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2023年11月4日 14:19

1 Number of different binary trees

(From *CLRS* 12-4)

Let b_n denote the number of different binary trees with n nodes. In this problem, you will find a formula for b_n, as well as an asymptotic estimate.

1. Show that $b_0 = 1$ and that, for $n \geq 1$,

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k}$$

显然 $b_0 = 1$
那么 b_n的情况有 左子树有k个节点,右子树有$n-1-k$个节点,
k可以从0取到$n-1$,且两边独立,使用乘法原则,
故而

$$\mathrm{b_n} = \sum_{\mathrm{k=0}}^{\mathrm{n-1}} b_k b_{n-1-k}$$

2. Let $B(x)$ be the generating function

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

Show that $B(x) = xB(x)^2 + 1$, and hence one way to express $B(x)$ in closed form is

$$B(x) = \frac{1}{2x}(1 - \sqrt{1-4x})$$

$$B(x)^2 = \sum_{0 \leq i,j}^{\infty} b_i b_j x^{i+j} = 1 + \sum_{n=1}^{\infty}\left(x^n \sum_{k=0}^{\infty} b_k b_{n-k}\right) = 1 + \sum_{n=1}^{\infty}\left(b_{n+1}x^n\right)$$

$$xB(x)^2 + 1 = x\left(1 + \sum_{n=1}^{\infty}\left(b_{n+1}x^n\right)\right) + 1 = \sum_{n=1}^{\infty}\left(b_n x^n\right) + 1$$

$$= \sum_{n=0}^{\infty}\left(b_n x^n\right) = B(x)$$

当我们得到上述等式后,根据求根公式得

$$\mathrm{B(x)} = \frac{1 \pm \sqrt{1-4x}}{2x}$$

又 $\lim\limits_{\mathrm{x\to 0}} \mathrm{B(x)} = 0$,

$$\therefore \mathrm{B(x)} = \frac{1}{2x}\left(1 - \sqrt{1-4x}\right)$$

3. Show that

$$b_n = \frac{1}{n+1}\binom{2n}{n}$$

(the nth **Catalan number**) by using the Taylor expansion of $\sqrt{1-4x}$ around $x=0$.

p.s. If you wish, instead of using the Taylor expansion, you may use the generalization of the binomial theorem (where n can be any real number) to noninteger exponents.

方法一:

$b_0 = 1 = \frac{1}{1}\binom{0}{0}; b_1 = 1 = \frac{1}{2}\binom{2}{1}$

我们假设 $n \le k: b_n = \frac{1}{n+1}\binom{2n}{n}$;

现需要证明: $n = k+1, b_n = \frac{1}{n+1}\binom{2n}{n}$

我们有: $\mathrm{b}_n = \sum_{\mathrm{i}=0}^{\mathrm{n}-1} b_i b_{n-1-i}$

后面暂时没想出来 ...

TBC ...

方法二:

$$\sqrt{1-4x} = 1 + \frac{-4x}{2} - \sum_{n=2}^{\infty}\frac{(2n-3)!!}{2^n * n!}(4x)^n = 1 - 2x - \sum_{n=2}^{\infty} 2 * \frac{(4n-6)!!}{n!}x^n$$

$$B(x) = \frac{1}{2x}\left(1-\sqrt{1-4x}\right) = \frac{1}{2x}\left(2x + 2\sum_{n=2}^{\infty}\frac{(4n-6)!!}{n!}x^n\right) = 1 + \sum_{n=2}^{\infty}\frac{(4n-6)!!}{n!}x^{n-1}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(4n-2)!!}{(n+1)!}x^n = 1 + \sum_{n=1}^{\infty} b_n x^n$$

$$\therefore b_n = \frac{(4n-2)!!}{(n+1)!}\ (n>0) = \frac{1}{n+1}\frac{2^n(2n-1)!!}{n!} = \frac{1}{n+1}\frac{(2n)!!\,(2n-1)!!}{n!\,n!}$$

$$= \frac{1}{n+1}\frac{(2n)!!\,(2n-1)!!}{n!\,n!} = \frac{1}{n+1}\left(\frac{(2n)!}{n!\,n!}\right) = \frac{1}{n+1}\binom{2n}{n}$$

4. Show that

$$b_n = \frac{4^n}{\sqrt{\pi}n^{3/2}}(1+O(1/n))$$

$$b_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\frac{(2n)!}{n!\,n!}\right) = \frac{1}{n+1}\left(\frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right)$$

$$= \frac{1}{n+1}\left(\frac{(2)^{2n}}{\sqrt{\pi n}}\right) = \frac{1}{n}\left(\frac{n}{1+n}\right)\left(\frac{(2)^{2n}}{\sqrt{\pi n}}\right) = \frac{1}{n}\left(\frac{4^n}{\sqrt{\pi}n^{\frac{1}{2}}}\right)\left(1-\left(\frac{1}{n+1}\right)\right) = \frac{4^n}{\sqrt{\pi}n^{\frac{3}{2}}}(1+O\left(\frac{1}{n}\right))$$

2 AVL trees

(From CLRS 13-3)

An **AVL tree** is a binary search tree that is **height balanced**: for each node x, the heights of the left and right subtrees of differ by at most 1. To implement an AVL tree, maintain an extra attribute h in each node such that $x.h$ is the height of node x. As for any other binary search tree T, assume that node $T.root$ points to the root node. Prove that an AVL tree with n nodes has height $O(\lg n)$.

(*Hint:* Prove that an AVL tree of height h has at least F_h nodes, where F_h is the hth Fibonacci number.)

对h进行归纳

1. h = 1 时，nodeNumber = 1 $\geq F_1 = 1$
2. 假设 $h \leq k$ 时, $\mathrm{nodeNumber_h} \geq F_h$
3. 下证: $h = k + 1$ 时 $nodeNumber_h \geq F_h$

   k+1高的树 至少有一个k高的子树, 且不存在高大于k的子树, 那么根据ALV的性质,我们知道另一个子树的高度为k或k-1,

   所以 $\mathrm{nodeNumber_{k+1}} \geq \mathrm{nodeNumber_k} + \mathrm{nodeNumber_{k-1}} \geq \mathrm{F_k} + \mathrm{F_{k-1}} = \mathrm{F_{k+1}}$

而$F_h = O\left(\frac{\varphi^{\mathrm{h}} - (1-\varphi)^h}{\sqrt{5}}\right)$ 其中$\varphi = \frac{1+\sqrt{5}}{2}$
$F_h = \mathrm{O}\left(a^{\mathrm{h}}\right)$ 其中a为某个常数
故而当有n个节点时，其高度为$\lg(n)$