Abstract

In this pieces of notes, we will go through the concepts related to functions and the meaning of graphs of the functions.

Contents

1 Function and its graph

'Functions describes the world!', one Professor in Mathematics of Massachusetts Institute of Technology (a.k.a. MIT) said that. His speech was greatly influential, as I have never heard such conclusive thinking about functions. In fact, in the past few years, whenever I was studying in schools, my thought about functions is always only about projecting elements from oen set to another set, but what he said had a big impact to my knowledge about functions.

What function talks about, is a subjection of one elements to one another. It can be thought of as a pointing action started from element A to element B, which not so far away, if we could think of subjecting a lot of, a bunch of, or a list of, whatever, objects from one collection to another collection, and they can all be matched under this pointing action, then it is a so-called function.

For example, in an Indian factory, the production of food undergoes many different process. Those can all be called functions. Let say a raw material comes to the factory first, it then be put to a machine to chop into many small pieces. It is the chopping function inside the factory. Next, the chopped material will be put into a pool of yellowish-brownish liquid and be stirred by dirty hands. It is the Mixing function in the factory. After that, the liquid will be drained on the dirty floor and be stepped on by Indian workers so that they can be smell freshed. It is the flavouring function in the factory. Finally, it will be sold to stores, which is the selling function.

Another example is what our body does. We eat and drink, going down the digestive system, and we sit on a toilet. Although we stupid human knows nothing about how the digestive system works, we could still name the conversion from food to poops a digestion, which means the digestive function representing the process in our body.

So we know that function as an english word represents the naming of a process of conversion, it is the time to explore how Math functions works.

1.1 What is a Function?

A function is defined as follow:

Definition (Function)**.** *Given an input x and an output y, a **function** is a relation between x and y so that we can write $y = f(x)$ to represent the relationship.*

Essential Practice 1.1.1. *Write down functions for the following input-output variables:*

1. *u as input and v as output;*

2. *b as input and a as output;*

3. *n as input and* 1 *as output (which we call it a constant function);*
4. *x^2 as input and y as output;*
5. *xy as input and z as output;*
6. *2^x as input and k as output;*
7. *$\sqrt{p}$ as input and q as output;*

Remark. *It is notable that we may write functions as $y = g(x)$, $y = h(x)$, $y = d(x)$, $\ldots$ as we want. The 'naming' of a function is always definitive and up to user's construction.*

We can also apply functions after functions. To do so, we have to talk about the following:

Definition (Composite functions)**.** *Let f and g be functions such that f takes x as input and y as output, and g takes y as input and z as output. Then we can say there is a function h that takes x as input and z as output. In other words, h is a **composite function** such that $z = h(x) = g(f(x))$.*

Essential Practice 1.1.2. *Let f and g be functions such that $b = f(a), c = g(b)$. Write a function for a as input and c as output.*

1.1.3 Function as an input-output pair

1.1.4 Defining a function with variables

1.1.5 Domain, Codomain and Range

1.2 Graph of a function

1.2.1 Blobs-and-arrows diagram for discrete functions

1.2.2 The pairing table and xy-coordination

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1.4 Solving equations using graphs of functions

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1.5.1 Translation

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2 Linear functions

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2.2.1 Slope of a line

2.2.2 Two-point form

2.2.3 Point-slope form

2.2.4 Slope-intercept form

2.2.5 General form

2.2.6 Special: Intercept-form

2.3 Parallel lines and Perpendicular lines

2.3.1 Point of intersection

2.3.2 Parallel lines

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2.5 Additional content: Point-line distance

By definition, the distance between two points on a $\mathbb{R}^2$ plane is

Definition (Distance between two points). *Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on $\mathbb{R}^2$-plane. The distance between A and B is computed by the formula*

$$\operatorname{dist}(A, B) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Now let $L : ax + by + c = 0$ be a straight line and $P(x_0, y_0)$ be a point on $\mathbb{R}^2$-plane. Note that with the following axiom

Axiom. *The distance between a point P and a line L is defined by the shortest distance between P and a point on L*

$$\operatorname{dist}(P, L) := \inf\{\operatorname{dist}(P, Q) : Q \in L\}$$

we can choose the perpendicular displacement of P from L to define the distance. Using the line Γ perpendicular to L passing through P, we can find such Q by following computation:

$$\begin{cases} L : ax + by + c = 0 \\ \Gamma : bx - ay + (ay_0 - bx_0) = 0 \end{cases} \implies Q(\frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2})$$

Therefore, the distance between P an Q is

$$\begin{aligned} \operatorname{dist}(P, Q) &= \frac{1}{a^2 + b^2}\sqrt{[(a^2 + b^2)x_0 - (b^2x_0 - aby_0 - ac)]^2 + [(a^2 + b^2)y_0 - (a^2y_0 - abx_0 - bc)]^2} \\ &= \frac{1}{a^2 + b^2}\sqrt{(a^2x_0 + aby_0 + ac)^2 + (b^2y_0 + abx_0 + bc)^2} \\ &= \frac{\sqrt{a^2 + b^2}\sqrt{(ax_0 + by_0 + c)^2}}{a^2 + b^2} \\ &= \frac{\sqrt{(ax_0 + by_0 + c)^2}}{\sqrt{a^2 + b^2}} \\ &= \left|\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\right| \end{aligned}$$

2.6 Linear inequalities

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4.2 Divisibility of polynomials

4.2.1 Division algorithm

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5.1 Exponential functions

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5.1.3 Solving equations using exponetiation method

5.2 Logarithmic functions

5.2.1 What is Logarithm?

5.2.2 Law of Logarithmic algebra

5.2.3 Solving equations using Logarithmic method

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6.2.1 General form

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6.3 Geometric sequence

6.3.1 General form

6.3.2 Summation on Geometric sequence

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6.4 Additional content: Arithmetic-Geometric sequence

6.5 Challenging questions