Exercise 4 – Traffic Density Waves

The goal of this exercise is to model traffic flow, and to use the model to make predictions from data about a real motorway. The project we will develop a **PDE model** in a form of scalar conservation law, and study basic techniques for linear advection equations. As in the previous exercises you should

- Formulate briefly a relevant mathematical model
- Use the model to derive information from the data
- Discuss the result and the shortcomings of the model

1 Modelling Unidirectional Traffic - a PDE model -

In this project we will examine a simple model of traffic on a unidirectional road, for instance a single direction of a motorway.

Many details will be ignored and simplified. Our job is to distinguish the most important factors of the phenomenon and to find a way to study it in a way which is sufficiently simple (so that it is easy to work with), and sufficiently concise (so that it can be applied). Traffic on a motorway is a phenomenon of discrete particles or cars. However, we will, however, work with a concept of a *local density* of cars, as if the stream of cars formed a continuous fluid and therefore the velocity field is also given, under this *continuum hypothesis*. We will, in this project, also ignore the important, sometimes decisive, existence of exit- and entry ramps, and merge points where the road meets with other motorways.

Thus, the mathematical abstraction is simply a one-dimensional space (a line, the x-axis), upon which various functions that describe the traffic, are defined. These functions will in general be functions of two variables, space x and time t.

The Traffic Density Field and Velocity Field Traffic is a phenomenon of individual cars or discrete particles but we employ the continuum hypothesis for the traffic flow. Hence it is important to understand how a discrete reality is interpreted in terms of a continuum model. In gas dynamics the mass density $\rho(x)$ is defined as the limit of mass per volume, as the volume includes the point x and goes to zero. However, the volume can not be as small as the atomic level (why?) and should be at least much larger than the atomic scale. We can do very similar considerations in a macroscopic situation, like the motorway (see exercises below).

We expect the density ρ to depend on both position and time, so we have a function of two variables,

$$\rho = \rho(x, t)$$

A useful descriptor of the situation is the concept of the *local velocity field*. At a given position x on the road and time $t > 0$, we will denote the average velocity of traffic by $u(x, t)$, which is also a function of two variables. We will treat in Sections 1 and 2 u and ρ as if they are continuous, even differentiable, functions of x and t .

Let $x = \zeta(t)$ be the position of the car at time t with initial position $\zeta(0) = \beta$. The image of ζ, or sometime the function ζ itself, is called a *particle trajectory*. We may consider the graph of a particle trajectory in x-t plane, with space x on the horizontal axis, and time t on the vertical axis as in the following diagram. The value of the velocity field u at a given position and a given time should be the velocity of the car that passes the given point at the give time, i.e.,

$$\zeta'(t) = u(\zeta(t), t). \tag{1}$$

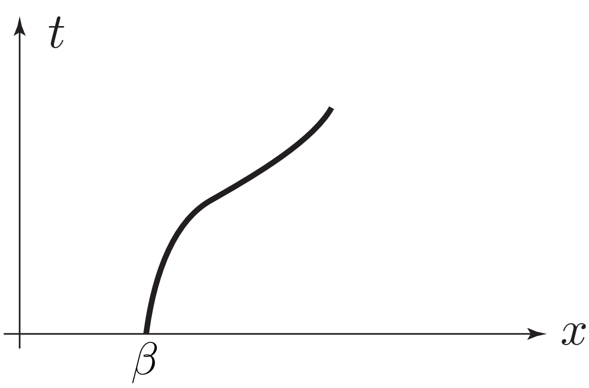

Figure 1: Graph of a particle trajectory. Initially, near $x = \beta$ the particle is moving slowly. Then it picks up speed, moving in the positive x-direction, and towards the end of the trajectory it slow down again. Notice that $x = \zeta(t)$, so that the argument is on the vertical axis and the value is on the horizontal axis.

1. In a traffic model cars play the role of atoms in gas dynamics. The unit length of our model should be bigger than certain scale to make the model valid. Estimate the spatial scale over which car density makes sense. Considering the length of a car and your unit length of the model estimate the maximum density. What is the dimension of the car density in this model?

2. How is the slope of this position curve related to the value of the local velocity field at some point (x_1, t_1) ?

3. Consider cars initially distributed between $x = 0$ and $x = 1$, and let β denote the initial position of a car. Suppose that the car moves with velocity $40 + 15\beta$ (km/h).

   (a) Show that there is a car in the interval $[0, 1]$ with any given velocity between 40 km/h and 55 km/h. What is the initial position of the car?

   (b) Show that, for a car with initial position β, the position at time t is determined by

   $$x = (40 + 15\beta)t + \beta$$

   and that at this (x, t), $u = 40 + 15\beta$, i.e., $u((40 + 15\beta)t + \beta, t) = 40 + 15\beta$.

   (c) Eliminate β and derive the general expression for the velocity field:

   $$u(x, t) = \frac{15x + 40}{15t + 1} \ .$$

Traffic Flux and a Conservation Law. In the present model, cars are neither created nor vanish anywhere on the road. In other words, the total number of cars are conserved and there is an equation expressing this fact.

The number N of cars between two points a and b is given by $N = \int_a^b \rho(x, t)\mathrm{d}x$ and can change only by a difference in inflow at the left endpoint a and outflow at the right endpoint b; thus

$$q(x_1, t) - q(x_2, t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} \rho(x, t)\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(x, t)\mathrm{d}x, \tag{2}$$

where the flux $q(x_i, t),\ i = 1, 2$, measures the amount of cars that cross the boundary point $x = x_i$ per unit time to the positive direction (from left to right). If ρ and q are differentiable, this equation becomes

$$\frac{\partial}{\partial t}\rho(x, t) + \frac{\partial}{\partial x}q(x, t) = 0. \tag{3}$$

Equation (3) is the fundamental dynamical equation for traffic flow.

Using the relation $q = \rho u$, we can rewrite (3) as

$$\frac{\partial}{\partial t}\rho(x, t) + \frac{\partial}{\partial x}\left(\rho(x, t)u(x, t)\right) = 0. \tag{4}$$

This is the differential form of the conservation law; it says that time variations in the density must be matched by spatial variations of the flow. It is basis for our study of the evolution of the traffic density from some initial state.

A word about making equations dimensionless. This is generally a good idea, since it may reveal the relative size of terms, uncover decisive parameters, and ensure for numerical computations that the dimensionless dependent variables have numerical values near one, where machine precision is always highest.

We do this by introducing characteristic values for the variables, and then measuring each variable relative to its characteristic value. For instance if q is a physical quantity (with units), and Q is a characteristic value for q, (having the same units) then $\bar{q} := q/Q$ will be dimensionless, and will have numerical values near 1. Differential equations involving physical variables q can, using the chain rule, be written as differential equations for the dimensionless variables $\bar{q}$, and it is these dimensionless equations which should be used for numerical studies.

In the present case, we would introduce a characteristic length X and a characteristic tie T. Then, a characteristic density would be $1/X$, and a characteristic velocity would be $U = X/T$. Equation (4) would then become, in dimensionless variables:

$$\frac{\partial}{\partial \bar{t}}\bar{\rho}(\bar{x}, \bar{t}) + \frac{\partial}{\partial}\bar{x}\left(\bar{\rho}(\bar{x}, \bar{t})\bar{u}(\bar{x}, \bar{t})\right) = 0.$$

Revealing in this no new dimensionless parameters. We can therefore assume the scaling performed, drop the overbars, and then treat equation (4) as a dimensionless equation.

Nevertheless, we still only have *one* equation for *two* unknown functions: $\rho(x, t)$ and $u(x, t)$.

1. Explain why the two equalities in the relation (2) should hold.
2. Derive PDE (3) from the integral equation (2). (Hint: Let $x_1 = x$ and $x_2 = x + h$, divide (2) by h, and take the limit as $h \to 0$. The mean value theorem is useful for this.)

To proceed, we must make an assumption or a relation that ties together the two functions. This is the subject of the next section.

Velocity-Density Relation. It is a common experience that there is a relation between traffic density ρ and traffic velocity u. If the density is low, cars travel faster (up to the speed limit). But if the density is high, cars slow down, drivers sensibly ajusting their speed to be able to brake without running into the car ahead of them. And if the density becomes very high, cars stop, creating a traffic 'plug', a section of cars that are not moving at all.

We translate this into the following basic assumption: The velocity field u is a function of the density ρ, i.e., $u = u(\rho)$, and furthermore, this function is of a type which

- has some positive value $\rho_{\max}$ at $\rho = 0$
- decreases with ρ, i.e., $u'(\rho) \leq 0$, and **(H)**
- vanishes for $\rho \geq \rho_{\max}$.

Pictorially, the function $u(\rho)$ must look something like figure 1.

The specific form of $u(\rho)$ may depend on details and condition of the road, for instance the number of lanes, and of course on the speed limits imposed.

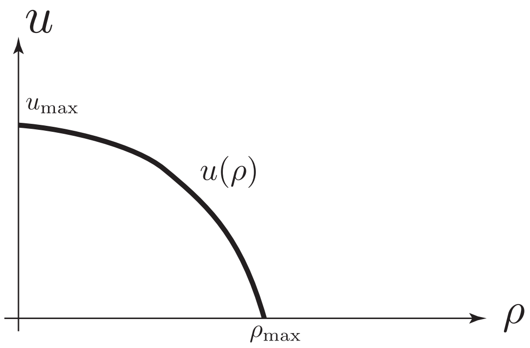

Figure 2: The assumed form of the velocity-density relation. At low densities the velocity approaches a maximal value, but for higher densities the velocity is smaller, and goes to zero at the maximal density ρ_{max} .

We may consider the flux q as a function of density. Since $q = \rho u$, we have that $q(\rho) = \rho u(\rho)$, and then the function $q(\rho)$ will then have the qualitative behavior seen on Figure 2.

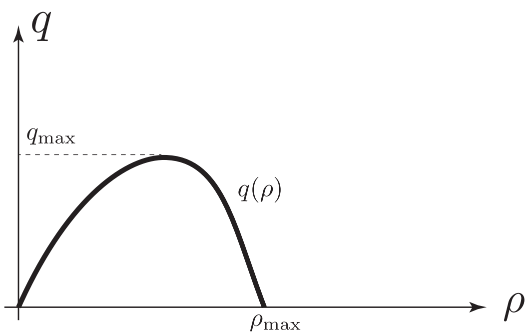

Figure 3: The subsequent form of the flow-density relation. At low densities the flow q is zero (because there are few cars). For higher densities the flow increases until it reaches a maximal value, q_{max}. As the density grows even higher, the velocity goes to zero, and thus also the flow goes to zero, until, at ρ_{max}, the flow becomes zero.

TO BE CONTINUED