Dynamical Systems 1
Course 01617
E21

Mads Peter Sørensen and Christian Henriksen
Department of Applied Mathematics and Computer Science
Technical University of Denmark

Assignment 2

Note: Please include references when applying results from the textbook [Perko].

Problem 1: We shall analyse the system of differential equations

$$\dot{x_1} = f_1(x_1, x_2) = x_2 \;, \tag{1}$$

$$\dot{x_2} = f_2(x_1, x_2) = \mu x_1 - \frac{1}{2} x_1^2 \;. \tag{2}$$

Here the two variables x_1 and x_2 depend on time t. A dot above a time dependent variable denotes differentiation with respect to t. The parameter μ is a real number.

a) Find all critical points of the two dimensional vectorfield $\mathbf{f} = (f_1, f_2)^T$ in the system (1-2).

b) Determine the stability of all critical points for the cases of $\mu > 0$ and $\mu < 0$. Classify these critical points according to the list: saddles, centers, cusps, nodes and foci.

c) Sketch the flow in a phase plane plot for $\mu > 0$ and $\mu < 0$. Use pencil and paper and support your findings using pplane or Maple (hint: you may use the command *plotfield* in Maple).

d) For $\mu \neq 0$ there are homoclinic orbits. Similar to finding center manifolds we can determine the homoclinic orbits by introducing a function $h \in C^\infty$ mapping $\mathbb{R} \to \mathbb{R}$ according to

$$x_2 = h(x_1) \;. \tag{3}$$

Which condition must h satisfy? Now show that h is given by

$$h(x_1)^2 = a_1 x_1^2 + b x_1^3 + c \;. \tag{4}$$

and determine the parameters a, b and c for $\mu > 0$ and $\mu < 0$.

e) Plot your analytical expression for $h(x_1)$ together with the vectorfield. Does the figure show what you expect?

Problem 2: The results from problem 1 indicate that a bifurcation occurs at $\mu = 0$ and that μ is a bifurcation parameter. Secondly, for $\mu = 0$ the critical point $(x_1, x_2) = (0, 0)$ is different compared to the cases where $\mu \neq 0$.

a) Construct a bifurcation diagram for the system in (1-2). In doing so sketch in a figure x_1 as function of μ, for each critical point (x_1, x_2).

b) Determine the type of bifurcation you observe.

c) Is the critical point $(x_1, x_2) = (0, 0)$ hyperbolic or nonhyperbolic for $\mu = 0$?

d) For $\mu = 0$ the system in (1-2) becomes

$$\dot{x_1} = x_2 \;, \tag{5}$$

$$\dot{x_2} = -\frac{1}{2} x_1^2 \;. \tag{6}$$

At the critical point $(0,0)$ the system possesses locally a center manifold $W^c(0,0)$. Determine this center manifold by introducing the mapping

$$x_2 = h(x_1) \; , \tag{7}$$

and find an analytical expression for h. If we strictly follow Theorem 1, p155, in the textbook by Perko, we would choose a function h_c, where $x_1 = h_c(x_2)$ instead of the expression in (7). However, the calculations using the mapping h is more straight forward than using the mapping defined by h_c. By the way they are each others inverse.

e) Plot your analytical expression for $h(x_1)$ together with the vectorfield $\mathbf{f}$ for $\mu = 0$ by using Maple or pplane. Determine the type of the critical point $(0,0)$ at $\mu = 0$ and compare to Theorem 3 p151 in section 2.11 of the textbook by Perko.