b) For simplicity, let's assume d is even.

First, we observe (recall) that $\underline{w}' = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ can be used to count the number of ones in $\underline{x}$, via $K_0 = \underline{w}'^T \underline{x}$.

This immediately suggests the following as a <u>possible</u> solution strategy:

1) Choose $\ell^* = \underset{\ell \in \{0,1,\dots d\}}{\arg\min} (\underline{w}'^T \underline{x} - \ell)^2$

2) Take the parity of ℓ^* as the decision result.

This method certainly works, but unfortunately is not based on <u>linear</u> discriminants, but rather <u>quadratic</u> discriminants (expand the square above to see this clearly).

However, something related that <u>is</u> based on linear discriminants will in fact work.

First, just consider the <u>even</u> integers $0, 2, 4, \dots d$.

$$|\underline{w}'^T \underline{x} - 2m|^2 = |K_0 - 2m|^2 = K_0^2 - 4mK_0 + 4m^2$$

Clearly, $\underset{m \in \{0,1,\dots d/2\}}{\arg\min} |K_0 - 2m|^2 = \underset{m \in \{0,1,\dots d/2\}}{\arg\min} -4mK_0 + 4m^2$

$$= \underset{m \in \{0,1,\dots d/2\}}{\arg\max} 4mK_0 - 4m^2$$

$$= \underset{m \in \{0,1,\dots d/2\}}{\arg\max} 4m\underline{w}'^T \underline{x} - 4m^2 \triangleq m_e^* \quad (\square)$$

(obvious from above, but you can also take derivatives w.r.t. m, set to zero, + solve for m_e^*).

It's easy to verify in <u>several ways</u> that $m_e^* = \frac{K_0}{2}$, i.e. the even integer result is $2m_e^*$.

Further recognize that $(\square)$ specifies a set of $d/2+1$ <u>linear</u> discriminant functions, and a way of selecting the nearest even integer to K_0.

Next, consider the odd integers:

$$(\underline{w}^T\underline{x} - (2m+1))^2 = (K_o - (2m+1))^2 = K_o^2 - (4m+2)K_o + 4m^2 + 4m + 1$$

$$\underset{m \in \{0,1,\ldots \frac{d}{2}-1\}}{\arg\min} (K_o - (2m+1))^2 = \underset{m}{\arg\min} -(4m+2)K_o + 4m^2 + 4m + 1$$

$$= \underset{m}{\arg\max} (4m+2)K_o - 4m^2 - 4m - 1$$

(again, ~~take~~ derivatives + solve for maximum)

$$= \underset{m}{\arg\max} (4m+2)\underline{w}^T\underline{x} - 4m^2 - 4m - 1 \quad (\Delta)$$

→ class 1 eq., and the the null set.

The last quantity is maximized @ $m_o^* = \frac{K_o}{2} - \frac{1}{2}$ -- this is integer when K_o is odd, and non-integer otherwise.
In the odd case, note that we get the desired equation $2m_o^* + 1 = K_o$.
In the even case, (Δ) still gives a result (the nearest odd integer to the even K_o...).

OK, so $(\square)$ defines a set of LDFs that are compared to pick the best even integer.
(Δ) defines a set of LDFs compared to pick the best odd integer.

Now, we need to choose between the best even & odd selections. For this, simply recognize that in deriving LDFs in both the even and odd cases, we started from the same sqd. distance expression, and we made the function linear by ignoring the same constant term K_o^2. Thus, in both the even and odd cases, the LDFs amount to the sqd. distance minus the same quantity, K_o^2. Thus, clearly, the (best) even + odd discriminants can simply be compared, with the (even/odd) argument of the largest of the two giving the parity, i.e.

Choose even if + only if
$$4m_e^* \underline{w}^T\underline{x} - 4m_e^{*2} \geq (4m_o^* + 2)\underline{w}^T\underline{x} - 4m_o^{*2} - 4m_o^* - 1$$