Danmarks Tekninske Universitet

Function spaces and mathematical analysis
Course number: 01325

Homework 2

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March 1, 2022

Exercise 4.22

i)

First we show linearity. Let $\alpha, \beta \in \mathrm{C}$ and $v, w \in \ell^n(\mathrm{N})$

$$\begin{aligned} S(\alpha v + \beta w) &= S(\alpha v_1 + \beta w_1, \alpha v_2 + \beta v_2, \ldots) \\ &= (\alpha v_2 + \beta w_2, \alpha v_3 + \beta v_3, \ldots) \\ \alpha S v + \beta S w &= \alpha(v_2, v_3, \ldots) + \beta(w_2, w_3, \ldots) \\ &= (\alpha v_2 + \beta w_2, \alpha v_3 + \beta v_3, \ldots) \end{aligned}$$

So S *is* linear.

Notice we say it is bounded iff $||Sv|| \leq K||v||$ or alternatively $||Sv||^2 \leq K^2||v||^2$ (notice this is only allowed since both sides are real positives). Since we are in $\ell^2(\mathrm{N})$ we have

$$||v||^2 = \sum_{k=1}^{\infty} |v_k|^2 < \infty$$

So

$$\begin{aligned} ||Sv||^2 &= ||(v_2, v_3, \ldots)||^2 \\ &= \sum_{k=2}^{\infty} |v_k|^2 \\ &\leq \sum_{k=1}^{\infty} |v_k|^2 = ||v||^2 \\ &< \infty \end{aligned}$$

So we are certain to have $||Sv||^2 \leq K^2||v^2||$ whenever $K \geq 1$. So S *is* bounded.

ii)

We say that S is an isometry if it holds that

$$||Sv|| = ||v||, \quad \forall v \in \ell^2(\mathrm{N})$$

Since it must hold for *all* v, let us choose $v = (1, 0, 0, \ldots)$, then

$$\begin{aligned} ||v|| &= 1 \\ ||Sv||^2 &= \sum_{k=2}^{\infty} |v_k|^2 = 0 \\ \Rightarrow \quad & ||Sv|| = 0 \end{aligned}$$

Hence S cannot be an isometry.

iii)

We must find S^* that satisfies

$$< Sv, w > = < v, S^* w >$$

So let us start by finding $< Sv, w >$

$$< Sv, w >= \sum_{k=1}^{\infty} v_{k+1}\bar{w}_k$$

Notice this corresponds to the pairs $(v_1, 0), (v_2, w_1), (v_3, w_2)....$ Now let us guess $S^*w = (0, w_1, w_2, ...)$, and let us test it

$$< v, S^*w >= v_1 \cdot 0 + \sum_{k=1}^{\infty} v_{k+1}\bar{w}_k = \sum_{k=1}^{\infty} v_{k+1}\bar{w}_k$$

Via theorem 4.4.3 (Riesz) we can now safely conclude our guess was correct and $S^*w = (0, w_1, w_2, ...)$. Notice this means that the adjoint to the left shift operator is in fact the right shift operator.

Exercise 4.25 [equation (4.35)]

We start by re-writing the expression slightly where $N \in \mathrm{N}$

$$\begin{aligned}\left\langle \sum_{k=1}^{\infty} c_k v_k, v \right\rangle &= \left\langle \sum_{k=1}^{N} c_k v_k, v \right\rangle + \left\langle \sum_{k=N+1}^{\infty} c_k v_k, v \right\rangle \\ &= \sum_{k=1}^{N} c_k \langle v_k, v\rangle + \left\langle \sum_{k=N+1}^{\infty} c_k v_k, v \right\rangle\end{aligned}$$

This follows directly from the definition of inner products. The question is then, what we mean by $\left\langle \sum_{k=N+1}^{\infty} c_k v_k, v \right\rangle$. We do know that $\sum_{k=1}^{\infty} c_k v_k$ is convergent, so it follows

$$\lim_{N\to\infty} \sum_{k=N+1}^{\infty} c_k v_k = \mathbf{0}$$

By Cauchy-Schwartz (theorem 4.1.4) we have

$$\left| \left\langle \sum_{k=N+1}^{\infty} c_k v_k, v \right\rangle \right| \leq \left\| \sum_{k=N+1}^{\infty} c_k v_k \right\| \cdot ||v||$$

Hence

$$\lim_{N\to\infty} \left| \left\langle \sum_{k=N+1}^{\infty} c_k v_k, v \right\rangle \right| = 0$$

Using 4.1.1 (iii) we can now conclude

$$\left\langle \sum_{k=1}^{\infty} c_k v_k, v \right\rangle = \sum_{k=1}^{\infty} c_k \langle v_k, v\rangle$$

So inner product is linear in the limit.