

1 Quaternion

Definition (Quaternion). *Let $a, b, c, d \in \mathbb{R}$ and unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be pointing positively along 3 spatial axes such that a **quaternion** q can be written in the form of*

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

*The scalar part of q is denoted by $\text{Re}\{q\} := a$ and the vector part of q is denoted by $\text{Im}\{q\} := b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. The space of quaternion is denoted by \mathbb{H} , called the **Hamiltonian**.*

To type it simple, I may sometimes denote $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ as (a, b, c, d) . For readability, it may sometimes be a column vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

Definition (Quaternion Arithmetic). *Let $p := (p_0, p_1, p_2, p_3), q := (q_0, q_1, q_2, q_3) \in \mathbb{H}$. Then the arithmetic of quaternion is defined by*

- *Addition: $p + q := (p_0 + q_0, p_1 + q_1, p_2 + q_2, p_3 + q_3)$;*
- *Scalar multiplication: $\lambda p := (\lambda p_0, \lambda p_1, \lambda p_2, \lambda p_3)$ for $\lambda \in \mathbb{R}$.*
- *$\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}, \mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}, \mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}, \mathbf{i}\mathbf{j}\mathbf{k} = -1$.*

Proposition (Identity element). *$1 \in \mathbb{H}$ is the only identity element.*

Proof. Suppose the identity element $e \in \mathbb{H}$ is in the form of (e_0, e_1, e_2, e_3) . Then

$$(q_0, q_1, q_2, q_3)(e_0, e_1, e_2, e_3) = \begin{pmatrix} q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 \\ q_0 e_1 + q_1 e_0 + q_2 e_3 - q_3 e_2 \\ q_0 e_2 + q_2 e_0 + q_3 e_1 - q_1 e_3 \\ q_0 e_3 + q_3 e_0 + q_1 e_2 - q_2 e_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Solving equation yields $e_0 = 1$ and $e_1 = e_2 = e_3 = 0$. □

Definition (Modulus). *The modulus of a quaternion $q = (q_0, q_1, q_2, q_3) \in \mathbb{H}$ is defined as*

$$|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Proposition (Conjugation). *For $q = (q_0, q_1, q_2, q_3)$, the algebraic conjugation \bar{q} is*

$$(q_0, -q_1, -q_2, -q_3)$$

Proof. Consider $\bar{q} := (\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$, we have to have

$$(q_0, q_1, q_2, q_3)(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3) = \begin{pmatrix} q_0\bar{q}_0 - q_1\bar{q}_1 - q_2\bar{q}_2 - q_3\bar{q}_3 \\ q_0\bar{q}_1 + q_1\bar{q}_0 + q_2\bar{q}_3 - q_3\bar{q}_2 \\ q_0\bar{q}_2 + q_2\bar{q}_0 + q_3\bar{q}_1 - q_1\bar{q}_3 \\ q_0\bar{q}_3 + q_3\bar{q}_0 + q_1\bar{q}_2 - q_2\bar{q}_1 \end{pmatrix} = \begin{pmatrix} q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which yields $\bar{q}_0 = q_0$, $\bar{q}_1 = -q_1$, $\bar{q}_2 = -q_2$, $\bar{q}_3 = -q_3$. □

2 Rubik's cube with quaternion