Eigenvalues

Eigenvalue Decomposition

An eigenvalue and eigenvector of a square matrix A are, respectively, a scalar λ and a nonzero vector v that satisfy

$$Av = \lambda v$$

With the eigenvalues on the diagonal matrix $\wedge$ and the corresponding eigenvectors forming the columns of matrix V, you have

$$AV = V \wedge$$

If V is nonsingular, this becomes the eigenvalue decomposition

$$A = V \wedge V^{-1}$$

A good example is the coefficient matrix of the differential equation $dx/dt = Ax'$.

```
A = [0  -6  -1
     6   2  -16
     -5  20 -10]
```

```
A = 3×3
     0    -6    -1
     6     2   -16
    -5    20   -10
```

The solution to this equation is expressed in terms of the matrix exponential $x(t) = e^{tA}x(0)$. The statement

```
lambda = eig(A)
```

```
lambda = 3×1 complex
  -3.0710 + 0.0000i
  -2.4645 +17.6008i
  -2.4645 -17.6008i
```

Produces a column vector containing the eigenvalues of A. For this matrix, the eigenvalues are complex.

The real part of each of the eigenvalues is negative, so $e^{\lambda t}$ approaches zero as t increases. The nonzero imaginary part of rtwo of the eigenvalues, $\pm w$, contributes to oscillary component, $sin(wt)$, to the slution of the differential equation.

With two output arguments, eig computes the eigenvectors and stores the eigenvalue in a diagonal matrix:

```
[V,D]=eig(A)
```

```
V = 3×3 complex
  -0.8326 + 0.0000i   0.2003 - 0.1394i   0.2003 + 0.1394i
  -0.3553 + 0.0000i  -0.2110 - 0.6447i  -0.2110 + 0.6447i
  -0.4248 + 0.0000i  -0.6930 + 0.0000i  -0.6930 + 0.0000i
D = 3×3 complex
  -3.0710 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

```
    0.0000 + 0.0000i  -2.4645 +17.6008i   0.0000 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i  -2.4645 -17.6008i
```

There s a double eigenvalue at $\lambda = 1$. The first and second columns of V are the same. For this matrix, a full set of linearly independent eigenvectors does not exist.

Schur Decomposition

Many advanced matrix computations do not require eigenvalue decomposition. They are based, instead, on the Schur decomposition

$A = USU'$,

where U is an orthogonal matrix and S is a block upper-triangular matrix with 1-by-1 and 2-by-2 blocks on the diagonal. The eigenvalues are revealed by the diagonal elements and blocks of S, while the columns U provide an orthogonal basis, which has much better numerical properties than a set of eigenvectors.

For example, compare the eigenvalue and Schur decomposition of this defective matrix:

```
A = [6   12   19
    -9 -20  -33
     4    9   15];
[V,D]=eig(A)
```

```
V = 3×3 complex
  -0.4741 + 0.0000i  -0.4082 - 0.0000i  -0.4082 + 0.0000i
   0.8127 + 0.0000i   0.8165 + 0.0000i   0.8165 + 0.0000i
  -0.3386 + 0.0000i  -0.4082 + 0.0000i  -0.4082 - 0.0000i
D = 3×3 complex
  -1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 - 0.0000i
```

```
[U,S]=schur(A)
```

```
U = 3×3
   -0.4741    0.6648    0.5774
    0.8127    0.0782    0.5774
   -0.3386   -0.7430    0.5774
S = 3×3
   -1.0000   20.7846  -44.6948
         0    1.0000   -0.6096
         0    0.0000    1.0000
```

The matris A is defective since it does not have a full set of linearly independent eigenvectors (the second and third columns of V are the same). Since not all columns of V are linearly independent, it has a large condition number about ~1e8. However, schur is able to calcuate three different basis vectors in U. Since U is orthogonal, cond(U)=1.

The matrix S has the real eigenvalue as the first entry on the diagonal and the repeated eigenvalue represented by the lower right 2-by-2 block. The eigenvalues of the 2-by-2 block are also eigenvalues of A:

```
eig(S(2:3,2:3))
```

```
ans = 2×1 complex
   1.0000 + 0.0000i
   1.0000 - 0.0000i
```