Stochastic Processes - Probability 2

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1 Part 1

1.1 Question 1

Let N_t denote the number capelin in the stomach just before the meal opportunity and A_t the number of capelin available for consumption at time t. Let D_t denote the number of capelin leaving the stomach between eating possibility at t and $t+1$. Since every capelin in the stomach has a probability of leaving, q, the number of capelin leaving follow a binomial distribution. We then have

$$A_t \sim \text{Geo}(p) \tag{1}$$
$$D_t \sim \text{Binorm}(N_t, q) \tag{2}$$

N_t is modelled as a discrete Markov chain. Specifically, by a 7×7 markov chain, where state $i \in \{0, .., 6\}$ corresponds to $N_t = i$.
Notice N_{t+1} is determined by N_t, A_t and D_t. Let $\rho_{i,j}$ denote the probability of jumping from state i to j. We then have

$$N_{t+1} = \min\{N_t + A_t, K\} - D_t \tag{3}$$
$$\rho_{i,j} = \begin{cases} \mathbb{P}\{A_t - D_t = j - i\} & j < K \\ \mathbb{P}\{A_t - D_t \geq j - i\} & j = K \\ 0 & otherwise \end{cases} \tag{4}$$

Specifically

$$\rho_{i,j} = \begin{cases} \sum_{x=0}^{i} \varphi_{A_t}(j-x)\varphi_{D_t}(i-x) & j < K \\ \sum_{x=0}^{i}(1 - \Phi_{A_t}(j-x))\varphi_{D_t}(i-x) & j = K \\ 0 & otherwise \end{cases} \tag{5}$$

Where

$$\varphi_{A_t}(x) = p(1-p)^x \tag{6}$$
$$\Phi_{A_t}(x) = \sum_{i=0}^{x} p(1-p)^i \tag{7}$$
$$\varphi_{D_t}(x) = \binom{N_t}{x} q^x (1-q)^{N_t - x} \tag{8}$$

Notice it is assumed that whenever more capelin are avaiable than the cod can eat, it will fill its stomach. The probability transition matrix of the markov process $\{N_t : t \geq 0\}$ then becomes

$$\boldsymbol{P}_{(K+1)\times(K+1)} = \begin{bmatrix} \rho_{0,0} & \cdots & \rho_{0,K} \\ \vdots & \ddots & \vdots \\ \rho_{K,0} & \cdots & \rho_{K,K} \end{bmatrix} \tag{9}$$

1.2 Question 2

The parameters $K = 6$, $p = \frac{1}{3}$ and $q = \frac{1}{2}$ are now inserted and the probability matrix, $\boldsymbol{P}$, computed.
By mean number of capelin, $\mathbb{E}\{N_t\}$, it is understood that the long term rate is desired, i.e. the mean after sufficiently long time, not under any specific starting assumption. We therefore wish to determine the limiting distribution of the process, $\boldsymbol{\pi}$.

$$\boldsymbol{\pi}_i = \lim_{t\to\infty} \mathbb{P}\{N_t = i\} \tag{10}$$

This is found by solving the linear equation cf theorem 4.3 and 4.4 in [1]

$$\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi} \tag{11}$$

We now have

$$\mathbb{E}\{N_t\} = \sum_{i=0}^{K} i \cdot \boldsymbol{\pi}_i \tag{12}$$
$$\mathbb{E}\{N_t\} = 1.61 \tag{13}$$

1.3 Question 3

Let R_t denote the number of capelin rejected (the portion that did not fit into the stomach of the cod) at some time t. It is desired to determine $\mathbb{E}\{R_t\}$. To do this we apply the law of total expectation. Notice $R_t > 0$ exactly when $A_t > K - N_t$.

$$\mathbb{E}\{R_t\} = \sum_{s=0}^{6} \mathbb{E}\{R_t|N_t = s\} \cdot \mathbb{P}\{N_t = s\} \tag{14}$$
$$= \sum_{s=0}^{6} \mathbb{E}\{R_t|N_t = s\} \cdot \boldsymbol{\pi}_s \tag{15}$$
$$\mathbb{E}\{R_t|N_t = s\} = \sum_{i=7-s}^{\infty} \mathbb{P}\{A_t = i\} \cdot (i - (K - s)) \tag{16}$$
$$= \sum_{i=7-s}^{\infty} \varphi_{A_t}(i) \cdot (i - (K - s)) \tag{17}$$

Notice $(i - (K - s))$ is exactly the number of capelin not eaten. Specifically this becomes

$$\mathbb{E}\{R_t\} = 0.394 \tag{18}$$

1.4 Question 4

Let M_t denote the meal size the cod had at some time t. Via the law of total probability we have

$$\mathbb{P}\{M_t = j\} = \sum_{s=0}^{K} \mathbb{P}\{M_t = j | K - N_t = s\} \cdot \mathbb{P}\{K - N_t = s\} \tag{19}$$

$$\mathbb{P}\{M_t = j | K - N_t = s\} = \begin{cases} \varphi_{A_t}(j) & j < s \\ 1 - \Phi_{A_t}(j-1) & j = s \end{cases} \tag{20}$$

Notice $K - N_t$ describes the room in the stomach of the cod. The interest is of the long run probability, i.e. the limit case. Therefore, $\mathbb{P}\{K - N_t = s\}$ is in fact described by $\boldsymbol{\pi}$

$$\mathbb{P}\{K - N_t = s\} = \boldsymbol{\pi}_{K-s} \tag{21}$$

And thus

$$\mathbb{P}\{M_t = j\} = \sum_{s=0}^{K} \mathbb{P}\{M_t = j | K - N_t = s\} \cdot \boldsymbol{\pi}_{K-s} \tag{22}$$

Let Y denote the meal size a randomly selected capelin from the stomach of the cod was part of. Since a meal of size j is weighted by j capelin we have

$$\mathbb{P}\{Y = j\} = \mathbb{P}\{M_t = j\} \cdot \frac{j}{c} \tag{23}$$

Where c is a normalization constant s.t. $\sum_{j=0}^{K} \mathbb{P}\{Y = j\} = 1$.
Since an alternative solution to the mean stomach contents of the cod would have been $\mathbb{E}\{N_t\} = \sum_{j=0}^{K} \mathbb{P}\{M_t = j\} \cdot j$, it is obvious that $c = \mathbb{E}\{N_t\}$

$$\mathbb{P}\{Y = j\} = \mathbb{P}\{M_t = j\} \cdot \frac{j}{\mathbb{E}\{N_t\}} \tag{24}$$

The probabilities are determined as

$$\begin{align} \mathbb{P}\{Y = 0\} &= 0 \tag{25} \\ \mathbb{P}\{Y = 1\} &= 0.144 \tag{26} \\ \mathbb{P}\{Y = 2\} &= 0.206 \tag{27} \\ \mathbb{P}\{Y = 3\} &= 0.221 \tag{28} \\ \mathbb{P}\{Y = 4\} &= 0.201 \tag{29} \\ \mathbb{P}\{Y = 5\} &= 0.152 \tag{30} \\ \mathbb{P}\{Y = 6\} &= 0.076 \tag{31} \end{align}$$

1.5 Question 5

By 3 capelin in the stomach of the cod, it is understood that the stomach contents must be exactly 3 after 5 days. By theorem 3.1 in [1] we have

$$\begin{align} \mathbb{P}\{N_5 = 3 | N_0 = 0\} &= \boldsymbol{P}^5_{0,3} \tag{32} \\ &= 0.144 \tag{33} \end{align}$$

1.6 Question 6

A modified transition probability matrix, $\boldsymbol{Q}_{4\times 4}$, is defined. The difference between $\boldsymbol{P}$ and $\boldsymbol{Q}$ is that in $\boldsymbol{Q}$ state 3 through 6 are combined as one absorbing state.

$$\boldsymbol{Q}_{i,j} = \begin{cases} \boldsymbol{P}_{i,j} & i < 3 \wedge j < 3 \\ 1 - \sum_{n=0}^{2} \boldsymbol{P}_{i,n} & i < 3 \wedge j = 3 \\ 1 & i = 3 \wedge j = 3 \end{cases} \tag{34}$$

We then have

$$\mathbb{P}\{\max\{N_t : 0 \leq t \leq 5 | N_0 = 0\} \geq 3\} = \boldsymbol{Q}^5_{0,3} \tag{35}$$
$$\mathbb{P}\{\max\{N_t : 0 \leq t \leq 5 | N_0 = 0\} \geq 3\} = 0.560 \tag{36}$$

1.7 Question 7

In general we have

$$\mathbb{P}\{X_2 = 0 | X_0 = 4, X_1 = 4\} = 1 \tag{37}$$
$$\mathbb{P}\{X_2 = 0 | X_1 = 4\} \neq 1 \tag{38}$$

Cf the definition of a markovian process on page 72 in [1] the process $\{X_t : t > 0\}$ is not markovian. Notice that due to the deterministic behaviour of the process, it can be required to know the value of *all* previous states. Therefore, it cannot be redefined as a markov model.
Let C_t be the number of capelin consumed by the cod at time t. Notice $C_t = \min\{A_t, K - X_t\}$. Then

$$X_{t+1} = X_t + C_t - C_{t-2} \tag{39}$$

In particular, notice it is not possible to fully determine $\{C_t : t \geq 0\}$ from $\{X_t : t \geq 0\}$ (C_t is not measurable wrt the filtration of X_t, $\mathcal{F}_N$).

1.8 Question 8

Since the model defined in question 7 is not markovian, and this question relates to markovian models, it is assumed that the process $\{N_t : t \geq 0\}$ is defined as in questions 2-6, i.e. by a markov chain.
Let τ_i denote the return time to state i, such that

$$\tau_i = \min\{s : N_s = i | N_0 = i\} \tag{40}$$

Since the markov chain is fully connected, $\boldsymbol{P}_{i,j} > 0 \; \forall (i,j)$, the model is obviously aperiodic. Therefore, theorem 4.3 from [1] applies

$$\boldsymbol{\pi}_i = \frac{1}{m_i} \tag{41}$$

Where m_i is the mean return time of state i. So

$$\mathbb{E}\{\tau_i\} = m_i = \frac{1}{\boldsymbol{\pi}_i} \tag{42}$$

$$\mathbb{E}\{\tau_0\} = 4.32 \tag{43}$$

2 Part 2

Let N_t denote the number capelin in the stomach at some time t. The mean digestion time is 18 hours. Since is follows a $\chi^2(1)$ distribution which does not have the memoryless property, the system cannot be modelled by a M/M/1 system. Instead a M/G/1 is needed.

2.1 Question 9

Mean number of capelin in the stomach of the cod, is understood as the expected number of capelin after sufficiently long time of observation, i.e. at stationarity.
Let $\{Y_t : t \geq 0\}$ denote the digestion process then

$$Y_t/18 \sim \chi^2(1) \tag{44}$$
$$\mathbb{E}\{Y_t\} = \frac{18}{24} \tag{45}$$
$$\mathbb{V}\{Y_t\} = 2 \cdot 1 \cdot \frac{18}{24} = \frac{3}{2} \tag{46}$$

Cf section 9.3.1 in [1] it holds that

$$\nu = \mathbb{E}\{Y_t\} = \frac{18}{24} \tag{47}$$
$$\tau^2 = \mathbb{V}\{Y_t\} = \frac{3}{2} \tag{48}$$
$$\rho = \lambda\nu = \frac{3}{4} \tag{49}$$
$$L = \rho + \frac{\lambda^2(\tau^2 + \nu^2)}{2(1-\rho)} = \frac{39}{8} \tag{50}$$

$\mathbb{E}\{N_t\}$ is determined using the fact that the limiting behaviour of the M/G/1 system is described by its embedded markov chain. It follows

$$\mathbb{E}_{t\to\infty}\{N_t\} = L = \frac{39}{8} = 4.875 \tag{51}$$

2.2 Question 10

No time is given, hence it is assumed that sufficiently long time has passed s.t. the variance has converged.
To determine the variance of the number of capelin in the stomach of a cod, the limiting distribution will be approximated. Cf [1] section 9.3.1 the M/G/1 system, $\{N_t : t \geq 0\}$, can be described by its *embedded markov chain*, N_t'. Specifically it holds that

$$\lim_{t\to\infty} \mathbb{P}\{N_t = j\} = \lim_{t\to\infty} \mathbb{P}\{N_t' = j\} \tag{52}$$

Let the transition probability matrix for N_t' be denoted by $\boldsymbol{P'}$. Further let $\rho_{i,j}$ denote the probability of transition from state i to j, then

$$\rho_{i,j} = \begin{cases} \alpha_{j-i+1} & i \leq 1, j \leq i+1 \\ \alpha_j & i = 0 \end{cases} \tag{53}$$

$$\alpha_k = \int_0^\infty \frac{(\lambda y)^k e^{-\lambda y}}{k!} d\Phi_Y(y) \tag{54}$$

Where $\int_0^\infty \frac{(\lambda y)^k e^{-\lambda y}}{k!} d\Phi_Y(y)$ is the Riemann–Stieltjes integral wrt the CDF of the digestion distribution, since the PDF of $\chi^2(1)$ is well defined, it becomes

$$\alpha_k = \int_0^\infty \frac{(\lambda y)^k e^{-\lambda y}}{k!} \frac{\varphi_Y\left(\frac{y}{\sqrt{\frac{24}{18}}}\right)}{\sqrt{\frac{24}{18}}} dy \tag{55}$$

With the PDF

$$\varphi_Y(y) = \frac{\sqrt{2} e^{\frac{y}{2}}}{2\Gamma(\frac{1}{2})\sqrt{y}} \tag{56}$$

Using the limiting distribution, the moments of N_t' is determined. Since it is approximated with finite states, it will only be an approximation

$$\boldsymbol{\pi} = (lim_{t\to\infty}(\boldsymbol{P'})^t)_{0,j} \qquad j \in \{0, 1, ...\} \tag{57}$$

$$\mathbb{E}\{N_t'\} = \sum_{i=0}^{\infty} i \cdot \boldsymbol{\pi}_i = 4.84 \tag{58}$$

$$\mathbb{E}\{(N_t')^2\} = \sum_{i=0}^{\infty} i^2 \cdot \boldsymbol{\pi}_i = 25.51 \tag{59}$$

$$\mathbb{V}\{N_t'\} = 25.51 - 4.84^2 = 2.05 \tag{60}$$

So

$$\mathbb{V}\{N_t\} \approx \mathbb{V}\{N_t'\} = 2.05 \tag{61}$$

2.3 Question 11

The process $\{N_t : t \geq 0\}$ can be modelled as a birth and death by a continuous time markov chain, so a M/M/1 system. Hence the birth rate and death rate (as defined on page 296 in [1]) are needed.
The birth rate is given readily given as $\lambda = 1$ as in accordance with its criteria stated on page 295 in [1]. It is known that knowing a capelin has been in the stomach for x time units does not provide us with further information of its total time in the

stomach. This translates to a memoryless distribution, i.e. exponential, which is consistent with a birth and death process. The death rate is then given by

$$\mu_k = \min\left\{\frac{k}{4}, \frac{6}{4}\right\} \tag{62}$$

The generator of $\{N_t : t \geq 0\}$ is given by

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu_1 & -(\lambda+\mu_1) & \lambda & \cdots \\ 0 & \mu_2 & -(\lambda+\mu_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{63}$$

Cf [1] page 306, the process will have a limiting distribution only if

$$\frac{\lambda}{\mu} < 1 \tag{64}$$

$$\frac{\lambda}{\frac{6}{4}} < 1 \tag{65}$$

$$\lambda < \frac{6}{4} \tag{66}$$

Notice $max\{\mu_k\}$ is used instead μ since that will be the death rate for diverging processes.
The bound of the growth rate is therefore $\frac{6}{4}$.

2.4 Question 12

Cf chapter 6.4 in [1] it hold that

$$\boldsymbol{\pi}_0 = \left(\sum_{k=0}^{\infty} \boldsymbol{\theta}_k\right)^{-1} \tag{67}$$

$$\boldsymbol{\pi}_k = \boldsymbol{\theta}_k \boldsymbol{\pi}_0 \tag{68}$$

$$\boldsymbol{\theta}_k = \prod_{j=0}^{k-1} \frac{\lambda_k}{\mu_{k+1}} \tag{69}$$

Where $\boldsymbol{\pi}$ is the limiting distribution. We have that

$$\mu_k = \min\left\{\frac{k}{4}, \frac{6}{4}\right\} \tag{70}$$

The mean number of capelin in the stomach of the large cod is then found by

$$\mathbb{E}\{N_t\} = \sum_{k=0}^{\infty} k \cdot \boldsymbol{\pi}_k \tag{71}$$

$$= 4.57 \tag{72}$$

2.5 Question 13

A modified generator, $\boldsymbol{Q}_{6\times 6}$, is defined. The difference between $\boldsymbol{G}$ and $\boldsymbol{Q}$ is that in $\boldsymbol{Q}$ state 5 is an absorbing state.

$$\boldsymbol{Q}_{i,j} = \begin{cases} \boldsymbol{G}_{i,j} & i < 5 \\ 0 & otherwise \end{cases} \tag{73}$$

$$\boldsymbol{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda+\mu_1) & \lambda & 0 & 0 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2) & \lambda & 0 & 0 \\ 0 & 0 & \mu_3 & -(\lambda+\mu_3) & \lambda & 0 \\ 0 & 0 & 0 & \mu_4 & -(\lambda+\mu_4) & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{74}$$

We then have

$$\mathbb{P}\{\max\{N_\tau : 0 \leq \tau \leq t | N_0 = 0\} < 5\} = 1 - \left(e^{\boldsymbol{Q}\cdot t}\right)_{0,5} \tag{75}$$

A nice analytical expression does not exist, so the expansion will be omitted here. Figure 1 shows the behaviour of the found expression.

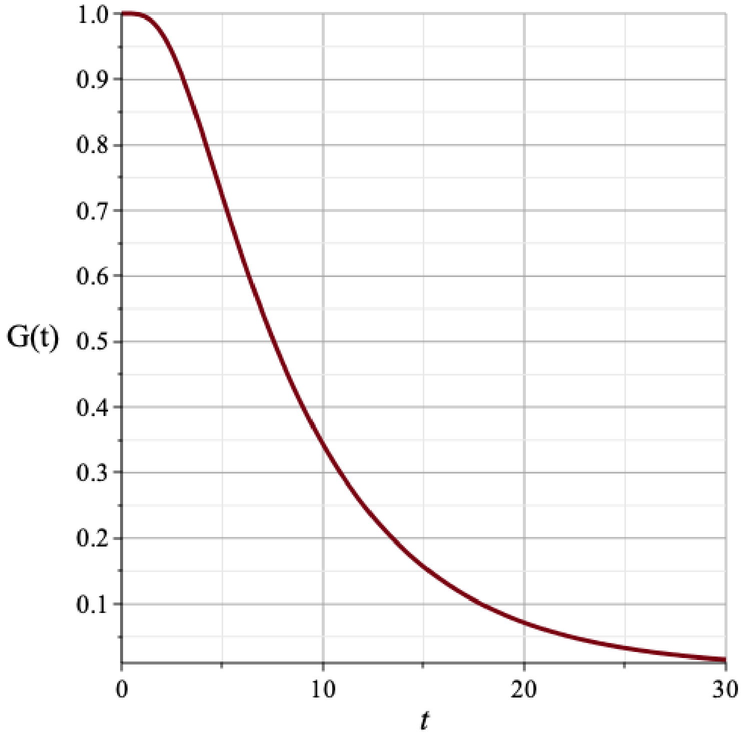

Figure 1 – $\mathbb{P}\{\max\{N_\tau : 0 \leq \tau \leq t | N_0 = 0\} < 5\}$

2.6 Question 14

The first time the stomach of the cod reaches a level with a least 5 capelin is described by $\tau = \inf\{t : N_t \geq 5\}$.

It is immediately seen that the answer to the previous exercise was in fact a survival function, the CDF and PDF is therefore given as

$$\Phi_\tau(t) = \left(e^{\boldsymbol{Q}\cdot t}\right)_{0,5} \tag{76}$$

$$\varphi_\tau(t) = \frac{\partial\left(e^{\boldsymbol{Q}\cdot t}\right)_{0,5}}{\partial t} \tag{77}$$

The moments are now easily found

$$\mathbb{E}\{\tau\} = \int_0^\infty t \cdot \varphi_\tau(t) dt \tag{78}$$

$$= 9.31 \tag{79}$$

$$\mathbb{E}\{\tau^2\} = \int_0^\infty t^2 \cdot \varphi_\tau(t) dt \tag{80}$$

$$= 130.0 \tag{81}$$

And the variance can be determined to

$$\mathbb{V}\{\tau\} = \mathbb{E}\{\tau^2\} - \mathbb{E}\{\tau\}^2 = 130.0 - 9.31^2 \tag{82}$$

$$= 43.3 \tag{83}$$

2.7 Question 15

The capelin contents of the stomach of the cod is best described using a finite continuous time markov chain. Again, let $\{N_t : t \geq 0\}$ describe the stomach contents of the cod. It is no longer a pure birth and death process, since there is a non-zero possibility of jumping across several states.
The generator, $\boldsymbol{G}_{21\times 21}$, is given by

$$\boldsymbol{G} = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & 0 \\ \mu_1 + q & -(\lambda + \mu_1 + q) & \lambda & \cdots & 0 \\ q & \mu_2 & -(\lambda + \mu_2 + q) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q & 0 & \cdots & \mu_{20} & -(\mu_{20} + q) \end{bmatrix} \tag{84}$$

Where $\mu_k = \min\left\{\frac{k}{4}, \frac{6}{4}\right\}$ as before, and q is the sea-sickness rate, given by $q = \frac{1}{14}$.

2.8 Question 16

The long run fraction of time where $N_t = j$ is in fact $\mathbb{P}\{N_t = j\} = \boldsymbol{\pi}_j$, i.e. the stationary/limiting distribution.
Cf the teaching notes from week 6, the limiting distribution of a finite state continuous time markotv chain satisfies

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{G} \Rightarrow \tag{85}$$

$$\boldsymbol{0} = \boldsymbol{\pi}\boldsymbol{G} \tag{86}$$

It then holds

$$\mathbb{P}\{N_t = j\} = \boldsymbol{\pi}_j \tag{87}$$

$$\boldsymbol{\pi} = \begin{bmatrix} 0.0997 & 0.1416 & 0.1748 & 0.1775 & 0.1485 & 0.1041 & ... \\ 0.0621 & 0.0370 & 0.0221 & 0.0132 & 0.0078 & 0.0047 & ... \\ 0.0028 & 0.0017 & 0.0010 & 0.0006 & 0.0004 & 0.0002 & ... \\ 0.0001 & 0.0001 & 0.0001 & & & & \end{bmatrix} \tag{88}$$

The mean number of capelin in the stomach of the cod is found by

$$\mathbb{E}\{N_t\} = \sum_{j=0}^{20} j \cdot \boldsymbol{\pi}_j \tag{89}$$

$$= 3.29 \tag{90}$$

2.9 Question 17

Let E_s be the *exit rate* when the cod has s capelin in the stomach, such that

$$E_s = \mu_s + q \cdot s \tag{91}$$

Now let T_t be the time spent inside the cod for some random capelin at time t. Then

$$\mathbb{E}\{T_t | N_t = s\} = s \cdot \frac{1}{E_s}, \qquad s > 0 \tag{92}$$

Notice the time to exit, T_t, is not defined in when there is no capelin in the stomach. The law of total expectation dictates

$$\mathbb{E}\{T_t\} = \sum_{s=1}^{20} \mathbb{E}\{T_t | N_t = s\} \cdot \mathbb{P}\{N_t = s | s \in \{1, 2, ..., 20\}\} \tag{93}$$

$$\mathbb{P}\{N_t = s | s \in \{1, 2, ..., 20\}\} = \frac{\boldsymbol{\pi}_s}{1 - \boldsymbol{\pi}_0} \tag{94}$$

The mean time spent in the digestion system of the cod then becomes

$$\mathbb{E}\{T_t\} = \sum_{s=1}^{20} s \cdot \frac{1}{\mu_s + q \cdot s} \cdot \frac{\boldsymbol{\pi}_s}{1 - \boldsymbol{\pi}_0} \tag{95}$$

$$\mathbb{E}\{T_t\} = 3.20 \tag{96}$$

So on average a capelin spends 3.2 days in the digestion system of the cod.

3 Part 3

3.1 Question 18

Let N_t denote the number of capelin ingested by the cod at time t. Let α_i denote the time it took to encounter capelin i. Let β_i denote the time took to handle capelin i, s.t. $\mathbb{E}\{\beta_i\} = \frac{1}{3}$. Since the encounter of capelin has constant intensity with $\frac{3}{2}$ per day it must hold that

$$\mathbb{E}\{\alpha_i\} = \left(\frac{3}{2}\right)^{-1} = \frac{2}{3} \tag{97}$$

Notice that the time spent to ingest capelin i is given by $\alpha_i + \beta_i$. Naturally it follows

$$\mathbb{E}\{N_t\} = \frac{t}{\mathbb{E}\{\alpha_i\} + \mathbb{E}\{\beta_i\}} = t \tag{98}$$

Hence

$$\mathbb{E}\{N_{10}\} = 10 \tag{99}$$

3.2 Question 19

Since the cod encounters capelin with constant intensity, the waiting time between each encounter is best described with an exponential distribution. Since this distribution is memory-less cf [2] the expected waiting time is unchanged.
The cod would therefore expect to wait $\mathbb{E}\{\alpha_i\} = \frac{2}{3}$ days.

3.3 Question 20

Remember that N_{10} denotes the number of capelin ingested by the cod at time 10. Further, let B_i denote the biomass of the ith capelin. It then follows

$$\mathbb{E}\{N_{10}\} = 10 \tag{100}$$
$$\mathbb{V}\{N_{10}\} = 5.6543 \tag{101}$$
$$\mathbb{E}\{B_i\} = 10 \tag{102}$$
$$\mathbb{V}\{B_i\} = 5 \tag{103}$$

It is desired to determine $\mathbb{E}\{\sum_{i=1}^{N_{10}} B_i\}$ and $\mathbb{V}\{\sum_{i=1}^{N_{10}} B_i\}$. Assuming the number of capelin caught is independent of the capelin size, $N_t \perp\!\!\!\perp B_i$, it is known via [2] that

$$\mathbb{E}\{\sum_{i=1}^{N_{10}} B_i\} = \mathbb{E}\{N_{10}\} \cdot \mathbb{E}\{B_i\} = 100 \tag{104}$$

$$\begin{aligned} \mathbb{V}\{\sum_{i=1}^{N_{10}} B_i\} &= (\mathbb{V}\{N_{10}\} + \mathbb{E}\{N_{10}\}^2)(\mathbb{V}\{B_i\} + \mathbb{E}\{B_i\}^2) - \mathbb{E}\{N_{10}\}^2 \cdot \mathbb{E}\{B_i\}^2 && (105) \\ &= (5.6543 + 10^2)(5 + 10^2) - 100 \cdot 100 && (106) \\ &= 1093.7 && (107) \end{aligned}$$

4 Part 4

4.1 Question 21

It is immediately realized that the position of the cod is described as 2D brownian motion. Let $\{X_t : t \geq 0\}$ describe the x-coordinate of the cod, and let $\{Y_t : t \geq 0\}$ describe the y-coordinate. Without loss of generality, the starting position is assumed to be $(X_0, Y_0) = \mathbf{0}$. Notice the standard deviation is 2km per day. Then

$$X_t \sim \mathcal{N}(0, 2^2 t) \tag{108}$$
$$Y_t \sim \mathcal{N}(0, 2^2 t) \tag{109}$$
$$X_t \perp\!\!\!\perp Y_t \tag{110}$$

The probability that the cod is within a distance of 5km of the place where it was dropped three days earlier is then given by

$$\mathbb{P}\left\{\sqrt{X_3^2 + Y_3^2} \leq 5\right\} \tag{111}$$

This is recognized as the Rayleigh distribution, so the CDF becomes

$$F(x, \sigma) = 1 - e^{-x^2/(2\sigma^2)} \tag{112}$$

And

$$\mathbb{P}\left\{\sqrt{X_3^2 + Y_3^2} \leq 5\right\} = F(5, 2^2 \cdot 3) = 1 - e^{-5^2/(2 \cdot 2^2 \cdot 3)} \tag{113}$$
$$= 0.647 \tag{114}$$

4.2 Question 22

It is desired to determine $\mathbb{P}\{(X_1 > 2, X_2 > 2) \vee (X_1 < 2, X_2 < 2) | X_0 = X_3 = 0\}$. Due to the symmetry of the problem it holds that

$$\mathbb{P}\{(X_1 > 2, X_2 > 2) \vee (X_1 < 2, X_2 < 2) | X_0 = X_3 = 0\} = \tag{115}$$
$$2 \cdot \mathbb{P}\{X_1 > 2, X_2 > 2 | X_0 = X_3 = 0\} \tag{116}$$

This problem is recognized as a brownian bridge, for which it holds

$$\mathbb{E}\{X_t : t_1 \leq t \leq t_2 | X_{t_1} = a \wedge X_{t_2} = b\} = a + \frac{t - t_1}{t_2 - t_1}(b - a) \tag{117}$$
$$\mathbb{V}\{X_t : t_1 \leq t \leq t_2 | X_{t_1} = a \wedge X_{t_2} = b\} = \frac{(t_2 - t)(t - t_1)}{t_2 - t_1} \tag{118}$$

The law of total probability dictates (for the sake of readability the condition $X_0 = X_3 = 0$ will be omitted below, s.t. $\mathbb{P}\{X_1 > 2, X_2 > 2\} = \mathbb{P}\{X_1 > 2, X_2 > 2 | X_0 = X_3 = 0\}$, but is should be considered a condition until further notice)

$$\mathbb{P}\{X_1 > 2, X_2 > 2\} = \mathbb{P}\{X_1 > 2\} \cdot \mathbb{P}\{X_2 > 2 | X_1 > 2\} \tag{119}$$

It then holds

$$\mathbb{E}\{X_1\} = 0 \tag{120}$$

$$\mathbb{V}\{X_1\} = 2^2 \cdot \left(\frac{(3-1)(1-0)}{3-0}\right) = \frac{8}{3} \tag{121}$$

$$\mathbb{E}\{X_2 | X_1 = s\} = s + \frac{2-1}{3-1}(0-s) = \frac{s}{2} \tag{122}$$

$$\mathbb{V}\{X_2 | X_1 = s\} = 2^2 \cdot \left(\frac{(3-2)(2-1)}{3-1}\right) = 2 \tag{123}$$

Let φ denote the PDF for a normal distribution

$$\varphi(x, \sigma, \mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{124}$$

So

$$\mathbb{P}\{X_1 > 2\} \cdot \mathbb{P}\{X_2 > 2 | X_1 > 2\} = \int_2^\infty \mathbb{P}\{X_2 > 2 | X_1 = s\} \cdot \varphi(s, \sqrt{8/3}, 0) ds \tag{125}$$

Then

$$\mathbb{P}\{X_2 > 2 | X_1 = s\} = \int_2^\infty \varphi(x, \sqrt{2}, s/2) dx \tag{126}$$

Hence

$$\mathbb{P}\{X_1 > 2, X_2 > 2\} = \int_2^\infty \int_2^\infty \varphi(x, \sqrt{2}, s/2) dx \cdot \varphi(s, \sqrt{8/3}, 0) ds \tag{127}$$

$$= 0.03725 \tag{128}$$

It is now no longer assumed that the condition $X_0 = X_3 = 0$ is present.
Finally

$$\mathbb{P}\{(X_1 > 2, X_2 > 2) \vee (X_1 < 2, X_2 < 2) | X_0 = X_3 = 0\} = 2 \cdot 0.03725 \tag{129}$$

$$= 0.0745 \tag{130}$$

4.3 Question 23

Let $\{Z_t : t \geq 0\}$ denote the vertical position of the cod. Let τ denote the hitting time given by

$$\tau = \inf\{t \geq 0 : Z_t = 20 \vee Z_t = 0\} \tag{131}$$

It is desired to find $\mathbb{P}\{Z_\tau = 0 | Z_0 = 4\}$ given it has a drift parameter, $\mu = 0.5$, and standard deviation, $\sigma = 1.5$. The problem is immediately recognized as a gambler's ruin problem.
Cf theorem 8.1 in [1] it holds

$$\mathbb{P}\{Z_\tau = 0 | Z_0 = 4\} = \frac{e^{-2\cdot\mu\cdot 4/\sigma^2} - e^{-2\cdot\mu\cdot 20/\sigma^2}}{e^{-2\cdot\mu\cdot 0/\sigma^2} - e^{-2\cdot\mu\cdot 20/\sigma^2}} \tag{132}$$
$$= 0.169 \tag{133}$$

4.4 Question 24

$Z(t)$ and $X(t)$ are given by

$$X(t) = \mu \cdot t + \sigma \cdot B(t) \tag{134}$$
$$Z(t) = (\mu \cdot t + \sigma \cdot B(t))e^{\mu\cdot t + \sigma\cdot B(t)} \tag{135}$$

Where $B(t)$ is ordinary brownian motion. The PDF for $B(t)$ is given by

$$\varphi_{B(t)}(x) = \frac{1}{\sqrt{2 \cdot t \cdot \pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{t}} \tag{136}$$

The expectation of $Z(t)$ is then found by

$$\mathbb{E}\{Z(t)\} = \int_{-\infty}^{\infty} (\mu \cdot t + \sigma \cdot s)e^{\mu\cdot t+\sigma\cdot s} \cdot \varphi_{B(t)}(s)ds \tag{137}$$
$$= t/2, \qquad t \in \mathbb{R}^+ \tag{138}$$

4.5 Question 25

Let S_i describe the ith sampling time of the cod. Then $\{S_i : i \geq 0\}$ is a poisson process with parameter λ. Cf page 242 in [1] the PDF of the first sampling is exponentially distributed

$$S_1 \sim \exp(\lambda) \tag{139}$$
$$\varphi_{S_1}(t) = \lambda e^{-\lambda\cdot t} \tag{140}$$

Let X_t denote the x-coordiante of the cod at time t.
It is mentioned that the parameters are as in question 23. However; since that was about vertical movement, it is assumed that $\{X_t : t \geq 0\}$ follows a simple browninan motion motion without drift as given in question 21.
The PDF of the x-coordinate of the cod at time, S_1 is found using the law of total probability

$$\varphi_{X_{S_1}}(x) = \int_0^\infty \varphi_{X_{S_1}|S_1=s}(x)\varphi_{S_1}(s)ds \tag{141}$$

Where

$$\varphi_{X_{S_1}|S_1=s}(x) = \varphi_{X_s}(x) = \frac{1}{\sqrt{2 \cdot 2^2 \cdot t \cdot \pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{2^2 \cdot t}} \tag{142}$$

Notice the standard deviation is assumed to be 2km per day.
It then holds

$$\varphi_{X_{S_1}}(x) = \frac{\sqrt{2}\sqrt{\lambda}}{4} e^{-\frac{\sqrt{\lambda}\sqrt{2}\sqrt{x^2}}{2}} \tag{143}$$

$$= \frac{1}{2 \cdot 2 \cdot (\sqrt{2}\sqrt{\lambda})^{-1}} e^{-\frac{|x-0|}{2 \cdot (\sqrt{2}\sqrt{\lambda})^{-1}}} \tag{144}$$

Remembering the PDF of a laplacian random variable, L, is given by

$$\varphi_L(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} \tag{145}$$

So X_{S_1} follows a laplacian distribution with parameters $\mu = 0$, $b = 2(\sqrt{2}\sqrt{\lambda})^{-1}$.

4.6 Question 26

Again, let X_t be the x-coordinate of the cod at some time t. From the explanation the following SDE is formulated

$$dX_t = X_t \, \gamma \, dt + \phi \, dB_t \tag{146}$$

Where B_t is ordinary brownaian motion. The SDE is immediately recognized as the SDE corresponding to the Ornstein-Uhlenbeck process. The solution is therefore given in terms of an Ito integral

$$X_t = X_0 e^{\gamma \cdot t} + \int_0^t \phi \cdot e^{\gamma(t-s)} dB_s, \qquad \gamma < 0 \tag{147}$$

4.7 Question 27

Let X_t be as before. Cf [1] page 439 it holds (in the stationary case)

$$Cov(X_s, X_t) = \frac{\phi^2}{2 \cdot \gamma} e^{\gamma|t-s|} \tag{148}$$

Then the auto-correlation is given by

$$Cor(X_s, X_t) = e^{\gamma|t-s|} \tag{149}$$

Since

$$Cor(X_s, X_t) = \frac{Cov(X_s, X_t)}{\sqrt{Cov(X_s, X_s)}\sqrt{Cov(X_t, X_t}} = \frac{\frac{\phi^2}{2\cdot\gamma} e^{\gamma|t-s|}}{\sqrt{\frac{\phi^2}{2\cdot\gamma}}^2} \tag{150}$$

The correlation for a time step of 1 day is know to be $\frac{1}{2}$, hence

$$e^{\gamma} = \frac{1}{2} \tag{151}$$

$$\gamma = -ln(2) \tag{152}$$

It is also known that the standard deviation during 1 day is 2km, so

$$\mathbb{V}\{X_t\} = Cov(X_t, X_t) = \frac{\phi^2}{-2\gamma} = 2^2 \tag{153}$$

$$\phi = 2\sqrt{2}\sqrt{ln(2)} \tag{154}$$

Cf [1] page 433 is holds

$$\mathbb{E}\{X_t\} = X_0 e^{\gamma t} \tag{155}$$

$$\mathbb{V}\{X_t\} = \phi^2 \left(\frac{1 - e^{2\cdot\gamma t}}{-2 \cdot \gamma} \right) \tag{156}$$

Therefore

$$\mathbb{P}\{|X_t| \leq x\} = \mathbb{P}\{-x \leq X_t \leq x\} \tag{157}$$

$$= \varPhi \left(\frac{x}{\sqrt{\phi^2 \left(\frac{1-e^{2\cdot\gamma\cdot t}}{-2\cdot\gamma} \right)}} \right) - \varPhi \left(\frac{-x}{\sqrt{\phi^2 \left(\frac{1-e^{2\cdot\gamma\cdot t}}{-2\cdot\gamma} \right)}} \right) \tag{158}$$

Where $\varPhi$ is the CDF of a standard normal variable.
Figure 2 shows $\mathbb{P}\{|X_t| \leq 2\}$. Notice it converges quite quickly at

$$\lim_{t\to\infty} \mathbb{P}\{|X_t| \leq 2\} = 0.683 \tag{159}$$

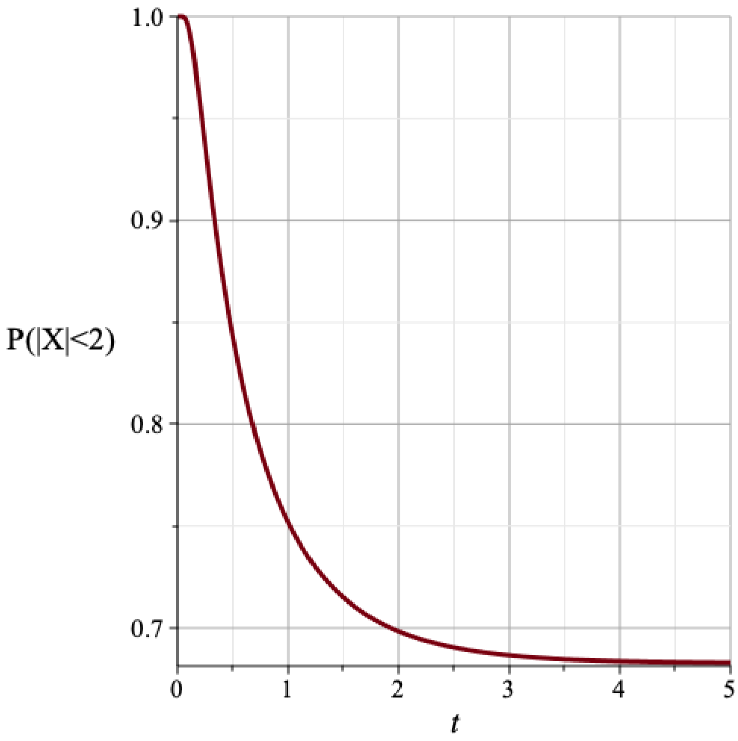

Figure 2 – $\mathbb{P}\{|X_t| \leq 2\}$

Additionally, an approximate solution could have been attained in terms of simulation. Figure 3 shows a simulated sample path of an OU process with the parameters from this exercise. Via simulation, the stationary probability, $\lim_{t\to\infty} \mathbb{P}\{|X_t| \leq 2\}$, was determined to be ≈ 0.69.

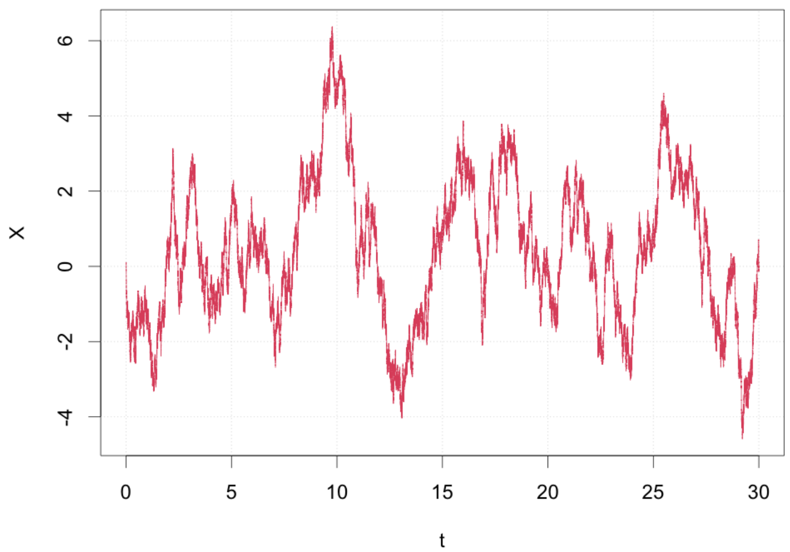

Figure 3 – Simulation of the OU process

4.8 Question 28

Let X_t be as before. It is desired to compute

$$\mathbb{P}\{|X_t| < 2, |X_{t+1}| < 2\} = \mathbb{P}\{|X_t| < 2\} \cdot \mathbb{P}\{|X_{t+1}| < 2||X_t| < 2\} \tag{160}$$

The law of total probability dictates

$$\mathbb{P}\{|X_t| < 2\} \cdot \mathbb{P}\{|X_{t+1}| < 2||X_t| < 2\} = \int_{-2}^{2} \varphi_{X_t}(z)\mathbb{P}\{|X_{t+1}| < 2|X_t = z\}dz \tag{161}$$

Where $\varphi_{X_t}(z)$ is the pdf for X_t.
Further, it holds that

$$\mathbb{P}\{|X_{t+1}| < 2|X_t = z\} = \int_{-2}^{2} \varphi_{X_{t+1}|X_t=z}(x)dx \tag{162}$$

Let the general PDF for at normal variable be

$$\varphi(x, \sigma, \mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{163}$$

Remembering that the following holds

$$\mathbb{E}\{X_t\} = X_0 e^{\gamma t} \tag{164}$$

$$\mathbb{V}\{X_t\} = \phi^2 \left(\frac{1 - e^{2\cdot\gamma t}}{-2\cdot\gamma}\right) \tag{165}$$

It follows

$$\varphi_{X_t}(z) = \varphi\left(z, \sqrt{\phi^2 \left(\frac{1 - e^{2\cdot\gamma t}}{-2\cdot\gamma}\right)}, 0\right) \tag{166}$$

$$\varphi_{X_{t+1}|X_t=z}(x) = \varphi\left(x, \sqrt{\phi^2 \left(\frac{1 - e^{2\cdot\gamma}}{-2\cdot\gamma}\right)}, z \cdot e^{\gamma}\right) \tag{167}$$

So

$$\mathbb{P}\{|X_t| < 2, |X_{t+1}| < 2\} = \tag{168}$$

$$\int_{-2}^{2} \varphi\left(z, \sqrt{\phi^2 \left(\frac{1 - e^{2\cdot\gamma t}}{-2\cdot\gamma}\right)}, 0\right) \int_{-2}^{2} \varphi\left(x, \sqrt{\phi^2 \left(\frac{1 - e^{2\cdot\gamma}}{-2\cdot\gamma}\right)}, z \cdot e^{\gamma}\right) dxdz \tag{169}$$

Figure 4 shows the probability as a function of time. Notice the probability converges quite quickly towards

$$\lim_{t\to\infty} \mathbb{P}\{|X_t| < 2, |X_{t+1}| < 2\} = 0.498 \tag{170}$$

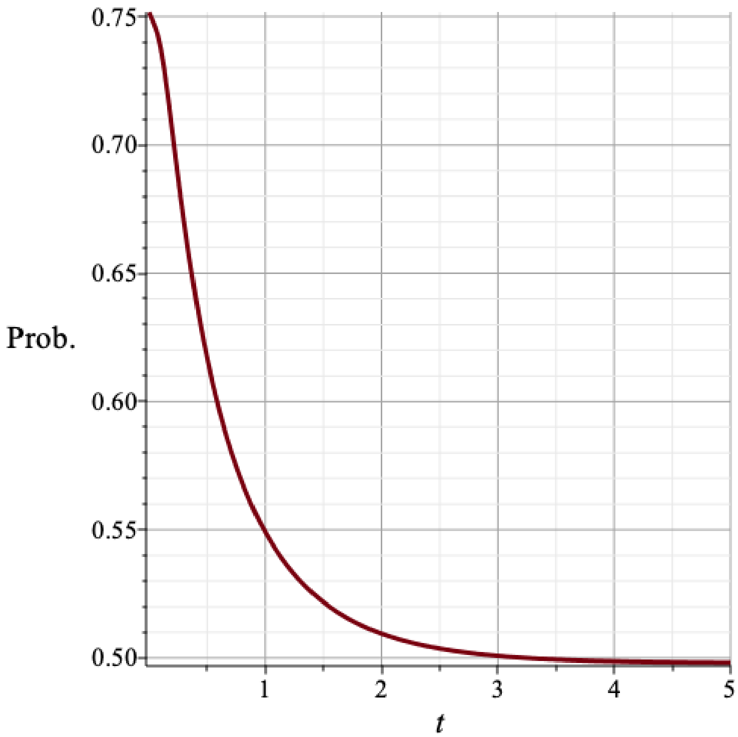

Figure 4 – $\mathbb{P}\{|X_t| < 2, |X_{t+1}| < 2\}$

References

[1] S. K. Mark A Pinsky, *An introduction to Stochastic modeling*, 4th ed., 2011.

[2] J. Pitmann, *Probability*, 1st ed., 1993.