Envelopes, Support Functions and the Convex Hull

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We define both envelopes and convex hulls in terms of *support functions* and show that the convex hull is a (possibly proper) subset of the envelope. We then make some observations about envelopes.

1. Envelope vs Convex Hull

We start with the definition of support functions on $\mathbb{R}^n$. [1]

Definition 1 (Support Function)**.** *Let S be a set in $\mathbb{R}^n$. We define the support function q_S as*

$$q_S(v) = sup_{s \in S} \langle s, v \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes inner product.

Note that $q_S(v)$ may be ∞ for some v.

In contrast to diagrams we will not distinguish between points and vectors, taking the definition of both to be elements in $\mathbb{R}^n$. This allows us to fix the origin at 0 and to consider two diagrams that are identical except for their origins to be translations of one another.

From the diagrams documentation and Definition 1 we can rewrite the definition of an envelope in terms of its support function q_d.

Definition 2 (Envelope)**.** *The envelope of a diagram d in direction v is*

$$e_d(v) = sup_{u \in d} \frac{\langle u, v \rangle}{\|v\|} \tag{1}$$

$$= sup_{u \in d} \langle u, v / \|v\| \rangle \tag{2}$$

$$= q_d(v / \|v\|) \tag{3}$$

We can create an intensional representation of the envelope by taking the union over all v of the sets

$$\{av : a \leq e_d(v)\} \cap \{bv : b \leq e_d(-v)\}$$

[1] The definition of a support function can be generalized to arbitrary vector spaces. Let S be any set in a vector space X. We define the support function q_S on the dual X' of X as $q_S(\ell) = sup_{s \in S} \ell(s)$ for any linear function ℓ. (see *Linear Algebra*, by Peter Lax, 1997 or just about any text book on linear functional analysis).

Therefore if we can represent the convex hull as the union over all v of the sets

$$\{av : a \leq c_d(v)\} \cap \{bv : b \leq c_d(-v)\}$$

where $c_d(v) \leq e_d(v)$, then we will have shown the convex hull of d is a subset of its envelope.

To define the convex hull in terms of the support function we use the following theorem. [2]

Theorem 1 (Convex Hull)**.** *The closed convex hull of any set S is the set of points x satisfying $\langle u, x\rangle \leq q_S(u)$ for all u in $\mathbb{R}^n$.*

Theorem 2. *The convex hull of a diagram is a subset of its envelope*

Proof: We consider all points in the convex hull of diagram d that lie on some vector v, these points have the form $c_d(v)v$ for some scalar $c_d(v)$. By Theorem 1, for all u, these points must satisfy

$$\langle u, c_d(v)v\rangle \leq q_d(u) \tag{4}$$

$$c_d(v)\langle u, v\rangle \leq q_d(u) \tag{5}$$

$$c_d(v) \leq \frac{q_d(u)}{\langle u, v\rangle} \tag{6}$$

Since the last inequality hold for all u we have,

$$c_d(v) \leq \inf_u \frac{q_d(u)}{\langle u, v\rangle} \tag{7}$$

$$\leq \frac{q_d(v)}{\langle v, v\rangle} \tag{8}$$

$$= q_d(v/\|v\|) = e_d(v) \tag{9}$$

and hence $c_d(v) \leq e_d(v)$ which is suffiicient to prove the Theorem as explained above. □

Finally, in order to show that the convex hull may be a proper subset of the envelope it is enough to find some diagram d and vector v such that $c_d(v) < e_d(v)$. We take d equal to the horizontal line from $(0, 0)$ to $(1, 0)$ in $\mathbb{R}^2$ and $v = (1, 1)$. Then

$$e_d((1, 1)) = sup_{x\in[0,1]} \frac{\langle (x, 0), (1, 1)\rangle}{\sqrt{2}} \tag{10}$$

$$= sup_{x\in[0,1]} \frac{x}{\sqrt{2}} \tag{11}$$

$$= \frac{1}{\sqrt{2}} \tag{12}$$

The convex hull of d is d since a line segment is convex, and the only point in d lying on $(1, 1)$ is $(0, 0)$, hence $c_d((1, 1)) = 0$ which is less than $1/\sqrt{2}$, thus $c_d((1, 1)) < e_d((1, 1))$.

[2] *Linear Algebra*, by Peter Lax, 1997, page 162

2. Observations

Almost all diagrams have an envelope which is a proper subset of its convex hull, Figures 1,2 and 5. In some cases the bounding box is a better approximation to the convex hull than the envelope, a square for example.

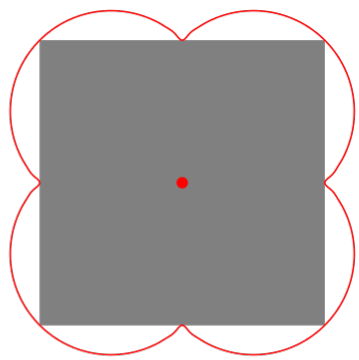

Figure 1. Envelope of unit square centered at $(0, 0)$

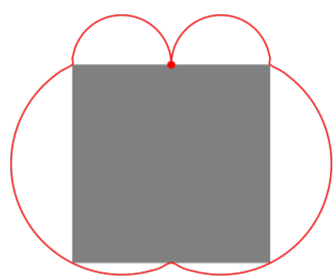

Figure 2. Envelope of unit square centered at $(0, -\frac{1}{2})$

The envelope of a diagram is highly dependent on its origin, Figures 3 and 4. As the origin moves farther away from the center of the figure, the envelope becomes a worse approximation to the convex hull.

Like the convex hull the shape of the bounding box is independent of the placement of the origin. This is not true of envelopes.

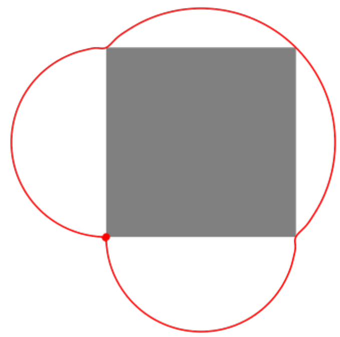

FIGURE 3. Envelope of unit square centered at $(\frac{1}{2}, \frac{1}{2})$

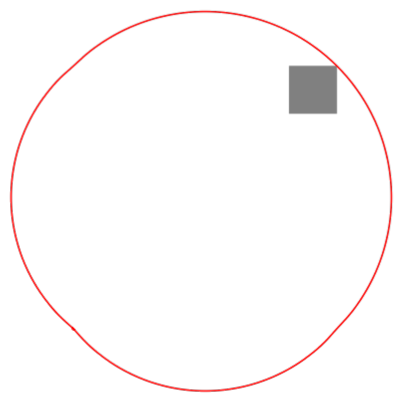

FIGURE 4. Envelope of unit square centered at $(5, 5)$

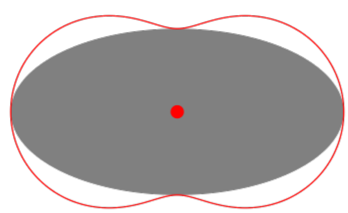

FIGURE 5. Envelope of an ellipse