Chem237: Lecture 2

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ODE; Variable Coefficients

Variable Coefficients; Homogenous

Now consider Variable coefficients to our differential equations.

$$\frac{dx}{dt} + a(t)x = 0 \tag{1}$$

Cosider the linear first order variable coeficient equation above. This is simple because the problem is seperable, and can easily be solved by direct integration.

$$\frac{dx}{dt} + a(t)x = 0$$
$$\int dx \frac{1}{x} = -\int dt \ a(t) \tag{2}$$
$$x(t) = x_0 \exp\left[-\int_{t_0}^{t} dt \ a(t)\right]$$

The last line is the homogeneous solution to the first order variable coefficient ODE above. Although the answer is in terms of an integral, an integral is much simplier than a differential equaiton, and we call this a solution, because we have made progress.

Variable Coefficients; Non-Homogenous

Now consider the non-homonogeous variable coefficient differential equation

$$\frac{dx}{dt} + a(t)x = f(t) \tag{3}$$

Where f(t) is a known (given) function. We know the general solution to a non-hoogenous equation is the summation of a particular solution (x_1), and the homogeneous equation.

$$x(t) = x_1(t) + C \exp\left[-A(t)\right]$$
$$A(t) \equiv \int_0^t dt \ a(t) \tag{4}$$

We will assume the particular solution is of the form

$$x_1(t) = e^{-A(t)} g(t) \tag{5}$$

Where g is an unknown function, and we will try to solve for it.

$$\frac{dx_1(t)}{dt} + a(t)x_1(t) = f(t) \Rightarrow$$
$$\frac{d}{dt} e^{-A(t)} g(t) + a(t) e^{-A(t)} g(t) = f(t)$$
$$e^{-A(t)} \frac{d}{dt} g(t) + g(t) \frac{d}{dt} e^{-A(t)} + a(t) e^{-A(t)} g(t) = f(t) \tag{6}$$
$$e^{-A(t)} \frac{d}{dt} g(t) = f(t) \Rightarrow$$
$$g(t) = \int^t dt \ e^{A(t)} f(t) \Rightarrow$$

NonLinear ODE

We will start a new, more complicated problem, the nonlinear ordinary differential equations. There is no general solution to solve a non-linear ODE, however, there are classes that can be solved.

Consider first a generalized form for the first order ODE

$$a(x)dx + b(y)dy = 0 \Rightarrow$$
$$b(y)\frac{dy}{dx} + a(x) = 0 \tag{7}$$

These two lines are equivalent, but the second line is much more convenient because it is seperable and can be solved by integration. This is the simpliest form of a non-linear equation, because there isn't a y depedence in every term.

An even more general form would be

$$A(x,y)dx + B(x,y)dy = 0 \tag{8}$$

We will see that this is a more preferable form.

If this is an exact differential (the LHS), calling it U, than consider

$$U(x,y), \quad dU(x,y) = Adx + Bdy \tag{9}$$

With both of these statements the solution is then

$$U(x,y) + c = 0 \tag{10}$$

So if we have an exact differential, than if we just find the exact differential than we can get the solution.

Consider exact differentials (the second line we simply give the partial derivitives names).

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$
$$dU = Adx + Bdy$$
$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \tag{11}$$

And the last line is a requirement for a total differential.

So we have seen that if we have an exact differential we can solve, and we have conditions that must be satisfied.

As an example consider

$$U(x,y) = x^2 + xy + y^3$$
$$\frac{\partial^2 U}{\partial x \partial y} = 1$$
$$\frac{\partial^2 U}{\partial y \partial x} = 1 \tag{12}$$

We see this is an exact differential, therefore the solution to the problem is given by

$$x^2 + xy + y^3 + C = 0 \tag{13}$$

Integrating Factor

If we do not have an exact differential the probem is no longer easy, we have a few methods we can use to solve. A well known method for addressing a non-linear ODE with is the **Integrating Factor**. Consider Adx + Bdy, such that it is not an exact differntial. Introduce the integrating factor $\lambda(x,y)$, such that

$$dU = \lambda(Adx + Bdy) \tag{14}$$

Is now transformed into an exact differential. We therefore need to find a λ such that the inexact differential becomes exact. A theorem exists stating that an integrating factor always exists, but there is no general method for finding it! But λ always exists.

There are known forms for A and B that we can recognize to solve for the integrating factor.

As an example consider

$$y' + f(x)y = g(x) \Rightarrow dy + f(x)ydx = g(x)dx \tag{15}$$

We can always consider an integrating factor

$$\lambda\left[dy + f(x)ydx\right] = \lambda g(x)dx \tag{16}$$

The idea is now to integrate each side, however, this will only be useful if the RHS is only a function of x. If it turns out that $\lambda \to \lambda(x)$ than the RHS is always a function of x. If $\lambda(x,y)$ than we may not be able to solve the LHS. So we will assume $\lambda \to \lambda(x)$ and see if it works. The whole point of the method is to get an exact differntial, therefore assume the LHS is exact.

$$\frac{\partial \lambda}{\partial x} = \frac{\partial(\lambda f y)}{\partial y} = \lambda(x)f(x) \tag{17}$$

This new equation is seperable and we can solve by integration.

$$\int d\lambda \frac{1}{\lambda} = \int dx f(x) \Rightarrow \lambda(x) = \exp\left[\int dx f(x)\right] \tag{18}$$

So we see that we can find the integrating factor with $\lambda(x)$. Now that we have an exact differential we need to confirm

$$dU \equiv \lambda(dy + f(x)ydx)$$
$$\frac{\partial U}{\partial x} = \lambda \tag{19}$$
$$U(x,y) = \lambda(x)y$$

So we can solve this problem by integration, and find a general equation for any f(x), g(x).

Another example consider

$$xy' + (1+x)y = e^x \Rightarrow y' + \left(\frac{1+x}{x}\right)y = \frac{e^x}{x}$$
$$\lambda(x) = \exp\left[\int dx \frac{1+x}{x}\right] = xe^x \tag{20}$$

Now if we multipy the LHS and RHS by λ

$$\int xe^x\left[y' + \frac{1+x}{x}y\right] = \int dx e^{2x} \Rightarrow U(x,y) = xe^x y = \frac{1}{2}e^{2x} + c \Rightarrow y = \frac{e^x}{2x} + \frac{c}{x}e^{-x} \tag{21}$$

So we see that a non-linear first order ODE can be solved with the integrating factor method. If we have an equation of the form

$$y' + f(x)y = g(x) \tag{22}$$

Than the integrating factor is a useful approach, using a non-trivial transfomration to generate a simple solution.

New Method

We can look at other classes of equations we can solve for.

$$A(x,y)dx + B(x,y)dy = 0 \tag{23}$$

Consider an equation of the above from with A and B both being homogeneous functions of degree r. Recall a homogeneous function obeys

$$A(cx, cy) = c^r A(x,y) \tag{24}$$

As an example

$$A = x^2 + yx, \qquad B = y^2 \tag{25}$$

Both functions are homogeneous functions of degree 2.

For this type of equation for we can generate a seperable equation through the substitution

$$y = vx \tag{26}$$

Consider our example above we find

$$ydx + (2\sqrt{xy} - x)dy = 0 \tag{27}$$

In this example r=1, let y=vx and substitute in dy = vdx + xdv (assume an exact differential).

$$vxdx + (2x\sqrt{v} - x)(vdx + xdv) = 0 \Rightarrow \frac{2\sqrt{v} - 1}{2v^{3/2}}dv = -\frac{1}{x}dx \tag{28}$$

And this equation is seperable and can be solved with integration!

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Final Example

Another example of the same class of problem.

$$(ax + by + c)dx + (ex + fy + gdy) = 0 \tag{29}$$

Where a,b,c,e,f,g are all constants. But ax + by + c is not homogeneous because c is a lone constant (same with g). Consider x = X + α and y = Y + β, dx = dX, and dy = dY. Substitute in these definitions we find

$$(aX + a\alpha + bY + b\beta + c)dX + (eX + e\alpha + fY + f\beta + g)dY = 0 \tag{30}$$

We can find α and β,

$$a\alpha + b\beta + c = 0$$
$$e\alpha + f\beta + g = 0 \tag{31}$$

This generates

$$(aX + bY)dX + (eX + fY)dY = 0 \tag{32}$$

Where we have a homogeneous equation, and we can solve using the methods from before.