Extra exercises – 01325 Mathematics 4

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Problem 1: Let V denote the subset of $C[-\pi,\pi]$ consisting of all finite linear combinations of functions

$$1, \cos x, \cos 2x, \ldots, \cos nx, \ldots, \sin x, \sin 2x, \ldots \sin nx, \ldots.$$

(i) Show that V is a subspace of $C[-\pi,\pi]$.

(ii) Is V closed in $C[-\pi,\pi]$? Hint: from Fourier analysis (e.g., example 6.17 in the MAT 2 book from 2009) it is known that for $x \in [-\pi,\pi]$,

$$\left| x^2 - \left(\frac{\pi^2}{3} + 4\sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos nx \right) \right| \le 4 \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

You may use this result without proof.

Problem 4: Consider the following functions defined on $\mathbb{R}$ (see p.20 for the definition of $\chi_{[a,b]}$):

$$\begin{aligned} f_1(x) &= x, \\ f_2(x) &= x^2 - 1, \\ f_3(x) &= (x^2-1)\chi_{[-1,2]}(x), \\ f_4(x) &= (x^2-1)\chi_{[-1,1]}(x), \\ f_5(x) &= e^{-|x|}, \end{aligned}$$

(i) Make a rough sketch of the graph of each of the functions.

(ii) Which functions have compact support? Determine the support for these functions.

(iii) Which functions belong to $C_0(\mathbb{R})$?

(iv) Which functions belong to $C_c(\mathbb{R})$?

(v) Which functions belong to $L^1(\mathbb{R})$?

Problem 6:

(i) Show that we can define an equivalence relation on $L^1(\mathbb{R})$ by

$$f \sim g \Leftrightarrow \int_{-\infty}^{\infty} |f(x) - g(x)| \, dx = 0.$$

Now let $\sim$ denote an equivalence relation on any set V.

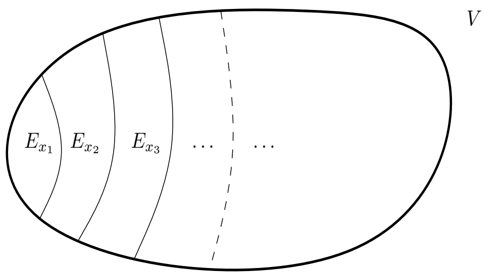
(ii) Show that for any $x_1, x_2 \in V$ we have

$$E_{x_1} = E_{x_2} \quad \text{or} \quad E_{x_1} \cap E_{x_2} = \emptyset.$$

(iii) Argue that there exists a collection $\{E_{x_i}\}_{i \in I}$ of elements in V such that

$$V = \bigcup_{i \in I} E_{x_i} \quad \text{and} \quad E_{x_i} \cap E_{x_j} = \emptyset \text{ if } i \neq j.$$

In words, the result in (iii) shows that an equivalence relation splits the set V into a disjoint union of equivalence classes. In the case of $L^1(\mathbb{R})$ we obtain a normed space if we identify all elements within a given equivalence class. Thus, strictly speaking $L^1(\mathbb{R})$ is not a space of functions, but a space of equivalence classes of functions.



Problem 7 Let $p \in [1, \infty[$ and consider the mapping

$$T : L^p(-2, 2) \to L^p(-2, 2), \; (Tf)(x) := xf(x).$$

(i) Show that T indeed maps $L^p(-2, 2)$ into $L^p(-2, 2)$.

(ii) Show that T is linear and bounded.

Our aim is now to find the precise value of the number $||T||$.

(iii) Let $f = \chi_{[a,2]}$ for some $a \in]0, 2[$ and show that

$$a \, ||f||_p \leq ||Tf||_p \leq 2 \, ||f||_p.$$

Hint: Do not calculate $||f||_p$ and $||Tf||_p$ explicitly, but look at the expression for $||Tf||_p$ for the given function $f = \chi_{[a,2]}$ and make estimates, using that only the values of x belonging to the interval $[a, 2]$ are relevant.

(iv) Look at the inequalities you derived in (iii). Is it possible that, for example, $||T|| = 1.8$?

(v) Find the exact value of the number $||T||$.

Problem 8 Consider the function

$$f(x) = \begin{cases} \frac{1}{|x|^{1/3}}, & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(i) Make a draft of the function f.

(ii) Check whether there exists any $p > 0$ for which $f \in L^p(\mathbb{R})$.

Problem 9 Consider the vector space $\mathbb{C}^2$. For $\mathbf{x} \in \mathbb{C}^2$, we write $\mathbf{x} = (x_1, x_2),\ x_1, x_2 \in \mathbb{C}$.

(i) Does the expression

$$||(x_1, x_2)|| := |x_1|$$

define a norm on $\mathbb{C}^2$?

(ii) Does the expression

$$||(x_1, x_2)|| := |x_1| - |x_2|$$

define a norm on $\mathbb{C}^2$?

(iii) Does the expression

$$||(x_1, x_2)|| := \max\left(|x_1|, |x_2|\right)$$

define a norm on $\mathbb{C}^2$?

Problem 10. (i) Let D be a dense subset of a Hilbert space H. Show that if a subspace V of H satisfies $D \subset \overline{V}$, then V is dense in H, i.e., $\overline{V} = H$.

(ii) Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal sequence in $L^2(\mathbb{R})$. Show that if $C_c(\mathbb{R}) \subset \overline{\operatorname{span}\{e_k\}_{k\in\mathbb{N}}}$, then $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Problem 11. The aim of this exercise is to give a direct proof (without using the results of Chapter 9) of the fact that the Haar MRA $\{V_j\}_{j\in\mathbb{Z}}$ considered in Example 8.2.3 and Exrecise 8.1 satisfies

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

To prove this, you may use the following hints/strategy:

1) Note that $g \in \bigcup_{j\in\mathbb{Z}} V_j$ means that $g \in V_{j'}$ for some $j' \in \mathbb{Z}$.

2) We need to show that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, see Def. 2.3.1. Hence (argue why), we need to show :

$$\forall f \in L^2(\mathbb{R})\ \forall \varepsilon > 0\ \exists j' \in \mathbb{Z} : \|f - g\|_2 < \varepsilon \quad \text{for some } g \in V_{j'}.$$

3) Use Problem 10 to argue that it suffices to show

$$\forall f \in C_c(\mathbb{R})\ \forall \varepsilon > 0\ \exists j' \in \mathbb{Z} : \|f - g\|_2 < \varepsilon \quad \text{for some } g \in V_{j'}.$$

4) You may use Heine-Cantor theorem without proof. This theorem implies that $f \in C_c(\mathbb{R})$ is *uniformly* continuous. Use this property to show that given $\varepsilon > 0$ and $f \in C_c(\mathbb{R})$, you can find $j' \in \mathbb{Z}$ such that $\|f - g\|_2 < \varepsilon$ for some $g \in V_{j'}$

5) For a given $f \in C_c(\mathbb{R})$ the approximation $g \in V_{j'}$ can be chosen as the average of f on each $[2^{-j'}k, 2^{-j'}(k+1)[$. This way g is piecewise constant and in $V_{j'}$

Problem 12. We will consider the continuous wavelet transform C_ψ on $L^2(\mathbb{R})$. Let $f \in L^2(\mathbb{R})$ be given. For each $(a, b) \in]0, \infty[\times\mathbb{R}$, we define $C_\psi(f)(a, b)$ by

$$C_\psi(f)(a, b) = \langle f, T_b D_a \psi \rangle .$$

Hence, $C_\psi(f)$ is a function of $(a, b) \in]0, \infty[\times\mathbb{R}$ given by:

$$C_\psi(f) = (a, b) \mapsto \langle f, T_b D_a \psi \rangle .$$

Written out, the definition reads for $(a, b) \in]0, \infty[\times\mathbb{R}$:

$$C_\psi(f)(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \overline{\psi\Big(\frac{x - b}{a}\Big)} dx$$

In this exercise we will show that the continuous wavelet transform resolves (locates) jump discontinuities.

We use the following variant of the Haar wavelet:

$$\psi(x) = \begin{cases} 1 & x \in [-\frac{1}{2}, 0], \\ -1 & x \in]0, \frac{1}{2}], \\ 0 & x \notin [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

(i) Show that $\operatorname{supp} T_b D_a \psi = [-\frac{a}{2} + b, \frac{a}{2} + b]$.

(ii) We assume that the signal $f \in L^2(\mathbb{R})$ is piecewise constant and has compact support. Write up a general expression for such a signal f. E.g., denote the points of jump discontinuities by x_k, $k = 0, \ldots, N$, and let f_k, $k = 0, \ldots, N-1$, denote the function value of f on the interval $[x_k, x_{k+1}[$.

(iii) Find an expression for $C_\psi(f)(a, b)$ for given (a, b), where you use the definition of $T_b D_a \psi$, i.e., $f(x)$, but not ψ, is allowed to appear in the final expression.

(iv) Show that for $a > 0$ and $b \in \mathbb{R}$ such that $|b - x_k| > \frac{a}{2}$ for each $k = 0, \ldots, N$, we have $C_\psi(f)(a, b) = 0$.

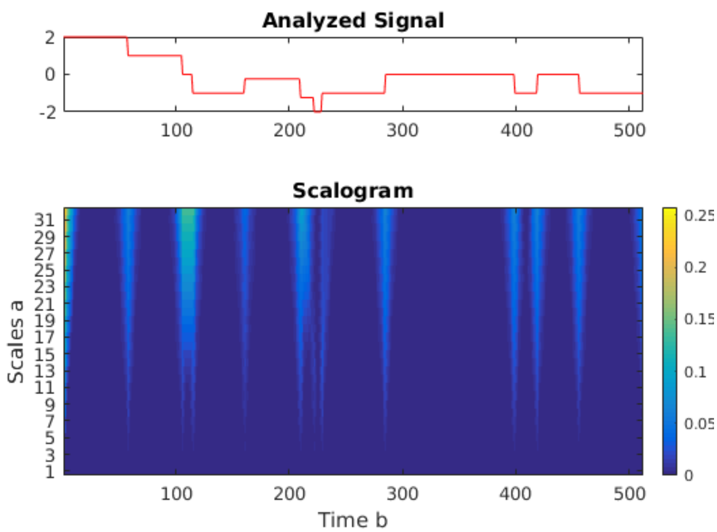
(v) We now let $b \in \mathbb{R}$ be fixed (but arbitrary) and consider $C_\psi(f)(a, b)$ as $a \to 0$. Define $\ell := \min_k |b - x_k|$. Show that when $b \neq x_k$ $(k = 0, \ldots, N)$, then $C_\psi(f)(a, b) = 0$ for small enough a (i.e., $a < 2\ell$). On the other hand, when $b = x_k$ for some $k = 0, \ldots, N$, then

$$C_\psi(f)(a, b) = c\, a^s \quad \text{for } a \to 0. \tag{1}$$

Show this and determine c (which depends on the function values of f) and s. Sketch $|C_\psi(f)(a, b)|$ as a function of $a > 0$ for the two values of b considered in the first part of this question.

(vi) Explain in your own words the output of the Matlab script:

```
%% CWT of piecewise constant signal
L=512; t=linspace(0,1,L); int1=randi(L/2,2,1);
% Construct random piecewise constant signal
signal=round(cos(t*3*pi))+round(cos(t*randn*pi));
signal=signal+randn * [zeros(1,int1(1,1)),ones(1,int1(2,1)),
                       zeros(1,L-int1(1,1)-int1(2,1))];
% Compute CWT using Haar wavelet
[cw1,sc] = cwt(signal,1:32,'haar','scal');
title('Scalogram')
```



Problem 13. Let $L > 0$. Assume $\lambda \in \mathbb{R}$.

(i) Find all eigenvalues and eigenfunctions for the eigenvalue problem:

$$y'' + \lambda y = 0, \quad x \in [0, L], \quad y(0) = 0, \quad y(L) = 0.$$

(ii) Prove that $\{\frac{1}{c_n} \sin(\frac{n\pi x}{L})\}_{n=1}^{\infty}$ is an ONB for $L^2([0, L])$ for some choice of c_n. Determine c_n.

(iii) Write down formula (4.26) explicitly for this ONB.

Problem 14. Let $r(x) = \mathrm{e}^{-x^2}$. Note that $\int_{\infty}^{\infty} r(x)dx = \sqrt{\pi}$.

(i) Find the first three Hermite polynomials H_0, H_1, and H_2, see Section 11.4. Verify in CAS (e.g., Maple) that they are orthogonal in $L_r^2(\mathbb{R})$. Compute (also in Maple) the norm in $L_r^2(\mathbb{R})$ for H_0, H_1, and H_2.

(ii) Let $f : \mathbb{R} \to \mathbb{C}$ be given by

$$f(x) = \max(0, 1 - \left|x - \tfrac{1}{2}\right|) \quad \text{for all } x \in \mathbb{R}.$$

Argue that $f \in L^2(\mathbb{R})$.

(iii) Let $\{e_\ell\}_{\ell=0}^{\infty}$ denote the Hermite ONB from Theorem 11.4.3. Compute and plot (in Maple) an N-term approximation $P_N f$, for $N = 1, \dots, 15$, given by

$$P_N f := \sum_{\ell=0}^{N-1} \langle f, e_\ell \rangle_{L^2} e_\ell$$

with f as in (ii). Argue that $\|f - P_N f\|_{L^2} \to 0$ as $N \to \infty$.

(iv) (optional) Compute (approximately) the absolute L^2-error

$$\|f - P_N f\|_{L^2}$$

as a function of $N = 1, \dots, 10$. Hint: Use Parseval's identity

Problem 101 (extension of Problem 221 from the Exam 2011) Consider the function

$$f(x) := e^x \chi_{[0,1]}(x).$$

(i) Calculate the convolution $(f * f)(x)$, $x \in \mathbb{R}$.

(ii) Calculate the Fourier transform

$$\widehat{f * f}(\gamma), \ \gamma \in \mathbb{R}.$$

Problem 102 Let $r : [a, b] \to]1, \infty[$ be a continuous function and consider the weighted L^2-space

$$L^2_r(a, b) = \{f : [a, b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 r(x)\, dx < \infty\}.$$

We know (you are not expected to show this) that $L^2_r(a, b)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_r = \int_a^b f(x)\overline{g(x)}r(x)\, dx.$$

(i) Show that $L^2_r(a, b)$ is a subspace of $L^2(a, b)$.

(ii) Show that if $\{e_k(x)\}_{k=1}^{\infty}$ is an orthonormal system in $L^2_r(a, b)$, then $\{e_k(x)\sqrt{r(x)}\}_{k=1}^{\infty}$ is an orthonormal system in $L^2(a, b)$.

Problem 103 Consider the Legendre differential equation, which, for a given and fixed parameter $\lambda \in \mathbb{R}$, is

$$(1 - x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \lambda u = 0. \tag{1}$$

Assume that (1) has a power series solution,

$$u(x) = \sum_{k=0}^{\infty} c_k x^k.$$

(i) Show that for $k \geq 2$,

$$c_{k+2}(k + 2)(k + 1) - c_k(k(k + 1) - \lambda).$$

(ii) Show that if (1) has a solution $u \neq 0$ which is a polynomial, then necessarily the parameter λ has the form

$$\lambda = \ell(\ell + 1)$$

for some $\ell \in \{0, 1, 2, \dots\}$.

Problem 104 Consider the mappings acting on differentiable functions $f : \mathbb{R} \to \mathbb{C}$ by

$$(Df)(x) := f'(x),$$

and

$$(Mf)(x) := xf(x).$$

(i) Show that D and M are linear mappings

(ii) Calculate the operator

$$DM - MD.$$

The operator $DM - MD$ is called the *commutator* of the operators D and M, and are used extensively in mathematical analysis and physics. It is usually denoted by $[D, M]$.

Problem 105 We will consider the modulation operator E_b for complex values of the parameter b. That is, for $b \in \mathbb{C}$ we let E_b act on the function $f : \mathbb{R} \to \mathbb{C}$ by

$$(E_b f)(x) := e^{2\pi i b x} f(x), \; x \in \mathbb{R}.$$

(i) Show that the function

$$f(x) := \frac{1}{\sqrt{1 + x^2}}$$

belongs to $L^2(\mathbb{R})$.

(ii) For which values of the parameter $b \in \mathbb{C}$ does the operator E_b define a bounded operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$?

Hint: Write $b = b_1 + ib_2$ and find the values of $b \in \mathbb{C}$ for which $E_b f \in L^2(\mathbb{R})$ for the function f in (i).

Problem 106 (Extension of Problem 222, Exam 2011) Consider a complex Hilbert space $\mathcal{H}$ and a bounded linear operator

$$T : \mathcal{H} \to \mathcal{H}.$$

We say that a number $\lambda \in \mathbb{C}$ is an eigenvalue for T if there exists an eigenvector, i.e., a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$T\mathbf{v} = \lambda\,\mathbf{v}.$$

(i) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue for T, and let $\mathbf{v}$ denote a corresponding eigenvector with $||\mathbf{v}|| = 1$. Calculate the numbers

$$\langle T\mathbf{v}, \mathbf{v}\rangle \quad \text{and} \quad \langle \mathbf{v}, T\mathbf{v}\rangle.$$

(ii) Assume that T is self-adjoint. Show that all the eigenvalues are real numbers.

Now assume that $\mathcal{H}$ has an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and that for each $k \in \mathbb{N}$ there exists a number $\lambda_k \in \mathbb{R}$ such that

$$Te_k = \lambda_k e_k.$$

(iii) Show that for any $f \in \mathcal{H}$,

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k\rangle\, e_k.$$

Problem 107 Assume that a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx}, \tag{2}$$

where the series $\sum_{n=-\infty}^{\infty} d_n$ is absolutely convergent. The goal of this exercise is to show that then the coefficients d_n are necessarily the well known Fourier coefficients on complex form.

(i) Show that the series in (2) has a convergent majorant series.

(ii) Calculate for each $m, n \in \mathbb{Z}$ the integral

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx.$$

Hint: consider the cases $m = n$ and $m \neq n$ separately.

(iii) Prove using (ii) that for any $m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} d_n e^{inx}\, e^{-imx}\, dx = 2\pi d_m.$$

Hint: The order of integration and summation can be interchanged by Theorem 5.34 in the MAT 2 book from 2012 (Theorem 5.33 in the version from 2009), or see Theorem 5.3.7 in the MAT4 book.

(iv) Conclude using (iii) that

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \ \forall n \in \mathbb{Z}.$$

Problem 108 (*This is a quite unusual exercise.*)

Consider the function

$$f(x) := \frac{1}{1+x^2}, \ x \in \mathbb{R}.$$

(i) Show that $f \in L^2(\mathbb{R})$.
Hint: By Exercise 5.3 we know that $f \in L^1(\mathbb{R})$.

(ii) Does the function f have any vanishing moments?

(iii) Argue that there exists a function $\phi \in L^2(\mathbb{R})$ such that

$$\widehat{\phi}(\gamma) = \frac{1}{1+\gamma^2}, \gamma \in \mathbb{R}. \tag{3}$$

Hint: Use a result in Section 7.2

(iv) One can show that the function ϕ in (iii) belongs to $L^1(\mathbb{R})$. Does the function ϕ have any vanishing moments?

(v) Does the function ϕ in (iii) generates a multiresolution analysis?

Problem 109 Does the expression

$$||\{x_k\}_{k=1}^{\infty}|| = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

define a norm on $\ell^p(\mathbb{N})$ for $p \in]0,1[$?

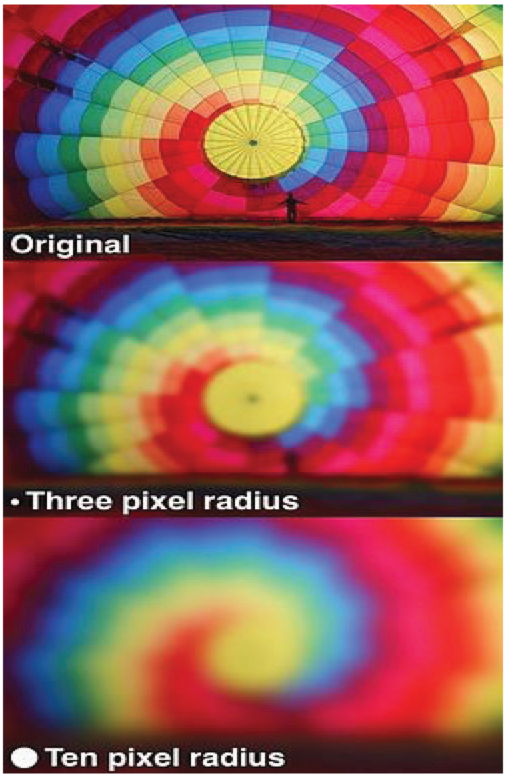
Problem 110 In image analysis, a *blur* is used to reduce the details in an image, or remove noise. Mathematically, a blur is described via an integral operator of the type discussed in Example 2.4.3 (however, with the interval $[a,b]$ replaced by a 2-dimensional set because an image is 2-dimensional).

Typically, a *Gaussian blur* is used. This means that the relation between the given signal ("image") x and the resulting blurred signal y is given by

$$y(t) = \int_a^b k(s,t)\, x(s)\, ds,$$

where

$$k(s,t) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(s-t)^2}{2\sigma^2}}$$



for a suitable chosen parameter $\sigma > 0$. The effect of the blurring is illustrated on the Figure for, respectively, a small and a larger value of σ.

Motivated by the blurring example, we want to consider a linear operator K, that maps a given signal x onto a new signal y, given by

$$y(t) = (Kx)(t) := \int_{-\infty}^{\infty} e^{-(s-t)^2} \, x(s) \, ds$$

We would like to make sure that the integral defining the function y actually is well defined for all $t \in \mathbb{R}$. The purpose of this exercise is to check whether this is the case for various vector spaces. Note that in order to show that an operator is *not* well defined on a certain vector space, it is enough to find just *one* function in the vector space, for which it is not well defined.

(i) Is $y(t) := (Kx)(t)$ well defined for all $x \in C(\mathbb{R})$ and all $t \in \mathbb{R}$?

(ii) Is $y(t) := (Kx)(t)$ well defined for all $x \in C_0(\mathbb{R})$ and all $t \in \mathbb{R}$?
Hint: you might use that

$$\int_{-\infty}^{\infty} e^{-u^2} \, du = \sqrt{\pi}.$$

(iii) Is $y(t) := (Kx)(t)$ well defined for all $x \in L^1(\mathbb{R})$ and all $t \in \mathbb{R}$?

Problem 111 In this exercise we give the formal definition of the composition of two operators, as used, e.g., in Exercise 3.14 and Exercise 4.17. Let $\mathcal{H}$ denote a Hilbert space and consider two bounded linear operators

$$S, T : \mathcal{H} \to \mathcal{H}.$$

We define the composed operator $ST : \mathcal{H} \to \mathcal{H}$ by

$$ST\mathbf{v} := S(T\mathbf{v}), \ \mathbf{v} \in \mathcal{H}.$$

(i) Show that the operator ST is linear.

(ii) Show that ST is bounded, and that

$$||ST|| \leq ||S|| \, ||T||.$$

Problem 112 Does $||f|| := \sup\{|f(x)| \bigm| x \in \mathbb{R}\}$ define a norm on the vector space $C(\mathbb{R})$?

Problem 113

(i) Give at least two arguments that the differential operator $P : L^1(\mathbb{R}) \to L^1(\mathbb{R}), Pf := f'$ is not well defined.

(ii) Let

$$V := \{f \in L^1(\mathbb{R}) \bigm| f \text{ is differentiable, } f' \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})\},$$

and argue that

$$P : V \to L^1(\mathbb{R}), Pf := f'$$

is well defined and linear.

(iii) Show that the operator P defined in (ii) is unbounded.

Problem 114 Let $\mathcal{H}$ denote a Hilbert space. Show that any closed subspace V of $\mathcal{H}$ itself forms a Hilbert space.

Problem 115 Look at the proof of Proposition 8.2.6. Why does $H_0 \in L^2(0,1)$?

Problem 116 Exercise 2.13 shows that if T is a bounded linear operator on a normed vector space V and W is a subspace of V, then

$$T(\overline{W}) \subseteq \overline{T(W)}. \tag{4}$$

It also shows that if T is invertible and T^{-1} is bounded, then we actually have that

$$T(\overline{W}) = \overline{T(W)}. \tag{5}$$

Intuitively, one could believe that the result in (5) always hold, but this is not true. In order to make a concrete example, it is enough to find an operator with non-closed range (think about this), and this is exactly what we will do now.

Let $\{e_k\}_{k=1}^{\infty}$ denote an orthonormal basis for a Hilbert space $\mathcal{H}$, and consider the vectors $v_k, k \in \mathbb{N}$, defined via

$$v_k := e_k + e_{k+1}.$$

In Problem 227 it is shown (you might do this) that $\overline{span}\{v_k\}_{k=1}^{\infty} = \mathcal{H}$, and that there does not exist complex numbers c_k such that $e_1 = \sum_{k=1}^{\infty} c_k v_k$.

Now, consider the operator T defined by

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, \ T\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k v_k.$$

(i) Show that T is well defined, linear, and bounded.

(ii) Let $W := \ell^2(\mathbb{N})$ and show that (5) does NOT hold.

Problem 117 We recall Taylors theorem. Assume that $f : I \to \mathbb{R}$ and that there exists a constant $C > 0$ such that $|f^n(x)| \leq C$ for all $n \in \mathbb{N}$ and all $x \in I$. Let $x_0 \in I$. Then, for $x \in I$ and $N \in \mathbb{N}$,

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^n(x_0)}{n!}(x - x_0)^n \right| \leq \frac{C}{(N+1)!} |x - x_0|^{N+1}.$$

Now consider the function $f(x) = e^x$, $x \in [0, 50]$.

(i) Find $N \in \mathbb{N}$ such that there exists a polynomial P of degree N such that

$$|f(x) - P(x)| \leq 0.1, \ \forall x \in [0, 50].$$

(ii) Find $N \in \mathbb{N}$ such that there exists a polynomial P of degree N such that

$$|f(x) - P(x)| \leq 0.1, \ \forall x \in [40, 50].$$

(iii) Is it realistic to approximate exponential functions by polynomials over large intervals?

(iv) Can you describe a more convenient approach to approximate exponential functions, or a more general class of functions? Check how your idea works on the function f above.

Problem 118 (extension of Problem in the exam 2014) Given two functions $f, g \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$, consider the function $\Phi(x) := f(x)g(x - a)$.

(i) Show that $\Phi \in L^1(\mathbb{R})$.

The short-time Fourier transform of $f \in L^2(\mathbb{R})$ is defined as a function of two variables, denoted by $V(f)(\cdot, \cdot)$, and given by

$$V(f)(a, \gamma) := \int_{-\infty}^{\infty} f(x)\chi_{[-3,3]}(x - a)e^{-2\pi i x\gamma}\, dx, \ a, \gamma \in \mathbb{R}.$$

(ii) Argue that $V(f)$ is well defined.

(iii) Show that V is a linear operator.

We will consider the function $f(x) := \cos(20\pi x)\, \chi_{[-3,3]}(x)$ and calculate the function $V(f)(a, \gamma)$ for two special choices of the parameter a.

(iv) Let $a = 0$ and calculate the function $V(f)(0, \gamma)$
Hint: You are allowed to use the results in Example 7.1.4, without repeating the calculations.

(v) Let $a = 6$ and calculate the function $V(f)(6, \gamma)$.

Problem 119 Let $\mathcal{H}$ denote a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. Define the *null space* of T by

$$\mathcal{N}_T := \{x \in \mathcal{H} \mid Tx = 0\}.$$

Show that $\mathcal{N}_T$ is a closed subspace of $\mathcal{H}$.

Problem 120: Show that the expression in (3.13) in the book actually defines a norm on $\ell^\infty(\mathbb{N})$.

Problem 121 Consider the linear operator T acting on differentiable functions f by

$$(Tf)(x) := xf'(x).$$

(i) Let

$$V := \{f \in L^2(\mathbb{R}) \,\big|\, f \text{ is differentiable and } [x \to xf'(x)] \in L^2(\mathbb{R})\}.$$

Is T well-defined as a linear operator $T : V \to V$?

(ii) Define a subspace W of $L^2(\mathbb{R})$ such that the operator $T : W \to W$ is well-defined.

(iii) Check that the space you have defined satisfy that $W \neq \{0\}$.

Problem 122 The purpose of this exercise is to return to the sampling issue, and provide a connection between the Hilbert space $L^2(\mathbb{R})$ and the Hilbert space

$$\ell^2(\mathbb{Z}) = \{\{x_k\}_{k\in\mathbb{Z}} \,|\, \sum_{k\in\mathbb{Z}} |x_k|^2 < \infty\}.$$

We will consider the *sampling operator* $\mathcal{S}$, which, for a given function $f \in L^2(\mathbb{R})$, yields the sequence

$$\mathcal{S}f := \{f(n)\}_{n\in\mathbb{Z}}.$$

Thus, the sampling operator applied to a function f simply gives us the sequence of function values $f(n)$, where $n \in \mathbb{Z}$. We will also consider the *average sampling operator* $\mathcal{A}$, which, for a given function $f \in L^2(\mathbb{R})$, yields the sequence

$$\mathcal{A}f := \{\mathcal{A}_n(f)\}_{n\in\mathbb{Z}}, \text{ where } \mathcal{A}_n(f) = \int_{n-1/2}^{n+1/2} f(x)\, dx. \tag{6}$$

Note that as we have seen in Exercise 8.7, the number $\mathcal{A}_n(f)$ is actually the average of the function f over the interval $[n - 1/2, n + 1/2]$.

(i) Make a draft of the function

$$f(x) := \sum_{j=2}^{\infty} \sqrt{j}\, \chi_{[j,j+j^{-3}]}(x) = \sqrt{2}\, \chi_{[2,2\frac{1}{8}]}(x) + \sqrt{3}\, \chi_{[3,3\frac{1}{27}]}(x) + \cdots, \ x \in \mathbb{R}$$

and show that $f \in L^2(\mathbb{R})$.

(ii) Is the sampling operator $\mathcal{S}$ well defined as an operator from $L^2(\mathbb{R})$ into $\ell^2(\mathbb{Z})$? *Hint:* Look at the function in (i).

(iii) Show that the nth coordinate in (6), $\mathcal{A}_n(f)$, actually is well defined for $f \in L^2(\mathbb{R})$, e.g., by showing that

$$\int_{n-1/2}^{n+1/2} |f(x)|\, dx \leq \left(\int_{n-1/2}^{n+1/2} |f(x)|^2\, dx\right)^{1/2} < \infty.$$

(iv) Show that the average sampling operator $\mathcal{A}$ is well defined as an operator from $L^2(\mathbb{R})$ into $\ell^2(\mathbb{Z})$.

(v) Show that $\mathcal{A} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z})$ is linear and bounded.

Similar results as in (iii)-(v) holds if we take averages over intervals of any fixed length T. That is, given any $T > 0$, the operator

$$\mathcal{A} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}),\ \mathcal{A}f := \{\mathcal{A}_n(f)\}_{n\in\mathbb{Z}},\ \text{where } \mathcal{A}_n(f) = \frac{1}{T}\int_{Tn-T/2}^{Tn+T/2} f(x)\,dx$$

is well-defined, linear, and bounded. The result of this exercise sounds like "bad news" because the sampling operator $\mathcal{S}$ does not map $L^2(\mathbb{R})$ into $\ell^2(\mathbb{Z})$. But in fact it is "good news," in the sense that even a "real world sampling is actually an "average sampling"! In fact, any measurement in any physically realizable experiment will take place over a (very small) time interval, and therefore in reality correspond to an average sampling with a very small value of T. Thus, the natural operator to consider is actually $\mathcal{A}$ and not $\mathcal{S}$.

Technical University of Denmark

Written exam. Date: June 6, 2008

Course name: Mathematics 4: Real Analysis, Course no. 01325

Part I : Problems 201 – 204 (6 questions)
Part II: Problems 5 – 7 (6 questions)

The exam consists of 7 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 201 (contained in Ex. 3.9) Consider a sequence $\{w_k\}_{k=1}^{\infty}$ of positive real numbers, and define the weighted ℓ^1-space $\ell^1_w(\mathbb{N})$ by

$$\ell^1_w(\mathbb{N}) := \left\{ \{x_k\}_{k=1}^{\infty} \mid x_k \in \mathbb{C}, \ \ \sum_{k=1}^{\infty} |x_k|\, w_k < \infty \right\}.$$

(i) Show that the expression $||\cdot||$ given by

$$||\{x_k\}_{k=1}^{\infty}|| := \sum_{k=1}^{\infty} |x_k|\, w_k$$

defines a norm on $\ell^1_w(\mathbb{N})$.

We now consider the special choice

$$w_k := 2^k, \ \ k \in \mathbb{N}.$$

(ii) Show that $\ell^1_w(\mathbb{N})$ is a subspace of $\ell^1(\mathbb{N})$.

The set of problems continues!

Problem 202 (Contained in Ex. 6.10) Consider the linear operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Uf)(x) = f(2x-2),\ x \in \mathbb{R}.$$

(i) Show that U is bounded on $L^2(\mathbb{R})$.

(ii) Compute the adjoint operator U^*.

Problem 203 (Ex. 10.6) Consider the symmetric B-splines B_2 and B_3. Find an expression for the Fourier transform of the convolution $B_2 * B_3$, i.e., determine the function

$$\widehat{B_2 * B_3}.$$

Problem 204 (Ex. 4.2) Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|| \cdot ||$. Show that for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$,

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right).$$

Problem 5-7 Taken out, as they are not relevant for Mathematics 4 in its present version.

The set of problems is completed!

Technical University of Denmark

TEST!! Date: May 6, 2009

Course name: Mathematics 4, Course no. 01325

Allowed aids: All aids.

The test consists of 6 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 205 (Ex. 1.7) Consider the function

$$f(x) := -xe^{-x},\ x \in [0, \infty[.$$

Calculate the number

$$\inf\{f(x) \mid x \in [0, \infty[\}.$$

Can "infimum" be replaced by "minimum"?

Problem 206 (Ex. 6.15) For $f \in L^2(-\pi, \pi)$, the complex Fourier coefficients are defined by

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx,\ k \in \mathbb{Z}.$$

Show that the integral defining c_k is well defined for $f \in L^2(-\pi, \pi)$, i.e., that

$$\int_{-\pi}^{\pi} \left| f(x)e^{-ikx} \right| dx < \infty.$$

The set of problems continues!

Problem 207 (Ex. 4.15) Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ denote a sequence in a Hilbert space $\mathcal{H}$. The purpose is to prove Lemma 4.4.4. in the book, which claims that (a) and (b) below are equivalent:

(a) $\{\mathbf{v}_k\}_{k=1}^{\infty}$ is complete;

(b) If $\mathbf{v} \in \mathcal{H}$ and $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$, then $\mathbf{v} = \mathbf{0}$.

Do the following:

(i) Show that if $\mathbf{v} \in \mathcal{H}$ and $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$.
Hint: show first that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \text{span}\{\mathbf{v}_k\}_{k=1}^{\infty}$.

(ii) Use (i) to show that if (a) holds, then (b) holds.

(iii) Prove that if (a) does not hold, then (b) does not hold.
Hint: Let $W := \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$, and use Theorem 4.3.5 to argue that if (a) does not hold, then $W^{\perp} \neq \{\mathbf{0}\}$.

Problem 208 (Ex. 5.17) Let $p \in [1, \infty[$ and consider the mapping

$$T : L^p(\mathbb{R}) \to L^p(\mathbb{R}),\ (Tf)(x) := f(3x + 2).$$

(i) Show that T indeed maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$.

(ii) Show that T is linear and bounded.

(iii) Consider the function

$$f(x) := \frac{1}{x^2}\, \chi_{[1,\infty[}(x),$$

and show that $f \in L^p(\mathbb{R})$ for all $p \in [1, \infty[$.

The set of problems continues!

Problem 209 Let ψ denote the Haar wavelet. We know that ψ can be constructed from the multiresolution analysis generated by the function $\phi = \chi_{[0,1[}$.

(i) Show that for $f \in L^2(\mathbb{R})$,

$$\langle f, T_k\phi\rangle = \int_0^1 f(x+k)\,dx.$$

(ii) Consider a compactly supported function $f \in L^2(\mathbb{R})$ and the expansion (8.14) in the book. Show that the term

$$\sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi$$

actually is a finite sum. *Hint:* Assume that $\operatorname{supp} f \subseteq [-N, N]$ and consider $c_k := \langle f, T_k\phi\rangle$ for $|k| \geq N+1$.

Problem 210 (Ex. 8.11) Let $\phi \in L^2(\mathbb{R})$, and denote its Fourier transform by $\hat{\phi}$.

(i) Show that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\langle \phi, T_k\phi\rangle = \begin{cases} 1 & \text{if } k = 0; \\ 0 & \text{if } k \neq 0. \end{cases}$$

Let

$$\Phi(\gamma) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\gamma+n)|^2.$$

One can show **(you are not expected to do this)** that

$$\langle \phi, T_k\phi\rangle = \int_0^1 \Phi(\gamma) e^{2\pi i k\gamma}\,d\gamma.$$

(ii) Show that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system if and only if $\Phi(\gamma) = 1$ for $\gamma \in \mathbb{R}$.

Hint: By Example 6.4.3 in the book, the numbers $\int_0^1 \Phi(\gamma)e^{2\pi i k\gamma}\,d\gamma$ are Fourier coefficients for the 1-periodic function Φ.

The set of problems is completed!

Technical University of Denmark

Written exam. Date: June 4, 2009

Course name: Mathematics 4, Course no. 01325

Allowed aids: All aids.

The exam consists of 5 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 211 Consider the function

$$f(x) = x - x^2.$$

Determine the number

$$\sup\{f(x) \big|\ x \in [0,4]\}.$$

Problem 212 (Ex. 4.6) Consider the vector space $\ell^p(\mathbb{N})$ for some $p \in [1,\infty[$, equipped with the usual norm

$$||\{x_k\}_{k=1}^{\infty}||_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}.$$

(i) Consider the vectors

$$\mathbf{x} = (1,0,0,\dots), \quad \mathbf{y} = (0,1,0,0,\dots),$$

and show that

$$||\mathbf{x}||_p = ||\mathbf{y}||_p = 1, \quad ||\mathbf{x}+\mathbf{y}||_p = ||\mathbf{x}-\mathbf{y}||_p = 2^{1/p}.$$

(ii) Assume that $p \neq 2$. Show that the norm $||\cdot||_p$ does not come from an inner product. *Hint: use Theorem 4.1.4.*

Problem 213 Consider a Hilbert space $\mathcal{H}$ and let W be a closed subspace of $\mathcal{H}$. Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be an orthonormal basis for W. Define the operator $P : \mathcal{H} \to \mathcal{H}$ by

$$P\mathbf{v} := \sum_{k=1}^{\infty} \langle \mathbf{v}, \mathbf{e}_k \rangle \mathbf{e}_k, \;\; \mathbf{v} \in \mathcal{H}.$$

(i) Show that if $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$, then

$$P(\mathbf{w} + \mathbf{u}) = \mathbf{w}.$$

(ii) Show that if $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$, then

$$||\mathbf{w} + \mathbf{u}||^2 = ||\mathbf{w}||^2 + ||\mathbf{u}||^2.$$

(iii) Show that $||P\mathbf{v}|| \leq ||\mathbf{v}||$ for all $\mathbf{v} \in \mathcal{H}$.
Hint: By Theorem 4.3.5, any $\mathbf{v} \in \mathcal{H}$ can be written

$$\mathbf{v} = \mathbf{w} + \mathbf{u} \text{ for some } \mathbf{w} \in W, \;\; \mathbf{u} \in W^{\perp}.$$

Problem 214 (Ex. 5.20) Let $p \in [1, \infty[$ and consider the linear mapping

$$T : L^p(0,2) \to L^p(0,2), \;\; (Tf)(x) := xf(x).$$

One can show **(you are not expected to do this)** that for $0 \leq a \leq b$,

$$a^p \int_a^b |f(x)|^p \, dx \leq \int_a^b |xf(x)|^p \, dx \leq b^p \int_a^b |f(x)|^p \, dx. \tag{7}$$

(i) Use (7) to argue that T indeed maps $L^p(0,2)$ into $L^p(0,2)$ and is bounded with

$$||T|| \leq 2.$$

(ii) Calculate the norm of the operator T.
Hint: let $f = \chi_{[\epsilon,2]}$ for some $\epsilon \in [0,2[$ and use (7) to show that

$$\epsilon \leq \frac{||Tf||_p}{||f||_p} \leq 2.$$

Problem 215 (Contained in Ex. 10.11) Let $m \in \mathbb{N}$, and consider the B-spline N_m. One can show **(you are not expected to do this)** that there exist constants $A, B > 0$ such that

$$A \leq \sum_{k \in \mathbb{Z}} |\widehat{N_m}(\gamma + k)|^2 \leq B, \ \gamma \in \mathbb{R}.$$

(i) Show that the function

$$G(\gamma) := \sum_{k \in \mathbb{Z}} |\widehat{N_m}(\gamma + k)|^2$$

is 1-periodic.

Define the function $\varphi \in L^2(\mathbb{R})$ by its Fourier transform $\widehat{\varphi}$ via

$$\widehat{\varphi}(\gamma) := \frac{1}{\sqrt{G(\gamma)}} \widehat{N_m}(\gamma).$$

(ii) Show that

$$\sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2 = 1, \ \gamma \in \mathbb{R}.$$

The set of problems is completed!

Technical University of Denmark

Written exam. Date: June 1, 2010.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

The exam consists of 4 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 216 Consider the set of functions

$$V := \{f : \mathbb{R} \to \mathbb{C} \mid f(-\pi) = f(\pi) = 0\}.$$

(i) Show that V is a vector space with respect to the usual operations of addition and scalar multiplication.

(ii) Give an example of a linear operator that does not map V into V.

The set of problems continues!

Problem 217 Consider the linear mapping

$$T : L^1(0,2) \to L^1(0,2),\ (Tf)(x) := \int_0^2 e^{-x^2y^2} f(y)\, dy.$$

(i) Show that for any $x \in [0,2]$,

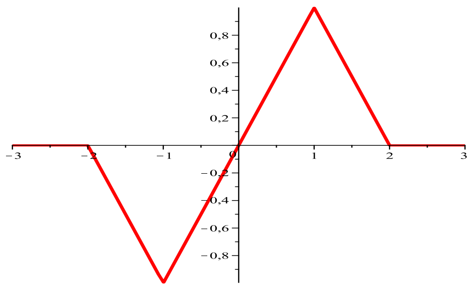
$$|(Tf)(x)| \le ||f||_{L^1(0,2)}.$$

(ii) Use the result in (i) to show that T indeed maps $L^1(0,2)$ into $L^1(0,2)$ and is bounded with

$$||T|| \le 2.$$

Problem 218 Consider the B-spline N_2 and its translate $T_{-2}N_2$. Define the function f by

$$f(x) = N_2(x) - T_{-2}N_2(x).$$



The graph shows the function f. The following questions can be answered without calculating the function f explicitly.

(i) Show that the Fourier transform of the function f is

$$\hat{f}(\gamma) = \left(1 - e^{4\pi i\gamma}\right)\left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^2.$$

(ii) Find the number of vanishing moments for the function f.
Hint: Instead of calculating the involved integrals you can use symmetry properties of the functions $f(x)$ and $xf(x)$.

Problem 219 Consider a Hilbert space $\mathcal{H}$ and let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Consider a bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$ for which the inverse T^{-1} is bounded as well.

(i) Show that if $\mathbf{v} \in \mathcal{H}$, then

$$\sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k \rangle|^2 = ||T^*\mathbf{v}||^2.$$

(ii) Show that the adjoint operator T^* is invertible, and that

$$(T^*)^{-1} = (T^{-1})^*.$$

Hint: Use the result in Exercise 4.17 (ii).

(iii) Show that for all $\mathbf{v} \in \mathcal{H}$

$$||\mathbf{v}|| \leq ||(T^*)^{-1}|| \, ||T^*\mathbf{v}||.$$

Hint: Use that $\mathbf{v} = (T^*)^{-1}T^*\mathbf{v}$.

(iv) Show that for all $\mathbf{v} \in \mathcal{H}$

$$\frac{1}{||T^{-1}||^2} \, ||\mathbf{v}||^2 \leq \sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k \rangle|^2 \leq ||T||^2 \, ||\mathbf{v}||^2.$$

The set of problems is completed!

Technical University of Denmark

Written exam. Date: May 31, 2011.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 220 Consider the function

$$f(x) := \sum_{k=2}^{\infty} k\, \chi_{[k,k+k^{-3}]}(x) = 2\, \chi_{[2,2\frac{1}{8}]}(x) + 3\, \chi_{[3,3\frac{1}{27}]}(x) + \cdots,\ x \in \mathbb{R}.$$

(i) Make a draft of the function f.

(ii) Is f a bounded function?

(iii) Does $f \in L^1(\mathbb{R})$?

Problem 221 Consider the function

$$f(x) := e^x \chi_{[0,1]}(x),\ x \in \mathbb{R}.$$

(i) Calculate the convolution $(f * f)(x)$ for $x \in [0,1]$.

The set of problems continues!

Problem 222 Consider a complex Hilbert space $\mathcal{H}$ and a bounded, linear, and self-adjoint operator

$$T : \mathcal{H} \to \mathcal{H}.$$

We say that a number $\lambda \in \mathbb{C}$ is an eigenvalue for T if there exists an eigenvector, i.e., a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$T\mathbf{v} = \lambda\,\mathbf{v}.$$

(i) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue for T, and let $\mathbf{v}$ denote a corresponding eigenvector with $||\mathbf{v}|| = 1$. Calculate the numbers

$$\langle T\mathbf{v}, \mathbf{v}\rangle \text{ and } \langle \mathbf{v}, T\mathbf{v}\rangle.$$

(ii) Show that all the eigenvalues are real numbers.

Now assume that $\mathcal{H}$ has an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and that for each $k \in \mathbb{N}$ there exists a number $\lambda_k \in \mathbb{R}$ such that

$$Te_k = \lambda_k e_k.$$

(iii) Show that for any $f \in \mathcal{H}$,

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k\rangle\, e_k.$$

Problem 223 We will consider the modulation operator E_b for complex values of the parameter b. That is, for $b \in \mathbb{C}$ we let E_b act on the function $f : \mathbb{R} \to \mathbb{C}$ by

$$(E_b f)(x) := e^{2\pi i b x} f(x),\ x \in \mathbb{R}.$$

(i) Show that the mapping E_b is linear for any $b \in \mathbb{C}$.

(ii) Show that for any $b \in \mathbb{C}$, the operator E_b maps $L^2(0,1)$ into $L^2(0,1)$ and is bounded.

Hint: Write $b = \alpha + i\beta$ for some $\alpha, \beta \in \mathbb{R}$, and use that every exponential function is bounded on $[0,1]$.

(iii) Determine the values of the parameter $b \in \mathbb{C}$ for which the operator $E_b : L^2(0,1) \to L^2(0,1)$ is self-adjoint.

The set of problems is completed!

Technical University of Denmark

Written exam. Date: May 29, 2012.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 11 questions. The 11 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 224 Let $\omega : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function, and consider the mapping

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Uf)(x) = \omega(x)f(x),\ x \in \mathbb{R}.$$

(i) Show that U indeed maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

(ii) Show that U is linear and bounded.

(iii) Compute the adjoint operator U^*.

(iv) Show that U is unitary if and only if

$$|\omega(x)| = 1,\ \forall\, x \in \mathbb{R}.$$

The set of problems continues!

Problem 225 We consider the space $L^p(0,\pi/2)$ for $p = 1/2$, i.e.,

$$L^{1/2}(0,\pi/2) := \{f :]0,\pi/2[\to \mathbb{C} \mid \int_0^{\pi/2} |f(x)|^{1/2}\,dx < \infty\}.$$

(i) Does the function

$$h(x) := \frac{1}{x},\ x \in]0,\pi/2[$$

belong to $L^{1/2}(0,\pi/2)$?

(ii) Show that every bounded function $f :]0,\pi/2[\to \mathbb{C}$ belongs to $L^{1/2}(0,\pi/2)$.

One can show (**you are not expected to do this**) that the functions

$$f(x) := \sin^2(x), \quad g(x) := \cos^2(x), \quad x \in]0,\pi/2[,$$

satisfy that

$$\begin{aligned} \int_0^{\pi/2} |f(x)|^{1/2}\,dx &= 1, \\ \int_0^{\pi/2} |g(x)|^{1/2}\,dx &= 1, \\ \int_0^{\pi/2} |f(x)+g(x)|^{1/2}\,dx &= \frac{\pi}{2}. \end{aligned}$$

(iii) Does the expression

$$||f|| := \left(\int_0^{\pi/2} |f(x)|^{1/2}\,dx \right)^2$$

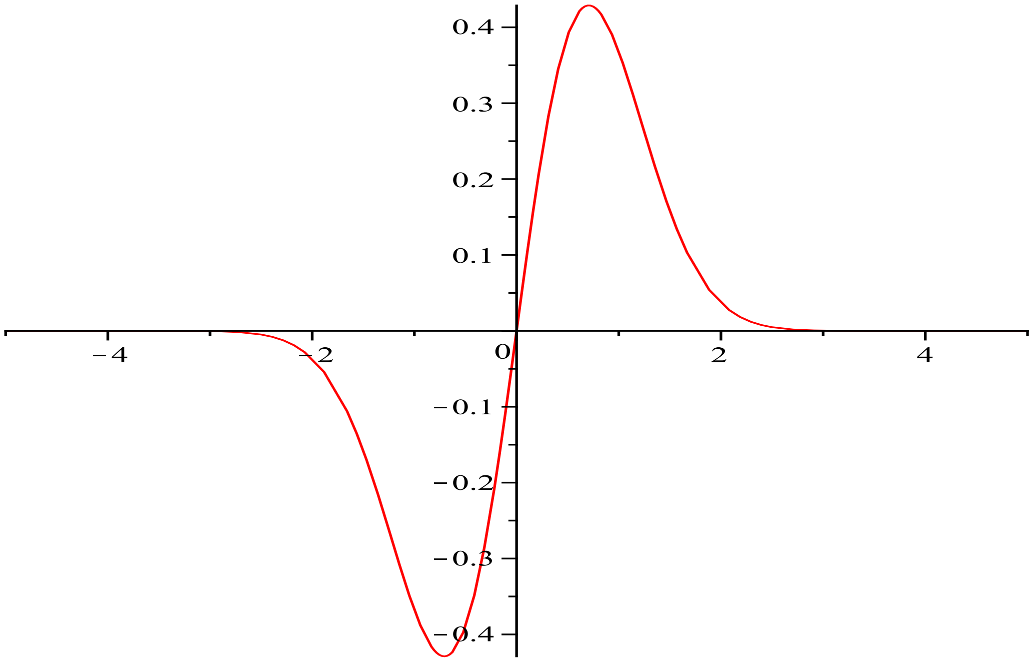
define a norm on $L^p(0,\pi/2)$ for $p = 1/2$?

The set of problems continues!

Problem 226 The figure shows the graph of the function

$$f(x) := xe^{-x^2}, \ x \in \mathbb{R}.$$

Find the number of vanishing moments for the function f. (An argument based on the properties of the function f is sufficient in order to obtain full credit.)



The set of problems continues!

Problem 227 Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ denote an orthonormal basis for a Hilbert space $\mathcal{H}$, and let

$$\mathbf{v}_k := \mathbf{e}_k + \mathbf{e}_{k+1}, \ k \in \mathbb{N}.$$

(i) Let $\mathbf{v} \in \mathcal{H}$ and assume that $\langle \mathbf{v}, \ \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$. Show that then

$$|\langle \mathbf{v}, \mathbf{e}_k \rangle| = |\langle \mathbf{v}, \mathbf{e}_{k+1} \rangle|, \ \forall k \in \mathbb{N}.$$

(ii) Show that $\overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty} = \mathcal{H}$.

Hint: Use (i) and Theorem 4.7.2 (iv) to show that if $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$, $k \in \mathbb{N}$, then $\mathbf{v} = \mathbf{0}$.

One can show (**you are not expected to do this**) that for any $N \in \mathbb{N}$ and complex numbers $c_k \in \mathbb{C}$ for $k = 1, \dots, N$,

$$\left|\left| \mathbf{e}_1 - \sum_{k=1}^{N} c_k \mathbf{v}_k \right|\right|^2 = |1 - c_1|^2 + \sum_{k=1}^{N-1} |c_k + c_{k+1}|^2 + |c_N|^2. \tag{8}$$

(iii) Show that it is impossible to choose $c_k \in \mathbb{C}$ such that

$$\mathbf{e}_1 = \sum_{k=1}^{\infty} c_k \mathbf{v}_k.$$

Hint: Use (8) to argue that we only need to consider the special sequence $c_k = (-1)^{k-1}, \ k \in \mathbb{N}$.

The set of problems is completed!

Technical University of Denmark

Written exam. Date: June 7, 2013.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 228 Consider the set of functions given by

$$V := \{f : \mathbb{R} \to \mathbb{C} \mid f(2) = f(-2) = 0\}.$$

(i) Show that V is a subspace of the vector space that consists of all functions $f : \mathbb{R} \to \mathbb{C}$.

For a given $b \in \mathbb{R}$, we now would like to consider the modulation operator E_b on the space V, i.e.,

$$E_b : V \to V,\ (E_b f)(x) := e^{2\pi i b x} f(x). \tag{9}$$

(ii) Show that E_b actually maps V into V.

(iii) Show that E_b is linear.

(iv) Is the operator $E_b : V \to V$ surjective?

The set of problems continues!

Problem 229 Consider the subspace of $L^1(\mathbb{R})$ consisting functions that only assume real values, i.e., let

$$W := \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \right\}.$$

(i) Denote the Fourier transform of the function $f \in W$ by $\widehat{f}$ and show that

$$\widehat{f}(-\gamma) = \overline{\widehat{f}(\gamma)},\ \forall\, \gamma \in \mathbb{R}.$$

(ii) Make a draft of the function $f(x) := e^{-x}\, \chi_{[0,2]}(x)$ and the function $g(x) := f(2-x)$.

(iii) Generalizing the idea in (ii), consider a function $f \in W$ with $\operatorname{supp} f \subseteq [0, N]$ for some $N \in \mathbb{N}$, and let

$$g(x) := f(N - x).$$

Show that

$$|\widehat{g}(\gamma)| = \left|\widehat{f}(\gamma)\right|,\ \forall\, \gamma \in \mathbb{R}.$$

Problem 230 Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) := \begin{cases} \sqrt{24}\, x & \text{for } x \in [0, 1/2], \\ 0 & \text{for } x \notin [0, 1/2]. \end{cases}$$

As usual, define the translation operator T_a, $a \in \mathbb{R}$, by

$$T_a f(x) = f(x - a),\ f \in L^2(\mathbb{R}).$$

(i) Show that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{R})$.

(ii) Is $\{T_k\phi\}_{k\in\mathbb{Z}}$ an orthonormal basis for $L^2(\mathbb{R})$?
Hint: Check one of the conditions in Theorem 4.7.2.

The set of problems continues!

Problem 231 Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ denote an orthonormal basis for a Hilbert space $\mathcal{H}$, and consider the vectors $\mathbf{v}_k, k \in \mathbb{N}$, defined via

$$\mathbf{v}_k := \mathbf{e}_k + \mathbf{e}_{k+1}.$$

Now consider the bounded linear operator (**you are not asked to prove that T has these properties**)

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k \mathbf{v}_k.$$

(i) Show that T is injective.
Hint: show first that for $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$,

$$\sum_{k=1}^{\infty} c_k \mathbf{v}_k = c_1 \mathbf{e}_1 + \sum_{k=2}^{\infty} (c_{k-1} + c_k)\, \mathbf{e}_k$$

(ii) Given $N \in \mathbb{N}$, consider the finite sequence $\{c_k\}_{k=1}^{\infty}$ given by

$$c_k = \begin{cases} (-1)^k & \text{for } k = 1, 2, \dots, N, \\ 0 & \text{for } k = N+1, N+2, \dots. \end{cases}$$

Show that $||\{c_k\}_{k=1}^{\infty}||_2 = \sqrt{N}$ and $||T\{c_k\}_{k=1}^{\infty}||_{\mathcal{H}} = \sqrt{2}$.

(iii) Denote the range of the operator T by $\mathcal{R}(T)$. The result in (i) implies that T has an inverse operator,

$$T^{-1} : \mathcal{R}(T) \to \ell^2(\mathbb{N}).$$

Prove that T^{-1} is an unbounded operator.

The set of problems is completed!

Technical University of Denmark

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Written exam. Date: June 3, 2014.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 232 Define the operator T that acts on functions $f : \mathbb{R} \to \mathbb{C}$, by

$$(Tf)(x) := \chi_{[0,\infty[}(x) f(x).$$

(i) Make a sketch of the functions $f(x) := e^{-x^2}$ and $Tf(x) = \chi_{[0,\infty[}(x)\, e^{-x^2}$.

(ii) Does the operator T map the space $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$?

(iii) Show that the operator T maps $L^1(\mathbb{R})$ into $L^1(\mathbb{R})$.

(iv) Show that T is linear and bounded as operator from $L^1(\mathbb{R})$ into $L^1(\mathbb{R})$.

(v) Is $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ an isometry?

The set of problems continues!

Problem 233 Given two functions $f, g \in L^2(\mathbb{R})$ and $b \in \mathbb{R}$, consider the function $\Phi(x) := f(x)g(x-b)$.

(i) Show that $\Phi \in L^1(\mathbb{R})$.

The short-time Fourier transform of $f \in L^2(\mathbb{R})$ is defined as a function of two variables, denoted by $V(f)(\cdot,\cdot)$, and given by

$$V(f)(b,\gamma) := \int_{-\infty}^{\infty} f(x)\chi_{[-3,3]}(x-b)e^{-2\pi i x\gamma}\,dx, \quad b,\gamma \in \mathbb{R}.$$

The result in (i) shows that $V(f)(b,\gamma)$ is well defined for all $b,\gamma \in \mathbb{R}$ (you do not need to argue for this).

We will consider the function $f(x) := \cos(20\pi x)\,\chi_{[-3,3]}(x)$ and calculate the function $V(f)(b,\cdot)$ for two special choices of the parameter b.

(ii) Let $b = 0$ and calculate $V(f)(0,\gamma)$ for $\gamma \in \mathbb{R}$. You are allowed to use the results in Example 7.1.4, without repeating the calculations.

(iii) Let $b = 6$ and calculate $V(f)(6,\gamma)$ for $\gamma \in \mathbb{R}$.
Hint: Make a sketch of the function $f(x)\chi_{[-3,3]}(x-b)$ for $b = 6$.

The set of problems continues!

Problem 234 Consider the B-spline N_2, given by

$$N_2(x) \;=\; \begin{cases} x & \text{if } x \in [0,1], \\ 2-x & \text{if } x \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

We will consider the set of translates $\{T_{2k}N_2\}_{k\in\mathbb{Z}}$.

(i) Show that the function $\phi = \chi_{[0,1[} - \chi_{[1,2[}$ satisfies that

$$\langle \phi, T_{2k}N_2\rangle = 0, \;\forall\, k \in \mathbb{Z}, \tag{10}$$

where $\langle \cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R})$.
Hint: split into the two cases $k = 0$ and $k \neq 0$. A geometric argument for (10) is sufficient.

(ii) Does the functions $\{T_{2k}N_2\}_{k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$?

Problem 235 Let $\mathcal{H}$ denote a Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator, with adjoint operator T^*. Define the *null space* of T^* by

$$\mathcal{N}_{T^*} := \{x \in \mathcal{H} \;\big|\; T^*x = 0\}.$$

Also, define the *range* of T by

$$\mathcal{R}_T = \{Tx \;\big|\; x \in \mathcal{H}\}.$$

The purpose of the problem is to show that

$$\mathcal{N}_{T^*} = \mathcal{R}_T^{\perp}.$$

(i) Show that $\mathcal{N}_{T^*} \subseteq \mathcal{R}_T^{\perp}$.
Hint: Assume that $T^*x = 0$ and calculate $\langle x, Ty\rangle$ for $y \in \mathcal{H}$.

(ii) Show that $\mathcal{R}_T^{\perp} \subseteq \mathcal{N}_{T^*}$.

The set of problems is completed!

Technical University of Denmark

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Written exam. Date: June 2, 2015.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids

Duration: 2 Hours.

The exam consists of 4 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 236 Consider the function $v(x) := e^{|x|}$, $x \in \mathbb{R}$, and let

$$L_v^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |f(x)| \, e^{|x|} \, dx < \infty\}.$$

(i) Show that $L_v^1(\mathbb{R})$ is a vector space.

(ii) Show that $L_v^1(\mathbb{R})$ is a subspace of $L^1(\mathbb{R})$.

(iii) Find a function $f \neq 0$ that belongs to $L_v^1(\mathbb{R})$.

(iv) Show that the expression

$$||f||_{L_v^1(\mathbb{R})} := \int_{-\infty}^{\infty} |f(x)| \, e^{|x|} \, dx$$

defines a norm on $L_v^1(\mathbb{R})$.

(v) Let $a \in \mathbb{R}$, and show that the translation operator T_a [as usual given by $T_a f(x) = f(x-a)$] maps $L_v^1(\mathbb{R})$ into $L_v^1(\mathbb{R})$.

(vi) Let $a \in \mathbb{R}$. Show that the translation operator T_a is bounded from $L_v^1(\mathbb{R})$ into $L_v^1(\mathbb{R})$ and satisfies that

$$||T_a f||_{L_v^1(\mathbb{R})} \leq e^{|a|} \, ||f||_{L_v^1(\mathbb{R})}, \; \forall f \in L_v^1(\mathbb{R}).$$

Problem 237 Consider the constant function $f(x) := 1$, $x \in \mathbb{R}$.

(i) Consider any $g \in C_c(\mathbb{R})$, and show that

$$||f - g||_\infty \geq 1.$$

(ii) Is the space $C_c(\mathbb{R})$ dense in $L^\infty(\mathbb{R})$?

Problem 238 Let $\mathcal{H}$ denote a complex Hilbert space, and consider a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$. Let $T^* : \mathcal{H} \to \mathcal{H}$ denote the adjoint operator.

The following two questions are independent.

(i) Show that the operator TT^* is self-adjoint.

(ii) Assume that $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v} \in \mathcal{H}$. Show that then

$$\langle T\mathbf{v}, T\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle, \; \forall\, \mathbf{v}, \mathbf{w} \in \mathcal{H}.$$

The set of problems continues!

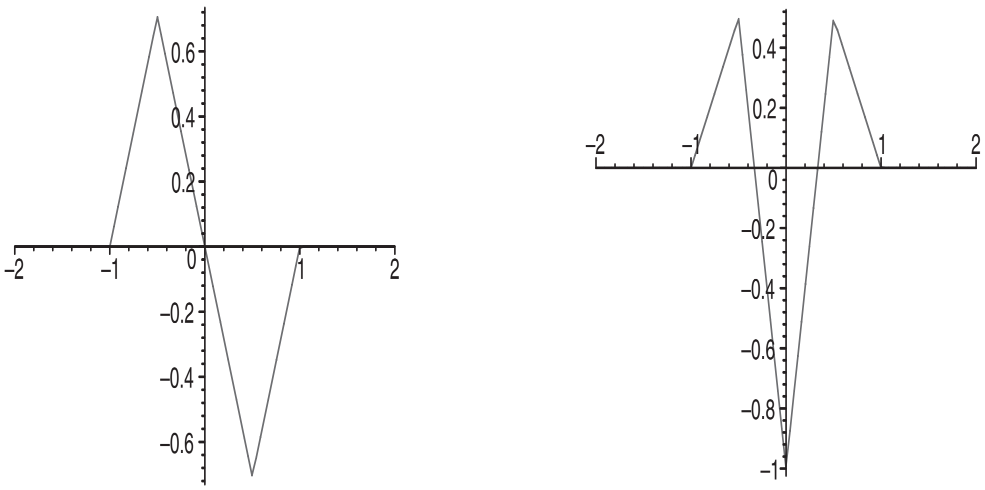

Figure 1: The figure shows two functions, ψ_1 (left) and ψ_2 (right). Both functions are supported on $[-1, 1]$.

Problem 239 In this Problem we will consider a wavelet system generated by *two* functions ψ_1 and ψ_2, i.e., a system of functions of the form $\{D^jT_k\psi_\ell\}_{\ell=1,2;j,k\in\mathbb{Z}}$. One can show (you are *not* expected to do this) that with ψ_1 and ψ_2 chosen as on the figure, each function $f \in L^2(\mathbb{R})$ has the expansion

$$f = \sum_{\ell=1}^{2} \sum_{j,k\in\mathbb{Z}} \langle f, D^jT_k\psi_\ell\rangle D^jT_k\psi_\ell.$$

(i) Do the functions $\{D^jT_k\psi_\ell\}_{\ell=1,2;j,k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$?
Hint: Without doing any calculations, consider the number $\langle \psi_1, T_1\psi_1\rangle$.

(ii) Show that for all $f \in L^2(\mathbb{R})$,

$$||f||^2 = \sum_{\ell=1}^{2} \sum_{j,k\in\mathbb{Z}} |\langle f, D^jT_k\psi_\ell\rangle|^2.$$

The set of problems is completed!

Technical University of Denmark

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Written exam. Date: June 1, 2016.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids

Duration: 2 Hours.

The exam consists of 4 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 240 Consider the set of functions given by

$$V := \{f : [-1, 1] \to \mathbb{C} \mid f \text{ is continuous and } f(1) = f(-1)\}.$$

(i) Show that V is a subspace of the vector space that consists of all bounded functions $f : [-1, 1] \to \mathbb{C}$.

For a given integer $b \in \mathbb{Z}$, we now would like to consider the modulation operator E_b on the space V, i.e.,

$$E_b : V \to V, \ (E_b f)(x) := e^{2\pi ibx} f(x). \tag{11}$$

(ii) Show that E_b actually maps V into V for $b \in \mathbb{Z}$.

(iii) Show that E_b is linear.

(iv) Is E_b surjective?

(v) We would now like to consider the operator E_b as in (11), but with $b \in \mathbb{C}$. Find the values of $b \in \mathbb{C}$, for which the operator E_b actually maps V into V.

The set of problems continues!

Problem 241 Let $\{\mathbf{v}_k\}_{k=1}^N$ denote a basis for a finite-dimensional vector space V. For $\mathbf{v} \in V$, write $\mathbf{v} = \sum_{k=1}^N c_k \mathbf{v}_k$ for suitable coefficients $c_k \in \mathbb{C}$, and define the norm of $\mathbf{v}$ by

$$||\mathbf{v}|| = \sum_{k=1}^{N} |c_k|. \tag{12}$$

(i) Show that (12) indeed defines a norm on V.

(ii) Let $T : V \to V$ denote an arbitrary linear operator, and equip V with the norm (12). Show that T is bounded with

$$||T|| \le \max_{k \in \{1,\dots,N\}} ||T\mathbf{v}_k||.$$

Problem 242 This Problem deals with functions $f \in L^1(\mathbb{R})$, and we will consequently use the norm

$$||f|| = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

(i) Show that the centered B-splines $B_m, m \in \mathbb{N}$, are even functions.

Hint: apply the formula $B_{m+1}(x) = \int_{-1/2}^{1/2} B_m(x-t)\, dt$.

Consider now the centered B-spline B_3, and define the functions f_k, $k \in \mathbb{N}$, by

$$f_k(x) := kB_3(kx),\ x \in \mathbb{R}. \tag{13}$$

(ii) Show that $||f_k|| = 1$ for all $k \in \mathbb{N}$.

Consider now the vector space

$$V := \{f \in L^1(\mathbb{R}) \,\big|\, f \text{ is differentiable and } f' \in L^1(\mathbb{R})\}.$$

(iii) Show that the differential operator $D : V \to L^1(\mathbb{R})$, given by $Df := f'$, is unbounded.

Hint: Use the properties of B_3 and that $B_3'(x) \ge 0$ for $x \in [-3/2, 0]$ to prove that

$$||Df_k|| = k \int_{-\infty}^{\infty} |B_3'(y)|\, dy = 2k \int_{-3/2}^{0} B_3'(y)\, dy = 2B_3(0)k.$$

The set of problems continues!

Problem 243 Let $\mathcal{H}$ denote a Hilbert space. Consider a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ and assume that there exist a constant $A > 0$ such that

$$A\,||\mathbf{v}|| \leq ||T\mathbf{v}||,\ \forall \mathbf{v} \in \mathcal{H}.$$

(i) Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ denote a sequence of vectors in $\mathcal{H}$, and assume that the sequence $\{T\mathbf{v}_k\}_{k=1}^{\infty}$ is convergent, i.e.,

$$T\mathbf{v}_k \to \mathbf{w} \text{ as } k \to \infty$$

for some $\mathbf{w} \in \mathcal{H}$. Show that then the sequence $\{\mathbf{v}_k\}_{k=1}^{\infty}$ is convergent as well, i.e.,

$$\mathbf{v}_k \to \mathbf{v} \text{ as } k \to \infty$$

for some $\mathbf{v} \in \mathcal{H}$.
Hint: Show first that $\{\mathbf{v}_k\}_{k=1}^{\infty}$ is a Cauchy sequence.

(ii) Show that then the range of T, i.e., the set

$$\mathcal{R}_T := \{T\mathbf{v} \,\big|\, \mathbf{v} \in \mathcal{H}\},$$

is closed.

Hint: Apply Lemma 2.2.3.

The set of problems is completed.

Danmarks Tekniske Universitet

Written exam, May 30, 2017
Course name: Mathematics 4
Course number: 01325
Aids: All aids allowed
Duration: 2 hours
Weighting: Exercise 1: 25%, Exercise 2: 25%, Exercise 3: 20%, Exercise 4: 30%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, notion, etc.) are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Exercise 1. We consider the spaces $L^p(-2,3)$ for $p = 1$ and $p = 2$. Define the linear operator $T : L^2(-2,3) \to L^1(-2,3)$ by $Tf = f$ for $f \in L^2(-2,3)$.

(a) Prove that

$$L^2(-2,3) \subset L^1(-2,3)$$

and that the norms on these spaces satisfy $\|f\|_1 \le 5^{1/2}\,\|f\|_2$ for all $f \in L^2(-2,3)$.

(b) Show that T is well-defined and bounded. Is T invertible?

(c) Compute $\left\|T\chi_{]-2,3[}\right\|_1$. Find then the norm $\|T\|$.

Exercise 2. Consider a complex Hilbert space $\mathcal{H}$ and a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$. If there exist a vector $\mathbf{v} \in \mathcal{H}$, $\mathbf{v} \neq \mathbf{0}$, and a number $\lambda \in \mathbb{C}$ such that

$$T\mathbf{v} = \lambda\,\mathbf{v},$$

we say that λ is an eigenvalue with eigenvector $\mathbf{v}$ for the operator T.

(a) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$ for the operator T. Show that

$$\langle T\mathbf{v}, T\mathbf{v}\rangle = |\lambda|^2\,\|\mathbf{v}\|^2 \quad \text{for all } \mathbf{v} \in \mathcal{H}.$$

(b) Let $U : \mathcal{H} \to \mathcal{H}$ be a *unitary* linear operator. Assume that $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$ for the operator U. Show that λ satisfies $|\lambda| = 1$.

(c) The Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has eigenvalues $1, -1, i$ and $-i$. Let $f \in L^2(\mathbb{R})$ be an even function satisfying $f, \hat{f} \in L^1(\mathbb{R})$. Show that $g := f + \mathcal{F}f$ is an eigenvector (eigenfunction) for $\mathcal{F}$ and state the associated eigenvalue. You may use without proof that $\mathcal{F}^2(f)(x) = f(-x)$, $\forall x \in \mathbb{R}$, holds for any function $f \in L^2(\mathbb{R})$ satisfying $f, \hat{f} \in L^1(\mathbb{R})$. Find an eigenvector of $\mathcal{F}$ that has an eigenvalue not equal to 1.

The set of problems continues!

Exercise 3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be complex normed vector spaces. Define the product $V \otimes W$ of the vector spaces by giving the set

$$V \otimes W = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\}$$

the obvious entrywise defined vector space structure:

$$(\mathbf{x}, \mathbf{y}) + (\mathbf{v}, \mathbf{w}) := (\mathbf{x} + \mathbf{v}, \mathbf{y} + \mathbf{w})$$
$$\alpha(\mathbf{v}, \mathbf{w}) := (\alpha\mathbf{v}, \alpha\mathbf{w})$$

for $\mathbf{x}, \mathbf{v} \in V$, $\mathbf{y}, \mathbf{w} \in W$, and $\alpha \in \mathbb{C}$. It can be shown (you are not supposed to do so) that $V \otimes W$ is a vector space. Define the mapping $\|\cdot\|_1 : V \otimes W \to \mathbb{R}$ by

$$\|(\mathbf{v}, \mathbf{w})\|_1 = \|\mathbf{v}\|_V + \|\mathbf{w}\|_W \text{ for } \mathbf{v} \in V \text{ and } \mathbf{w} \in W.$$

(a) Show that $\|\cdot\|_1$ defines a norm on $V \otimes W$.

(b) Now assume that $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are Banach spaces. Prove that $V \otimes W$ equipped with the norm $\|\cdot\|_1$ is a Banach space. Hint: Consider a Cauchy sequence $\{(\mathbf{v}_k, \mathbf{w}_k)\}_{k=1}^{\infty}$ in $V \otimes W$.

Exercise 4. Let $\mathcal{H}$ be a complex Hilbert space $\mathcal{H}$. For a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ we define the *null space* of T by

$$\mathcal{N}_T := \{\mathbf{x} \in \mathcal{H} \mid T\mathbf{x} = 0\}$$

and the *range* of T by

$$\mathcal{R}_T := \{T\mathbf{x} \mid \mathbf{x} \in \mathcal{H}\}.$$

You may use the following extension of Lemma 4.5.6 without proof: a bounded linear operator $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection P (onto $\mathcal{R}_P$, see Definition 4.5.5) if and only if P is self-adjoint (that is, $P = P^*$) and idempotent (that is, $P^2 = P$).

(a) Assume that P is an orthogonal projection. Show that $\mathcal{R}_P \perp \mathcal{N}_P$ (that is, show that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{x} \in \mathcal{R}_P$ and $\mathbf{y} \in \mathcal{N}_P$).

(b) Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ be an orthonormal system in $\mathcal{H}$. Define

$$P : \mathcal{H} \to \mathcal{H}, \quad P\mathbf{x} = \sum_{k=1}^{\infty} \langle \mathbf{x}, \mathbf{v}_k \rangle \mathbf{v}_k$$

It can be used without proof that P is a well-defined linear operator. Show that P is an orthogonal projection, i.e., that P is bounded, self-adjoint, and idempotent.

(c) Let ψ denote the Haar wavelet. It can be shown (you are not supposed to do so) that

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad Qf = \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \langle f, D^j T_k \psi \rangle D^j T_k \psi$$

defines a linear operator on $L^2(\mathbb{R})$. Argue that Q is an orthogonal projection.

(d) Let Q be defined as in (c). Find a function $f \in L^2(\mathbb{R})$, $f \neq 0$, for which $Qf = 0$ and a function $g \in L^2(\mathbb{R})$, $g \neq 0$, for which $Qg = g$.

The set of problems is completed.

Danmarks Tekniske Universitet

Written exam, May 29, 2018
Course name: Mathematics 4
Course number: 01325
Aids: All aids allowed by DTU (mobile phones and internet access not allowed)
Duration: 2 hours
Weighting: Exercise 1: 25%, Exercise 2: 25%, Exercise 3: 20%, Exercise 4: 30%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, notion, etc.) are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Exercise 1. Let p be a real number satisfying $1 \le p < \infty$. Define the linear operator $T : L^p(0,2) \to L^p(0,2)$ by

$$Tf(x) = x\, f(x) \quad \text{for } x \in \,]0,2[$$

for $f \in L^p(0,2)$. To clarify notation: Tf is the function $x \mapsto x \cdot f(x)$ on $]0,2[$.

(a) Show that T is a well-defined, linear operator.

(b) Show that T is bounded and that $\|T\| \le 2$.

(c) Show that T is also bounded as a linear operator from $L^2(0,2)$ to $L^1(0,2)$.

Exercise 2. Let $\mathcal{H}$ be a Hilbert space, and let $T : \mathcal{H} \to \mathcal{H}$ be an isometry.

(a) Show that $\|T\| = 1$. Give an example of a linear, bounded operator $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ that is *not* an isometry and that satisfies $\|S\| = 1$.

(b) Show that the range $\mathcal{R}(T)$ is a closed set in $\mathcal{H}$. Recall that the *range* of T is defined by

$$\mathcal{R}(T) := \left\{ T\mathbf{x} \,\middle|\, \mathbf{x} \in \mathcal{H} \right\}.$$

The set of problems continues!

Exercise 3. The Shannon wavelet $\psi \in L^2(\mathbb{R})$ and scaling function $\phi \in L^2(\mathbb{R})$ are given by, respectively,

$$\hat{\psi} = \chi_{[-1,-1/2[\cup[1/2,1[} \quad \text{and} \quad \hat{\phi} = \chi_{[-1/2,1/2[}, \tag{1}$$

where χ denotes the characteristic function and $\hat{}$ denotes the Fourier transform on $L^2(\mathbb{R})$.

(a) Use the inverse Fourier transform to obtain an expression for $\psi(x)$ and $\phi(x)$ for almost all $x \in \mathbb{R}$.

(b) Argue that ψ and ϕ are continuous functions and compute $\psi(1/2)$ and $\phi(1/2)$.

Exercise 4. We consider again the function $\phi \in L^2(\mathbb{R})$ defined in (1). In this exercise $\mathcal{F}f$ and $\hat{f}$ also denote the Fourier transform on $L^2(\mathbb{R})$; note that Exercise 4 can be solved independently of Exercise 3 and vice versa. Define the Paley-Wiener space PW_I, where I is a closed and bounded interval in $\mathbb{R}$, by

$$PW_I = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \operatorname{supp} \hat{f} \subseteq I \right\}.$$

For each $j \in \mathbb{Z}$, let V_j denote the closed subspace $PW_{[-2^{j-1},2^{j-1}]}$.

(a) Let $j, k \in \mathbb{Z}$. Show that the function $\widehat{D^j T_k \phi}$ (that is, the function $\mathcal{F}D^j T_k \phi$) can be written as:

$$\widehat{D^j T_k \phi}(\gamma) = \begin{cases} 2^{-j/2} \mathrm{e}^{-2\pi i k 2^{-j}\gamma} & \text{for } \gamma \in [-2^{j-1}, 2^{j-1}[\,, \\ 0 & \text{for } \gamma \in \mathbb{R} \setminus [-2^{j-1}, 2^{j-1}[\,. \end{cases} \tag{2}$$

(b) Show that $PW_{[-1/2,1/2]} = \overline{\operatorname{span}\{T_k\phi\}_{k\in\mathbb{Z}}}$. *Hint*: Use properties of Fourier series and Formula (2) with $j = 0$.

(c) Find a 1-periodic function H_0 satisfying equation (8.5) in the book. Recall that equation (8.5) reads

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma) \quad \text{for } \gamma \in \mathbb{R}.$$

(d) Argue that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis (MRA).

The set of problems is completed.

Danmarks Tekniske Universitet

Written exam, May 29, 2019
Course name: Mathematics 4
Course number: 01325
Aids: All aids allowed by DTU (mobile phones and internet access not allowed)
Duration: 4 hours
Weighting: Exercise 1: 15%, Exercise 2: 20%, Exercise 3: 25%, Exercise 4: 20%, Exercise 5: 20%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, notion, etc.) are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Exercise 1. Let $\mathcal{H}$ be a complex Hilbert space, let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator, and let V be a closed subspace of $\mathcal{H}$. Suppose that $\{\boldsymbol{e}_k\}_{k=1}^{\infty}$ is an orthonormal basis for V.

(a) For any $\boldsymbol{v} \in \mathcal{H}$ and $k \in \mathbb{N}$ show that

$$\langle \boldsymbol{v}, \boldsymbol{e}_k \rangle = \langle U\boldsymbol{v}, U\boldsymbol{e}_k \rangle .$$

(b) Prove that $\{U\boldsymbol{e}_k\}_{k=1}^{\infty}$ is an orthonormal system.

(c) Prove that for any $\boldsymbol{w} \in U(V) := \{U\boldsymbol{v} \mid \boldsymbol{v} \in V\}$ we have the expansion formula:

$$\boldsymbol{w} = \sum_{k=1}^{\infty} \langle \boldsymbol{w}, U\boldsymbol{e}_k \rangle U\boldsymbol{e}_k.$$

(d) Argue that $\{U\boldsymbol{e}_k\}_{k=1}^{\infty}$ is an orthonormal basis for $U(V)$.

Exercise 2. Let $\mathcal{H}$ be a complex Hilbert space, and let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.

(a) Show that if T is an isometry, then

$$\langle T\boldsymbol{v}, T\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in \mathcal{H}.$$

(b) Show that T is an isometry if, and only if, for any orthonormal basis $\{\boldsymbol{e}_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ the sequence $\{T\boldsymbol{e}_k\}_{k=1}^{\infty}$ is an orthonormal system.

(c) Give an example of a Hilbert space $\mathcal{H}$, an isometry T, and an orthonormal basis $\{\boldsymbol{e}_k\}_{k=1}^{\infty}$ such that $\{T\boldsymbol{e}_k\}_{k=1}^{\infty}$ is not a basis.

Exercise 3. Consider a sequence $\{w_k\}_{k=1}^{\infty}$ of positive real numbers, and define the weighted ℓ^1-space $\ell^1_w(\mathbb{N})$ by

$$\ell^1_w(\mathbb{N}) := \left\{ \{x_k\}_{k=1}^{\infty} \;\middle|\; x_k \in \mathbb{C}, \sum_{k=1}^{\infty} |x_k|\, w_k < \infty \right\}.$$

(a) Show that the expression $\|\cdot\|_w$ given by

$$\|\{x_k\}_{k=1}^{\infty}\|_w := \sum_{k=1}^{\infty} |x_k|\, w_k$$

defines a norm on $\ell_w^1(\mathbb{N})$.

In the rest of the exercise, we consider the special choice $w_k := 2^k,\ k \in \mathbb{N}$.

(b) Show that $\ell_w^1(\mathbb{N})$ is a subspace of $\ell^1(\mathbb{N})$ and that $\ell_w^1(\mathbb{N}) \neq \ell^1(\mathbb{N})$.

(c) Show that the right-shift operator T maps $\ell_w^1(\mathbb{N})$ into $\ell_w^1(\mathbb{N})$. As usual the right-shift operator T maps $\{x_1, x_2, \ldots\}$ into $\{0, x_1, x_2, \ldots\}$.

(d) Show that the right-shift operator T is bounded from $\ell_w^1(\mathbb{N})$ to $\ell_w^1(\mathbb{N})$ and satisfies

$$\|T\,\{x_k\}_{k=1}^{\infty}\|_w \leq 2\,\|\{x_k\}_{k=1}^{\infty}\|_w$$

for all $\{x_k\}_{k=1}^{\infty} \in \ell_w^1(\mathbb{N})$.

(e) Argue that $\ell_w^1(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$ with respect to the norm $\|\cdot\|_1$ (as usual given by $\|\{x_k\}_{k=1}^{\infty}\|_1 := \sum_{k=1}^{\infty} |x_k|$). Is the vector space $\ell_w^1(\mathbb{N})$ equipped with the norm $\|\cdot\|_1$ a Banach space?

Exercise 4. Define $f_n \in L^2(\mathbb{R})$ by $f_n = \mathrm{sinc} \cdot \chi_{[-n,n]}$ for each $n \in \mathbb{N}$, where χ denotes the characteristic function. When speaking of the Fourier transform in the questions below, we refer to the Fourier transform on $L^2(\mathbb{R})$.

(a) Show that $\|\mathrm{sinc}\|_2 < \infty$. You may use without proof that $\|f_1\|_2 \leq 2$. Here $\|\cdot\|_2$ denotes the standard norm on $L^2(\mathbb{R})$.

(b) Show that $\mathrm{sinc} \notin L^1(\mathbb{R})$.

(c) Show that $f_n \to \mathrm{sinc}$ in the $\|\cdot\|_2$-norm as $n \to \infty$. Explain then how one can compute the Fourier transform of sinc.

(d) Find the exact value of $\|\mathrm{sinc}\|_2$. You may here use without proof that the Fourier transform of sinc is $\chi_{]-1/2,1/2[}$.

Exercise 5. Suppose $\phi \in L^2(\mathbb{R})$ is a compactly supported function that generates a multiresolution analysis. Let ψ denote the associated wavelet given by formula (8.9).

(a) Show that the right hand side of the scaling equation

$$D^{-1}\phi = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k T_{-k}\phi$$

is a finite sum.

(b) Show that there exists a finite collection of functions from $\{D^j T_k \phi\}_{j,k\in\mathbb{Z}}$ that is linearly dependent.

(c) Show that any finite collection of functions from $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ is linearly independent.

The set of problems is completed.

TECHNICAL UNIVERSITY OF DENMARK

Written 4 hour exam, 29 May 2020
Course: Mathematics 4 01325
Allowed aids: All aids allowed by DTU.
Weights: Problem 1: 25%, Problem 2: 25%, Problem 3: 25%, and Problem 4: 25%.

The weights are only guidelines. The exam is evaluated as a whole.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, notion, etc.) are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 1

Let V and W be normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively, and let $T : V \to W$ be a linear operator. Show that the following is equivalent:

(a) T is bounded, i.e., there exists $C \geq 0$ such that $\|Tx\|_W \leq C\|x\|_V$ for all $x \in V$.

(b) T is uniformly continuous, i.e.,

$$\forall \epsilon > 0 \, \exists \delta > 0 : \forall x, y \in V : \|x - y\|_V \leq \delta \Rightarrow \|Tx - Ty\|_W \leq \epsilon \, .$$

(c) T is continuous, i.e.,

$$\forall x \in V \, \forall \epsilon > 0 \, \exists \delta > 0 : \forall y \in V : \|x - y\|_V \leq \delta \Rightarrow \|Tx - Ty\|_W \leq \epsilon \, .$$

(d) T is continuous at the zero vector in V, i.e.,

$$\forall \epsilon > 0 \, \exists \delta > 0 : \forall y \in V : \|0 - y\|_V \leq \delta \Rightarrow \|T0 - Ty\|_W \leq \epsilon \, .$$

Hint 1: It is enough to show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).
Hint 2: For (d) $\Rightarrow$ (a) try to let $\epsilon = 1$ in (d) and use that if $\lambda > 0$ then $\|x\|_V \leq c \iff \|\lambda x\|_V \leq \lambda c$.

Problem 2

Let $C^0([0,1])$ be the space of continuous functions $[0,1] \to \mathbb{C}$ and let $C^2([0,1])$ be the space of twice differentiable functions $[0,1] \to \mathbb{C}$ with continuous second derivative, i.e., if $f \in C^2([0,1])$ then f'' exists and is continuous.

Let $\|\cdot\|_{L^2}$ denote the 2-norm on $C^0([0,1])$, i.e., for $f \in C^0([0,1])$ we have

$$\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x\right)^{1/2}.$$

For $f \in C^2([0,1])$ we define

$$\|f\|_{H^2} = \left(|f(0)|^2 + |f(1)|^2 + \int_0^1 |f''(x)|^2 \,\mathrm{d}x\right)^{1/2},$$

and $D^2 f$ by

$$(D^2 f)(x) = f''(x)\,.$$

You can without a proof use that $D^2 : C^2([0,1]) \to C^0([0,1])$ is a linear operator.

(i) Show that $\|\cdot\|_{H^2}$ is a norm on $C^2([0,1])$.

(ii) Equip $C^0([0,1])$ with the L^2-norm $\|\cdot\|_{L^2}$ and $C^2([0,1])$ with the H^2-norm $\|\cdot\|_{H^2}$. Show that D^2 is a bounded operator.

(iii) Find the kernel of D^2.

(iv) Find the operator norm of D^2.

Problem 3

Let $f \in L^2(\mathbb{R})$

(i) Show that if $x, y \in \mathbb{R}$ with $x < y$ then there exists a constant $C \geq 0$ such that

$$\left| \int_x^y f(t)\,\mathrm{d}t \right| \leq C\sqrt{|y-x|}\,.$$

Hint: Consider the product $f\,\chi_{[x,y]}$ and use Cauchy-Schwarz inequality.

Let $c > 0$ and consider the function $g_c = \dfrac{1}{2c}\chi_{[-c,c]}$, i.e.,

$$g_c(x) = \begin{cases} \dfrac{1}{2c}\,, & -c \leq x \leq c\,, \\ 0\,, & |x| > c\,. \end{cases}$$

(ii) Show that if $y > x$ and $y - x < c$ then

$$(g_c * f)(x) - (g_c * f)(y) = \frac{1}{2c}\left(\int_{x-c}^{y-c} f(t)\,\mathrm{d}t - \int_{x+c}^{y+c} f(t)\,\mathrm{d}t \right).$$

(iii) Show that $g_c * f$ is a uniformly continuous function.
Hint: Combine (i) and (ii).

Problem 4

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -e^{x}\,, & x < 0\,, \\ 0\,, & x = 0\,, \\ e^{-x}\,, & x > 0\,. \end{cases}$$

(i) Show that f is an odd function.

(ii) Show that $f \in L^p$ for any $p \geq 1$.

(iii) Find the Fourier transform $\mathcal{F}f = \widehat{f}$.

(iv) Find the Fourier transform of the Fourier transform $\mathcal{F}^2 f = \mathcal{F}\widehat{f} = \widehat{\widehat{f}}\,$.

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End of exam.

TECHNICAL UNIVERSITY OF DENMARK

Written 4 hour exam, 31 May 2021
Course: Mathematics 4 01325
Allowed aids: All aids allowed by DTU.
Weights: Problem 1: 30%, Problem 2: 30%, Problem 3: 20%, and Problem 4: 20%.

The weights are only guidelines. The exam is evaluated as a whole.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, notion, etc.) are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 1

Let V and W be normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Define addition and scalar multiplication in $V \times W$ by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)\,, \qquad \lambda(v, w) = (\lambda v, \lambda w)\,.$$

These operations make $V \times W$ into a vector space. (You do not need to prove this).

Define $\|\cdot\|_1 : V \times W \to \mathbb{R}$ and $\|\cdot\|_{\max} : V \times W \to \mathbb{R}$ by

$$\|(v, w)\|_1 = \|v\|_V + \|w\|_W\,, \qquad \|(v, w)\|_{\max} = \max\{\|v\|_V, \|w\|_W\}\,.$$

(i) Show that $\|\cdot\|_1$ is a norm on $V \times W$.

(ii) Show that $\|\cdot\|_{\max}$ is a norm on $V \times W$.

Let $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ be inner products on V and W, respectively and define

$$\langle\cdot,\cdot\rangle : (V \times W) \times (V \times W) \to \mathbb{C}\,, \quad \text{by} \quad \langle(v_1, w_1), (v_2, w_2)\rangle = \langle v_1, v_2\rangle_V + \langle w_1, w_2\rangle_W\,.$$

(iii) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $V \times W$.

(iv) Is $f : V \times V \to \mathbb{C}$ defined by $f(v_1, v_2) = \langle v_1, v_2\rangle_V$ linear? (Give a complete argument for your answer).

The exam continues - Please turn over!

Problem 2

Let $g \in C_c(\mathbb{R})$ be a continuous function with compact support. Define for $f \in C_c(\mathbb{R})$ the function $P_g f$ by

$$P_g f(x) = \left(\int_{-\infty}^{\infty} f(t) g(t) \, \mathrm{d}t \right) g(x) \, .$$

(i) Show that $P_g f$ is well-defined and that $P_g f \in C_c(\mathbb{R})$.

(ii) Show that $P_g : C_c(\mathbb{R}) \to C_c(\mathbb{R}) : f \mapsto P_g f$ is a linear operator.

Equip $C_c(\mathbb{R})$ with the 2-norm:

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t \right)^{1/2} .$$

(iii) Show that $P_g : (C_c(\mathbb{R}), \|\cdot\|_2) \to (C_c(\mathbb{R}), \|\cdot\|_2)$ is bounded.

(iv) Find the operator norm of $P_g : (C_c(\mathbb{R}), \|\cdot\|_2) \to (C_c(\mathbb{R}), \|\cdot\|_2)$.

Let $p \geq 1$ and equip $C_c(\mathbb{R})$ with the p-norm:

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t \right)^{1/p} .$$

(v) Show that $P_g : (C_c(\mathbb{R}), \|\cdot\|_p) \to (C_c(\mathbb{R}), \|\cdot\|_p)$ is bounded.

(vi) Find the operator norm of $P_g : (C_c(\mathbb{R}), \|\cdot\|_p) \to (C_c(\mathbb{R}), \|\cdot\|_p)$.

Problem 3

Consider a Hilbert space $\mathcal{H}$ and let $(\mathbf{e}_k)_{k \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Consider the linear operator $T : \mathcal{H} \to \mathcal{H}$ defined by $T\mathbf{e}_k = \mathbf{e}_{2k}$. (If $\mathbf{v} = \sum_{k=1}^{\infty} c_k \mathbf{e}_k$ then $T\mathbf{v} = \sum_{k=1}^{\infty} c_k \mathbf{e}_{2k}$, you do not need to show this or that T is linear).

(i) Show that T is bounded.

(ii) Find the operator norm of T.

(iii) Find the adjoint operator T^*.

(iv) Show that T is an isometry.

The exam continues!

Problem 4

Recall that $\mathcal{F}\chi_{[-1,1]}(y) = \operatorname{sinc}(2y)$. Let α and β be real numbers with $\alpha < \beta$. Use the translation operator T_a, the modulation operator E_b, the dilation operator D_c, and their relations with the Fourier transform to calculate the Fourier transform of $\chi_{[\alpha,\beta]}$ without explicit integration.

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End of exam.