

# Envelopes, Support Functions and the Convex Hull

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We define both envelopes and convex hulls in terms of *support functions* and show that the convex hull is a (possibly proper) subset of the envelope. It turns out that our notion of envelope is intimately related to the diagram's support function as we will make precise below. In fact if we restrict ourselves to normalized vectors, there is no difference between the two concepts. Finally we make some observations about envelopes.

## 1. ENVELOPE VS CONVEX HULL

We start with the definition of support functions on  $\mathbb{R}^n$ .<sup>1</sup>

**Definition 1** (Support Function). *Let  $S$  be a set in  $\mathbb{R}^n$ . We define the support function  $q_S$  as*

$$q_S(v) = \sup_{s \in S} \langle s, v \rangle$$

where  $\langle \cdot, \cdot \rangle$  denotes inner product.

Note that  $q_S(v)$  may be  $\infty$  for some  $v$ . The following theorem and example will help us gain some intuition about support functions.<sup>2</sup>

**Theorem 1.** *Let  $S$  be a nonempty compact convex set and let  $q$  be its support function. If  $v$  is any fixed point other than the origin, then the following hold:*

- i. There exists a point  $x_v$  in  $S$  such that  $q(v) = \langle x_v, v \rangle$ .*
- ii. The hyperplane  $H \equiv \{x : \langle x, v \rangle = q(v)\}$  supports  $S$  at  $x_v$ .*
- iii. The signed distance from  $H$  to the origin is  $q(v)/\|v\|$ .*

**EXAMPLE.** Let  $d$  be the diamond in  $\mathbb{R}^2$  with vertices  $(0, \pm 1)$  and  $(\pm 1, 0)$  (see Figure 1). To calculate  $q(v)$  for  $v = (2, 1)$  we need to calculate  $\sup_{s \in d} \langle s, v \rangle$ . This maximum is easily seen to be attained at the point  $(1, 0)$  and hence  $q(v) = \langle (1, 0), (2, 1) \rangle = 2$ . In general we have  $q((x, y)) = \max(|x|, |y|)$ .

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<sup>1</sup>The definition of a support function can be generalized to arbitrary normed linear spaces over the reals. Let  $S$  be any set in a vector space  $X$ . We define the support function  $q_S$  on the dual  $X'$  of  $X$  as  $q_S(\ell) = \sup_{s \in S} \ell(s)$  for any linear function  $\ell$ . (see *Functional Analysis*, by Peter Lax, 2002, p. 84).

<sup>2</sup>The proof of the theorem can be found in *Convex Sets and Their Applications* by Steven R Lay, 1982. pp. 206-209.

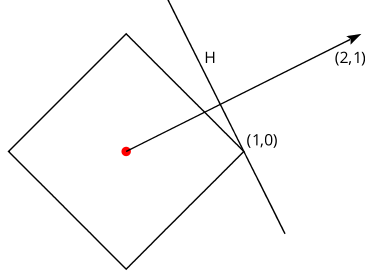


FIGURE 1. diamond

In contrast to diagrams we will not distinguish between points and vectors, taking the definition of both to be elements in  $\mathbb{R}^n$ . This allows us to fix the origin at 0 and to consider two diagrams that are identical except for their origins to be translations of one another.

From the diagrams documentation and Definition 1 we can rewrite the definition of an envelope in terms of its support function  $q_d$ .

**Definition 2** (Envelope). *The envelope of a diagram  $d$  in direction  $v$  is*

$$\begin{aligned}
 (1) \quad e_d(v) &= \sup_{u \in d} \frac{\langle u, v \rangle}{\|v\|^2} \\
 (2) \quad &= \sup_{u \in d} \langle u, v / \|v\|^2 \rangle \\
 (3) \quad &= q_d(v / \|v\|^2)
 \end{aligned}$$

We can create an intensional representation of the envelope by taking the union over all  $v$  of the sets

$$\{av : a \leq e_d(v)\} \cap \{bv : b \leq e_d(-v)\}$$

2 Therefore if we can represent the convex hull as the union over all  $v$  of the sets

$$\{av : a \leq c_d(v)\} \cap \{bv : b \leq c_d(-v)\}$$

where  $c_d(v) \leq e_d(v)$ , then we will have shown the convex hull of  $d$  is a subset of its envelope.

To define the convex hull in terms of the support function we use the following theorem. <sup>3</sup>

**Theorem 2** (Convex Hull). *The closed convex hull of any set  $S$  is the set of points  $x$  satisfying  $\langle u, x \rangle \leq q_S(u)$  for all  $u$  in  $\mathbb{R}^n$ .*

**Theorem 3.** *The convex hull of a diagram is a subset of its envelope*

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<sup>3</sup>*Linear Algebra*, by Peter Lax, 1997, page 162. This theorem can also be generalized to arbitrary normed linear spaces over  $\mathbb{R}$ . It is interesting to note that the proof of the general case requires Zorn's lemma - which is equivalent to the axiom of choice.

PROOF: We consider all points in the convex hull of diagram  $d$  that lie on some vector  $v$ , these points have the form  $c_d(v)v$  for some scalar  $c_d(v)$ . By Theorem 2, for all  $u$ , these points must satisfy

$$(4) \quad \langle u, c_d(v)v \rangle \leq q_d(u)$$

$$(5) \quad c_d(v)\langle u, v \rangle \leq q_d(u)$$

$$(6) \quad c_d(v) \leq \frac{q_d(u)}{\langle u, v \rangle}$$

Since the last inequality holds for all  $u$  we have,

$$(7) \quad c_d(v) = \inf_{u \in \mathbb{R}^n} \frac{q_d(u)}{\langle u, v \rangle}$$

$$(8) \quad \leq \frac{q_d(v)}{\langle v, v \rangle}$$

$$(9) \quad = q_d(v/\|v\|^2) = e_d(v)$$

and hence  $c_d(v) \leq e_d(v)$  which is sufficient to prove the Theorem as explained above.  $\square$

Finally, in order to show that the convex hull may be a proper subset of the envelope it is enough to find some diagram  $d$  and vector  $v$  such that  $c_d(v) < e_d(v)$ . We take  $d$  equal to the horizontal line from  $(0, 0)$  to  $(1, 0)$  in  $\mathbb{R}^2$  and  $v = (1, 1)$ . Then

$$(10) \quad e_d((1, 1)) = \sup_{x \in [0, 1]} \frac{\langle (x, 0), (1, 1) \rangle}{2}$$

$$(11) \quad = \sup_{x \in [0, 1]} \frac{x}{2}$$

$$(12) \quad = \frac{1}{2}$$

The convex hull of  $d$  is  $d$  since a line segment is convex, and the only point in  $d$  lying on  $(1, 1)$  is  $(0, 0)$ , hence  $c_d((1, 1)) = 0$  which is less than  $1/2$ , thus  $c_d((1, 1)) < e_d((1, 1))$ .

## 2. OBSERVATIONS

Almost all diagrams have an envelope which is a proper subset of its convex hull, Figures 2,3 and 6. In some cases the bounding box is a better approximation to the convex hull than the envelope, a square for example.

The envelope of a diagram is highly dependent on its origin, Figures 4 and 5. As the origin moves farther away from the center of the diagram, the envelope becomes a worse approximation to the convex hull.

Like the convex hull the shape of the bounding box is independent of the placement of the origin. This is not true of envelopes. Finally, I wonder if there is an efficient algorithm to

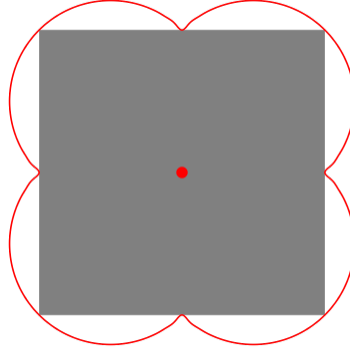


FIGURE 2. Envelope of unit square centered at  $(0,0)$

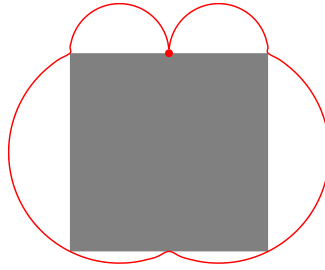


FIGURE 3. Envelope of unit square centered at  $(0, -\frac{1}{2})$

calculate and extensional form of the convex hull using the variational form

$$c_d(v) = \inf_{u \in \mathbb{R}^n} \frac{q_d(u)}{\langle u, v \rangle}$$

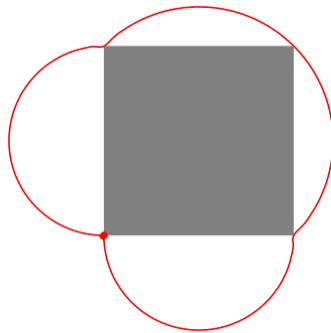


FIGURE 4. Envelope of unit square centered at  $(\frac{1}{2}, \frac{1}{2})$

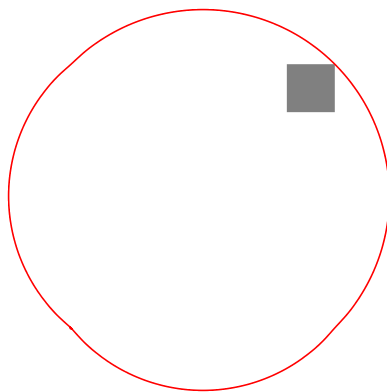


FIGURE 5. Envelope of unit square centered at  $(5, 5)$

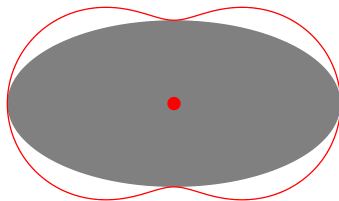


FIGURE 6. Envelope of an ellipse