

Chem237: Lecture 4

Shane Flynn

4/10/18

More Series

The alternating series has some interesting convergence properties, it takes the form

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(1 + 1) = 2 \quad (1)$$

If we start to look at some of the explicit values of the series we see

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (2)$$

With general numbers we know that addition and subtraction commute i.e. $A + B = B + A$, so we would expect that if we re-arrange the order in which we take our infinite summation we would still produce the same result at the end. So consider a fancy summation scheme for the alternating series where we group terms

$$\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \frac{1}{2} + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}\right) - \frac{1}{4} \dots \quad (3)$$

So we break the summation into positive and negative terms, defining a set of partial summations over the positive and then negative terms. If you numerically evaluate the summations in this way you will find that the summation appears to approach 1.5 as n goes to ∞ . In fact it turns out that you can re-arrange the terms in any sort of pattern, and find the series converges to any value you want. This is a consequence of the series not being absolutely convergent! The commutation of terms applies to finite numbers, it is not true for an infinite series, meaning $A + B \neq B + A$ over an infinite interval, it is only true for finite numbers. If the series is absolutely convergent however, then the series converges to the same value no matter what, and changing the order of the summation will not effect the result.

Power Series

The purpose of the power series is to expand a function in terms of x .

$$\sum_{n=0}^{\infty} a_n x^n = f(x) \quad (4)$$

If we have a Taylor series of a known function x then things are simple for example

Geometric

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (5)$$

Exponential

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad (6)$$

Sine

$$\sum_{n=0}^{\infty} \frac{x^{2k+1}(-1)^{k+1}}{(2k+1)!} = \sin(x) \quad (7)$$

Cosine

$$\sum_{n=0}^{\infty} \frac{x^{2k}(-1)^k}{(2k)!} = \cos(x) \quad (8)$$

Logarithm

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x) \quad (9)$$

If we are given a series we do not recognize, we can try applying a transformation to make it into a recognized form, such as taking a derivative or an integral of the original series.

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \quad (10)$$

We can write a new series, using the derivative.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (11)$$

Now consider an integral

$$\int_1^x dx f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad (12)$$

Consider for example

$$\sum_{n=1}^{\infty} n x^n \quad (13)$$

This is similar to a geometric series in form.

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \Rightarrow \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \frac{d}{dx} \left(\frac{1}{1-x} \right) \\ \sum_{n=0}^{\infty} n x^{n-1} &= \frac{1}{(1-x)^2} \end{aligned} \quad (14)$$

To make this last line look like our example we just need to multiply by x and change the summation bounds

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad (15)$$

As another example this time consider

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad (16)$$

This expression again looks similar to a geometric series. Consider what happens if we take an integral of a geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \int dx \frac{1}{1-x} = \quad (17)$$

If we divide by x we reproduce the problem.

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \Rightarrow \int dx \frac{1}{1-x} \Rightarrow \\ \int dx \left(\frac{1}{1-x} \right) \left(\frac{1}{x} \right) &= \int dx \sum_{n=0}^{\infty} x^n \left(\frac{1}{x} \right) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n} \left(\frac{1}{x} \right) \\ \sum_{n=1}^{\infty} \frac{x^n}{n} &= \int dx \left(\frac{1}{1-x} \right) \frac{1}{x} \Rightarrow \ln(1-x) \end{aligned} \quad (18)$$

Another example, we can start using power series, consider a function of x .

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (19)$$

Where $S(1)$ would be evaluated as

$$S(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (20)$$