ECE 7810 ASSIGNMENT REPORT
(Solution of Fields by Num. Mtds I)

Author: Jamiu Babatunde Mojolagbe

Department: Electrical and Computer Engineering

Student ID: #7804719

Email: mojolagm@myumanitoba.ca

Course: ECE 7810

Homework: 3

Sub. Date: November 14, 2016

QUESTION A

Transmission line equation:

$$\frac{\partial V(x,t)}{\partial x} + L\frac{\partial I(x,t)}{\partial t} + RI(x,t) = 0 \quad \text{——— (i)}$$

$$\frac{\partial I(x,t)}{\partial x} + C\frac{\partial V(x,t)}{\partial t} + GV(x,t) = 0 \quad \text{——— (ii)}$$

where L = inductance, C = capacitance
V = voltage, I = current
R = impedance, G = conductance

Equations (i) and (ii) can be written as:

$$C\frac{\partial V}{\partial t} = -GV - \frac{\partial I}{\partial x} \quad \text{——— (1)}$$

$$L\frac{\partial I}{\partial t} = -RI - \frac{\partial V}{\partial x} \quad \text{——— (2)}$$

Creating vector-matrix equation from (1) and (2):

$$\frac{\partial}{\partial t}\begin{bmatrix} V \\ I \end{bmatrix} = -\begin{bmatrix} \frac{G}{C} & \frac{1}{C}\frac{\partial}{\partial x} \\ \frac{1}{L}\frac{\partial}{\partial x} & \frac{R}{L} \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix} \quad \text{——— (3)}$$

It can be conveniently expressed as a single equation:

$$\frac{\partial F}{\partial t} = \bar{S}F \quad \text{——— (4)}$$

where F = voltage-current vector
$\bar{S}$ = operator matrix

$$F = \begin{bmatrix} V & I \end{bmatrix}^T \quad \text{——— (5)}$$

$$\bar{S} = -\begin{bmatrix} \frac{G}{C} & \frac{1}{C}\frac{\partial}{\partial x} \\ \frac{1}{L}\frac{\partial}{\partial x} & \frac{R}{L} \end{bmatrix} \quad \text{——— (6)}$$

Equation (4) can be solved by finding the propagator matrix $\bar{K}(x,t)$

$$\frac{\partial \bar{K}(x,t)}{\partial t} = \bar{S}\bar{K}(x,t) \qquad (7)$$

with initial condition:

$$\lim_{t \to 0} \bar{K}(x,t) = \bar{I}\delta(x-x') \qquad (8)$$

where δ = Dirac delta function
x' = initial position of the voltage
$\bar{I}$ = Identity matrix

$$\bar{K} = e^{\bar{S}t}\delta(x-x') \qquad (9)$$

where Dirac delta function can be expressed interms of Fourier integral:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jk(x-x')}\,dk$$

$\therefore$ equation (9) becomes, with exponential expressed in power series, as:

$$\bar{K} = \left(1 + \bar{S}t + \frac{\bar{S}^2t^2}{2!} + \cdots\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jk(x-x')}\,dk \qquad (10)$$

$$\bar{K} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\bar{S}_f t}\exp[jk(x-x')]\,dk \qquad (11)$$

$$\bar{S}_f = -\begin{bmatrix} \frac{G}{C} & \frac{jk}{C} \\ \frac{jk}{L} & \frac{R}{L} \end{bmatrix} \qquad (12)$$

The exponential $e^{\bar{S}_f t}$ can be converted to a 2x2 transition matrix $\bar{A}$, then the eigenvalue λ_1, λ_2 of $\bar{S}_f$ are found by extracting the roots of $\det[\lambda I - \bar{S}_f] = 0$

$$\det\begin{bmatrix} \lambda + \frac{G}{C} & \frac{jK}{C} \\ \frac{jK}{L} & \frac{R}{L} \end{bmatrix}$$

Giving that:

$$\lambda_1, \lambda_2 = -\frac{G}{2C}\left[1 \pm \sqrt{1-\gamma^2}\right] - \frac{R}{2L}\left[1 \pm \sqrt{1-\gamma^2}\right]$$

where $\gamma = \dfrac{V_p K}{\left(\frac{G}{2C} - \frac{R}{2L}\right)}$ $\quad V_p$ = velocity of propagation

Eigenvector can be constructed by selecting one column of $\mathrm{adj}[\lambda_1 \bar{I} - \bar{S}_f]$ and one column of $\mathrm{adj}[\lambda_2 \bar{I} - \bar{S}_f]$. Therefore, the respective first and second columns of the modal matrix is given by:

$$\bar{M} = \begin{bmatrix} -Z_0 & -Z_0 \\ \frac{1-\sqrt{1-\gamma^2}}{j\gamma} & \frac{1+\sqrt{1-\gamma^2}}{j\gamma} \end{bmatrix}$$

where $Z_0 = \sqrt{\frac{L}{C}}$

$V_p = \dfrac{1}{\sqrt{LC}}$

$\bar{M}$ = modal matrix

The modal matrix and its inverse along with a diagonalized exponential eigenvalue matrix are combined to form the matrix product:

$$\bar{A} = e^{\bar{S}t} = \bar{M}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\bar{M}^{-1}$$

$\Rightarrow$

$$A_{11} = e^{-\alpha t}\left[\frac{-b}{V_p}\frac{\sin\left[tV_p\sqrt{K^2-K_*^2}\right]}{\sqrt{K^2-K_*^2}} + \cos\left[tV_p\sqrt{K^2-K_*^2}\right]\right]$$

$$A_{12} = -jKZ_0 e^{-at}\left[\frac{\sin\left[tV_p\sqrt{K^2-K_*^2}\right]}{\sqrt{K^2-K_*^2}}\right]$$

$$A_{21} = \frac{A_{12}}{Z_0^2}$$

$$A_{22} = e^{-at}\left[\frac{b}{V_p}\,\frac{\sin\left[tV_p\sqrt{K^2-K_*^2}\right]}{\sqrt{K^2-K_*^2}} + \cos\left[tV_p\sqrt{K^2-K_*^2}\right]\right]$$

where $K_* = \frac{G}{2CV_p} - \frac{R}{2LV_p}$, $a = \frac{1}{2}\left(\frac{G}{C}+\frac{R}{L}\right)$, $b = \frac{1}{2}\left(\frac{G}{C}-\frac{R}{L}\right)$

The propagator equation expressed in terms of matrix $\bar{A}$ becomes:

$$\bar{K} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\exp[jK\bar{x}]\,dK$$

where $\bar{x} = x - x'$

Integrating each integral above yields:

$$\bar{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}$$

$$K_{11} = \frac{e^{-at}}{2}\left[\delta(\bar{x}+tV_p) + \delta(\bar{x}-tV_p) - \frac{b}{V_p}J_0\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] + \frac{bt\,J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)}\right]}{\sqrt{\bar{x}^2-(tV_p)^2}}\right]$$

$$K_{12} = -Z_0\frac{e^{-at}}{2}\left[\delta(\bar{x}+tV_p) - \delta(\bar{x}-tV_p) - \bar{x}\frac{b}{V_p}\frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right]}{\sqrt{\bar{x}^2-(tV_p)^2}}\right]$$

$$K_{21} = \frac{K_{12}}{Z_0^2}$$

$$K_{22} = \frac{e^{-at}}{2}\left[\delta(\bar{x}+tV_p) + \delta(\bar{x}-tV_p) + \frac{b}{V_p} J_0\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] + bt\frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right]}{\sqrt{\bar{x}^2-(tV_p)^2}}\right] \quad (13)$$

Propagator equation ~~eqn~~ with initial voltage and current can be expressed as:

$$\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} = \int_{-\infty}^{\infty} \bar{K} \cdot \begin{bmatrix} V(x') \\ I(x') \end{bmatrix} dx' \quad (14)$$

Insert (13) into (14):

$$V(x,t) = \frac{e^{-at}}{2}\Bigg[V(x+tV_p) + V(x-tV_p) - Z_0\left[I(x+tV_p) - I(x-tV_p)\right] + bt\int_{x-tV_p}^{x+tV_p} \frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2+(tV_p)^2}\right]}{\sqrt{\bar{x}^2-(tV_p)^2}} V(x')\,dx'$$

$$- \frac{b}{V_p}\int_{x-tV_p}^{x+tV_p} J_0\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] V(x')\,dx'$$

$$+ Z_0\frac{b}{V_p}\int_{x-tV_p}^{x+tV_p} \bar{x}\frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right]}{\sqrt{\bar{x}^2-(tV_p)^2}} I(x')\,dx'\Bigg] \quad (15)$$

$$I(x,t) = \frac{e^{-at}}{2}\Bigg[I(x+tV_p) + I(x-tV_p) - \frac{1}{Z_0}\left[V(x+tV_p) - V(x-tV_p)\right]$$

$$+ bt \int_{x-tV_p}^{x+tV_p} \frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] I(x')\,dx'}{\sqrt{\bar{x}^2-(tV_p)^2}}$$

$$+ \frac{b}{V_p}\int_{x-tV_p}^{x+tV_p} J_0\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] I(x')\,dx'$$

$$+ \frac{1}{Z_0}\frac{b}{V_p}\int_{x-tV_p}^{x+tV_p} \frac{J_1\left[\frac{b}{V_p}\sqrt{\bar{x}^2-(tV_p)^2}\right] V(x')\,dx'}{\sqrt{\bar{x}^2-(tV_p)^2}} \Bigg] \quad \text{— (16)}$$

(15) and (16) can be integrated using Simpson's rule and properties of Bessel functions to give:

$$V(x,t) = \frac{e^{-at}}{2}\Bigg[\left[1-\frac{(bt)}{3}+\frac{(bt)^2}{6}\right]\left[V(x+tV_p)+V(x-tV_p)\right]$$

$$- Z_0\left[1+\frac{(bt)^2}{6}\right]\left[I(x+tV_p)-I(x-tV_p)\right]$$

$$+ \frac{4}{3}(bt)\left[I_1(bt) - I_0(bt)\right]V(x)\Bigg] \quad \text{— (17)}$$

$$I(x,t) = \frac{e^{-at}}{2}\Bigg[\left[1+\frac{(bt)}{3}+\frac{(bt)^2}{6}\right]\left[I(x+tV_p)+I(x-tV_p)\right]$$

$$- \frac{1}{Z_0}\left[1+\frac{(bt)^2}{6}\right]\left[V(x+tV_p)-V(x-tV_p)\right]$$

$$+ \frac{4(bt)}{3}\left[I_1(bt) + I_0(bt)\right]I(x)\Bigg] \quad \text{— (18)}$$

where I_0 = 0 order modified Bessel function of the first kind

I_1 = 1st order modified Bessel function of the first kind

Equations (17) and (18) are for lossy transmission line.

For lossless transmission line:

$R, G, a, b = 0$, equations (17) and (18) becomes:

$$V(x,t) = \frac{1}{2}\Big[[V(x+tV_p) + V(x-tV_p)] - Z_0[I(x+tV_p) - I(x-tV_p)] \Big]$$

$$I(x,t) = \frac{1}{2}\Big[[I(x+tV_p) + I(x-tV_p)] - \frac{1}{Z_0}[V(x+tV_p) - V(x-tV_p)] \Big]$$

QUESTION B

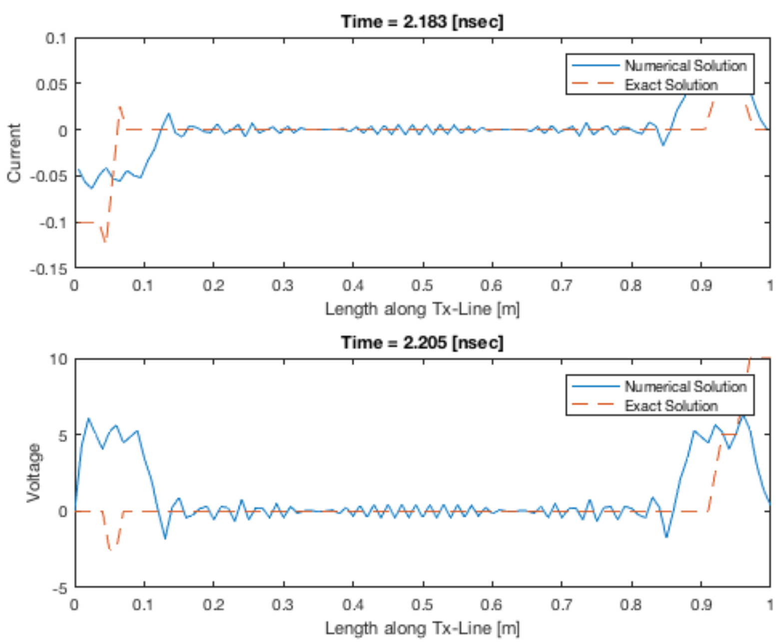
This question was implemented in the file named "**fdtdLosslessVsExact.m**". The exact solution was implemented as derived in the solution to the "Question A" above. The code is commented and documented.

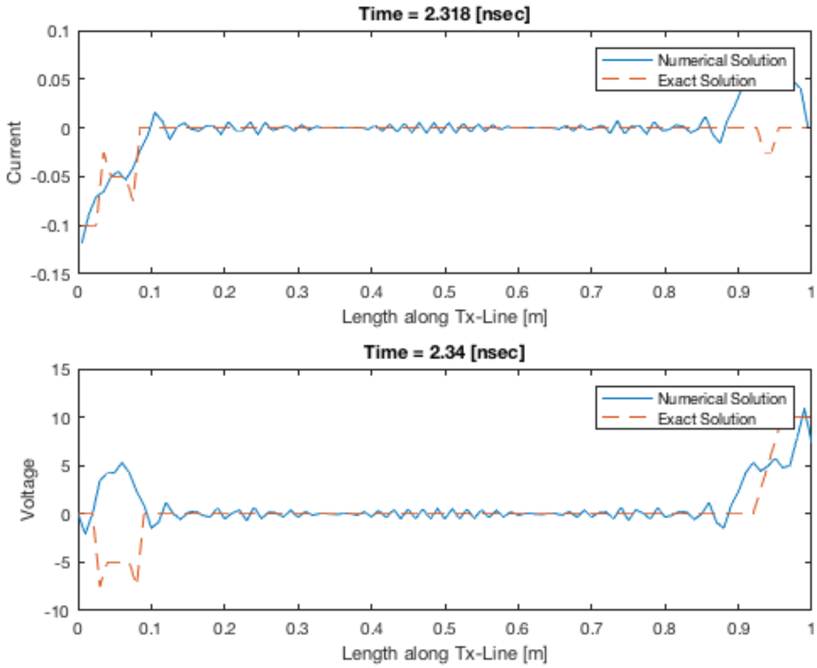
The following results were obtained:

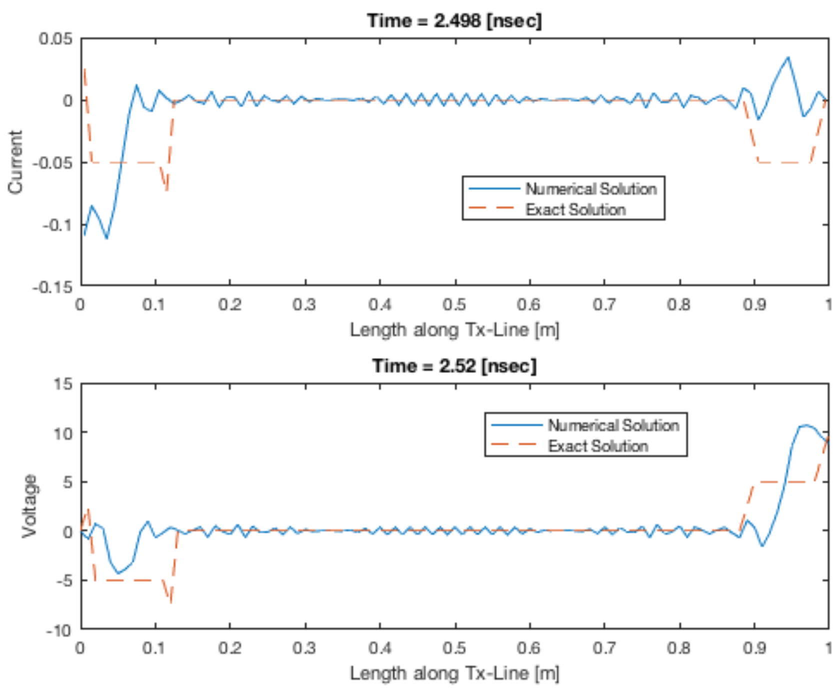
Given that C=50pF/m and L=0.5μH/m, the velocity of propagation obtained, c0 was

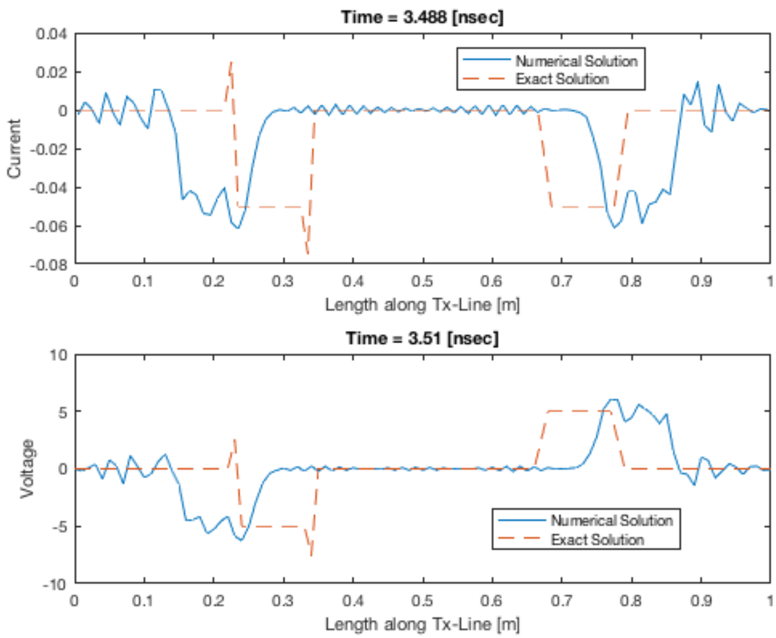
$$c0 = \frac{1}{\sqrt{LC}} = 200000000\ m/s$$

Case 1: For Courant limit, CFL = 0.9, the following graphs were obtained at voltage times: 2.2, 2.3, 2.5 and 3.5ns

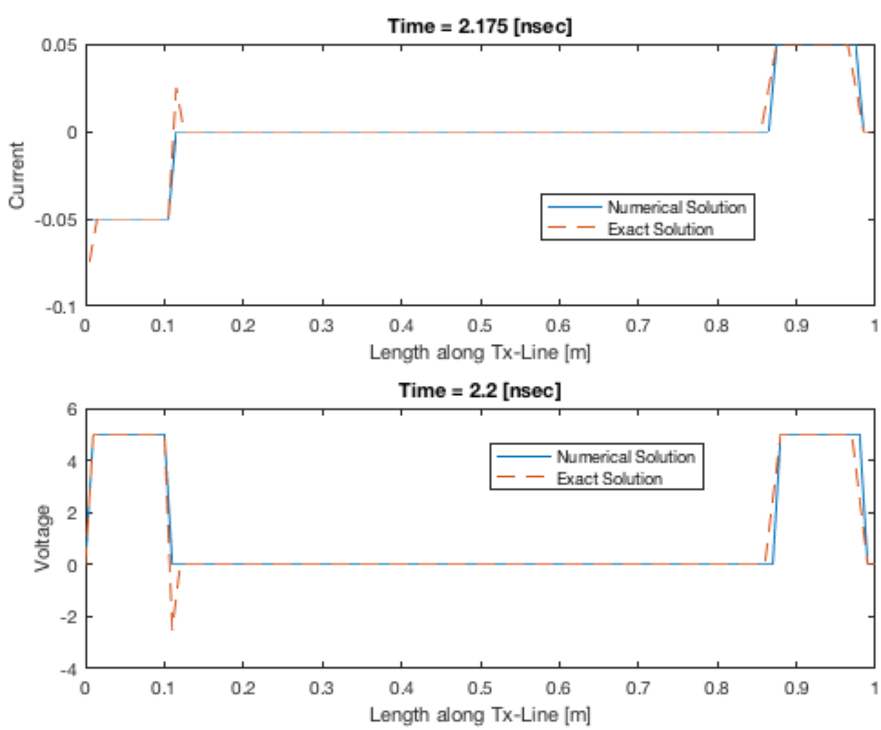


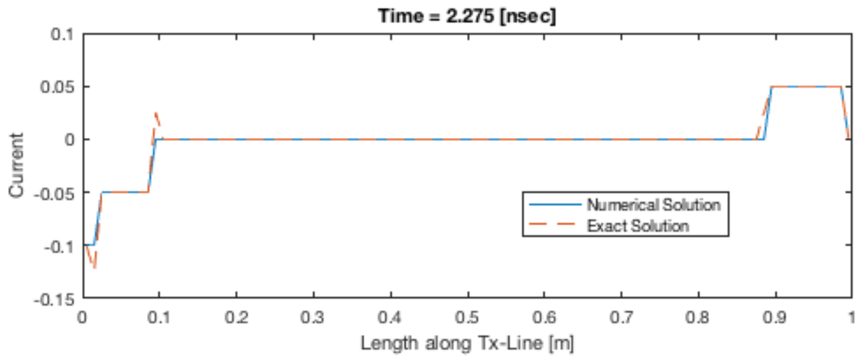

Time = 2.318 [nsec]
Current
Numerical Solution
Exact Solution
Length along Tx-Line [m]
Time = 2.34 [nsec]
Voltage
Numerical Solution
Exact Solution
Length along Tx-Line [m]

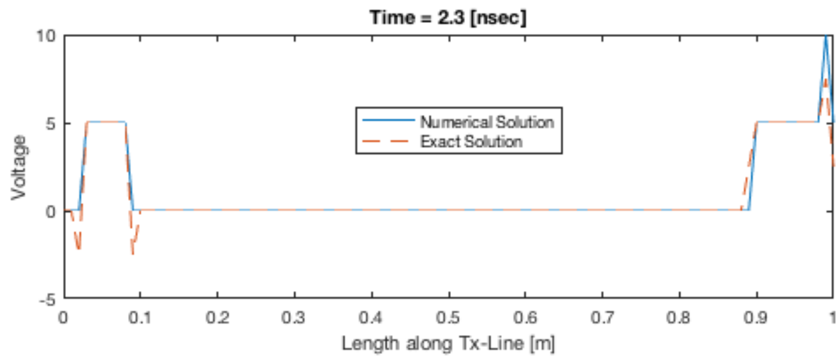

Time = 2.498 [nsec]
Current
Numerical Solution
Exact Solution
Length along Tx-Line [m]
Time = 2.52 [nsec]
Voltage
Numerical Solution
Exact Solution
Length along Tx-Line [m]

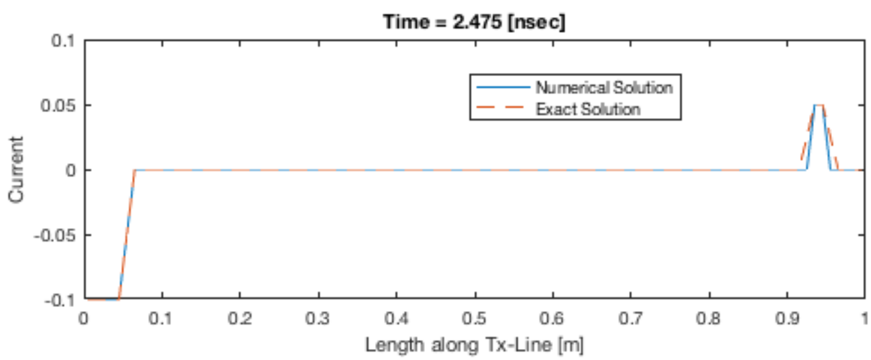


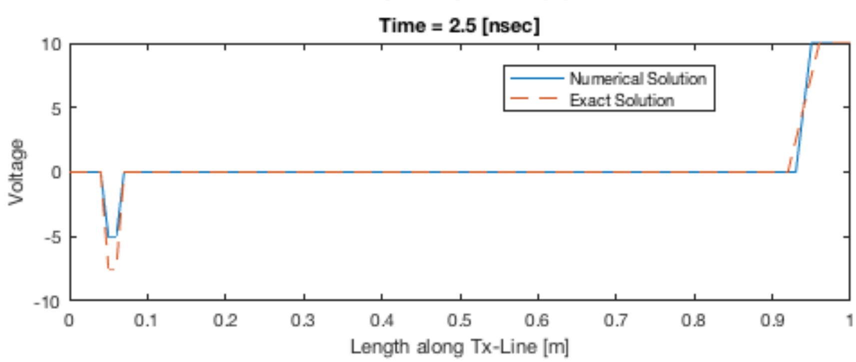
Case 2: For Courant limit, CFL = 1.0, the following graphs were obtained at voltage times: 2.2, 2.3, 2.5 and 3.5ns

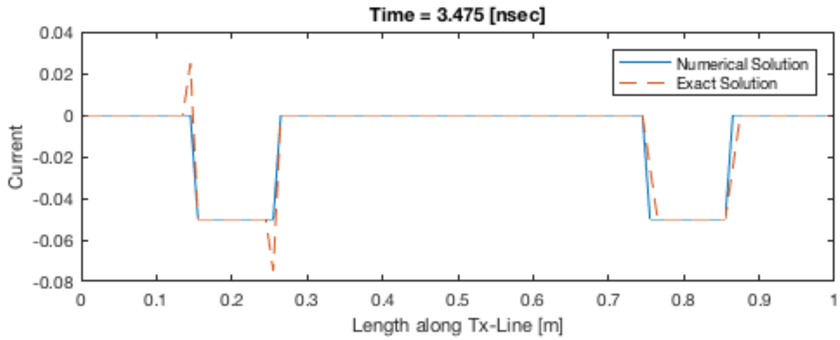


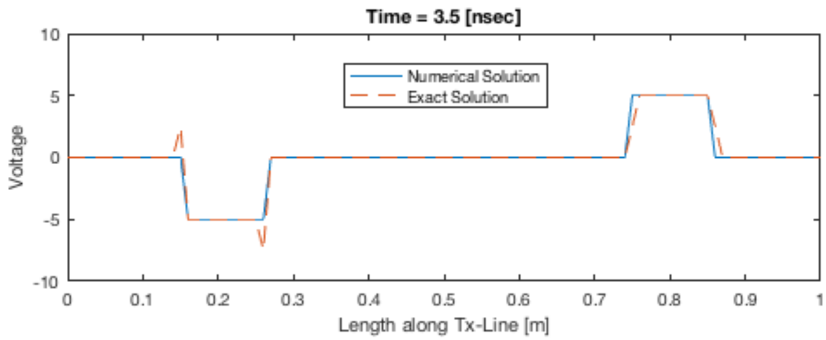

Time = 2.275 [nsec]
Current
Numerical Solution
Exact Solution
Length along Tx-Line [m]


Time = 2.3 [nsec]
Voltage
Numerical Solution
Exact Solution
Length along Tx-Line [m]

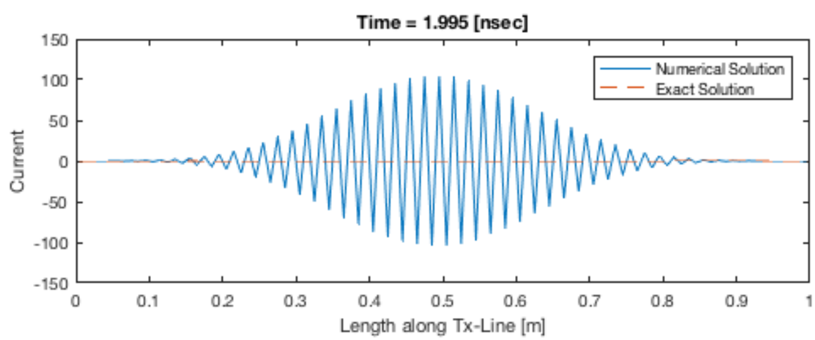

Time = 2.475 [nsec]
Current
Numerical Solution
Exact Solution
Length along Tx-Line [m]

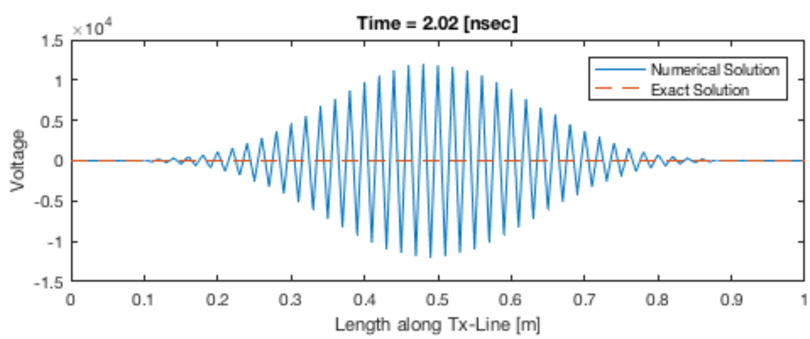

Time = 2.5 [nsec]
Voltage
Numerical Solution
Exact Solution
Length along Tx-Line [m]

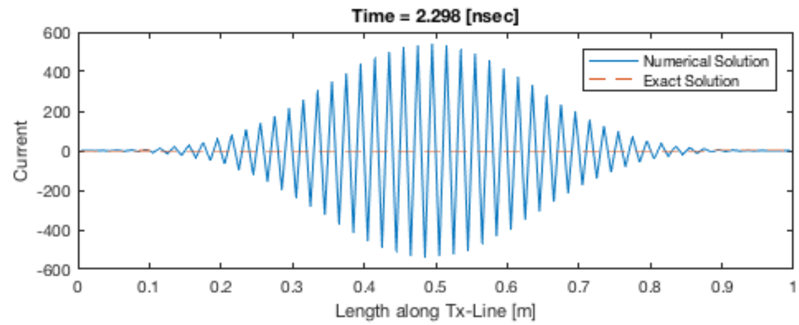


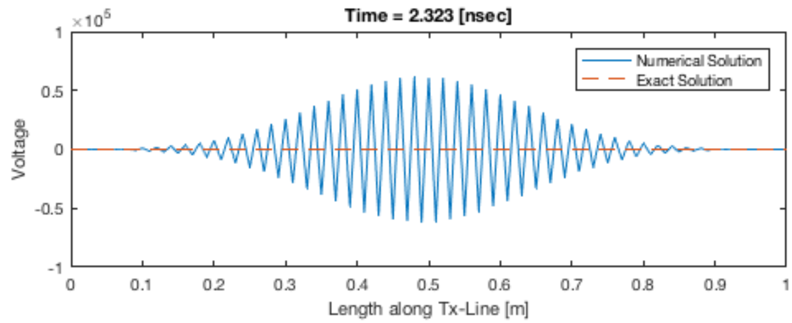


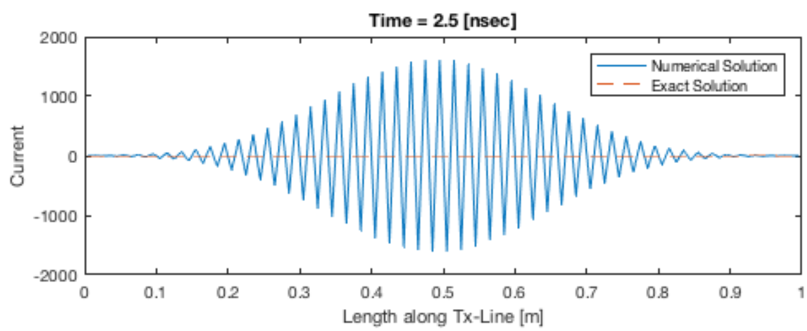
Case 3: For Courant limit, CFL = 1.01, the following graphs were obtained at voltage times: 2.2, 2.3, 2.5 and 3.5ns

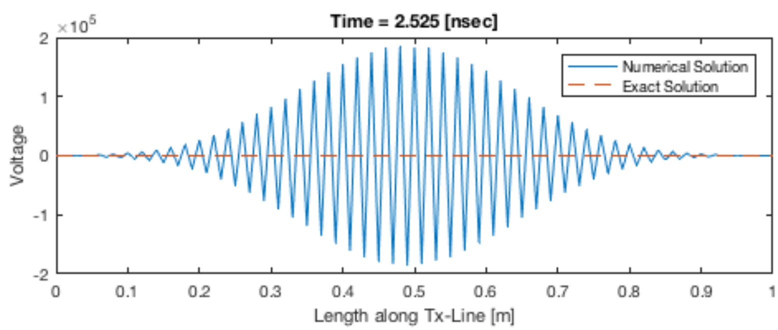


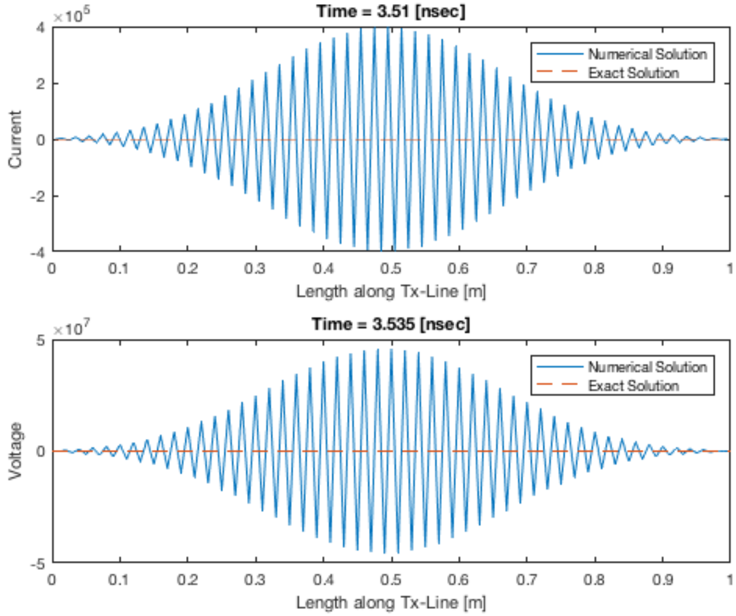



Time = 2.298 [nsec]
Numerical Solution
Exact Solution
Current
Length along Tx-Line [m]


Time = 2.323 [nsec]
×10^5
Numerical Solution
Exact Solution
Voltage
Length along Tx-Line [m]


Time = 2.5 [nsec]
Numerical Solution
Exact Solution
Current
Length along Tx-Line [m]


Time = 2.525 [nsec]
×10^5
Numerical Solution
Exact Solution
Voltage
Length along Tx-Line [m]



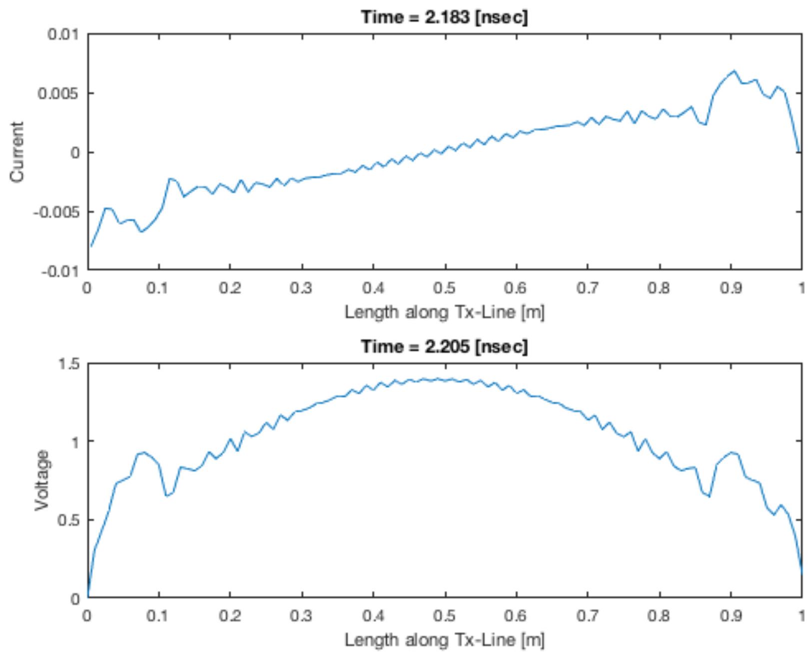
Observation:

As shown above, when the Courant value was less than 1.0, precisely 0.9, there seems to be partial numerical dispersion; however, when CFL was 1.0 there was stability in the wave. With Courant number greater than 1.0 the dispersion was great in the wave high instability in the wave. Again, observing the exact solution also showed that the numerical solution tends to be closer to the exact solution.

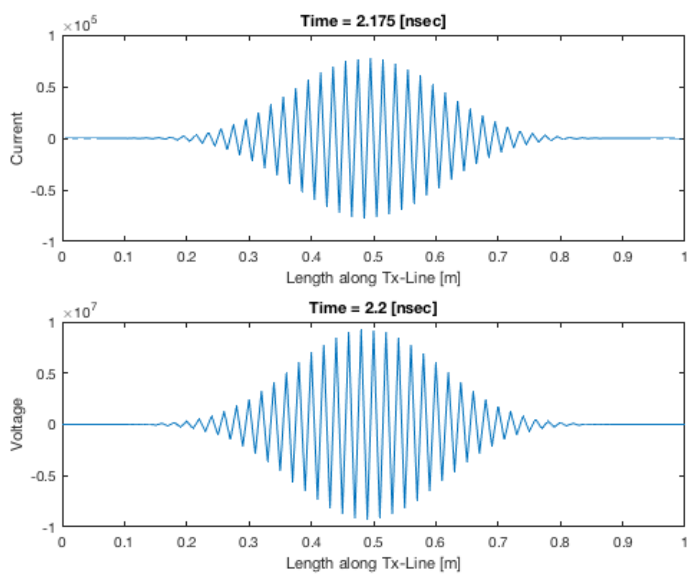
QUESTION C

This question's implementation file is named "**fdtdLossy.m**". Different cases were tested based on the different Courant numbers given. This code is well implemented and documented to easy readability. The following results were obtained:

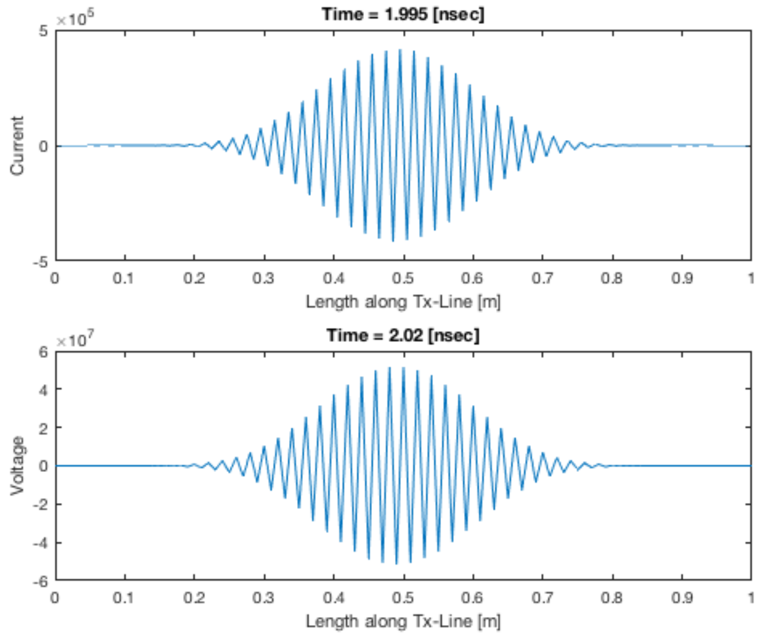
Case 1: For Courant limit, CFL = 0.9, the following graph was obtained at voltage time: 2.2ns



Case 2: For Courant limit, CFL = 1.0, the following graph was obtained at voltage time: 2.2ns



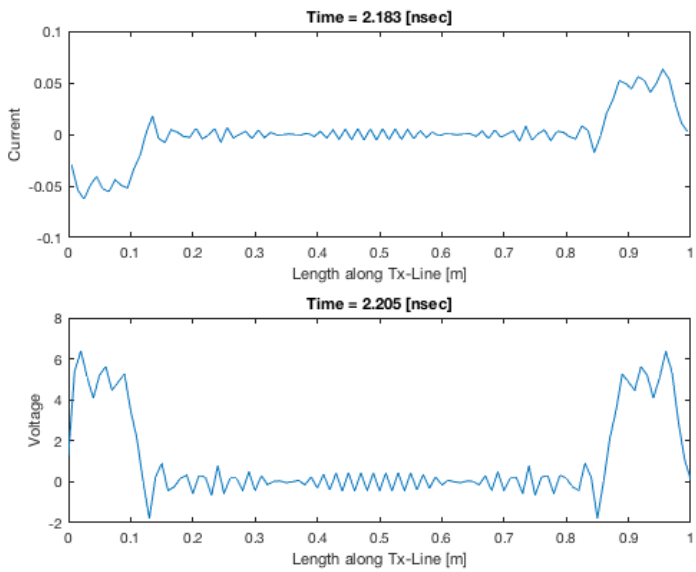
Case 3: For Courant limit, CFL = 1.01, the following graph was obtained at voltage time: 2.2ns



QUESTION D

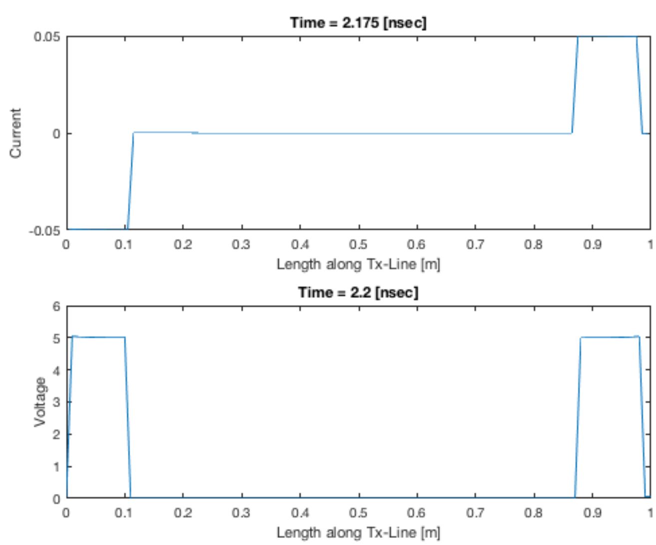
The implementation file for this question is named “**QuestionD.m**”. Like the previous codes written, this code is well documented. The following results were obtained for different Courant values.

Case 1: For Courant limit, CFL = 0.9, the following graph was obtained at voltage time: 2.2ns



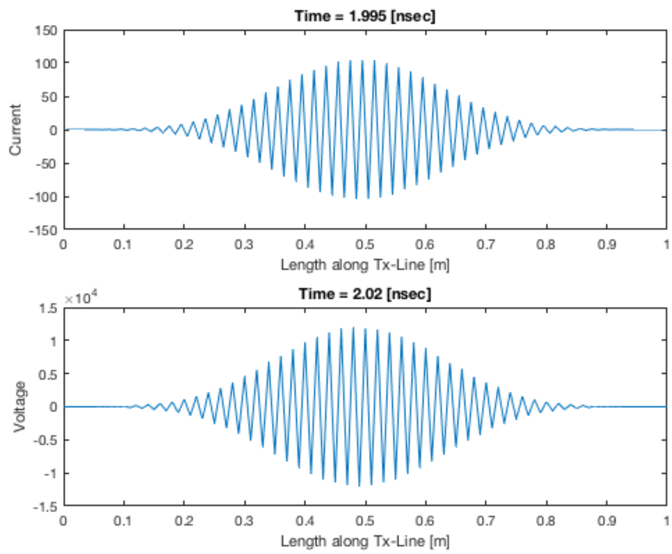
Vt_near = 0.0798 Vt_far = 0.0798

Case 2: For Courant limit, CFL = 1.0, the following graph was obtained at voltage time: 2.2ns



Vt_near = 0.0773 Vt_far = 0.0773

Case 3: For Courant limit, CFL = 1.01, the following graph was obtained at voltage time: 2.2ns



Vt_near = 0.0215 Vt_far = 0.0215

Observation:

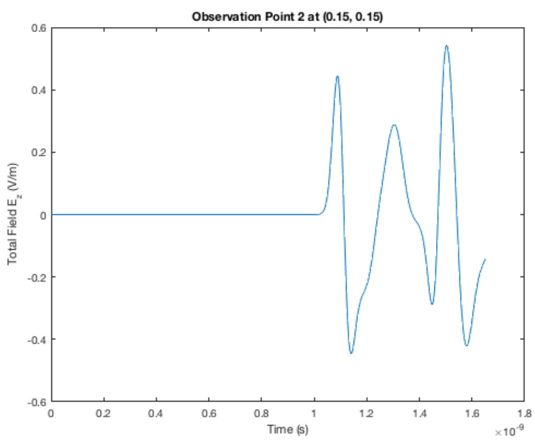
As shown above, when the Courant value was less than 1.0, precisely 0.9, there seems to be partial numerical dispersion; however, when CFL was 1.0 there was stability in the wave. With Courant number greater than 1.0 the dispersion was great in the wave high instability in the wave.

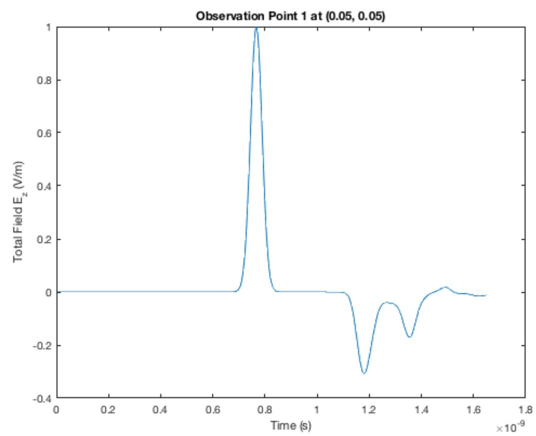
QUESTION E

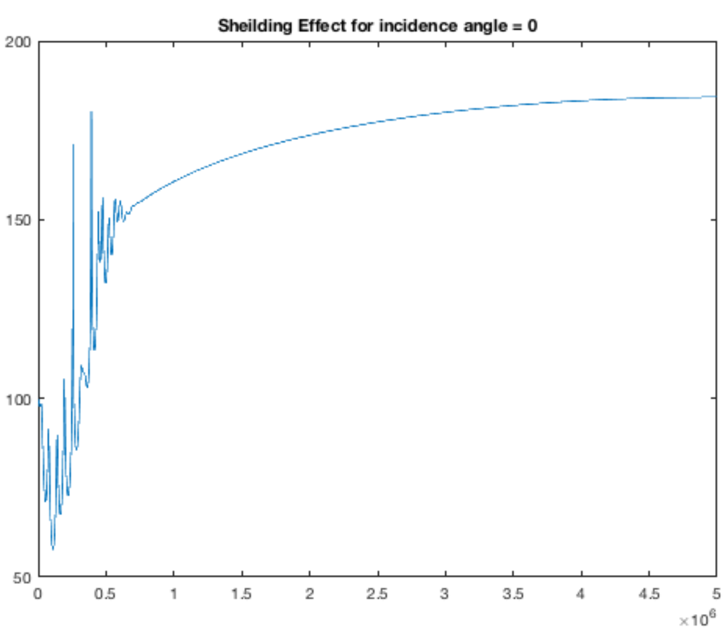
As contained in the question, the shielding was opened as given in the question as the observation points are two: one inside the shield and the other outside the shield.

For the TM case, the code was properly commented to show that the code is well understood. The following graphs were obtained from the two observation points for different incidence angles:

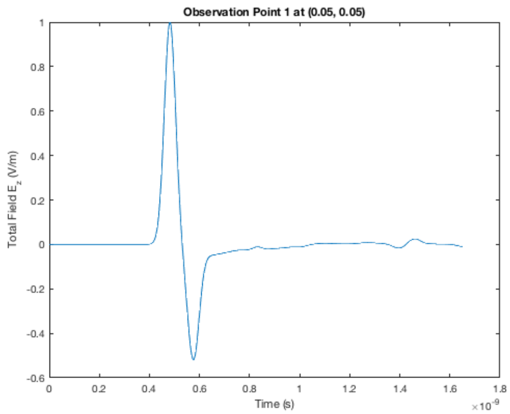
Case 1: For angle of incidence, $\theta = 0$

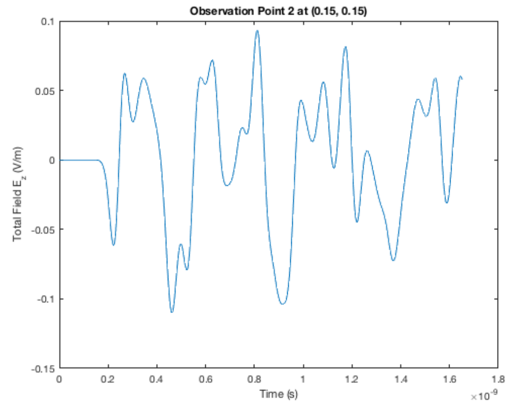


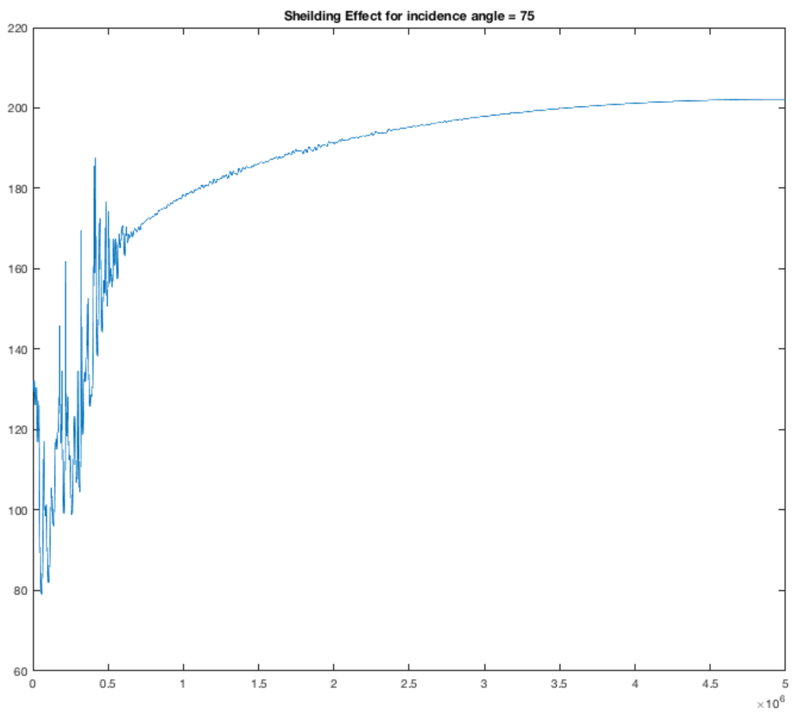




Case 2: For angle of incidence, $\theta = 75$







Case 3: For angle of incidence, $\theta = 180$