VECTOR AND TENSOR ANALYSIS

Chapter I

ELEMENTARY OPERATIONS

1. *Definitions.* Quantities which have magnitude only are called *scalars*. The following are examples: mass, distance, area, volume. A scalar can be represented by a number with an associated sign, which indicates its magnitude to some convenient scale.

There are quantities which have not only magnitude but also direction. The following are examples: force, displacement of a point, velocity of a point, acceleration of a point. Such quantities are called *vectors* if they obey a certain law of addition set forth in § 2 below. A vector can be represented by an arrow. The direction of the arrow indicates the direction of the vector, and the length of the arrow indicates the magnitude of the vector to some convenient scale.

Let us consider a vector represented by an arrow running from a point P to a point Q, as shown in Figure 1. The straight line through P and Q is called the *line of action* of the vector, the point P is called the *origin* of the vector, and the point Q is called the *terminus* of the vector.

To denote a vector we write the letter indicating its origin followed by the letter indicating its terminus, and place a bar over the two letters. The vector represented in Figure 1 is then represented by the symbols $\overline{PQ}$. In this book the superimposed bar will not be used in any capacity other than the above, and hence its presence can always

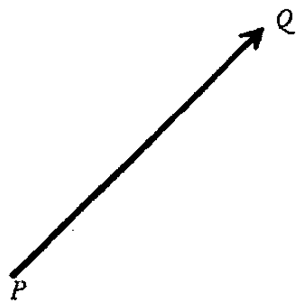

Figure 1

be interpreted as denoting vector character. This notation for vectors is somewhat cumbersome. Hence when convenient we shall use a simpler notation which consists in denoting a vector by a single symbol in bold-faced type. Thus, the vector in Figure 1 might be denoted by the symbol $\mathbf{a}$. In this book no mathematical symbols will be printed in bold-faced type except those denoting vectors.*

The magnitude of a vector is a scalar which is never negative. The magnitude of a vector $\overline{PQ}$ will be denoted by either PQ or $|\overline{PQ}|$. Similarly, the magnitude of a vector $\mathbf{a}$ will be denoted by either a or $|\mathbf{a}|$.

Two vectors are said to be equal if they have the same magnitudes and the same directions. To denote the equality of two vectors the usual sign is employed. Hence, if $\mathbf{a}$ and $\mathbf{b}$ are equal vectors, we write

$$\mathbf{a} = \mathbf{b}.$$

A vector $\mathbf{a}$ is said to be equal to zero if its magnitude a is equal to zero. Thus $\mathbf{a} = 0$ if $a = 0$. Such a vector is called a zero vector.

2. *Addition of vectors.* In § 1 it was stated that vectors are quantities with magnitude and direction, and which obey a certain law of addition. This law, which is called the *law of vector addition*, is as follows.

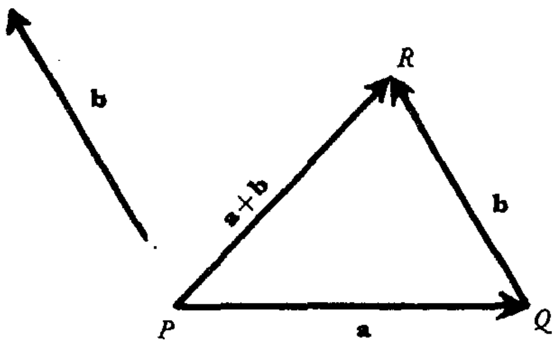

Figure 2

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors, as shown in Figure 2. The origin and terminus of $\mathbf{a}$ are P and Q. A vector equal to $\mathbf{b}$ is constructed with

* It is difficult to write bold-faced symbols on the blackboard or in the exercise book. When it is desired to write a single symbol denoting a vector, the reader will find it convenient to write the symbol in the ordinary manner, and to place a bar over it to indicate vector character.

its origin at Q. Its terminus falls at a point R. The sum $\mathbf{a}+\mathbf{b}$ is the vector $\overline{PR}$, and we write

$$\mathbf{a}+\mathbf{b} = \overline{PR}.$$

Theorem 1. Vectors satisfy the commutative law of addition; that is,

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}.$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the two vectors shown in Figure 2. Then

$$\mathbf{a}+\mathbf{b} = \overline{PR}. \tag{2.1}$$

We now construct a vector equal to $\mathbf{b}$, with its origin at P. Its terminus falls at a point S. A vector equal to $\mathbf{a}$ is then constructed with its origin at S. The terminus of this vector will fall at R, and Figure 3 results. Hence

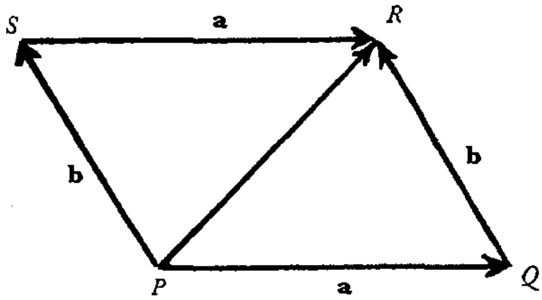

Figure 3

$$\mathbf{b}+\mathbf{a} = \overline{PR}. \tag{2.2}$$

From (2.1) and (2.2) it follows that $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$.

Theorem 2. Vectors satisfy the associative law of addition; that is,

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

Proof. Let us construct the polygon in Figure 4 having the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as consecutive sides. The corners of this polygon are labelled P, Q, R and S. It then appears that

$$\begin{aligned}(\mathbf{a}+\mathbf{b})+\mathbf{c} &= \overline{PR}+\mathbf{c}\\ &= \overline{PS},\\ \mathbf{a}+(\mathbf{b}+\mathbf{c}) &= \mathbf{a}+\overline{QS}\\ &= \overline{PS}.\end{aligned}$$

Hence the theorem is true.

According to Theorem 2 the sum of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is independent of the order in which they are added. Hence we can write $\mathbf{a}+\mathbf{b}+\mathbf{c}$ without ambiguity.

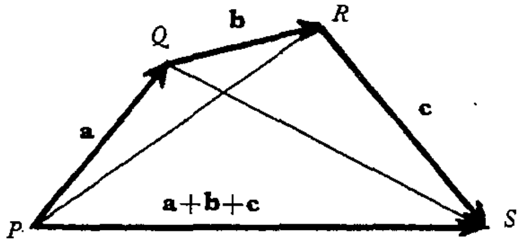

Figure 4

Figure 4 shows the construction of the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$. The sum of a larger number of vectors can be constructed similarly. Thus, to find the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ it is only necessary to construct the polygon having $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ as consecutive sides. The required vector is then the vector with its origin at the origin of $\mathbf{a}$, and its terminus at the terminus of $\mathbf{d}$.

3. *Multiplication of a vector by a scalar.* By definition, if m is a positive scalar and $\mathbf{a}$ is a vector, the expression $m\mathbf{a}$ is a vector with magnitude ma and pointing in the same direction as $\mathbf{a}$; and if m is negative, $m\mathbf{a}$ is a vector with magnitude $|m|\ a$, and pointing in the direction opposite to $\mathbf{a}$.

We note in particular that $-\mathbf{a}$ is a vector with the same magnitude as $\mathbf{a}$ but pointing in the direction opposite to $\mathbf{a}$. Figure 5 shows this vector, and as further examples of the multiplication of a vector by a scalar, the vectors $2\mathbf{a}$ and $-2\mathbf{a}$.

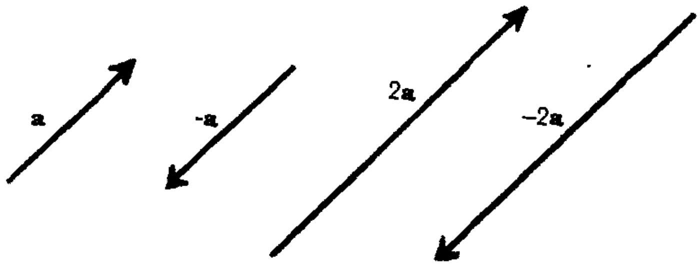

Figure 5

Theorem. The multiplication of a vector by a scalar satisfies the distributive laws; that is, (3.1) $$(m+n)\mathbf{a} = m\mathbf{a}+n\mathbf{a},$$

(3.2) $$m(\mathbf{a}+\mathbf{b}) = m\mathbf{a}+m\mathbf{b}.$$

Proof of (3.1). If $m+n$ is positive, both sides of (3.1) represent a vector with magnitude $(m+n)a$ and pointing in the same direction as $\mathbf{a}$. If $m+n$ is negative, both sides of (3.1) represent a vector with magnitude $|m+n|a$ and pointing in the direction opposite to $\mathbf{a}$.

Proof of (3.2). Let m be positive, and let $\mathbf{a}$, $\mathbf{b}$, $m\mathbf{a}$ and $m\mathbf{b}$ be as shown in Figures 6 and 7. Then

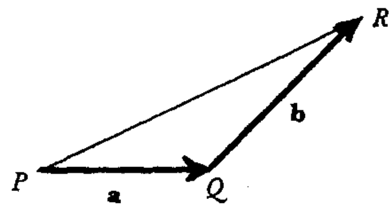

Figure 6

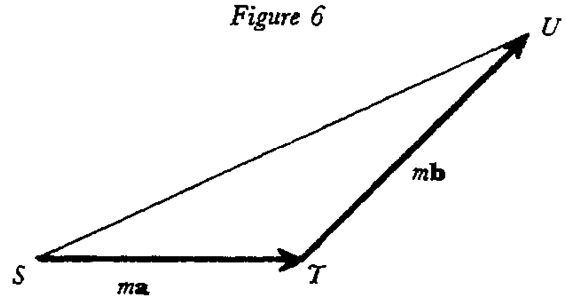

Figure 7

(3.3) $$m(\mathbf{a}+\mathbf{b}) = m\overline{PR}, \qquad m\mathbf{a}+m\mathbf{b} = \overline{SU}.$$

The two triangles PQR and STU are similar. Corresponding sides are then proportional, the constant of proportionality being m. Thus

(3.4) $$mPR=SU.$$

Since $\overline{PR}$ and $\overline{SU}$ have the same directions, and since m is positive, then $m\overline{PR} = \overline{SU}$. Substitution in both sides of this equation from (3.3) yields (3.2).

Now, let m be negative. Then Figure 7 is replaced by Figure 8. Equations (3.3) apply in this case also. The triangles PQR and STU are again similar, but the constant of proportionality is $|m|$, so $|m|PR = SU$. Since $\overline{PR}$ and $\overline{SU}$ have opposite directions and m is negative,

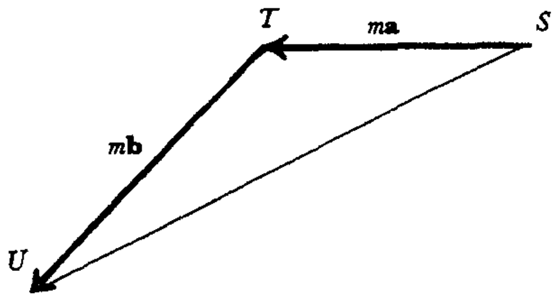

Figure 8

then $m\overline{PR} = \overline{SU}$. Substitution in both sides of this equation from (3.3) again yields (3.2).

4. *Subtraction of vectors.* If **a** and **b** are two vectors, their difference **a** – **b** is defined by the relation

$$\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b}),$$

where the vector –**b** is as defined in the previous section. Figure 9 shows two vectors **a** and **b**, and also their difference **a** – **b**.

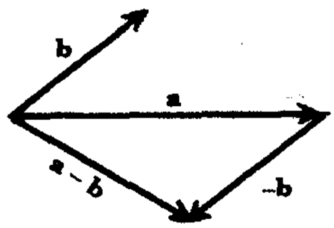

Figure 9

5. *Linear functions.* If **a** and **b** are any two vectors, and m and n are any two scalars, the expression $m\mathbf{a}+n\mathbf{b}$ is called a linear function of **a** and **b**. Similarly, $m\mathbf{a}+n\mathbf{b}+p\mathbf{c}$ is a linear function of **a**, **b**, and **c**. The extension of this to the cases involving more than three vectors follows the obvious lines.

Theorem 1. If **a** and **b** are any two nonparallel vectors in a plane, and if **c** is any third vector in the plane of **a** and **b**, then **c** can be expressed as a linear function of **a** and **b**.

Proof. Since **a** and **b** are not parallel, there exists a parallelogram with **c** as its diagonal and with edges parallel to **a** and **b**. Figure 10 shows this parallelogram. We note from this figure that

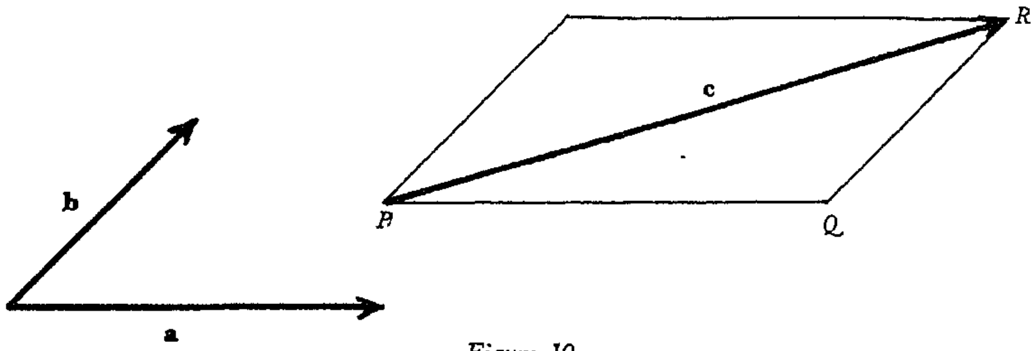

Figure 10

$$\mathbf{c} = \overline{PQ} + \overline{QR}. \tag{5.1}$$

But $\overline{PQ}$ is parallel to $\mathbf{a}$, and $\overline{QR}$ is parallel to $\mathbf{b}$. Thus there exist scalars m and n such that

$$\overline{PQ} = m\mathbf{a}, \qquad \overline{QR} = n\mathbf{b}.$$

Substitution from these relations in (5.1) yields

$$\mathbf{c} = m\mathbf{a} + n\mathbf{b}.$$

Theorem 2. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors not all parallel to a single plane, and if $\mathbf{d}$ is any other vector, then $\mathbf{d}$ can be expressed as a linear function of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Proof. This theorem is the extension of Theorem 1 to space. Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not parallel to a single plane, there exists a parallelepiped with $\mathbf{d}$ as its diagonal and with edges parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Hence there exist scalars m, n and p such that

$$\mathbf{d} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}.$$

6. *Rectangular cartesian coordinates.* In much of the theory and application of vectors it is convenient to introduce a set of rectangular cartesian coordinates. We shall *not* denote these by the usual symbols x, y and z, however, but shall use instead the symbols x_1, x_2 and x_3. These coordinates are said to have "right-handed orientation" or to be "right-handed" if when the thumb of the right hand is made to point in the direction of the positive x_3 axis, the fingers point in the direction of the 90° rotation which carries the positive x_1 axis into coincidence

with the positive x_2 axis. Otherwise the coordinates are "left-handed". In Vector Analysis it is highly desirable to use the same orientation always, for certain basic formulas are changed by a change in orientation. In this book we shall follow the usual practise of using right-handed coordinates throughout. Figure 11 contains the axes of such a set of coordinates.

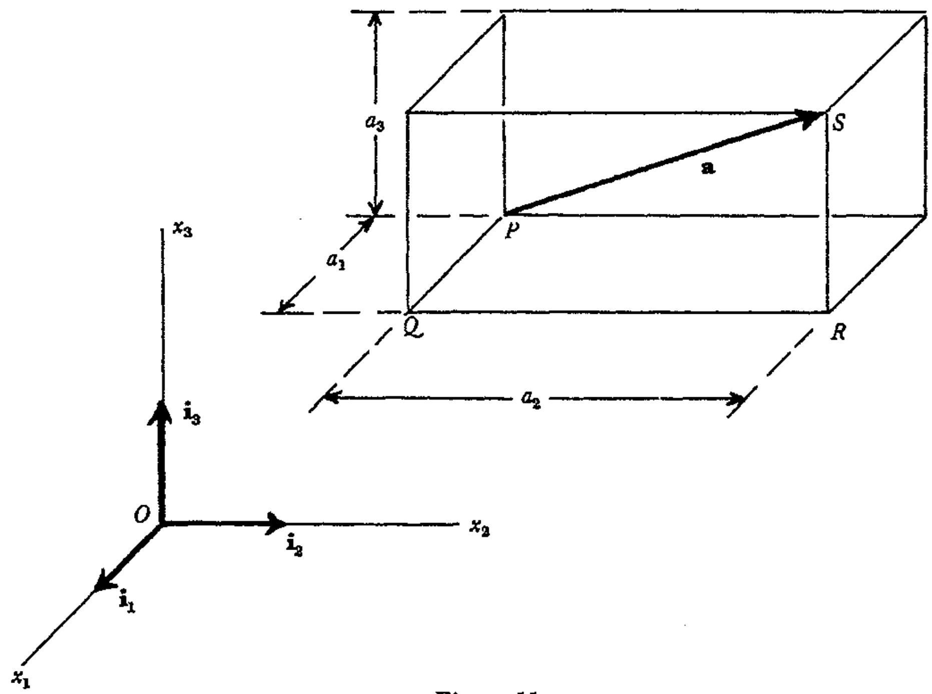

Figure 11

It is also convenient to introduce three vectors of unit magnitude, one pointing in the direction of each of the three positive coordinate axes. These vectors are denoted by $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, and are shown in Figure 11.

Let us consider a vector $\mathbf{a}$. It has orthogonal projections in the directions of the positive coordinate axes. These are denoted by a_1, a_2 and a_3, as shown in Figure 11. They are called the components of $\mathbf{a}$. It should be noted that they can be positive or negative. Thus, for example, a_1 is positive when the angle between $\mathbf{a}$ and the direction of the positive x_1 axis (the angle QPS in the figure) is acute, and is negative when this angle is obtuse.

From Figure 11 it also appears that $\mathbf{a}$ is the diagonal of a rectangular parallelepiped whose edges have lengths $|a_1|$, $|a_2|$ and $|a_3|$. Hence the magnitude a of the vector $\mathbf{a}$ is given by the relation

$$a = \sqrt{a_1^2+a_2^2+a_3^2}. \tag{6.1}$$

From the figure it also appears that

$$\mathbf{a} = \overline{PQ}+\overline{QR}+\overline{RS}. \tag{6.2}$$

Now the vector $\overline{PQ}$ is parallel to $\mathbf{i}_1$. Because of the definitions of a_1 and of the product of a scalar by a vector, we then have the relation $\overline{PQ} = a_1\mathbf{i}_1$. Similarly $\overline{QR} = a_2\mathbf{i}_2$ and $\overline{RS} = a_3\mathbf{i}_3$. Substitution in (6.2) from these relations yields

$$\mathbf{a} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3. \tag{6.3}$$

This relation expresses the vector $\mathbf{a}$ as a linear function of the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$. We note that the coefficients are the components of $\mathbf{a}$.

Theorem. The components of the sum of a number of vectors are equal to the sums of the components of the vectors.

Proof. We consider two vectors $\mathbf{a}$ and $\mathbf{b}$ with components a_1, a_2, a_3, b_1, b_2 and b_3. Then

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3, \\ \mathbf{b} &= b_1\mathbf{i}_1+b_2\mathbf{i}_2+b_3\mathbf{i}_3. \end{aligned}$$

Addition of both sides of these equations leads to the relation

$$\mathbf{a}+\mathbf{b} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3+b_1\mathbf{i}_1+b_2\mathbf{i}_2+b_3\mathbf{i}_3.$$

Now the sum of a number of vectors is independent of the order in which the vectors are added, by Theorem 1 of § 2. Hence we may write the above equation in the form

$$\mathbf{a}+\mathbf{b} = a_1\mathbf{i}_1+b_1\mathbf{i}_1+a_2\mathbf{i}_2+b_2\mathbf{i}_2+a_3\mathbf{i}_3+b_3\mathbf{i}_3.$$

By the theorem in § 3 we may then write this in the form

$$\mathbf{a}+\mathbf{b} = (a_1+b_1)\mathbf{i}_1+(a_2+b_2)\mathbf{i}_2+(a_3+b_3)\mathbf{i}_3.$$

Hence the components of $\mathbf{a}+\mathbf{b}$ are a_1+b_1, a_2+b_2 and a_3+b_3. This proves the theorem when two vectors are added. The proof is similar when more than two vectors are added.

7. *The scalar product.* Let us consider two vectors **a** and **b** with magnitudes a and b, respectively. Let α be the smallest nonnegative angle between **a** and **b**, as shown in Figure 12. Then $0° \leqslant \alpha \leqslant 180°$.

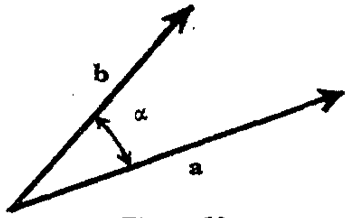

Figure 12

The scalar $ab \cos \alpha$ arises quite frequently, and hence it is convenient to give it a name. It is called the scalar product of **a** and **b**. It is also denoted by the symbols $\mathbf{a} \cdot \mathbf{b}$, and hence we have

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \alpha. \tag{7.1}$$

The scalar product is sometimes referred to as the dot product.

If the components of **a** and **b** are denoted by a_1, a_2, a_3, b_1, b_2 and b_3 in the usual manner, the direction cosines of the directions of **a** and **b** are respectively

$$\frac{a_1}{a}, \frac{a_2}{a}, \frac{a_3}{a}; \quad \frac{b_1}{b}, \frac{b_2}{b}, \frac{b_3}{b}.$$

By a formula of analytic geometry, we then have

$$\cos \alpha = \frac{a_1}{a}\frac{b_1}{b} + \frac{a_2}{a}\frac{b_2}{b} + \frac{a_3}{a}\frac{b_3}{b}.$$

Substitution in (7.1) of this expression for $\cos \alpha$ yields

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{7.2}$$

This relation expresses the scalar product of two vectors in terms of the components of the vectors.

Theorem 1. The scalar product is commutative; that is,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

Proof. Because of (7.2), we have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\ \mathbf{b} \cdot \mathbf{a} &= b_1 a_1 + b_2 a_2 + b_3 a_3. \end{aligned}$$

Since $a_1b_1 = b_1a_1$, etc., the truth of the theorem follows immediately.

Theorem 2. The scalar product is distributive; that is,

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

Proof. According to the theorem in § 6, the components of $\mathbf{b}+\mathbf{c}$ are b_1+c_1, b_2+c_2 and b_3+c_3. Hence, by (7.2) we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3)\\ &= a_1b_1+a_2b_2+a_3b_3+a_1c_1+a_2c_2+a_3c_3\\ &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.\end{aligned}$$

This completes the proof.

If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then

$$\mathbf{a}\cdot\mathbf{b} = 0.$$

However, if it is given that $\mathbf{a}\cdot\mathbf{b} = 0$, it does not necessarily follow that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. It can be said only that at least one of the following must be true: $a = 0$; $b = 0$; $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Similarly, if it is given that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c},$$

it does not necessarily follow that $\mathbf{b} = \mathbf{c}$. For this relation can be written in the form $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = 0$, and hence it can be said only that at least one of the following is true: $a = 0$; $\mathbf{b} = \mathbf{c}$; $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}-\mathbf{c}$.

We note the following expressions, in which $\mathbf{a}$ is any vector and $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the unit vectors introduced in § 6:

$$\mathbf{a}\cdot\mathbf{a} = a^2,$$

$$\begin{array}{lll}\mathbf{i}_1\cdot\mathbf{i}_1 = 1, & \mathbf{i}_1\cdot\mathbf{i}_2 = 0, & \mathbf{i}_1\cdot\mathbf{i}_3 = 0,\\ \mathbf{i}_2\cdot\mathbf{i}_1 = 0, & \mathbf{i}_2\cdot\mathbf{i}_2 = 1, & \mathbf{i}_2\cdot\mathbf{i}_3 = 0,\\ \mathbf{i}_3\cdot\mathbf{i}_1 = 0, & \mathbf{i}_3\cdot\mathbf{i}_2 = 0, & \mathbf{i}_3\cdot\mathbf{i}_3 = 1.\end{array} \tag{7.3}$$

8. *The vector product.* Let us again consider two vectors $\mathbf{a}$ and $\mathbf{b}$, the smallest nonnegative angle between then being denoted by α, as shown in Figure 12. Then $0° \leqslant \alpha \leqslant 180°$. The vector product of $\mathbf{a}$ and $\mathbf{b}$ is a third vector $\mathbf{c}$ defined in terms of $\mathbf{a}$ and $\mathbf{b}$ by the following three conditions:

(i) $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$;

(ii) the direction of $\mathbf{c}$ is that indicated by the thumb of the right hand when the fingers point in the sense of the rotation α from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$;

(iii) $c = ab \sin \alpha$.

These conditions define $\mathbf{c}$ uniquely. Figure 13 shows $\mathbf{c}$. The vector

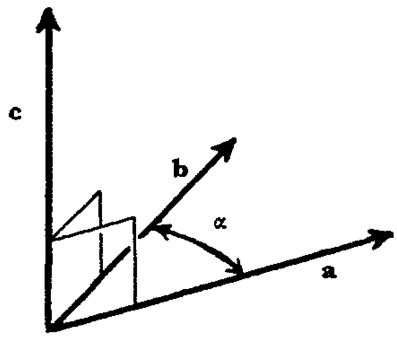

Figure 13

product of $\mathbf{a}$ and $\mathbf{b}$ is also denoted by $\mathbf{a}\times\mathbf{b}$. Hence

(8.1) $$\mathbf{c} = \mathbf{a}\times\mathbf{b}.$$

The vector product is also called the cross product.

Theorem 1. The area A of the parallelogram with the vectors $\mathbf{a}$ and $\mathbf{b}$ forming adjacent edges is given by the relation

(8.2) $$A = |\mathbf{a}\times\mathbf{b}|.$$

Proof. Figure 14 shows the parallelogram. If p is the perpendicular distance from the terminus of $\mathbf{b}$ to the line of action of $\mathbf{a}$, then $A = ap$. But $p = b \sin \alpha$. Hence

$$\begin{aligned} A &= ab \sin \alpha \\ &= |\mathbf{a}\times\mathbf{b}|. \end{aligned}$$

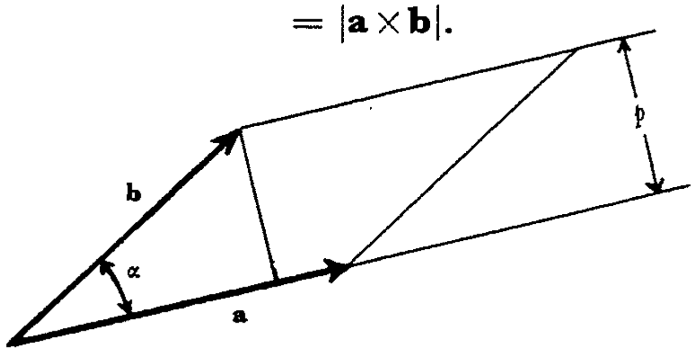

Figure 14

We shall now determine the components of the vector product **c** in (8.1) in terms of the components of **a** and **b**. Because of condition (i) above, we have $\mathbf{a} \cdot \mathbf{c} = 0$, $\mathbf{b} \cdot \mathbf{c} = 0$. Because of (7.2), these equations can take the form

$$\begin{aligned} a_1c_1+a_2c_2+a_3c_3 &= 0, \\ b_1c_1+b_2c_2+b_3c_3 &= 0. \end{aligned}$$

If these equations are solved for c_1 and c_2 in terms of c_3, it is found that

$$\frac{c_1}{a_2b_3-a_3b_2} = \frac{c_2}{a_3b_1-a_1b_3} = \frac{c_3}{a_1b_2-a_2b_1}.$$

In order to preserve symmetry, we denote the common value o these three fractions by K, whence we have

$$\begin{aligned} c_1 &= K(a_2b_3-a_3b_2), \\ c_2 &= K(a_3b_1-a_1b_3), \\ c_3 &= K(a_1b_2-a_2b_1). \end{aligned} \tag{8.3}$$

Now $c^2 = c_1^2+c_2^2+c_3^2$. Hence

$$\begin{aligned} c^2 &= K^2[(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2] \\ &= K^2[a_1^2(b_2^2+b_3^2)+a_2^2(b_3^2+b_1^2)+a_3^2(b_1^2+b_2^2) \\ &\quad -2(a_2b_2a_3b_3+a_3b_3a_1b_1+a_1b_1a_2b_2)]. \end{aligned}$$

The first term inside the square brackets can be written in the form $a_1^2(b^2-b_1^2)$. If the second and third terms are treated similarly it is found that

$$\begin{aligned} c^2 &= K^2[(a_1^2+a_2^2+a_3^2)\ b^2-(a_1b_1+a_2b_2+a_3b_3)^2] \\ &= K^2[a^2b^2-(ab\cos\alpha)^2] \\ &= K^2a^2b^2(1-\cos^2\alpha) \\ &= K^2a^2b^2\sin^2\alpha. \end{aligned}$$

But by condition (iii) above, $c^2 = a^2b^2 \sin^2 \alpha$. Thus $K = \pm 1$. If these two values of K are inserted in (8.3) two vectors **c** result with the same magnitude but pointing in opposite directions. Only one of these vectors satisfies condition (ii) above. Now both values of K are numerical, and are hence independent of **a** and **b**. Thus the same value of K will satisfy condition (ii) for all vectors **a** and **b**. Hence it is only necessary to find K for any one special case in which $\mathbf{c}$ can be found directly and with ease from conditions (i) – (iii) above. If we take $\mathbf{a} = \mathbf{i}_1$ and $\mathbf{b} = \mathbf{i}_2$, it is found from these conditions that $\mathbf{c} = \mathbf{i}_3$. Thus $a_1 = b_2 = c_3 = 1, a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0$, and substitution in (8.3) yields $K = 1$. From (8.3) we then have in general,

$$c_1 = a_2b_3 - a_3b_2,\ c_2 = a_3b_1 - a_1b_3,\ c_3 = a_1b_2 - a_2b_1. \tag{8.4}$$

Thus *the components of the vector product* $\mathbf{a}\times\mathbf{b}$ *are* $a_2b_3 - a_3b_2$, $a_3b_1 - a_1b_3$, $a_1b_2 - a_2b_1$. *Hence*

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i}_1 + (a_3b_1 - a_1b_3)\mathbf{i}_2 + (a_1b_2 - a_2b_1)\mathbf{i}_3, \tag{8.5}$$

or, in determinant form

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{8.6}$$

Theorem 2. The vector product is *not* commutative, because

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}. \tag{8.7}$$

Proof. By (8.6) it follows that

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}.$$

Since this determinant differs from the determinant in (8.6) only in that two rows are interchanged, the two determinants differ only in sign. Hence (8.7) is true. The truth of this theorem can also be seen easily by examining the three conditions which define the vector product. According to these conditions the effect of interchanging the order of $\mathbf{a}$ and $\mathbf{b}$ is only to reverse the direction of the vector product.

Theorem 3. The vector product is distributive; that is,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}. \tag{8.8}$$

Proof. Let us write

$$\mathbf{d} = \mathbf{a}\times(\mathbf{b}+\mathbf{c}),\quad \mathbf{e} = \mathbf{a}\times\mathbf{b},\quad \mathbf{f} = \mathbf{a}\times\mathbf{c}.$$

Then

$$\begin{aligned} d_1 &= a_2(b_3+c_3) - a_3(b_2+c_2) \\ &= (a_2b_3-a_3b_2)+(a_2c_3-a_3c_2) \\ &= e_1+f_1. \end{aligned}$$

Similarly $d_2 = e_2+f_2$ and $d_3 = e_3+f_3$. Hence $\mathbf{d} = \mathbf{e}+\mathbf{f}$, and so (8.8) is true.

We note the following expressions, in which $\mathbf{a}$ is any vector and $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the unit vectors introduced in § 6:

$$\mathbf{a}\times\mathbf{a} = 0,$$

$$(8.9)\qquad \begin{array}{lll} \mathbf{i}_1\times\mathbf{i}_1 = 0, & \mathbf{i}_1\times\mathbf{i}_2 = \mathbf{i}_3, & \mathbf{i}_1\times\mathbf{i}_3 = -\mathbf{i}_2, \\ \mathbf{i}_2\times\mathbf{i}_1 = -\mathbf{i}_3, & \mathbf{i}_2\times\mathbf{i}_2 = 0, & \mathbf{i}_2\times\mathbf{i}_3 = \mathbf{i}_1, \\ \mathbf{i}_3\times\mathbf{i}_1 = \mathbf{i}_2, & \mathbf{i}_3\times\mathbf{i}_2 = -\mathbf{i}_1, & \mathbf{i}_3\times\mathbf{i}_3 = 0. \end{array}$$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then

$$\mathbf{a}\times\mathbf{b} = 0.$$

Also, if it is given that $\mathbf{a}\times\mathbf{b} = 0$, then at least one of the following must be true: $a = 0$; $b = 0$; $\mathbf{a}$ is parallel to $\mathbf{b}$. Similarly, if

$$\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c},$$

then $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = 0$ and at least one of the following must be true:

$$a = 0; \quad \mathbf{b} = \mathbf{c}; \quad \mathbf{a} \text{ is parallel to } \mathbf{b}-\mathbf{c}.$$

9. *Multiple products of vectors.* Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any three vectors. The expression

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

is a scalar, and is called a *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

If the components of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are denoted in the usual way, then the components of $\mathbf{b}\times\mathbf{c}$ are $b_2c_3-b_3c_2$, $b_3c_1-b_1c_3$, $b_1c_2-b_2c_1$, and we have by (7.2)

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1),$$

or

$$(9.1)\qquad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Theorem 1. The permutation theorem for scalar triple products. If the vectors in a scalar triple product are subjected to an odd number of permutations, the value of this product is changed only in sign; and if the number of permutations is even the value of the product is not changed.

Proof. A permutation of the vectors in a scalar triple product is defined as the interchange of any two vectors which appear in the product. From (9.1) it appears that a single permutation produces an interchange of two rows in the determinant. Since such an interchange of rows results in a change of sign only, the truth of the theorem is established.

Because of this theorem we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\\ &= -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}) = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}).\end{aligned}$$

Theorem 2. The volume V of the parallelepiped with the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ forming adjacent edges is given by the relation

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|, \tag{9.2}$$

where the vertical lines here denote the absolute value.

Proof. Figure 15 shows the parallelepiped. Let $\mathbf{d} = \mathbf{b}\times\mathbf{c}$. Then

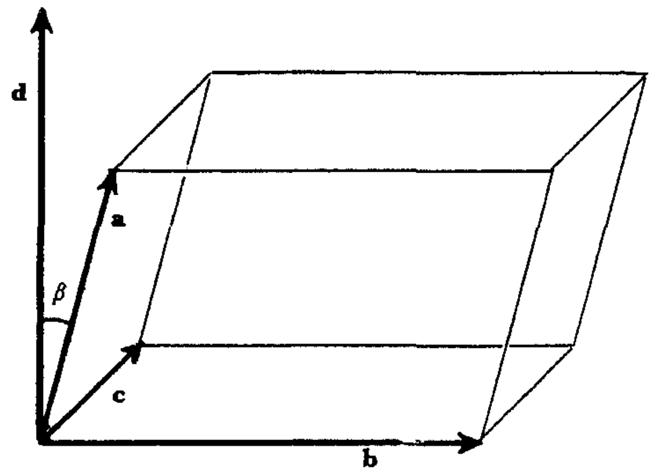

Figure 15

$d = |\mathbf{b}\times\mathbf{c}|$, and by Theorem 1 of § 8 the area of the parallelogram forming the base of the parallelepiped is then d. Hence $V = hd$, where h is the altitude of the parallelepiped. But $\mathbf{d}$ is perpendicular to the base, and if β is the angle between $\mathbf{a}$ and $\mathbf{d}$, then $h = a \mid \cos \beta \mid$. (The absolute value signs are necessary here, since β lies in the range $0° \leqslant \beta \leqslant 180°$ and hence $\cos \beta$ may be negative.) Thus

$$\begin{aligned} V &= \mid ad \cos \beta \mid \\ &= \mid \mathbf{a} \cdot \mathbf{d} \mid \\ &= \mid \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \mid. \end{aligned}$$

The expression

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

is a vector, and is called a *vector triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Let us write

$$\mathbf{d} = \mathbf{b} \times \mathbf{c}, \qquad \mathbf{e} = \mathbf{a} \times \mathbf{d}.$$

Then $\mathbf{e}$ is equal to the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. By (8.5) we have

$$\begin{aligned} e_1 &= a_2 d_3 - a_3 d_2 \\ &= a_2(b_1 c_2 - b_2 c_1) - a_3(b_3 c_1 - b_1 c_3) \\ &= b_1(a_2 c_2 + a_3 c_3) - c_1(a_2 b_2 + a_3 b_3). \end{aligned}$$

Because of (7.2), this can be written in the form

$$\begin{aligned} e_1 &= b_1(\mathbf{a} \cdot \mathbf{c} - a_1 c_1) - c_1(\mathbf{a} \cdot \mathbf{b} - a_1 b_1) \\ &= b_1(\mathbf{a} \cdot \mathbf{c}) - c_1(\mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

Similarly

$$\begin{aligned} e_2 &= b_2(\mathbf{a} \cdot \mathbf{c}) - c_2(\mathbf{a} \cdot \mathbf{b}), \\ e_3 &= b_3(\mathbf{a} \cdot \mathbf{c}) - c_3(\mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

Hence $\mathbf{e} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, and since $\mathbf{e} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{9.3}$$

This is a rather important identity. It will be used frequently.

We note that the right side of (9.3) is a vector in the plane of $\mathbf{b}$ and $\mathbf{c}$. This is to be expected, since the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to the vector $\mathbf{b} \times \mathbf{c}$ which is itself perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$.

Let us now consider the expression

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}).$$

It is a vector. If we regard it as a vector triple product of $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, then by (9.3),

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\,[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]-\mathbf{d}\,[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]. \tag{9.4}$$

Since an interchange of the order of the vectors in a vector product only changes the sign,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = -(\mathbf{c}\times\mathbf{d})\times(\mathbf{a}\times\mathbf{b}).$$

If we regard the right side of this equation as the vector triple product of $\mathbf{c} \times \mathbf{d}$, $\mathbf{a}$ and $\mathbf{b}$, then by (9.3),

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = -\mathbf{a}\,[(\mathbf{c}\times\mathbf{d})\cdot\mathbf{b}]+\mathbf{b}\,[(\mathbf{c}\times\mathbf{d})\cdot\mathbf{a}]. \tag{9.5}$$

We next consider the expression

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}).$$

It is a scalar. If we consider it as the scalar triple product of $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, and subject these three vectors to two permutations, then according to Theorem 1 of § 9, we have

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot[\mathbf{d}\times(\mathbf{a}\times\mathbf{b})].$$

If the vector triple product on the right-hand side of this equation is expanded by the identity in (9.3), we obtain

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{c}\cdot\mathbf{a})(\mathbf{d}\cdot\mathbf{b})-(\mathbf{c}\cdot\mathbf{b})(\mathbf{d}\cdot\mathbf{a}),$$

or in a form more easily recalled,

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}). \tag{9.6}$$

There are many other multiple products of vectors. In general these can be simplified by means of the theorems and formulas above. In dealing with multiple products of vectors care must be exercised to avoid writing down expressions which are ambiguous or have not been defined. Thus, for example, the expression $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ is ambiguous since $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, and the following expressions have not been defined:

$$\mathbf{ab},\quad \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}),\quad \mathbf{a}\times(\mathbf{b}\cdot\mathbf{c}).$$

10. *Moment of a vector about a point.* Let $\mathbf{a}$ be a vector with origin at

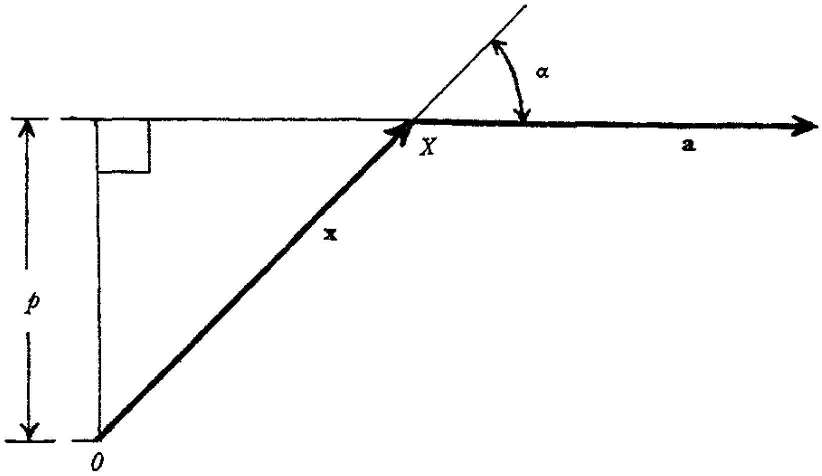

Figure 16

a point X, and let $\mathbf{x}$ be a vector with origin at a point 0 and terminus at X, as shown in Figure 16. The moment of $\mathbf{a}$ about the point O is by definition the vector $\mathbf{P}$ given by the relation

$$\mathbf{P} = \mathbf{x} \times \mathbf{a}.$$

Theorem 1. If $\mathbf{P}$ is the moment of $\mathbf{a}$ about a point 0, then

$$P = pa,$$

where p is the perpendicular distance from 0 to the line of action of $\mathbf{a}$.

Proof. Now $P = xa \sin \alpha$, where α is the angle between $\mathbf{x}$ and $\mathbf{a}$. But $p = x \sin \alpha$. Hence $P = pa$.

Theorem 2. The moments about a point of any two equal vectors with the same line of action are equal.

Proof. Let $\mathbf{a}$ and $\mathbf{a}'$ be two equal vectors with the same line of ac-

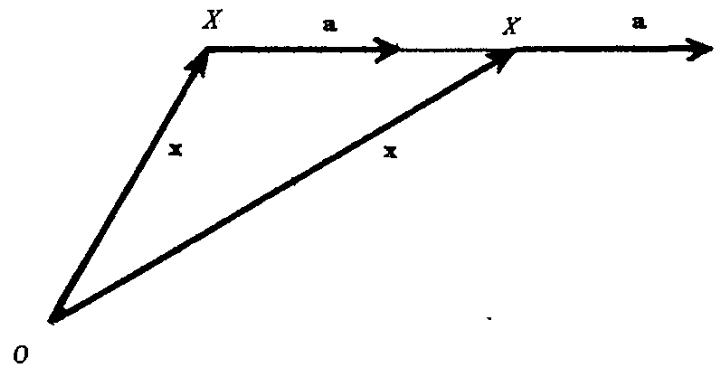

Figure 17

tion, and with origins X and X', as shown in Figure 17. Let O be any point, and let $\overline{OX} = \mathbf{x}$, $\overline{OX}' = \mathbf{x}'$. Let $\mathbf{P}$ and $\mathbf{P}'$ be the moments of $\mathbf{a}$ and $\mathbf{a}'$ about O. Then

$$\mathbf{P} = \mathbf{x} \times \mathbf{a}\,, \quad \mathbf{P}' = \mathbf{x}' \times \mathbf{a}'\,.$$

But $\mathbf{x}' = \mathbf{x} + \overline{XX}'$. Hence

$$\begin{aligned} \mathbf{P}' &= (\mathbf{x} + \overline{XX}') \times \mathbf{a}' \\ &= \mathbf{x} \times \mathbf{a}' + \overline{XX}' \times \mathbf{a}'. \end{aligned}$$

Since $\overline{XX}'$ is parallel to $\mathbf{a}'$, $\overline{XX}' \times \mathbf{a}' = 0$. Hence, since $\mathbf{a}' = \mathbf{a}$ we have finally

$$\mathbf{P}' = \mathbf{x} \times \mathbf{a} = \mathbf{P}.$$

11. *Moment of a vector about a directed line.* Each line defines two directions which are opposite. A line is said to be directed when one of these directions is labelled the positive direction and the other the negative direction.

Let us consider a directed line L, and let $\mathbf{b}$ denote a unit vector pointing in the positive direction of the line, as shown in Figure 18. We

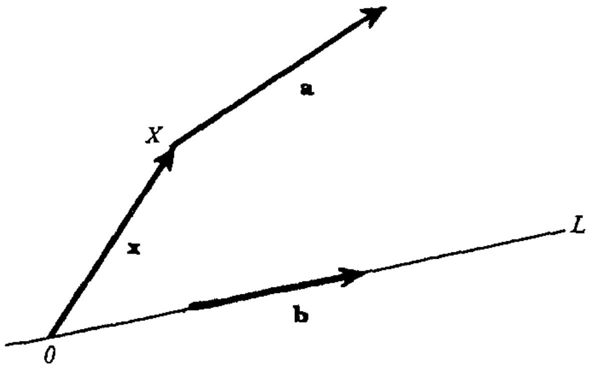

Figure 18

also introduce a point O on L and a vector $\mathbf{a}$ with origin at a point X. If $\mathbf{P}$ denotes the moment of $\mathbf{a}$ about O, the moment of $\mathbf{a}$ about L is by definition the orthogonal projection of $\mathbf{P}$ on L. It is a scalar, and if it is denoted by Q we have

$$Q = P \cos \Phi,$$

where Φ is the angle between $\mathbf{P}$ and the unit vector $\mathbf{b}$. Hence $Q = \mathbf{b}\cdot\mathbf{P}$. But $\mathbf{P} = \mathbf{x}\times\mathbf{a}$, where $\mathbf{x} = \overline{OR}$. Thus

$$Q = \mathbf{b}\cdot(\mathbf{x}\times\mathbf{a}). \tag{11.1}$$

Theorem 1. The above definition of the moment of a vector about a directed line L is independent of the position of the point O on L.

Proof. Let O' be a second point on L, as shown in Figure 19. Also,

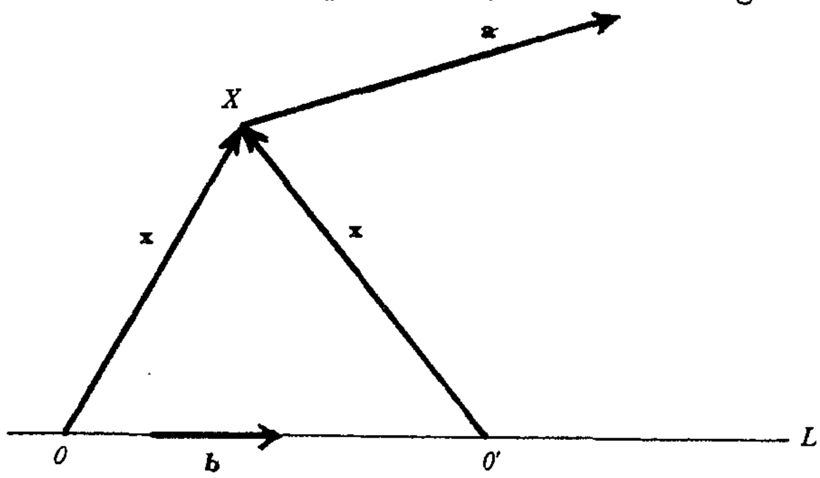

Figure 19

let $\mathbf{P}'$ be the moment of $\mathbf{a}$ about O', and let Q' be the corresponding moment of $\mathbf{a}$ about L. Then

$$Q = \mathbf{b}\cdot(\mathbf{x}\times\mathbf{a})\,, \quad Q' = \mathbf{b}\cdot(\mathbf{x}'\times\mathbf{a})\,,$$

where $\mathbf{x}'$ is as shown. But $\mathbf{x}' = \overline{O''O}+\mathbf{x}$. Thus

$$\begin{aligned} Q' &= \mathbf{b}\cdot[(\overline{O''O}+\mathbf{x})\times\mathbf{a}] \\ &= \mathbf{b}\cdot(\overline{O''O}\times\mathbf{a})+Q. \end{aligned}$$

Since $\mathbf{b}$ and $\overline{O''O}$ have the same line of action L, then $\mathbf{b}\cdot(\overline{O''O}\times\mathbf{x}) = 0$ by Theorem 2 of § 9. Thus $Q' = Q$, which proves the theorem.

Theorem 2. If $\mathbf{P}$ denotes the moment of a vector $\mathbf{a}$ about the origin of the coordinates, then the three components of $\mathbf{P}$ are equal respectively to the moments of $\mathbf{a}$ about the three coordinate axes.

Proof. The truth of this theorem follows immediately from the above definitions of the moments of a vector about a point and about a line.

12. *Differentiation with respect to a scalar variable.* Let u be a scalar variable. If there is a value of a vector $\mathbf{a}$ corresponding to each value of the scalar u, $\mathbf{a}$ is said to be a function of u. When it is desired to indicate such a correspondence, we write $\mathbf{a}(u)$.

Let us consider a general value of the scalar u and the corresponding vector $\mathbf{a}(u)$. Let the vector $\overline{OP}$ in Figure 20 denote this vector. We

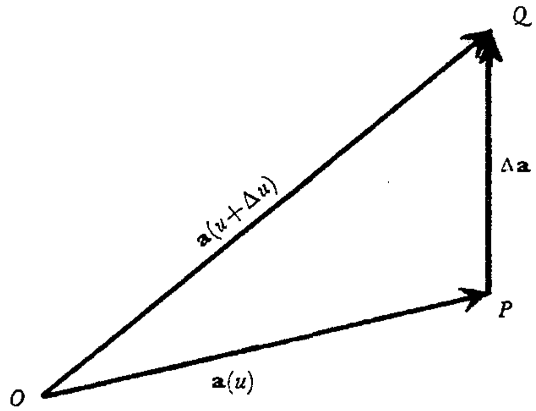

Figure 20

now increase the scalar u by an amount Δu. The vector corresponding to the scalar $u+\Delta u$ is $\mathbf{a}(u+\Delta u)$. Let the vector $\overline{OQ}$ in Figure 20 denote this vector. The change in $\mathbf{a}(u)$ corresponding to the change Δu in u is then $\mathbf{a}(u+\Delta u)-\mathbf{a}(u)$. In the usual notation of calculus we denote it by $\Delta\mathbf{a}$, so that

$$\Delta\mathbf{a} = \mathbf{a}(u+\Delta u) - \mathbf{a}(u).$$

From the figure it is seen that $\Delta\mathbf{a} = \overline{PQ}$. Since Δu is a scalar, the vector $\dfrac{\Delta\mathbf{a}}{\Delta u}$ has the same direction as $\overline{PQ}$. The vector

$$\operatorname*{limit}_{\Delta u \to 0} \frac{\Delta\mathbf{a}}{\Delta u}$$

is the rate of change of $\mathbf{a}$ with respect to u. It is also called the derivative of $\mathbf{a}$ with respect to u, and is denoted by the symbol $\dfrac{d\mathbf{a}}{du}$, so that

$$\frac{d\mathbf{a}}{du} = \operatorname*{limit}_{\Delta u \to 0} \frac{\Delta\mathbf{a}}{\Delta u}.$$

In precisely the same way, we define the derivative with respect to u of the vector $\frac{d\mathbf{a}}{du}$. This vector is denoted by

$$\frac{d}{du}\left(\frac{d\mathbf{a}}{du}\right) \text{ or } \frac{d^2\mathbf{a}}{du^2}.$$

Higher derivatives of $\mathbf{a}$ with respect to u are defined similarly.

Let $\mathbf{a}(u)$ and $\mathbf{b}(u)$ be any two vectors which are functions of a scalar u, and let m be a scalar function of u. We shall now derive the following formulas:

$$\frac{d}{du}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{du}+\frac{d\mathbf{b}}{du}, \tag{12.1}$$

$$\frac{d}{du}(m\mathbf{a}) = m\frac{d\mathbf{a}}{du}+\frac{dm}{du}\mathbf{a}, \tag{12.2}$$

$$\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{du}+\frac{d\mathbf{a}}{du}\cdot\mathbf{b}, \tag{12.3}$$

$$\frac{d}{du}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{du}+\frac{d\mathbf{a}}{du}\times\mathbf{b}. \tag{12.4}$$

Proof of (12.1). When u increases by an amount Δu, the change in the sum $\mathbf{a}+\mathbf{b}$ is

$$\Delta(\mathbf{a}+\mathbf{b}) = (\mathbf{a}+\Delta\mathbf{a}+\mathbf{b}+\Delta\mathbf{b})-(\mathbf{a}+\mathbf{b}). \tag{12.5}$$

According to Theorem 2 of § 2 the sum of a number of vectors is independent of the order of summation. Thus the right side of (12.5) can be written in the form $\mathbf{a}-\mathbf{a}+\mathbf{b}-\mathbf{b}+\Delta\mathbf{a}+\Delta\mathbf{b}$, which reduces to $\Delta\mathbf{a}+\Delta\mathbf{b}$. Thus

$$\Delta(\mathbf{a}+\mathbf{b}) = \Delta\mathbf{a}+\Delta\mathbf{b}.$$

If both sides of this equation are divided by Δu, and if Δu is then made to approach zero, (12.1) is obtained.

Proof of (12.2). When u increases by an amount Δu, the change in $m\mathbf{a}$ is

$$\Delta(m\mathbf{a}) = (m+\Delta m)(\mathbf{a}+\Delta\mathbf{a})-m\mathbf{a}. \tag{12.6}$$

According to the theorem in § 3 the multiplication of a vector by a scalar satisfies the distributive laws, as exemplified by Equations (3.1) and (3.2). Because of the law exemplified by (3.1) we can then write (12.6) in the form

$$(12.7) \qquad \Delta(m\mathbf{a}) = m(\mathbf{a}+\Delta\mathbf{a})+\Delta m(\mathbf{a}+\Delta\mathbf{a}) - m\mathbf{a},$$

and because of the law exemplified by (3.2), we can then write (12.7) in the form

$$\begin{aligned}\Delta(m\mathbf{a}) &= m\mathbf{a}+m\Delta\mathbf{a}+\Delta m\ \mathbf{a}+\Delta m\ \Delta\mathbf{a} - m\mathbf{a}\\ &= m\ \Delta\mathbf{a}+\Delta m\ \mathbf{a}+\Delta m\ \Delta\mathbf{a}.\end{aligned}$$

If both sides of this equation are divided by Δu, and if Δu is then made to approach zero, (12.2) results.

Proof of (12.3). When u increases by an amount Δu, the change in $\mathbf{a}\cdot\mathbf{b}$ is

$$\Delta(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}+\Delta\mathbf{a})\cdot(\mathbf{b}+\Delta\mathbf{b}) - \mathbf{a}\cdot\mathbf{b}.$$

Since the scalar product is distributive, this equation may be written in the form

$$\begin{aligned}\Delta(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\Delta\mathbf{b}+\Delta\mathbf{a}\cdot\mathbf{b}+\Delta\mathbf{a}\cdot\Delta\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\Delta\mathbf{b}+\Delta\mathbf{a}\cdot\mathbf{b}+\Delta\mathbf{a}\cdot\Delta\mathbf{b}.\end{aligned}$$

If both sides of this equation are divided by the scalar Δu, we have

$$\frac{\Delta(\mathbf{a}\cdot\mathbf{b})}{\Delta u} = \mathbf{a}\cdot\frac{\Delta\mathbf{b}}{\Delta u}+\frac{\Delta\mathbf{a}}{\Delta u}\cdot\mathbf{b}+\frac{\Delta\mathbf{a}}{\Delta u}\cdot\Delta\mathbf{b}.$$

If we now let Δu approach zero, (12.3) results.

Proof of (12.4). This proof follows exactly the same pattern as the proof of (12.3), and hence will not be given here.

It is important to note that in (12.4) the order in which the vectors $\mathbf{a}$ and $\mathbf{b}$ appear must be the same in all terms, since $\mathbf{a}\times\mathbf{b}$ is not equal to $\mathbf{b}\times\mathbf{a}$.

If $\mathbf{a}(u)$ is a vector with components a_1, a_2 and a_3, then

$$\mathbf{a} = a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3.$$

By (12.1) and (12.2) we then have

$$\begin{aligned}\frac{d\mathbf{a}}{du} &= \frac{d}{du}(a_1\mathbf{i}_1)+\frac{d}{du}(a_2\mathbf{i}_2)+\frac{d}{du}(a_3\mathbf{i}_3)\\ &= a_1\frac{d\mathbf{i}_1}{du}+a_2\frac{d\mathbf{i}_2}{du}+a_3\frac{d\mathbf{i}_3}{du}+\frac{da_1}{du}\mathbf{i}_1+\frac{da_2}{du}\mathbf{i}_2+\frac{da_3}{du}\mathbf{i}_3.\end{aligned}$$

Now a_1, a_2 and a_3 are scalar functions of u. Also $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are unit vectors pointing in the directions of the positive coordinate axis. If they are the same for all values of u, then

$$\frac{d\mathbf{i}_1}{du} = \frac{d\mathbf{i}_2}{du} = \frac{d\mathbf{i}_3}{du} = 0,$$

and so

$$\frac{d\mathbf{a}}{du} = \frac{da_1}{du}\mathbf{i}_1 + \frac{da_2}{du}\mathbf{i}_2 + \frac{da_3}{du}\mathbf{i}_3.$$

From this equation we see that the components of the derivative of a vector are equal to the derivatives of the components, provided the directions of the coordinate axes are independent of the variable of differentiation.

13. *Integration with respect to a scalar variable.* Let $\mathbf{a}$ be a given function of a scalar u. We introduce orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ with directions independent of u. Then

$$\mathbf{a}(u) = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3.$$

We make the definition

$$\int \mathbf{a}(u)\, du = \mathbf{i}_1 \int a_1(u)\, du + \mathbf{i}_2 \int a_2(u)\, du + \mathbf{i}_3 \int a_3(u)\, du. \tag{13.1}$$

From this definition it follows that

$$\frac{d}{du}\int \mathbf{a}(u)\, du = \mathbf{i}_1\, a_1(u) + \mathbf{i}_2\, a_2(u) + \mathbf{i}_3\, a_3(u) = \mathbf{a},$$

as expected, since integration is the inverse of differentiation. It is to be noted that each integral on the right side of (13.1) gives rise to a constant of integration.

Theorem. If $\mathbf{a}(u)$ is a linear function of constant vectors, with coefficients which are functions of u, then $\int \mathbf{a}(u)\, du$ can be obtained by formal integration in which constant vectors are treated as are constants in ordinary integration, and arbitrary constant vectors are inserted where arbitrary constants would appear in ordinary integration.

Proof. We have

$$\mathbf{a}(u) = \mathbf{p} f(u) + \mathbf{q}\, g(u) + \cdots,$$

where $\mathbf{p}$, $\mathbf{q}$, $\cdots$ are constant vectors and $f(u)$, $g(u)$, $\cdots$ are given functions of u. By (13.1) it then follows that

$$\begin{aligned}\int \mathbf{a}(u)\, du &= \mathbf{i}_1 \left[p_1 \int f(u)\, du + q_1 \int g(u)\, du + \cdots \right] \\ &\quad + \mathbf{i}_2 \left[p_2 \int f(u)\, du + \cdots \right] + \mathbf{i}_3 \left[p_3 \int f(u)\, du + \cdots \right] \\ &= \mathbf{p} \int f(u)\, du + \mathbf{q} \int g(u)\, du + \cdots .\end{aligned}$$

If k, l, $\cdots$ denote the integration constants of the integrals in the last line, then the total contribution of these constants to $\int \mathbf{a}(u)\, du$ is the single arbitrary constant vector $\mathbf{c}$ such that

$$\mathbf{c} = \mathbf{p}k + \mathbf{q}l + \cdots .$$

The following examples illustrate the above theorem:

$$\begin{aligned}\int (\mathbf{p}u + \mathbf{q})\, du &= \tfrac{1}{2}\mathbf{p}u^2 + \mathbf{q}u + \mathbf{c}, \\ \int \mathbf{p} \cos u\, du &= \mathbf{p} \sin u + \mathbf{c}.\end{aligned}$$

14. *Linear vector differential equations.* The equation

$$\left(p_0 \frac{d^n}{du^n} + p_1 \frac{d^{n-1}}{du^{n-1}} + \cdots + p_{n-1} \frac{d}{du} + p_n \right) \mathbf{x} = \mathbf{a}, \tag{14.1}$$

in which $\mathbf{a}$ and $p_0, p_1, \cdots, p_n$ are given functions of the scalar u and $\mathbf{x}$ is an unknown vector, is a linear vector differential equation of order n. Let F denote the differential operator in (14.1). Then (14.1) can be expressed in the form

$$F[\mathbf{x}] = \mathbf{a}. \tag{14.2}$$

Theorem. The general solution of the linear vector differential equation $F[\mathbf{x}] = \mathbf{a}$ is $\mathbf{x} = \mathbf{Y} + \mathbf{A}$, where $\mathbf{A}$ is a particular solution of this differential equation, and

$$\mathbf{Y} = \mathbf{c}_1 y_1 + \mathbf{c}_2 y_2 + \mathbf{c}_3 y_3 + \cdots + \mathbf{c}_n y_n,$$

$\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \cdots, \mathbf{c}_n$ *being arbitrary constant vectors and* $y_1, y_2, y_3, \cdots, y_n$ *being* n *linearly independent solutions* of the homogeneous scalar differential equation $F[y] = 0$.

Proof. Let us introduce the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ with directions independent of u. Then

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3, \\ \mathbf{x} &= x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3, \\ F[\mathbf{x}] &= F[x_1]\mathbf{i}_1 + F[x_2]\mathbf{i}_2 + F[x_3]\mathbf{i}_3. \end{aligned}$$

Hence, from (14.2) we have

$$(14.3) \qquad F[x_1] = a_1, \quad F[x_2] = a_2, \quad F[x_3] = a_3.$$

Let A_1, A_2 and A_3 denote particular solutions of these three scalar differential equations, and let $y_1, y_2, \cdots, y_n$ denote u linearly independent particular solutions of the scalar differential equation $F[y] = 0$. Then the general solutions of Equations (14.3) are

$$\begin{aligned} x_1 &= c_{11}y_1 + c_{12}\,y_2 + c_{13}y_3 + \cdots + c_{1n}y_n + A_1, \\ x_2 &= c_{21}y_1 + c_{22}y_2 + c_{23}y_3 + \cdots + c_{2n}y_n + A_2, \\ x_3 &= c_{31}y_1 + c_{32}y_2 + c_{33}y_3 + \cdots + c_{3n}y_n + A_3, \end{aligned}$$

where the c's are arbitrary constants. Let us multiply these three equations by $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively, and then add. The result can be written in the form

$$(14.4) \qquad \mathbf{x} = \mathbf{Y} + \mathbf{A},$$

where

$$\begin{aligned} \mathbf{Y} &= \mathbf{c}_1y_1 + \mathbf{c}_2y_2 + \mathbf{c}_3y_3 + \cdots + \mathbf{c}_ny_n, \\ \mathbf{A} &= A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3, \end{aligned}$$

the vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\cdots$, $\mathbf{c}_n$ being arbitrary constant vectors. Equation (14.4) gives the general solution of Equation (14.1). We note that $\mathbf{Y}$ is the general solution of the homogeneous equation $F[\mathbf{x}] = 0$, and that $\mathbf{A}$ is a particular solution of Equation (14.1). The particular solution $\mathbf{A}$ can be found by procedures very similar to those used to find particular solutions of linear scalar differential equations. This is demonstrated below.

As an example, let us find the general solution of the differential equation

$$\frac{d^2\mathbf{x}}{du^2}-\frac{d\mathbf{x}}{du}-2\mathbf{x} = 10\mathbf{p}\sin u+\mathbf{q}(2u+1), \tag{14.5}$$

where $\mathbf{p}$ and $\mathbf{q}$ are constant vectors. We must first find two linearly independent solutions of the equation

$$\frac{d^2y}{du^2}-\frac{dy}{du}-2y = 0. \tag{14.6}$$

The auxiliary equation of this differential equation is

$$m^2-m-2 = 0.$$

It has roots -1, 2, whence the required solutions of (14.6) are e^{-u} and e^{2u}. Thus

$$\mathbf{Y} = \mathbf{c}_1e^{-u}+\mathbf{c}_2e^{2u}.$$

We now use the method of undetermined coefficients to find a particular solution $\mathbf{A}$ of Equation (14.5). The function on the right side of (14.5), and the derivatives of this function, are linear functions of $\sin u$, $\cos u$, u, 1, none of which are particular solutions of (14.6). Hence, we look for a particular solution $\mathbf{A}$ in the form

$$\mathbf{A} = \mathbf{b}\sin u+\mathbf{c}\cos u+\mathbf{d}u+\mathbf{e},$$

where $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ are constant vectors. By substitution in (14.5) we readily find by equating coefficients that

$$3\mathbf{b}-\mathbf{c} = -10\mathbf{p}, \qquad \mathbf{b}+3\mathbf{c} = 0,$$
$$\mathbf{d} = -\mathbf{q}, \qquad \mathbf{d}+2\mathbf{e} = -\mathbf{q}.$$

Solving these four equations for $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, we find that

$$\mathbf{b} = -3\mathbf{p}, \qquad \mathbf{c} = \mathbf{p}, \qquad \mathbf{d} = -\mathbf{q}, \qquad \mathbf{e} = 0.$$

The general solution of Equation (14.5) is then

$$\mathbf{x} = \mathbf{c}_1e^{-u}+\mathbf{c}_2e^{2u}+\mathbf{p}(-3\sin u+\cos u)-\mathbf{q}u.$$

Problems

1. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ all lie in a horizontal plane. Their

magnitudes are 1, 2, 3 and 2, and their directions are east, northeast, north and northwest, respectively. Construct these vectors.

2. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are defined as in Problem 1, construct the vectors $(\mathbf{a}+\mathbf{b})+\mathbf{c}$, $(\mathbf{b}+\mathbf{a})+\mathbf{c}$, $\mathbf{c}+(\mathbf{a}+\mathbf{b})$, and by measuring their magnitudes and directions verify that they are equal.

3. If $\mathbf{a}$ and $\mathbf{b}$ are defined as in Problem 1, contruct the vectors $\mathbf{a}+2\mathbf{b}$, $2\mathbf{a}+\mathbf{b}$, $3\mathbf{a}-\mathbf{b}$, $-2\mathbf{a}-2\mathbf{b}$.

4. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are defined as in Problem 1, express each of these vectors as a linear function of the other two, determining the coefficients graphically to two decimal places in each case.

5. Given that

$$\mathbf{a}+2\mathbf{b} = \mathbf{m}, \qquad 2\mathbf{a}-\mathbf{b} = \mathbf{n},$$

where $\mathbf{m}$ and $\mathbf{n}$ are known vectors, solve for $\mathbf{a}$ and $\mathbf{b}$.

6. If $\mathbf{a}$ and $\mathbf{b}$ are vectors with a common origin 0 and terminuses A and B, in terms of $\mathbf{a}$ and $\mathbf{b}$ find the vector $\overline{OC}$, where C is the middle point of AB.

7. The vectors $\mathbf{a}$ and $\mathbf{b}$ form consecutive sides of a regular hexagon, the terminus of $\mathbf{a}$ coinciding with the origin of $\mathbf{b}$. In terms of $\mathbf{a}$ and $\mathbf{b}$ find the vectors forming the other four sides.

8. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ have a common origin and terminuses A, B, C and D, and if $\mathbf{b}-\mathbf{a} = \mathbf{c}-\mathbf{d}$, show that $ABCD$ is a parallelogram.

9. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have a common origin and form adjacent edges of a parallelepiped. Show that $\mathbf{a}+\mathbf{b}+\mathbf{c}$ forms a diagonal.

10. Vectors are drawn from the center of a regular pentagon to its vertices. Show that their sum is zero.

11. Consider the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ defined in Problem 1. Introduce rectangular cartesian coordinate axes such that the four vectors lie in the x_1x_2 plane with the x_1 axis pointing east and the x_2 axis pointing north. Find the components of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{a}+2\mathbf{b}$, and $3\mathbf{a}-2\mathbf{b}$; also, express these vectors in terms of their components and the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$.

12. Do Problem 4, making an exact determination of the coefficients analytically by the use of components.

13. Given that

$$\begin{aligned}\mathbf{a} &= \mathbf{i}_1+2\mathbf{i}_2+\mathbf{i}_3,\\ \mathbf{b} &= 2\mathbf{i}_1+\mathbf{i}_2,\\ \mathbf{c} &= 3\mathbf{i}_1-4\mathbf{i}_2-5\mathbf{i}_3,\end{aligned}$$

verify that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$, $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$. Also, find $|\mathbf{a}\times\mathbf{b}|$, $|\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}|$.

14. Show that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = a^2-b^2$.

15. Show that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\times\mathbf{b}$.

16. The two vectors $\mathbf{a} = \mathbf{i}_1+3\mathbf{i}_2-2\mathbf{i}_3$, $\mathbf{b} = 3\mathbf{i}_1+2\mathbf{i}_2-2\mathbf{i}_3$ have a common origin. Show that the line joining their terminuses is parallel to the x_1x_2 plane, and find its length.

17. Show that the vectors $\mathbf{a} = \mathbf{i}_1+4\mathbf{i}_2+3\mathbf{i}_3$, $\mathbf{b} = 4\mathbf{i}_1+2\mathbf{i}_2-4\mathbf{i}_3$ are perpendicular.

18. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are as defined in Problem 13, find $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, $(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}$, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$.

19. If the vectors drawn from the origin to three points A, B and C are respectively equal to the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ defined in Problem 13, find a unit vector $\mathbf{n}$ perpendicular to the plane ABC. Hence find the distance from the origin to this plane.

20. Show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0.$$

21. Show that

$$\begin{aligned}\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] &= [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\ (\mathbf{c}\times\mathbf{d})\\ &= (\mathbf{b}\cdot\mathbf{d})\ (\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\ (\mathbf{a}\times\mathbf{d}).\end{aligned}$$

22. Show that

$$[\mathbf{a}\times\mathbf{b}]\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2.$$

23. Show that

$$\mathbf{a}[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]-\mathbf{b}[\mathbf{c}\cdot(\mathbf{d}\times\mathbf{a})]+\mathbf{c}[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]-\mathbf{d}[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = 0.$$

24. Show that

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})](\mathbf{f}\times\mathbf{g}) = \begin{vmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\\ \mathbf{f}\cdot\mathbf{a} & \mathbf{f}\cdot\mathbf{b} & \mathbf{f}\cdot\mathbf{c}\\ \mathbf{g}\cdot\mathbf{a} & \mathbf{g}\cdot\mathbf{b} & \mathbf{g}\cdot\mathbf{c}\end{vmatrix}.$$

25. Show that

$$\mathbf{n} = \frac{\mathbf{n}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\mathbf{a} + \frac{\mathbf{n}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}\mathbf{b} + \frac{\mathbf{n}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}\mathbf{c}\,.$$

This formula can be used to express any vector $\mathbf{n}$ as a linear function of any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ not lying in the same plane.

26. Express the vector $\mathbf{n} = \mathbf{i}_1+2\mathbf{i}_2+3\mathbf{i}_3$ as a linear function of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ defined in Problem 13. (See Problem 25.)

27. Express the vector $\mathbf{n} = 2\mathbf{i}_1-2\mathbf{i}_2-3\mathbf{i}_3$ as a linear function of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ defined in Problem 13. (See Problem 25.)

28. Show that

$$\begin{aligned}\mathbf{a}\times[(\mathbf{f}\times\mathbf{b})\times(\mathbf{g}\times\mathbf{c})]+\mathbf{b}\times[(\mathbf{f}\times\mathbf{c})\times(\mathbf{g}\times\mathbf{a})]\\ +\mathbf{c}\times[(\mathbf{f}\times\mathbf{a})\times(\mathbf{g}\times\mathbf{b})] = 0.\end{aligned}$$

29. If 0 is the origin of the coordinates and A, B and C are three points such that

$$\begin{aligned}\overline{OA} &= 2\mathbf{i}_1+2\mathbf{i}_2-\mathbf{i}_3,\\ \overline{AB} &= \mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3,\\ \overline{BC} &= -2\mathbf{i}_1+2\mathbf{i}_2-3\mathbf{i}_3,\end{aligned}$$

find (i) the moment of $\overline{BC}$ about A, (ii) the moment of $\overline{CB}$ about O, (iii) the moment of $\overline{BC}$ about the directed line OA, (iv) the moments of $\overline{BC}$ about the coordinate axes.

30. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, prove that a times the moment of $\mathbf{b}$ about the line of action of $\mathbf{a}$ is equal to b times the moment of $\mathbf{a}$ about the line of action of $\mathbf{b}$.

31. If $\mathbf{a}(u)$ has a constant magnitude, show that

$$\mathbf{a}\cdot\frac{d\mathbf{a}}{du} = 0\,.$$

32. If $\mathbf{a} = \mathbf{p}\cos u+\mathbf{q}\sin u$, where $\mathbf{p}$ and $\mathbf{q}$ are constant vectors and u is a variable, show that

$$\mathbf{a}\cdot\left[\frac{d\mathbf{a}}{du}\times\frac{d^2\mathbf{a}}{du^2}\right] = 0\,.$$

33. If $\mathbf{a}$ is a function of a variable u, show that

$$\frac{d}{du}\left[\mathbf{a}\cdot\left(\frac{d\mathbf{a}}{du}\times\frac{d^2\mathbf{a}}{du^2}\right)\right] = \left(\mathbf{a}\times\frac{d\mathbf{a}}{du}\right)\cdot\frac{d^3\mathbf{a}}{du^3}\,.$$

34. Given that the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are independent of a variable u, evaluate $\int \mathbf{a}\, du$ when

(i) $$\mathbf{a} = \mathbf{i}_1 + 2u\mathbf{i}_2 + 8u^3\mathbf{i}_3,$$

(ii) $$\mathbf{a} = \mathbf{i}_1 \cos u + \mathbf{i}_2 \sin u - \mathbf{i}_3\, e^u,$$

(iii) $$\mathbf{a} = \frac{2\mathbf{i}_1}{4-u^2} + \frac{2\mathbf{i}_2}{4+u^2}.$$

35. Evaluate the following integrals, in which $\mathbf{p}$ and $\mathbf{q}$ are constant vectors:

(i) $$\int (\mathbf{p} + \mathbf{q}u^2)\, du,$$

(ii) $$\int (\mathbf{p} \cos u + \mathbf{q} \sec^2 u)\, du,$$

(iii) $$\int \frac{\mathbf{p}+\mathbf{q}u}{4-u^2}\, du.$$

36. Find the vector $\mathbf{x}(u)$ in each of the following cases, given that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are constant vectors:

(i) $$\frac{d\mathbf{x}}{du} = \mathbf{p}\, u^2 + \mathbf{q}\, e^{2u},$$

(ii) $$\frac{d^2\mathbf{x}}{du^2} = \mathbf{p} \cos u + \mathbf{q} \sin u,$$

(iii) $$\frac{d^2\mathbf{x}}{du^2} = (\mathbf{p} \sin u - \mathbf{q} \cos u) \times \mathbf{r}.$$

37. Find the general solutions of the differential equations

(i) $$\frac{d^2\mathbf{x}}{du^2} - \frac{d\mathbf{x}}{du} - 6\mathbf{x} = 0,$$

(ii) $$\frac{d^2\mathbf{x}}{du^2} + 4\frac{d\mathbf{x}}{du} + 4\mathbf{x} = 0,$$

(iii) $$\frac{d^2\mathbf{x}}{du^2} - 2\frac{d\mathbf{x}}{du} + 5\mathbf{x} = 0,$$

(iv) $$\frac{d^4\mathbf{x}}{du^4} - 6\frac{d^3\mathbf{x}}{du^3} + 11\frac{d^2\mathbf{x}}{du^2} - 6\frac{d\mathbf{x}}{du} = 0.$$

38. Find the general solutions of the following differential equations, given that $\mathbf{p}$ and $\mathbf{q}$ are constant vectors:

(i) $$\frac{d\mathbf{x}}{du} - 3\mathbf{x} = \mathbf{p}(3u^2+1),$$

(ii) $$\frac{d^2\mathbf{x}}{du^2} - 4\mathbf{x} = 16\mathbf{p}\cos 2u,$$

(iii) $$\frac{d^2\mathbf{x}}{du^2} + 2\frac{d\mathbf{x}}{du} = -6\mathbf{p}e^u + 5\mathbf{q}\sin u.$$

39. Find the general solutions of the following differential equations, given that $\mathbf{p}$ and $\mathbf{q}$ are constant vectors:

(i) $$\frac{d^2\mathbf{x}}{du^2} - 2\frac{d\mathbf{x}}{du} + \mathbf{x} = 2\mathbf{p}e^u,$$

(ii) $$\frac{d^2\mathbf{x}}{du^2} - 3\frac{d\mathbf{x}}{du} = 2\mathbf{p}e^u + 18\mathbf{q}u^2,$$

(iii) $$u^2\frac{d^2\mathbf{x}}{du^2} - u\frac{d\mathbf{x}}{du} - 3\mathbf{x} = 6\mathbf{p}.$$