Envelopes, Support Functions and the Convex Hull

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We define both envelopes and convex hulls in terms of *support functions* and show that the convex hull is a (possibly proper) subset of the envelope. We start with the definition of support functions on $\mathbb{R}^n$. [1]

Definition 1 (Support Function)**.** *Let S be a set in $\mathbb{R}^n$. We define the support function q_S as*

$$q_S(v) = sup_{s \in S}\langle s, v\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes inner product.

Note that $q_S(v)$ may be ∞ for some v.

In contrast to diagrams we will not distinguish between points and vectors, taking the definition of both to be elements in $\mathbb{R}^n$. This allows us to fix the origin at 0 and to consider two diagrams that are identical except for their origins to be translations of one another.

From the diagrams documentation and Definition 1 we can rewrite the definition of an envelope in terms of its support function q_d.

Definition 2 (Envelope)**.** *The envelope of a diagram d in direction v is*

$$\begin{align}
e_d(v) &= sup_{u \in d} \frac{\langle u, v\rangle}{\|v\|} \tag{1}\\
&= sup_{u \in d}\langle u, v/\|v\|\rangle \tag{2}\\
&= q_d(v/\|v\|) \tag{3}
\end{align}$$

We can create an intensional representation of the envelope by taking the union over all v of the sets

$$\{av : a \leq e_d(v)\} \cap \{bv : b \leq e_d(-v)\}$$

Therefore if we can represent the convex hull as the union over all v of the sets

$$\{av : a \leq c_d(v)\} \cap \{bv : b \leq c_d(-v)\}$$

where $c_d(v) \leq e_d(v)$, then we will have shown the convex hull of d is a subset of its envelope.

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[1] The definition of a support function can be generalized to arbitrary vector spaces. Let S be any set in a vector space X. We define the support function q_S on the dual X' of X as $q_S(\ell) = sup_{s \in S}\ell(s)$ for any linear function ℓ. (see *Linear Algebra*, by Peter Lax, 1997 or just about any text book on linear functional analysis).

To define the convex hull in terms of the support function we use the following theorem. [2]

Theorem 1 (Convex Hull)**.** *The closed convex hull of any set S is the set of points x satisfying $\langle u, x\rangle \leq q_S(u)$ for all u in $\mathbb{R}^n$.*

Theorem 2. *The convex hull of a diagram is a subset of its envelope*

PROOF: We consider all points in the convex hull of diagram d that lie on some vector v, these points have the form $c_d(v)v$ for some scalar $c_d(v)$. By Theorem 1, for all u, these points must satisfy

$$\langle u, c_d(v)v\rangle \leq q_d(u) \tag{4}$$

$$c_d(v)\langle u, v\rangle \leq q_d(u) \tag{5}$$

$$c_d(v) \leq \frac{q_d(u)}{\langle u, v\rangle} \tag{6}$$

Since the last inequality hold for all u we have,

$$c_d(v) \leq \inf_u \frac{q_d(u)}{\langle u, v\rangle} \tag{7}$$

$$\leq \frac{q_d(v)}{\langle v, v\rangle} \tag{8}$$

$$= q_d(v/\|v\|) = e_d(v) \tag{9}$$

and hence $c_d(v) \leq e_d(v)$ which is suffiicient to prove the Theorem as explained above. $\square$

Finally, in order to show that the convex hull may be a proper subset of the envelope it is enough to find some diagram d and vector v such that $c_d(v) < e_d(v)$. We take d equal to the horizontal line from $(0, 0)$ to $(1, 0)$ in $\mathbb{R}^2$ and $v = (1, 1)$. Then

$$e_d((1, 1)) = sup_{x\in[0,1]} \frac{\langle (x, 0), (1, 1)\rangle}{\sqrt{2}} \tag{10}$$

$$= sup_{x\in[0,1]} \frac{x}{\sqrt{2}} \tag{11}$$

$$= \frac{1}{\sqrt{2}} \tag{12}$$

The convex hull of d is d since a line segment is convex, and the only point in d lying on $(1, 1)$ is $(0, 0)$, hence $c_d((1, 1)) = 0$ which is less than $1/\sqrt{2}$, thus $c_d((1, 1)) < e_d((1, 1))$.

[2] *Linear Algebra*, by Peter Lax, 1997, page 162