Danmarks Tekniske Universitet

Project 3 - 02526 Mathematical Modeling

Sunday 19th April

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1 Introduction

The local traffic authorities have issued two problems concerning traffic and density waves that must be investigated to better understand how traffic on motorways work. The first problem is understanding the on-ramp section of the motorway, and investigate how driver acceleration influences the velocity and density of cars. The second problem is related to road capacity and sets out to determine the critical car density for the motorway based on data measuring the number of cars passing a specific road segment in a given time interval of the day. The analysis will be performed using a simplified model of traffic on a unidirectional road, including an on-ramp located prior to the unidirectional road.

2 Methodology

2.1 The model

The simple model is a one-dimensional space that represents the road segment in space x. The road segment is expanded to include an on-ramp from $x = -1$ to $x = 0$, where a traffic light is located and the real motorway begins until $x = +inf$. This one-dimensional space will be the base for modeling traffic through density, flux and velocity functions describing traffic in space x as well as time t.

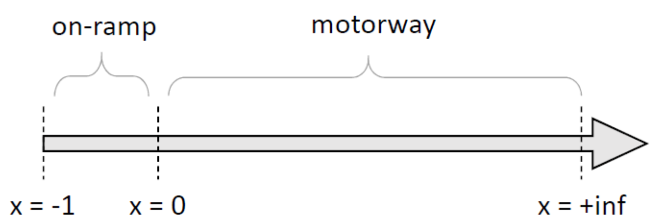

Figure 1: Illustration of the simple model.

At time $t = 0$ the on-ramp is assumed to be uniformly filled with cars, and at the same time $t = 0$ the lights turn green and all cars begin to uniformly accelerate until they hit the speed limit $u_m = 110\frac{km}{h}$, from which they continue with constant velocity. It is assumed that the magnitude of the acceleration is dependent on the position of the car in the initial line-up on the on-ramp x_0. The acceleration can then be expressed as:

$$a(x_0) = \alpha \cdot (1 + x_0), \quad -1 < x_0 < 0 \tag{1}$$

This expression will prove to be useful when the path of a car starting at x_0 must be determined, as well as the velocity field of cars.
We want to determine the road capacity and here we use the relationship between flux and density. The relation between flux q and density ρ of cars on the road segment is expected to be on the form of equation[1] 2, where a, p_m and b can be determined from the measured data4.

$$q(p) = a\rho(\rho_m - \rho)^b, \quad 0 \leq \rho \leq \rho_m \tag{2}$$

With the fitted model we can determine the slope of the flux-density, $q'(\rho)$. This is very important because it is a component of a linear equation which we use to model the change of density of cars. Seen below.
For the equation below, let us assume that we have a high-way where most cars are uniformly distributed, ρ_0 and a section where the cars distributed with a function $\rho_1(x,t) = f(x - ct)$ hence the combined density is $\rho_0 + \rho_1(x,t)$. The linear transport equation then states:

$$\frac{\delta}{\delta t}\rho_1 + c\frac{\delta}{\delta x}\rho_1 = 0, \qquad \rho_1(x,0) = f(x)$$

[1]Exercise 4 - Traffic Density Waves. PGHJ, Mathematical Modelling, 02526 DTU.

and here we have the $c = q'(\rho_0)$. Wee see that $\rho_0+\rho_1(x,t)$ is a solution to the equation above. With $c = q'(\rho_0)$ and the linear transport equation, we can derive that as times goes forward the density of cars can have the following cases. If:

- $q'(\rho_0) < 0$: the density of cars would move backwards in time
- $q'(\rho_0) = 0$: the density of cars will not move - this is called the critical car density
- $q'(\rho_0) > 0$: the density of cars will move forward

Therefore if ρ_0 is too great, i.e., the traffic is heavy, the density would move backward through the traffic.

3 Results

3.1 The on-ramp

The path of a car $x(t, x_0)$ initially at x_0 on the on-ramp has been illustrated and is shown in figure 2.

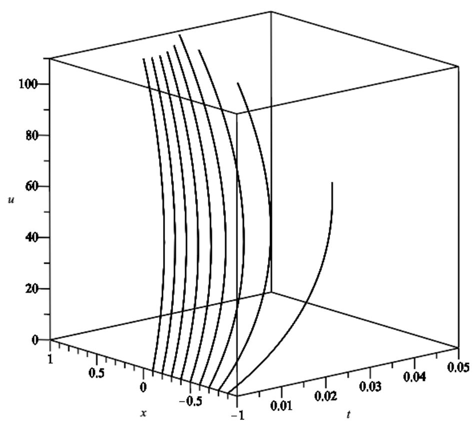

Figure 2: Path of a car starting at different x_0 values shown against time (t) and velocity (u).

Further the velocity field $u(x,t)$ for $x > -1, t > 0$ can be determined from the acceleration equation 1 and knowing that once cars hit $u_m = 110km/h$ they will continue at constant velocity.

$$u(x,t) = \begin{cases} \int \alpha \cdot (1+x_0)dt, & \int \alpha \cdot (1+x_0)dt < 110 \\ 110, & \int \alpha \cdot (1+x_0)dt \geq 110 \end{cases} \tag{3}$$

If we assume the cars uniformly distributed for $t > 0$, then they must be uniformly distributed from the start. Furthermore, the cars must follow one of two scenarios. The first scenario is where they maintain the exact same distance. This would imply that their acceleration has been the same for all cars. However, we know the acceleration is dependent of their position, this can therefore not be the case. The other scenario is where the cars accelerate at different speeds. The distances between the cars will now change, however, they can still remain uniformly distributed if the distance is the same between all cars. This can happen if e.g. car 1 moves 1 meter, when car 2 moves 1,5m, and car 3 2m etc. At some point the last car - which is also the one going fastest - will reach $110\frac{km}{t}$. At this point the last car cannot have reached the full $110\frac{km}{t}$, and assuming it intends to, it will still be accelerating. This means that the car in front of it must accelerate slightly faster, and the one in front of that one likewise, etc. In turn, the last car must also accelerate at some speed larger than 0. Since it has already reached $110\frac{km}{t}$ it cannot do so, which is a contradiction. We can, therefore, conclude that the cars cannot be uniformly distributed.

3.2 Road capacity

The parameters a, pm and b from equation 2 have been determined using Matlabs curvefitting package. The fit and the data is show below:

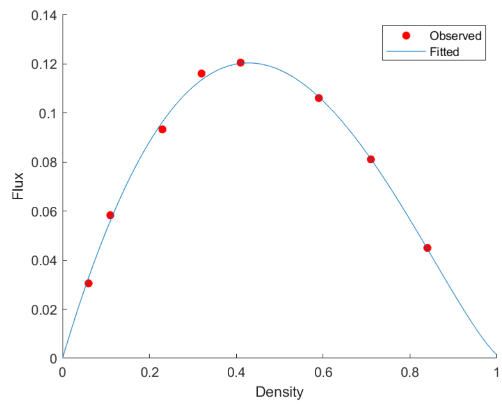

Figure 3: Illustration of measured data and a best fit of the form from equation 2.

The parameters with 95% confidence interval are determined to be:

$$\begin{aligned} a &= 0.586 \quad (0.5453,\ 0.6267) \\ b &= 1.364 \quad (1.118,\ 1.611) \\ pm &= 1.012 \quad (0.9403\ 1.085) \end{aligned}$$

we now see that we have a plot equivalent to figure 5 [2] and with the determined flow-density function, we can now find the critical car density p_c on this given stretch of road. We simply do this be solving:

$$q'(\rho) = 0, \qquad 0 \leq \rho \leq \rho_m$$

which gives the solution $p_c = 0.4281$ with an equivalent flux of $q(p) = 0.120$. This matches quite well with the top in figure 3. If the car density in a given period is above the critical car density, we will observe perturbation in the traffic flow because the traffic moved forward but the density wave will mode backward. We will see that in the periods:
06:30 - 07:00, 7:30 - 8:00 and 8:30 - 09:00.
For some reason, it seems during the period from 8.00 to 8.30 there are no measurements. Based on pure speculation, this data-point would most likely be similar to the one before and the one after, i.e. high density and sub-optimal flux.

4 Discussion

A lot of assumptions were made to create a simple model investigating how traffic on motorways work. The first assumption that sets the model apart from reality is that traffic is unidirectional, and that the on-ramp is an extension of the motorway. This allows the model to describe the behavior of traffic starting from velocity equal to zero, and continue until $u_m = 110\frac{km}{h}$ from were the velocity is constant. On-ramps for motorways often merge into existing traffic, which would require a more advanced model to describe. The model can still give the local authorities an idea of how acceleration influences the velocity and density of cars, but would not be sufficient for more advanced analysis. Another big assumption for the on-ramp analysis is that drivers comply with traffic

[2] Exercise 4 - Traffic Density Waves, PGHJ. Mathematical Modelling, 02526 DTU.

rules and are observant. It is not always the case that drivers stay below the max speed, as well as drivers being observant of when to accelerate at a green light. Both of these cases would diverge from our analysis and create slightly different results than the ones modeled. In addition to this, it is assumed that acceleration is constant, which might not always be the case. Some drivers might tailgate the driver in front, which would result in non-constant acceleration.

5 Conclusion

Our task was to model local traffic, i.e. the on-ramp for a motorway. We did so using the assumption that all cars accelerate constantly until they reach a speed limit, which in our case is $110\frac{km}{t}$. Furthermore we assumed that cars starting farther from the motorway accelerated slower than the ones who starter closer to it. Under these conditions we found the paths cars follow and their velocity at several starting positions. We further found that the cars cannot be uniformly distributed for $t > 0$, the main reason why is simply because the first cars cannot accelerate indefinitely, which means the slowest will at some point "catch up" and reach the speed limit as well. We then looked at the road capacity. We here found that the flux of cars depends on the density, i.e. the *amount of traffic*. We found the critical density of cars to be at 0.4281. We further saw that for our data, the density was above critical, i.e. traffic jams, in the periods 06:30 - 07:00, 7:30 - 8:00 and 8:30 - 09:00. The period 8.00 - 8.30 was not included in the data, we therefore do not know anything about the traffic flow in this period.

Appendices

Time Interval	Number of cars	Average Density
05:30-06:00	55	0.06
06:00-06:30	209	0.32
06:30-07:00	146	0.71
07:00-07:30	217	0.41
07:30-08:00	81	0.84
08:30-09:00	191	0.59
09:00-09:30	168	0.23
09:30-10:00	105	0.11

Figure 4: Counts of cars on the motorway (a single direction) throughout the day.