Danmarks Tekninske Universitet

Discrete mathematics 2: algebra
Course number: 01018

Assignment 1

Author:
Mads Esben Hansen, s174434

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Part 1

Let $Mat_{m\times n}(\mathbb{R})$ be the set of $m \times n$ matrices with coefficients in $\mathbb{R}$. For $M, N \in Mat_{m\times n}(\mathbb{R})$, we say that M is equivalent to N, if there exists an invertible $n \times n$ matrix P in $Mat_{n\times n}(\mathbb{R})$ and an invertible matrix Q in $Mat_{m\times m}(\mathbb{R})$ such that $N = Q^{-1} \cdot M \cdot P$. Show being equivalent is indeed an equivalence relation on the set $Mat_{m\times n}(\mathbb{R})$.

According to definition 6 in the course material it holds that for an equivalence relation, $\sim$, is a relation that satisfies the following:

- $\forall a \in A$ we have $a \sim a$ (reflexivity).
- $\forall a, b \in A$ we have $a \sim b \Rightarrow b \sim a$ (symmetry).
- $\forall a, b, c \in A$ we have $a \sim b \wedge b \sim c \Rightarrow a \sim c$ (transitivity).

We will now show that all 3 things are satisfied for our equivalence relation. But before we begin let us remember that a square matrix is invertible if and only if it is non-singular, i.e. its determinant is different from zero.

Reflexivity

Let us define $M \in Mat_{m\times m}(\mathbb{R})$. To be relexive there must exist and Q and P such that:

$$M = Q^{-1} \cdot M \cdot P \tag{1}$$

We can freely choose these Q and P, so we can choose these to both be the idendity matrix on dimension M, $Q = P = I_m$. Since it generally holds that the identity matrix is invertible and $I_m = I_m^{-1}$, we have:

$$(I_m^{-1} \cdot M) \cdot I_m = I_m \cdot M = M \tag{2}$$

We know there exist at least one pair Q and P such that equation 1 is satisfied. In turn the equivalence relation is closed under reflexivity.

Symmetry

Let us define $M \in Mat_{m\times m}(\mathbb{R})$ and $N \in Mat_{n\times n}(\mathbb{R})$. We want to show:

$$\exists Q_1, P_1 \ : \ N = Q_1^{-1} \cdot M \cdot P_1 \ \wedge \ Det(Q_1) \neq 0 \ \wedge \ Det(P_1) \neq 0 \ \Rightarrow \tag{3}$$
$$\exists Q_2, P_2 \ : \ M = Q_2^{-1} \cdot N \cdot P_2 \ \wedge \ Det(Q_2) \neq 0 \ \wedge \ Det(P_2) \neq 0 \tag{4}$$

Now let us assume that Q_1 and P_1 exists and are both invertible, $\exists Q_1, P_1 \ : \ N = Q_1^{-1} \cdot M \cdot P_1 \ \wedge \ Det(Q_1) \neq 0 \ \wedge \ Det(P_1) \neq 0$. We then have:

$$N = Q_1^{-1} \cdot M \cdot P_1 \ \Leftrightarrow \ Q_1 \cdot N \cdot P_1^{-1} = Q_1 \cdot Q_1^{-1} \cdot M \cdot P_1 \cdot P_1^{-1} = M \ \Leftrightarrow \tag{5}$$
$$M = Q_1 \cdot N \cdot P_1^{-1} \tag{6}$$

Now it is quite easy to see that whenever equation 3 is satisfied we know that exists Q_2 and P_2 that satisfies equation 4, namely when $Q_2 = Q_1^{-1}$ and $P_2 = P_1^{-1}$. In turn the equivalence relation is closed under symmetry.

Transitivity

Let us define $M \in Mat_{m\times m}(\mathbb{R})$, $N \in Mat_{n\times n}(\mathbb{R})$, and $O \in Mat_{o\times o}(\mathbb{R})$. We want to show:

$$\exists Q_1, P_1 \ : \ N = Q_1^{-1} \cdot M \cdot P_1 \ \wedge \ Det(Q_1) \neq 0 \ \wedge \ Det(P_1) \neq 0 \ \wedge \tag{7}$$
$$\exists Q_2, P_2 \ : \ O = Q_2^{-1} \cdot N \cdot P_2 \ \wedge \ Det(Q_2) \neq 0 \ \wedge \ Det(P_2) \neq 0 \ \Rightarrow \tag{8}$$
$$\exists Q_3, P_3 \ : \ O = Q_3^{-1} \cdot M \cdot P_3 \ \wedge \ Det(Q_3) \neq 0 \ \wedge \ Det(P_3) \neq 0 \tag{9}$$

By simple insertion we get:

$$O = Q_2^{-1} \cdot (Q_1^{-1} \cdot M \cdot P_1) \cdot P_2 \;\Leftrightarrow \tag{10}$$

$$O = (Q_2^{-1} \cdot Q_1^{-1}) \cdot M \cdot (P_1 \cdot P_2) \tag{11}$$

Now we remember that it generally holds for matrices that $Det(A \cdot B) = Det(A) \cdot Det(B)$. We can now define Q_3 and P_3:

$$Q_3^{-1} = Q_2^{-1} \cdot Q_1^{-1} \quad \Leftrightarrow \quad Q_3 = Q_2 \cdot Q_1 \tag{12}$$

$$P_3 = P_1 \cdot P_2 \tag{13}$$

We know Q_3 exist and is invertible iff. the same holds for Q_1 and Q_2 since $Det(Q_3) = Det(Q_2) \cdot Det(Q_1)$. The same goes for P_3 since $Det(P_3) = Det(P_2) \cdot Det(P_1)$. In turn the equivalence relation is closed under transitivity.
We can finally further conclude that the relation, $\sim$, is in fact an equivalence relation in accordance with definition 6.

Part 2

A

Let n, m be positive integers such that m is less than or equal to n. Further denote by $(S_n, \circ)$ the symmetric group on n letters. How many distinct m-cycles does S_n contain?

We define $n, m \in \mathbb{Z} \,:\, m \leq n$. Let $(S_n, \circ)$ define the symmetric group on n letters. From basic combinatorics we know that the number of ways to choose m elements from $\{1, 2, ..., n\}$ is given by:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m) = \frac{n!}{(n-m)!} \tag{14}$$

Now we have found the number of all m-cycles in S_n, but we only wanted the number of distinct ones. All non-distinct cycles will be the same cycle starting at different places, e.g. abcd, bcda, cdab, and dabc. For all m-cycles this will happen in m different ways. We therefore find the number of distinct m-cycles by dividing the total number of m-cycles by m.

$$\frac{1}{m} \cdot \frac{n!}{(n-m)!} = \frac{n!}{m \cdot (n-m)!} \tag{15}$$

B

How many distinct permutations of the form $(ab)(cd)$, with (ab) and (cd) disjoint 2-cycles, does S_n contain?

For this to make any sense, n must be larger than 4, we therefore assume $n \geq 4$. We then find the number of possibilities for (ab), we can use equation 15 to do this. By insertion we get:

$$\frac{n!}{2 \cdot (n-2)!} \tag{16}$$

We must now find the number of ways we can choose (cd). We remember that (ab) and (cd) must be disjoint 2-cycles, there is therefore only $n-2$ letter to choose from this time. Inserting into equation 15 we get:

$$\frac{(n-2)!}{2 \cdot ((n-2)-2)!} = \frac{(n-2)!}{2 \cdot (n-4)!} \tag{17}$$

It is now tempting to simple multiply equation 16 and 17 together. However, remembering that (ab) and (cd) are distinct, this means that the cycles *commute*, meaning $(ab)(cd) = (cd)(ab)$. For all pairs (ab) and (cd) this will happen 2 times, meaning all permutations of the form $(ab)(cd)$ will be counted twice. In turn we can get the result by multiplying equation 16 and 17 and dividing by 2:

$$\frac{1}{2} \cdot \frac{n!}{2 \cdot (n-2)!} \cdot \frac{(n-2)!}{2 \cdot (n-4)!} = \frac{n!}{8 \cdot (n-4)!} \tag{18}$$