Course 01257 Advanced Modelling - Applied Mathematics.
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Project 3. Nonlinear wave dynamics in sine-Gordon systems.

Project description. In the 1960s and 1970s a major breakthrough in solving nonlinear partial differential equations (PDEs) was made [1]. The first three central PDEs to be solved analytically was the Korteweg - de Vries (KdV) equation, the sine Gordon (sG) equation and the nonlinear Schrödinger (NLS) equation. These equations model a broad class of physical phenomena as waterwaves, superconducting Josephson tunnel junctions, nonlinear optical pulses and a rich variety of physical phenomena from nano scale to astronomical scales [2, 3]. However, the analytical methods are complicated and difficult. A major issue is that the analytical solution methods need to be adopted specifically to each PDE in a non trivial way. This despite we have methods based on the same fundamental solution principles as in the inverse scattering technique, the Bäcklund transformation and the Hirota's bilinear operator method [1, 2, 4].

The solutions of the above mentioned PDEs are particular and has properties much more rich than seen in linear PDEs. The solutions are localized wave forms in space, which do not interact during collision. They are very stable and keep their shape. The shapes can be localized waves, kinks, that is fronts raising from one level to another, and localized oscillating wave forms coined breathers. Due to the stability of these solutions they are named solitons. Single solitons can often be found relatively easy by assuming a travelling wave form with a speed depending on the wave amplitude. This reduces the problem from solving a PDE to solving an ordinary differential equation. Many nonlinear PDEs have localized single travelling waves, which are not solitons but close to having the same properties. We call these waves for solitary waves or quasi solitons. They will loss energy in the form of nearly linear radiation waves during collision with other solitary waves. Common for solitons and solitary waves are that they result from a balance between the steepening effect of pure nonlinearity and dispersion.

In many physical experiments or technical devices it is relevant to consider energy loss, energy input, various boundary conditions, higher order dispersion effects, higher order nonlinearities, delay phenomena and more. Adding such perturbative terms to the soliton equations tranfers the solitons into solitary waves, which only in part retain the soliton properties. Such problems are mostly studied by perturbation analysis, that is energy balance considerations and multiple scale perturbation analysis, and by numerical methods. However, for small perturbations we utilize that a solution to first order is a soliton and that the added second order terms governs equations, which are easy to solve.

The aim of this project work is to investigate elastically coupled physical pendula in the gravitational field. We shall allow the pendulas to make full 2π turns. Let $\varphi_n(t)$, $n = 1, 2, ..., N$, be the rotation angle of pendulum number n measured from the downward position. Furthermore, the pendula have a moment of inertia I with respect to their rotation axis. For a pendulm consisting of a nearly massless rod with a mass m mounted at one end and the other end fixed to the rotation axis the moment of inertia is $I = \ell^2 m$, where ℓ is the length of the pendula. If the pendula are coupled with linear springs, with spring constant κ_0, the resulting dynamical equation becomes [3]

$$m\ell^2 \frac{d^2\varphi_n}{dt^2} - \kappa_0(\varphi_{n-1} - 2\varphi_n + \varphi_{n+1}) + mg\ell \sin(\varphi_n) = 0 \ . \tag{1}$$

Here g is the gravitational constant. Any damping of the pendula are neglected but can be added. Furthermore, One can exert external driving forces on the pendula. If the spacing between the pendula are Δx the position of pendulum n can be set to x=$n\Delta x$ and the rotation angle at x is now denoted $\varphi(x, t)$. Introducing the mass density $\rho = m/\Delta x$ and replace the spring constant κ_0 with $\kappa/\Delta x$, we arrive at the continuum limit of Eq. (1) [3, 4]

$$\rho\ell^2 \frac{\partial^2\varphi}{\partial t^2} - \kappa\frac{\partial^2\varphi}{\partial x^2} + \rho g\ell \sin(\varphi) = 0 \ . \tag{2}$$

In the continuum limit we have expanded $\varphi_{n\pm1}$=$\varphi(x \pm \Delta x)$ in Taylor series and taking the limit $\Delta x \to 0$. The equation in (2) is called the sine-Gordon equation, and it possesses soliton solutions. It is completely integrable, meaning we for any sufficiently smooth initial condition can determine analytically the evolution of this initial condition.

The sine-Gordon equation can also model a superconducting tunnel diode called a Josephson junction. It consists of two superconducting thin films separated by a nano scale insulating barrier. Superconducting electrons at low temperatures can tunnel through the barrier and a magnetic current vortex is formed over the barrier. Josephson junctions are used for voltage standard and calibration of voltmeters, measurement of ultra weak magnetic fields, various electronic devices, including computers. The latter only as prototypes.

In this project we shall use the pendula model in (1) to study a system with varying length of the pendula. Specifically we shall look at solitons and breathers being transmitted or reflected at a localized area at the center of the pendula line, where the pendula length vary. Outside this area the pendula length is constant. Questions to be answered are: can varying pendula lengths lead to a trapping potential for solitons or a barrier? As for linear waves will part of a soliton pass through the barrier and part bee reflected? The same questions for breathers. We shall conduct similar studies for Josephson junctions with varying parameters.

Bibliography

References

[1] Lamb, G.L., Elements of Soliton Theory (Pure and Applied Mathematics), Wiley-Blackwell, 1980.

[2] Dodd,R.K, Eilbeck, J.C., Gibbon, J.D. and Morris, H.C., Solitons and Nonlinear Wave Equations, Academic Press, 1984.
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[3] Dauxois, T. and Peyrard, M., Physics of solitons, Cambridge University Press, 2006.

[4] Scott, A.C., Nonlinear Science, Emergence and Dynamics of Coherent Structures, Oxford University Press, 2ed. 2003.

Learning goals and tasks

The overall learning goals and tasks in this project are listed below as a guide for the project. It is up to you how to reach these goals and at the same time conduct studies and completing a report on the system of equations in (1) and (2), its solutions and applications. The plan should be considered dynamic and may change as you progress. The means of reaching the learning goals include among others studying texts and lecture notes provided in Learn DTU, reading in books, reading scientific papers, call for assistance of your supervisor, call for lectures by your supervisor on specific subjects. Furthermore, you may read reports and presentations on the subject, seek help from other students, the internet and the DTU library.

1) Learning goal: Understand how to derive the models in (1) and (2) for coupled pendulas in the gravitational field [3].

Task: Derive a model, which includes variation of the pendula lengths.

2) Learning goal: Learn how to find travelling wave solutions for a nonlinear partial differential equation, specifically for the sine-Gordon equation in (2). [3, 4].

Task: Find a travelling wave solution of the sine-Gordon equation by inserting the solution ansatz $\varphi(x,t) = f(\xi)$, $\xi = x - x_0 - ct$, into the sine-Gordon equation (2) and solve the resulting ordinary differential equation. In this case the travelling wave is a soliton.

3) Learning goal: Learn about the effect of dispersion in linear partial differential equations.

Task: Find the dispersion relation for the linarized sine-Gordon equation, that is in the case where the nonlinear term $\sin(\varphi)$ is replaced by φ resulting in a linear dispersive partial differential equation, actually called the Klein-Gordon equation [4].

4) Learning goal: How to add perturbative terms like damping and energy input and their physical origin. Learn how to use the energy method to find approximate solutions of the perturbed sine-Gordon equation.

Task: Use the energy method to find approximate travelling wave solutions in a pendula chain with varying pendula lengths. Extend to the case of added damping and energy input.

5) Learning goal: Derive the perturbed sine-Gordon model for the dynamics of fluxons in long Josephson junctions. Derive also suitable boundary conditions[3, 4].

Task: Derive the model equation for a long Josephson junction using coupled single point Josephson junction circuits. Add perturbative terms and physically relevant boundary conditions.

6) Learning goal: Learn how to solve a nonlinear partial differential equation using the semi difference method. Here semi difference means discretize in the space variable x and keep the time t as a continues variable. The resulting set of coupled ordinary differential equations is solve by a Runge-Kutta method. Note that the pendula model in (2) is already discrete in space due to the original physical set up.

Task: Discretize a perturbed version of equation (2) in the space variable and implement the resulting ordinary coupled equations in Matlab using a Runge-Kutta method. Conduct simulations of travelling wave solution and compare with the approximate analytical expressions for the solitary waves found by the energy method. Investigate the interaction of solitary waves with localized potentials (varying length pendulas).

7) Learning goal: Learn how to find two soliton solutions and breather solutions of the sine-Gordon equation (2).

Task: Conduct simulations of breather and two soliton solutions and conduct simulations of head on colliding travelling waves. Investigate their interaction with localized potentials. Discuss your findings.