IMM - DTU

02407 Stochastic Processes
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DTU MASTER OF SCIENCE EXAMINATION

Course: **Stochastic Processes** Course no.: **02407**

Final exercise

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Capelin is a small but numerous fish which is an important food species for seals, cods and many other animals in the Arctic. It belongs to the smelt family (these often smell like cucumber) and has a very high content of lipids. In fact a close relative (candlefish) has been used dried as candles in the Pacific. We will consider different models for cod preying on capelin which is important information that fisheries biologists need in order to regulate the fishery so that it becomes sustainable.

A proper solution of the following problems will mention the assumptions necessary and state additional assumptions whenever required. For some of the questions access to some kind of computer facilities is needed; MATLAB, Python or R should be sufficient tools here, while MAPLE might be helpful although not strictly needed for other questions.

Part 1

In the first, rather primitive, model we assume that the cod has a daily meal opportunity at dawn and that the size of this meal at dawn expressed in number of capelin can be described by a geometrically distributed random variable with probability mass function $p(1-p)^i$, where $i = 0, 1, \ldots$. Each food item (capelin) present in the stomach at the beginning of a day has a probability q of being digested (leave the stomach) before the meal opportunity at dawn the following day. The cod stomach has finite capacity of K capelin. If the combined meal size and stomach content exceeds this value the stomach content will be K.

Question 1

Formulate a model describing the daily content of capelin in a cod stomach at dawn, immediately before the meal opportunity.

Problem set continues ...

We now consider specific values of the model parameters $K = 6, p = \frac{1}{3}$, and $q = \frac{1}{2}$.

Question 2

What is the mean number of capelins you will expect to find in a cod stomach?

Question 3

What is the expected amount of capelin being part of a potential meal, which is actually not ingested due to lack of stomach capacity?

We say that a cod had a meal of size j if the actual number of capelin ingested was j irrespective of the size of the original or potential meal.

Question 4

What is the probability that a capelin was part of a meal of size j?

Fisheries scientists on R/V DANA are studying cod in the Barents Sea. During the night they catch cod, empty their stomachs (flush the stomach with water) and equip cod with electronic devices that are able to register their feeding activities. Since the devices are very expensive a trade off has to be made between the amount of data and the probability of recuperating the device. Five days is the maximum time that the scientists are willing to wait for the results, considering the severe weather conditions that can occur in the Arctic. Now suppose a cod is caught, examined and released at sea with an empty stomach.

Question 5

What is the probability that there will be 3 capelin in the stomach after 5 days?

Question 6

What is the probability that the cod has had at least 3 capelin in the stomach during the first 5 days?

We now assume that each capelin stays exactly two days in the cod stomach. We will denote the process of the number of capelin in the cod stomach right after dawn, immediately before the meal opportunity, at day t by X_t.

Question 7

Is the process X_t Markovian? Depending on your answer: Specify the model or formulate a Markovian model that describes X_t.

Problem set continues ...

Question 8

What is the mean time (in days) between periods, where the stomach is empty?

Part 2

During the polar summer it is perhaps more suitable to assume that food is accessible throughout the day and the capelins are caught individually on a continuous basis. The mean number of encountered capelin is 1 per day, and we assume that the probability of encountering one is constant over the day and proportional to the length of a small time interval—this is a standard assumption in population dynamics. The digestion time can be described by a scaled $\chi^2(1)$-distribution with mean 18 hours. In order to simplify the analysis we assume that there are no size constraints on the stomach. The capelins are assumed to be digested sequentially.

Question 9

What is the mean number of capelin in a cod stomach?

Question 10

What is the variance of the number of capelin in a cod stomach?

Large migrating cod (*skrei* in Norwegian) is the most important fish in the northern part of Norway. The local communities are severely hit by recent collapses of the cod population due to over fishing of the stock of capelin. We are asked by fellow scientists at R/V G. O. Sars to make a more accurate model of the dynamics of the stomach of a large cod in order to refine the estimates of the daily ration. Recently it has been shown that the digestion rate depends in a nonlinear way on the number of capelin in the stomach. However, we will make the approximation that the digestion rate increases linearly with number of capelin in the stomach until 6 capelin are found in the stomach. Whenever 6 or more capelin are present in the stomach the digestion rate is at its maximum (6 times the digestion rate of one). Since a large *skrei* is about 1.20 m and capelin are very small in relation to this we assume that the stomach capacity is unlimited. The mean digestion time is 4 days, for one capelin. Knowing that it has been in the stomach for x time units does not provide us with further information of its total time in the stomach. The encounter process is as above.

Question 11

Determine a bound for the feeding rate λ such that the stomach content will not grow without bounds.

We return to the estimate of a stock assessment group aboard R/V DANA who estimated that the encounter rate is 1 capelin per day.

Question 12

What is the mean number of capelin in the stomach of a large cod?

Problem set continues ...

Question 13

Suppose that at some point in time a cod has an empty stomach. Find an expression for the probability that the stomach has not reached a level (for the first time) with at least 5 capelin present.

Question 14

Suppose, as in the previous question, that at some point in time a cod has an empty stomach. Calculate the expectation and the variance of the time it takes for the stomach to reach a level (for the first time) with at least 5 capelin present.

It has been known for some time that even fish can become seasick. This happens when there are large waves (as with many people). We are asked to refine our model to incorporate this knowledge. The consequences to the cod is that sometimes it will vomit thus empty the stomach instantly and completely. With the harsh weather conditions in the Barents Sea it has been estimated that such bad weather may occur randomly with a rate of 1 every 2 weeks. For the remaining questions in this part you can assume that the stomach capacity is finite at 20 capelin. The digestion process is considered to be the same as the one used for questions 12-14.

Question 15

Formulate a model to describe this scenario of the stomach content.

Question 16

Determine the long run fraction of time with j capelin in the stomach, and the mean number of capelin in the stomach.

Question 17

Calculate the mean time a capelin will be in the digestion system of a cod.

Part 3

We are asked to consider yet another scenario. Cod encounters individual capelin with constant intensity of $\frac{3}{2}$ per day, but immediately after an encounter the cod has to handle the caught capelin for some exponentially distributed random time. During this time the cod is prevented from catching a new capelin during this process and the mean handling time is $\frac{1}{3}$ day.

Question 18

What is the mean number of capelin ingested during a 10 day period?

Problem set continues ...

Question 19

A cod has been caught. If the cod hadn't been disturbed (caught), how long would it have had to wait until the next capture of a capelin?

The stock assessment group asks for the biomass consumed by a cod in a 10 day period. We are given the information that there is variation in the size of capelin. This variation can be described by a log-normal distribution with mean $\mu = 10$ cm and variance $\sigma^2 = 5$ cm^2. The variation in size does not alter our previous assumptions on handling times and digestion times. We are told by a computer wizard on board R/V Akademik Feodorov that the variation in the arrival process after 10 days is 5.6543, we trust her.

Question 20

What is the mean and the variance of the biomass of capelin ingested by a cod during a 10 day period, starting immediately after a cod was caught?

Part 4

Consider again the case, where cods are caught, equipped with electronic devices and released. Initially we assume that a cod will roam horizontally and direction-less. The distance moved in any two or more time intervals will be independent from the movement in any other interval. If we consider space described by a coordinate system then the standard deviation of the daily movement in each of the x and y directions is 2km.

Question 21

What is the probability that the cod is within a distance of 5km of the place where it was dropped three days earlier?

Question 22

A cod is caught three days later at the same x-coordinate at which it was released. What is the probability that the cod was more than 2 kilometres away in the same x-direction at both one and two days after its release.

The biologists are also interested in the behaviour of the cods with respect to vertical movements immediately upon release. Suppose that these movements can be described in a similar way as the horizontal movements with a downward drift of 0.5m per half hour with standard deviation of 1.5m per half hour.

Question 23

What is the probability that a cod released at a depth of 4m will reach the surface before the sea floor, when the water depth is 20m?

Problem set continues ...

We will consider an equation related to an enzymatic process in the cods digestion system. This process can be written as $Z(t) = X(t)e^{X(t)}$, where $X(t)$ is a Brownian motion with drift -0.5 and variance parameter 1.

Question 24

Calculate the expected value of $Z(t)$.

We now observe the horizontal movements of the cod according to a Poisson process with rate λ, i.e. at discrete but random time epochs. We are interested in the position of the fish in the x direction only, and suppose that the parameters are as in Question 23.

Question 25

Show that the value of the position process at the time of the first observation is given by a modified Laplace distribution i.e. a distribution that conditioned on the variable being positive is exponential, and where the absolute value is exponentially distributed conditioned on the variable being negative. Find the parameters of the distribution.

Although most cod migrate, particluarly during the spawning season some cod tend to revert around the same position, lets call it the place of return. For simplicity we will (again) only consider movements in one (e.g. the x) direction. The movements occur according to a Gaussian process such that the mean of the change during a small time interval is approximately equal to the product of the value of the current posistion, the length of the time interval and a certain constant γ. The variance of the change during a small interval is approximately equal to the length of the time interval multiplied with the value ϕ.

Question 26

Formulate a model for the movement of the cod in the x-direction.

Suppose the standard deviation of the change in the location of the cod in the x direction during one day is 2 km and the correlation between the location of the cod in the x-direction from any day to another is $\frac{1}{2}$.

Question 27

Calculate the probability that the cod is less than 2 km away from its place of return at an arbitray point in time.

Question 28

Calculate the probability that the cod is less than 2 km away from its place of return at an arbitray point in time and also exactly one day after.

Problem set ends.