Danmarks Tekninske Universitet

Discrete mathematics 2: algebra
Course number: 01018

Assignment 2

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Part 1

Consider the group $(G, \circ)$ of all the maps $f_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by $f_{a,b}(x) := ax + b$, where $a, b \in \mathbb{R}$, $a \neq 0$ and $\circ$ is the composition operation. Let $\phi : G \to \mathbb{R}_*$ with $\phi(f_{a,b}) := a$. Here $\mathbb{R}_*$ is the set of all non-zero real numbers.

A) Let $a, b, c, d \in \mathbb{R}$ and suppose that $a \neq 0$ and $c \neq 0$. Show that $f_{a,b}^{-1} = f_{a^{-1},-a^{-1}b}$ and $f_{a,b} \circ f_{c,d} = f_{ac,ad+b}$.

Let us start by showing $f_{a,b}^{-1} = f_{a^{-1},-a^{-1}b}$:

$$\begin{aligned}
&f_{a,b}(x) = ax + b \quad \Leftrightarrow \quad f_{a,b}(x) - b = ax \quad \Leftrightarrow (a \neq 0) \quad a^{-1} \cdot f_{a,b}(x) - a^{-1} \cdot b = x \quad \Rightarrow \\
&f_{a,b}^{-1}(x) = a^{-1} \cdot x - a^{-1} \cdot b, \quad if \;\; a \neq 0 \quad \Rightarrow \\
&f_{a,b}^{-1}(x) = f_{a^{-1},-a^{-1}b}(x)
\end{aligned}$$

Now let us show $f_{a,b} \circ f_{c,d} = f_{ac,ad+b}$:

$$\begin{aligned}
&f_{a,b} \circ f_{c,d}(x) = f_{a,b}(f_{c,d}(x)) = f_{a,b}(cx + d) = a(cx + d) + b = acx + ad + b \quad \Rightarrow \\
&f_{a,b} \circ f_{c,d} = f_{ac,ad+b}
\end{aligned}$$

B) Define $T := \{f_{1,b} \mid b \in \mathbb{R}\} \subset G$. Show that T is a subgroup of $(G, \circ)$.

According to Definition 55 $H \subset G$ is a subset iff:

- $e \in H$.
- for any $f \in H$ also $f^{-1} \in H$.
- for any $f, g \in H$ also $f \circ g \in H$.

We have $e_T = f_{1,0}$ since $f_{1,b} \circ f_{1,0} = f_{1,b}$. Since $0 \in \mathbb{R}$ we have that $f_{1,0} \in T$, so $e \in T$.

For any $f_{1,b} \in T$ we have that $f_{1,b} \circ f_{1,-b} = f_{1,-b} \circ f_{1,b} = e_T$, i.e. $f_{1,b}^{-1} = f_{1,-b}$ and since $b \in \mathbb{R}$ it must hold that $-b \in \mathbb{R}$, and in turn $f_{1,b}^{-1} \in T$. So $f \in T \quad \Rightarrow \quad f^{-1} \in T$.

Let $f_{1,b_1}, f_{1,b_2} \in T$ and $f_{1,b_1} \circ f_{1,b_2} = f_{1,b_1+b_2}$, since $b_1, b_2 \in \mathbb{R}$ it must hold that $b_1 + b_2 \in \mathbb{R}$ which means that $f_{1,b_1+b_2} \in T$. So $f, g \in T \quad \Rightarrow \quad f \circ g \in T$.

We can therefore conclude that T is a subgroup of $(G, \circ)$.

C) Show that for any $g \in G$ it holds that $gT = Tg$.

Let us choose a $g \in G$ by $f_{a',b'}$ then we have:

$$\begin{aligned}
gT &= \{f_{a',b'} \circ f_{1,b} \mid b \in \mathbb{R}\} = \{f_{a',a'b+b'} \mid b \in \mathbb{R}\} \\
Tg &= \{f_{1,b} \circ f_{a',b'} \mid b \in \mathbb{R}\} = \{f_{a',b+b'} \mid b \in \mathbb{R}\}
\end{aligned}$$

We note that $gT = Tg$ if there exists some $b_1, b_2 \in \mathbb{R}^2$ s.t. $f_{a',a'b_1+b'} = f_{a',b_2+b'}$ i.e. $a'b_1 + b' = b_2 + b'$:

$$\begin{aligned}
&\exists b_1, b_2 \in \mathbb{R}^2 \ : \ a'b_1 + b' = b_2 + b' \quad \Rightarrow \quad gT = Tg \\
&a'b_1 + b' = b_2 + b' \quad \Leftrightarrow \quad a'b_1 = b_2 \\
&a', b_1 \in \mathbb{R}^2 \quad \Rightarrow \quad a' \cdot b_1 \in \mathbb{R} \quad \Leftrightarrow \quad b_2 \in \mathbb{R} \quad \Rightarrow \\
&gT = Tg
\end{aligned}$$

So the left and right coset of T given any g will always be the same.

Part 2

Consider the permutations $g_1 = (132)$ and $g_2 = (1345)$ from $S5$.

A) Compute the order and the sign of the permutation $g_1 \circ g_2$.

Order
We have that:

$$g_1 \circ g_2 = (132)(1345) = (12)(345)$$

We now have two disjoint cycles and we know these commute, i.e. $(12)(345) = (345)(12)$. The only way for $((12)(345))^n = id$ is when $(12)^n = id$ and $(345)^n = id$. We know that for any m-cycle it holds that the cycle to the power of a multiple of m will equal id. In other words $(12)^n = id$ whenever n is a multiple of 2, and $(345)^n = id$ whenever n is a multiple of 3. This means that the order of (12)(345) is the greatest common multiple of 2 and 3.

$$ord(g_1 \circ g_2) = ord((12)(345)) = GCM(2, 3) = 6.$$

Sign
Lemma 38 from the book states that:

$$\text{The sign of an m-cycle is } (-1)^{m-1}.$$

Further we know (this does not have a 'nice' name in the book) that:

$$sign(f_1 \circ f_2) = sign(f_1) \cdot sign(f_2).$$

Now we get:

$$sign(g_1 \circ g_2) = sign(g_1) \cdot sign(g_2) = (-1)^2 \cdot (-1)^3 = -1$$

Note: We could also have used the fact that $(132)(1345) = (12)(345)$, then we would have taken $sign((12)) \cdot sign((345)) = (-1)^1 \cdot (-1)^2 = -1$ and got the same result.

B) It is given that the set of permutations $H := \{id, g_1, g_1^2\} \subset S5$ is a subgroup of $(S_5, \circ)$. You may use this fact without proving it. Now compute the cosets $g_2 \circ H$ and $H \circ g2$.

$$\begin{aligned}
&g_1^2 = (132)(132) = (123) \Rightarrow \\
&H = \{id, (132), (123)\}
\end{aligned}$$

$g_2 \circ H$:

$$\begin{aligned}
&(1345) \circ id = (1345) \\
&(1345)(132) = (145)(23) \\
&(1345)(123) = (1245) \quad \Rightarrow \\
&g_2 \circ H = \{(1345), (145)(23), (1245)\}
\end{aligned}$$

$H \circ g_2$:

$$
\begin{aligned}
&id \circ (1345) = (1345) \\
&(132)(1345) = (12)(345) \\
&(123)(1345) = (2345) \quad \Rightarrow \\
&H \circ g_2 = \{(1345), (12)(345), (2345)\}
\end{aligned}
$$

C) How many distinct left cosets does H have in S_5?

According to theorem 75 in the book we have that for $(G, \circ)$, $H \subset G$:

$$\forall f, g \in G \quad f \circ H = g \circ H \quad \vee \quad f \circ H \cap g \circ H = \emptyset$$

We also know that any set of finite order can be written as a finite union of cosets:

$$G = \cup_{i=1}^{n} f_i H, \quad H \subset G, \ f_i \in G$$

Further we know that the coset is always of the same order as the subset:

$$|f \circ H| = |H|, \quad H \subset G, \ f_i \in G$$

As a natural consequence of this the order of G can be written as a multiple of the order of H, this is also Lagrange's Theorem (one of the many).

$$|G| = n \cdot |H| \quad \Leftrightarrow \quad \frac{|G|}{|H|} = n$$

Where n is the number of coset required to unionize G. Via theorem 75 we know that all these n cosets will be distinct, i.e. the the distinct number of left (and right) cosets H has in G is given by $\frac{|G|}{|H|}$. In our case of H and S_5, this yields:

$$\frac{|S_5|}{|H|} = \frac{5!}{3} = 5 \cdot 4 \cdot 2 = 40$$

So H has 40 distinct left cosets in S_5.