Danmarks Tekninske Universitet

Function spaces and mathematical analysis
Course number: 01325

Homework 1

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Exercise 2.2

Let us first consider the left hand side of the inequality

$$A \sum_{k=1}^{N} |c_k|^2$$

Let us assume

$$A \sum_{k=1}^{N} |c_k|^2 = 0$$

We have that $|\cdot| : \mathrm{C} \to \mathrm{R}^+$ (including 0). Since $A > 0$ it follows

$$\begin{aligned} A \sum_{k=1}^{N} |c_k|^2 = 0 \quad &\Rightarrow \\ c_k = 0, \quad &\forall k \end{aligned}$$

We further know that

$$0 \leq A \sum_{k=1}^{N} |c_k|^2 \leq \left\| \sum_{k=1}^{N} c_k v_k \right\|^2$$

It then follows logically

$$\begin{aligned} \left\| \sum_{k=1}^{N} c_k v_k \right\|^2 = 0 \quad &\Rightarrow \\ A \sum_{k=1}^{N} |c_k|^2 = 0 \quad &\Rightarrow \\ c_k = 0, \quad &\forall k \end{aligned}$$

Now via definition 1.2.4 we conclude $\{v_k\}_{k=1}^{N}$ are linearly independent.

Problem 1

i)

We assume it is known that $C[-\pi, \pi]$ is a vector space when equipped with regular addition and multiplication (shown in exercise 1.3).
V is clearly a non-empty subset of the vector space of continuous function. To show that V is in fact a subspace, we will use lemma 1.2.7.
Let $f, g \in V$ and let $\alpha, \beta \in \mathrm{C}$, then

$$\alpha f + \beta g \in V, \quad \forall f, g \in V, \ \ \alpha, \beta \in \mathrm{C}$$

This follows directly since f and g are both chosen from V and we know that V consists of *all* linear combination.
We therefore conclude that V *is* a subspace of $C[-\pi, \pi]$.

ii)

We notice that the hint actually tells us that x^2 can be approximated by linear combinations of elements we know are in V, which mean that this will again be in V (simply by definition). However, since x^2 is not in V we can see that we can approximate (with arbitrarily good precision) an element outside of V only using elements in V. Therefore V cannot be closed. Mathematically we write this as:

$$\left| x^2 - \left(\frac{\pi^2}{3} + 4 \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos(nx) \right) \right| \leq 4 \sum_{N+1}^{\infty} \frac{1}{n^2}$$

So if we choose $f(x) = \left(\frac{\pi^2}{3} + 4 \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos(nx) \right) \in V, x \in [-\pi, \pi]$, then

$$\lim_{N\to\infty} \left| x^2 - f(x) \right| = 0, \; x \in [-\pi, \pi] \quad \Rightarrow$$
$$\lim_{N\to\infty} f = x^2 \notin V, \; x \in [-\pi, \pi]$$

So V does not contain its entire boundary, and via definition 1.4.1 it is not closed.