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Assignment 1

Note: Please include references when applying results of, say, [Per00].

Problem 1

Non-dimensionalization is a technique that can reduce the dimension of the space of parameters, thus making it easier to analyze a family of differential equations. In practice, one make a linear change of coordinates in each of the space variables, and in time, resulting in dimensionless new coordinates. If the change of variables are chosen well, the new system can often be made simpler than the original.

To make things concrete, we look at the logistic differential equation.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{1}$$

Here P is a population that can multiply, but whose growth limited (because of limited food, space, etc). The limit to growth is given by the carrying capacity K. The parameter r (measured in, say, per day, is called the growth rate, and it is easy to see that wen P is small, the growth is approximately exponential with coefficient r.

Notice that we have two parameters $r > 0$ and $K > 0$, so in principle there is two-infinities of dynamical systems to analyze. However, we can simplify greatly by changing coordinates.

Let t_c and P_c be constants to be determined later, and introduce a new variable p by letting $P = P_c p$. Also rescale time by setting $t = t_c\tau$. Having unit of P_c is the same as that of P, and the unit of t_c is the same as that of t, implies that p and τ are dimensionless quantities.

Question (a) Show that using the new coordinate and time, the dynamical system (1) takes the form.

$$\frac{dp}{d\tau} = t_c r p\left(1 - \frac{P_c p}{K}\right).$$

Question (b) Now choose values for the constant t_c and P_c so the system simplifies to

$$\frac{dp}{d\tau} = p(1-p). \tag{2}$$

This means that to understand the behavior of the logistic differential equation, we need only analyze one system.

Question (c) Analyse (2) graphically, and describe the behavior of solutions. You need only consider $p \geq 0$.

Think about (but don't write about it), that knowing the constant P_c, allows you to describe the behavior of an arbitrary solution to the original equation (1).

In general, in a n-dimensional system, you can usually get rid of $n+1$ parameters. There is some choice in which parameters to get rid of.

Question (d) Consider the system

$$\dot{X} = aX - bXY$$
$$\dot{Y} = -dY + cXY$$

where a, b, c, d all are positive parameters. Show that you can non-dimensionalize to obtain the system

$$\frac{dx}{d\tau} = x - xy$$
$$\frac{dy}{d\tau} = -\alpha y + xy$$

and determine α in terms of (some of) the parameters a, b, c and d. Show that you can also non-dimensionalize to obtain the system

$$\frac{dx}{d\tau} = \beta x - xy$$
$$\frac{dy}{d\tau} = -y + xy$$

and determine the value of β.

Problem 2

The dynamics of piecewise linear vector fields is an active area of research. In this problem, we look at one such vector field.

Let

$$\mathbf{A} = \begin{pmatrix} 2a & -1 \\ 1 & 0 \end{pmatrix},$$

where $|a| < 1$, and consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Question (a) What kind of equilibrium point is the origin?

Question (b) Let $\omega = \sqrt{1-a^2}$, and show that $\lambda = a + i\omega$ is an eigenvalue, and that $\mathbf{w} = (\lambda, 1)$ is a corresponding eigenvector.

Question (c) Letting

$$\mathbf{P} = \begin{pmatrix} \omega & a \\ 0 & 1 \end{pmatrix},$$

use the theorem in [Per00] Section 1.6 to find the matrix $\mathbf{R}$ that satisfy

$$\mathbf{A} = \mathbf{PRP}^{-1}.$$

Question (d) Use Corollary 3 in Section 1.3 of [Per00] to find $\exp(t\mathbf{R})$.

If we at time $t = 0$, start following the flow of the vector field $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$ from a point $(0, y)$, $y \neq 0$, there is a first time $T = T(a) > 0$, where the orbit hits the y-axis.

Question (e) Show that $T(a) = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{1-a^2}}$. Hint 1: For each t the flow is linear and thus maps lines through the origin to lines through the origin. Hint 2: It is easier to work in **vu** coordinates in which the flow becomes the flow of the linear vector field given by $\mathbf{R}$.

Question (f) Show that the flow ϕ satisfies

$$\phi_{T(a)}(0, y) = (0, -\mathrm{e}^{\frac{\pi a}{\sqrt{1-a^2}}}\, y).$$

Finally, let

$$\mathbf{B} = \begin{pmatrix} 2b & -1 \\ 1 & 0 \end{pmatrix},$$

with $|b| < 1$ and consider the vector field

$$\mathbf{f}(x, y) = \begin{cases} \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} & x \leq 0, \\ \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix} & x \geq 0. \end{cases}$$

The vector field is continuous but not C^1, one can however show that existence and uniqueness still holds, so $\mathbf{f}$ defines a flow Φ. Notice that the flow of $\mathbf{f}$ carries the positive y-axis onto the negative y-axis in time $T(a)$, and the negative y-axis onto the positive one in time $T(b)$.

Question (g) By considering the flow $\Phi_{T(b)+T(a)}$ on the positive y-axis, explain why the origin is stable if and only if

$$\frac{a}{\sqrt{1-a^2}} + \frac{b}{\sqrt{1-b^2}} \leq 0.$$

(This condition can be shown to be equal to $a + b \leq 0$.)

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.