Exercise 4 – Traffic Density Waves

Mathematical Modelling
02526

Here is a list of brief answers (not detailed solutions) to the questions posed in the text

Question 1
This question does not have a sharp or unique answer; it is about understanding what 'local density' of discrete objectsis. - If the length of one car ℓ is taken as the unit length, then all density calculations should be done over a length no less than, say, 30-50 cars, such that the addition of a single car does not change the local value of the density by a large fraction. The dimension of density here is number of cars per unit length, i.e., (distance)$^{-1}$. The maximum possible density in these units is 1, since there can at most be one car per car length (when they are tightly packed).

Question 2
The slope of the t vs x, or (dt/dx), at some point (x_1, t_1) on the curve, is the reciprocal value of the local velocity field at (x_1, t_1). This is

Question 3
(a): Cars with $\beta = 0$ have velocity 40 km/h, and cars with $\beta = 1$ have velocity 40+15 = 55 km/h.

(b): The constant-velocity position curve x vs t will be: $x = (40 + 15\beta)t + \beta$. This means that $u((40 + 15\beta)t + \beta, t) = 40 + 15\beta$.

(c): Since u is the slope of the curve (in the above equation), $u = 40 + 15\beta$. Re-expressing (solving for β),

$$\beta = \frac{x - 40t}{1 + 15t}$$

and thus:

$$u(x,t) = 40 + 15 \cdot \frac{x - 40t}{1 + 15t} = \frac{15x + 40}{1 + 15t}$$

Question 4
Flux = number of cars passing a certain point per second = (number of cars at the point) $\times$ (the velocity at that point). Equation (2) expresses that the change per time, (d/dt) of the number of cars between x_1 and x_2 equals the flux *into* the interval $q(x_1, t)$ minus the flux out of the interval $q(x_2, t)$).

The second equality is due to the fact that the only explicit time dependence is in the second term of the integrand.

Question 5
To derive (3) from (2), notice that to first order in Δx,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_x^{x+\Delta x} \rho(x,t)\mathrm{d}x = \frac{\partial}{\partial t}\rho(x,t)\Delta x$$

and since both sides equal $q(x,t)-q(x+\Delta x,t)$, we can divide both sides by Δx, take the limit $\Delta x \to 0$, and we have (3). One can (but does not have to) invoke the mean value theorem.

Part 2 - Linear Theory for Transport

Question 1.
How to prove that an equation is nonlinear? Just provide one example where it fails to be linear. Consider what happens, for instance, if $u(\rho) = \rho$. Then the equation reads

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho^2) = 0$$

What happens when we replace ρ with the a linear combination $\alpha\rho_1 + \beta\rho_2$ of densities. Then the left hand side would read:

$$\frac{\partial}{\partial t}(\alpha\rho_1 + \beta\rho_2) + \frac{\partial}{\partial x}((\alpha\rho_1 + \beta\rho_2)^2) = 0$$

or

$$\alpha\frac{\partial}{\partial t}\rho_1 + \beta\frac{\partial}{\partial t}\rho_2 + \frac{\partial}{\partial x}(\alpha^2\rho_1^2) + \frac{\partial}{\partial x}(\beta^2\rho_2^2) + 2\alpha\beta\frac{\partial}{\partial x}(\rho_1\rho_2) = 0$$

The last term makes this equation **not** the sum of the terms for $\alpha\rho_1$ and $\beta\rho_2$, and we are not guaranteed that this would be equal to zero.

The most important consequence is that the *superposition principle* does not hold for the equation; in other words, we cannot just add any two solutions and reach a new solution. This makes nonlinear equations more challenging to solve.

Question 2.
For constant density ρ_0 both the time derivative of ρ_0, and the space derivative of $\rho_0 u(\rho_0)$ vanish, and so ρ_0 is a solution.

Question (6)
Now we have to prove that any linear combination $\alpha\rho_1 + \beta\rho_2$ of two solutions ρ_1 and ρ_2 is also a solution. Since there are no product terms however, it follows from the lineariy of the derivative; inserting the linear combination we have:

$$\frac{\partial}{\partial t}(\alpha\rho_1 + \beta\rho_2) + c\frac{\partial}{\partial x}(\alpha\rho_1 + \beta\rho_2) = \alpha\left[\frac{\partial}{\partial t}(\rho_1) + c\frac{\partial}{\partial x}(\rho_1)\right] + \beta\left[\frac{\partial}{\partial t}(\rho_2) + c\frac{\partial}{\partial x}(\rho_2)\right] = 0$$

because each term on the left hand side is equal to zero.

Question (7)
The terms

$$\frac{\partial}{\partial t}\rho_0 + \frac{\partial}{\partial x}(\rho_0 u(\rho_0))$$

were dropped because, since ρ_0 is a solution, they are equal to zero.

Question (8)
Plug $f(x-ct)$ into the equation; one gets

$$\frac{\partial}{\partial t}f(x-ct) + c\frac{\partial}{\partial x}f(x-ct) = -cf'(x-ct) + cf'(x-ct) = 0$$

where f' denotes the derivative of the function f .

Question (9)
In the (t,x)-diagram (meaning: if the t-axix is the first axis) the line $x = x_0 + ct$ has slope c. Then in the (x,t)-diagram, with x as the first axis, the same line has slope $1/c$.

Question (10)
(A right parenthesis was missing in the question. Here is how to apply the chain rule:

$$\begin{aligned}
\frac{\partial g}{\partial x} &= \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial g}{\partial y} \\
\frac{\partial g}{\partial t} &= \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial t} = -c\frac{\partial g}{\partial y} + 1\cdot\frac{\partial g}{\partial s}
\end{aligned}$$

where one uses the transformation relations at the bottom of page 7.

Question (11)
At $\rho = \rho_c$ the slope c is equal to zero, and the perturbation will not travel – it remains in its initial position.

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