Danmarks Tekniske Universitet

Function spaces and mathematical analysis
Course number: 01325

Homework 3

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Exercise 3.8

i)

We have

$$V = \left\{ \{x_k\}_{k=1}^{\infty} | x_k \in \mathcal{C}, \forall k \in \mathcal{N},\ finitely\ many\ x_k\ non-zero \right\} \tag{1}$$

$$\ell^{\infty}(\mathcal{N}) = \left\{ \{x_k\}_{k=1}^{\infty} | x_k \in \mathcal{C}, \sup_{k \in \mathcal{N}} |x_k| < \infty \right\} \tag{2}$$

Consider $\boldsymbol{x} = \{x_k\}_{k=1}^{\infty} \in V$. Then there must exists a $N \in \mathcal{N}$ such that

$$x_k = 0, \qquad \forall k \geq N \tag{3}$$

Since $|x_k| < \infty,\ \forall x_k$ it follows trivially that

$$\max_{k \in \mathcal{N}} |x_k| = \sup_{k \in \mathcal{N}} |x_k| < \infty \tag{4}$$

Since $\boldsymbol{x} \in V$ was chosen freely, we conclude that $V \subset \ell^{\infty}(\mathcal{N})$.

ii)

Let $\boldsymbol{x} = \{x_k\}_{k=1}^{\infty} \in V$ and $\boldsymbol{y} = \{y_k\}_{k=1}^{\infty} \in V$. Then $\exists N \in \mathcal{N}$ s.t.

$$x_k = y_k = 0, \quad \forall k \geq N. \tag{5}$$

Let $\alpha, \beta \in \mathcal{C}$, then

$$|\alpha x_k + \beta y_k| \leq |\alpha x_k| + |\beta y_k|. \tag{6}$$

We know that all $x_k \in \mathcal{C}$ and $y_k \in \mathcal{C}$, so

$$|\alpha x_k| + |\beta y_k| \leq |\alpha| \sup |x_k| + |\beta| \sup |y_k| < \infty. \tag{7}$$

Specifically we have

$$\sup |\alpha \boldsymbol{x} + \beta \boldsymbol{y}| < \infty \tag{8}$$

Hence $\alpha \boldsymbol{x} + \beta \boldsymbol{y} \in V$. We now conclude that V is in fact a subspace (via lemma 1.2.7).

iii)

Consider $\boldsymbol{x} \in \ell^2(\mathcal{N})$ s.t. $x_k = 1,\ \forall k$. Further consider $\boldsymbol{y} \in V$. Since $\boldsymbol{y}$ consists of finitely many non-zeros entries we have

$$\exists N \in \mathcal{N}\ :\ y_k = 0,\ \forall k \geq N. \tag{9}$$

Then

$$\sup_{k<N} |x_k - y_k| = 0, \tag{10}$$

$$\sup_{k \geq N} |x_k - y_k| \geq 1. \tag{11}$$

Naturally it follows

$$||\boldsymbol{x} - \boldsymbol{y}||_{\infty} \geq 1. \tag{12}$$

Hence V is not dense in $\ell^{\infty}(\mathcal{N})$.

iv)

Consider $\boldsymbol{x} \in \ell^2(\mathcal{N})$ s.t. $x_k = \frac{1}{k}$. Further consider $\boldsymbol{y}_n \in V$ s.t.

$$y_k = \begin{cases} \frac{1}{k} & k < n \\ 0 & k \geq n. \end{cases} \tag{13}$$

Now we can directly compute

$$||\boldsymbol{x} - \boldsymbol{y}_n||_\infty = \left\|(0, 0, .., \frac{1}{n}, \frac{1}{n+1}, ...)\right\|_\infty = \frac{1}{n}. \tag{14}$$

Notice we are now in a situation where

$$\lim_{n\to\infty} ||\boldsymbol{x} - \boldsymbol{y}_n||_\infty = 0, \tag{15}$$

but $\boldsymbol{x} \notin V$. By lemma 2.2.3 V cannot be closed.

Exercise 5.19

i)

Consider $f \in L^1(0,2)$, where

$$L^1(0,2) = \left\{ f :]0,2[\to C \mid \int_0^2 |f(x)| dx < \infty \right\} \tag{16}$$

Additionally

$$||f||_{L^1(0,2)} = \int_0^2 |f(x)| dx. \tag{17}$$

We want to show $||Tf||_{L^1(0,2)} < \infty$, i.e.

$$\int_0^2 |(Tf)(x)| dx < \infty, \tag{18}$$

hence $|(Tf)(x)|$ must be bounded! We have

$$|(Tf)(x)| = \left| \int_0^x t f(t) dt \right| \leq \int_0^2 |t f(t)| dt. \tag{19}$$

We now show that

$$\int_0^x |t f(t)| dt < \infty, \quad \forall f \in L^1(0,2),\ \forall x \in [0,2]. \tag{20}$$

We have

$$\int_0^x |t f(t)| dt \leq \int_0^x |t| \cdot |f(t)| dt \leq \int_0^2 2|f(t)| dt = 2 \int_0^2 |f(t)| dt, \tag{21}$$

since $x \in [0,2]$. Since $f \in L^1(0,2)$, we have per definition that

$$2 \int_0^2 |f(t)| dt = 2||f||_{L^1(0,2)} < \infty. \tag{22}$$

Inserting into the original expression, we get

$$|(Tf)(x)| \leq 2||f||_{L^1(0,2)}. \tag{23}$$

So,

$$||(Tf)(x)|| = \int_0^2 |(Tf)(x)| dx \leq \int_0^2 2||f|| dx = 4||f|| < \infty \tag{24}$$

Hence $Tf \in L^1(0,2)$ if $f \in L^1(0,2)$.

ii)

To show linearity, we consider $f, g \in L^1(0,2)$.

$$\begin{aligned} (T(f+g))(x) &= \int_0^x (t(f+g))(t) dt & (25) \\ &= \int_0^x t f(t) + t g(t) dt & (26) \\ &= \int_0^x t f(t) dt + \int_0^x t g(t) dt & (27) \\ &= (Tf)(x) + (Tg)(x). & (28) \end{aligned}$$

Now consider $\alpha \in C$, then

$$(T(\alpha f))(x) = \int_0^x t(\alpha f)(t)dt \tag{29}$$

$$= \alpha \int_0^x tf(t)dt \tag{30}$$

$$= \alpha Tf(x). \tag{31}$$

So T is linear. In **i)** we showed that $||Tf|| \leq 4||f||$, so T is a linear and bounded operator cf definition 2.4.1 with $||T|| \leq 4$.