Juni 2021

Technological University of Denmark

Assignment 1

The first assignment in the course 02953, convex optimization.

Mads Esben Hansen
s174434

A1.8

Convexity of some sets. Determine if each set below is convex.

A) $\{(x, y) \in R^2_{++} \quad | \quad x/y \leq 1\}$

We can re-write the set as:

$$\{(x, y) \in R^2_{++} \quad | \quad x \leq y\}$$

Which can be plotted as:

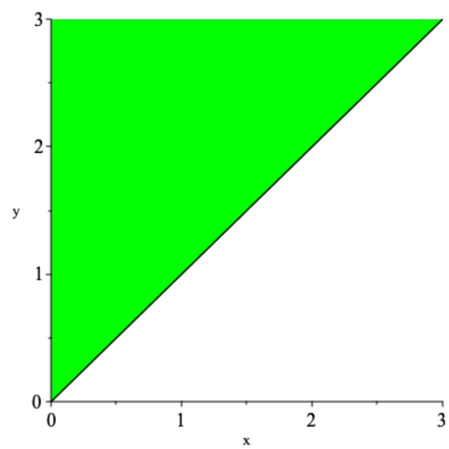

Figure 1: Illustration of $\{(x, y) \in R^2_{++} \quad | \quad x/y \leq 1\}$.

From figure 1 we clearly see that the set is convex.

B) $\{(x, y) \in R^2_{++} \quad | \quad x/y \geq 1\}$

We can re-write the set as:

$$\{(x, y) \in R^2_{++} \quad | \quad x \geq y\}$$

Which can be plotted as:

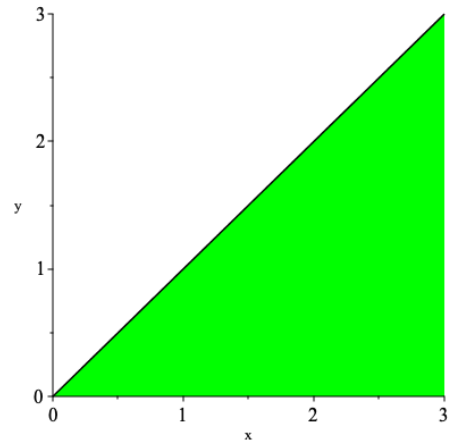

Figure 2: Illustration of $\{(x, y) \in R^2_{++} \quad | \quad x/y \geq 1\}$.

From figure 2 we clearly see that the set is convex.

C) $\{(x, y) \in R^2_{++} \quad | \quad xy \leq 1\}$

We can re-write the set as:

$$\{(x, y) \in R^2_{++} \quad | \quad x \leq 1/y\}$$

Which can be plotted as:

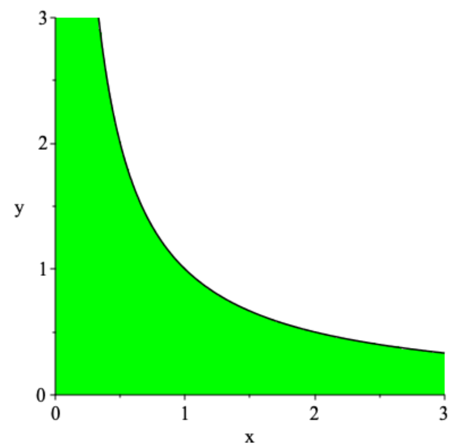

Figure 3: Illustration of $\{(x, y) \in R^2_{++} \quad | \quad x \leq 1/y\}$.

From figure 3 we clearly see that the set is not convex.

D) $\{(x, y) \in R^2_{++} \quad | \quad xy \geq 1\}$

We can re-write the set as:

$$\{(x, y) \in R^2_{++} \quad | \quad x \geq 1/y\}$$

Which can be plotted as:

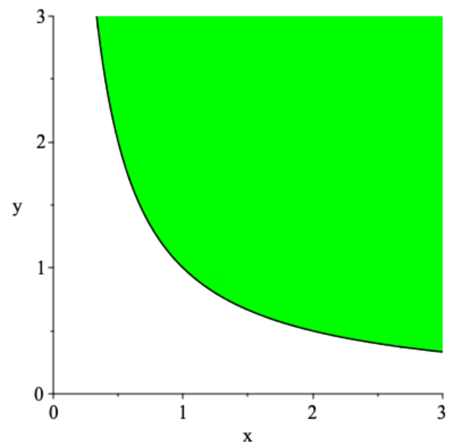

Figure 4: Illustration of $\{(x, y) \in R^2_{++} \quad | \quad x \geq 1/y\}$.

From figure 4 we clearly see that the set is convex.

A2.3

Logarithmic barrier for the second-order cone. The function $f(x,t) = -log(t^2 - x^T x)$, with $dom(f) = \{(x,t) \in R_n \times R \quad | \quad t > |x|_2\}$ (i.e., the second-order cone), is convex. (The function f is called the logarithmic barrier function for the second-order cone.) This can be shown many ways, for example by evaluating the Hessian and demonstrating that it is positive semidefinite. In this exercise you establish convexity of f using a relatively painless method, leveraging some composition rules and known convexity of a few other functions.

A) Explain why $t - (1/t)u^T u$ is a concave function on $dom(f)$. Hint. Use convexity of the quadratic over linear function.

Cf. the textbook p. 72, $f(x,y) = \frac{x^2}{y}, \quad y > 0$ is convex. Hence, $-\frac{u^T u}{t}$ is concave. $f(x,y) = t$ is concave (and convex), hence $f(x,t) = t + (-\frac{u^T u}{t})$ is the sum of concave functions, and is therefore also concave.

B) From this, show that $-log(t - (1/t)u^T u)$ is a convex function on $dom(f)$.

We know that $t > ||u||_2$, hence $\frac{u^T u}{t} < t$ which implies that $t - \frac{u^T u}{t} > 0$. Cf. the textbook p. 86, if g is concave and positive, then $log(g)$ is concave. Therefore $log(t - \frac{u^T u}{t})$ is concave, which means that $-log(t - \frac{u^T u}{t})$ is convex.

C) From this, show that f is convex.

We re-write $log(t^2 - x^T x)$ as $log(t - \frac{x^T x}{t}) + log(t)$. Notice that we just showed that $log(t - \frac{x^T x}{t})$ is concave. Notice also that $log(t)$ is concave. Hence $log(t^2 - x^T x)$ is the sum of 2 concave functions and is therefore also concave. This means that $-log(t^2 - x^T x)$ must be convex.

A3.3

Reformulating constraints in CVX*. Each of the following CVX code fragments describes a convex constraint on the scalar variables x, y, and z, but violates the CVX rule set, and so is invalid. Briefly explain why each fragment is invalid. Then, rewrite each one in an equivalent form that conforms to the CVX rule set. In your reformulations, you can use linear equality and inequality constraints, and inequalities constructed using CVX functions. You can also introduce additional variables, or use LMIs. Be sure to explain (briefly) why your reformulation is equivalent to the original constraint, if it is not obvious.
Check your reformulations by creating a small problem that includes these constraints, and solving it using CVX. Your test problem doesn't have to be feasible; it's enough to verify that CVX processes your constraints without error.
Remark. This looks like a problem about 'how to use CVX software', or 'tricks for using CVX'. But it really checks whether you understand the various composition rules, convex analysis, and constraint reformulation rules.

A) `norm([x+2y, x-y]) == 0`

We realize that the norm can only be 0 if all elements are 0, hence:

$$\begin{aligned} x + 2y = 0 \quad &\rightarrow \quad x = -2y \\ x - y = 0 \quad &\rightarrow \quad x = y \end{aligned}$$

This means that:

$$\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$$

B) `square(square(x+y)) <= x - y`

$$\begin{gathered} ((x+y)^2)^2 = (x+y)^2 \\ (x+y)^4 \leq x - y \end{gathered}$$

Since $f(x) = x^4$ is a convex function and $x + y$ is affine, we know that $(x + y)^4$ is convex, and it therefore works.

C) `1/x + 1/y <= 1; x >= 0; y >=0`

We can use the cvx function `inv_pos` to let the program know that the value is positive (and concave).

$$\begin{aligned} \texttt{inv\_pos(x) + inv\_pos(y)} &\leq 1, \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

D) `norm([max(x,1), max(y,2)] <= 3*x + y`
We can re-write it as:

```
    max(x,1) <= t1
    max(y,2) <= t2
norm(t1,t2) <= 3*x+y
```

Notice that max(x,1) and max(y,2) are composition of convex and affine functions, and therefore convex. Notice also that norm(t1,t2) is a convex function.

E) `x*y >= 1; x>= 0; y >= 0`

We re-write it is as:

```
x >= inv_pos(y)
```

Note that this implicitly constrains both x and y.

F) `(x+y)^2 / sqrt(y) <= x - y + 5`.
We write it using the cvx function by:

```
quad_over_lin(x+y, sqrt(y)) <= x - y + 5
```

G) `x^3 + y^3 <= 1; x >= 0; y >=0`
We write it as the composition (sum) of 2 convex cvx functions:

```
pow_pos(x,3) + pow_pos(y,3) <= 1
```

H) `x + z <= 1 + sqrt(x*y - z^2; x >= 0; y >=0`
We notice that we can reformulate:

$$\sqrt{xy - z^2} = \sqrt{y(x - \frac{z^2}{y})}$$

Which is the same as the geometric mean of $(y, x - \frac{z^2}{y})$. Hence the constraint can be reformulated as:

```
x + z <= 1 + geo_mean([y, x - quad_over_lin(z,y)])
```

We can test the re-formulation of the constraints using the following code:

```
cvx_begin
    variable x(5)
    minimize(norm(x,1))
    subject to
        x(1)+2*x(2) == 0;
        x(1)-x(2) == 0;

        (x(1)+x(2))^4 <= x(1)-x(2);

        inv_pos(x(1)) + inv_pos(x(2)) <= 1;
```

```
        max(x(1),1) <= x(4);
        max(x(2),2) <= x(5);
        norm([x(4),x(5)]) <= 3*x(1) + x(2);

        x(1) >= inv_pos(x(2));

        quad_over_lin(x(1)+x(2),sqrt(x(2)))<=x(1)-x(2)+5;

        pow_pos(x(1),3) + pow_pos(x(2),3) <= 1;

        x(1) + x(3) <= 1 + geo_mean([x(1) - quad_over_lin(x(3),x(2)), x

        x >= 0;
cvx_end
```

This problem is obviously not feasible, but it can be run correctly by cvx.

Problem 4

Consider the following linear optimization problem

$$\begin{aligned} &\textit{minimize} \quad && x_1 + 2x_2 \\ &\textit{subject to} \quad && x_1 - x_2 \leq 0 \\ & && -x_1 + \alpha x_2 \leq -1 \end{aligned}$$

With variables x_1 and x_2 and where α is a scalar parameter. Find the optimal value as a function of the parameter α. For which values of α is the problem unbounded?

First we notice that if $\alpha = 1$ we have that:

$$\begin{aligned} & & x_1 - x_2 \leq 0 \\ -x_1 + x_2 \leq -1 \quad & \rightarrow & x_1 - x_2 \geq 1 \end{aligned}$$

So the same thing must be both greater than 1 and smaller than 0, which is impossible.

We can now formulate the dual of the LP directly as done in the slides:

$$\begin{aligned} &\textit{maximize} \quad && [0 \;\; 1]\lambda \\ &\textit{subject to} \quad && \begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix} \lambda + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \\ & && \lambda \geq 0 \end{aligned}$$

From the constraints we get the following system of equations:

$$\begin{aligned} \lambda_1 - \lambda_2 + 1 = 0 \quad \rightarrow \quad \lambda_1 = \lambda_2 - 1 \\ -\lambda_1 + \alpha\lambda_2 + 2 = 0 \\ \lambda \geq 0 \end{aligned}$$

We can now eliminate λ_1 in the second expression:

$$\begin{aligned} -\lambda_1 + \alpha\lambda_2 + 2 = -(\lambda_2 - 1) + \alpha\lambda_2 + 2 = 0 \quad \rightarrow \\ (\alpha - 1)\lambda_2 = -3 \quad \rightarrow \quad \lambda_2 = \frac{-3}{(\alpha - 1)} \geq 0 \end{aligned}$$

We then get:

$$\lambda_1 = \frac{-3}{\alpha - 1} - 1 \geq 0 \quad \rightarrow \quad -2 \leq \alpha$$
$$\lambda_2 = \frac{-3}{(\alpha - 1)} \geq 0 \quad \rightarrow \quad \alpha < 1$$

We see that maximizing the problem is the same as maximizing λ_2, we therefore get:

$$d^*(x) = \begin{cases} \frac{-3}{(\alpha-1)} & -2 \leq \alpha < 1 \\ -\infty & otherwise \end{cases}$$

We know for LPs that $p^* = d^*$ unless they are both infeasible. We see that they are both infeasible for $\alpha = 1$. For $\alpha < -2$ the dual is infeasible, for $\alpha > 1$ the dual is also infeasible. Hence the solution to the primal problem becomes:

$$p^*(x) = \begin{cases} \frac{-3}{(\alpha-1)} & -2 \leq \alpha < 1 \\ \infty & \alpha = 1 \\ -\infty & otherwise \end{cases}$$

Problem 5

Consider the problem

$$\begin{array}{ll} \textit{minimize} & tr(A_0 Z) + b_0^T x \\ \textit{subject to} & tr(A_i Z) + b_i^T x + c_i = 0, \quad i = 1, ..., m, \\ & x \in R_+^n, Z \in S_+^p \end{array}$$

with variables $x \in R^n$ and $Z \in S^n$ and problem data $b_0, b_1, ...b_m \in R^n,\ A_0, A_1, ..., A_m \in S^p$, and $c_1, ..., c_m \in R$. Derive the dual.

We define the Lagrangian by:

$$\begin{aligned} L(x, Z, \lambda, \nu, W) &= tr(A_0 Z) + b_0^T x + \sum (tr(A_i Z) + b_i^T x + c_i)\lambda_i - \nu^T x - tr(WZ) \\ &= tr((A_0 - W + \sum (A_i \lambda_i))Z) + (b_0 + \sum b_i - \nu)^T x + \sum (c_i \lambda_i) \end{aligned}$$

Hence we get the dual function:

$$\inf_{x,Z}(L) = \begin{cases} c^T \lambda & tr(A_0 - W + \sum (A_i \lambda_i)) = 0 \ \wedge \ b_0 + \sum b_i - \nu = 0 \\ \infty & otherwise \end{cases}$$

We can re-write this to the dual problem by:

$$\begin{array}{ll} \textit{maximize} & c^T \lambda \\ \textit{subject to} & tr(A_0 - W + \sum (A_i \lambda_i)) = 0 \\ & b_0 + \sum b_i - \nu = 0 \\ & W \geq 0, \quad \lambda \geq 0, \quad \nu \geq 0 \end{array}$$