01325 Function spaces and mathematical analysis

Course Information (can be updated)

Jens Gravesen, 303B/152, ☎ 45253064, ✉jgra@dtu.dk

Modules

The course is placed in module E5B, i.e., Wednesdays 13:00–17:00. A course afternoon generally consists of a lecture from 13:00 to 15:00 and an exercise session from 15:15 to 17:00.
The lectures and exercises takes place in Aud. 43 in building 303A.

Evaluation

Three homeworks should be completed during the course. Grades for the course will be given in the 7 step scale and will be based on the 3 homeworks and the result of a 4-hour written exam. The submitted homeworks and the written examination will count in the relation 1:7. All homeworks are individual, i.e., only single authored homeworks are accepted.

Course book/material

The main textbook is:

O. Christensen: *Functions, Spaces, and Expansions*, Birkhäuser, 2010. The book can be downloaded chapter by chapter via the DTU Library `https://findit.dtu.dk`. Alternatively, the hard cover version is sold in Polyteknisk Boghandel: `https://www.polyteknisk.dk/home/dtu/detailed_view/9780817649791?z=Dtu-lyng`

The teaching material consists of the above mentioned textbook, the weekly work plans and other material posted on DTU Learn.
You can find lectures by Ole Christensen (the author of the text book) on YouTube:

Danish: `https://www.youtube.com/playlist?list=PLgowegO9Se5-nUTMTp58WbUdVbbqIlfed`

English: `https://www.youtube.com/playlist?list=PLgowegO9Se59A3kzc29t2FY4Fu3JfjzpU`

Teaching plan

Will be updated. All references are to the main textbook unless otherwise stated. The exercises named Problem 1–243 are from `extra_exercises_2021.pdf` on DTU Learn.

W	Content
1	*Title:* On functions of a real variable, normed vector spaces, and convergence. *Topics:* Motivating examples, topology, supremum and infimum, maximum and minimum, (normed) vector spaces, function spaces, convergence. *Reading:* §1.1-1.6, §2.1–2.2, *Class work:* Exercises 1.3, 1.4, 1.8, 2.1, 2.3, 2.4, 2.5, Problem 9. *Self-study:* §1.4 §2.2. *Hand-in*: No homework
2	*Title:* Banach spaces and linear operators on normed vector spaces. *Topics:* Proof by induction, linear operators on normed spaces, Banach spaces, $\ell^p(\mathbb{N})$-spaces, and Hölder's and Minkowski's inequalities. *Reading:* §Theorem 1.7.3, §1.9, §2.4, §3.1–3.2 (except Example 3.2.2). *Class work:* Exercises 1.24, 2.10, 2.12, 2.13, 2.14(i), 3.1, 3.3, 3.14 *Hand-in*: **HW1** (Exercise 2.2 and Problem 1)
3	*Title:* Hilbert spaces *Topics:* Hilbert spaces, inner product spaces, orthogonality, the Hilbert space $\ell^2(\mathbb{N})$, linear functionals on Hilbert spaces. *Reading:* §4.1–4.4 *Class work:* Exercises 1.15, 4.1, 4.3-4.4, 4.8, 4.9, 4.11 4.12. *Hand-in*: No homework
4	*Title:* Infinite series and linear operators on Hilbert spaces *Topics:* Infinite series (in normed vector spaces), (Schauder) basis, Riesz' representation theorem, (Linear) operators on Hilbert spaces, adjoint operator, self-adjoint operator, *Reading:* Definition 2.5.1, Definition 2.5.4, Example 3.2.2, §4.4–4.5 *Class work:* Exercises 4.21, 4.18, 4.25 [only (4.34)], 4.17, 1.16, 4.16, 4.19, 4.31, 4.15. *Hand-in*: No homework.
5	*Title:* Completion of normed vector spaces. *Topics:* Dual spaces, completion. *Reading:* The note `complete.pdf` uploaded to DTU Learn. (If you have had 01125 "Fundamental topological concepts and metric spaces" or similar you can have a look at `complete01330.pdf`, especially Exercise 1 in Section 1 and Exercise 3 in Section 2) *Class work:* Exercise 1–8 in the note `complete.pdf`. *Hand-in*: **HW2** (Exercises 4.22, 4.25 [only Equation (4.35)])
6	*Title:* Function spaces and integration. *Topics:* Integration, function spaces, $C_c(\mathbb{R})$, Banach spaces $C_0(\mathbb{R})$ and $L^p(\mathbb{R})$. *Reading:* §1.7, 5.1, 5.2, 5.4, 5.5 *Class work:* Exercises 5.3 (only for f_4, f_6 and f_7), 5.4, 5.8, 5.9, 3.7, Problems 8, 109 *Hand-in*: No homework

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series – continued from previous page –

W	Content
7	*Title:* Orthonormal bases & Sturm-Liouville theory *Topics:* Orthonormal bases in abstract Hilbert spaces, Parseval's identity, the Hilbert spaces $L^2(\mathbb{R})$ and $L^2(a,b)$, Fourier series in $L^2([-\pi,\pi])$, Sturm-Liouville theory & special functions. *Reading:* §4.7, Theorem 14 and Section 3 (Theorem 8) in the note on completion, §6.1, §6.3, §6.4, Chapter 11. *Class work:* Exercise 1.1, Problem 13 & 14 (you do not need to use Maple, you can calculate by hand if you want to), Exercises 6.12, 11.4, 6.15. *Hand-in*: No homework
8	*Title:* Approximation theory *Topics:* Projection operators, Bessel's inequality, convergence of expansions, dense subsets, extensions of bounded, linear operators. *Reading:* §2.3, §3.3, Lemma 4.6.4(i)-(ii). *Class work:* Exercise 2.6, 2.7, 2.14(ii), 3.5, 4.29 (modified), W8.1 (on Work Plan) *Hand-in*: No homework
9	*Title*: Operators on $L^2(\mathbb{R})$ & the Fourier transform on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. *Topics*: Operators on $L^2(\mathbb{R})$ and $L^p(\mathbb{R})$. The Fourier transform on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. *Reading*: §6.2 and §7.1 *Class work*: Exercises 5.18, 6.3, 6.4 (only for D_c), 6.5 (prove only one of the three relations), 7.1, 7.4 [only part of it]. *Hand-in*: **HW3** (Exercises 3.8, 5.19)
10	*Title*: Convolution, Fourier analysis, Fourier transformation. *Topics*: Convolution, completeness of the trigonometric polynomials, the inversion theorem. *Reading*: §7.3 and the note on applications of convolution: `convolution_application.pdf`. *Class work*: Exercises 7.5, 7.9, Problem 233, 7.11, and Exercise W10.1 (on Work Plan). *Hand-in*: No homework
	Easter vacation, 11– 18 April.
11	*Title*: Shannon's sampling theorem and wavelet analysis in $L^2(\mathbb{R})$. *Topics*: Shannon's sampling theorem, Nyquist and wavelet analysis in $L^2(\mathbb{R})$. *Reading*: §7.4 and §8.1–8.2 (until page 164). *Class work*: Exercise 6.10, 8.1, Problem 114, Exercises 7.13, 8.4 *Hand-in*: No homework
12	*Title*: Wavelet analysis in $L^2(\mathbb{R})$ *Topics:* Multi-resolution analysis, scaling function, scaling equation, wavelet. *Reading: Pages 164 – 170.* *Class work: Exercises 8.2, 8.6, 8.7 (i)-(vii), 8.11, Problem 115* *Hand-in*: No homework
13	*Title: Q&A session* *Topics: Everything (from the course)* *Reading: No reading, but think about questions to ask and try to solve the 2022 exam.* *Class work: Problem 106, 217, 119, Exercises 11.3 and 1.22, Problem 105, 122, 104.* *Hand-in*: No homework

Reading guide: **HW**: Homework, **W**: Week, **TBA**: To be announced.

The exam is on campus.

On 25 May, (the precise time and place has not been published)
(or I have not seen it)

Check yourself!

Curriculum/Syllabus (for the written exam)

a) From the main textbook [Christensen]:

- Sections 1.2, 1.4–1.5 (until Def. 1.5.6), 1.6–1.7, 1.9.
- Chapter 2.
- Chapter 3.
- Sections 4.1–4.5, Lemma 4.6.4, Lemma 4.6.5 and Section 4.7 (until Def. 4.7.3).
- Sections 5.1, 5.2, 5.4, 5.5.
- Chapter 6.
- Sections 7.1–7.4.
- Sections 8.1, 8.2.
- Chapter 11.

b) The notes on completion and on application of convolution.

c) The exercises on the weekly work plans.