

Chem231B: Lecture 2

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Operator Algebra

A cornerstone of this course will be the use of operators to solve Quantum Mechanics Problems. By using the operator formulation of QM we can completely ignore the wavefunction and instead compute observables from just the algebraic manipulations of operators. This means we can ignore solving the various integrals and differential equations commonly found in the wave function formulation.

The Ground State

We can consider the ground state energy of the HO, which has an eigenvector satisfying the following equation

$$a|\phi_0\rangle = 0 \quad (1)$$

This statement is really a differential equation, remember what the operators represent

$$\left\{ \frac{1}{\sqrt{2}} \sqrt{\frac{m\omega}{\hbar}} x + \frac{i}{\sqrt{m\hbar\omega}} p \right\} \phi_0 = 0 \quad (2)$$

In the position representation we have

$$\left(\frac{m\omega}{\hbar} x + \frac{d}{dx} \right) \phi_0(x) = 0 \quad (3)$$

The general solution to this differential equation is of the form

$$\phi(x) = ce^{-\frac{m\omega x^2}{2\hbar}} \quad (4)$$

This solution (which takes some effort) is just for the ground state. All of the solutions are actually proportional, therefore only 1 ket exists to describe the ground state (it is degenerate) giving an energy of

$$E_0 = \frac{\hbar\omega}{2} \quad (5)$$

First Excited State

We want more, how do we approach the excited states? To do this we will utilize the creation and annihilation operators. It is important to realize that a and a^\dagger do NOT conserve normalization, as we will see below.

We can show that all of the states are also non-degenerate. Suppose we have a single vector satisfying

$$N|\phi_n\rangle = n|\phi_n\rangle \quad (6)$$

Likewise we have an eigenvector associated with the $n+1$ eigenvalue.

$$N|\phi_{n+1}\rangle = (n+1)|\phi_{n+1}\rangle \quad (7)$$

We also know from Lecture 1 that our annihilation operator can be used to write

$$a|\phi_{n+1}\rangle = c|\phi_n\rangle \quad (8)$$

If we simply stick the a^\dagger operator in this expression we have a nice simplification

$$\begin{aligned}
a^\dagger a |\phi_{n+1}\rangle &= a^\dagger c |\phi_n\rangle \\
N |\phi_{n+1}\rangle &= a^\dagger c |\phi_n\rangle \\
(n+1) |\phi_{n+1}\rangle &= a^\dagger c |\phi_n\rangle \\
|\phi_{n+1}\rangle &= \frac{c}{(n+1)} a^\dagger |\phi_n\rangle
\end{aligned} \tag{9}$$

This result shows all $n+1$ vectors are proportional to $a^\dagger |\phi_n\rangle$ and therefore proportional to each other and the eigenvalues are not degenerate.

$$|\phi_n\rangle \rightarrow E_n = \left(n + \frac{1}{2}\right) \hbar\omega \tag{10}$$

Our result also shows that we need to know the pre-factors to get the wavefunctions.

$$a |\phi_0\rangle = 0, \quad |\phi_1\rangle = c_1 a^\dagger |\phi_0\rangle \dots \tag{11}$$

We can find our normalization by taking the scalar product

$$\begin{aligned}
\langle \phi_1 | \phi_1 \rangle &= |c_1|^2 \langle \phi_0 | a a^\dagger | \phi_0 \rangle \\
&= |c_1|^2 \langle \phi_0 | (a^\dagger a + 1) | \phi_0 \rangle
\end{aligned} \tag{12}$$

The last line follows because of the commutator

$$[a, a^\dagger] = 1 \rightarrow a a^\dagger - a^\dagger a = 1 \rightarrow a a^\dagger = 1 + a^\dagger a \tag{13}$$

If we require $|\phi_1\rangle$ to be normalized and have a constant c_1 to be real and positive (relative to the phase of $|\phi_0\rangle$). But $|\phi_0\rangle$ is a normalized eigenstate of N with an eigenvalue of zero as we have shown, therefore

$$\langle \phi_1 | \phi_1 \rangle = |c_1|^2 = 1, \quad c_1 = 1 \tag{14}$$

We can always have an arbitrary phase term associate with ϕ that is fine, it will just make ϕ complex, choosing c_1 to be 1 makes ϕ real.

Second Excited State

In the same manner we can construct the next state using our operators, assuming c_2 to be real and $|\phi_2\rangle$ to be normalized.

$$\begin{aligned}
|\phi_2\rangle &= c_2 a^\dagger |\phi_1\rangle \\
\langle \phi_2 | \phi_2 \rangle &= |c_2|^2 \langle \phi_1 | a a^\dagger | \phi_1 \rangle \\
&= |c_2|^2 \langle \phi_1 | (a^\dagger a + 1) | \phi_1 \rangle \\
&= |c_2|^2 \langle \phi_1 | (N + 1) | \phi_1 \rangle \\
&= |c_2|^2 \langle \phi_1 | (N \phi_1 + \phi_1) \rangle \\
&= |c_2|^2 \langle \phi_1 | \phi_1 + \phi_1 \rangle \\
&= |c_2|^2 2 \langle \phi_1 | \phi_1 \rangle \\
&= 2 |c_2|^2 = 1
\end{aligned} \tag{15}$$

We have therefore shown (taking the normalization into account)

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} a^\dagger |\phi_1\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |\phi_0\rangle \tag{16}$$

General Solution Harmonic Oscillator

Hopefully now you can realize that we can build all of the wavefunctions by multiplying with a^\dagger and finding the appropriate normalization. The general case works in the same manner

$$\begin{aligned}
|\phi_n\rangle &= c_n |\phi_{n-1}\rangle \\
\langle \phi_n | \phi_n \rangle &= |c_n|^2 \langle \phi_{n-1} | a a^\dagger | \phi_{n-1} \rangle \\
\langle \phi_n | \phi_n \rangle &= |c_n|^2 \langle \phi_{n-1} | a^\dagger a + 1 | \phi_{n-1} \rangle \\
&\rightarrow c_n = \frac{1}{\sqrt{n}}
\end{aligned} \tag{17}$$

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} a^\dagger |\phi_{n-1}\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |\phi_0\rangle \quad (18)$$

Where this last line represents the general solution for the Harmonic Oscillator!

Orthonormal and Closure

Because H is Hermitian the HO kets $|\phi_n\rangle$ are orthogonal. Each is individually normalized therefore we have an orthonormal set.

$$\langle \phi_{n'} | \phi_n \rangle = \delta_{nn'} \quad (19)$$

We also know that H is an observable and we therefore have closure over the basis.

$$\sum_n |\phi_n\rangle \langle \phi_n| = 1 \quad (20)$$

New Section

Any generic operator A (representing a physical observable) can be expressed in terms of a and a^\dagger because X and P are simply linear combinations of a and a^\dagger . This essentially means we can solve any expectation value by using a and a^\dagger .

We have already found general expressions relating a and a^\dagger to the HO $|\phi_n\rangle$. These expressions make it very convenient to operate on $|\phi_n\rangle$.

$$\begin{aligned} a^\dagger |\phi_n\rangle &= \sqrt{n+1} |\phi_{n+1}\rangle \\ a |\phi_n\rangle &= \sqrt{n} |\phi_{n-1}\rangle \end{aligned} \quad (21)$$

Which makes it clear where the names creation and annihilation operators come from. The second relationship follows from the commutator of $[a, a^\dagger]$

$$a |\phi_n\rangle = a \frac{1}{\sqrt{n}} a^\dagger |\phi_{n-1}\rangle = \frac{1}{\sqrt{n}} a a^\dagger |\phi_{n-1}\rangle = \frac{1}{\sqrt{n}} (a^\dagger a + 1) |\phi_{n-1}\rangle = \sqrt{n} |\phi_{n-1}\rangle \quad (22)$$

The creation and annihilation names are intuitive when acting on the ket, however, acting on the bra we actually find the inverse.

$$\begin{aligned} \langle \phi_n | a &= \sqrt{n+1} \langle \phi_{n+1} | \\ \langle \phi_n | a^\dagger &= \sqrt{n} \langle \phi_{n-1} | \end{aligned} \quad (23)$$

Matrix Elements

From our operator definitions we know

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}} X, \quad \hat{X} = \frac{1}{\sqrt{2}} (a^\dagger + a) \quad (24)$$

Substituting these expressions we can write the following

$$X |\phi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) |\phi_n\rangle \quad (25)$$

And in the analogous manner we can define the momentum operation as

$$P |\phi_n\rangle = \sqrt{m\hbar\omega} \frac{i}{\sqrt{2}} (a^\dagger - a) |\phi_n\rangle \quad (26)$$

With all of these expressions we can analyze various matrix elements (recall the HO vectors are orthonormal).

$$\langle \phi_{n'} | a | \phi_n \rangle = \langle \phi_{n'} | \sqrt{n} \phi_{n-1} \rangle = \sqrt{n} \langle \phi_{n'} | \phi_{n-1} \rangle = \sqrt{n} \delta_{n', n-1} \quad (27)$$

In the same manner we can easily write

$$\langle \phi_{n'} | a^\dagger | \phi_n \rangle = \sqrt{n+1} \delta_{n', n+1} \quad (28)$$

$$\langle \phi_{n'} | X | \phi_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} [\sqrt{n+1} \delta_{n', n+1} + \sqrt{n} \delta_{n', n-1}] \quad (29)$$

$$\langle \phi_{n'} | P | \phi_n \rangle = i \sqrt{\frac{m\hbar\omega}{2}} [\sqrt{n+1} \delta_{n', n+1} - \sqrt{n} \delta_{n', n-1}] \quad (30)$$

From these matrix elements it becomes clear that only a and a^\dagger are Hermitian Conjugates and the matrix $a^\dagger a$ only has non-zero elements for the diagonal which starts at 0 and increases by 1 along the diagonal.

Wavefunction

We can now start discussing the wavefunction itself. We know from before that the groundstate is proportional to a Gaussian. As stated before, we can get all of the expectation values using a and a^\dagger because they are proportional to linear combinations of X and P . The groundstate of the HO (well known) can be written explicitly in the $\{|x\rangle\}$ representation as

$$\phi_0(x) = \langle x|\phi_0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}x^2\right\} \quad (31)$$

To obtain the $\phi_n(x)$ associated with the other stationary states (eigenstates are stationary, linear combinations are not) we can use our general solution to the HO vectors and the x representation of a^\dagger

$$a^\dagger = \frac{1}{\sqrt{2}} \left(\hat{X} - i\hat{P} \right) = \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}} \frac{d}{dx} \right\} \quad (32)$$

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