Chem237: Lecture 2

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4/4/18

ODE; Variable Coefficients

We can now consider ODEs with variable coefficients. Unfortunately most variable coefficient ODEs cannot be solved analytically, except for certain special cases.

Variable Coefficients; Homogenous

$$\frac{dx}{dt} + a(t)x = 0 \tag{1}$$

Cosider the linear, first order variable coefficient ODE above. This example is trivial because the problem is seperable, and can easily be solved by direct integration.

$$\begin{aligned} \frac{dx}{dt} &+ a(t)x = 0 \\ \int dx\frac{1}{x} &= -\int dt \ \mathrm{a}(t) \\ \ln(x) - \ln(x_0) &= -\int dt \ \mathrm{a}(t) \\ \ln(x) &= \ln(x_0) - \int dt \ \mathrm{a}(t) \\ x(t) &= x_0 \exp\left[-\int_{t_0}^{t} dt \ \mathrm{a}(t)\right] + c \end{aligned} \tag{2}$$

The last line represents the homogeneous solution to the first order variable coefficient ODE (equation 1). Although the final answer is in terms of an integral, an integral is in general simpler than a differential equation to solve. We therefore call this a solution, because we have made some progress.

Differentials and Algebra

A quick side note that is worth mentioning. Solving a seperable differential equations (as we did above) seems to imply that the differential operators can be treated as algebraic elements (it looks like we just multiplied by dt in the above example). Although this is convenient it is not 'legitimate', and we should probably prove it works for these seperable equations. Consider a simple differential equation

$$\begin{aligned} f(y)\frac{dy}{dx} &= g(x) \\ \int dx f(y)\frac{dy}{dx} &= \int dx g(x) \end{aligned} \tag{3}$$

If we let F(y) define the anti-derivitive of f(y), meaning

$$F(y) = \int f(y)dy \qquad \Rightarrow \qquad F'(y) = f(y) \tag{4}$$

Applying the chain rule to F(y) we find

$$\frac{d}{dx}F(y) = f(y)\frac{dy}{dx} \tag{5}$$

We can now substitute in the chain rule into Equation 3 and we see find

$$\begin{aligned}\int dxF'(y) &= \int dxg(x)\\ F(y) &= \int dxg(x)\\ \int f(y)dy &= \int dxg(x)\end{aligned} \tag{6}$$

So we are not actually treating the operators algebraically, we can naturally re-arrange the equation through the chain rule. This result is true for the specific equation we analyzed, although most people will solve seperable ODEs by pretending the operators are algebraic elements, it is good to realize why this machiney works in this context (and you should not assume it will work for differnt problems).

Variable Coefficients; Non-Homogenous

Now consider the non-homonogeous variable coefficient differential equation

$$\frac{dx}{dt} + a(t)x = f(t) \tag{7}$$

Where f(t) is some known (given) function. The general solution to any non-homogenous differential equation is the summation of associated homogenous equations solution, and a particular solution (x_p).

We can therefore use the solution we found to Equation 1, for our homogenous solution, and then construct the solution to Equation 7.

$$\begin{aligned}x(t) &= x_p(t) + x_0 e^{-A(t)}\\ A(t) &\equiv \int_{t_0}^{t} dt\ \mathrm{a}(t)\end{aligned} \tag{8}$$

We now need to find a solution to the particular solution, and unfortunately there is no general strategy for this task. A decent guess for the form of the particular solution is an expotential function multiplied by some other function that we will solve for.

The original differential equation relates a function and its derivitive, therefore the expotential is a good candidate for our solution. But the particular solution needs to tell us something new it cannot just be an expotential like the homogenous solution. So we hope that some generic g(t) function exists and we try to solve for it.

So we will guess our particular solution is of the form

$$x_p(t) = e^{-A(t)}g(t) \tag{9}$$

Substituting our guess into Equation 7 we can try to solve for g(t).

$$\begin{aligned}\frac{dx_p(t)}{dt} + a(t)x_p(t) &= f(t)\\ \frac{d}{dt}e^{-A(t)}g(t) + a(t)e^{-A(t)}g(t) &= f(t)\\ e^{-A(t)}\frac{d}{dt}g(t) + g(t)\frac{d}{dt}e^{-A(t)} + a(t)e^{-A(t)}g(t) &= f(t)\\ e^{-A(t)}\frac{d}{dt}g(t) - a(t)e^{-A(t)}g(t) + a(t)e^{-A(t)}g(t) &= f(t)\\ e^{-A(t)}\frac{d}{dt}g(t) &= f(t) \Rightarrow\\ \int^{t} dt\ e^{A(t)}f(t) &= g(t)\end{aligned} \tag{10}$$

So it appears that we can in general find this mysterious g(t) function, the catch is that we need to compute an integral. And we have no guarantee that we can analytically solve that integral.

NonLinear ODE

Let's now consider a more complicated problem, the NonLinear Ordinary Differential Equations. A theme in Differential Equations; there is no general strategy for solving a non-linear ODE. However, there are classes that can be solved analytically, some of which we will explore.

Consider first a generalized form of the first order ODE

$$\begin{aligned} a(x)dx + b(y)dy = 0 \Rightarrow \\ b(y)\frac{dy}{dx} + a(x) = 0 \end{aligned} \tag{11}$$

These two lines are equivalent, but the second line is more convenient to work with. Notice it is separable and can therfore be solved by integration. This is one of the simplest forms of a non-linear equation, (there isn't a y dependence in every term, therefore it is non-linear).

We can generalize the above ODE further, consider

$$A(x, y)dx + B(x, y)dy = 0 \tag{12}$$

We will now introduce a well-known topic, the exact differntial (U). An exact differential can be written in the following form.

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\ dU(x, y) &\equiv Adx + Bdy \\ \frac{\partial^2 U}{\partial x \partial y} &= \frac{\partial^2 U}{\partial y \partial x} \end{aligned} \tag{13}$$

Please note: in Line 2 we simply give our partial derivitives names (A and B), no magic occuring here.

If we assume our generalized ODE is an exact differential we could imagine writing

$$U(x, y) + c = 0 \tag{14}$$

So if we have an exact differential, than if we just find the exact differential than we can get the solution.

Consider exact differentials (the second line we simply give the partial derivatives names).

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\ dU &= Adx + Bdy \\ \frac{\partial^2 U}{\partial x \partial y} &= \frac{\partial^2 U}{\partial y \partial x} \end{aligned} \tag{15}$$

And the last line is a requirement for a total differential.

So we have seen that if we have an exact differential we can solve, and we have conditions that must be satisfied.

As an example consider

$$\begin{aligned} (2x + y)dx + (x + 3y^2)dy = 0 \\ A(x, y) = 2x + y \\ \frac{\partial A}{\partial y} = 1 \\ B(x, y) = x + 3y^2 \\ \frac{\partial B}{\partial x} = 1 \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial^2 U}{\partial x \partial y} &= 1 \\ \frac{\partial^2 U}{\partial y \partial x} &= 1 \\ U(x, y) &= x^2 + xy + y^3 \end{aligned} \tag{17}$$

We see this is an exact differential, therefore the solution to the problem is given by

$$x^2 + xy + y^3 + C = 0 \tag{18}$$

Integrating Factor

If we do not have an exact differential the problem is no longer easy, we have a few methods we can use to solve. A well known method for addressing a non-linear ODE with is the **Integrating Factor**. Consider Adx + Bdy, such that it is not an exact differntial. Introduce the integrating factor λ(x,y), such that

$$dU = \lambda(Adx + Bdy) \tag{19}$$

Is now transformed into an exact differential. We therefore need to find a λ such that the inexact differential becomes exact.A theorem exists stating that an integrating factor always exists!

Proof

The proof for this theorem is as follows consider the above example :

$$\begin{aligned} A(x,y)dx + B(x,y)dy = 0 \\ \frac{dy}{dx} = -\frac{A}{B} \end{aligned} \tag{20}$$

Which has a general solution of :

$$f(x,y) = C \tag{21}$$

C is a constant. We then take a total differential:

$$\begin{aligned} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \\ \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \end{aligned} \tag{22}$$

Relating the two we can see :

$$\begin{aligned} \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{A}{B} \\ \frac{\frac{\partial f}{\partial x}}{A} = \frac{\frac{\partial f}{\partial y}}{B} \end{aligned} \tag{23}$$

We can then relate this ratio using a factor , which is the integrating factor $\lambda(x,y)$:

$$\begin{aligned} \frac{\partial f}{\partial x} = \lambda A \\ \frac{\partial f}{\partial y} = \lambda B \end{aligned} \tag{24}$$

We multiply are general differential equation by an integrating factor as follows :

$$\lambda A dx + \lambda B dy = 0 \tag{25}$$

Unfortunately there is no general method for finding it, but λ always exists! There are known forms for A and B that we can recognize to solve for the integrating factor. As an example consider:

$$y' + f(x)y = g(x) \Rightarrow dy + f(x)ydx = g(x)dx \tag{26}$$

We can always consider an integrating factor

$$\lambda\left[dy + f(x)ydx\right] = \lambda g(x)dx \tag{27}$$

The idea is now to integrate each side, however, this will only be useful if the RHS is only a function of x. If it turns out that $\lambda \to \lambda(x)$ than the RHS is always a function of x. If λ(x,y) than we may not be able to solve the LHS. So we will assume $\lambda \to \lambda(x)$ and see if it works. The whole point of the method is to get an exact differential, therefore assume the LHS is exact.

$$\frac{\partial \lambda}{\partial x} = \frac{\partial(\lambda f y)}{\partial y} = \lambda(x)f(x) \tag{28}$$

This new equation is separable and we can solve by integration.

$$\int d\lambda \frac{1}{\lambda} = \int dx f(x) \Rightarrow \lambda(x) = \exp\left[\int dx f(x)\right] \tag{29}$$

So we see that we can find the integrating factor with $\lambda(x)$. Now that we have an exact differential we need to confirm

$$\begin{aligned} dU &\equiv \lambda(dy + f(x)y dx) \\ \frac{\partial U}{\partial x} &= \lambda \\ U(x,y) = \lambda(x)y & \end{aligned} \tag{30}$$

So we can solve this problem by integration, and find a general equation for any f(x), g(x). Another example consider

$$\begin{aligned} xy' + (1+x)y = e^x \Rightarrow y' + \left(\frac{1+x}{x}\right) y = \frac{e^x}{x} \\ \lambda(x) = \exp\left[\int dx \frac{1+x}{x}\right] = xe^x \end{aligned} \tag{31}$$

Now if we multiply the LHS and RHS by λ

$$\int xe^x \left[y' + \frac{1+x}{x} y\right] = \int dx e^{2x} \Rightarrow U(x,y) = xe^x y = \frac{1}{2}e^{2x} + c \Rightarrow y = \frac{e^x}{2x} + \frac{c}{x}e^{-x} \tag{32}$$

So we see that a non-linear first order ODE can be solved with the integrating factor method. If we have an equation of the form

$$y' + f(x)y = g(x) \tag{33}$$

Than the integrating factor is a useful approach, using a non-trivial transformation to generate a simple solution.

New Method

We can look at other classes of equations we can solve for.

$$A(x,y)dx + B(x,y)dy = 0 \tag{34}$$

Consider an equation of the above from with A and B both being homogeneous functions of degree r. Recall a homogeneous function obeys

$$A(cx, cy) = c^r A(x,y) \tag{35}$$

As an example

$$A = x^2 + yx, \qquad B = y^2 \tag{36}$$

Both functions are homogeneous functions of degree 2.

For this type of equation for we can generate a separable equation through the substitution

$$y = vx \tag{37}$$

Consider our example above we find

$$ydx + (2\sqrt{xy} - x)dy = 0 \tag{38}$$

In this example r=1, let y=vx and substitute in dy = vdx + xdv (assume an exact differential).

$$vxdx + (2x\sqrt{v} - x)(vdx + xdv) = 0 \Rightarrow \frac{2\sqrt{v} - 1}{2v^{3/2}} dv = -\frac{1}{x}dx \tag{39}$$

And this equation is separable and can be solved with integration!

Final Example

Another example of the same class of problem.

$$(ax + by + c)dx + (ex + fy + gdy) = 0 \tag{40}$$

Where a,b,c,e,f,g are all constants. But ax + by + c is not homogeneous because c is a lone constant (same with g). Consider x = X + α and y = Y + β, dx = dX, and dy = dY. Substitute in these definitions we find

$$(aX + a\alpha + bY + b\beta + c)dX + (eX + e\alpha + fY + f\beta + g)dY = 0 \tag{41}$$

We can find α and β,

$$\begin{aligned} a\alpha + b\beta + c = 0 \\ e\alpha + f\beta + g = 0 \end{aligned} \tag{42}$$

This generates

$$(aX + bY)dX + (eX + fY)dY = 0 \tag{43}$$

Where we have a homogeneous equation, and we can solve using the methods from before.