Technical University of Denmark

02435: Decision-making under uncertainty

Group Assignment

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Part 1

Task 1

In the first task, we are given the historic paper demands for the last 10 years on a weekly level. Our goal is now to use bootstrapping to generate *new* scenarios. We start by assessing the data. Figure 1 shows the 10 years of data that we are given.

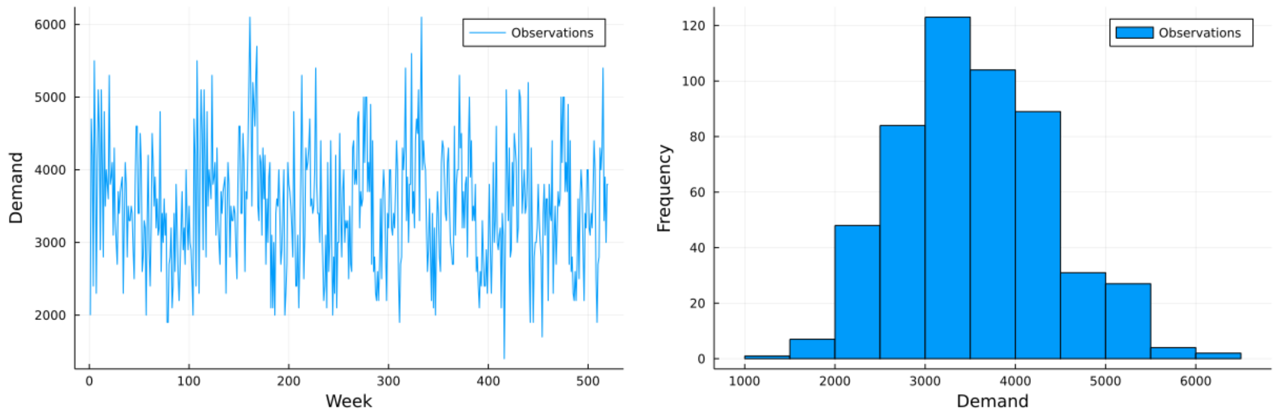

Figure 1: Historic paper demand.

We can now zoom in on a randomly selected narrow interval of the data. Figure 2 shows a narrow interval of the historic paper demand. It is evident that there is some kind of time correlation in the data, i.e., auto-correlation.

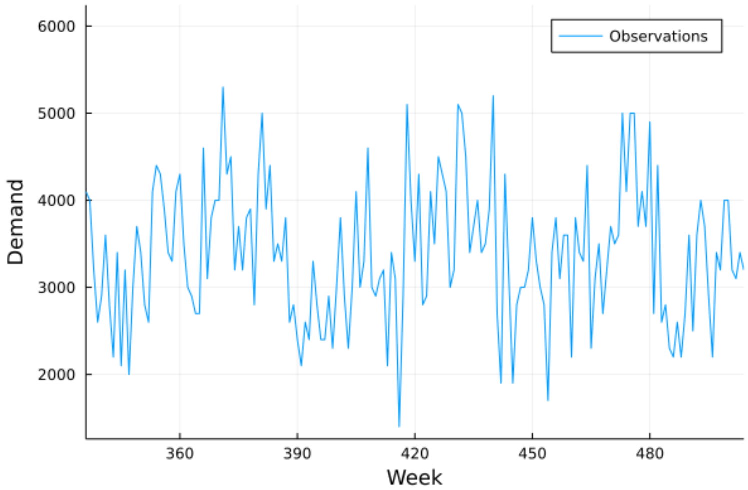

Figure 2: Narrow interval of the historic paper demand.

We can now take a look at the auto-correlation of the time series. Figure 3 shows that there is some auto-correlation present. In order to maintain the auto-correlation structure, it is necessary that we divide the time series into blocks before we bootstrap, i.e., select samples. Therefore, we need to select a suitable block size. This problem is a two-sided sword. On one hand, we want to choose a block size that is large enough such that the auto-correlation structure is kept, on the other hand, we want it small enough such that we can generate a reasonable amount of unique scenarios. In this regard, it seems that there is a couple of reasonable possibilities based on the auto-correlation, as the ACF has values above the confidence interval up to eight weeks, and again after a year. Thus, possibilities are: 4, 8 and 52 weeks, which corresponds to 1 month, 2 months and 1 year. For simplicity, we choose to use a block size of 4 weeks, in order to ensure variability in scenarios and retain some auto-correlation.[1]

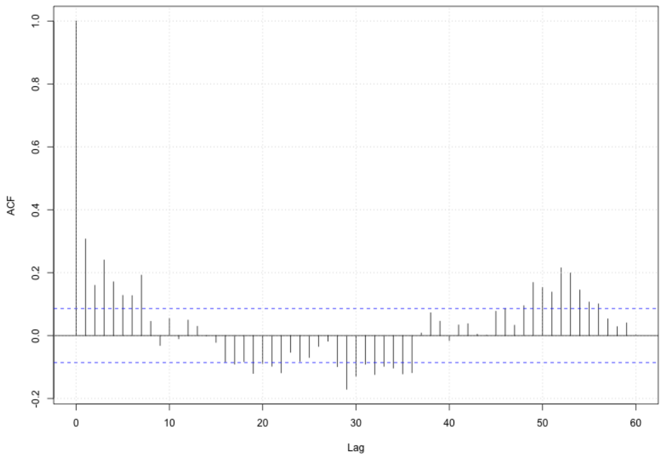

Figure 3: Auto-correlation of the demand.

We are now able to generate samples based on the 4 weeks blocks. We generate 1000 scenarios. We then compute the Euclidean distance between the scenarios, which enables us to use the *k-medoids* algorithm from lecture 2, to find the 5 most representative scenarios. Figure 4 show the selected scenarios in color. The grey lines show all the 1000 generated scenarios. The probabilities of each scenario are given in Equation 1.

$$\boldsymbol{\pi} = [0.171 \quad 0.224 \quad 0.115 \quad 0.25 \quad 0.24] \tag{1}$$

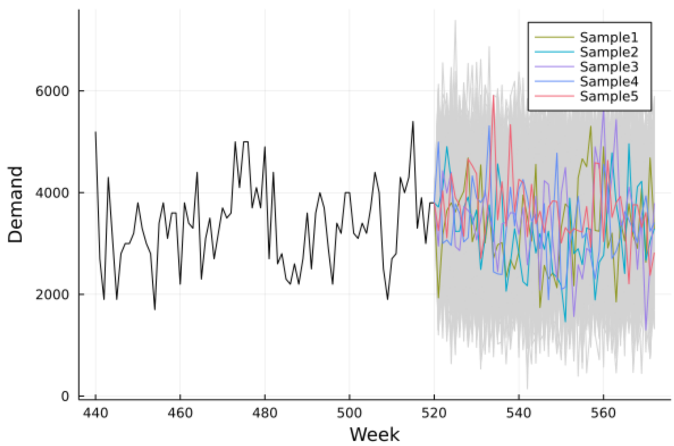

Figure 4: Selected representative scenarios found by bootstrapping and k-medoids.

[1]A further analysis returned the best BIC-score for an ARMA(1,1)-model

Task 2

General formulation

The paper manufacturers planning problem can be formulated as a two-stage stochastic program, as the procurement decisions are made before information about paper demand becomes available to the decision-maker. These are said to be first-stage decisions while decisions about production planning are referred to as second-stage decisions as they can be postponed until demand becomes certain. All sets, variables, and parameters are presented and described in the nomenclature presented in Figure 5. A version in higher resolution appears from Appendix II.

Nomenclature for Part 1

Abbreviations

P	Paper
W	Wood

Parameters

η	Yield coefficient [$\text{ton}_{\text{wood}}/\text{ton}_{\text{paper}}$]
$\overline{A}_w$	Maximum delivery amount from supplier w [ton/week]
ϕ	Cost of unsatisfied demand for paper [€/ton]
π_s	Probability of scenario s
$\underline{A}_w$	Minimum delivery amount from supplier w [ton/week]
c^P	Cost of storing paper from one time period to another [€/ton]
c_k^W	Cost of storing wood from one time period to another at site k [€/ton]
c_w^D	Cost of wood from supplier w [€/ton]
d_t	Demand for paper in time period t [ton/week]
U	Production capacity constraint of paper factory [ton/week]

Sets

K	Set of wood storage sites
S	Set of scenarios
T	Set of time periods
W	Set of suppliers

Variables

θ_w	Binary decision variable for selection of supplier w
$a_{w,t}$	Amount of wood delivered by supplier w in time period t [ton/week]
m_t	Amount of unsatisfied demand for paper in time period t [ton/week]
p_t^W	Amount of paper produced in time period t [ton/week]
$s_{k,t}^W$	Amount of wood stored at site k in time period t [ton/week]
s_t^P	Amount of paper stored in time period t [ton/week]

Figure 5: Nomenclature for Part 1.

Below the general formulation of the deterministic problem is presented:

$$\textbf{Min} \quad \sum_{w\in W}\sum_{t\in T} a_{w,t}\cdot c^D_w + \sum_{t\in T} s^P_t \cdot c^P + m_t\cdot\phi + \sum_{k\in K}\sum_{t\in T} s^W_{k,t}\cdot c^W_k \tag{2}$$

$$\begin{aligned}
\textbf{s.t.} \quad & \theta_w \cdot \underline{A}_w \le a_{w,t} \le \theta_w\cdot\overline{A}_w && \forall\, w\in W,\ t\in T && (3)\\
& p_t \le U && \forall\, t\in T && (4)\\
& s^P_t = s^P_{t-1} + p_t - d_t + m_t && \forall\, t\in T && (5)\\
& s^P_t \le L^P && \forall\, t\in T && (6)\\
& \sum_{k\in K} s^W_{k,t} = \sum_{w\in W} a_{w,t} + \sum_{k\in K} s^W_{k,t-1} - p_t\cdot\eta && \forall\, t\in T && (7)\\
& s^W_{k,t} \le L^W_k && \forall\, k\in K,\ t\in T && (8)\\
& s^W_{k,0} = s^P_0 = 0 && \forall\, k\in K && (9)\\
& \theta_w \in \{0,1\} && \forall\, w\in W && (10)\\
& a_{w,t} \ge 0 && \forall\, w\in W,\ t\in T && (11)\\
& p_t \ge 0 && \forall\, t\in T && (12)\\
& m_t \ge 0 && \forall\, t\in T && (13)\\
& s^P_t \ge 0 && \forall\, t\in T && (14)\\
& s^W_{k,t} \ge 0 && \forall\, k\in K,\ t\in T && (15)
\end{aligned}$$

The initial formulation is extended to account for uncertain demand by including the set of scenarios, S, and modifying the objective function by multiplying all terms relating to the second stage by the probability of each scenario π_s. Hereby, the general formulation of the two-stage stochastic program becomes:

$$\textbf{Min} \quad \sum_{w\in W}\sum_{t\in T} a_{w,t}\cdot c^D_w + \pi_s\left[\sum_{t\in T} s^P_{t,s}\cdot c^P + m_{t,s}\cdot\phi + \sum_{k\in K}\sum_{t\in T} s^W_{k,t,s}\cdot c^W_k\right] \tag{16}$$

$$\begin{aligned}
\textbf{s.t.} \quad & \theta_w \cdot \underline{A}_w \le a_{w,t} \le \theta_w\cdot\overline{A}_w && \forall\, w\in W,\ t\in T\backslash\{0\} && (17)\\
& p_{t,s} \le U && \forall\, t\in T,\ s\in S && (18)\\
& s^P_{t,s} = s^P_{t-1,s} + p_{t,s} - d_{t,s} + m_{t,s} && \forall\, t\in T,\ s\in S && (19)\\
& s^P_{t,s} \le L^P && \forall\, t\in T,\ s\in S && (20)\\
& \sum_{k\in K} s^W_{k,t,s} = \sum_{w\in W} a_{w,t} + \sum_{k\in K} s^W_{k,t-1,s} - p_{t,s}\cdot\eta && \forall\, t\in T,\ s\in S && (21)\\
& s^W_{k,t,s} \le L^W_k && \forall\, k\in K,\ t\in T,\ s\in S && (22)\\
& s^W_{k,0,s} = s^P_{0,s} = 0 && \forall\, k\in K,\ s\in S && (23)\\
& a_{w,0} = 0 && \forall\, w\in W && (24)\\
& \theta_w \in \{0;1\} && \forall\, w\in W && (25)\\
& a_{w,t} \ge 0 && \forall\, w\in W,\ t\in T && (26)\\
& p_{t,s} \ge 0 && \forall t\in T,\ s\in S && (27)\\
& m_{t,s} \ge 0 && \forall t\in T,\ s\in S && (28)\\
& s^P_{t,s} \ge 0 && \forall t\in T,\ s\in S && (29)\\
& s^W_{k,t,s} \ge 0 && \forall k\in K,\ t\in T,\ s\in S && (30)
\end{aligned}$$

Objective function:

Equation 16 presents the objective function of the procurement and production planning problem. From this, it is evident that total costs of supplying the desired amount of paper is minimized by optimizing the sum of costs related to procuring raw materials (first term), storing raw materials (second term), storing finished products (third term), and penalty for unsatisfied demand (fourth term). The three last terms are multiplied with the probability of each scenario π_s.

Constraints:
Equation 17 ensures that the amount delivered by supplier w in week t, $a_{w,t}$, stays within the interval defined for said supplier $[\underline{A}_w, \overline{A}_w]$ in case the supplier in question has been selected in the first stage. Whether a supplier has been selected or not is governed by the binary decision variables θ_w.

Equation 18 ensures that the amount of paper produced in time period t denoted P_t, never exceeds the production capacity constraint of the factory expressed by the parameter U.

Equation 19 updates the amount of paper stored in time period t while Equation 21 has a similar function for the amount of wood stored at storage site k at time period t.

Equation 20 guarantees that the amount of paper stored in time period t never exceeds the paper storage capacity L^P while Equation 22 fulfills the same purpose for the wood storages.

Equation 23 specifies that the paper storage, as well as all wood storage sites, are initially empty. Equation 25 ensures that each supplier is at most selected once, by specifying that the selection of wood suppliers is governed by a binary variable.

Equation 26-Equation 30 enforces non-negativity for all continuous decision variables.

Solution to the problem

Utilizing the data provided in the file `TaskB2-data.jl` the problem is solved by a Julia implementation. This results in an objective value of $1.80 \cdot 10^7$.

Figure 6 shows a graphical representation of the wood supply from the 15 suppliers. Notice that suppliers 8 and 13 are exactly 0. These suppliers have not been selected, the rest have been. Notice that in general, we can say that the model finds it advantageous to get a high initial supply of wood. This is done largely to avoid unsatisfied demand, as this is penalized with a high cost, given by the penalty parameter, ϕ, in the objective function. After a short period of time (approx. 3 weeks), all deliveries are at the lower bound of what the contracts allow.

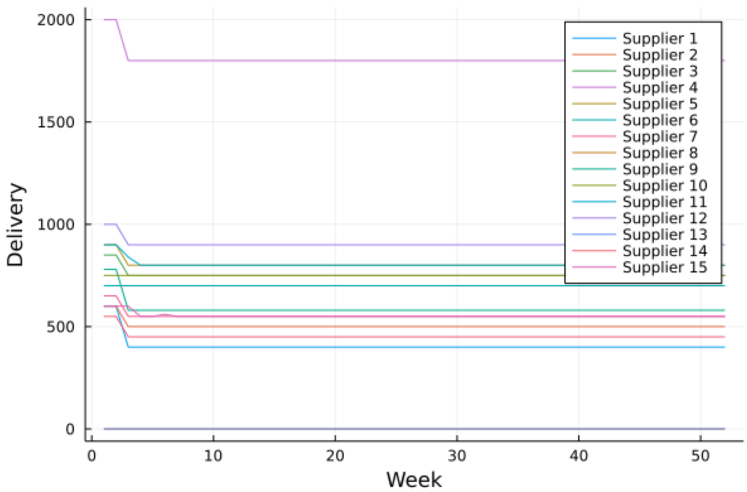

Figure 6: Weekly supply of wood from the 15 suppliers.

Figure 7 shows the wood on stock for the 4 sites over the whole period. Notice that only one warehouse is used. This is simply because it has enough capacity to handle all the wood and has the lowest cost per ton. Further note that the wood on stock seems to increase throughout the period. This indicates that even with the supply being at the lower bounds for all contracts, the wood delivery still exceeds the production need in the long run. Thus, the inflexibility resulting from Equation 17 leads to a build-up of raw materials.

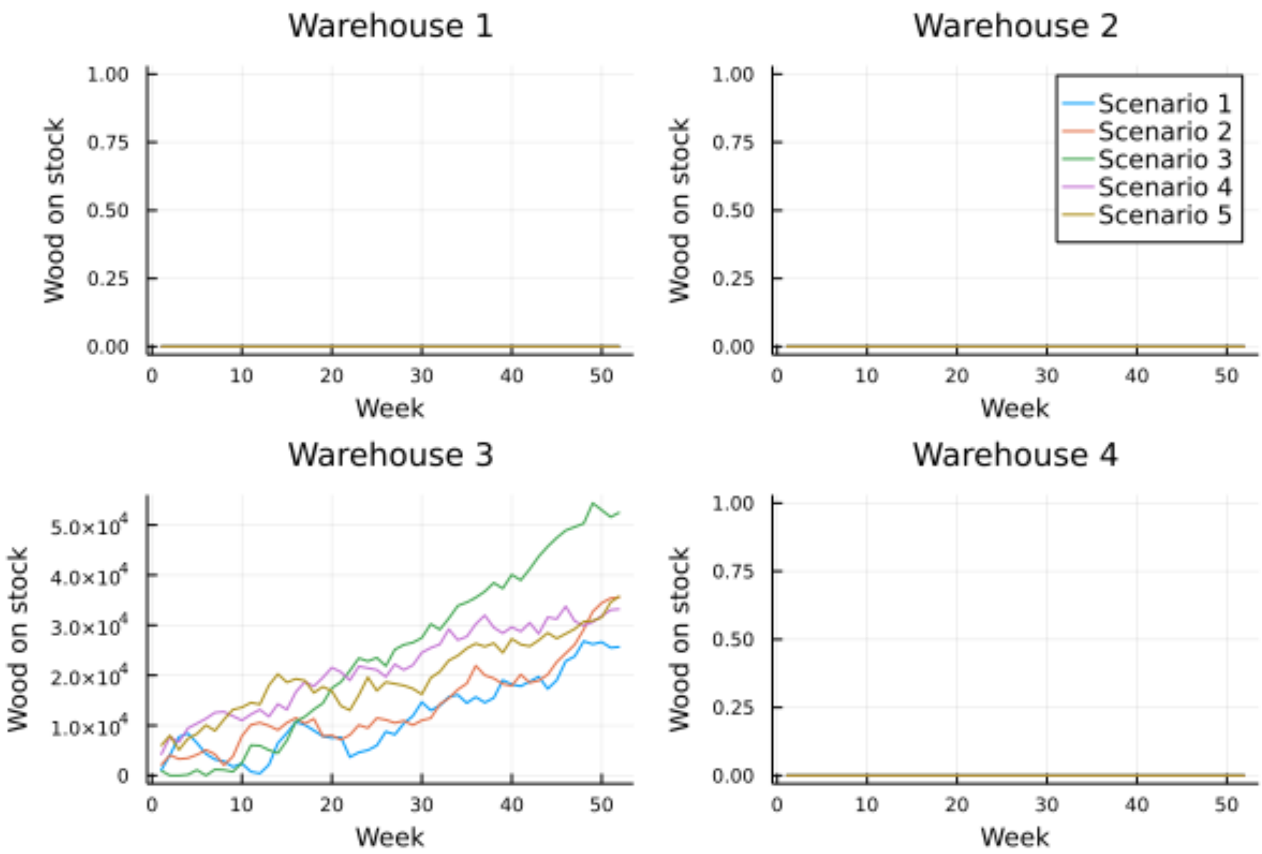

Figure 7: Wood on stock at the 4 sites.

Figure 8 shows the paper on stock. Notice that there is no paper on stock. This is simply because there is no delay when converting from wood to paper, and it is cheaper to store the wood than the paper — even when accounting for the conversion rate, η.

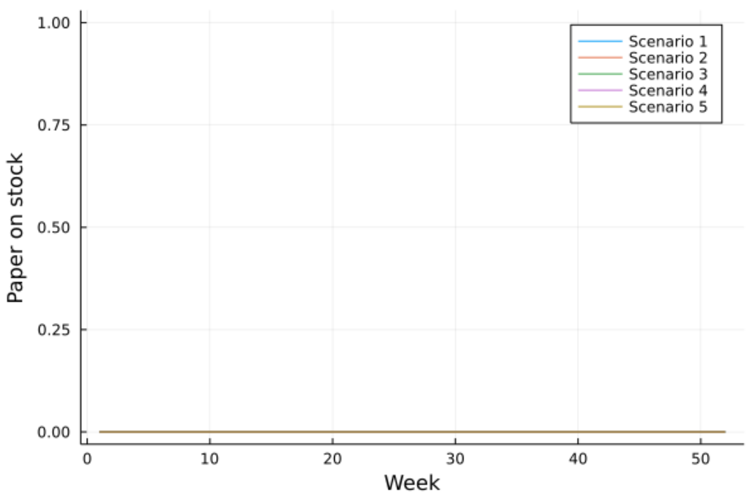

Figure 8: Paper on stock.

Figure 9 shows the production of paper in tons for the 5 scenarios over the whole period.

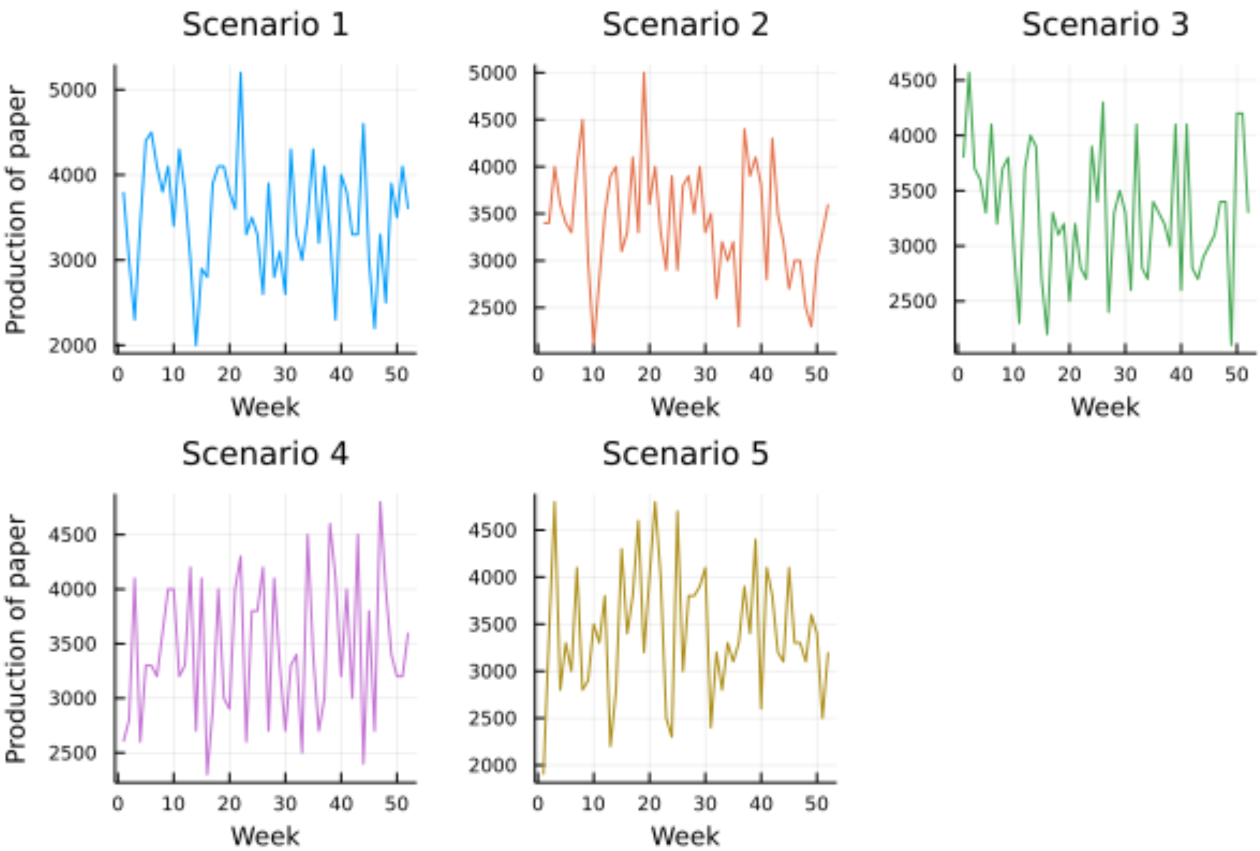

Figure 9: Production of paper in the 5 scenarios.

Figure 10 shows the deficit of paper in tons for the 5 scenarios over the whole period. Notice that only in scenario 3 there is a deficit in one week. This is because of the relatively high penalty associated with such a deficit; which then leads to an optimal solution where a quite large amount of wood is bought initially.

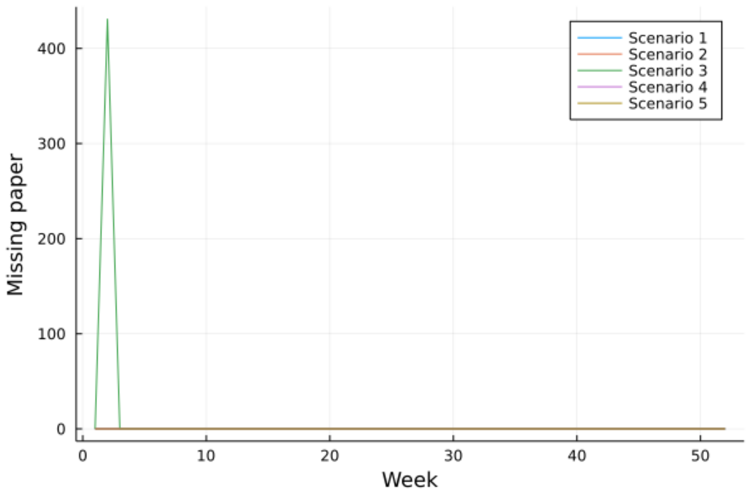

Figure 10: Deficit in paper in the 5 scenarios.

Figure 11 shows the demand of paper in tons for the 5 scenarios over the whole period. If we compare this figure with the previous two — the production and the deficit — we see that to a large extend the production is equal to the demand. The only exception to this is in scenario 3 in week 3, where there is a small deficit.

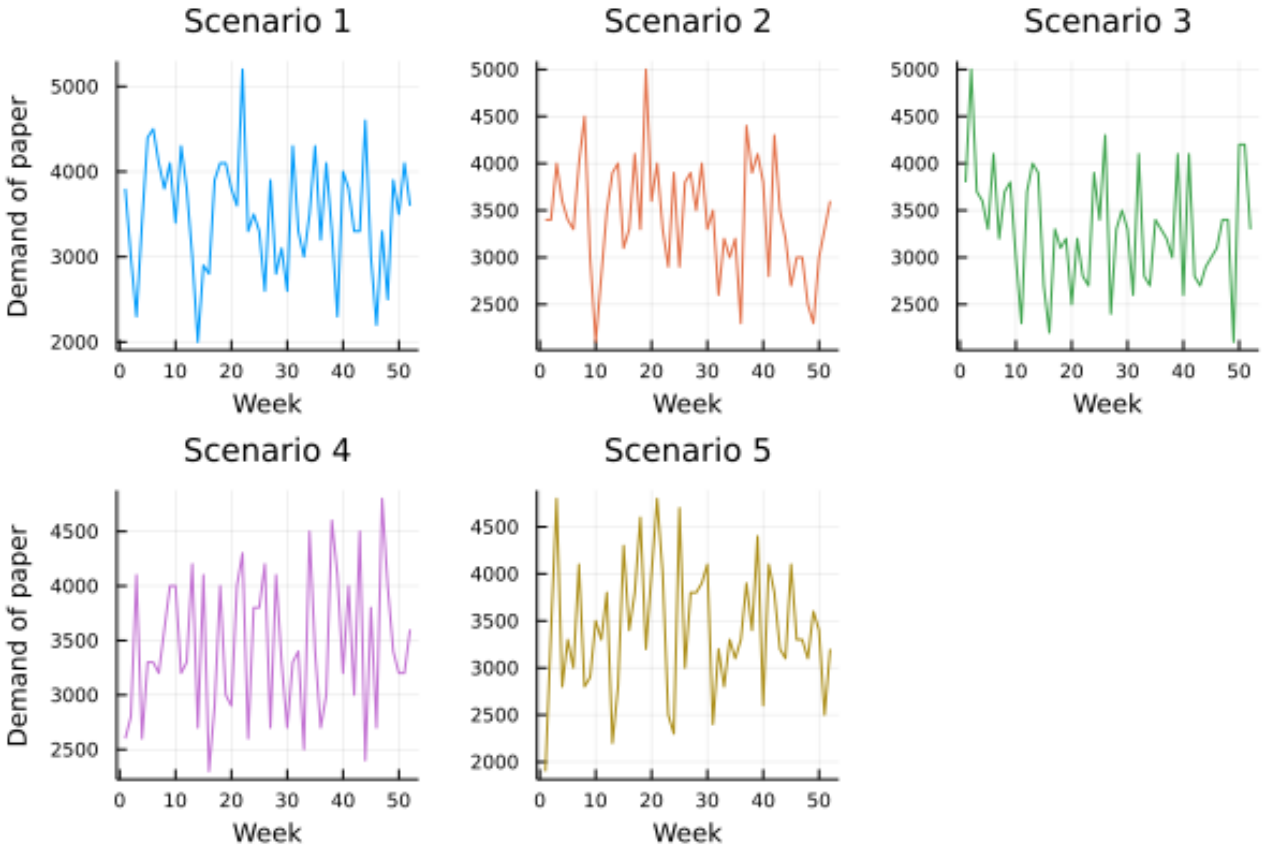

Figure 11: Demand of paper in the 5 scenarios.

What are the important decisions for the company?

The most important decisions the company are faced with are the first-stage decisions. Selecting which suppliers to buy from, governed by the binary decision variable θ_w, is the most restrictive decision as it limits which suppliers to consider for the remaining part of the planning horizon. These decisions also affects the constraint in Equation 17, which states that if a supplier is selected the company is obligated to sources at least $\underline{A}_w$ units of wood per week from this supplier. This leaves only a small range of flexibility between what the company is obligated to take delivery of and the maximum amount a supplier is able to deliver $\overline{A}_w$.

In the second stage the decision-maker has limited management autonomy as it is restricted by the procurement decisions made in the first stage. This reduces the complexity of the production decision, such that the sum of production, unfulfilled demand, and paper taken from storage must equal demand in each time period.

The solution is very sensitive to the value of ϕ. Given the high initial value for this parameter the model chooses to source sufficient wood to avoid unsatisfied demand as this is penalized heavily. This results in a high amount of wood in storage throughout the entire planning period which is associated with storage costs. Thus, the high penalty for unsatisfied demand does not directly affect the objective value, however, it forces the decision-maker to accept paying higher storage costs than would be accepted if assuming a lower value for ϕ.

Task 3

Table 1 presents the stochastic solution compared to the wait-and-see and expected value solutions. The delivery variables a, have been listed for every supplier, but summed for all time periods for simplicity. The deliveries of the wait-and-see solution are different for each scenario thus they are not listed here. The expected value solution chooses to source the wood from fewer but cheaper suppliers thus not relying on the flexibility of deliveries when choosing more suppliers. This can be an issue in the recourse problem when the demand fluctuates. The objective values in each of the five scenarios are also listed, where it can be seen that the three models have their highest costs in scenario 3. Unsurprisingly, the wait-and-see approach produces the lowest costs of each scenario, where as the expected value solution have much higher costs which is due to poor procurement planning and thus the solutions for all scenarios are penalized heavily when the model cannot meet the demand. This only happen in scenario 3 for the stochastic solution and the deficit demand is only 431 for all time periods, and it is even lower in the wait-and-see solution at only 7.69 tons of unmet demand.

Table 1: Tabulated results of the stochastic solution, wait and see solution and the expected value solution, respectively.

	Stochastic Solution	**Wait-And-See Solution**	**Expected Value Solution**
Agg. deliveries for each supllier w:			
$\sum_{t\in T} a_{1,t}$ (tons)	$2.12 \cdot 10^4$	Individual	$2.55 \cdot 10^4$
$\sum_{t\in T} a_{2,t}$ (tons)	$2.62 \cdot 10^4$	Individual	$2.72 \cdot 10^4$
$\sum_{t\in T} a_{3,t}$ (tons)	$3.92 \cdot 10^4$	Individual	$3.97 \cdot 10^4$
$\sum_{t\in T} a_{4,t}$ (tons)	$9.40 \cdot 10^4$	Individual	$9.43 \cdot 10^4$
$\sum_{t\in T} a_{5,t}$ (tons)	$4.18 \cdot 10^4$	Individual	$4.63 \cdot 10^4$
$\sum_{t\in T} a_{6,t}$ (tons)	$3.64 \cdot 10^4$	Individual	$3.64 \cdot 10^4$
$\sum_{t\in T} a_{7,t}$ (tons)	$2.88 \cdot 10^4$	Individual	0
$\sum_{t\in T} a_{8,t}$ (tons)	0	Individual	0
$\sum_{t\in T} a_{9,t}$ (tons)	$3.06 \cdot 10^4$	Individual	0
$\sum_{t\in T} a_{10,t}$ (tons)	$3.90 \cdot 10^4$	Individual	$3.90 \cdot 10^4$
$\sum_{t\in T} a_{11,t}$ (tons)	$4.18 \cdot 10^4$	Individual	$4.68 \cdot 10^4$
$\sum_{t\in T} a_{12,t}$ (tons)	$4.70 \cdot 10^4$	Individual	$5.20 \cdot 10^4$
$\sum_{t\in T} a_{13,t}$ (tons)	0	Individual	0
$\sum_{t\in T} a_{14,t}$ (tons)	$2.36 \cdot 10^4$	Individual	$2.41 \cdot 10^4$
$\sum_{t\in T} a_{15,t}$ (tons)	$2.88 \cdot 10^4$	Individual	$3.12 \cdot 10^4$
Unsatisfied Demand Per Scenario:			
$\sum_{t\in T} m_{t,1}$ (tons)	0	0	$4.68 \cdot 10^3$
$\sum_{t\in T} m_{t,2}$ (tons)	0	0	$3.72 \cdot 10^3$
$\sum_{t\in T} m_{t,3}$ (tons)	$4.31 \cdot 10^2$	$7.69 \cdot 10^0$	$2.29 \cdot 10^3$
$\sum_{t\in T} m_{t,4}$ (tons)	0	0	$1.12 \cdot 10^3$
$\sum_{t\in T} m_{t,5}$ (tons)	0	0	$1.15 \cdot 10^3$
Result Per Scenario:			
s_1 (€) Probability: 0.32	$1.68 \cdot 10^7$	$1.54 \cdot 10^7$	$6.14 \cdot 10^7$
s_2 (€) Probability: 0.06	$1.69 \cdot 10^7$	$1.52 \cdot 10^7$	$5.18 \cdot 10^7$
s_3 (€) Probability: 0.26	$2.13 \cdot 10^7$	$1.84 \cdot 10^7$	$3.76 \cdot 10^7$
s_4 (€) Probability: 0.28	$1.70 \cdot 10^7$	$1.47 \cdot 10^7$	$2.58 \cdot 10^7$
s_5 (€) Probability: 0.08	$1.69 \cdot 10^7$	$1.46 \cdot 10^7$	$2.61 \cdot 10^7$
Expected Value	$1.80 \cdot 10^7$	$1.59 \cdot 10^7$	$4.19 \cdot 10^7$

Table 2 presents quality metrics for the stochastic program using the scenarios in the file `reduced-scenarios.csv` as input.

Optimising the planning problem with a stochastic program instead of an expected value program results in a cost reduction for the manufacturer of $VSS = 2.38 \cdot 10^7$ €. This is due to the stochastic program being able to respond to the uncertainty given in the five scenarios. The expected value program does not consider uncertainty and only relies on the expected demand for each time period.

The wait-and-see solution describes the best possible result that can be achieved in the theoretical situation where all information is available to the decision-maker at the time of the first-stage decisions. This is rather irrelevant for practical applications due to the uncertain nature of the stochastic problem where parts of the information is not realized until the second stage. If the stochastic programming is a minimisation problem, then the Expected Value of Perfect Information (EVPI) is a lower bound for the stochastic problem. The EVPI is $-2.14 \cdot 10^6$ € and is negative since it is a minimisation problem. The EVPI can be seen as is the maximum amount of resources the planner should be willing to spend on obtaining perfect insight to what will happen in the future. Thus, a more accurate forecast would have a value between 0 and EVPI depending on how much it reduces the uncertainty.

The wait-and-see solution does not have any direct value to the decision-maker, but it can be used to evaluate the recourse decisions via EVPI.

Table 2: Summary of quality metrics

Name	**Abbreviation**	**Value (€)**
Expectation of expected value solution	EEV	$4.19 \cdot 10^7$
Recourse Problem	RP	$1.80 \cdot 10^7$
Wait-and-see	WS	$1.59 \cdot 10^7$
Expected value of perfect information	EVPI	$-2.14 \cdot 10^6$
Value of stochastic solution	VSS	$-2.38 \cdot 10^7$

Task 4

In the previous task, we looked at EVPI and VSS, both of which are in-sample measures. We will now turn our attention to out-of-sample tests. We are given 100 out of sample demand scenarios. We can now compute the first stage decision variables using both stochastic programming and mean value solution using the in-sample scenarios, i.e., the 5 scenarios given initially.

Figure 12 shows the objective value for both in- and out-of-sample demands using the first stage variables found by solving the stochastic program. This illustration gives an indication of how well the stochastic program generalizes; and consequently, how representative the in-sample scenarios really were. Notice that we see a significant difference between the expected objective values (indicated by the vertical lines) between the in- and the out-of-sample. This indicates that the initial scenarios do not completely represent the distribution of the out-of-sample scenarios, and is therefore over-optimistic regarding the expected objective value.

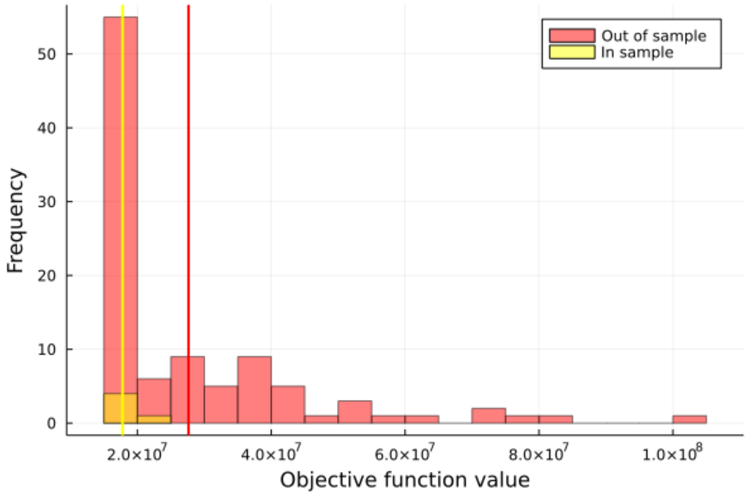

Figure 12: In sample and out of sample comparison of RP.

Figure 13 shows the out-of-sample objective values using both WS, RP and EV. Notice that WS is not truly an out-of-sample solution, since always assumes perfect information. Notice also that even though we just saw that RP is over-optimistic it seems that the mean value approach performs much worse than the out-of-sample as well.

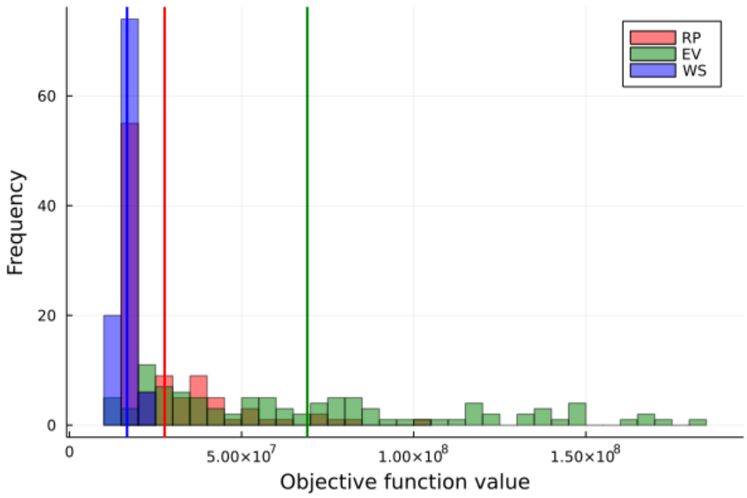

Figure 13: In sample and out of sample comparison of RP.

As mentioned, EVPI and VSS are in-sample tests, as such, they will be sensitive to whether the chosen scenarios are representative. When the chosen in-sample scenarios are representative of the data, we would expect the EVPI and VSS to give a good indication of the performance of the stochastic solution. However, when performing out-of-sample tests, we test the overall performance. Therefore, this approach does not require the in-sample scenarios to be representative. If not, the performance would most likely be poor, but the test would work.

When we access the in-sample tests, i.e., EVPI and VSS, together with the out-of-sample test, we see a clear picture. Obviously, we would prefer the wait-and-see method, but since we cannot predict the future with 100% certainty, this is unfortunately not a viable option. We, therefore, have to choose between the mean value solution and the solution from the stochastic program. Based both on the in-sample VSS and the difference in the out-of-sample test, we would strongly suggest solving the stochastic program above using the mean value solution.

Part 2

Task 5

In Task 5 we will work with the company *Lovely Cards*, which is a customer of *Paper2000*. Here in Task 5, we are to formulate a planning problem which minimizes the production costs while still meeting the demand. All sets, variables, and parameters used in Task 5 are presented and described in the nomenclature presented in Figure 14. A version in higher resolution appears from Appendix III.

Nomenclature for Part 2

Parameters

γ	Ad hoc purchase price [€]
$\overline{K}$	Maximum production capacity [units/week]
σ	Conversion factor for new paper
$\underline{K}$	Minimum production capacity [units/week]
C^K	Production cost per capacity [€/unit of capacity]
C^P	Cost per produced unit of paper [€/unit of paper]
L_t	Demand for birthday cards in time period t [units/week]
U	Maximum amount of purchase [units/week]

Sets

Z	Uncertainty set, conversion factor between paper and birthday cards $\mathbb{Z} = \{z \in \mathbb{R} \mid 400 \leq z \leq 600\}$
T	Set of time periods [weeks] $T = \{t \in \mathbb{Z} \mid 1 \leq t \leq 4\}$

Variables

α_t	Help variable
β	Help variable
ϕ_t	Help variable
θ_t	Help variable
Q_t	Slope for affine decision rule
q_t^0	Intercept for affine decision rule
x_t	Amount of paper processed [units/week]
y	Production capacity [units/week]

Figure 14: Nomenclature for Part 2.

5.1 Robust Model

We have the following mathematical model describing the planning problem.

$$
\begin{aligned}
\underset{x_t,y}{\textbf{Min}} \quad & \sum_{t\in T} C^P x_t + C^K y && && (31)\\
\textbf{s.t.} \quad & x_t \le y, && \forall t\in T && (32)\\
& z x_t \ge L_t, && \forall t\in T,\ z\in\mathcal{Z} && (33)\\
& \underline{K} \le y \le \overline{K} && && (34)\\
& x_t \in \mathbb{R}_+, && \forall t\in T && (35)\\
& y \in \mathbb{R}_+ && && (36)
\end{aligned}
$$

where the objective, 31, describes the cost of processing paper per week, $\sum_{t\in T} C^P x_t$, plus the setup cost of the production, $C^K y$. The first constraint, 32, makes sure the production each week does not exceed the capacity for which the production is set up for all 4 weeks. The next constraint, 33, makes sure that the demand L_t is met every week. Constraint 34 restricts the capacity of the production to be within the given limits, while the last two constraints enforce the variables x_t and y to be non-negative integers.

We now see that the uncertainty is described by a box uncertainty. We reformulate constraint 33 according to slide 34 in lecture 3.

$$
\begin{aligned}
x_t \min_z\{z\in\mathbb{R}|\ 400\le z\le 600\} \ge L_t, &\quad \forall t\in T \Rightarrow\\
x_t(500 + 100\min_z\{z\in\mathbb{R}|\ -1\le z\le 1\}) \ge L_t, &\quad \forall t\in T \Rightarrow\\
500x_t - |100x_t| \ge L_t, &\quad \forall t\in T \Rightarrow\\
500x_t - 100x_t \ge L_t, &\quad \forall t\in T
\end{aligned}
$$

The first 3 steps are standard steps but the last step where we remove the absolute signs without any auxiliary variables is only possible because x_t is non-negative. We can now substitute the reformulated constraint back into the model and obtain the linear robust optimization problem.

$$
\begin{aligned}
\underset{x_t,y}{\textbf{Min}} \quad & \sum_{t\in T} C^P x_t + C^K y && && (37)\\
\textbf{s.t.} \quad & x_t \le y, && \forall t\in T && (38)\\
& 400 x_t \ge L_t, && \forall t\in T && (39)\\
& \underline{K} \le y \le \overline{K} && && (40)\\
& x_t \in \mathbb{R}_+, && \forall t\in T && (41)\\
& y \in \mathbb{R}_+ && && (42)
\end{aligned}
$$

5.2 Solving the Robust Model

We have implemented the robust model in Julia and obtained a cost of 65329.15 €. In Figure 15 the weekly consumption of paper, x_t, is plotted together with the total capacity y.

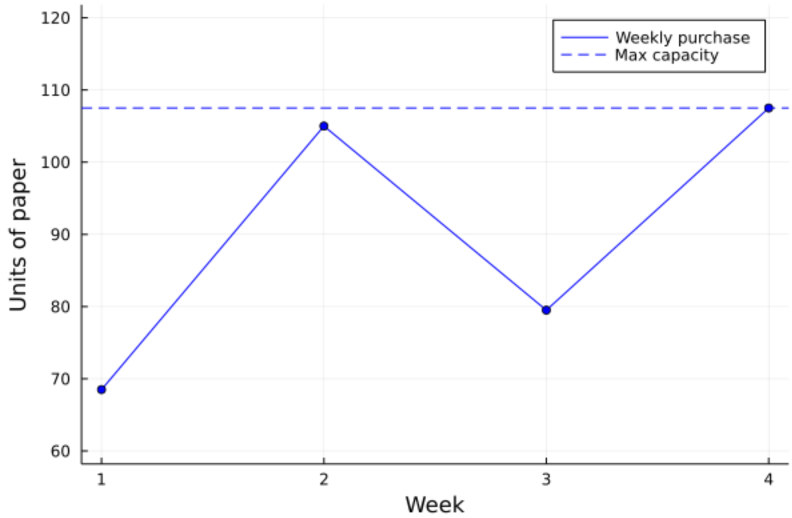

Figure 15: Production for the robust model.

5.3 Adjustable Robust Model

We now introduce an ad hoc purchase possibility. This can be incorporated by the use of adjustable robust optimization. All additional symbols that are added compared to the robust model are presented and described in Figure 14. We now extend the robust model with the possibility to buy extra paper dependent on the realization of the uncertainty.

$$\mathbf{Min}_{x_t, y, x_t^{\text{ad hoc}}(z)} \quad \sum_{t \in T} C^P x_t + C^K y + \sum_{t \in T} \gamma x_t^{\text{ad hoc}}(z) \tag{43}$$

$$\begin{aligned}
\textbf{s.t.} \quad & x_t + x_t^{\text{ad hoc}}(z) \leq y, && \forall t \in T,\ -1 \leq z \leq 1 && (44)\\
& (500 + 100z)x_t + \sigma x_t^{\text{ad hoc}}(z) \geq L_t, && \forall t \in T,\ -1 \leq z \leq 1 && (45)\\
& 0 \leq x_t^{\text{ad hoc}}(z) \leq U, && \forall t \in T,\ -1 \leq z \leq 1 && (46)\\
& \underline{K} \leq y \leq \overline{K} && && (47)\\
& x_t, x_t^{\text{ad hoc}}(z) \in \mathbb{R}_+, && \forall t \in T && (48)\\
& y \in \mathbb{R}_+ && && (49)
\end{aligned}$$

Our recourse variable $x_t^{\text{ad hoc}}(z)$ is not easy to work with so we make it into an affine decision rule.

$$x_t^{\text{ad hoc}}(z) = q_t^0 + Q_t z \tag{50}$$

$x_t^{\text{ad hoc}}(z)$ is substituted into the problem;

$$\underset{x_t, y, q_t^0, Q_t}{\textbf{Min}} \quad \sum_{t \in T} C^P x_t + C^K y + \sum_{t \in T} \gamma(q_t^0 + Q_t z) \tag{51}$$

$$\textbf{s.t.} \quad x_t + q_t^0 + Q_t z \le y, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{52}$$
$$(500 + 100z)x_t + \sigma(q_t^0 + Q_t z) \ge L_t, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{53}$$
$$0 \le q_t^0 + Q_t z \le U, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{54}$$
$$\underline{K} \le y \le \overline{K} \tag{55}$$
$$x_t \in \mathbb{R}_+, \qquad \forall t \in T \tag{56}$$
$$q_t^0, Q_t \in \mathbb{R}, \qquad \forall t \in T \tag{57}$$
$$y \in \mathbb{R}_+ \tag{58}$$

Before we can convert the above into a solvable LP we need to remove the uncertainty from the objective function.

$$\underset{x_t, y, \beta, q_t^0, Q_t}{\textbf{Min}} \quad \sum_{t \in T} C^P x_t + C^K y + \beta \tag{59}$$

$$\textbf{s.t.} \quad \sum_{t \in T} \gamma(q_t^0 + Q_t z) \le \beta, \qquad -1 \le z \le 1 \tag{60}$$
$$x_t + q_t^0 + Q_t z \le y, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{61}$$
$$(500 + 100z)x_t + \sigma(q_t^0 + Q_t z) \ge L_t, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{62}$$
$$0 \le q_t^0 + Q_t z \le U, \qquad \forall t \in T,\ -1 \le z \le 1 \tag{63}$$
$$\underline{K} \le y \le \overline{K} \tag{64}$$
$$x_t \in \mathbb{R}_+, \qquad \forall t \in T \tag{65}$$
$$q_t^0, Q_t \in \mathbb{R}, \qquad \forall t \in T \tag{66}$$
$$\beta \in \mathbb{R} \tag{67}$$
$$y \in \mathbb{R}_+ \tag{68}$$

We can now convert it into a solvable LP. We start by converting constraint 60.

$$\sum_{t \in T} \gamma q_t^0 + \gamma Q_t z \le \beta \Rightarrow \tag{69}$$
$$\sum_{t \in T} \gamma q_t^0 + \max_z \{\gamma Q_t z \mid -1 \le z \le 1\} \le \beta \Rightarrow \tag{70}$$
$$\sum_{t \in T} \gamma q_t^0 + |\gamma Q_t| \le \beta \Rightarrow \tag{71}$$
$$\sum_{t \in T} \gamma q_t^0 + \alpha_t \le \beta \tag{72}$$
$$-\alpha_t \le \gamma Q_t \le \alpha_t, \qquad \forall t \in T \tag{73}$$

We now continue with constraint 61.

$$x_t + q_t^0 + Q_t z \leq y \Rightarrow \tag{74}$$
$$x_t + q_t^0 + \max_z\{Q_t z \mid -1 \leq z \leq 1\} \leq y \Rightarrow \tag{75}$$
$$x_t + q_t^0 + |Q_t z| \leq y \Rightarrow \tag{76}$$
$$x_t + q_t^0 + \phi_t \leq y, \qquad \forall t \in T \tag{77}$$
$$-\phi_t \leq Q_t \leq \phi_t, \qquad \forall t \in T \tag{78}$$

The next constraint is 62.

$$500x_t + 100zx_t + \sigma q_t^0 + \sigma Q_t z \geq L_t \Rightarrow \tag{79}$$
$$500x_t + \sigma q_t^0 + \min_z\{100zx_t + \sigma Q_t z \mid -1 \leq z \leq 1\} \geq L_t \Rightarrow \tag{80}$$
$$500x_t + \sigma q_t^0 - |100x_t + \sigma Q_t| \geq L_t \Rightarrow \tag{81}$$
$$500x_t + \sigma q_t^0 - \theta_t \geq L_t, \qquad \forall t \in T \tag{82}$$
$$-\theta \leq 100x_t + \sigma Q_t \leq \theta_t, \qquad \forall t \in T \tag{83}$$

Lastly, we have constraint 63.

$$q_t^0 + Q_t z \leq U \Rightarrow \tag{84}$$
$$q_t^0 + \max_z\{Q_t z \mid -1 \leq z \leq 1\} \leq U \Rightarrow \tag{85}$$
$$q_t^0 + \phi_t \leq U, \qquad \forall t \in T \tag{86}$$
$$\tag{87}$$

and

$$q_t^0 + Q_t z \geq 0 \Rightarrow \tag{88}$$
$$q_t^0 + \min_z\{Q_t z \mid -1 \leq z \leq 1\} \geq 0 \Rightarrow \tag{89}$$
$$q_t^0 - \phi_t \geq 0, \qquad \forall t \in T \tag{90}$$
$$\tag{91}$$

We collect it all in one model.

$$\underset{x_t, y, \beta, q_t^0, Q_t, \alpha_t, \phi_t, \theta_t}{\textbf{Min}} \quad \sum_{t \in T} C^P x_t + C^K y + \beta \tag{92}$$

$$\textbf{s.t.} \quad \sum_{t \in T} \gamma q_t^0 + \alpha_t \leq \beta \tag{93}$$

$$-\alpha_t \leq \gamma Q_t \leq \alpha_t, \quad \forall t \in T \tag{94}$$

$$x_t + q_t^0 + \phi_t \leq y, \quad \forall t \in T \tag{95}$$

$$-\phi_t \leq Q_t \leq \phi_t, \quad \forall t \in T \tag{96}$$

$$500 x_t + \sigma q_t^0 - \theta_t \geq L_t, \quad \forall t \in T \tag{97}$$

$$-\theta \leq 100 x_t + \sigma Q_t \leq \theta_t, \quad \forall t \in T \tag{98}$$

$$q_t^0 + \phi_t \leq U, \quad \forall t \in T \tag{99}$$

$$q_t^0 - \phi_t \geq 0, \quad \forall t \in T \tag{100}$$

$$\underline{K} \leq y \leq \overline{K} \tag{101}$$

$$x_t, \phi_t, \alpha_t, \theta_t \in \mathbb{R}_+, \quad \forall t \in T \tag{102}$$

$$q_t^0, Q_t \in \mathbb{R}, \quad \forall t \in T \tag{103}$$

$$\beta \in \mathbb{R} \tag{104}$$

$$y \in \mathbb{R}_+ \tag{105}$$

Solving the Adjustable Robust Model

We have implemented the adjustable robust model in Julia and obtain a cost of 57810.15 €. In Figure 16 the weekly purchase of paper is plotted together with the total capacity.

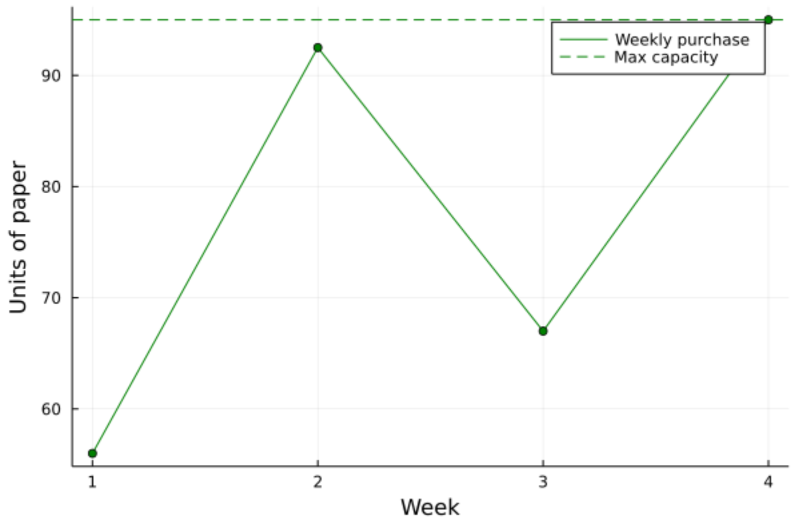

Figure 16: Production for the adjustable robust model.

Comparison

We see that the robust model achieves an objective value of 65329.15 € while the adjustable robust model is almost 10000 € cheaper at 57810.15 €. To understand why we see such a large difference we start by inspecting Figure 17.

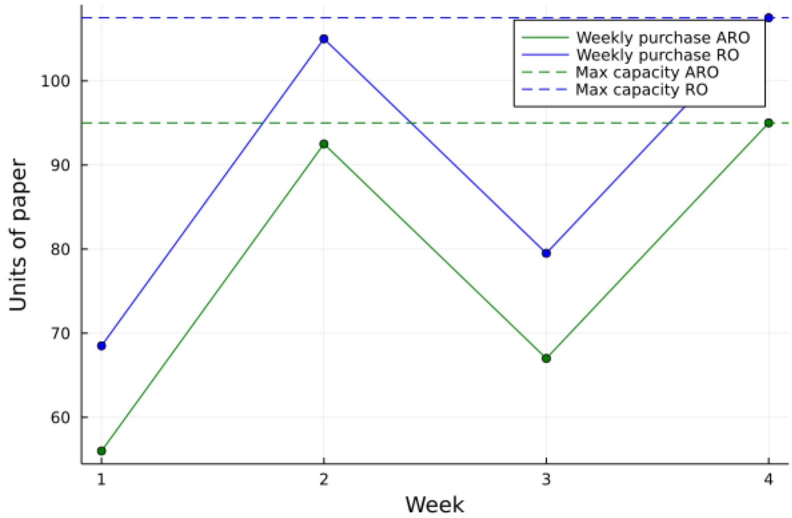

Figure 17: Comparison of the robust model and the adjustable robust model.

We see that when given the ad hoc purchase possibility with a very favourable conversion rate between paper and birthday cards we can keep the production capacity lower. Furthermore, we can also meet the same demand for a smaller amount of paper. To understand how these two differences impact the operational costs we look at Figure 18 and Figure 19.

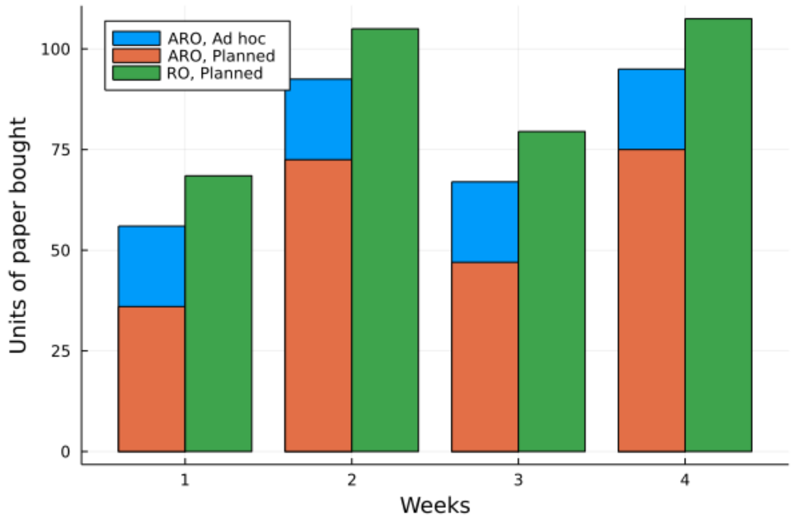

Figure 18: Amount of purchased paper.

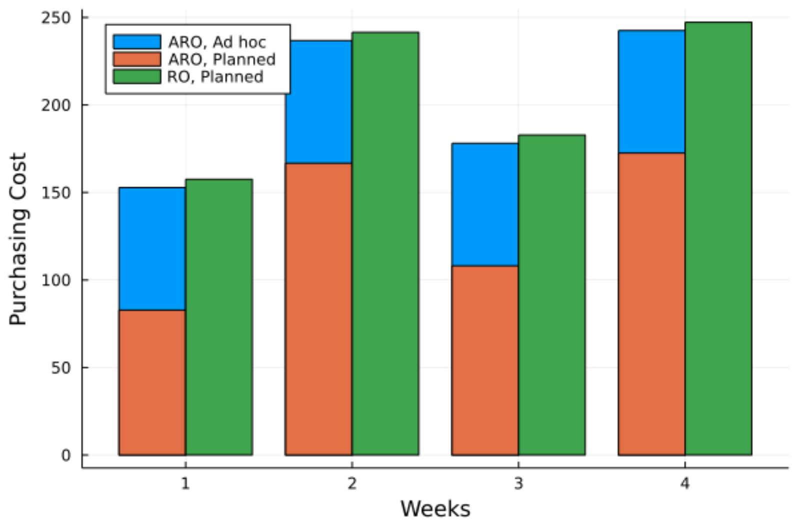

Figure 19: Cost of purchasing paper.

First, we see in Figure 18 that the ad hoc purchase option is fully utilized every week in the adjustable robust model. Next, we see from Figure 19 that the difference in purchasing cost is very small so we inspect how the different production capacities impact the costs.

The robust model needs a production capacity of 107.5 packs of paper per week while the adjustable robust model only needs a production capacity of 95 packs of paper per week. The cost per unit of production capacity is 600 € and hence the robust model uses 64500 € on setting up the production while the adjustable robust model only uses 57000 €. We see that having a higher conversion rate between paper and birthday cards is very favourable and hence the adjustable robust model can produce much cheaper.

Task 6

In this task we will work with the problem:

$$\textbf{Max} \quad 10x_1 + 20x_2 + 15x_3 \tag{106}$$

$$\begin{aligned} \textbf{s.t.} \quad 5x_1 + 3x_2 + 3x_3 &\leq 10 && (107)\\ 7x_1 - 2x_2 - 2x_3 &\geq 5 && (108)\\ -2 \leq x_1 &\leq 10 && (109)\\ 0 \leq x_2 &\leq 15 && (110)\\ -10 \leq x_3 &\leq 10 && (111) \end{aligned}$$

In the following sub-tasks, we will introduce uncertainty to the model and reformulate it to a robust linear model. This can be handled with three different approaches:

- The objective can be multiplied by negative one, making it into a minimization problem, similar to the slides.
- The constraint with uncertainty can likewise be multiplied by negative one, to produce a minimization sub-problem.
- Finally, the maximization can be handled directly, avoiding the use of duality.

In Task 6 we will only apply the last two approaches but in the appendix ”Task 6 Alternative” we have shown the first methodology for both the sub-tasks.

6.1 Polyhedral uncertainty set

In this sub-task, we have the below version of the problem, where $2 \leq \tilde{a}_2 + \tilde{a}_3 \leq 8$, and $\tilde{a}_2, \tilde{a}_3 \geq 0$.

$$\textbf{Max} \quad 10x_1 + 20x_2 + 15x_3 \tag{112}$$

$$\begin{aligned} \textbf{s.t.} \quad 5x_1 + \tilde{a}_2x_2 + \tilde{a}_3x_3 &\leq 10 && (113)\\ 7x_1 - 2x_2 - 2x_3 &\geq 5 && (114)\\ -2 \leq x_1 &\leq 10 && (115)\\ 0 \leq x_2 &\leq 15 && (116)\\ -10 \leq x_3 &\leq 10 && (117) \end{aligned}$$

In constraint Equation 113, parameters $\tilde{a}_2$ and $\tilde{a}_3$ are uncertain.

In order to handle this sensible constraint using the standard approach, the factors are multiplied by negative one, making it a bizarre constraint.

$$-5x_1 - \tilde{a}_2x_2 - \tilde{a}_3x_3 \geq -10 \tag{118}$$

In order to find the worst case, we need to minimize the contribution from the two uncertain parts.

$$-5x_1 - \min_{\{\tilde{a}_2+\tilde{a}_3\leq 8, \tilde{a}_2+\tilde{a}_3\geq 2, \tilde{a}_2, \tilde{a}_3 \geq 0\}} \{\tilde{a}_2x_2 + \tilde{a}_3x_3\} \geq -10 \tag{119}$$

Which leads to the sub-problem:

$$\textbf{Min} \quad \tilde{a}_2x_2 + \tilde{a}_3x_3 \tag{120}$$

$$\begin{aligned} \textbf{s.t.} \quad \tilde{a}_2 + \tilde{a}_3 &\leq 8 \quad : \lambda_1 && (121)\\ \tilde{a}_2 + \tilde{a}_3 &\geq 2 \quad : \lambda_2 && (122)\\ \tilde{a}_2, \tilde{a}_3 &\geq 0 && (123) \end{aligned}$$

Again the constraint Equation 126 is multiplied by negative one to get the standard form.

$$-\tilde{a}_2 - \tilde{a}_3 \geq -8 \quad : \lambda_1 \tag{124}$$

Leading to the dual problem:

$$\textbf{Max} \quad -8\lambda_1 + 2\lambda_2 \tag{125}$$

$$\begin{aligned} \textbf{s.t.} \quad -\lambda_1 + \lambda_2 &\geq x_2 && (126)\\ -\lambda_1 + \lambda_2 &\geq x_3 && (127)\\ \lambda_1, \lambda_2 &\leq 0 && (128) \end{aligned}$$

Where the objective function has the parameters of the right-hand side of the constraints in the primal problem. As the variables in the primal problem are $a_i \geq 0$, the associated constraints will likewise be $\geq$ constraints. Likewise; $\lambda_i \leq 0$ as the primal constraints are greater-than-or-equal-to constraints.

This can then be inserted back into the original problem:

$$\textbf{Max} \quad 10x_1 + 20x_2 + 15x_3 \tag{129}$$

$$\textbf{s.t.} \quad -5x_1 + 8\lambda_1 - 2\lambda_2 \geq -10 \tag{130}$$
$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{131}$$
$$-\lambda_1 + \lambda_2 \geq x_2 \tag{132}$$
$$-\lambda_1 + \lambda_2 \geq x_3 \tag{133}$$
$$\lambda_1, \lambda_2 \leq 0 \tag{134}$$
$$-2 \leq x_1 \leq 10 \tag{135}$$
$$0 \leq x_2 \leq 15 \tag{136}$$
$$-10 \leq x_3 \leq 10 \tag{137}$$

We have implemented the program in Julia and obtained an objective value of 31.25.

6.2 Budget of uncertainty

We now introduce a budgeted uncertainty set to the below problem.

$$\textbf{Max} \quad 10x_1 + 20x_2 + 15x_3 \tag{138}$$
$$\textbf{s.t.} \quad \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 x_3 \leq 10 \tag{139}$$
$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{140}$$
$$x_1 \geq -2 \tag{141}$$
$$x_1 \leq 10 \tag{142}$$
$$0 \leq x_2 \leq 15 \tag{143}$$
$$-10 \leq x_3 \leq 10 \tag{144}$$

Where $\tilde{a}_1 \sim \mathcal{U}(1,8), \tilde{a}_2 \sim \mathcal{U}(2,9), \tilde{a}_3 \sim \mathcal{U}(1,5)$

A budget is set up, where 2 of the three can deviate from their means, thus:

$\Gamma = 2$

With means $\bar{a}_j$ and deviations $\hat{a}_j$:

$\bar{a}_1 = 4.5, \bar{a}_2 = 5.5, \bar{a}_3 = 3$

$\hat{a}_1 \sim \mathcal{U}(-3.5, 3.5), \hat{a}_2 \sim \mathcal{U}(-3.5, 3.5), \hat{a}_3 \sim \mathcal{U}(-2, 2)$

Inserting this into constraint 139, gives:

$$\sum_{j \in J} \bar{a}_j x_j + \max_{\{S | S \subseteq J, |S| \leq \lfloor \Gamma \rfloor\}} \left\{ \sum_{j \in S} \hat{a}_{ij} \left| x_j \right| \right\} \leq 10 \tag{145}$$

In order to find the max, we can change the formulation to:

$\max \{ |P_1 x_1| + |P_2 x_2|, |P_1 x_1| + |P_3 x_3|, |P_2 x_2| + |P_3 x_3| \}$

Where P_j is the extreme value, which $\hat{a}_j$ can assume.

Introducing a new variable u, enable the finding the maximum of the three possible value.

$$\sum_{j \in J} \bar{a}_j x_j + u \leq 10 \tag{146}$$
$$-u \leq P_1 x_1 + P_2 x_2 \leq u \tag{147}$$
$$-u \leq P_1 x_1 - P_2 x_2 \leq u \tag{148}$$
$$-u \leq P_1 x_1 + P_3 x_3 \leq u \tag{149}$$
$$-u \leq P_1 x_1 - P_3 x_3 \leq u \tag{150}$$
$$-u \leq P_2 x_2 + P_3 x_3 \leq u \tag{151}$$
$$-u \leq P_2 x_2 - P_3 x_3 \leq u \tag{152}$$
$$u \geq 0 \tag{153}$$

Thus, the robust problem formulation with a budget of 2 uncertain parameters becomes:

$$\textbf{Max} \quad 10x_1 + 20x_2 + 15x_3 \tag{154}$$
$$\textbf{s.t.} \quad \sum_{j \in J} \bar{a}_j x_j + u \leq 10 \tag{155}$$
$$-u \leq P_1 x_1 + P_2 x_2 \leq u \tag{156}$$
$$-u \leq P_1 x_1 - P_2 x_2 \leq u \tag{157}$$
$$-u \leq P_1 x_1 + P_3 x_3 \leq u \tag{158}$$
$$-u \leq P_1 x_1 - P_3 x_3 \leq u \tag{159}$$
$$-u \leq P_2 x_2 + P_3 x_3 \leq u \tag{160}$$
$$-u \leq P_2 x_2 - P_3 x_3 \leq u \tag{161}$$
$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{162}$$
$$x_1 \geq -2 \tag{163}$$
$$x_1 \leq 10 \tag{164}$$
$$x_2 \geq 0 \tag{165}$$
$$x_2 \leq 15 \tag{166}$$
$$x_3 \geq -10 \tag{167}$$
$$x_3 \leq 10 \tag{168}$$
$$u \geq 0 \tag{169}$$

We have implemented the program in Julia and obtained an objective value of 18.4345 when rounded to 4 decimal places.

Appendix

Appendix I: Task 6 Alternative

In task 6 three methods where listed where only two where demonstrated. We will here show the last one. We restate the given problem.

$$\underset{x_1,x_2,x_3}{\textbf{Max}} \quad 10x_1 + 20x_2 + 15x_3 \tag{170}$$

$$\begin{aligned} \textbf{s.t.} \quad 5x_1 + 3x_2 + 3x_3 &\leq 10 && (171)\\ 7x_1 - 2x_2 - 2x_3 &\geq 5 && (172)\\ -2 \leq x_1 &\leq 10 && (173)\\ 0 \leq x_2 &\leq 15 && (174)\\ -10 \leq x_3 &\leq 10 && (175) \end{aligned}$$

For the rest of this appendix we will work with the minimization version of the problem.

$$\underset{x_1,x_2,x_3}{\textbf{Min}} \quad -10x_1 - 20x_2 - 15x_3 \tag{176}$$

$$\begin{aligned} \textbf{s.t.} \quad 5x_1 + 3x_2 + 3x_3 &\leq 10 && (177)\\ 7x_1 - 2x_2 - 2x_3 &\geq 5 && (178)\\ -2 \leq x_1 &\leq 10 && (179)\\ 0 \leq x_2 &\leq 15 && (180)\\ -10 \leq x_3 &\leq 10 && (181) \end{aligned}$$

6.1 Polyhedral uncertainty set

In this sub-task we work with a polyhedral uncertainty set given by

$$(a_2, a_3) \in \mathcal{A} := \left\{ \mathbf{a} \in \mathbb{R}^2_+ \mid \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{a} \leq \begin{bmatrix} 8 \\ -2 \end{bmatrix} \right\} \tag{182}$$

The uncertainty is given in the constraint 177 as

$$5x_1 + a_2x_2 + a_3x_3 \leq 10, \qquad (a_2, a_3) \in \mathcal{A} \tag{183}$$

We rewrite the uncertainty variables in 183 according to slide 16 from lecture 4.

$$\begin{aligned} 5x_1 + a_2x_2 + a_3x_3 \leq 10, \qquad (a_2, a_3) \in \mathcal{A} \Rightarrow && (184)\\ 5x_1 + \max_{a_2,a_3}\{a_2x_2 + a_3x_3 \ : \ (a_2, a_3) \in \mathcal{A}\} \leq 10 && (185) \end{aligned}$$

We now work explicitly with the maximization problem.

$$\underset{a_2,a_3}{\textbf{Max}} \quad a_2x_2 + a_3x_3 \tag{186}$$

$$\textbf{s.t.} \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \leq \begin{bmatrix} 8 \\ -2 \end{bmatrix} \tag{187}$$

$$a_2, a_3 \geq 0 \tag{188}$$

We now take the dual.

$$\underset{\lambda_2,\lambda_3}{\textbf{Min}} \quad 8\lambda_2 - 2\lambda_3 \tag{189}$$

$$\textbf{s.t.} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_3 \end{bmatrix} \geq \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{190}$$

$$\lambda_2, \lambda_3 \geq 0 \tag{191}$$

We now have a minimization problem as the outer optimization problem and can hence insert it into the outer problem. We hence obtain the following problem.

$$\underset{x_1,x_2,x_3,\lambda_1,\lambda_2}{\textbf{Min}} \quad -10x_1 - 20x_2 - 15x_3 \tag{192}$$

$$\textbf{s.t.} \quad 5x_1 + 8\lambda_2 - 2\lambda_3 \leq 10 \tag{193}$$

$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{194}$$

$$\lambda_2 - \lambda_3 \geq x_2 \tag{195}$$

$$\lambda_2 - \lambda_3 \geq x_3 \tag{196}$$

$$-2 \leq x_1 \leq 10 \tag{197}$$

$$0 \leq x_2 \leq 15 \tag{198}$$

$$-10 \leq x_3 \leq 10 \tag{199}$$

$$\lambda_2, \lambda_3 \geq 0 \tag{200}$$

We have also implemented this version into Julia and obtain 31.25 as the other formulation.

6.2 Budgeted uncertainty set

In this sub task we work with a polyhedral uncertainty set given by

$$(a_1, a_2, a_3) \in \mathcal{A} := \left\{ \mathbf{a} \in \mathbb{R}^3 \;|\; -1 \leq \mathbf{a} \leq 1, \; \sum_{i=1}^{3} \|a_i\|_1 \leq \Gamma \right\} \tag{201}$$

where $\Gamma = 2$. The uncertainty is again given in the constraint 177.

$$(4.5 + 3.5a_1)x_1 + (5.5 + 3.5a_2)x_2 + (3 + 2a_3)x_3 \leq 10, \quad (a_1, a_2, a_3) \in \mathcal{A} \tag{202}$$

$$\Longleftrightarrow \tag{203}$$

$$4.5x_1 + 5.5x_2 + 3x_3 + 3.5a_1x_1 + 3.5a_2x_2 + 2a_3x_3 \leq 10, \quad (a_1, a_2, a_3) \in \mathcal{A} \tag{204}$$

We rewrite 204 according to slide 31 from lecture 4.

$$4.5x_1 + 5.5x_2 + 3x_3 + \max_{a_1,a_2,a_3}\{3.5a_1x_1 + 3.5a_2x_2 + 2a_3x_3 \;:\; (a_1,a_2,a_3) \in \mathcal{A}\} \leq 10 \tag{205}$$

$$\iff \tag{206}$$

$$4.5x_1 + 5.5x_2 + 3x_3 + 2\lambda + \sum_{i=1}^{3}\mu_i \leq 10 \tag{207}$$

$$\lambda + \mu_1 \geq 3.5y_1 \tag{208}$$

$$\lambda + \mu_2 \geq 3.5y_2 \tag{209}$$

$$\lambda + \mu_3 \geq 2y_3 \tag{210}$$

$$-y_i \leq x_i \leq y_i, \quad \forall i \in \{1,2,3\} \tag{211}$$

$$\lambda \geq 0 \tag{212}$$

$$\mu_i \geq 0, \quad \forall i \in \{1,2,3\} \tag{213}$$

$$y_i \geq 0, \quad \forall i \in \{1,2,3\} \tag{214}$$

As for the polyhedral set we will now work explicitly with the maximization problem.

$$\underset{a_1,a_2,a_3}{\textbf{Max}} \quad 3.5x_1a_1 + 3.5x_2a_2 + 2x_3a_3 \tag{215}$$

$$\textbf{s.t.} \quad \sum_{i=1}^{3}|a_i| \leq \Gamma, \quad \forall i \in \{1,2,3\} \;:\; \lambda \tag{216}$$

$$-1 \leq a_i \leq 1, \quad \forall i \in \{1,2,3\} \;:\; \mu_i \tag{217}$$

$$\tag{218}$$

$$\iff$$

$$\tag{219}$$

$$\underset{a_1,a_2,a_3}{\textbf{Max}} \quad 3.5|x_1|a_1 + 3.5|x_2|a_2 + 2|x_3|a_3$$

$$\textbf{s.t.} \quad \sum_{i=1}^{3}a_i \leq \Gamma, \quad \forall i \in \{1,2,3\} \;:\; \lambda \tag{220}$$

$$0 \leq a_i \leq 1, \quad \forall i \in \{1,2,3\} \;:\; \mu_i \tag{221}$$

$$\tag{222}$$

We now take the dual.

$$\underset{\lambda,\mu_1,\mu_2,\mu_3}{\textbf{Min}} \quad \Gamma\lambda + \sum_{i=1}^{3}\mu_i \tag{223}$$

$$\textbf{s.t.} \quad \lambda + \mu_1 \geq 3.5|x_1| \tag{224}$$

$$\lambda + \mu_2 \geq 3.5|x_2| \tag{225}$$

$$\lambda + \mu_3 \geq 2.0|x_3| \tag{226}$$

$$\lambda \geq 0 \tag{227}$$

$$\mu_i \geq 0, \quad \forall i \in \{1,2,3\} \tag{228}$$

We now have a minimization problem as the outer optimization problem and can hence insert it into the outer problem. We hence obtain the following problem.

$$\underset{x_1,x_2,x_3,\lambda,\mu_1,\mu_2,\mu_3}{\mathbf{Min}} \quad -10x_1 - 20x_2 - 15x_3 \tag{229}$$

$$\mathbf{s.t.} \quad 7x_1 - 2x_2 - 2x_3 \geq 5 \tag{230}$$

$$-2 \leq x_1 \leq 10 \tag{231}$$

$$0 \leq x_2 \leq 15 \tag{232}$$

$$-10 \leq x_3 \leq 10 \tag{233}$$

$$4.5x_1 + 5.5x_2 + 3x_3 + 2\lambda + \sum_{i=1}^{3} \mu_i \leq 10 \tag{234}$$

$$\lambda + \mu_1 \geq 3.5y_1 \tag{235}$$

$$\lambda + \mu_2 \geq 3.5y_2 \tag{236}$$

$$\lambda + \mu_3 \geq 2y_3 \tag{237}$$

$$-y_i \leq x_i \leq y_i, \quad \forall i \in \{1,2,3\} \tag{238}$$

$$\lambda \geq 0 \tag{239}$$

$$\mu_i \geq 0, \quad \forall i \in \{1,2,3\} \tag{240}$$

$$y_i \geq 0, \quad \forall i \in \{1,2,3\} \tag{241}$$

As for the polyhedral set we have also implemented this version into Julia and obtain 18.4348 rounded to 4 decimal places. This is of course the same optimum as for the other version given in task 5.

Appendix II: Nomenclature for Part 1

Nomenclature for Part 1

Abbreviations

P Paper

W Wood

Parameters

η Yield coefficient [$\text{ton}_{\text{wood}}/\text{ton}_{\text{paper}}$]

$\overline{A}_w$ Maximum delivery amount from supplier w [ton/week]

ϕ Cost of unsatisfied demand for paper [€/ton]

π_s Probability of scenario s

$\underline{A}_w$ Minimum delivery amount from supplier w [ton/week]

c^P Cost of storing paper from one time period to another [€/ton]

c_k^W Cost of storing wood from one time period to another at site k [€/ton]

c_w^D Cost of wood from supplier w [€/ton]

d_t Demand for paper in time period t [ton/week]

U Production capacity constraint of paper factory Tekst [ton/week]

Sets

K Set of wood storage sites

S Set of scenarios

T Set of time periods

W Set of suppliers

Variables

θ_w Binary decision variable for selection of supplier w

$a_{w,t}$ Amount of wood delivered by supplier w in time period t [ton/week]

m_t Amount of unsatisfied demand for paper in time period t [ton/week]

p_t^W Amount of paper produced in time period t [ton/week]

$s_{k,t}^W$ Amount of wood stored at site k in time period t [ton/week]

s_t^P Amount of paper stored in time period t [ton/week]

Appendix III: Nomenclature for Part 2

Nomenclature for Part 2

Parameters

γ Ad hoc purchase price [€]

$\overline{K}$ Maximum production capacity [units/week]

σ Conversion factor for new paper

$\underline{K}$ Minimum production capacity [units/week]

C^K Production cost per capacity [€/unit of capacity]

C^P Cost per produced unit of paper [€/unit of paper]

L_t Demand for birthday cards in time period t [units/week]

U Maximum amount of purchase [units/week]

Sets

Z Uncertainty set, conversion factor between paper and birthday cards
$\mathbb{Z} = \{z \in \mathbb{R} \,|\, 400 \leq z \leq 600\}$

T Set of time periods [weeks]
$T = \{t \in \mathbb{Z} \,|\, 1 \leq t \leq 4\}$

Variables

α_t Help variable

β Help variable

ϕ_t Help variable

θ_t Help variable

Q_t Slope for affine decision rule

q_t^0 Intercept for affine decision rule

x_t Amount of paper processed [units/week]

y Production capacity [units/week]