Chem237: Lecture 3

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Stuff

Series

Chapter 2 discusses tests for convergence of series and methods to obtain the sum of a series in closed form. A series is a sum of various numbers, or elements of a sequence. A sequence is defined by :

$$a_1, a_2, \dots a_n \tag{1}$$

Where n is the total number of terms in the sequence. a_n can be real or complex. A series is defined by :

$$S = \sum_{n=1}^{\infty} a_1 + a_2 \dots + a_n \tag{2}$$

A series is the sum of infinite number of terms possible in the sequence. A partial sum is the summation of parts of sequence:

$$S_N = \sum_{n=1}^{N} a_n \tag{3}$$

Where capital N is the highest number of the sequence. With these two definitions we can state that a series 'S' is the limit of a partial sum S_N as N approaches infinity

$$S = \lim_{N \to \infty} S_N \tag{4}$$

Convergence and Divergence

Absolute convergence occurs in a series if :

$$S = \sum_{n=1}^{\infty} |a_n| \tag{5}$$

The absolute value of the sequence equals a finite number. A necessary condition for convergence is that the sequence approaches 0 as the limit goes to infinity :

$$\lim_{N \to \infty} a_n = 0 \tag{6}$$

This means that the end of the sequence equals to 0 in order to get a finite number, and thus convergence. Consider the series :

$$S(x) = \sum_{n=0}^{\infty} x^n \tag{7}$$

If $x < 1$ then this series converges.
If $x > 1$ then this series diverges.

A divergent series does not converge to a finite number and neither does the limit. An example of a divergent series is :

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{8}$$

However if we consider a partial sum of these sequence it does converge :

$$S(N) = \sum_{n=1}^{N} \frac{1}{n} \approx \frac{1}{N} \tag{9}$$

If a series but diverges if taking the absolute value of the sequence but converges without the absolute value of the entire sequence, this is called **Conditional Convergence**. An example of this series is :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tag{10}$$

Geometric Series

A geometric series : A geometric series has a constant ratio between the terms.

$$\begin{aligned} &\sum_{n=0}^{\infty} x^n \\ S_N = \sum_{n=0}^{N} x^n = x^0 + x^1 + x^2 + x^3 \dots \end{aligned} \tag{11}$$

Consider the geometric series at 1 term higher which can be described one of two ways :

$$\begin{aligned} S_{N+1} &= x^{N+1} + S_N \\ S_{N+1} &= xS_N + 1 \end{aligned} \tag{12}$$

Solving the system of linear equation gives you :

$$S_N = \frac{1 + x^N}{1 - x} \tag{13}$$

Using this equation consider $|x| < 1$:

$$S = \lim_{N \to \infty} S_N = \frac{1}{1 - x} \tag{14}$$

If $|x| \leq 1$ then the series diverges.

D'Alembert-Lauche test

Sometimes known as the 'ratio test' , tests the convergence of real numbers. Consider the sequence :

$$\sum_{n=1}^{\infty} a_n \tag{15}$$

If :

$$\frac{a_{n+1}}{a_n} < 1 \tag{16}$$

The series **converges absolutely**. If > 1 diverges. If it $= 1$, then we must choose a different method.

Integral Test for Convergence

This test is used to test non-negative terms for convergence., consider a sequence of continuous function $f(n)$

$$\sum_{n=1}^{\infty} f(n) \tag{17}$$

with :

$$\int_1^{\infty} f(x)dx \tag{18}$$

Reinman Zeta Function

$$\mathcal{L}(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \sum_{n=1}^{\infty} = \frac{1}{n^s} \tag{19}$$

Where s is an integer of a simple real variable ... Consider a real s.

$$\begin{gathered} \frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^s n \to \infty : \left(1 + \frac{1}{n}\right)^{-s} = \left(1 - \frac{s}{n}\right) + \ldots \\ \left(1 + \frac{1}{n}\right)^{-s} = \exp\left[\ln\left(1 + \frac{1}{n}\right)^{-s}\right] = \exp\left[-s * \ln\left(1 + \frac{1}{n}\right)\right] = \exp\left[(-s) * \frac{1}{n}\right] = 1 - \frac{s}{n} \end{gathered} \tag{20}$$

Since this doesn't help us determine whether it converges or diverges we will try the Integral test. Consider the Reinman zeta function defined above as a function of x :

$$f(x) = \frac{1}{x^s} \tag{21}$$

We will now do the integral test :

$$\begin{gathered} \int_1^{\infty} \frac{1}{x^s} = \left(\frac{1}{1-s}\right)\left(\frac{1}{x^{1-s}}\right) \\ \text{If } s > 1 \text{ it converges} \\ \text{If } s < 1 \text{ it diverges} \end{gathered} \tag{22}$$

These rules for s can then be applied to the results of the ratio test.

Alternating Series

If a series is not absolute convergent :

$$\sum_n a_n = a_n = (-1)^n |a_n| \tag{23}$$

Consider the series :

$$S(+1) = S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln(2) \tag{24}$$

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = \ln(1+x) \tag{25}$$