W.B.U.T.CS-312, 2006, 2009]*

Solution. Let $f(x) = x^3 - 5x - 7$

We choose $x_0 = 2, x_1 = 3$ so that $f(x_0) = -9, f(x_1) = 5$. Then the regula-falsi iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}), n = 1, 2, \ldots \quad \ldots (1)$$

gives

$$x_2 = 3 - \frac{5}{5-(-9)}(3-2)$$

$$= 2.642857$$

Proceeding in this way, the iteration formula (1) gives the following table :

No of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)<0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	–9	5	2.642857	–1.754740
2	2.642857	3	–1.754740	5	2.735635	–0.205506
3	2.735635	3	–0.205506	5	2.746072	–0.022474
4	2.746072	3	–0.022474	5	2.747208	–0.002444
5	2.747208	3	–0.002444	5	2.747332	–0.000257
6	2.747332	3	–0.000257	5	2.747345	–0.000027

Hence a real root of the given equation correct to four decimal places is 2.7473.

Ex. 5. Find the root of the equation $xe^x - 3 = 0$ that lies between 1 and 2, correct to 4 significant figure using the method of False position.

Solution. Let $f(x) = xe^x - 3$

Here we choose $x_0 = 1, x_1 = 2$ so that $f(x_0) = -0.2817$,

$f(x_1) = 11.7781$

So the iteration formula for false position method

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}), \quad n = 1, 2, \ldots.$$

gives

$$x_2 = 2 - \frac{11.7781}{11.7781 - (-0.2817)}(2-1)$$

$$= 1.02336$$

Proceeding in this way, we obtain the following table :

No of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)>0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	2	–0.2817	11.7781	1.02336	–0.15247
2	1.02336	2	–0.15247	11.7781	1.03584	–0.08155
3	1.03584	2	–0.08155	11.7781	1.04247	–0.04333
4	1.04247	2	–0.04333	11.7781	1.04598	–0.02295
5	1.04598	2	–0.02295	11.7781	1.04802	–0.01105
6	1.04802	2	–0.01105	11.7781	1.04891	

Thus the required root correct to four significant figure is 1.049.

Ex.6. Find the root of the equation $e^x = 2x + 1$ correct to 4 places of decimal, using Newton-Raphson method near $x = 1$.

[*W.B.U.T., CS-312, 2005*]

Solution. The given equation is

$$e^x = 2x + 1$$

i.e., $e^x - 2x - 1 = 0$

Let $f(x) = e^x - 2x - 1$

$\therefore \; f'(x) = e^x - 2$

So the Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, \ldots$$

gives

$$x_{n+1} = x_n - \frac{e^{x_n} - 2x_n - 1}{e^{x_n} - 2}$$

$$= \frac{x_n e^{x_n} - e^{x_n} + 1}{e^{x_n} - 2} \quad \ldots \quad (1)$$

For quick convergence, we choose $x_0 = 1.5$

From (1), we get

$$x_1 = \frac{1.5e^{1.5} - e^{1.5} + 1}{e^{1.5} - 2} = 1.305903$$

$$x_2 = \frac{1.305903e^{1.305903} - e^{1.305903}}{e^{1.305903} - 2}$$

$$= 1.259059$$

Similarly $x_3 = 1.25906$

Hence the root of the given equation near $x = 1$ is 1.25906.

Ex. 7. **Using Newton - Raphson method, find the value of $\sqrt[4]{12}$.**

Solution. Let $x = \sqrt[4]{12}$

$\therefore\ x^4 = 12$

So let $f(x) = x^4 - 12$

$\therefore\ f'(x) = 4x^3$

Then the Newton - Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2,\cdots$$

gives

$$x_{n+1} = x_n - \frac{x_n^4 - 12}{4x_n^3}$$

$$= \frac{3x_n^4 + 12}{4x_n^3},\ n = 0,1,2,\cdots$$

Now $f(0) = -12 < 0, f(1) = -11 < 0, f(2) = 4 > 0$

So a real root of $f(x) = 0$ lies between 1 and 2.

Taking $x_0 = 1.5$ for rapid convergence, we get

$$x_1 = \frac{3x_0^4 + 12}{4x_0^3}$$

$$= \frac{3(1\cdot5)^4 + 12}{4(1\cdot5)^3}$$

$$= 2.0138$$

$$x_2 = \frac{3(2\cdot0138)^4 + 12}{4(2\cdot0138)^3}$$

$$= 1.8777$$

Similarly, $x_3 = 1.8614,\ x_4 = 1.8612$.

Hence $\sqrt[4]{12} \simeq 1.861$, correct to four significant figures.

Ex. 8. **Using Newton - Raphson method, obtain iteration formula for the reciprocal of a number N and hence find the value of $\frac{1}{22}$, correct to three significant figures.**

Solution. Let $\frac{1}{N} = x$

Then $\frac{1}{x} - N = 0$.

Let $f(x) = \frac{1}{x} - N$

$\therefore\ f'(x) = -\frac{1}{x^2}$

So the Newton - Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2,\cdots$$

gives

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n(1 - Nx_n)$$

$$= (2 - Nx_n)x_n, \quad n = 0, 1, 2, \cdots \qquad \ldots (1)$$

which is the required iterative formula.

As the value of $\frac{1}{22}$ lies between $\frac{1}{25}$ and $\frac{1}{20}$.

i.e. between 0.04 and 0.05, so we take $N = 22, x_0 = 0.04$.

Then, from (1),we have

$$x_1 = (2 - 22 \times 0.04)0.04 = 0.0448$$

$$\therefore \quad x_2 = (2 - 22 \times 0.0448)0.0448 = 0.04544.$$

$$x_3 = 0.04545$$

$$x_4 = 0.04545.$$

Thus the required value of $\frac{1}{22}$ is 0.0454, correct to three significant figures.

Ex. 9. Evaluate $\sqrt{12}$ to three places of decimal by Newton-Raphson method [W.B.U.T., CS-312, 2007]

Solution. Let $x = \sqrt{12}$

$$\therefore x^2 - 12 = 0$$

Let $f(x) = x^2 - 12$

$\therefore f'(x) = 2x$

So the Newton-Raphson's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

gives

$$x_{n+1} = x_n - \frac{x_n^2 - 12}{2x_n}$$

$$= \frac{x_n^2 + 12}{2x_n}, \quad n = 0, 1, 2, \ldots$$

Now $f(3) = -3$, $f(4) = 4$

So a root lies between 3 and 4.

Choose $x_0 = 3$.

$$\therefore x_1 = \frac{3^2 + 12}{2 \times 3} = 3.5$$

$$\therefore x_2 = \frac{(3.5)^2 + 12}{2 \times 3.5} = 3.46421$$

Similarly, $x_3 = 3.46410$

$x_4 = 3.46410$

$\therefore \sqrt{12} = 3.464$, correct upto 3 places of decimal.

Exercise

I. Short Answer Questions

1. Explain bisection method
2. Show that the convergence of bisection method is linear
3. Discuss the advantage and disadvantage of bisection method
4. Find out an expression of the inherent error in Newton-Raphson method. [W.B.U.T., CS-312, 2003]
5. Show that Newton-Raphson method has quadratic rate of convergence. [W.B.U.T.,CS-312, 2003, 2007]
6. Give a geometrical interpretation of Newton Raphson method
7. Evaluate $\sqrt{12}$ to three places of decimal by Newton-Raphson method. [W.B.U.T., CS-312, 2003]
8. Discuss the advantage and disadvantage of Regula-Falsi method. [W.B.U.T., CS-312, 2003]
9. Find the positive root of $x^2 - \sin x = 0$ correct upto three significant digits by the method of false position.

10. Interpret Regula-Falsi method geometrically

[W.B.U.T., CS-312, 2003]

Answers

7.3.464 9.0.877

II. Long Answer Questions

1. Compute a real root of the following equations by bisection method correct to five significant figures :

(a) $x^3-3x-5=0$

(b) $x^4-x-10=0$

(c) $x^3-9x+1=0$ in [2, 3]

(d) $\cos x = xe^x$

(e) $\tan x + x = 0$

(f) $3x-\cos x-1=0$ in [0,1]

2. Using the method of bisection to compute a root of $x^3-4x-1=0$ between 2 and 3 upto four significant digits.

3. Find the positive real root of $x^3-x^2-1=0$ using the bisection method of 4 iterations. *[W.B.U.T., CS-312, 2010]*

4. Using the method of bisection to compute a root of $x^3-x-1=0$ correct upto two significant digits.

[W.B.U.T.,CS-312, 2000]

5. Use Regula-Falsi method to evaluate the smallest real root of each of the following equations

(a) $x^3+x^2-1=0$ (b) $x^3-4x+1=0$

(c) $2x^3-3x-6=0$ (d) $xe^x=\cos x$

(e) x^3-3x-5 *[W.B.U.T.,CS-312, 2003]*

6. Find out the root of the following equation using Regula Falsi method:

$3x-\cos x-1=0$ that lies between 0 and 1 correct to four decimal places) *[W.B.U.T., CS-312, 2007,2008]*

7. Using Newton-Raphson method, find a real root of the following equations correct to three decimal places

(i) $x^4-x-1=0$ (ii) $3x^2+2x-9=0$

(iii) $2x-3\sin x-5=0$ (iv) $x+e^x=0$

8. Construct a iterative formula to evaluate the following using Newton-Raphson method and hence evaluate

(i) $\sqrt[9]{15}$ (ii) $\sqrt[7]{125}$ (iii) $\sqrt[3]{21}$ (iv) $\sqrt[4]{13}$

9. Prove that Newton-Raphson's iteration formula for $\sqrt{N}$ is

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right).$$ Hence find $\sqrt{21}$.

10. Find the Newton-Raphson iteration formula to find the p^{th} root of positive number N and hence find the cube root of 17.

[W.B.U.T.,CS-312, 2009]

11. Prove that Newton-Raphson's iteration formula for $\frac{1}{N}$ is $x_{n+1}=x_n(2-Nx_n)$. Hence find the value of $\frac{1}{13}$ correct upto two significant figures.

12. Find the root of $e^x-x=0$, near $x=0$ correct to three significant figures. *[W.B.U.T.,CS.312, 2002]*

Answers

1. (a) 2.278 (b) 1.855 (c) 2.953 (d)0.5177
(e) 2.0289 (f)0.6071

2. 2.687 3.1.467 4.1.3

5. (a) 0.7548 (b) 0.254 (iii) 0.732 (d) 0.5177

6. 0.6071 7. (i) 1.221 (ii) 1.430 (iii) 2.883 (iv) –0.567

8. (i) 1.35106 (ii) 1.993 (iii) 2.7589 (iv) 1.8988

10. 2.5113

III. MULTIPLE CHOICE QUESTIONS

1. A transcendental equation may have either no root or one or more

(a) True (b) False

2. Which of the following does not always guarantee the convergence

(a) secant method (b) Bisection method

(c) Regula-Falsi method (d) Newton-Raphson method

[W.B.U.T.,CS-312, 2006]

3. The rate of convergence of Newton-Raphson method is

(a) linear (b) quadratic (c) cubic (d) none of these

4. The order of convergence of the Newton-Raphson's method is

(a) 0 (b) 1 (c) 2 (d) 3

5. The Newton-Raphson's method is fail when

(a) $f'(x)=1$ (b) $f'(x)=0$ (c) $f'(x)=-1$ (d) $f''(x)=0$

6. The condition of convergence of Newton-Raphson method when applied to an equation $f(x)=0$ in an interval is

(a) $f'(x)\neq 0$ (b) $|f'(x)|<1$

(c) $\{f'(x)\}^2>|f(x)f''(x)|$ (d) $\{f''(x)\}^2>|f(x)f'(x)|$

[W.B.U.T.,CS-312, 2007, 2008]

7. The accuracy attainable with Newton-Raphson method does not depend upon the value of the derivative $f'(x)$

(a) True (b) False [W.B.U.T., CS-312, 2006]

8. One of the roots of $x^3-17x+5=0$ lies in between

(a) 1 and 2 (b) 0 and 1 (c) 2 and 3 (d) none of these

9. One of the roots of $x^2+5x-3=0$ lies in between

(a) 1 and 2 (b) 0 and 1

(c) 2 and 3 (d) none of these

10. For the Newton-Raphson's method, the iterative formula is

(a) $x_{n+1}=\dfrac{f(x_n)}{f'(x_n)}$ (b) $x_{n+1}=\dfrac{f'(x_n)}{f(x_n)}$

(c) $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ (d) $x_{n+1}=x_n+\dfrac{f(x_n)}{f'(x_n)}$

11. Newton-Raphson's method is also known as

(a) normal method (b) tangent method

(c) Parallel method (d) none of these

12. The rate of convergence of bisection method is

(a) linear (b) quadratic (c) cubic (d) none of these

[W.B.U.T.,CS-312, 2007]

13. Bisection method used for finding the real root of a transcendental equation is

(a) an analytical method (b) graphical method

(c) iterative method (d) none of these

14. Bisection method is

(a) conditionally and surely convergent

(b) unconditionally and surely convergent

(c) conditionally convergent

(d) none of these

15. The root of a transcendental equation $f(x)=0$ is obtained more rapidly by $N-R$ method when $f'(x)$ in the neighbourhood of the root is

(a) zero (b) small (c) large (d) none of these

16. Newton-Raphson's iterative formula for finding the square root of a positive real number R is

(a) $x_{n+1} = x_n + \dfrac{R}{x_n}$ (b) $x_{n+1} = x_n - \dfrac{R}{x_n}$

(c) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{R}{x_n}\right)$ (d) $x_{n+1} = \dfrac{1}{2}\left(x_n - \dfrac{R}{x_n}\right)$

17. Regula-Falsi method is

(a) Conditionally convergent (b) linearly convergent

(c) divergent (d) none of these

[W.B.U.T.,CS-312, 2009]

18. Regula Falsi method used for finding the real roots of a numerical equation is

(a) an analytical method (b) graphical method

(c) iterative method (d) none of these

[W.B.U.T.,CS-312, 2004, 2006]

19. Newton-Raphson method even does not fail when $f'(x)=0$ in the neighbourhood of the real root

(a) True (b) False

20. If $f(a_0)f(b_0)<0$ and $a_0<b_0$, then the first approximation of one of the roots of $f(x)=0$ by Regula-Falsi method is

(a) $\dfrac{a_0 f(b_0)+b_0 f(a_0)}{f(b_0)+f(a_0)}$ (b) $\dfrac{a_0 f(a_0)-b_0 f(b_0)}{f(a_0)-f(b_0)}$

(c) $\dfrac{a_0 f(b_0)-b_0 f(a_0)}{f(b_0)-f(a_0)}$ (d) $\dfrac{a_0 f(a_0)+b_0 f(b_0)}{f(a_0)+f(b_0)}$

Answers

1.a	2.a	3. b	4.c	5.b	6.c	7. b	8. b,c	9.b	10.c
11.b	12. a	13.c	14.b	15.c	16.c	17.a	18.c	19.b	20.c

7 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

7.1 Introduction:

Differential equation are involved in many problems in Engineering and Science. In this chapter, we discuss various numerical methods for solving ordinary differential equations. Our aim is to study the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y),\ y_0 = y(x_0) \quad \ldots \quad (1)$$

in which $f(x, y)$ is a continuous function of x and y in some domain D of the xy-plane and (x_0, y_0) is a given point in D. The condition $y_0 = f(x_0)$ is known as the initial condition. Sufficient conditions for the existence and uniqueness of the solution of the equation (1) are the well known Lipschitz conditions given by

(i) $f(x, y)$ is defined and continuous in D, the region containing (x_0, y_0)

(ii) there exists a constant L such that

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2| \quad \ldots \quad (2)$$

for all $(x, y_1), (x, y_2) \in D$.

We now proceed to consider numerical techniques for solving (1) at a sequence of points $x_i = x_0 + ih$, called the *mesh points*, h being the step length. Let y_i be the approximation to the exact solution $y(x_i)$ of (1). A continuous approximation to y is then obtained by interpolating the data points (x_i, y_i).

7.2. Euler's method.

We shall now describe a method, known as Euler's method, which gives the solution in the form of a set of tabulated values. In single step method, we determine a function $\phi(x, y; h)$ of x, y and h (the step length) depending on $f(x, y)$ and its derivatives such that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0(h^{p+1}), \quad \ldots \quad (3)$$

where p is a positive integer, called the order of the method.