Advanced Modelling - Applied Mathematics

E21 01257
January 19, 2022

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1 Abstract

In this project, we have derived a model for an infinite countable number of pendula coupled by torsion springs. When we let the distance between the pendula tend towards 0, the resulting model is a nonlinear partial differential equation known as sine-Gordon. It has various applications such as describing the phase differences in a Josephson junction. For the case without dampening and rotational bias we derive an analytic travelling wave solution, which turns out to be a soliton. In the presence of dampening and rotational bias we derive the stationary soliton speed. We compare the analytical solution with numerical approximations. In the case of varying pendula lengths we limit ourselves to consider approximate numerical solutions by the Runge-Kutta-Fehlberg Method. The analytical and numerical solution show close correspondence. The findings might have interesting applications in the design of Long Josephson Diodes.

Contents

2 Introduction

This project is done as a part of the course *Advanced modelling - applied mathematics* at DTU. The aim of this project is to investigate elastically coupled physical pendula in the gravitational field. This leads to a dynamic system that with suitable coefficients is described by the sine-Gordon equation. The sine-Gordon equation can also be used to model a *Josephson junction*. Hence, we will model and gain insights into the systems of pendula to infer insights of the Josephson junction. Specifically, a system of pendula of varying lengths will be investigated.

2.1 Long Josephson Diode

The Josephson effect occurs when two superconductors are placed next to each other with a barrier between them. This makes the effects of quantum mechanics, which usually occur at an atomic scale, visible at a macroscopic scale. The macroscopic scale makes it much easier to observe the effects of quantum mechanics, and is also used to make extremely accurate measurements of volts. According to *The Emerging Science of Spontaneous Order* [1], the standard volt is measured using 20,000 Josephson junctions in a series. A long Josephson diode uses the Josephson effect by moving the super-conductors through a magnetic field to generate an electric current and power a diode.

Further, they have behaviour similar to a traditional transistor, making them very interesting within the field of computer science. In fact, they have a richer behavior and are used in e.g. quantum computers.

The behaviour of these long Josephson diodes is governed by the same dynamics as a system of torsionally coupled pendula, making the study of these a relevant topic.

3 Model for Coupled Pendula

In this section, we wish to model a series of N pendula connected to a rod at uniform distance, where each pendulum is connected by a rotational spring to its neighbors as seen in figure 1.
We define θ_n to be the radial angle of pendulum n relative to is ground state position.
Using Lagrangian mechanics, the energy in the system can be described by the kinetic energies, $\frac{m\ell^2}{2}\dot{\phi}_n$, the potential spring energies, $\frac{C}{2}(\phi_n - \phi_{n-1})^2$, and the potential gravitational energies, $mg\ell(1 - cos(\phi_n))$, where n indicates the pendulum number ordered from 1 to N.

By definition, the Lagrangian density is $\mathcal{L} = K - V$, where K is the kinetic energy, and V is the potential energy.

$$\mathcal{L} = \frac{m\ell^2}{2}\dot{\phi}_n^2 - \frac{C}{2}(\phi_n - \phi_{n-1})^2 - mg\ell(1 - \cos(\phi_n))$$

By the principle of least action, we can solve the Euler-Lagrange equation to find the action functional ϕ_n that describes the equations of motion for the system.

$$\frac{\partial L}{\partial \phi_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_n}\right) = 0$$
$$m\ell^2\ddot{\phi}_n - C(\phi_{n+1} - 2\phi_n + \phi_{n-1}) + mg\ell\sin(\phi_n) = 0 \tag{1}$$

In the case where the pendula are of varying length, we get

$$m\ell_n^2\ddot{\phi}_n - C(\phi_{n+1} - 2\phi_n + \phi_{n-1}) + mg\ell_n\sin(\phi_n) = 0$$

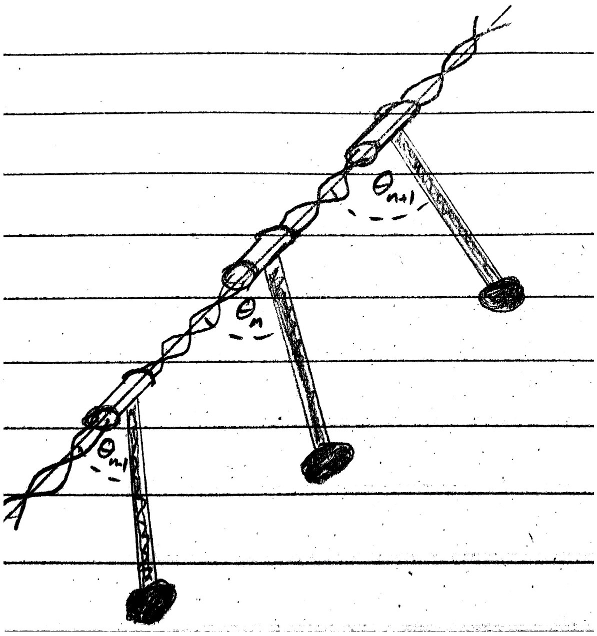


Figure 1 – Graphical representation of part of the series of pendula.

According to *Physics of Solitons* [2] a Taylor expansion of the middle term yields the following.

$$\phi_{n+1} + \phi_{n-1} - 2\phi_n = a^2 \frac{\partial^2 \phi}{\partial x^2} + \mathcal{O}\left(a^4 \frac{\partial^4 \phi}{\partial x^4}\right)$$

To simplify notation, let $\phi_{tt} = \frac{\partial^2 \phi}{\partial t^2}$ and $\phi_{xx} = \frac{\partial^2 \phi}{\partial x^2}$. Now let the distance between the pendula, a, approach 0. Furthermore, in the continuum ℓ_n will approach a continuous function $\ell(x)$. In the continuum $\frac{m}{a}$ will hence be a measure of the mass density, denoted by ρ. Further, notice that the spring constant C becomes larger when $a \to 0$, s.t. $\lim_{a \to 0} \frac{C}{a} = \kappa$.

$$\rho \ell(x)^2 \phi_{tt}(x,t) - \kappa \phi_{xx}(x,t) + \rho g \ell(x) \sin(\phi(x,t)) = 0 \tag{2}$$

Let $\ell(x) = \ell(0) + \Delta\ell(x)$. By inserting this into equation (2), we obtain the following expression for the model. We will refer to this as the pendulum equation.

$$\begin{aligned} &\rho l(0)^2 \phi_{tt}(x,t) - \kappa \phi_{xx}(x,t) + \rho g l(0) \sin(\phi(x,t)) \\ &= -\Delta\ell(x)\rho \left(g \sin(\phi(x,t) + (\Delta\ell(x) + 2l(0))\, \phi_{tt}(x,t)\right) \end{aligned}$$

Notice that $\lim_{\Delta\ell \to 0}$ we arrive at the same result as before.

4 Travelling Wave Solution for sine-Gordon

In the following section, we wish to derive a solution to the sine-Gordon equation under a travelling wave ansatz.

$$\rho \ell^2 \frac{\partial^2 \phi}{\partial t^2} - \kappa \frac{\partial^2 \phi}{\partial x^2} + \rho g \ell \sin(\phi) = 0 \tag{3}$$

To reduce the number of dimensions and obtain a more simplistic model, we start by non-dimensionalization.

4.1 Non-dimensionalization

In order to simplify the system, we can introduce unitless variables.

$$x = k_x \cdot \tilde{x}$$
$$t = k_t \cdot \tilde{t}$$

The units for x and k_x are meters, and the units for t and k_t are seconds.

$$\phi = \phi(x,t) = \phi(\tilde{x},\tilde{t}) = \tilde{\phi} \tag{4}$$

By the relationship in (4), we use the chain rule to compute the partial derivatives. Inserting these into equation (3) yields the following.

$$\rho\ell^2 \frac{\partial^2\tilde{\phi}}{\partial\tilde{t}^2}\frac{1}{k_t^2} - \kappa\frac{\partial^2\tilde{\phi}}{\partial\tilde{x}^2}\frac{1}{k_x^2} + \rho g\ell\sin(\phi) = 0 \tag{5}$$

Dividing by $\rho\ell^2\frac{1}{k_t^2}$ leaves us with two coefficients.

$$\frac{\partial^2\tilde{\phi}}{\partial\tilde{t}^2} - \frac{k_t^2\kappa}{\rho\ell^2}\frac{\partial^2\tilde{\phi}}{\partial\tilde{x}^2}\frac{1}{k_x^2} + \frac{k_t^2 g}{\ell}\sin(\phi) = 0 \tag{6}$$

Since we can choose the parameters freely, it is suitable to let them balance out the the other parameters.

$$\frac{k_t^2 g}{\ell} = 1 \leftrightarrow k_t = \sqrt{\frac{\ell}{g}} \tag{7}$$

$$\frac{k_t^2\kappa}{\rho\ell^2}\frac{1}{k_x^2} = 1 \leftrightarrow k_x = \sqrt{\frac{\kappa}{\rho\ell g}} \tag{8}$$

Inserting into equation (6) yields the sine-Gordon with no parameters. For the rest of the derivation, $\tilde{\phi}(x,t) = \phi(x,t)$.

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial x^2} + \sin(\phi) = 0 \tag{9}$$

4.2 Solution

We now introduce a function $f : \xi \to \mathbb{R}$. We furthermore assume that $f \in C^k$, where $k \geq 2$. Let $\phi(x,t) = f(\xi)$, where $\xi = x - x_0 - ct$. Notice that this solution ansatz corresponds to a function centered around x_0, whose range is displaced over time. This is why it is called a *travelling wave solution.*

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}(\frac{df}{d\xi}\frac{\partial\xi}{\partial t}) = \frac{d^2 f}{d\xi^2}(\frac{\partial\xi}{\partial t})^2 = c^2 f^{(2)} \tag{10}$$

Similarly the partial derivative with respect to x is found.

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{df}{d\xi}\frac{\partial\xi}{\partial x}) = \frac{d^2 f}{d\xi^2}(\frac{\partial\xi}{\partial x})^2 = f^{(2)} \tag{11}$$

We insert the results into the non-dimensionalized sine-Gordon.

$$(c^2 - 1)f^{(2)} - \sin(f) = 0 \tag{12}$$

To obtain a first order ODE, we can integrate both sides.

$$\int (c^2-1)f^{(2)} - \sin(f)d\xi = \int d\xi$$
$$(c^2-1)\frac{1}{2}(f^{(1)})^2 - \cos(f) = k \tag{13}$$

Now we define some desirable properties of the solution. Since we wish to describe a travelling wave, we introduce the following conditions in the limit.

$$\begin{aligned} &f(\xi) \to 0 \text{ for } \ x \to -\infty \\ &f(\xi) \to 2\pi \text{ for } \ x \to +\infty \\ &f^{(1)}(\xi) \to 0 \text{ for } \ \xi \to \pm\infty \end{aligned}$$

The above requirements describe the profile of a wave that is flat with function value 0 to the left and then at some point on the x-axis increases to a flat area with function value 2π.

$$\lim_{\xi\to\pm\infty}(c^2-1)\frac{1}{2}(f^{(1)})^2 - \cos(f) = -\cos(0) = -1$$

Hence, it follows that $k = -1$. Now we introduce a scaling $f(\xi) = 4h(\xi)$ to cancel a constant that will arise in the derivation. It is inserted into equation (13).

$$(c^2-1)\frac{1}{2}(4h^{(1)})^2 = \cos(4h) - 1 \tag{14}$$

By the double angle formula and the Pythagorean identity, we get

$$\cos(4h) - 1 = 1 - 2\sin^2(2h) - 1 \tag{15}$$
$$= -2\sin^2(2h) \tag{16}$$

Then we insert equation (16) into equation (14) and divide both sides by -2.

$$(1-c^2)4(h^{(1)})^2 = \sin^2(2h) \tag{17}$$

By the double-angular relationship we have

$$\sin(2h) = \frac{2\tan(h)}{1+\tan^2(2h)} \tag{18}$$

Then we take the square root on both sides of equation (17) and replace the right hand side with equation (18).

$$\pm 2\sqrt{1-c^2}(h^{(1)}) = \frac{2\tan(h)}{1+\tan^2(2h)} \leftrightarrow \frac{1+\tan^2(2h)}{\tan(h)}h^{(1)} = \frac{\pm 1}{\sqrt{1-c^2}} \tag{19}$$

Then we substitute $y(\xi) = \tan(h)$ and $h(\xi) = \arctan(y(\xi))$. At first, we find $h^{(1)}$.

$$h^{(1)} = \frac{1}{1+y^2}y^{(1)}$$

Then we find $y^{(1)}$.

$$y^{(1)} = \pm\frac{1}{\sqrt{1-c^2}}y \tag{20}$$

Equation (20) is a first order linear differential equation with the solution

$$y(\xi) = Ae^{\pm\frac{1}{\sqrt{1-c^2}}\xi}$$

Since $y(\xi) = \tan(h)$, $f(\xi) = 4h(\xi)$ and $\xi = x - x_0 - ct$ we get that

$$\phi(x,t) = f(\xi) = 4\arctan(Ae^{\pm\frac{x-x_0-ct}{\sqrt{1-c^2}}}) \tag{21}$$

If we let $A = 1$, the solution for $t = 0$ will be symmetric around x_0. The solution profile is presented in figure 2. The kink can be graphically seen as where the curve goes from 0 to 2π.

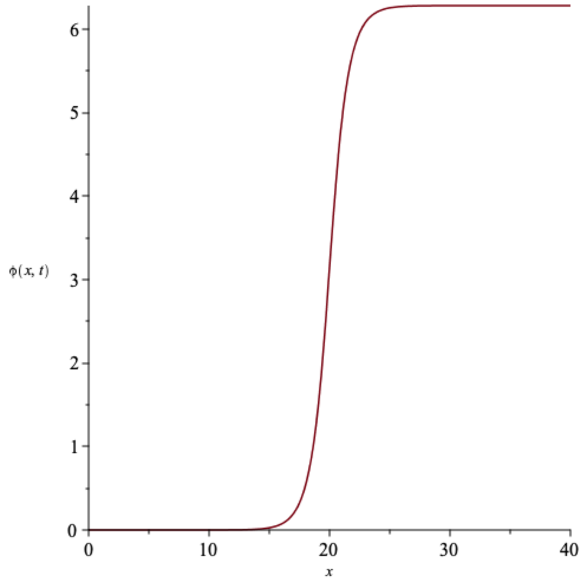


Figure 2 – Solution profile at $t = 0$. The animation can be seen here.

4.3 Velocity of travelling wave

To derive an expression for the velocity of the wave, the partial derivative of equation (21) with respect to t is found.

$$\frac{\partial\phi}{\partial t} = -\frac{4c}{A^2 e^{2\frac{x-x_0-ct}{\sqrt{1-c^2}}}+1}Ae^{\frac{x-x_0-ct}{\sqrt{1-c^2}}}$$

In order to find the mean angular velocity of the soliton, we can integrate over all the angular velocities and normalize with the length of the bar.

$$\begin{aligned}\bar{\nu}(t) &= \frac{1}{L}\int_0^L \frac{\partial\phi}{\partial t}dx \\ &= \frac{1}{L}\int_0^L -\frac{4c}{A^2 e^{2\frac{x-x_0-ct}{\sqrt{1-c^2}}}+1}Ae^{\frac{x-x_0-ct}{\sqrt{1-c^2}}}dx\end{aligned}$$

Then we can investigate the plot profile. From figure 3 it is seen that the analytical solution moves at a constant speed. This is because the system is conserving energy due to the absence of perturbations. The reason why it is not immediately at a steady state is that the soliton is centered around $x_0 = 0$, which is the very left side in the starting position, and L in the final position, which is the very right side. As it starts at the left side, only half of the wave exists at $t = 0$, but as time goes on, more of the wave exists until eventually the wave is completely present in the range $x \in [0, L]$.

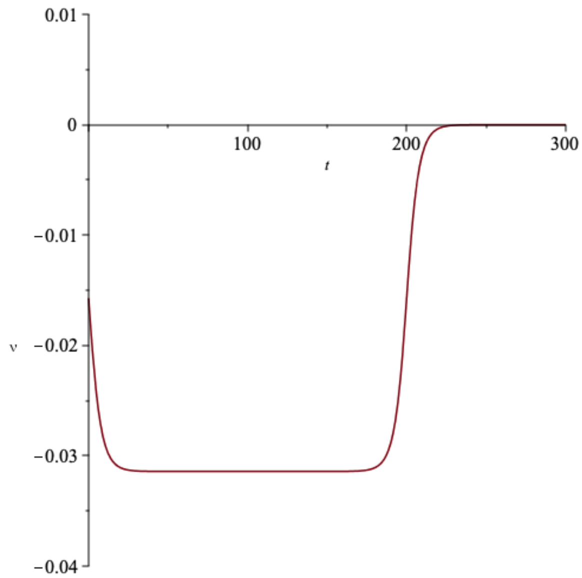


Figure 3 – Average velocity of all pendula over time

From this the period can be computed.

$$T = \frac{2\pi}{\bar{\nu}} \tag{22}$$

Using the parameters $A = 1$, $x_0 = 0$, $L = 40$ and $c = 0.2$, we arrive at the following expression.

$$T = \frac{2\pi}{|-0.0314159|} = 200 \tag{23}$$

It turns out that our numerical approximations of the pendulum equation will appear to have a shorter period. This discrepancy is discussed in section 8.

5 Dispersion in the Klein-Gordon

The linearized version of the sine-Gordon equation, known as the Klein-Gordon equation, is found by using a small-angle approximation $\sin(\phi) \approx \phi$. Replacing the sine function with a linear term ϕ gives us the simplified model.

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \phi = 0 \tag{24}$$

We assume the solution ϕ is of the following form.

$$\phi(x,t) = e^{i(kx-\omega t)}, k \in \mathbb{R} \tag{25}$$

In the above, k is the wave number and ω the angular frequency. Inserting the solution back into equation (24) results in the following.

$$-\omega^2 e^{i(kx-\omega t)} + k^2 e^{i(kx-\omega t)} + e^{i(kx-\omega t)} = 0 \tag{26}$$

As for all z, $e^z \neq 0$, we can simplify the equation to

$$-\omega^2 + k^2 + 1 = 0 \tag{27}$$

Isolating ω and simplifying gives

$$\omega = \omega(k) = \pm\sqrt{k^2 + 1} \tag{28}$$

The wave length of $\phi(x,t)$, or the period in relation to x based on the complex solution of ϕ is $\lambda = 2\pi/k$ and the time period of $\phi(x,t)$ in relation to t is given as $\tau = 2\pi/\omega$.

The dispersion relation for the linearized sine-Gordon equation is thus given by equation (28). This gives us a wave that disperses over time, in contrast to the sine wave, which has a counteracting force wanting to make the wave narrower, which balances out and causes it to keep its shape.

6 Perturbed sine-Gordon

In this section, we wish consider how perturbations affect the variations of the total energy [2]. This perturbation accounts for the loss due to friction and increase due to power inputs to the system. Furthermore, we impose boundary conditions corresponding to a parameter η proportional to strength of an externally applied magnetic field.

We consider the perturbed sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \sin(\phi) = \epsilon R \tag{29}$$

where

$$\epsilon R = -\alpha\phi_t - \gamma \tag{30}$$

The constant α is a dampening term proportional to the angular velocity. The energy input γ can add or remove energy from the system. By definition, the Lagrangian $\mathcal{L} = K - V$, where K is the kinetic energy, and V is the potential energy.

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - (1 - \cos(\phi)) \tag{31}$$

Then we can construct the Hamiltonian density.

$$\mathcal{H} = \phi_t \frac{\partial \mathcal{L}}{\partial \phi_t} - \mathcal{L} \tag{32}$$

As the momentum $\pi = \frac{\partial \mathcal{L}}{\partial \phi_t} = \phi_t$, we can insert the Lagrangian and reduce to find the Hamiltonian density.

$$\begin{aligned}\mathcal{H} &= \phi_t(\phi_t) - \frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + (1 - \cos(\phi)) \\ &= \frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + (1 - \cos(\phi))\end{aligned}$$

The Hamiltonian is the total energy of the system, which is given as the integral of the Hamiltonian density.

$$H = \int_0^L \mathcal{H} dx \tag{33}$$

Then we find the partial derivative with respect to time to investigate the effects of the dampening term α and the input term γ.

$$\frac{\partial H}{\partial t} = \int_0^L (-\alpha\phi_t + \gamma)\phi_t dx + [\phi_x\phi_t]_0^L \tag{34}$$

We then introduce η representing a magnetic force applied on the outermost part of the bar.

$$\phi_x(0,t) = \phi_x(L,t) = \eta$$

Under these conditions, the following integral can be established.

$$\frac{\partial H}{\partial t} = \int_0^L (-\alpha\phi_t + \gamma)\phi_t dx + \eta[\phi_t]_0^L \tag{35}$$

From equation (35) it is seen that the dampening coefficient α is negatively related to the square of the phase difference, which means it will decrease the overall energy of the system and thereby slow down the soliton over time. The bias current γ will on the other hand add or remove energy from the system depending on the sign of ϕ_t. From the last term it is seen that the magnetic field force η will act opposite in the two boundaries. For the rest of the report we consider the case where $\eta = 0$. According to *Nonlinear Science* [3], the solution for H is the following.

$$H = \frac{8}{\sqrt{1-v^2}} \tag{36}$$

Taking the time derivative, we can construct a first order ODE for the soliton speed.

$$\frac{dv}{dt} = \frac{\pi\gamma}{4}(1-v^2)^{3/2} - \alpha v(1-v^2) \tag{37}$$

From this, we can see that α will slow down a soliton propogating in either direction, whereas γ drives the kink in a direction, either right or left, depending on the sign. When $\alpha \neq 0$ and $\gamma \neq 0$, it is possible to determine the equilibrium where $\frac{dv}{dt} = 0$, i.e. a stationary velocity, v_∞. The inputs 'balance' out.

$$v_\infty = \frac{1}{\sqrt{1 + \left(\frac{4\alpha}{\pi\gamma}\right)^2}} \tag{38}$$

7 Numerical Implementation of sine-Gordon

7.1 Discretized sine-Gordon

In order to implement the system numerically, we discretize the continuous sine-Gordon partially in x.

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} - \sin(\phi)$$

We introduce the following discretization.

$$\begin{aligned} &x = x_n = n\Delta x, \quad n = 1, 2, ..., N \\ &L = N\Delta x \end{aligned}$$

Now define the angle to depend on the discrete variable x_n by the above relation.

$$\phi(x,t) = \phi(x_n,t) = \phi_n(t)$$

According to *Physics of Solitons* [2], we can use the following approximation.

$$\phi_{n+1} + \phi_{n-1} - 2\phi_n = a^2\frac{\partial^2\phi}{\partial x^2} + \mathcal{O}\left(a^4\frac{\partial^4\phi}{\partial x^4}\right)$$

Now we insert the above in the sine-Gordon equation.

$$\frac{d^2\phi}{dt^2} = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{\Delta x^2} - \sin(\phi_n) \tag{39}$$

By introducing an auxiliary term, we can define a system of first order ODEs.

$$y_{1,n}(t) = \phi_n(t), \qquad y_{2,n}(t) = \frac{dy_{1,n}(t)}{dt} = \frac{\phi_n(t)}{dt} \tag{40}$$

$$\frac{dy_{1,n}}{dt} = y_{2,n} \tag{41}$$

$$\frac{dy_{2,n}}{dt} = \frac{y_{1,n+1} - 2y_{1,n} + y_{1,n-1}}{\Delta x^2} - \sin(y_{1,n}) \tag{42}$$

Before this system can be implemented, the boundary conditions need to be determined. A relevant boundary condition for our pendula/Josephson diode case is the given by

$$\frac{\partial\phi(0,t)}{\partial x} = \eta, \qquad \frac{\partial\phi(L,t)}{\partial x} = \eta \tag{43}$$

Notice, for pendula, $\eta = 0$ translates to the end pendula moving freely in one direction. We now wish to determine the dynamics at the end points by considering equation (42).

$$\begin{aligned} \frac{dy_{2,1}}{dt} &= \frac{y_{1,2} - 2y_{1,1} + y_{1,0}}{\Delta x^2} - \sin(y_{1,1}) \\ \frac{dy_{2,N}}{dt} &= \frac{y_{1,N+1} - 2y_{1,N} + y_{1,N-1}}{\Delta x^2} - \sin(y_{1,N}) \end{aligned}$$

Let us introduce two artificial points, $\phi_0(t)$ and $\phi_{N+1}(t)$, from the boundary condition.

$$\begin{gathered} \frac{d\phi_1(t)}{dx} = \frac{\phi_2(t) - \phi_0(t)}{2\Delta x} = \eta \Rightarrow \\ \phi_0 = \phi_2 - 2\Delta x\eta \end{gathered}$$

Hence

$$\frac{dy_{2,1}}{dt} = \frac{y_{1,2} - 2y_{1,1} + y_{1,2} - 2\Delta x\eta}{\Delta x^2} - \sin(y_{1,1})$$
$$\frac{dy_{2,1}}{dt} = \frac{2y_{1,2} - 2y_{1,1}}{\Delta x^2} - \sin(y_{1,1}) - \frac{2\eta}{\Delta x} \tag{44}$$

Similarly $\frac{dy_{2,N}}{dt}$ is determined to be

$$\frac{dy_{2,N}}{dt} = \frac{2y_{1,N-1} - 2y_{1,N}}{\Delta x^2} - \sin(y_{1,N}) + \frac{2\eta}{\Delta x} \tag{45}$$

Now we have everything we need in order to solve a nonlinear partial differential equation using the semi difference method in e.g. matlab.

7.2 Solitary Wave Solution to sine-Gordon

At first, we present the numerical solution to the sine-Gordon equation presented in section (4).

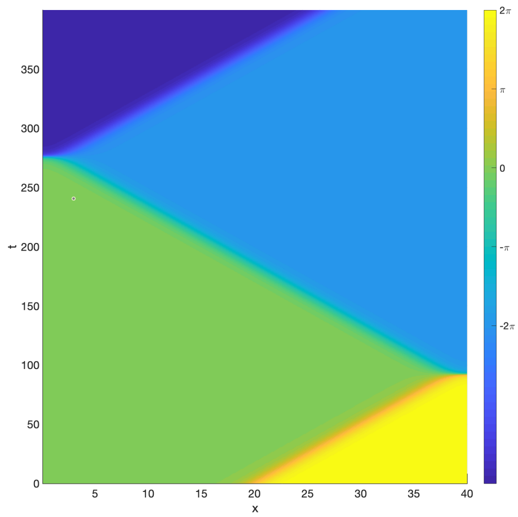


Figure 4 – Numerical simulation of traveling soliton wave.

In figure 4, we see a wave made by rotating all pendula on the right half a full 2π. The wave's movement can be visualized as occurring along the "borders" of the different colors with the slope of these lines being the velocity. The soliton travels to the right, "bounces" off of the last pendulum and then goes to the left at the same velocity in the opposite direction, bounces off the left side, and travels to the right, and so on.

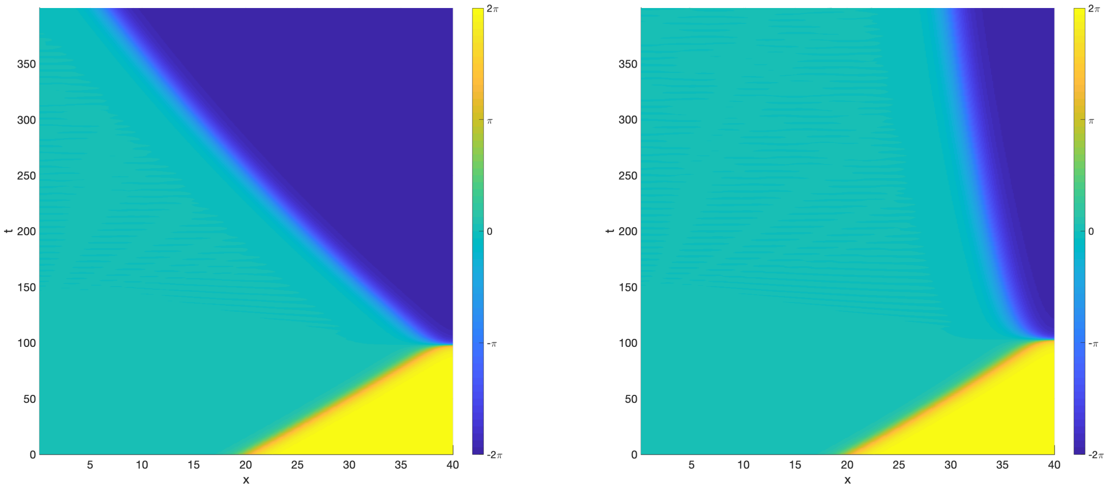
The simulation shown in figure 4 yielded a period (mean time to do one rotation) of 184.31. The mean angular velocity can then be found by

$$\frac{\partial\bar{\phi}}{\partial t} = \frac{-2\pi}{184.31} = -0.034 \tag{46}$$

In equation (23), we found the analytical period to be 200 time units. We see that the numerical simulation showed a slightly lower period. This is due to the fact that the analytical method assumes that the boundary *is* connected beyond the definition interval. In the simulation, we see acceleration of the soliton as it approaches the boundary because the same energy is concentrated in less mass; this is the difference between the two methods.

7.2.1 Solitary Wave Solution to Perturbed sine-Gordon

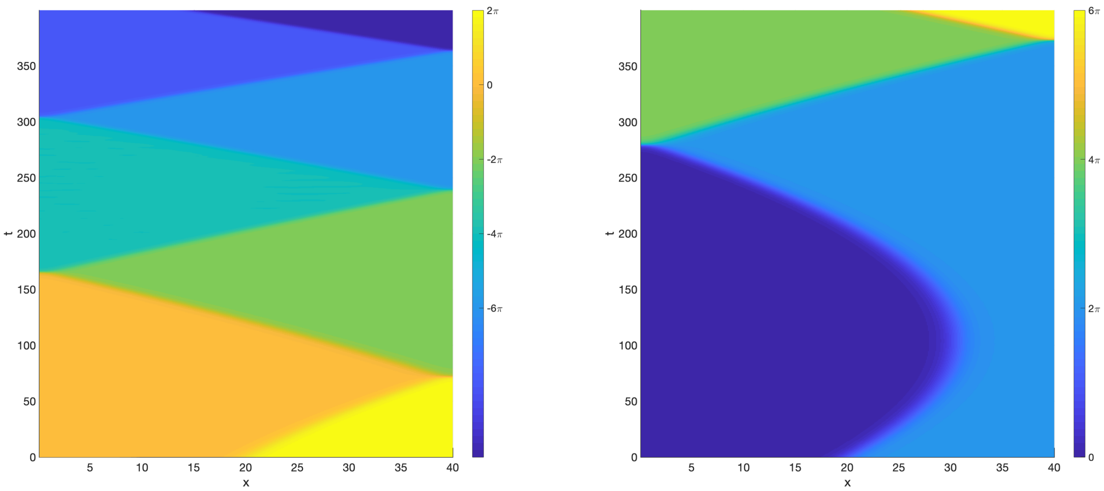
Here we present our numerical results in the presence of dampening term α and $\gamma = 0$.



(a) $\alpha = 0.0015$ **(b)** $\alpha = 0.0025$

Figure 5 – Soliton for different dampening terms

In figure 5, we see the same soliton wave as before, but the wave loses energy as it travels due to dampening. In figure 5a, a low dampening term is observed, where the wave after bouncing off the right wall very slowly reduces in velocity. However, in figure 5b, we see a slightly higher dampening term, which greatly increases the rate at which the wave's velocity shrinks.
Here we present the profiles of the perturbed sine-Gordon with bias term γ and $\alpha = 0$.



(a) $\gamma = -0.0025$ **(b)** $\gamma = 0.0025$

Figure 6 – Soliton under different bias forces.

In figure 6a, we see the velocity of the wave increases over time, as the γ is working with the rotational force. On the other hand, in figure 6b, we see that γ is working against the rotation, and thus slows down the wave until it reaches a critical point where the system is allowed to rotate in the opposite direction, and thus switch direction, where it slowly accelerates just like in figure 6a.

7.2.2 Stationary speed of soliton

We now have the necessary framework to compare the analytic expression for stationary speed of a solitary wave (when $\alpha \neq 0$ and $\gamma \neq 0$) given in equation (38) with numerical simulation. Choosing $\alpha = 0.1$, multiple simulations have been run with varying values of γ.
Figure 7 shows a comparison of the numerical and analytically obtained stationary speeds for constant α and varying γ. The analytical solution assumes constant qualitative behaviour, i.e. a soliton traveling from side to side. However, in reality for very low values of γ the soliton might not be able to move. For large values of γ the dynamics changes s.t. rotation will occur across the entire definition interval simultaneously, which can be seen in the sudden change of the numerical solution.

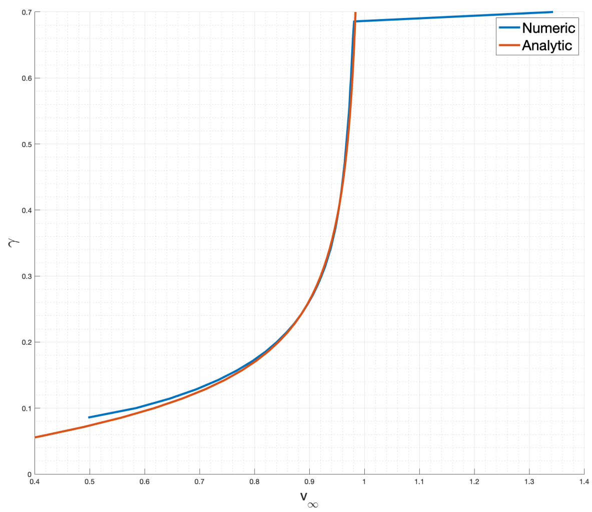


Figure 7 – Stationary speed of a traveling wave.

8 Numerical Implementation of Pendulum Equation

8.1 Pendulum Equation

Remember the pendulum equation is given by

$$\phi_{tt}(x,t) = \frac{\kappa}{\rho \ell(x)^2} \phi_{xx}(x,t) - \frac{g}{\ell(x)} \sin(\phi(x,t)) \tag{47}$$

The system can now be semi-discretized with respect to x. We introduce N pendula distanced Δx from each other such that.

$$\phi_n(t) = \phi(n \cdot \Delta x, t), \quad n \in \{1, ..., N\} \tag{48}$$

$$\ell_n = \ell(n \cdot \Delta x), \quad n \in \{1, ..., N\} \tag{49}$$

Then

$$\frac{d^2\phi_n}{dt^2} = \frac{\kappa}{\rho\ell_n^2}\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{\Delta x^2} - \frac{g}{\ell_n}\sin(\phi_n) \tag{50}$$

As with sine-Gordon, we can re-write the system by

$$y_{1,n} = \phi_n, \quad y_{2,n} = \frac{d\phi_n}{dt} \tag{51}$$

$$\frac{dy_{1,n}}{dt} = y_{2,n} \tag{52}$$

$$\frac{dy_{2,n}}{dt} = \frac{\kappa}{\rho\ell_n^2}\frac{y_{1,n+1} - 2y_{1,n} + y_{1,n-1}}{\Delta x^2} - \frac{g}{\ell_n}\sin(y_{1,n}) \tag{53}$$

With perturbations we get

$$\frac{dy_{2,n}}{dt} = \frac{\kappa}{\rho\ell_n^2}\frac{y_{1,n+1} - 2y_{1,n} + y_{1,n-1}}{\Delta x^2} - \frac{g}{\ell_n}\sin(y_{1,n}) + \gamma - \alpha y_{2,n} \tag{54}$$

Introducing the boundary condition

$$\frac{d\phi_1(t)}{dx} = \eta, \qquad \frac{d\phi_N(t)}{dx} = \eta \tag{55}$$

Introducing artificial points at $n = 0$ and $n = N + 1$ as we did for the sine-Gordon equation, we get the pendulum equation with perturbation terms.

$$\frac{dy_{2,1}}{dt} = \frac{\kappa}{\rho\ell_1^2}\frac{2y_{1,2} - 2y_{1,1}}{\Delta x^2} - \frac{g}{\ell_1}\sin(y_{1,1}) - \frac{\kappa}{\rho\ell_1^2}\frac{2\eta}{\Delta x} + \gamma - \alpha y_{2,n} \tag{56}$$

$$\frac{dy_{2,N}}{dt} = \frac{\kappa}{\rho\ell_n^2}\frac{2y_{1,N-1} - 2y_{1,N}}{\Delta x^2} - \frac{g}{\ell_n}\sin(y_{1,N}) + \frac{\kappa}{\rho\ell_n^2}\frac{2\eta}{\Delta x} + \gamma - \alpha y_{2,n} \tag{57}$$

8.2 Varying pendulum length

The system given above can be implemented in e.g. matlab, which enables numerical simulations. Figure 8 shows 3 selected variations of $\ell_n,\ n \in \{1, ..., N\}$.

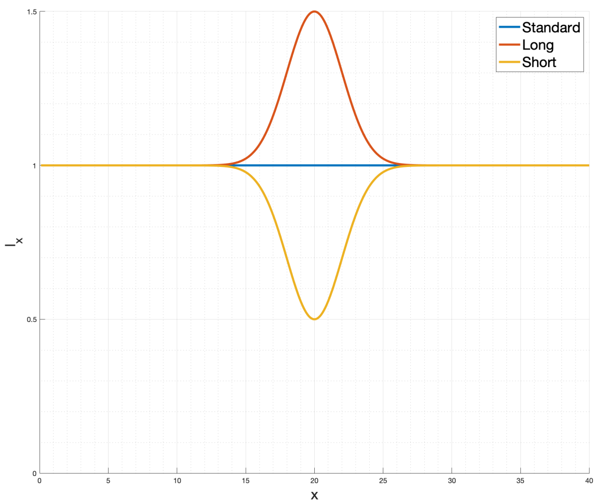


Figure 8 – The length of the pendula across the rod.

We now wish to investigate the relationship between the mean angular velocity, $\frac{d\bar{\phi}}{dt}$, and the bias perturbation, γ. To this end numerical simulations will be run, starting with a travelling wave with negative angular velocity and the pendulum parameters:

$$
\begin{aligned}
g &= 9.82 \\
\rho &= 1 \\
\kappa &= 10 \\
\eta &= 0 \\
\alpha &= 0.1
\end{aligned}
$$

Figure 9 shows the result of numerical simulations for the pendulum equation with varying pendula lengths. Notice the mean angular velocity increases very dramatically at some point around $\frac{d\bar{\phi}}{dt} \approx 0.5$ in all cases. In fact, the sudden shift marks a change in the dynamics of the overall system. Before the shift the system consists of a rod of pendula with a kink going back and forth. After the shift *all* pendula start to rotate simultaneously; hence the dramatic change in mean angular velocity across all pendula.

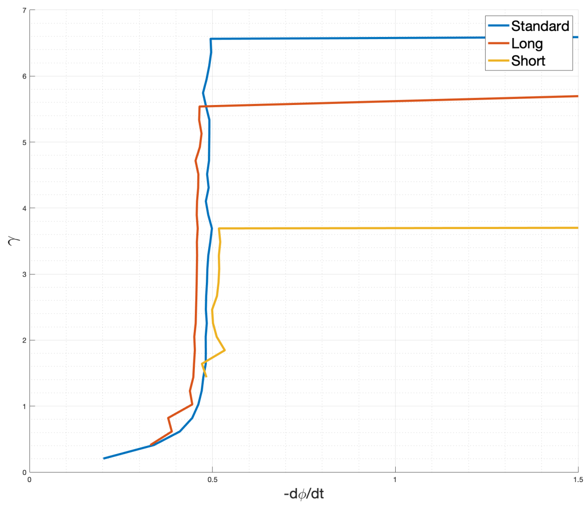


Figure 9 – γ as a function of mean angular velocity $\frac{d\bar{\phi}}{dt}$.

9 Discussion

Solving nonlinear partial differential equations is no trivial task. However, by assuming a travelling wave solution we have been able to find an analytical expression in accordance with the results found in the literature and simulate them numerically with acceptable precision. While we observe deviations between our analytical and numerical results, we find that they share the same underlying behavior. We believe the deviations are explained by our constrained setup, since we allow for the soliton to be reflected, while the pure sine-Gordon travelling wave solution moves in an unconstrained space. Furthermore, computational constraints prevent us from producing simulations of smaller increments, because the discretized systems results in relatively demanding computations of many dimensions. These constraints were especially significant for simulations of pendula of varying lengths.

Of particular interest is the results regarding varying pendula lengths. To the best of our knowledge no previous work has been done in relation to the $(\frac{d\bar{\phi}}{dt}, \gamma)$-characteristics of varying pendula lengths. As mentioned this is analogue to the IV characteristic of Josephson junctions, hence the implications of these findings might extend into fields such as quantum computing.

10 Conclusion

We have derived a model for a simple setup of coupled pendula connected by torsion springs by using the principle of least action. By the use of appropriate solution ansatz, we were able to determine a travelling wave solution in the sine-Gordon equation. By finding the dispersion relations of the linear Klein-Gordon, we were able to justify that the existence of soliton solutions for the sine-Gordon only occurs in the presence of a nonlinear sinusoidal term.

We have used the energy method to investigate the effect of adding external forces to the sine-Gordon equation. Specifically, we were able to determine the stationary velocity for a soliton and the change in total system energy over time. We found that the analytical solution aligned very well with the results obtained numerically with deviation only happening close to the boundary of the system.

The extension of the perturbed sine-Gordon to include varying length pendula displays interesting features that could provide a foundation for research in regards to designing improved Long Josephson Diodes.

References

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11 Appendix

A Matlab code - sine-Gordon

```
clear all

%% Define A
N = 400;
dx = 0.1;

A = -eye(N)*2;
A = A + diag(ones(N-1,1),-1);
A = A + diag(ones(N-1,1),1);
A(1,2) = 2;
A(end,end-1) = 2;

A = A/dx.^2;

%% Soliton solution
y0 = zeros(N*2,1);
x0 = N*dx/2;
c = 0.2;

[y0(1:N), y0(N+1:end)] = soliton(1,x0,c,0.1:0.1:40,0);

eta = 0;
g1 = 0.0;
g2 = 0;
omega = 0.5;
alpha = 0.0;

auxFun = @(t,y) odefun(y,t,N,A,2*eta/dx,g1,g2,omega,alpha);
options = odeset('RelTol',1e-8,'AbsTol',ones(2*N,1)*1e-8);
[T,Y] = ode45(auxFun,[0,600],y0);


%% Multiple gamma
gammaN = 50;
period = zeros(2,gammaN);
gamma = linspace(0,0.7, gammaN);

eta = 0;
g2 = 0;
omega = 0.5;
alpha = 0.1;

for i=1:gammaN
    y0 = zeros(N*2,1);
    x0 = N*dx/2;
    c = 0.2;
```

```
    [y0(1:N), y0(N+1:end)] = soliton(1,x0,c,0.1:0.1:40,0);
    auxFun = @(t,y) odefun(y,t,N,A,2*eta/dx,-gamma(i),g2,omega,
        alpha);
    [T,Y] = ode45(auxFun,[0,600],y0);

    maxn = 100;
    j = 1;
    n = 1;
    epsilon = 1;
    period0 = Inf;
    period1 = 0;

    while(n < maxn)
        % positive rotation
        %[~, tmin1] = min(abs(Y(:,N/2)-pi*j));
        %[~, tmin2] = min(abs(Y(:,N/2)-pi*(j+2)));
        %period1 = T(tmin2) - T(tmin1);

        % negative rotation
        [~, tmin1] = min(abs(Y(:,N/2)+pi*j));
        [~, tmin2] = min(abs(Y(:,N/2)+pi*(j+2)));
        period1 = T(tmin2) - T(tmin1);


        if (norm(period0-period1,'inf')<abs(period0-period1)/10)
            period0 = period1;
            break
        else
             period0 = period1;
        end
        n = n+1;
    end
    period(1,i) = gamma(i);
    period(2,i) = period0;
end

%% Angular velocity plot
v_inf = @(gamma,alpha) 1./sqrt(1+(4*alpha./(pi*gamma)).^2);

hold on
figure(1)
set(gcf,'Position',[100 100 1000 800])
plot(40./period(2,:),period(1,:), 'LineWidth', 3)
plot(v_inf(gamma,alpha),gamma, 'LineWidth', 3)
ylabel('\gamma', 'FontSize',24);
xlabel('v_\infty', 'FontSize',20);
xlim([0.4,1.4])
legend('Numeric', 'Analytic', 'FontSize',20)
grid on
grid minor
hold off
```

```
%% 3D plot
x = (1:N)*dx;
surfc(x(1:1:N),T(1:floor(size(T,1)/1e3):size(T,1)),Y(1:floor(size(T,1)/1e3):size(T,1),1:1:N), 'EdgeColor', 'none')
xlabel('x','FontSize',15);
ylabel('t','FontSize',15);
zlabel('\phi(x,t)','FontSize',15);
set(gca,'FontSize',16,...
        'TickDir','in',...
        'TickLength',[.02,.02],...
        'LineWidth',1)
colorbar('Ticks', [-6*pi,-4*pi,-2*pi, 0, 2*pi, 4*pi,6*pi], ....
     'Ticklabels',{'-6\pi','-4\pi','-2\pi', '0', '2\pi','4\pi','6\pi'})
%saveas(gcf,'figur1.eps','epsc')


%% FUNCTIONS
function dy = odefun(y,t,N,A,eta,g1,g2,omega,alpha)
    dy = zeros(N*2,1);
    dy(1:N) = y(N+1:end);
    dy(N+1:end) = A*y(1:N) - sin(y(1:N)) + g1 + g2*sin(omega*t) - alpha*y(N+1:end);
    dy(N+1) = dy(N+1) - eta;
    dy(end) = dy(end) + eta;
end

function [solx, soldx] = soliton(A,x0,c,x,t)
    solx = 4*atan(A*exp((x-x0-c*t)/sqrt(1-c^2)));
    soldx = -4*A*c*exp((-c*t + x - x0)./sqrt(-c^2 + 1))./(sqrt(-c^2 + 1)*(A^2*exp((-c*t + x - x0)./sqrt(-c^2 + 1)).^2 + 1));
end
```

B Matlab code - pendulum equation

```
clear all

%% Define A
N = 400;
dx = 0.1;

A = -eye(N)*2;
A = A + diag(ones(N-1,1),-1);
A = A + diag(ones(N-1,1),1);
A(1,2) = 2;
A(end,end-1) = 2;

A = A/dx.^2;

%% Define l
gaus = @(x,mu,sig) exp(-(((x-mu).^2)/(2*sig.^2)));
l = ones(N,1) - 0.5*gaus(0.1:0.1:40, 20, 2)';

hold on
figure(1)
set(gcf,'Position',[100 100 1000 800])
ylim([0,1.5])
l = ones(N,1) + 0.0*gaus(0.1:0.1:40, 20, 2)';
plot(0.1:0.1:40, l, 'LineWidth', 3)

l = ones(N,1) + 0.5*gaus(0.1:0.1:40, 20, 2)';
plot(0.1:0.1:40, l, 'LineWidth', 3)

l = ones(N,1) - 0.5*gaus(0.1:0.1:40, 20, 2)';
plot(0.1:0.1:40, l, 'LineWidth', 3)

legend('Standard', 'Long', 'Short', 'FontSize',20)
xlabel('x','FontSize',20);
ylabel('l_x','FontSize',20);
grid on
grid minor
hold off

%% Soliton solution
y0 = zeros(N*2,1);
x0 = N*dx/2;
c = 0.2;
[y0(1:N), y0(N+1:end)] = soliton(1,x0,c,0.1:0.1:40,0);
g = 9.82;
rho = 1;
kappa = 10;
eta = 0;
gamma1 = 0.5;
```

```
gamma2 = 0;
omega = 0;
alpha = 0.1;

auxFun = @(t,y) pendfun(y,t,N,A,l,g,rho,kappa,eta,-gamma1,gamma2,
    omega,alpha);
options = odeset('RelTol',1e-8,'AbsTol',ones(2*N,1)*1e-8);
[T,Y] = ode45(auxFun,[0,600],y0);

%% Multiple gamma
gammaN = 40;
period = zeros(2,gammaN);
gamma = linspace(0,8,gammaN);

for i=1:gammaN
    auxFun = @(t,y) pendfun(y,t,N,A,l,g,rho,kappa,eta,-gamma(i),
        gamma2,omega,alpha);
    [T,Y] = ode45(auxFun,[0,600],y0);

    maxn = 20;
    j = 1;
    epsilon = 1;
    period0 = Inf;
    period1 = 0;

    while(j < maxn)
        % negative rotation
        [~, tmin1] = min(abs(Y(:,N/2)+pi*j));
        [~, tmin2] = min(abs(Y(:,N/2)+pi*(j+2)));
        period1 = T(tmin2) - T(tmin1);

        if (norm(period0-period1,'inf')<(period0+period1)/20)
            period0 = period1;
            if (norm(Y(tmin2,:),'inf')>1)
             y0 = Y(tmin2,:)';
             y0(1:N) = y0(1:N) - round(y0(N/2)/pi)*pi;
            end
            break
        else
             period0 = period1;
        end
        j = j+1;
    end
    period(1,i) = gamma(i);
    period(2,i) = period0;
end

%% Angular velocity plot
hold on
figure(1)
set(gcf,'Position',[100 100 1000 800])
```

```
plot(2*pi./period_standard(2,:),period_standard(1,:), 'LineWidth',
    3)
plot(2*pi./period_large(2,:),period_large(1,:), 'LineWidth', 3)
plot(2*pi./period_small(2,:),period_small(1,:), 'LineWidth', 3)
legend('Standard', 'Long','Short', 'FontSize',20)
ylabel('\gamma', 'FontSize',26);
xlabel('-d\phi/dt', 'FontSize',20);
xlim([0,1.5])
grid on
grid minor
hold off


%% 3D plot
x = (1:N)*dx;
surfc(x(1:1:N),T(1:floor(size(T,1)/1e3):size(T,1)),Y(1:floor(size(T
    ,1)/1e3):size(T,1) ,1:1:N), 'EdgeColor', 'none')
xlabel('x','FontSize',15);
ylabel('t','FontSize',15);
zlabel('\phi(x,t)','FontSize',15);
set(gca,'FontSize',16,...
        'TickDir','in',...
        'TickLength',[.02,.02],...
        'LineWidth',1)
colorbar('Ticks', [-6*pi,-4*pi,-2*pi, 0, 2*pi, 4*pi,6*pi], ....
    'Ticklabels',{'-6\pi','-4\pi','-2\pi', '0', '2\pi','4\pi','6\pi
        '})
%saveas(gcf,'figur1.eps','epsc')


%% FUNCTIONS
function dy = pendfun(y,t,N,A,l,g,rho,kappa,eta,gamma1,gamma2,omega
    ,alpha)
    dy = zeros(N*2,1);
    dy(1:N) = y(N+1:end);
    dy(N+1:end) = (kappa./(rho*l.^2)).*A*y(1:N) - (g./l).*sin(y(1:N
        )) + gamma1 + gamma2*sin(omega*t) - alpha*y(N+1:end);
    dy(N+1) = dy(N+1) - (kappa/(rho*l(1)^2))*eta;
    dy(end) = dy(end) + (kappa/(rho*l(end)^2))*eta;
end

function [solx, soldx] = soliton(A,x0,c,x,t)
    solx = 4*atan(A*exp((x-x0-c*t)/sqrt(1-c^2)));
    soldx = -4*A*c*exp((-c*t + x - x0)./sqrt(-c^2 + 1))./(sqrt(-c^2
        + 1)*(A^2*exp((-c*t + x - x0)./sqrt(-c^2 + 1)).^2 + 1));
end
```