gradient

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1 Global epistatsis model

We assume that genotypes are stored in a $N \times L$ matrix, where N is the number of individuals and L is the number of loci. The model has the following parameters:

β, vector of effect sizes for different loci

α, power term that controls linearity

σ^2, variance of the Gaussian error term (which has mean 0)

μ, population-wide mean

We define the mean for individual i as

$$\mu_i = sgn(b_i)|b_i|^\alpha + \mu,$$

where $b_i = \beta^T X_i = \sum_j \beta_j X_{ij}$. Alternatively, we write the phenotype Y_i for individual i as

$$Y_i = sgn(b_i)|b_i|^\alpha + \mu + \epsilon_i,$$

where ϵ_i is the Gaussian error for individual i.

In this model, the loglikelihood is

$$\begin{aligned} LL(\theta) &= \sum_{i=1}^{n} \log P(Y_i \mid \theta) \\ &= \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_i-\mu_i)^2}{2\sigma^2}\right) \\ &= -\frac{n}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu_i)^2 . \end{aligned}$$

1.0.1 gradients

First, the gradient for the effect size β_l. Defining operator $D := \frac{\delta}{\delta\beta_l}$,

$$
\begin{aligned}
DLL(\theta) &= D\left(-n\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu_i)^2\right)\\
&= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}D(Y_i-\mu_i)^2\\
&= \frac{1}{2\sigma^2}\sum_{i=1}^{n}2(Y_i-\mu_i)D\mu_i
\end{aligned}
$$

We have

$$
D\mu_i = D\left[sgn(b_i)|b_i|^{\alpha}\right]
$$

where $b = \sum_{j=1}^{k}\beta_j G_{ij}$ and $sgn(x)$ is the sign function. We have $Dsgn(b_i) = 0$, so, by the product rule,

$$
\begin{aligned}
D\mu_i &= sgn(b)D\left(|b_i|^{\alpha}\right)\\
&= sgn(b_i)\alpha|b_i|^{\alpha-1}D|b_i|\\
&= sgn(b_i)\alpha|b_i|^{\alpha-1}sgn(b_i)G_{il}\\
&= \alpha|b_i|^{\alpha-1}G_{il}
\end{aligned}
$$

And thus

$$
= DLL(\theta) = \frac{\alpha}{\sigma^2}\sum_{i=1}^{n}(Y_i-\mu_i)\left|\sum_{j=1}^{k}\beta_j G_{ij}\right|^{\alpha-1}\cdot G_{il}.
$$

1.0.2 Now we need to calculate with respect to μ, σ^2 and α.

For α Defining $D := \frac{\delta}{\delta\alpha}$, we again have

$$
DLL(\theta) = \frac{1}{2\sigma^2}\sum_{i=1}^{n}2(Y_i-\mu_i)D\mu_i \tag{1}
$$

$$
\tag{2}
$$

We can calculate

$$
\begin{aligned}
D\mu_i &= D\big(sgn(b)|b|^{\alpha}+\mu\big)\\
&= sgn(b)|b|^{\alpha}\log|b|
\end{aligned}
$$

where again $b = \sum_{j=1}^{k}\beta_j G_{ij}$. (And all the log's are base e.) Thus

$$
\frac{\delta}{\delta\alpha}LL(\theta) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i-\mu_i)sgn(b_i)|b_i|^{\alpha}\log|b|.
$$

For μ Defining $D := \frac{\delta}{\delta\mu}$, we again have

$$DLL(\theta) = \frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(X_i - \mu_i)D\mu_i \tag{3}$$

$$\tag{4}$$

We can calculate

$$\begin{aligned} D\mu_i &= D\big(sgn(b)|b|^\alpha + \mu\big) \\ &= 1 \end{aligned}$$

and thus

$$\frac{\delta}{\delta\mu}LL(\theta) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu_i)$$

.

For σ^2 (remembering to ignore the squared, since it's just part of the parameter's name) Defining $D := \frac{\delta}{\delta\sigma^2}$, we again have

$$DLL(\theta) = D\left(-n\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_i)^2\right) \tag{5}$$

$$= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_i)^2 \tag{6}$$

2 checking the gradient numerically

In [69]:
```
import numpy as np
from scipy.optimize import approx_fprime
import numpy.random as npr
```

In [89]:
```
k = 10
G = npr.choice([-1,0,1], size = k, replace = True)
beta = npr.uniform(-1,1, size = k)
mu_0 = 4.5
alpha = 4
```

In [90]:
```
def mu_i(betas):
    b = np.sum(betas*G)
    absb = np.abs(b)
    sgnb = np.sign(b)
    return sgnb * absb**alpha + mu_0
```

In [91]:
```
def Dmu_i(betas):
    absb = np.abs(np.sum(betas*G))
    return alpha*(absb**(alpha-1.0))*G
```

```
In [92]: mu_i(beta)

Out[92]: -8.932148169982433

In [93]: approx_fprime(beta, mu_i, 1e-9)

Out[93]: array([  0.        , -28.06526389,  28.06526389, -28.06526389,
               -28.06526389,  28.06526389,  28.06526389, -28.06526389,
               -28.06526389,  28.06526389])

In [94]: Dmu_i(beta)

Out[94]: array([  0.        , -28.06526173,  28.06526173, -28.06526173,
               -28.06526173,  28.06526173,  28.06526173, -28.06526173,
               -28.06526173,  28.06526173])
```

Checks out!