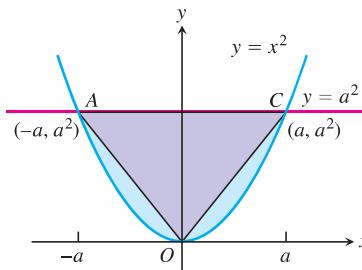


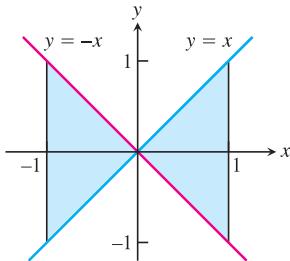
- 107.** The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.



- 108.** Suppose the area of the region between the graph of a positive continuous function f and the x -axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

- 109.** Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\int_{-1}^1 (x - (-x)) dx = \int_{-1}^1 2x dx$
b. $\int_{-1}^1 (-x - (x)) dx = \int_{-1}^1 -2x dx$



- 110.** True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

$$\int_a^b [f(x) - g(x)] dx.$$

Give reasons for your answer.

Theory and Examples

- 111.** Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x} dx$$

in terms of F .

- 112.** Show that if f is continuous, then

$$\int_0^1 f(x) dx = \int_0^1 f(1-x) dx.$$

- 113.** Suppose that

$$\int_0^1 f(x) dx = 3.$$

Find

$$\int_{-1}^0 f(x) dx$$

if **a.** f is odd, **b.** f is even.

- 114. a.** Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x) dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

- 115.** If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x) dx}{f(x) + f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I .

- 116.** By using a substitution, prove that for all positive numbers x and y ,

$$\int_x^{xy} \frac{1}{t} dt = \int_1^y \frac{1}{t} dt.$$

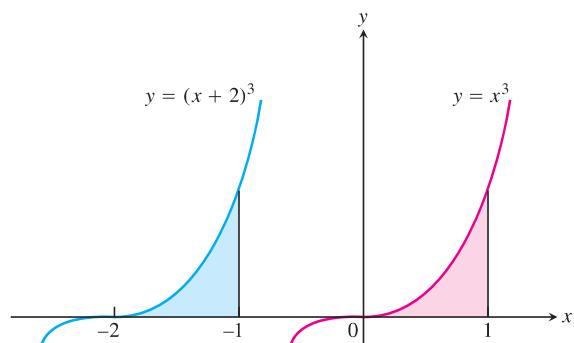
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation.

$$\int_a^b f(x) dx = \int_{a-c}^{b-c} f(x+c) dx. \quad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x . For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3 dx = \int_0^1 x^3 dx$$

because the areas of the shaded regions are congruent.



- 117.** Use a substitution to verify Equation (1).

- 118.** For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2$, $a = 0$, $b = 1$, $c = 1$

b. $f(x) = \sin x$, $a = 0$, $b = \pi$, $c = \pi/2$

c. $f(x) = \sqrt{x-4}$, $a = 4$, $b = 8$, $c = 5$

COMPUTER EXPLORATIONS

In Exercises 119–122, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

- Plot the curves together to see what they look like and how many points of intersection they have.
- Use the numerical equation solver in your CAS to find all the points of intersection.
- Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

- Sum together the integrals found in part (c).

119. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$, $g(x) = x - 1$

120. $f(x) = \frac{x^4}{2} - 3x^3 + 10$, $g(x) = 8 - 12x$

121. $f(x) = x + \sin(2x)$, $g(x) = x^3$

122. $f(x) = x^2 \cos x$, $g(x) = x^3 - x$

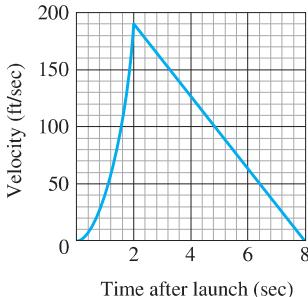
Chapter 5 Questions to Guide Your Review

- How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
- What is sigma notation? What advantage does it offer? Give examples.
- What is a Riemann sum? Why might you want to consider such a sum?
- What is the norm of a partition of a closed interval?
- What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
- What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
- What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
- Describe the rules for working with definite integrals (Table 5.4). Give examples.
- What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
- What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?
- Discuss how the processes of integration and differentiation can be considered as “inverses” of each other.
- How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
- How is integration by substitution related to the Chain Rule?
- How can you sometimes evaluate indefinite integrals by substitution? Give examples.
- How does the method of substitution work for definite integrals? Give examples.
- How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

Chapter 5 Practice Exercises

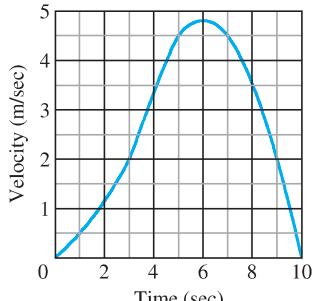
Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.



- a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

- b. Sketch a graph of the rocket's height aboveground as a function of time for $0 \leq t \leq 8$.
2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s -axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?
- b. Sketch a graph of s as a function of t for $0 \leq t \leq 10$ assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

a. $\sum_{k=1}^{10} \frac{a_k}{4}$
 b. $\sum_{k=1}^{10} (b_k - 3a_k)$
 c. $\sum_{k=1}^{10} (a_k + b_k - 1)$
 d. $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

a. $\sum_{k=1}^{20} 3a_k$
 b. $\sum_{k=1}^{20} (a_k + b_k)$
 c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$
 d. $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P .

5. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (2c_k - 1)^{-1/2} \Delta x_k$, where P is a partition of $[1, 5]$

6. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k (c_k^2 - 1)^{1/3} \Delta x_k$, where P is a partition of $[1, 3]$

7. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\sin c_k)(\cos c_k) \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^2 3f(x) dx = 12$, $\int_{-2}^5 f(x) dx = 6$, and $\int_{-2}^5 g(x) dx = 2$, find the values of the following.

a. $\int_{-2}^2 f(x) dx$
 b. $\int_2^5 f(x) dx$
 c. $\int_5^{-2} g(x) dx$
 d. $\int_{-2}^5 (-\pi g(x)) dx$
 e. $\int_{-2}^5 \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_0^2 f(x) dx = \pi$, $\int_0^2 7g(x) dx = 7$, and $\int_0^1 g(x) dx = 2$, find the values of the following.

a. $\int_0^2 g(x) dx$
 b. $\int_1^2 g(x) dx$
 c. $\int_2^0 f(x) dx$
 d. $\int_0^2 \sqrt{2} f(x) dx$
 e. $\int_0^2 (g(x) - 3f(x)) dx$

Area

In Exercises 11–14, find the total area of the region between the graph of f and the x -axis.

11. $f(x) = x^2 - 4x + 3$, $0 \leq x \leq 3$

12. $f(x) = 1 - (x^2/4)$, $-2 \leq x \leq 3$

13. $f(x) = 5 - 5x^{2/3}$, $-1 \leq x \leq 8$

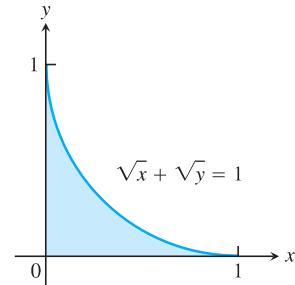
14. $f(x) = 1 - \sqrt{x}$, $0 \leq x \leq 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

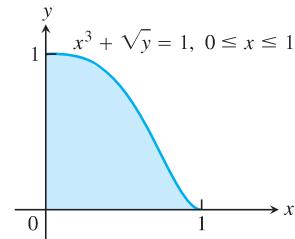
15. $y = x$, $y = 1/x^2$, $x = 2$

16. $y = x$, $y = 1/\sqrt{x}$, $x = 2$

17. $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, $y = 0$



18. $x^3 + \sqrt{y} = 1$, $x = 0$, $y = 0$, for $0 \leq x \leq 1$



19. $x = 2y^2$, $x = 0$, $y = 3$

20. $x = 4 - y^2$, $x = 0$

21. $y^2 = 4x$, $y = 4x - 2$

22. $y^2 = 4x + 4$, $y = 4x - 16$

23. $y = \sin x$, $y = x$, $0 \leq x \leq \pi/4$

24. $y = |\sin x|$, $y = 1$, $-\pi/2 \leq x \leq \pi/2$

25. $y = 2 \sin x$, $y = \sin 2x$, $0 \leq x \leq \pi$

26. $y = 8 \cos x$, $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

27. Find the area of the “triangular” region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

28. Find the area of the “triangular” region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

29. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x -axis.

30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 3\pi/2$.

33. **Area** Find the area between the curve $y = 2(\ln x)/x$ and the x -axis from $x = 1$ to $x = e$.

34. a. Show that the area between the curve $y = 1/x$ and the x -axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x -axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x -axis from ka to kb is the same as the area between the curve and the x -axis from $x = a$ to $x = b$ ($0 < a < b, k > 0$).

Initial Value Problems

35. Show that $y = x^2 + \int_1^x \frac{1}{t} dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

36. Show that $y = \int_0^x (1 + 2\sqrt{\sec t}) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x} \tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 37 and 38 in terms of integrals.

37. $\frac{dy}{dx} = \frac{\sin x}{x}, \quad y(5) = -3$

38. $\frac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Solve the initial value problems in Exercises 39–42.

39. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$

40. $\frac{dy}{dx} = \frac{1}{x^2+1} - 1, \quad y(0) = 1$

41. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1; \quad y(2) = \pi$

42. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 43–72.

43. $\int 2(\cos x)^{-1/2} \sin x dx$

44. $\int (\tan x)^{-3/2} \sec^2 x dx$

45. $\int (2\theta + 1 + 2 \cos(2\theta + 1)) d\theta$

46. $\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2 \sec^2(2\theta - \pi) \right) d\theta$

47. $\int \left(t - \frac{2}{t} \right) \left(t + \frac{2}{t} \right) dt$

48. $\int \frac{(t+1)^2 - 1}{t^4} dt$

49. $\int \sqrt{t} \sin(2t^{3/2}) dt$

50. $\int (\sec \theta \tan \theta) \sqrt{1 + \sec \theta} d\theta$

51. $\int e^x \sec^2(e^x - 7) dx$

52. $\int e^y \csc(e^y + 1) \cot(e^y + 1) dy$

53. $\int (\sec^2 x) e^{\tan x} dx$

54. $\int (\csc^2 x) e^{\cot x} dx$

55. $\int_{-1}^1 \frac{dx}{3x - 4}$

56. $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

57. $\int_0^4 \frac{2t}{t^2 - 25} dt$

58. $\int \frac{\tan(\ln v)}{v} dv$

59. $\int \frac{(\ln x)^{-3}}{x} dx$

60. $\int \frac{1}{r} \csc^2(1 + \ln r) dr$

61. $\int x 3^{x^2} dx$

62. $\int 2^{\tan x} \sec^2 x dx$

63. $\int \frac{3 dr}{\sqrt{1 - 4(r-1)^2}}$

64. $\int \frac{6 dr}{\sqrt{4 - (r+1)^2}}$

65. $\int \frac{dx}{2 + (x-1)^2}$

66. $\int \frac{dx}{1 + (3x+1)^2}$

67. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2 - 4}}$

68. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 25}}$

69. $\int \frac{e^{\sin^{-1} \sqrt{x}} dx}{2\sqrt{x-x^2}}$

70. $\int \frac{\sqrt{\sin^{-1} x} dx}{\sqrt{1-x^2}}$

71. $\int \frac{dy}{\sqrt{\tan^{-1} y}(1+y^2)}$

72. $\int \frac{(\tan^{-1} x)^2 dx}{1+x^2}$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 73–112.

73. $\int_{-1}^1 (3x^2 - 4x + 7) dx$

74. $\int_0^1 (8s^3 - 12s^2 + 5) ds$

75. $\int_1^2 \frac{4}{v^2} dv$

76. $\int_1^{27} x^{-4/3} dx$

77. $\int_1^4 \frac{dt}{t\sqrt{t}}$

78. $\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}} du$

79. $\int_0^1 \frac{36 dx}{(2x+1)^3}$

80. $\int_0^1 \frac{dr}{\sqrt[3]{(7-5r)^2}}$

81. $\int_{1/8}^1 x^{-1/3} (1 - x^{2/3})^{3/2} dx$

82. $\int_0^{1/2} x^3 (1 + 9x^4)^{-3/2} dx$

83. $\int_0^\pi \sin^2 5r dr$

84. $\int_0^{\pi/4} \cos^2 \left(4t - \frac{\pi}{4} \right) dt$

85. $\int_0^{\pi/3} \sec^2 \theta d\theta$

86. $\int_{\pi/4}^{3\pi/4} \csc^2 x dx$

87. $\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx$

88. $\int_0^\pi \tan^2 \frac{\theta}{3} d\theta$

89. $\int_{-\pi/3}^0 \sec x \tan x dx$

90. $\int_{\pi/4}^{3\pi/4} \csc z \cot z dz$

91. $\int_0^{\pi/2} 5(\sin x)^{3/2} \cos x dx$

92. $\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x dx$

93. $\int_0^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1+3 \sin^2 x}} dx$

94. $\int_0^{\pi/4} \frac{\sec^2 x}{(1+7\tan x)^{2/3}} dx$

95. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x} \right) dx$

96. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2} \right) dx$

97. $\int_{-2}^{-1} e^{-(x+1)} dx$

98. $\int_{-\ln 2}^0 e^{2w} dw$

99. $\int_0^{\ln 5} e^r (3e^r + 1)^{-3/2} dr$

100. $\int_0^{\ln 9} e^\theta (e^\theta - 1)^{1/2} d\theta$

101.

$\int_1^e \frac{1}{x} (1 + 7 \ln x)^{-1/3} dx$

102. $\int_1^3 \frac{(\ln(v+1))^2}{v+1} dv$

103. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta$

104. $\int_1^e \frac{8 \ln 3 \log_3 \theta}{\theta} d\theta$

105. $\int_{-3/4}^{3/4} \frac{6 \, dx}{\sqrt{9 - 4x^2}}$

107. $\int_{-2}^2 \frac{3 \, dt}{4 + 3t^2}$

109. $\int \frac{dy}{y\sqrt{4y^2 - 1}}$

111. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$

106. $\int_{-1/5}^{1/5} \frac{6 \, dx}{\sqrt{4 - 25x^2}}$

108. $\int_{\sqrt{3}}^3 \frac{dt}{3 + t^2}$

110. $\int \frac{24 \, dy}{y\sqrt{y^2 - 16}}$

112. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

Average Values

113. Find the average value of $f(x) = mx + b$

- a. over $[-1, 1]$
- b. over $[-k, k]$

114. Find the average value of

- a. $y = \sqrt{3x}$ over $[0, 3]$
- b. $y = \sqrt{ax}$ over $[0, a]$

115. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

116. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

117. a. Verify that $\int \ln x \, dx = x \ln x - x + C$.

- b. Find the average value of $\ln x$ over $[1, e]$.

118. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

T 119. Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. (See Exercise 98, Section 3.6.) This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F , which is slightly higher than the average value of $f(x)$.

T 120. **Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C , measured in units of cal/deg-mole (calories per degree gram molecule). The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20^\circ \leq T \leq 675^\circ\text{C}$ and the temperature at which it is attained.

Differentiating Integrals

In Exercises 121–128, find dy/dx .

121. $y = \int_2^x \sqrt{2 + \cos^3 t} \, dt$

122. $y = \int_2^{7x^2} \sqrt{2 + \cos^3 t} \, dt$

123. $y = \int_x^1 \frac{6}{3 + t^4} \, dt$

124. $y = \int_{\sec x}^2 \frac{1}{t^2 + 1} \, dt$

125. $y = \int_{\ln x^2}^0 e^{\cos t} \, dt$

126. $y = \int_1^{e^{\sqrt{x}}} \ln(t^2 + 1) \, dt$

127. $y = \int_0^{\sin^{-1} x} \frac{dt}{\sqrt{1 - 2t^2}}$

128. $y = \int_{\tan^{-1} x}^{\pi/4} e^{\sqrt{t}} \, dt$

Theory and Examples

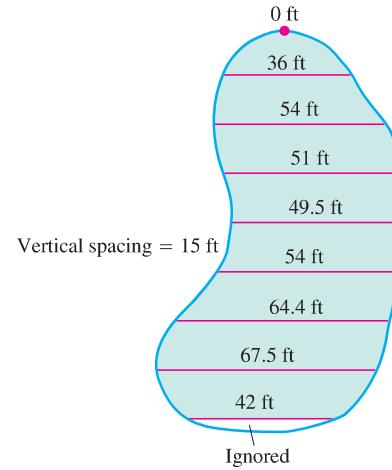
129. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

130. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4} \, dx$ in terms of F and give a reason for your answer.

131. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2} \, dt$. Explain the main steps in your calculation.

132. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2)) \, dt$. Explain the main steps in your calculation.

133. **A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



134. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

- a. At what altitude does A's parachute open?
- b. At what altitude does B's parachute open?
- c. Which skydiver lands first?

Chapter 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x) dx = 7$, does $\int_0^1 f(x) dx = 1$?

- b. If $\int_0^1 f(x) dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)} dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^2 f(x) dx = 4$, $\int_2^5 f(x) dx = 3$, $\int_{-2}^5 g(x) dx = 2$.

Which, if any, of the following statements are true?

- a. $\int_5^2 f(x) dx = -3$ b. $\int_{-2}^5 (f(x) + g(x)) dx = 9$

- c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. **Initial value problem** Show that

$$y = \frac{1}{a} \int_0^x f(t) \sin a(x-t) dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \quad \frac{dy}{dx} = 0 \text{ and } y = 0 \text{ when } x = 0.$$

(Hint: $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. **Proportionality** Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

- a. $\int_0^{x^2} f(t) dt = x \cos \pi x$ b. $\int_0^{f(x)} t^2 dt = x \cos \pi x$.

6. Find $f(\pi/2)$ from the following information.

- i) f is positive and continuous.

- ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a.$$

7. The area of the region in the xy -plane enclosed by the x -axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t) dt \right) du = \int_0^x f(u)(x-u) du.$$

(Hint: Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x .)

9. **Finding a curve** Find the equation for the curve in the xy -plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. **Shoveling dirt** You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval I** if f has only finitely many discontinuities in I , the limits

$$\lim_{x \rightarrow c^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x)$$

exist and are finite at every interior point of I , and the appropriate one-sided limits exist and are finite at the endpoints of I . All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1-x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.32) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^3 f(x) dx &= \int_{-1}^0 (1-x) dx + \int_0^2 x^2 dx + \int_2^3 (-1) dx \\ &= \left[x - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^2 + \left[-x \right]_2^3 \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

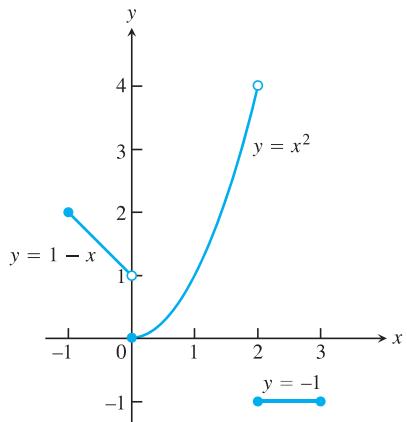


FIGURE 5.32 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx) \int_a^x f(t) dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule (see Exercises 31–38).

Graph the functions in Exercises 11–16 and integrate them over their domains.

$$11. f(x) = \begin{cases} x^{2/3}, & -8 \leq x < 0 \\ -4, & 0 \leq x \leq 3 \end{cases}$$

$$12. f(x) = \begin{cases} \sqrt{-x}, & -4 \leq x < 0 \\ x^2 - 4, & 0 \leq x \leq 3 \end{cases}$$

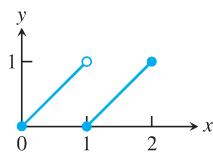
$$13. g(t) = \begin{cases} t, & 0 \leq t < 1 \\ \sin \pi t, & 1 \leq t \leq 2 \end{cases}$$

$$14. h(z) = \begin{cases} \sqrt{1-z}, & 0 \leq z < 1 \\ (7z-6)^{-1/3}, & 1 \leq z \leq 2 \end{cases}$$

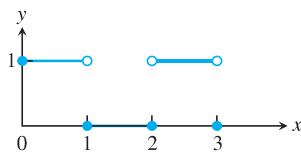
$$15. f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1-x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$$

$$16. h(r) = \begin{cases} r, & -1 \leq r < 0 \\ 1-r^2, & 0 \leq r < 1 \\ 1, & 1 \leq r \leq 2 \end{cases}$$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



Limits

Find the limits in Exercises 19–22.

$$19. \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$20. \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \tan^{-1} t dt$$

$$21. \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$$

$$22. \lim_{n \rightarrow \infty} \frac{1}{n} \left(e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n} \right)$$

Approximating Finite Sums with Integrals

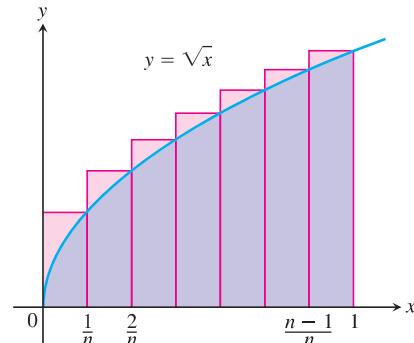
In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$S_n = \sqrt{\frac{1}{n}} \cdot \frac{1}{n} + \sqrt{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \sqrt{\frac{n}{n}} \cdot \frac{1}{n} = \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}.$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3} n^{3/2}.$$

The following table shows how good the approximation can be.

<i>n</i>	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

23. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5 dx$$

and evaluating the integral.

24. See Exercise 23. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} (1^3 + 2^3 + 3^3 + \cdots + n^3).$$

25. Let $f(x)$ be a continuous function. Express

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right]$$

as a definite integral.

26. Use the result of Exercise 25 to evaluate

- $\lim_{n \rightarrow \infty} \frac{1}{n^2} (2 + 4 + 6 + \dots + 2n),$
- $\lim_{n \rightarrow \infty} \frac{1}{n^{16}} (1^{15} + 2^{15} + 3^{15} + \dots + n^{15}),$
- $\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \dots + \sin \frac{n\pi}{n} \right).$

What can be said about the following limits?

- $\lim_{n \rightarrow \infty} \frac{1}{n^{17}} (1^{15} + 2^{15} + 3^{15} + \dots + n^{15})$
- $\lim_{n \rightarrow \infty} \frac{1}{n^{15}} (1^{15} + 2^{15} + 3^{15} + \dots + n^{15})$

27. a. Show that the area A_n of an n -sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2} \sin \frac{2\pi}{n}.$$

- b. Find the limit of A_n as $n \rightarrow \infty$. Is this answer consistent with what you know about the area of a circle?

28. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \dots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n \rightarrow \infty} S_n$, show that

$$S_n = \frac{1}{n} \left[\left(\frac{1}{n} \right)^2 + \left(\frac{2}{n} \right)^2 + \dots + \left(\frac{n-1}{n} \right)^2 \right]$$

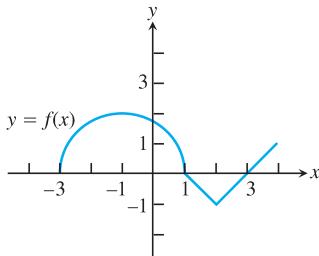
and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2 dx.$$

(Hint: Partition $[0, 1]$ into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

Defining Functions Using the Fundamental Theorem

29. A function defined by an integral The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t) dt$.



- Find $g(1)$.
- Find $g(3)$.
- Find $g(-1)$.
- Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.
- Write an equation for the line tangent to the graph of g at $x = -1$.

- f. Find the x -coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

- g. Find the range of g .

30. A differential equation Show that both of the following conditions are satisfied by $y = \sin x + \int_x^\pi \cos 2t dt + 1$:

- $y'' = -\sin x + 2 \sin 2x$
- $y = 1$ and $y' = -2$ when $x = \pi$.

Leibniz's Rule In applications, we sometimes encounter functions like

$$f(x) = \int_{\sin x}^{x^2} (1+t) dt \quad \text{and} \quad g(x) = \int_{\sqrt{x}}^{2\sqrt{x}} \sin t^2 dt,$$

defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. The first integral can be evaluated directly, but the second cannot. We may find the derivative of either integral, however, by a formula called **Leibniz's Rule**.

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

Figure 5.33 gives a geometric interpretation of Leibniz's Rule. It shows a carpet of variable width $f(t)$ that is being rolled up at the left at the same time x as it is being unrolled at the right. (In this interpretation, time is x , not t .) At time x , the floor is covered from $u(x)$ to $v(x)$. The rate du/dx at which the carpet is being rolled up need not be the same as the rate dv/dx at which the carpet is being laid down. At any given time x , the area covered by carpet is

$$A(x) = \int_{u(x)}^{v(x)} f(t) dt.$$

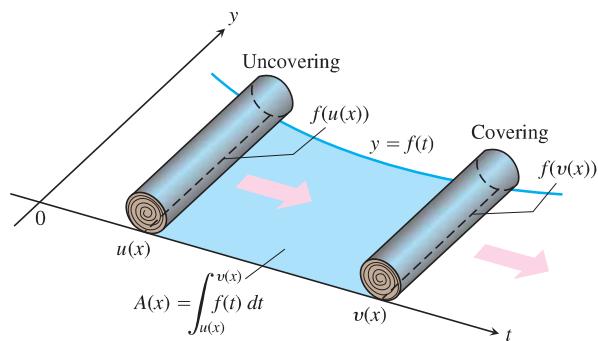


FIGURE 5.33 Rolling and unrolling a carpet: a geometric interpretation of Leibniz's Rule:

$$\frac{dA}{dx} = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

At what rate is the covered area changing? At the instant x , $A(x)$ is increasing by the width $f(v(x))$ of the unrolling carpet times the rate dv/dx at which the carpet is being unrolled. That is, $A(x)$ is being increased at the rate

$$f(v(x)) \frac{dv}{dx}.$$

At the same time, A is being decreased at the rate

$$f(u(x)) \frac{du}{dx},$$

the width at the end that is being rolled up times the rate du/dx . The net rate of change in A is

$$\frac{dA}{dx} = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx},$$

which is precisely Leibniz's Rule.

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t) dt = F(v(x)) - F(u(x)).$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned} \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt &= \frac{d}{dx} \left[F(v(x)) - F(u(x)) \right] \\ &= F'(v(x)) \frac{dv}{dx} - F'(u(x)) \frac{du}{dx} \quad \text{Chain Rule} \\ &= f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}. \end{aligned}$$

Use Leibniz's Rule to find the derivatives of the functions in Exercises 31–38.

31. $f(x) = \int_{1/x}^x \frac{1}{t} dt$

32. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2} dt$

33. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2 dt$

34. $g(y) = \int_{\sqrt{y}}^{y^2} \frac{e^t}{t} dt$

35. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t} dt$

36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t dt$

37. $y = \int_0^{\ln x} \sin e^t dt$

38. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t dt$

Theory and Examples

39. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

$$\int_x^{x+3} t(5-t) dt.$$

40. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

41. Find the areas between the curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$ and the x -axis from $x = 1$ to $x = e$. What is the ratio of the larger area to the smaller?

42. a. Find df/dx if

$$f(x) = \int_1^{e^x} \frac{2 \ln t}{t} dt.$$

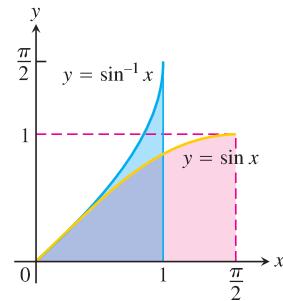
- b. Find $f(0)$.

- c. What can you conclude about the graph of f ? Give reasons for your answer.

43. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4} dt$.

44. Use the accompanying figure to show that

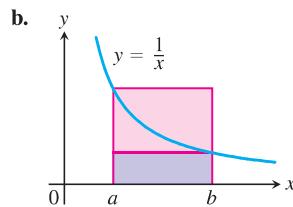
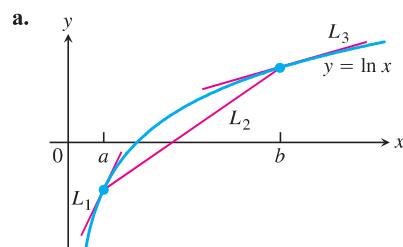
$$\int_0^{\pi/2} \sin x dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x dx.$$



45. Napier's inequality Here are two pictorial proofs that

$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}.$$

Explain what is going on in each case.



(Source: Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

Chapter 5 Technology Application Projects

Mathematica/Maple Modules:

Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves

Visualize and approximate areas and volumes in Part I.

Riemann Sums, Definite Integrals, and the Fundamental Theorem of Calculus

Parts I, II, and III develop Riemann sums and definite integrals. Part IV continues the development of the Riemann sum and definite integral using the Fundamental Theorem to solve problems previously investigated.

Rain Catchers, Elevators, and Rockets

Part I illustrates that the area under a curve is the same as the area of an appropriate rectangle for examples taken from the chapter. You will compute the amount of water accumulating in basins of different shapes as the basin is filled and drained.

Motion Along a Straight Line, Part II

You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among position, velocity, and acceleration. Figures in the text can be animated using this software.

Bending of Beams

Study bent shapes of beams, determine their maximum deflections, concavity, and inflection points, and interpret the results in terms of a beam's compression and tension.



6

APPLICATIONS OF DEFINITE INTEGRALS

OVERVIEW In Chapter 5 we saw that a continuous function over a closed interval has a definite integral, which is the limit of any Riemann sum for the function. We proved that we could evaluate definite integrals using the Fundamental Theorem of Calculus. We also found that the area under a curve and the area between two curves could be computed as definite integrals.

In this chapter we extend the applications of definite integrals to finding volumes, lengths of plane curves, and areas of surfaces of revolution. We also use integrals to solve physical problems involving the work done by a force, the fluid force against a planar wall, and the location of an object's center of mass.

6.1 Volumes Using Cross-Sections

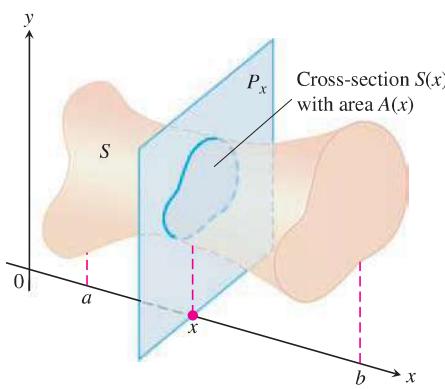


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x in the interval $[a, b]$.

In this section we define volumes of solids using the areas of their cross-sections. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1). We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of slicing, the disk method, and the washer method.

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h , then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylinders, like the one in Figure 6.1. If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $S(x)$ of area $A(x)$, and A is a continuous function of x , we can define and calculate the volume of the solid S as the definite integral of $A(x)$. We now show how this integral is obtained by the **method of slicing**.

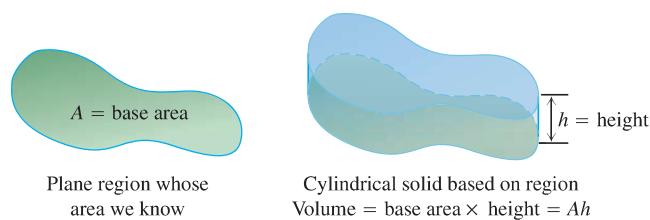


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

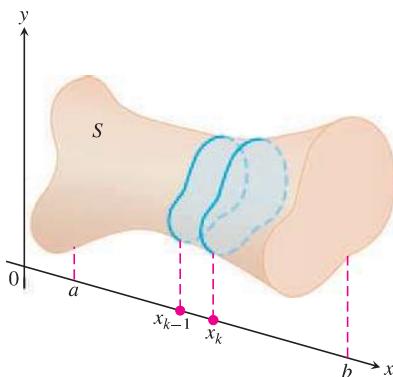


FIGURE 6.3 A typical thin slab in the solid S .

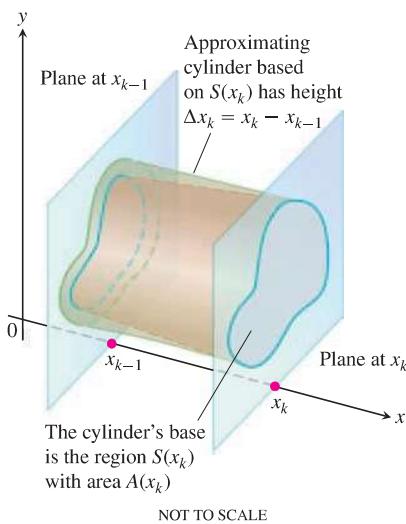


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x -axis at the partition points $a = x_0 < x_1 < \dots < x_n = b$. The planes P_{x_k} , perpendicular to the x -axis at the partition points, slice S into thin “slabs” (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k) \Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^n V_k = \sum_{k=1}^n A(x_k) \Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero. Taking a partition of $[a, b]$ into n subintervals with $\|P\| \rightarrow 0$ gives

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n A(x_k) \Delta x_k = \int_a^b A(x) dx.$$

So we define the limiting definite integral of the Riemann sum to be the volume of the solid S .

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

$$V = \int_a^b A(x) dx.$$

This definition applies whenever $A(x)$ is integrable, and in particular when it is continuous. To apply the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x -axis and its vertex at the origin and include a typical cross-section (Figure 6.5).

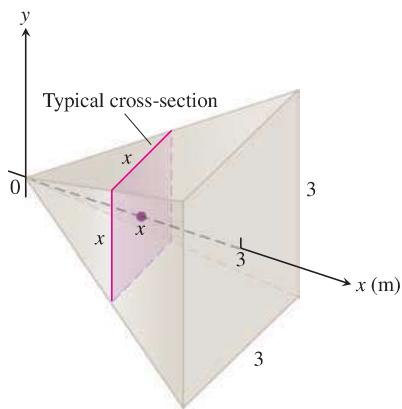


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

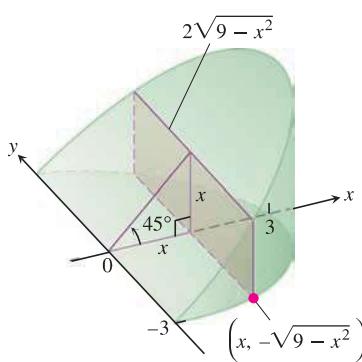


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x -axis. The cross-sections are rectangles.

2. *A formula for $A(x)$.* The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.

4. *Integrate to find the volume:*

$$V = \int_0^3 A(x) dx = \int_0^3 x^2 dx = \frac{x^3}{3} \Big|_0^3 = 9 \text{ m}^3. \quad \blacksquare$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x -axis (Figure 6.6). The base of the wedge in the figure is the semicircle with $x \geq 0$ that is cut from the circle $x^2 + y^2 = 9$ by the 45° plane when it intersects the y -axis. For any x in the interval $[0, 3]$, the y -values in this semicircular base vary from $y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$. When we slice through the wedge by a plane perpendicular to the x -axis, we obtain a cross-section at x which is a rectangle of height x whose width extends across the semicircular base. The area of this cross-section is

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)(2\sqrt{9 - x^2}) \\ &= 2x\sqrt{9 - x^2}. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x) dx = \int_0^3 2x\sqrt{9 - x^2} dx \\ &= -\frac{2}{3}(9 - x^2)^{3/2} \Big|_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$,
 $du = -2x dx$, integrate,
and substitute back. \blacksquare

EXAMPLE 3 Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.7). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

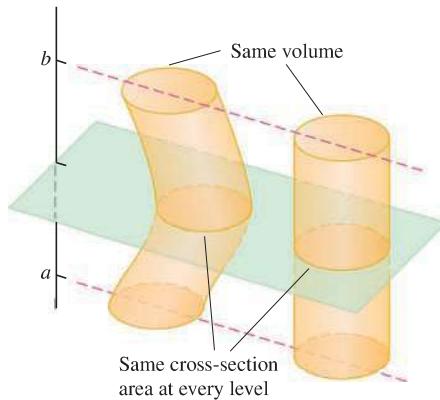


FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume, which can be illustrated with stacks of coins. \blacksquare

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

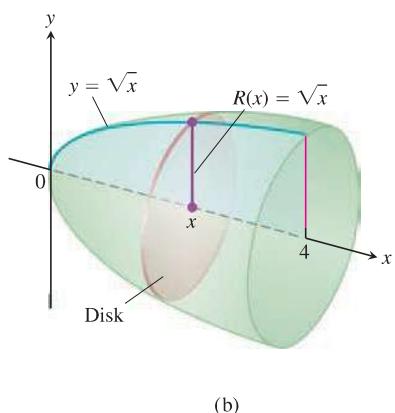
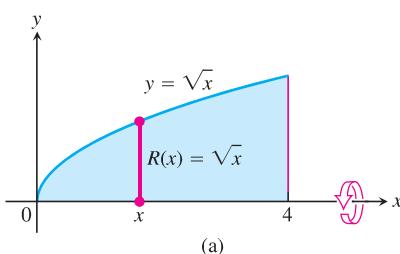


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume in this case gives

Volume by Disks for Rotation About the x -axis

$$V = \int_a^b A(x) dx = \int_a^b \pi[R(x)]^2 dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2 dx \\ &= \int_0^4 \pi[\sqrt{x}]^2 dx \\ &= \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned} \quad \begin{matrix} \text{Radius } R(x) = \sqrt{x} \text{ for} \\ \text{rotation around } x\text{-axis} \end{matrix}$$

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

is rotated about the x -axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x -axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2). \quad \begin{matrix} R(x) = \sqrt{a^2 - x^2} \text{ for} \\ \text{rotation around } x\text{-axis} \end{matrix}$$

Therefore, the volume is

$$V = \int_{-a}^a A(x) dx = \int_{-a}^a \pi(a^2 - x^2) dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{-a}^a = \frac{4}{3} \pi a^3. \quad \blacksquare$$

The axis of revolution in the next example is not the x -axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

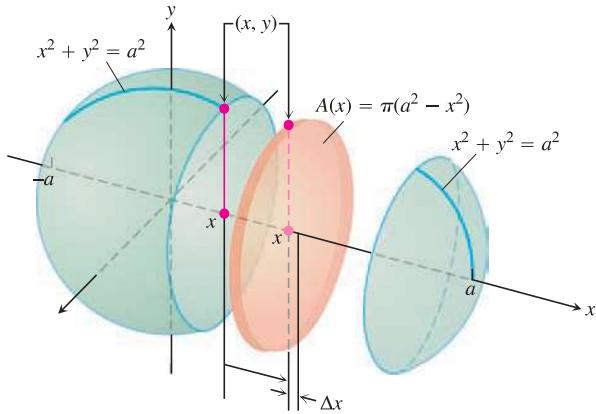


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x -axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2 dx \\ &= \int_1^4 \pi[\sqrt{x} - 1]^2 dx && \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1 \\ &= \pi \int_1^4 [x - 2\sqrt{x} + 1] dx && \text{Expand integrand.} \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. && \text{Integrate.} \end{aligned}$$

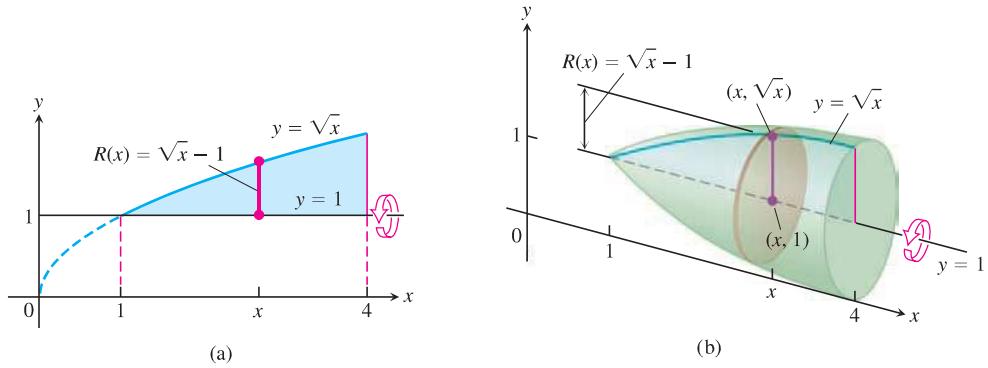


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6. ■

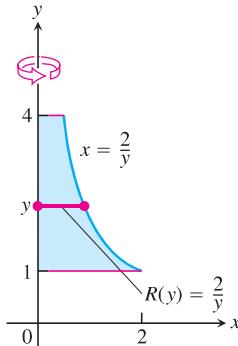
To find the volume of a solid generated by revolving a region between the y -axis and a curve $x = R(y)$, $c \leq y \leq d$, about the y -axis, we use the same method with x replaced by y . In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2,$$

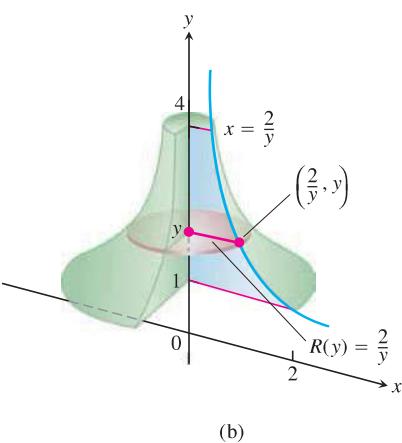
and the definition of volume gives

Volume by Disks for Rotation About the y -axis

$$V = \int_c^d A(y) dy = \int_c^d \pi[R(y)]^2 dy.$$



(a)



(b)

FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y -axis.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

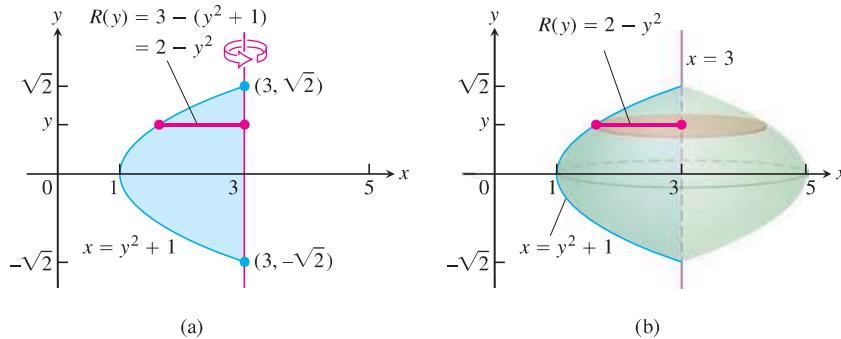
$$\begin{aligned} V &= \int_1^4 \pi[R(y)]^2 dy \\ &= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \quad \text{Radius } R(y) = \frac{2}{y} \text{ for} \\ &= \pi \int_1^4 \frac{4}{y^2} dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] = 3\pi. \end{aligned}$$

■

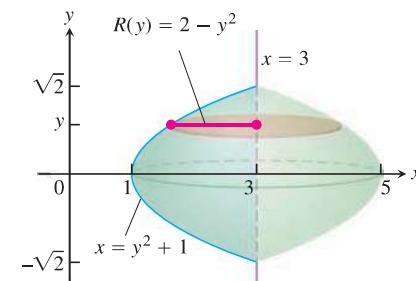
EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$ and have y -coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2 dy \quad y = \pm\sqrt{2} \text{ when } x = 3 \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2 dy \quad \text{Radius } R(y) = 3 - (y^2 + 1) \text{ for rotation around axis } x = 3 \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy \quad \text{Expand integrand.} \\ &= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \quad \text{Integrate.} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$



(a)



(b)

FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

■

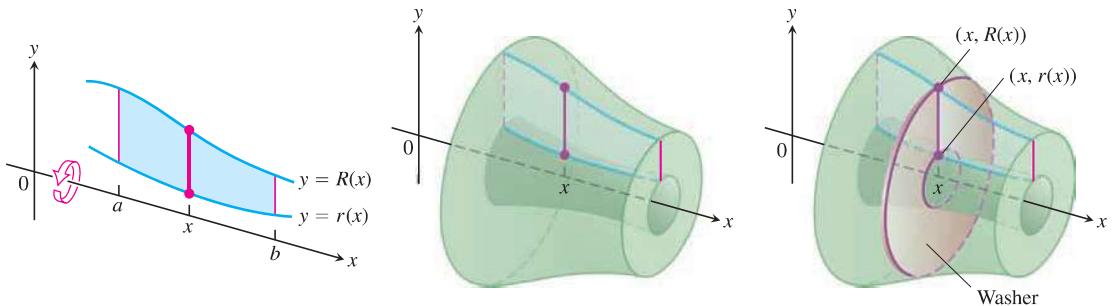
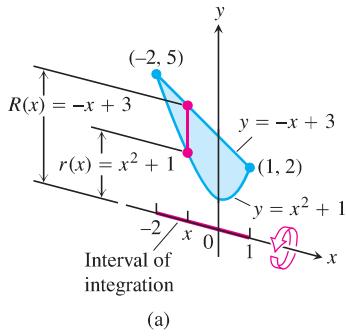


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x) dx$ leads to a slightly different formula.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are



$$\begin{aligned} \text{Outer radius: } & R(x) \\ \text{Inner radius: } & r(x) \end{aligned}$$

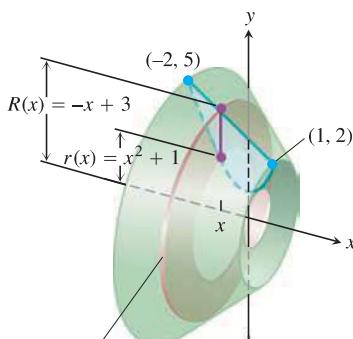
The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives

Volume by Washers for Rotation About the x -axis

$$V = \int_a^b A(x) dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx.$$



$$\begin{aligned} \text{Washer cross-section} \\ \text{Outer radius: } & R(x) = -x + 3 \\ \text{Inner radius: } & r(x) = x^2 + 1 \end{aligned}$$

FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x -axis, the line segment generates a washer.

This method for calculating the volume of a solid of revolution is called the **washer method** because a thin slab of the solid resembles a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

Solution We use the four steps for calculating the volume of a solid as discussed early in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14a).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x -axis along with the region.

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\begin{aligned} \text{Outer radius: } & R(x) = -x + 3 \\ \text{Inner radius: } & r(x) = x^2 + 1 \end{aligned}$$

3. Find the limits of integration by finding the x -coordinates of the intersection points of the curve and line in Figure 6.14a.

$$x^2 + 1 = -x + 3$$

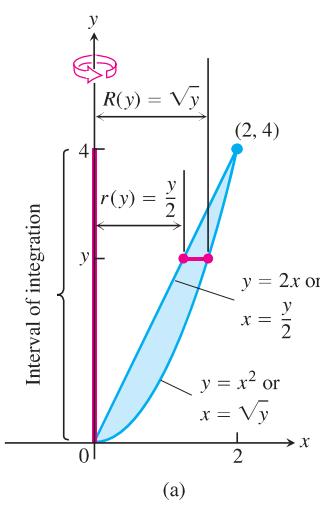
$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, \quad x = 1$$

Limits of integration

4. Evaluate the volume integral.



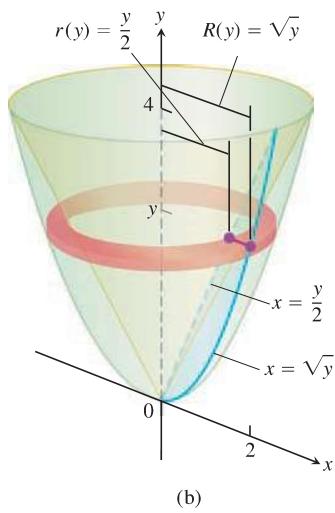
$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx \\ &= \int_{-2}^1 \pi((-x + 3)^2 - (x^2 + 1)^2) dx \\ &= \pi \int_{-2}^1 (8 - 6x - x^2 - x^4) dx \\ &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5} \end{aligned}$$

Rotation around x -axis

Values from Steps 2 and 3

Simplify algebraically. ■

To find the volume of a solid formed by revolving a region about the y -axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x . In this situation the line segment sweeping out a typical washer is perpendicular to the y -axis (the axis of revolution), and the outer and inner radii of the washer are functions of y .



EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y -axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2) dy \\ &= \int_0^4 \pi \left(\left[\sqrt{y} \right]^2 - \left[\frac{y}{2} \right]^2 \right) dy \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$

Rotation around y -axis

Substitute for radii and limits of integration. ■

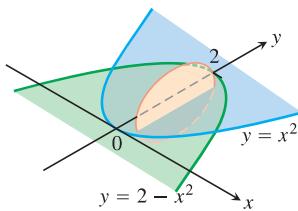
FIGURE 6.15 (a) The region being rotated about the y -axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

Exercises 6.1

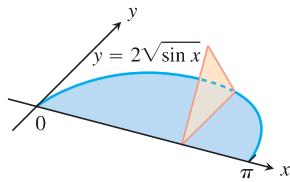
Volumes by Slicing

Find the volumes of the solids in Exercises 1–10.

- The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.
- The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

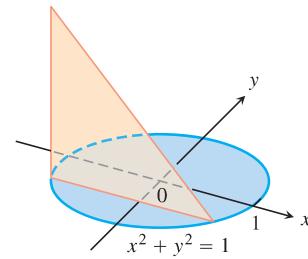


- The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
- The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
- The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x -axis. The cross-sections perpendicular to the x -axis are
 - equilateral triangles with bases running from the x -axis to the curve as shown in the accompanying figure.

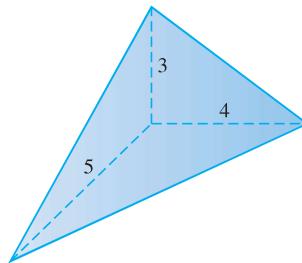


- squares with bases running from the x -axis to the curve.
- The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x -axis are
 - circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.
 - squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.
- The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the x -axis are
 - rectangles of height 10.
 - rectangles of perimeter 20.

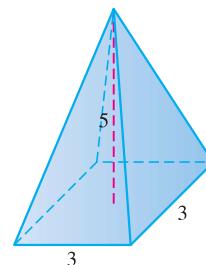
- The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x -axis are
 - isosceles triangles of height 6.
 - semi-circles with diameters running across the base of the solid.
- The solid lies between planes perpendicular to the y -axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y -axis are circular disks with diameters running from the y -axis to the parabola $x = \sqrt{5y^2}$.
- The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y -axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



- Find the volume of the given tetrahedron. (Hint: Consider slices perpendicular to one of the labeled edges.)

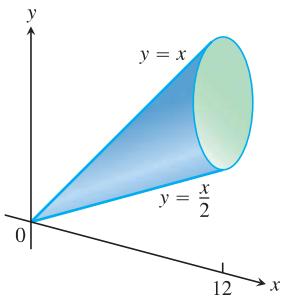


- Find the volume of the given pyramid, which has a square base of area 9 and height 5.



- A twisted solid** A square of side length s lies in a plane perpendicular to a line L . One vertex of the square lies on L . As this square moves a distance h along L , the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.
 - Find the volume of the column.
 - What will the volume be if the square turns twice instead of once? Give reasons for your answer.

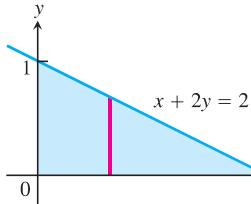
- 14. Cavalieri's principle** A solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x -axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



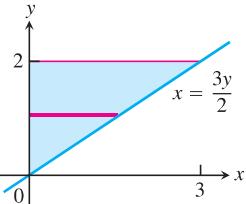
Volumes by the Disk Method

In Exercises 15–18, find the volume of the solid generated by revolving the shaded region about the given axis.

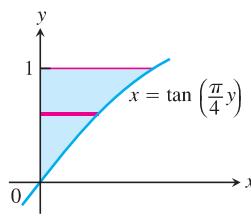
15. About the x -axis



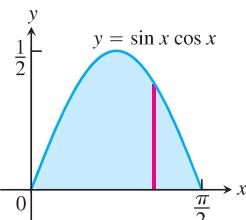
16. About the y -axis



17. About the y -axis



18. About the x -axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 19–28 about the x -axis.

19. $y = x^2$, $y = 0$, $x = 2$ 20. $y = x^3$, $y = 0$, $x = 2$

21. $y = \sqrt{9 - x^2}$, $y = 0$ 22. $y = x - x^2$, $y = 0$

23. $y = \sqrt{\cos x}$, $0 \leq x \leq \pi/2$, $y = 0$, $x = 0$

24. $y = \sec x$, $y = 0$, $x = -\pi/4$, $x = \pi/4$

25. $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$

26. The region between the curve $y = \sqrt{\cot x}$ and the x -axis from $x = \pi/6$ to $x = \pi/2$.

27. The region between the curve $y = 1/(2\sqrt{x})$ and the x -axis from $x = 1/4$ to $x = 4$.

28. $y = e^{x-1}$, $y = 0$, $x = 1$, $x = 3$

In Exercises 29 and 30, find the volume of the solid generated by revolving the region about the given line.

29. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y -axis, about the line $y = \sqrt{2}$

30. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \leq x \leq \pi/2$, and on the left by the y -axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 31–36 about the y -axis.

31. The region enclosed by $x = \sqrt{5}y^2$, $x = 0$, $y = -1$, $y = 1$

32. The region enclosed by $x = y^{3/2}$, $x = 0$, $y = 2$

33. The region enclosed by $x = \sqrt{2 \sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$

34. The region enclosed by $x = \sqrt{\cos(\pi y/4)}$, $-2 \leq y \leq 0$, $x = 0$

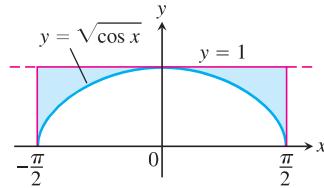
35. $x = 2/\sqrt{y+1}$, $x = 0$, $y = 0$, $y = 3$

36. $x = \sqrt{2y/(y^2+1)}$, $x = 0$, $y = 1$

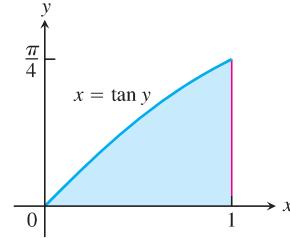
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 37 and 38 about the indicated axes.

37. The x -axis



38. The y -axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 39–44 about the x -axis.

39. $y = x$, $y = 1$, $x = 0$

40. $y = 2\sqrt{x}$, $y = 2$, $x = 0$

41. $y = x^2 + 1$, $y = x + 3$

42. $y = 4 - x^2$, $y = 2 - x$

43. $y = \sec x$, $y = \sqrt{2}$, $-\pi/4 \leq x \leq \pi/4$

44. $y = \sec x$, $y = \tan x$, $x = 0$, $x = 1$

In Exercises 45–48, find the volume of the solid generated by revolving each region about the y -axis.

45. The region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$, and $(1, 1)$

46. The region enclosed by the triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

47. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x -axis, and on the right by the line $x = 2$

48. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 49 and 50, find the volume of the solid generated by revolving each region about the given axis.

49. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x -axis, and on the right by the line $x = 1$, about the line $x = -1$

50. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x -axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

51. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about
- the x -axis.
 - the y -axis.
 - the line $y = 2$.
 - the line $x = 4$.
52. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about
- the line $x = 1$.
 - the line $x = 2$.
53. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about
- the line $y = 1$.
 - the line $y = 2$.
 - the line $y = -1$.
54. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about
- the x -axis.
 - the y -axis.

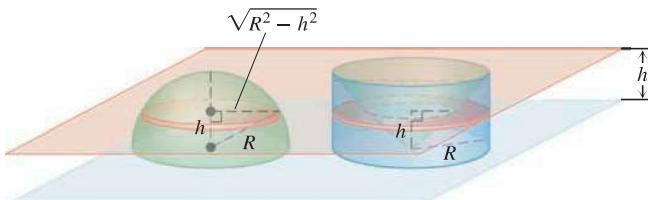
Theory and Applications

55. **The volume of a torus** The disk $x^2 + y^2 \leq a^2$ is revolved about the line $x = b$ ($b > a$) to generate a solid shaped like a doughnut and called a **torus**. Find its volume. (Hint: $\int_{-a}^a \sqrt{a^2 - y^2} dy = \pi a^2/2$, since it is the area of a semicircle of radius a .)
56. **Volume of a bowl** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y -axis.
- Find the volume of the bowl.
 - Related rates** If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

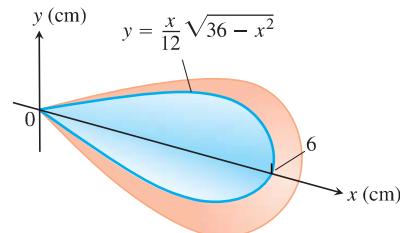
57. Volume of a bowl

- A hemispherical bowl of radius a contains water to a depth h . Find the volume of water in the bowl.
- Related rates** Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of $0.2 \text{ m}^3/\text{sec}$. How fast is the water level in the bowl rising when the water is 4 m deep?
- Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

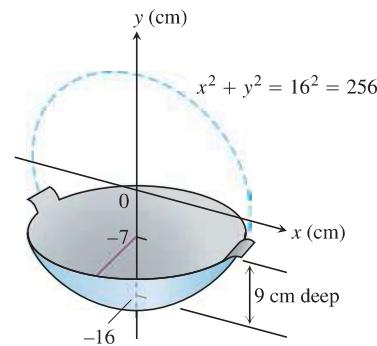
59. **Volume of a hemisphere** Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the accompanying figure.



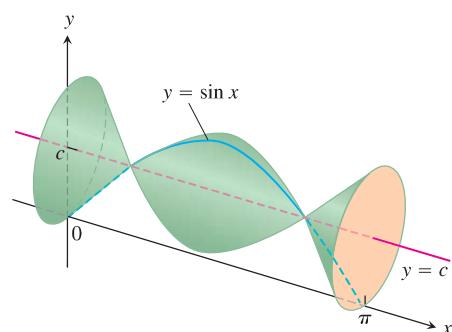
60. **Designing a plumb bob** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3 , how much will the plumb bob weigh (to the nearest gram)?



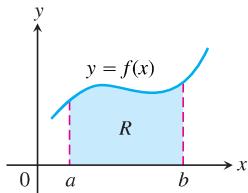
61. **Designing a wok** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3 .)



62. **Max-min** The arch $y = \sin x$, $0 \leq x \leq \pi$, is revolved about the line $y = c$, $0 \leq c \leq 1$, to generate the solid in the accompanying figure.
- Find the value of c that minimizes the volume of the solid. What is the minimum volume?
 - What value of c in $[0, 1]$ maximizes the volume of the solid?
 - Graph the solid's volume as a function of c , first for $0 \leq c \leq 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



63. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, $x = b > a$, and $y = 0$ (see accompanying figure). If the volume of the solid formed by revolving R about the x -axis is 4π , and the volume of the solid formed by revolving R about the line $y = -1$ is 8π , find the area of R .



64. Consider the region R given in Exercise 63. If the volume of the solid formed by revolving R around the x -axis is 6π , and the volume of the solid formed by revolving R around the line $y = -2$ is 10π , find the area of R .

6.2

Volumes Using Cylindrical Shells

In Section 6.1 we defined the volume of a solid as the definite integral $V = \int_a^b A(x) dx$, where $A(x)$ is an integrable cross-sectional area of the solid from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x -axis. However, this method of slicing is sometimes awkward to apply, as we will illustrate in our first example. To overcome this difficulty, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.

Slicing with Cylinders

Suppose we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid so that the axis of each cylinder is parallel to the y -axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx . This slab interpretation allows us to apply the same integral definition for volume as before. The following example provides some insight before we derive the general method.

EXAMPLE 1 The region enclosed by the x -axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (Figure 6.16). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x -values of the left and right sides of the parabola in Figure 6.16a in terms of y . (These x -values are the inner and outer radii for a typical washer, requiring us to solve $y = 3x - x^2$ for x , which leads to complicated formulas.) Instead of rotating a horizontal strip of thickness Δy , we rotate a *vertical strip* of thickness Δx . This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip and of thickness Δx . An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.17. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole. We then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of

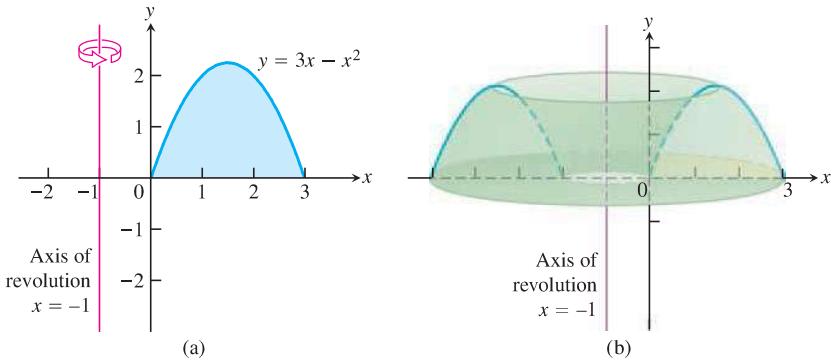


FIGURE 6.16 (a) The graph of the region in Example 1, before revolution.
(b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

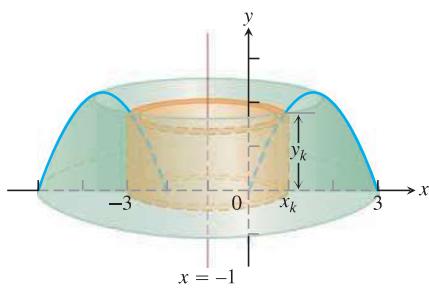


FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k , where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.16a).

Each slice is sitting over a subinterval of the x -axis of length (width) Δx_k . Its radius is approximately $(1 + x_k)$, and its height is approximately $3x_k - x_k^2$. If we unroll the cylinder at x_k and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx_k (Figure 6.18). The outer circumference of the k th cylinder is $2\pi \cdot$ radius $= 2\pi(1 + x_k)$, and this is the length of the rolled-out rectangular slab. Its volume is approximated by that of a rectangular solid,

$$\begin{aligned}\Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness} \\ &= 2\pi(1 + x_k) \cdot (3x_k - x_k^2) \cdot \Delta x_k.\end{aligned}$$

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum

$$\sum_{k=1}^n \Delta V_k = \sum_{k=1}^n 2\pi(x_k + 1)(3x_k - x_k^2) \Delta x_k.$$

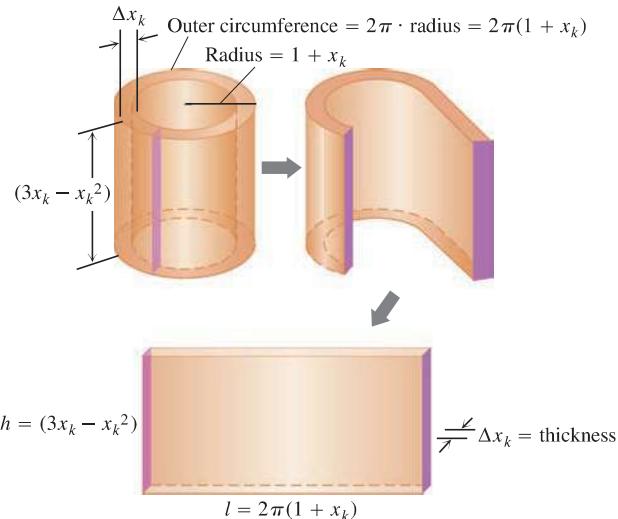


FIGURE 6.18 Cutting and unrolling a cylindrical shell gives a nearly rectangular solid (Example 1).

Taking the limit as the thickness $\Delta x_k \rightarrow 0$ and $n \rightarrow \infty$ gives the volume integral

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{k=1}^n 2\pi(x_k + 1)(3x_k - x_k^2) \Delta x_k \\ &= \int_0^3 2\pi(x + 1)(3x - x^2) dx \\ &= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2) dx \\ &= 2\pi \int_0^3 (2x^2 + 3x - x^3) dx \\ &= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4 \right]_0^3 = \frac{45\pi}{2}. \end{aligned}$$

■

We now generalize the procedure used in Example 1.

The Shell Method

Suppose the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x -axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (Figure 6.19a). We assume $a \geq L$, so the vertical line may touch the region, but not pass through it. We generate a solid S by rotating this region about the vertical line L .

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \dots < x_n = b$, and let c_k be the midpoint of the k th subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.19a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width $\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.19b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

The volume of a cylindrical shell of height h with inner radius r and outer radius R is

$$\pi R^2 h - \pi r^2 h = 2\pi \left(\frac{R+r}{2} \right) (h)(R-r)$$

$$\begin{aligned} \Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \end{aligned}$$

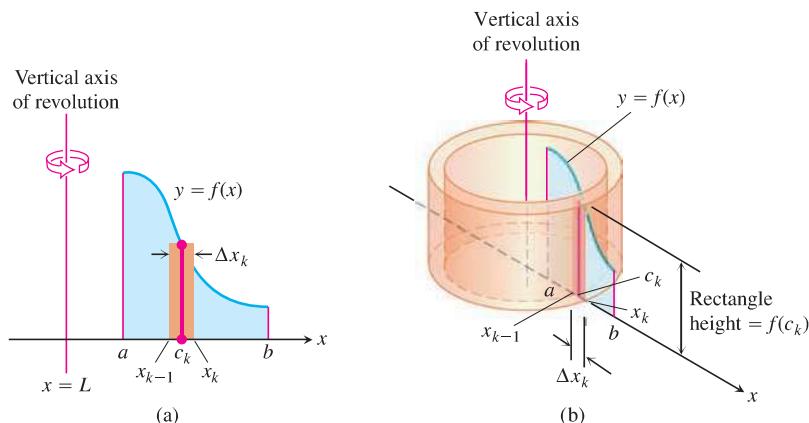


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles based on P :

$$V \approx \sum_{k=1}^n \Delta V_k$$

The limit of this Riemann sum as each $\Delta x_k \rightarrow 0$ and $n \rightarrow \infty$ gives the volume of the solid as a definite integral:

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta V_k = \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx. \\ &= \int_a^b 2\pi(x - L)f(x) dx. \end{aligned}$$

We refer to the variable of integration, here x , as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x -axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \left(\begin{matrix} \text{shell} \\ \text{(radius)} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{(height)} \end{matrix} \right) dx.$$

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

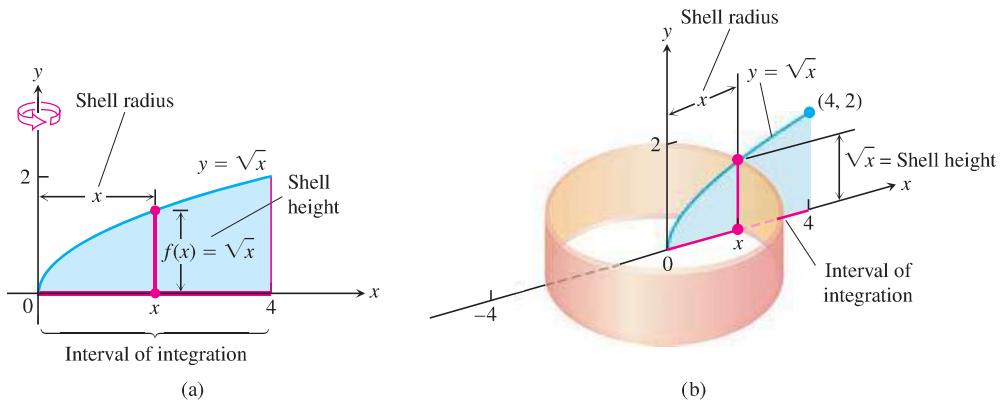


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx .

The shell thickness variable is x , so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi \left(\text{radius} \right) \left(\text{height} \right) dx \\ &= \int_0^4 2\pi(x)(\sqrt{x}) dx \\ &= 2\pi \int_0^4 x^{3/2} dx = 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

■

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x 's with y 's.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 4 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y , so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y -axis in Figure 6.21). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \left(\text{radius} \right) \left(\text{height} \right) dy \\ &= \int_0^2 2\pi(y)(4 - y^2) dy \\ &= 2\pi \int_0^2 (4y - y^3) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi. \end{aligned}$$

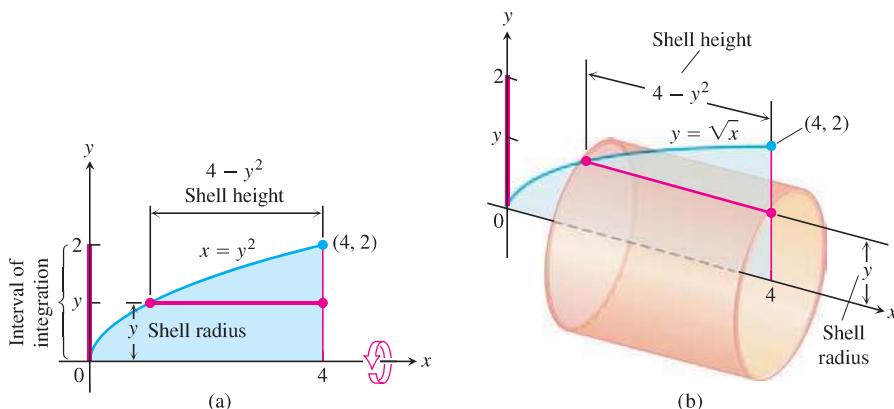


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3.
(b) The shell swept out by the horizontal segment in part (a) with a width Δy .

■

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. Find the limits of integration for the thickness variable.
3. Integrate the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

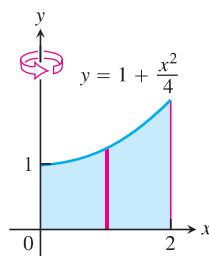
The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. (Exercise 45 outlines a proof.) Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

Exercises 6.2

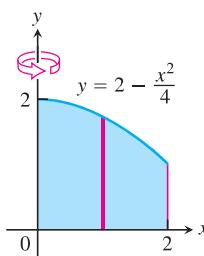
Revolution About the Axes

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

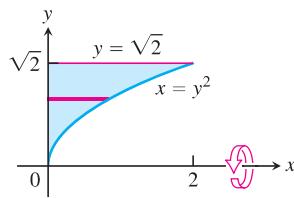
1.



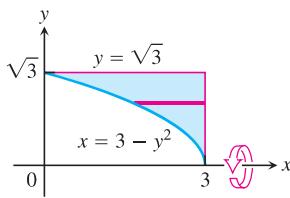
2.



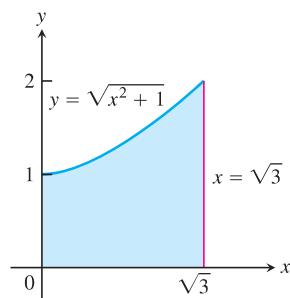
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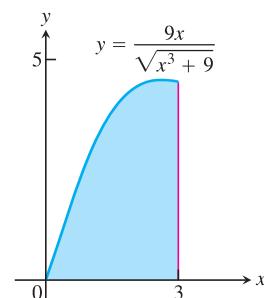
4.



5. The y-axis



6. The y-axis



Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y -axis.

7. $y = x$, $y = -x/2$, $x = 2$

8. $y = 2x$, $y = x/2$, $x = 1$

9. $y = x^2$, $y = 2 - x$, $x = 0$, for $x \geq 0$

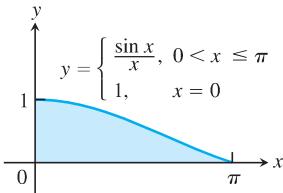
10. $y = 2 - x^2$, $y = x^2$, $x = 0$

11. $y = 2x - 1$, $y = \sqrt{x}$, $x = 0$

12. $y = 3/(2\sqrt{x})$, $y = 0$, $x = 1$, $x = 4$

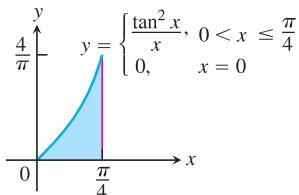
13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

- a. Show that $x f(x) = \sin x, 0 \leq x \leq \pi$.
- b. Find the volume of the solid generated by revolving the shaded region about the y -axis in the accompanying figure.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \leq \pi/4 \\ 0, & x = 0 \end{cases}$

- a. Show that $x g(x) = (\tan x)^2, 0 \leq x \leq \pi/4$.
- b. Find the volume of the solid generated by revolving the shaded region about the y -axis in the accompanying figure.



Revolution About the x -Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x -axis.

15. $x = \sqrt{y}, \quad x = -y, \quad y = 2$

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$

17. $x = 2y - y^2, \quad x = 0 \quad 18. \quad x = 2y - y^2, \quad x = y$

19. $y = |x|, \quad y = 1 \quad 20. \quad y = x, \quad y = 2x, \quad y = 2$

21. $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

22. $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

Revolution About Horizontal and Vertical Lines

In Exercises 23–26, use the shell method to find the volumes of the solids generated by revolving the regions bounded by the given curves about the given lines.

23. $y = 3x, \quad y = 0, \quad x = 2$

- a. The y -axis
- b. The line $x = 4$
- c. The line $x = -1$
- d. The x -axis
- e. The line $y = 7$
- f. The line $y = -2$

24. $y = x^3, \quad y = 8, \quad x = 0$

- a. The y -axis
- b. The line $x = 3$
- c. The line $x = -2$
- d. The x -axis
- e. The line $y = 8$
- f. The line $y = -1$

25. $y = x + 2, \quad y = x^2$

- a. The line $x = 2$
- b. The line $x = -1$
- c. The x -axis
- d. The line $y = 4$

26. $y = x^4, \quad y = 4 - 3x^2$

- a. The line $x = 1$
- b. The x -axis

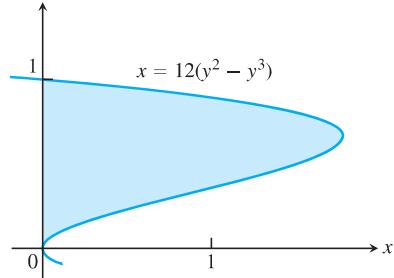
In Exercises 27 and 28, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

27. a. The x -axis

- b. The line $y = 1$

- c. The line $y = 8/5$

- d. The line $y = -2/5$

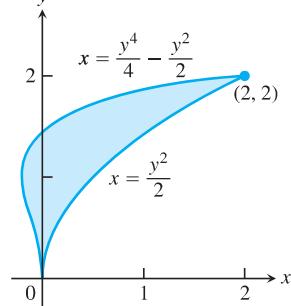


28. a. The x -axis

- b. The line $y = 2$

- c. The line $y = 5$

- d. The line $y = -5/8$



Choosing the Washer Method or Shell Method

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y -axis, for example, and washers are used, we must integrate with respect to y . It may not be possible, however, to express the integrand in terms of y . In such a case, the shell method allows us to integrate with respect to x instead. Exercises 29 and 30 provide some insight.

29. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

- a. the shell method.
- b. the washer method.

30. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

- a. the x -axis using the washer method.

- b. the y -axis using the shell method.

- c. the line $x = 4$ using the shell method.

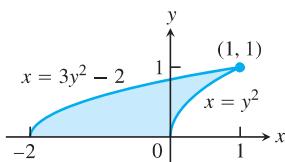
- d. the line $y = 8$ using the washer method.

In Exercises 31–36, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

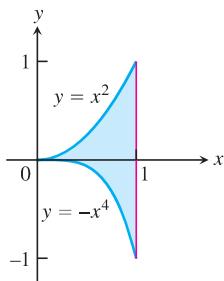
31. The triangle with vertices $(1, 1)$, $(1, 2)$, and $(2, 2)$ about
 a. the x -axis b. the y -axis
 c. the line $x = 10/3$ d. the line $y = 1$
32. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about
 a. the x -axis b. the y -axis
 c. the line $x = 4$ d. the line $y = 2$
33. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y -axis about
 a. the x -axis b. the line $y = 1$
34. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about
 a. the x -axis b. the y -axis
 c. the line $x = 1$ d. the line $y = 1$
35. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about
 a. the x -axis b. the y -axis
36. The region bounded by $y = 2x - x^2$ and $y = x$ about
 a. the y -axis b. the line $x = 1$
37. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the x -axis to generate a solid. Find the volume of the solid by
 a. the washer method. b. the shell method.
38. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y -axis to generate a solid. Find the volume of the solid by
 a. the washer method. b. the shell method.

Theory and Examples

39. The region shown here is to be revolved about the x -axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



40. The region shown here is to be revolved about the y -axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



41. A bead is formed from a sphere of radius 5 by drilling through a diameter of the sphere with a drill bit of radius 3.
 a. Find the volume of the bead.
 b. Find the volume of the removed portion of the sphere.
42. A Bundt cake, well known for having a ringed shape, is formed by revolving around the y -axis the region bounded by the graph of $y = \sin(x^2 - 1)$ and the x -axis over the interval $1 \leq x \leq \sqrt{1 + \pi}$. Find the volume of the cake.
43. Derive the formula for the volume of a right circular cone of height h and radius r using an appropriate solid of revolution.
44. Derive the equation for the volume of a sphere of radius r using the shell method.
45. **Equivalence of the washer and shell methods for finding volume** Let f be differentiable and increasing on the interval $a \leq x \leq b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1} . Revolve about the y -axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi((f^{-1}(y))^2 - a^2) dy = \int_a^b 2\pi x(f(b) - f(x)) dx.$$

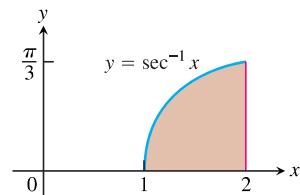
To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi((f^{-1}(y))^2 - a^2) dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x)) dx.$$

Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.8, Exercise 128, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (Source: "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly*, Vol. 98, No. 2, Feb. 1991, pp. 154–156.)

46. The region between the curve $y = \sec^{-1} x$ and the x -axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y -axis to generate a solid. Find the volume of the solid.



47. Find the volume of the solid generated by revolving the region enclosed by the graphs of $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$ about the y -axis.
48. Find the volume of the solid generated by revolving the region enclosed by the graphs of $y = e^{x/2}$, $y = 1$, and $x = \ln 3$ about the x -axis.

6.3

Arc Length

We know what is meant by the length of a straight line segment, but without calculus, we have no precise definition of the length of a general winding curve. If the curve is the graph of a continuous function defined over an interval, then we can find the length of the curve using a procedure similar to that we used for defining the area between the curve and the x -axis. This procedure results in a division of the curve from point A to point B into many pieces and joining successive points of division by straight line segments. We then sum the lengths of all these line segments and define the length of the curve to be the limiting value of this sum as the number of segments goes to infinity.

Length of a Curve $y = f(x)$

Suppose the curve whose length we want to find is the graph of the function $y = f(x)$ from $x = a$ to $x = b$. In order to derive an integral formula for the length of the curve, we assume that f has a continuous derivative at every point of $[a, b]$. Such a function is called **smooth**, and its graph is a **smooth curve** because it does not have any breaks, corners, or cusps.

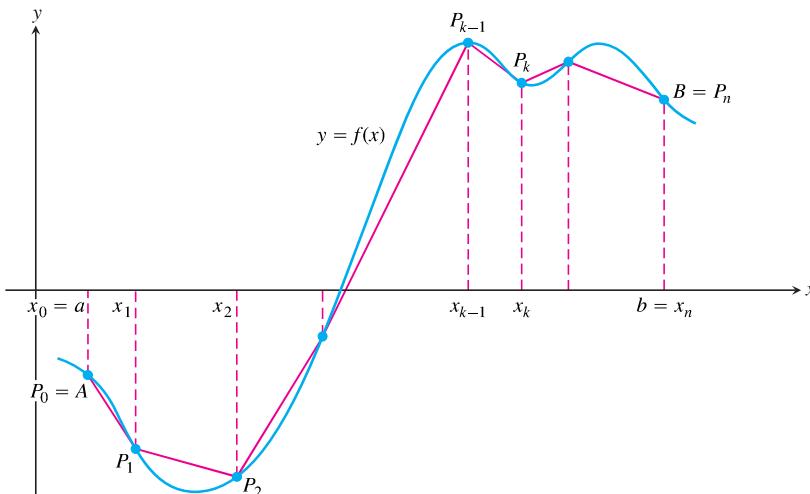


FIGURE 6.22 The length of the polygonal path $P_0P_1P_2 \cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B .

We partition the interval $[a, b]$ into n subintervals with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. If $y_k = f(x_k)$, then the corresponding point $P_k(x_k, y_k)$ lies on the curve. Next we connect successive points P_{k-1} and P_k with straight line segments that, taken together, form a polygonal path whose length approximates the length of the curve (Figure 6.22). If $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, then a representative line segment in the path has length (see Figure 6.23)

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2},$$

so the length of the curve is approximated by the sum

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \quad (1)$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. Now, by the Mean Value Theorem, there is a point c_k , with $x_{k-1} < c_k < x_k$, such that

$$\Delta y_k = f'(c_k) \Delta x_k.$$

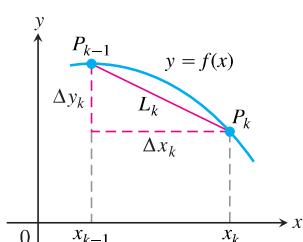


FIGURE 6.23 The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

With this substitution for Δy_k , the sums in Equation (1) take the form

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k. \quad (2)$$

Because $\sqrt{1 + [f'(x)]^2}$ is continuous on $[a, b]$, the limit of the Riemann sum on the right-hand side of Equation (2) exists as the norm of the partition goes to zero, giving

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n L_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

We define the value of this limiting integral to be the length of the curve.

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (3)$$

EXAMPLE 1 Find the length of the curve (Figure 6.24)

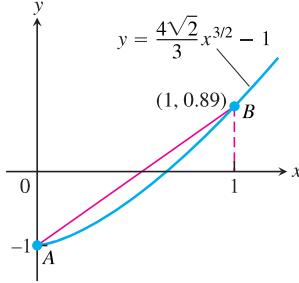


FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = (2\sqrt{2}x^{1/2})^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx \\ &= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big|_0^1 = \frac{13}{6} \approx 2.17. \end{aligned}$$

Eq. (3) with
 $a = 0, b = 1$
Let $u = 1 + 8x$,
integrate, and
replace u by
 $1 + 8x$.

Notice that the length of the curve is slightly larger than the length of the straight-line segment joining the points $A = (0, -1)$ and $B = (1, 4\sqrt{2}/3 - 1)$ on the curve (see Figure 6.24):

$$2.17 > \sqrt{1^2 + (1.89)^2} \approx 2.14 \quad \text{Decimal approximations} \blacksquare$$

EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \leq x \leq 4.$$

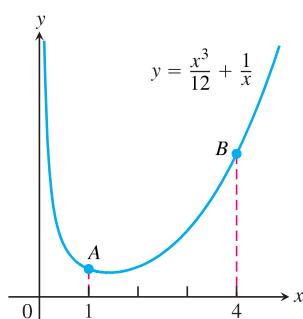


FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

Solution A graph of the function is shown in Figure 6.25. To use Equation (3), we find

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

so

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right) \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2. \end{aligned}$$

The length of the graph over $[1, 4]$ is

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + [f'(x)]^2} dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx \\ &= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6. \end{aligned}$$

■

EXAMPLE 3 Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \quad 0 \leq x \leq 2.$$

Solution We use Equation (3) with $a = 0$, $b = 2$, and

$$\begin{aligned} y &= \frac{1}{2}(e^x + e^{-x}) \\ \frac{dy}{dx} &= \frac{1}{2}(e^x - e^{-x}) \\ \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\ 1 + \left(\frac{dy}{dx}\right)^2 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \left[\frac{1}{2}(e^x + e^{-x})\right]^2. \end{aligned}$$

The length of the curve from $x = 0$ to $x = 2$ is

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^2 \frac{1}{2}(e^x + e^{-x}) dx && \text{Eq. (3) with } a = 0, b = 2 \\ &= \frac{1}{2} \left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.63. \end{aligned}$$

■

Dealing with Discontinuities in dy/dx

At a point on a curve where dy/dx fails to exist, dx/dy may exist. In this case, we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (3):

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy. \quad (4)$$

EXAMPLE 4 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (3).

We therefore rewrite the equation to express x in terms of y :

$$\begin{aligned} y &= \left(\frac{x}{2}\right)^{2/3} \\ y^{3/2} &= \frac{x}{2} && \text{Raise both sides to the power } 3/2. \\ x &= 2y^{3/2}. && \text{Solve for } x. \end{aligned}$$

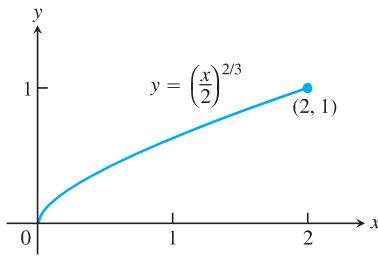


FIGURE 6.26 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.26).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (4) to find the curve's length:

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + 9y} dy && \text{Eq. (4) with } c = 0, d = 1. \\ &= \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1 && \text{Let } u = 1 + 9y, \\ &= \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27. && du/9 = dy, \\ &&& \text{integrate, and} \\ &&& \text{substitute back.} \end{aligned}$$

The Differential Formula for Arc Length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt. \quad (5)$$

From Equation (3) and Figure 6.22, we see that this function $s(x)$ is continuous and measures the length along the curve $y = f(x)$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$ for each $x \in [a, b]$. The function s is called the **arc length function** for $y = f(x)$. From the Fundamental Theorem, the function s is differentiable on (a, b) and

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (6)$$

A useful way to remember Equation (6) is to write

$$ds = \sqrt{dx^2 + dy^2}, \quad (7)$$

which can be integrated between appropriate limits to give the total length of a curve. From this point of view, all the arc length formulas are simply different expressions for the equation

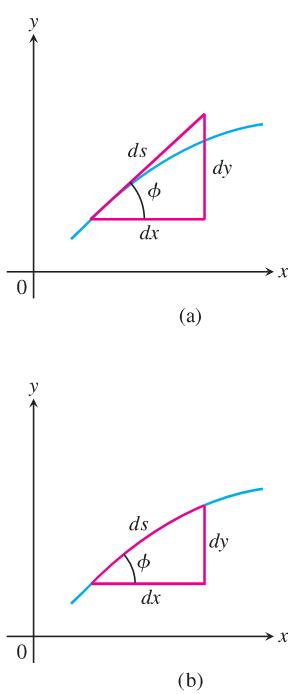


FIGURE 6.27 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

$L = \int ds$. Figure 6.27a gives the exact interpretation of ds corresponding to Equation (7). Figure 6.27b is not strictly accurate, but is to be thought of as a simplified approximation of Figure 6.27a. That is, $ds \approx \Delta s$.

EXAMPLE 5 Find the arc length function for the curve in Example 2 taking $A = (1, 13/12)$ as the starting point (see Figure 6.25).

Solution In the solution to Example 2, we found that

$$1 + [f'(x)]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

Therefore the arc length function is given by

$$\begin{aligned}s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2} dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt \\ &= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}.\end{aligned}$$

To compute the arc length along the curve from $A = (1, 13/12)$ to $B = (4, 67/12)$, for instance, we simply calculate

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$

This is the same result we obtained in Example 2. ■

Exercises 6.3

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–10. If you have a grapher, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = x^{3/2}$ from $x = 0$ to $x = 4$
3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
4. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
5. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
6. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
7. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5$, $1 \leq x \leq 8$
8. $y = (x^3/3) + x^2 + x + 1/(4x + 4)$, $0 \leq x \leq 2$
9. $x = \int_0^y \sqrt{\sec^4 t - 1} dt$, $-\pi/4 \leq y \leq \pi/4$
10. $y = \int_{-2}^x \sqrt{3t^4 - 1} dt$, $-2 \leq x \leq -1$

T Finding Integrals for Lengths of Curves

In Exercises 11–18, do the following.

- a. Set up an integral for the length of the curve.
- b. Graph the curve to see what it looks like.
- c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

11. $y = x^2$, $-1 \leq x \leq 2$
12. $y = \tan x$, $-\pi/3 \leq x \leq 0$
13. $x = \sin y$, $0 \leq y \leq \pi$
14. $x = \sqrt{1 - y^2}$, $-1/2 \leq y \leq 1/2$
15. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$
16. $y = \sin x - x \cos x$, $0 \leq x \leq \pi$
17. $y = \int_0^x \tan t dt$, $0 \leq x \leq \pi/6$
18. $x = \int_0^y \sqrt{\sec^2 t - 1} dt$, $-\pi/3 \leq y \leq \pi/4$

Theory and Examples

19. a. Find a curve through the point $(1, 1)$ whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}} dx.$$

- b. How many such curves are there? Give reasons for your answer.

20. a. Find a curve through the point $(0, 1)$ whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}} dy.$$

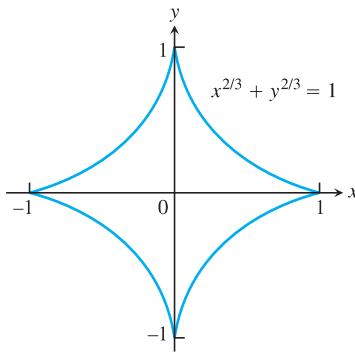
- b. How many such curves are there? Give reasons for your answer.

21. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t} dt$$

from $x = 0$ to $x = \pi/4$.

22. **The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not “asteroids”) because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \leq x \leq 1$, and multiplying by 8.



23. **Length of a line segment** Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \leq x \leq 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

24. **Circumference of a circle** Set up an integral to find the circumference of a circle of radius r centered at the origin. You will learn how to evaluate the integral in Section 8.3.

25. If $9x^2 = y(y - 3)^2$, show that

$$ds^2 = \frac{(y+1)^2}{4y} dy^2.$$

26. If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2} (5x^2 - 16) dx^2.$$

27. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \leq x \leq a$ is always $\sqrt{2a}$? Give reasons for your answer.

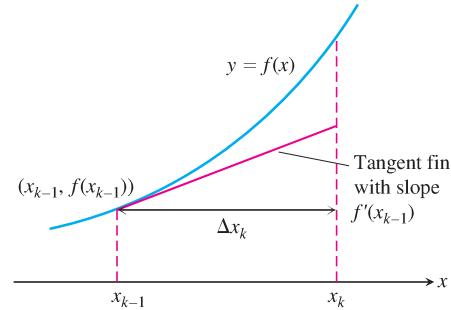
28. **Using tangent fins to derive the length formula for curves** Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

- a. Show that the length of the k th tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1}) \Delta x_k)^2}$.

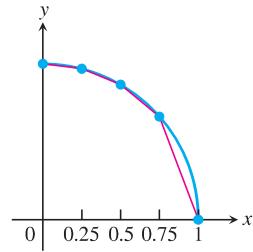
- b. Show that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2} dx,$$

which is the length L of the curve $y = f(x)$ from a to b .



29. Approximate the arc length of one-quarter of the unit circle (which is $\frac{\pi}{2}$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).



30. **Distance between two points** Assume that the two points (x_1, y_1) and (x_2, y_2) lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

31. Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using $(0, 0)$ as the starting point. What is the length of the curve from $(0, 0)$ to $(1, 2)$?

32. Find the arc length function for the curve in Exercise 8, using $(0, 1/4)$ as the starting point. What is the length of the curve from $(0, 1/4)$ to $(1, 59/24)$?

COMPUTER EXPLORATIONS

In Exercises 33–38, use a CAS to perform the following steps for the given graph of the function over the closed interval.

- Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)
- Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.
- Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

33. $f(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$

34. $f(x) = x^{1/3} + x^{2/3}$, $0 \leq x \leq 2$

35. $f(x) = \sin(\pi x^2)$, $0 \leq x \leq \sqrt{2}$

36. $f(x) = x^2 \cos x$, $0 \leq x \leq \pi$

37. $f(x) = \frac{x-1}{4x^2+1}$, $-\frac{1}{2} \leq x \leq 1$

38. $f(x) = x^3 - x^2$, $-1 \leq x \leq 1$

6.4

Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated in Chapter 16.

Defining Surface Area

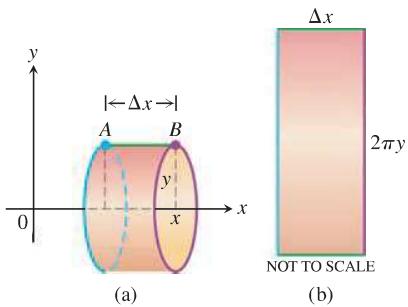


FIGURE 6.28 (a) A cylindrical surface generated by revolving the horizontal line segment AB of length Δx about the x -axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution, as we saw earlier in the chapter. However, if you revolve only the bounding curve itself, it does not sweep out any interior volume but rather a surface that surrounds the solid and forms part of its boundary. Just as we were interested in defining and finding the length of a curve in the last section, we are now interested in defining and finding the area of a surface generated by revolving a curve about an axis.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x -axis. If we rotate the horizontal line segment AB having length Δx about the x -axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x -axis.

Suppose the line segment AB has length L and is slanted rather than horizontal. Now when AB is rotated about the x -axis, it generates a frustum of a cone (Figure 6.29a). From classical geometry, the surface area of this frustum is $2\pi y^* L$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x -axis. This surface area is the same as that of a rectangle with side lengths L and $2\pi y^*$ (Figure 6.29b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x -axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \leq x \leq b$, about the x -axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc PQ and the band it sweeps out as part of the graph of f .

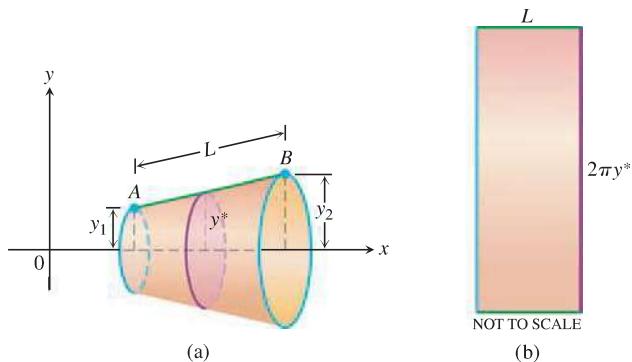


FIGURE 6.29 (a) The frustum of a cone generated by revolving the slanted line segment AB of length L about the x -axis has area $2\pi y^* L$. (b) The area of the rectangle for $y^* = \frac{y_1 + y_2}{2}$, the average height of AB above the x -axis.

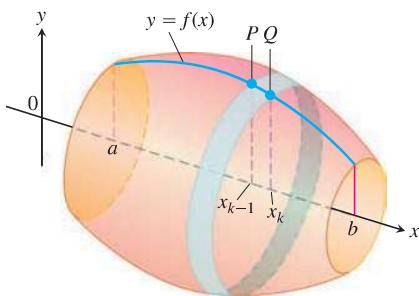


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x -axis. The surface is a union of bands like the one swept out by the arc PQ .

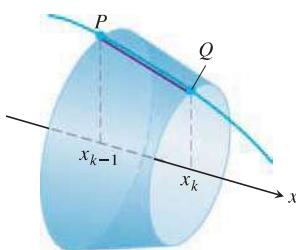


FIGURE 6.31 The line segment joining P and Q sweeps out a frustum of a cone.

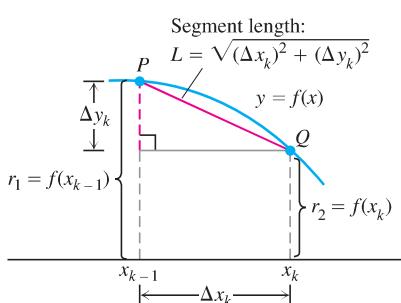


FIGURE 6.32 Dimensions associated with the arc and line segment PQ .

As the arc PQ revolves about the x -axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x -axis (Figure 6.31). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ . The surface area of the frustum of the cone shown in Figure 6.31 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q , and L is its length (just as before). Since $f \geq 0$, from Figure 6.32 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned} \text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ , is approximated by the frustum area sum

$$\sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \quad (1)$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. Moreover, if the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.33). At this point,

$$\begin{aligned} f'(c_k) &= \frac{\Delta y_k}{\Delta x_k}, \\ \Delta y_k &= f'(c_k) \Delta x_k. \end{aligned}$$

With this substitution for Δy_k , the sums in Equation (1) take the form

$$\begin{aligned} &\sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k) \Delta x_k)^2} \\ &= \sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2} \Delta x_k. \end{aligned} \quad (2)$$

These sums are not the Riemann sums of any function because the points x_{k-1} , x_k , and c_k are not the same. However, it can be proved that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b .

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx. \quad (3)$$

The square root in Equation (3) is the same one that appears in the formula for the arc length differential of the generating curve in Equation (6) of Section 6.3.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x -axis (Figure 6.34).

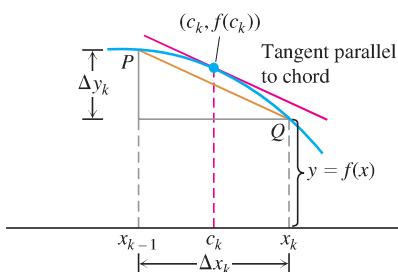


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ .

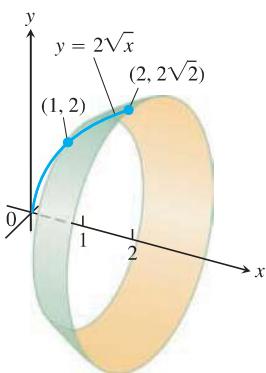


FIGURE 6.34 In Example 1 we calculate the area of this surface.

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}. \end{aligned}$$

With these substitutions, we have

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$

■

Revolution About the y -Axis

For revolution about the y -axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the y -Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y -axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

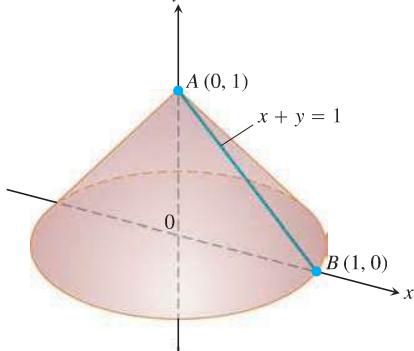


FIGURE 6.35 Revolving line segment AB about the y -axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y -axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$\begin{aligned} c &= 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1, \\ \sqrt{1 + \left(\frac{dx}{dy}\right)^2} &= \sqrt{1 + (-1)^2} = \sqrt{2} \end{aligned}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1-y)\sqrt{2} dy \\ &= 2\pi\sqrt{2} \left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2} \left(1 - \frac{1}{2}\right) \\ &= \pi\sqrt{2}. \end{aligned}$$

The results agree, as they should.

Exercises 6.4

Finding Integrals for Surface Area

In Exercises 1–8:

- a. Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.
 - T** b. Graph the curve to see what it looks like. If you can, graph the surface too.
 - T** c. Use your grapher's or computer's integral evaluator to find the surface's area numerically.
1. $y = \tan x, 0 \leq x \leq \pi/4;$ x-axis
 2. $y = x^2, 0 \leq x \leq 2;$ x-axis
 3. $xy = 1, 1 \leq y \leq 2;$ y-axis
 4. $x = \sin y, 0 \leq y \leq \pi;$ y-axis
 5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4);$ x-axis
 6. $y + 2\sqrt{y} = x, 1 \leq y \leq 2;$ y-axis
 7. $x = \int_0^y \tan t dt, 0 \leq y \leq \pi/3;$ y-axis
 8. $y = \int_1^x \sqrt{t^2 - 1} dt, 1 \leq x \leq \sqrt{5};$ x-axis

Finding Surface Area

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \leq x \leq 4,$ about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height.}$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \leq x \leq 4,$ about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height.}$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \leq x \leq 3,$ about the x-axis. Check your result with the geometry formula

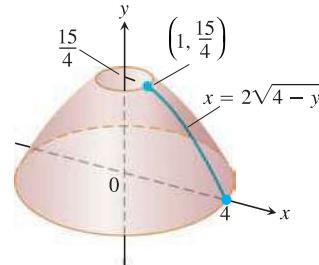
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height.}$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \leq x \leq 3,$ about the y-axis. Check your result with the geometry formula

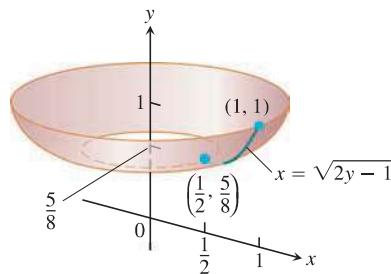
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height.}$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

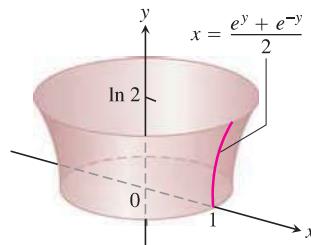
13. $y = x^3/9, 0 \leq x \leq 2;$ x-axis
14. $y = \sqrt{x}, 3/4 \leq x \leq 15/4;$ x-axis
15. $y = \sqrt{2x - x^2}, 0.5 \leq x \leq 1.5;$ x-axis
16. $y = \sqrt{x+1}, 1 \leq x \leq 5;$ x-axis
17. $x = y^3/3, 0 \leq y \leq 1;$ y-axis
18. $x = (1/3)y^{3/2} - y^{1/2}, 1 \leq y \leq 3;$ y-axis
19. $x = 2\sqrt{4 - y}, 0 \leq y \leq 15/4;$ y-axis



20. $x = \sqrt{2y - 1}, 5/8 \leq y \leq 1;$ y-axis



21. $x = (e^y + e^{-y})/2, 0 \leq y \leq \ln 2;$ y-axis



22. $y = (1/3)(x^2 + 2)^{3/2}$, $0 \leq x \leq \sqrt{2}$; y -axis (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx , and evaluate the integral $S = \int 2\pi x \, ds$ with appropriate limits.)

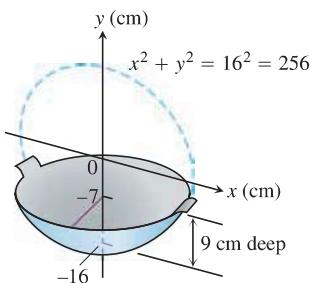
23. $x = (y^4/4) + 1/(8y^2)$, $1 \leq y \leq 2$; x -axis (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy , and evaluate the integral $S = \int 2\pi y \, ds$ with appropriate limits.)

24. Write an integral for the area of the surface generated by revolving the curve $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$, about the x -axis. In Section 8.4 we will see how to evaluate such integrals.

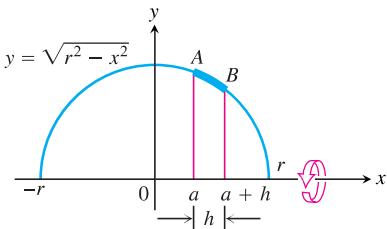
25. **Testing the new definition** Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}$, $-a \leq x \leq a$, about the x -axis.

26. **Testing the new definition** The lateral (side) surface area of a cone of height h and base radius r should be $\pi r \sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x$, $0 \leq x \leq h$, about the x -axis.

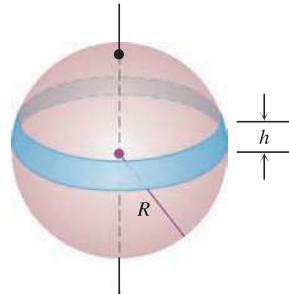
T 27. **Enameling woks** Your company decided to put out a deluxe version of a wok you designed. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See accompanying figure.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1 \text{ cm}^3 = 1 \text{ mL}$, so $1 \text{ L} = 1000 \text{ cm}^3$.)



28. **Slicing bread** Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x -axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x -axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



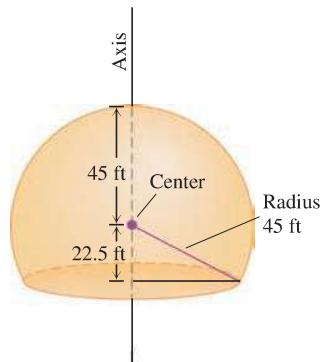
29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

- How much outside surface is there to paint (not counting the bottom)?

- Express the answer to the nearest square foot.

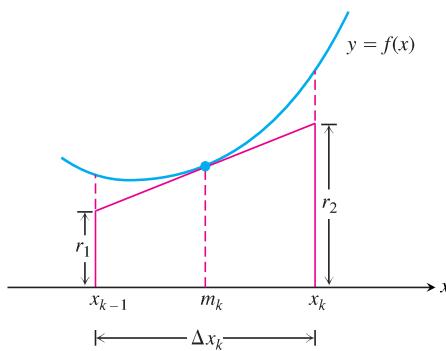


31. **An alternative derivation of the surface area formula** Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the k th subinterval $[x_{k-1}, x_k]$, construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the accompanying figure.

- Show that

$$r_1 = f(m_k) - f'(m_k) \frac{\Delta x_k}{2} \quad \text{and} \quad r_2 = f(m_k) + f'(m_k) \frac{\Delta x_k}{2}.$$

- Show that the length L_k of the tangent line segment in the k th subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k) \Delta x_k)^2}$.



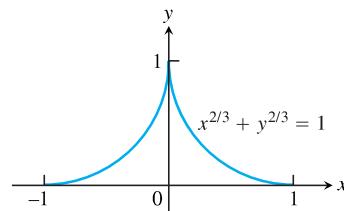
- c. Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x -axis is $2\pi f(m_k) \sqrt{1 + (f'(m_k))^2} \Delta x_k$.

- d. Show that the area of the surface generated by revolving $y = f(x)$ about the x -axis over $[a, b]$ is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\text{lateral surface area of } k\text{th frustum} \right) = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

32. **The surface of an astroid** Find the area of the surface generated by revolving about the x -axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown in the accompanying figure.

(Hint: Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \leq x \leq 1$, about the x -axis and double your result.)



6.5 Work and Fluid Forces

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body (or object) and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

Work Done by a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the body with the formula

$$W = Fd \quad (\text{Constant-force formula for work}). \quad (1)$$

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter ($N \cdot m$). This combination appears so often it has a special name, the **joule**. In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

Joules

The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

EXAMPLE 1 Suppose you jack up the side of a 2000-lb car 1.25 ft to change a tire. The jack applies a constant vertical force of about 1000 lb in lifting the side of the car (but because of the mechanical advantage of the jack, the force you apply to the jack itself is only about 30 lb). The total work performed by the jack on the car is $1000 \times 1.25 = 1250$ ft-lb. In SI units, the jack has applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ■

Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts on an object moving along a straight line, which we take to be the x -axis. We assume that the magnitude of the force is a continuous function F of the object's position x . We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, the continuous function F

will not vary much from x_{k-1} to x_k . The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k , the same as it would be if F were constant and we could apply Equation (1). The total work done from a to b is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^n F(c_k) \Delta x_k.$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b :

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n F(c_k) \Delta x_k = \int_a^b F(x) dx.$$

DEFINITION The **work** done by a variable force $F(x)$ in the direction of motion along the x -axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x) dx. \quad (2)$$

The units of the integral are joules if F is in newtons and x is in meters, and foot-pounds if F is in pounds and x is in feet. So the work done by a force of $F(x) = 1/x^2$ newtons in moving an object along the x -axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J.}$$

Hooke's Law for Springs: $F = kx$

Hooke's Law says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x . In symbols,

$$F = kx. \quad (3)$$

The constant k , measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

EXAMPLE 2 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x -axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.36). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \quad \text{to} \quad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb.}$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x dx = 8x^2 \Big|_0^{0.25} = 0.5 \text{ ft-lb.}$$

Eq. (2) with
 $a = 0, b = 0.25,$
 $F(x) = 16x$ ■

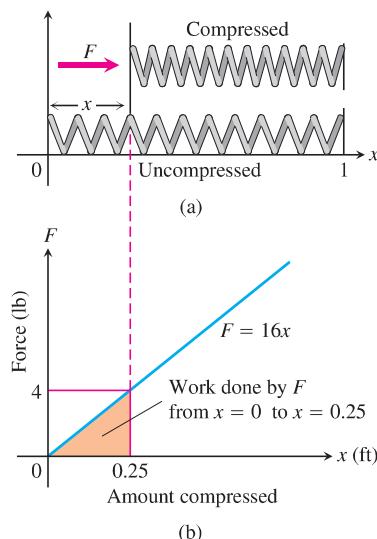


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

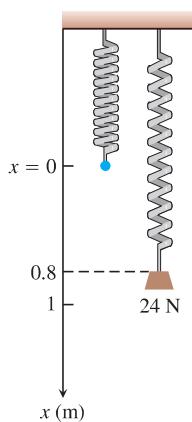


FIGURE 6.37 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

EXAMPLE 3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

- Find the force constant k .
- How much work will it take to stretch the spring 2 m beyond its natural length?
- How far will a 45-N force stretch the spring?

Solution

(a) *The force constant.* We find the force constant from Equation (3). A force of 24 N maintains the spring at a position where it is stretched 0.8 m from its natural length, so

$$\begin{aligned} 24 &= k(0.8) && \text{Eq. (3) with} \\ k &= 24/0.8 = 30 \text{ N/m.} && F = 24, x = 0.8 \end{aligned}$$

(b) *The work to stretch the spring 2 m.* We imagine the unstressed spring hanging along the x -axis with its free end at $x = 0$ (Figure 6.37). The force required to stretch the spring x m beyond its natural length is the force required to hold the free end of the spring x units from the origin. Hooke's Law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x \, dx = 15x^2 \Big|_0^2 = 60 \text{ J.}$$

(c) *How far will a 45-N force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \quad \text{or} \quad x = 1.5 \text{ m.}$$

A 45-N force will keep the spring stretched 1.5 m beyond its natural length. ■

The work integral is useful to calculate the work done in lifting objects whose weights vary with their elevation.

EXAMPLE 4 A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Figure 6.38). The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?

Solution The bucket has constant weight, so the work done lifting it alone is weight \times distance $= 5 \cdot 20 = 100$ ft-lb.

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging. When the bucket is x ft off the ground, the remaining proportion of the rope still being lifted weighs $(0.08) \cdot (20 - x)$ lb. So the work in lifting the rope is

$$\begin{aligned} \text{Work on rope} &= \int_0^{20} (0.08)(20 - x) \, dx = \int_0^{20} (1.6 - 0.08x) \, dx \\ &= [1.6x - 0.04x^2]_0^{20} = 32 - 16 = 16 \text{ ft-lb.} \end{aligned}$$

The total work for the bucket and rope combined is

$$100 + 16 = 116 \text{ ft-lb.} \quad \blacksquare$$

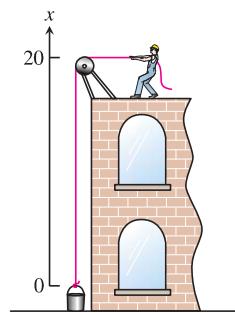


FIGURE 6.38 Lifting the bucket in Example 4.

Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? Engineers often need to know the answer in order to design or choose the right pump to transport water or some other liquid from one place to another. To find out how much work is required to pump the liquid, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next example shows what to do.

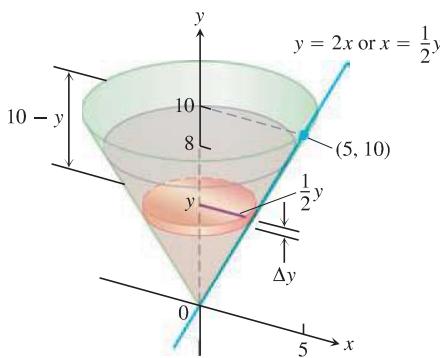


FIGURE 6.39 The olive oil and tank in Example 5.

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft³. How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y -axis at the points of a partition of the interval $[0, 8]$.

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2 \Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57 \Delta V = \frac{57\pi}{4}y^2 \Delta y \text{ lb.} \quad \text{Weight} = (\text{weight per unit volume}) \times \text{volume}$$

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2 \Delta y \text{ ft-lb.}$$

Assuming there are n slabs associated with the partition of $[0, 8]$, and that $y = y_k$ denotes the plane associated with the k th slab of thickness Δy_k , we can approximate the work done lifting all of the slabs with the Riemann sum

$$W \approx \sum_{k=1}^n \frac{57\pi}{4}(10 - y_k)y_k^2 \Delta y_k \text{ ft-lb.}$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero and the number of slabs tends to infinity:

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{57\pi}{4}(10 - y_k)y_k^2 \Delta y_k = \int_0^8 \frac{57\pi}{4}(10 - y)y^2 dy \\ &= \frac{57\pi}{4} \int_0^8 (10y^2 - y^3) dy \\ &= \frac{57\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 30,561 \text{ ft-lb.} \end{aligned}$$

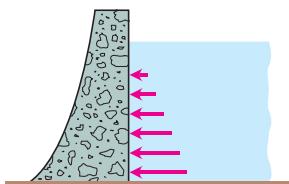


FIGURE 6.40 To withstand the increasing pressure, dams are built thicker as they go down.

Weight-density

A fluid's weight-density w is its weight per unit volume. Typical values (lb/ft³) are listed below.

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Freshwater	62.4

Fluid Pressures and Forces

Dams are built thicker at the bottom than at the top (Figure 6.40) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of freshwater in pounds per cubic foot. The pressure h feet below the surface of any fluid is the fluid's weight-density times h .

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h :

$$p = wh. \quad (4)$$

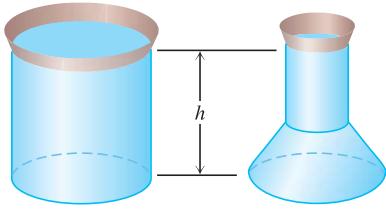


FIGURE 6.41 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.41.) If F , p , and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \end{aligned}$$

$p = wh$ from Eq. (4)

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \quad (5)$$

For example, the weight-density of freshwater is $62.4 \text{ lb}/\text{ft}^3$, so the fluid force at the bottom of a $10 \text{ ft} \times 20 \text{ ft}$ rectangular swimming pool 3 ft deep is

$$\begin{aligned} F &= whA = (62.4 \text{ lb}/\text{ft}^3)(3 \text{ ft})(10 \cdot 20 \text{ ft}^2) \\ &= 37,440 \text{ lb}. \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (5). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (5) no longer is usable in that form (because h varies).

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w . To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy -plane (Figure 6.42). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y -axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y .

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times (\text{strip depth})$. The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned} \Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \cdot (\text{strip depth}) \cdot L(y) \Delta y. \end{aligned}$$

Assume there are n strips associated with the partition of $a \leq y \leq b$ and that y_k is the bottom edge of the k th strip having length $L(y_k)$ and width Δy_k . The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^n (w \cdot (\text{strip depth})_k \cdot L(y_k)) \Delta y_k. \quad (6)$$

The sum in Equation (6) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (w \cdot (\text{strip depth})_k \cdot L(y_k)) \Delta y_k = \int_a^b w \cdot (\text{strip depth}) \cdot L(y) dy.$$

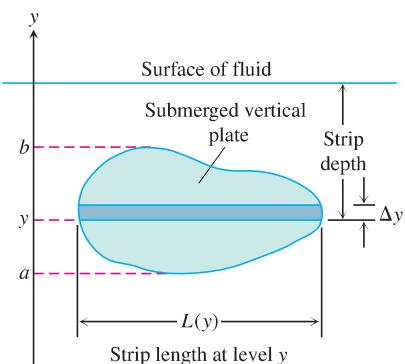


FIGURE 6.42 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F = \text{pressure} \times \text{area} = w \times (\text{strip depth}) \times L(y) \Delta y$.

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y -axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y . Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y) dy. \quad (7)$$

EXAMPLE 6 A flat isosceles right-triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y -axis upward along the plate's axis of symmetry (Figure 6.43). The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper-right vertex at $(3, 3)$. The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

$$\begin{aligned} F &= \int_a^b w \cdot \left(\frac{\text{strip}}{\text{depth}} \right) \cdot L(y) dy && \text{Eq. (7)} \\ &= \int_0^3 62.4(5 - y)2y dy \\ &= 124.8 \int_0^3 (5y - y^2) dy \\ &= 124.8 \left[\frac{5}{2}y^2 - \frac{y^3}{3} \right]_0^3 = 1684.8 \text{ lb.} \end{aligned}$$

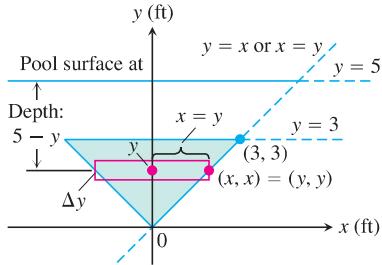


FIGURE 6.43 To find the force on one side of the submerged plate in Example 6, we can use a coordinate system like the one here.

Exercises 6.5

Springs

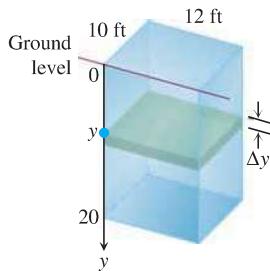
- Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.
- Stretching a spring** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.
 - Find the force constant.
 - How much work is done in stretching the spring from 10 in. to 12 in.?
 - How far beyond its natural length will a 1600-lb force stretch the spring?
- Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?
- Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?
- Subway car springs** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.
 - What is the assembly's force constant?
 - How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in.-lb.
- Bathroom scale** A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

Work Done by a Variable Force

7. **Lifting a rope** A mountain climber is about to haul up a 50 m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?
8. **Leaky sandbag** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)
9. **Lifting an elevator cable** An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?
10. **Force of attraction** When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2 . If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.
11. **Leaky bucket** Assume the bucket in Example 4 is leaking. It starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation x ft.)
12. (Continuation of Exercise 11.) The workers in Example 4 and Exercise 11 changed to a larger bucket that held 5 gal (40 lb) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

Pumping Liquids from Containers

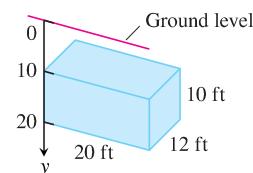
13. **Pumping water** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft³.
 - a. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?
 - b. If the water is pumped to ground level with a (5/11)-horsepower (hp) motor (work output 250 ft-lb/sec), how long will it take to empty the full tank (to the nearest minute)?
 - c. Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.
 - d. **The weight of water** What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft³? 62.59 lb/ft³?



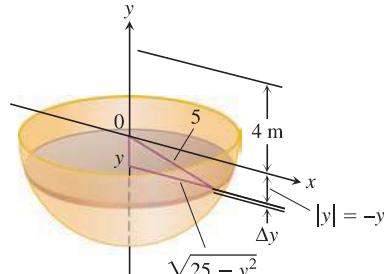
14. **Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown has its top 10 ft below ground level. The cistern,

currently full, is to be emptied for inspection by pumping its contents to ground level.

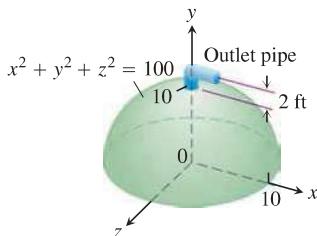
- a. How much work will it take to empty the cistern?
- b. How long will it take a 1/2-hp pump, rated at 275 ft-lb/sec, to pump the tank dry?
- c. How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)
- d. **The weight of water** What are the answers to parts (a) through (c) in a location where water weighs 62.26 lb/ft³? 62.59 lb/ft³?



15. **Pumping oil** How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?
16. **Pumping a half-full tank** Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 4 ft above the top of the tank?
17. **Emptying a tank** A vertical right-circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft³. How much work does it take to pump the kerosene to the level of the top of the tank?
18. a. **Pumping milk** Suppose that the conical container in Example 5 contains milk (weighing 64.5 lb/ft³) instead of olive oil. How much work will it take to pump the contents to the rim?
b. **Pumping oil** How much work will it take to pump the oil in Example 5 to a level 3 ft above the cone's rim?
19. The graph of $y = x^2$ on $0 \leq x \leq 2$ is revolved about the y -axis to form a tank that is then filled with salt water from the Dead Sea (weighing approximately 73 lbs/ft³). How much work does it take to pump all of the water to the top of the tank?
20. A right-circular cylindrical tank of height 10 ft and radius 5 ft is lying horizontally and is full of diesel fuel weighing 53 lbs/ft³. How much work is required to pump all of the fuel to a point 15 ft above the top of the tank?
21. **Emptying a water reservoir** We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m³.



22. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3 . A firm you contacted says it can empty the tank for $1/2\epsilon$ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?



Work and Kinetic Energy

23. **Kinetic energy** If a variable force of magnitude $F(x)$ moves a body of mass m along the x -axis from x_1 to x_2 , the body's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

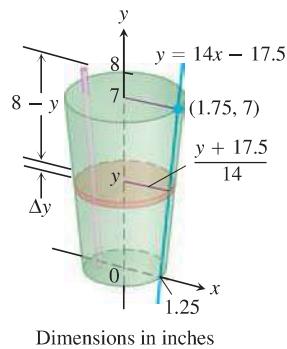
to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x) dx = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2,$$

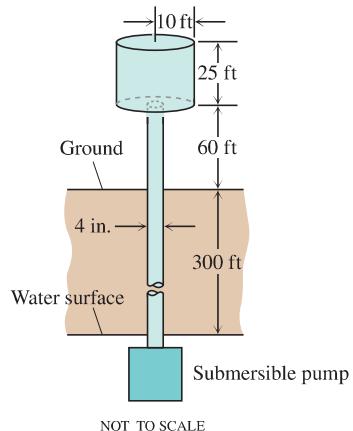
where v_1 and v_2 are the body's velocities at x_1 and x_2 . In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of a body of mass m moving with velocity v . Therefore, *the work done by the force equals the change in the body's kinetic energy*, and we can find the work by calculating this change.

In Exercises 24–28, use the result of Exercise 23.

24. **Tennis** A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2 , the acceleration of gravity.)
25. **Baseball** How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz, or 0.3125 lb.
26. **Golf** A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done on the ball getting it into the air?
27. On June 11, 2004, in a tennis match between Andy Roddick and Paradorn Srichaphan at the Stella Artois tournament in London, England, Roddick hit a serve measured at 153 mi/h. How much work was required by Andy to serve a 2-oz tennis ball at that speed?
28. **Softball** How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?
29. **Drinking a milkshake** The truncated conical container shown here is full of strawberry milkshake that weighs $4/9 \text{ oz/in}^3$. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.



30. **Water tower** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300 ft well through a vertical 4 in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft above ground. The pump is a 3 hp pump, rated at $1650 \text{ ft} \cdot \text{lb/sec}$. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 62.4 lb/ft^3 .



31. **Putting a satellite in orbit** The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24} \text{ kg}$ is Earth's mass, $G = 6.6720 \times 10^{-11} \text{ N} \cdot \text{m}^2 \text{ kg}^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2} dr \text{ joules.}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

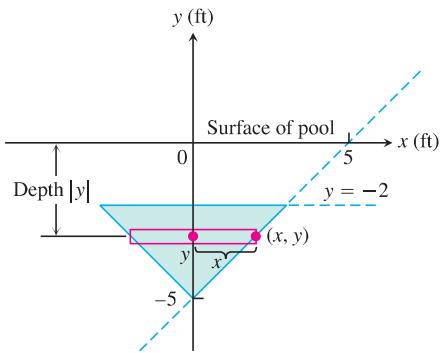
- 32. Forcing electrons together** Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons.}$$

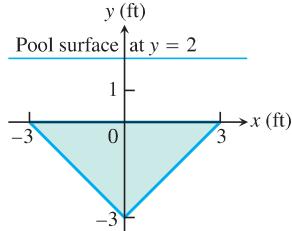
- Suppose one electron is held fixed at the point $(1, 0)$ on the x -axis (units in meters). How much work does it take to move a second electron along the x -axis from the point $(-1, 0)$ to the origin?
- Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x -axis from $(5, 0)$ to $(3, 0)$?

Finding Fluid Forces

- 33. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



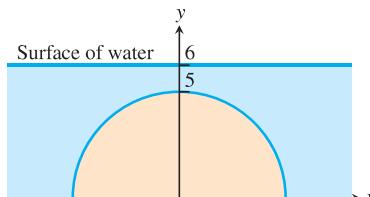
- 34. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



- 35. Rectangular plate** In a pool filled with water to a depth of 10 ft, calculate the fluid force on one side of a 3 ft by 4 ft rectangular plate if the plate rests vertically at the bottom of the pool

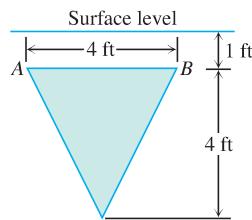
- on its 4-ft edge.
- on its 3-ft edge.

- 36. Semicircular plate** Calculate the fluid force on one side of a semicircular plate of radius 5 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 6 ft.

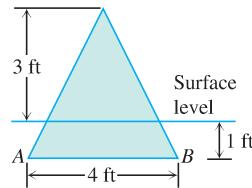


- 37. Triangular plate** The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.

- Find the fluid force against one face of the plate.
- What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



- 38. Rotated triangular plate** The plate in Exercise 37 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



- 39. New England Aquarium** The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 lb/ft^3 . (In case you were wondering, the glass is $\frac{3}{4}$ in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

- 40. Semicircular plate** A semicircular plate 2 ft in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

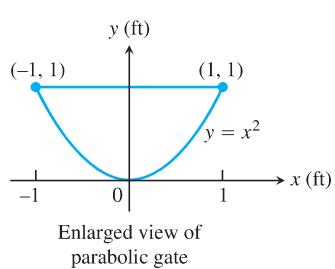
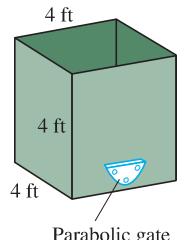
- 41. Tilted plate** Calculate the fluid force on one side of a 5 ft by 5 ft square plate if the plate is at the bottom of a pool filled with water to a depth of 8 ft and

- lying flat on its 5 ft by 5 ft face.
- resting vertically on a 5-ft edge.
- resting on a 5-ft edge and tilted at 45° to the bottom of the pool.

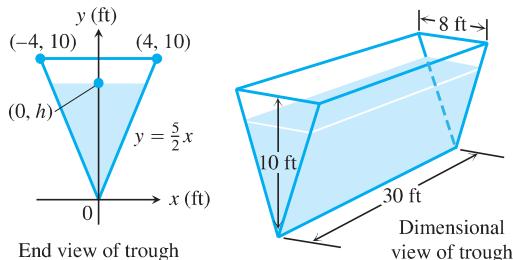
- 42. Tilted plate** Calculate the fluid force on one side of a right-triangular plate with edges 3 ft, 4 ft, and 5 ft if the plate sits at the bottom of a pool filled with water to a depth of 6 ft on its 3-ft edge and tilted at 60° to the bottom of the pool.

- 43. Cubical metal tank** The cubical metal tank shown here has a parabolic gate held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3 .

- What is the fluid force on the gate when the liquid is 2 ft deep?
- What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

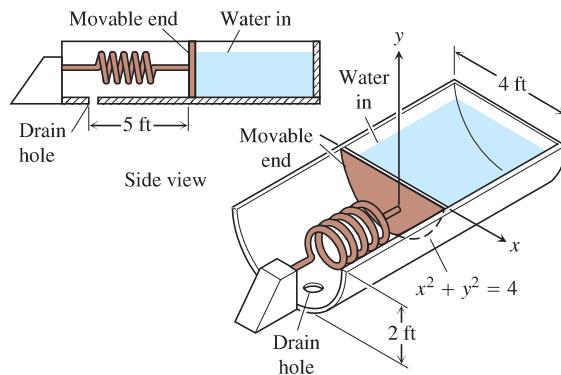


44. The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot. What is the value of h ?



45. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight-density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.
46. (Continuation of Exercise 45.) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 45) times the area of the plate.
47. Water pours into the tank shown here at the rate of $4 \text{ ft}^3/\text{min}$. The tank's cross-sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a

spring. The spring constant is $k = 100 \text{ lb/ft}$. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of $5 \text{ ft}^3/\text{min}$. Will the movable end reach the hole before the tank overflows?



48. **Watering trough** The vertical ends of a watering trough are squares 3 ft on a side.
- Find the fluid force against the ends when the trough is full.
 - How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?

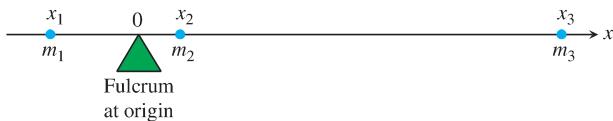
6.6

Moments and Centers of Mass

Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the *center of mass* (Figure 6.44). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. For the moment, we deal with one- and two-dimensional objects. Three-dimensional objects are best done with the multiple integrals of Chapter 15.

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1 , m_2 , and m_3 on a rigid x -axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not, depending on how large the masses are and how they are arranged along the x -axis.

Each mass m_k exerts a downward force $m_k g$ (the weight of m_k) equal to the magnitude of the mass times the acceleration due to gravity. Each of these forces has a tendency to turn the axis about the origin, the way a child turns a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \quad (1)$$

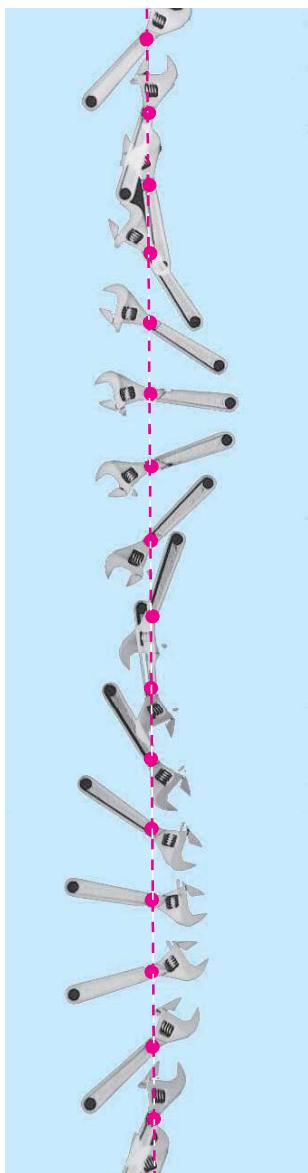


FIGURE 6.44 A wrench gliding on ice turning about its center of mass as the center glides in a vertical line.

The system will balance if and only if its torque is zero.

If we factor out the g in Equation (1), we see that the system torque is

$$\underbrace{g}_{\text{a feature of the environment}} \cdot \underbrace{(m_1x_1 + m_2x_2 + m_3x_3)}_{\text{a feature of the system}}.$$

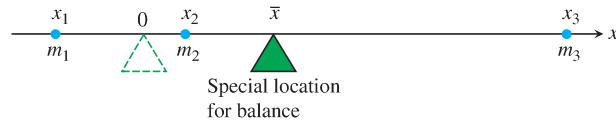
Thus, the torque is the product of the gravitational acceleration g , which is a feature of the environment in which the system happens to reside, and the number $(m_1x_1 + m_2x_2 + m_3x_3)$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed.

The number $(m_1x_1 + m_2x_2 + m_3x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** m_1x_1, m_2x_2, m_3x_3 of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_k x_k$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance, that is, at what point \bar{x} to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\begin{aligned} \text{Torque of } m_k \text{ about } \bar{x} &= \left(\begin{array}{l} \text{signed distance} \\ \text{of } m_k \text{ from } \bar{x} \end{array} \right) \left(\begin{array}{l} \text{downward} \\ \text{force} \end{array} \right) \\ &= (x_k - \bar{x})m_k g. \end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for \bar{x} :

$$\begin{aligned} \sum (x_k - \bar{x})m_k g &= 0 && \text{Sum of the torques equals zero.} \\ \bar{x} &= \frac{\sum m_k x_k}{\sum m_k}. && \text{Solved for } \bar{x} \end{aligned}$$

This last equation tells us to find \bar{x} by dividing the system's moment about the origin by the system's total mass:

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}. \quad (2)$$

The point \bar{x} is called the system's **center of mass**.

Masses Distributed over a Plane Region

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Figure 6.45). The mass of the system is

$$\text{System mass: } M = \sum m_k.$$

Each mass m_k has a moment about each axis. Its moment about the x -axis is $m_k y_k$, and its moment about the y -axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis: } M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis: } M_y = \sum m_k x_k.$$

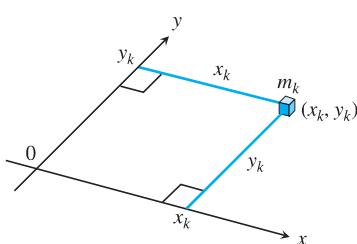


FIGURE 6.45 Each mass m_k has a moment about each axis.

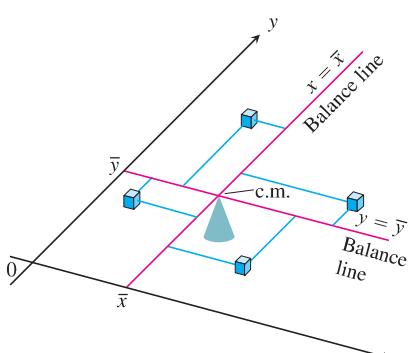


FIGURE 6.46 A two-dimensional array of masses balances on its center of mass.

The x -coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \quad (3)$$

With this choice of \bar{x} , as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.46).

The y -coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \quad (4)$$

With this choice of \bar{y} , the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point (\bar{x}, \bar{y}) . We call this point the system's **center of mass**.

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases, we assume the distribution of mass to be continuous, and the formulas we use to calculate \bar{x} and \bar{y} contain integrals instead of finite sums. The integrals arise in the following way.

Imagine that the plate occupying a region in the xy -plane is cut into thin strips parallel to one of the axes (in Figure 6.47, the y -axis). The center of mass of a typical strip is (\tilde{x}, \tilde{y}) . We treat the strip's mass Δm as if it were concentrated at (\tilde{x}, \tilde{y}) . The moment of the strip about the y -axis is then $\tilde{x} \Delta m$. The moment of the strip about the x -axis is $\tilde{y} \Delta m$. Equations (3) and (4) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x} \Delta m}{\sum \Delta m}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y} \Delta m}{\sum \Delta m}.$$

The sums are Riemann sums for integrals and approach these integrals as limiting values as the strips into which the plate is cut become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y} dm}{\int dm}.$$

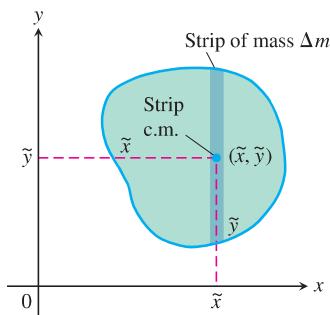


FIGURE 6.47 A plate cut into thin strips parallel to the y -axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass (\tilde{x}, \tilde{y}) .

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy -Plane

$$\text{Moment about the } x\text{-axis: } M_x = \int \tilde{y} dm$$

$$\text{Moment about the } y\text{-axis: } M_y = \int \tilde{x} dm$$

$$\text{Mass: } M = \int dm$$

$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

Density

A material's density is its mass per unit area. For wires, rods, and narrow strips, we use mass per unit length.

The differential dm is the mass of the strip. Assuming the density δ of the plate to be a continuous function, the mass differential dm equals the product δdA (mass per unit area times area). Here dA represents the area of the strip.

To evaluate the integrals in Equations (5), we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinate axes. We then express the strip's mass dm and the coordinates (\tilde{x}, \tilde{y}) of the strip's center of mass in terms of x or y . Finally, we integrate $\tilde{y} dm$, $\tilde{x} dm$, and dm between limits of integration determined by the plate's location in the plane.

EXAMPLE 1 The triangular plate shown in Figure 6.48 has a constant density of $\delta = 3 \text{ g/cm}^2$. Find

- (a) the plate's moment M_y about the y -axis.
- (b) the plate's mass M .
- (c) the x -coordinate of the plate's center of mass (c.m.).

Solution Method 1: Vertical Strips (Figure 6.49)

- (a) The moment M_y : The typical vertical strip has the following relevant data.

$$\begin{aligned}\text{center of mass (c.m.)}: & (\tilde{x}, \tilde{y}) = (x, x) \\ \text{length}: & 2x \\ \text{width}: & dx \\ \text{area}: & dA = 2x dx \\ \text{mass}: & dm = \delta dA = 3 \cdot 2x dx = 6x dx \\ \text{distance of c.m. from } y\text{-axis}: & \tilde{x} = x\end{aligned}$$

The moment of the strip about the y -axis is

$$\tilde{x} dm = x \cdot 6x dx = 6x^2 dx.$$

The moment of the plate about the y -axis is therefore

$$M_y = \int \tilde{x} dm = \int_0^1 6x^2 dx = 2x^3 \Big|_0^1 = 2 \text{ g} \cdot \text{cm}.$$

- (b) The plate's mass:

$$M = \int dm = \int_0^1 6x dx = 3x^2 \Big|_0^1 = 3 \text{ g}.$$

- (c) The x -coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y} = M_x/M$.

Method 2: Horizontal Strips (Figure 6.50)

- (a) The moment M_y : The y -coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x -coordinate is the x -coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x -value) and 1 (the strip's right-hand x -value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y+2}{4}.$$

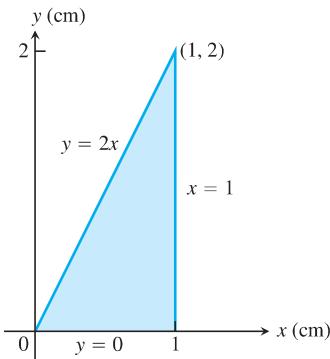


FIGURE 6.48 The plate in Example 1.

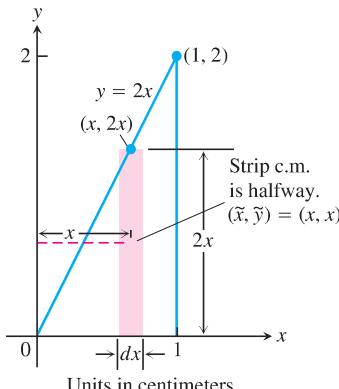


FIGURE 6.49 Modeling the plate in Example 1 with vertical strips.

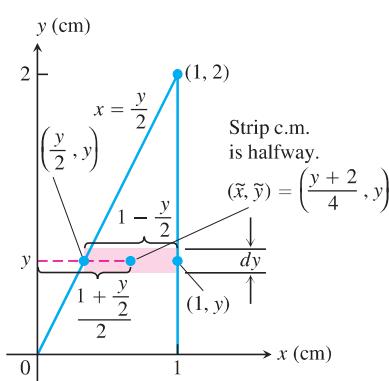


FIGURE 6.50 Modeling the plate in Example 1 with horizontal strips.

We also have

$$\begin{aligned} \text{length: } & 1 - \frac{y}{2} = \frac{2-y}{2} \\ \text{width: } & dy \\ \text{area: } & dA = \frac{2-y}{2} dy \\ \text{mass: } & dm = \delta dA = 3 \cdot \frac{2-y}{2} dy \\ \text{distance of c.m. to } y\text{-axis: } & \tilde{x} = \frac{y+2}{4}. \end{aligned}$$

The moment of the strip about the y -axis is

$$\tilde{x} dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2} dy = \frac{3}{8} (4-y^2) dy.$$

The moment of the plate about the y -axis is

$$M_y = \int \tilde{x} dm = \int_0^2 \frac{3}{8} (4-y^2) dy = \frac{3}{8} \left[4y - \frac{y^3}{3} \right]_0^2 = \frac{3}{8} \left(\frac{16}{3} \right) = 2 \text{ g} \cdot \text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2} (2-y) dy = \frac{3}{2} \left[2y - \frac{y^2}{2} \right]_0^2 = \frac{3}{2} (4-2) = 3 \text{ g}.$$

(c) The x -coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and \bar{y} . ■

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

EXAMPLE 2 Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x -axis (Figure 6.51). Assume the density of the plate at the point (x, y) is $\delta = 2x^2$, which is twice the square of the distance from the point to the y -axis.

Solution The mass distribution is symmetric about the y -axis, so $\bar{x} = 0$. We model the distribution of mass with vertical strips since the density is given as a function of the variable x . The typical vertical strip (see Figure 6.51) has the following relevant data.

$$\text{center of mass (c.m.): } (\tilde{x}, \tilde{y}) = \left(x, \frac{4-x^2}{2} \right)$$

$$\text{length: } 4 - x^2$$

$$\text{width: } dx$$

$$\text{area: } dA = (4 - x^2) dx$$

$$\text{mass: } dm = \delta dA = \delta(4 - x^2) dx$$

$$\text{distance from c.m. to } x\text{-axis: } \tilde{y} = \frac{4-x^2}{2}$$

The moment of the strip about the x -axis is

$$\tilde{y} dm = \frac{4-x^2}{2} \cdot \delta(4-x^2) dx = \frac{\delta}{2} (4-x^2)^2 dx.$$

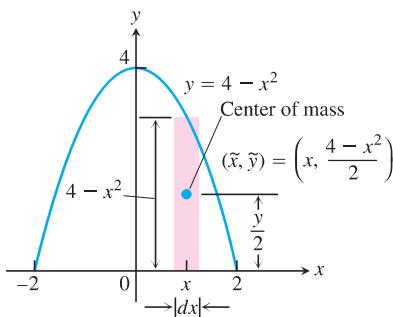


FIGURE 6.51 Modeling the plate in Example 2 with vertical strips.

The moment of the plate about the x -axis is

$$\begin{aligned} M_x &= \int \tilde{y} dm = \int_{-2}^2 \frac{\delta}{2} (4 - x^2)^2 dx = \int_{-2}^2 x^2(4 - x^2)^2 dx \\ &= \int_{-2}^2 (16x^2 - 8x^4 + x^6) dx = \frac{2048}{105} \\ M &= \int dm = \int_{-2}^2 \delta(4 - x^2) dx = \int_{-2}^2 2x^2(4 - x^2) dx \\ &= \int_{-2}^2 (8x^2 - 2x^4) dx = \frac{256}{15}. \end{aligned}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

■

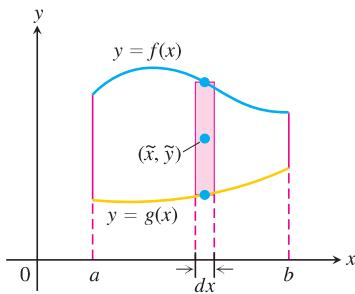


FIGURE 6.52 Modeling the plate bounded by two curves with vertical strips. The strip c.m. is halfway, so $\tilde{y} = \frac{1}{2} [f(x) + g(x)]$.

Plates Bounded by Two Curves

Suppose a plate covers a region that lies between two curves $y = g(x)$ and $y = f(x)$, where $f(x) \geq g(x)$ and $a \leq x \leq b$. The typical vertical strip (see Figure 6.52) has

center of mass (c.m.):	$(\tilde{x}, \tilde{y}) = (x, \frac{1}{2} [f(x) + g(x)])$
length:	$f(x) - g(x)$
width:	dx
area:	$dA = [f(x) - g(x)] dx$
mass:	$dm = \delta dA = \delta[f(x) - g(x)] dx$

The moment of the plate about the y -axis is

$$M_y = \int x dm = \int_a^b x \delta [f(x) - g(x)] dx,$$

and the moment about the x -axis is

$$\begin{aligned} M_x &= \int y dm = \int_a^b \frac{1}{2} [f(x) + g(x)] \cdot \delta [f(x) - g(x)] dx \\ &= \int_a^b \frac{\delta}{2} [f^2(x) - g^2(x)] dx. \end{aligned}$$

These moments give the formulas

$$\bar{x} = \frac{1}{M} \int_a^b x \delta [f(x) - g(x)] dx \quad (6)$$

$$\bar{y} = \frac{1}{M} \int_a^b \frac{\delta}{2} [f^2(x) - g^2(x)] dx \quad (7)$$

EXAMPLE 3 Find the center of mass for the thin plate bounded by the curves $g(x) = x/2$ and $f(x) = \sqrt{x}$, $0 \leq x \leq 1$, (Figure 6.53) using Equations (6) and (7) with the density function $\delta(x) = x^2$.

Solution We first compute the mass of the plate, where $dm = \delta[f(x) - g(x)] dx$:

$$M = \int_0^1 x^2 \left(\sqrt{x} - \frac{x}{2} \right) dx = \int_0^1 \left(x^{5/2} - \frac{x^3}{2} \right) dx = \left[\frac{2}{7}x^{7/2} - \frac{1}{8}x^4 \right]_0^1 = \frac{9}{56}.$$

Then from Equations (6) and (7) we get

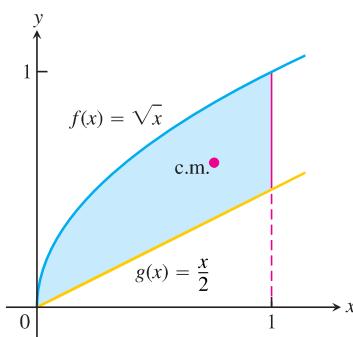


FIGURE 6.53 The region in Example 3.

$$\begin{aligned}\bar{x} &= \frac{56}{9} \int_0^1 x^2 \cdot x \left(\sqrt{x} - \frac{x}{2} \right) dx \\ &= \frac{56}{9} \int_0^1 \left(x^{7/2} - \frac{x^3}{2} \right) dx \\ &= \frac{56}{9} \left[\frac{2}{9}x^{9/2} - \frac{1}{10}x^5 \right]_0^1 = \frac{308}{405},\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{56}{9} \int_0^1 \frac{x^2}{2} \left(x - \frac{x^2}{4} \right) dx \\ &= \frac{28}{9} \int_0^1 \left(x^3 - \frac{x^4}{4} \right) dx \\ &= \frac{28}{9} \left[\frac{1}{4}x^4 - \frac{1}{20}x^5 \right]_0^1 = \frac{252}{405}.\end{aligned}$$

The center of mass is shown in Figure 6.53. ■

Centroids

When the density function is constant, it cancels out of the numerator and denominator of the formulas for \bar{x} and \bar{y} . Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape, as in “Find the centroid of a triangle or a solid cone.” To do so, just set δ equal to 1 and proceed to find \bar{x} and \bar{y} as before, by dividing moments by masses.

EXAMPLE 4 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a .

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.54). The distribution of mass is symmetric about the y -axis, so $\bar{x} = 0$. To find \bar{y} , we imagine the wire divided into short subarc segments. If (\tilde{x}, \tilde{y}) is the center of mass of a subarc and θ is the angle between the x -axis and the radial line joining the origin to (\tilde{x}, \tilde{y}) , then $\tilde{y} = a \sin \theta$ is a function of the angle θ measured in radians (see Figure 6.54a). The length ds of the subarc containing (\tilde{x}, \tilde{y}) subtends an angle of $d\theta$ radians, so $ds = a d\theta$. Thus a typical subarc segment has these relevant data for calculating \bar{y} :

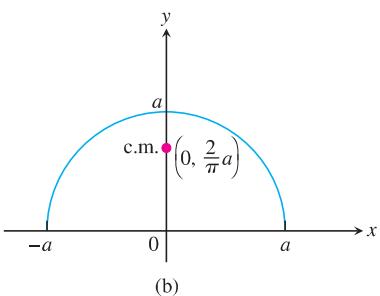
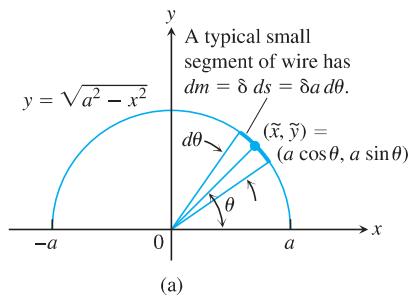


FIGURE 6.54 The semicircular wire in Example 4. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

length:	$ds = a d\theta$	Mass per unit length times length
mass:	$dm = \delta ds = \delta a d\theta$	
distance of c.m. to x -axis:	$\tilde{y} = a \sin \theta$	

Hence,

$$\bar{y} = \frac{\int \bar{y} dm}{\int dm} = \frac{\int_0^\pi a \sin \theta \cdot \delta a d\theta}{\int_0^\pi \delta a d\theta} = \frac{\delta a^2 [-\cos \theta]_0^\pi}{\delta a \pi} = \frac{2}{\pi} a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Figure 6.54b). Notice how δ cancels in the equation for \bar{y} , so we could have set $\delta = 1$ everywhere and obtained the same value for \bar{y} . ■

In Example 4 we found the center of mass of a thin wire lying along the graph of a differentiable function in the xy -plane. In Chapter 16 we will learn how to find the center of mass of wires lying along more general smooth curves in the plane (or in space).

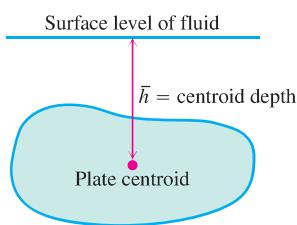


FIGURE 6.55 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

Fluid Forces and Centroids

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.55), we can take a shortcut to find the force against one side of the plate. From Equation (7) in Section 6.5,

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y) dy \\ &= w \int_a^b (\text{strip depth}) \times L(y) dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w , the distance \bar{h} from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \quad (8)$$

EXAMPLE 5 A flat isosceles triangular plate with base 6 ft and height 3 ft is submerged vertically, base up with its vertex at the origin, so that the base is 2 ft below the surface of a swimming pool. (This is Example 6, Section 6.5.) Use Equation (8) to find the force exerted by the water against one side of the plate.

Solution The centroid of the triangle (Figure 6.43) lies on the y -axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$ (where $y = 2$) since the pool's surface is $y = 5$. The triangle's area is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(6)(3) = 9.$$

Hence,

$$F = w\bar{h}A = (62.4)(3)(9) = 1684.8 \text{ lb.} \quad \blacksquare$$

The Theorems of Pappus

In the fourth century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

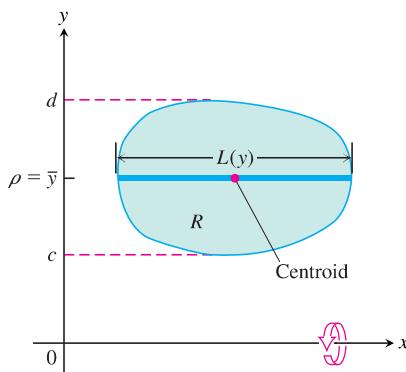


FIGURE 6.56 The region R is to be revolved (once) about the x -axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

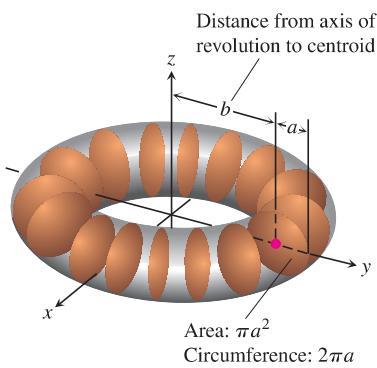


FIGURE 6.57 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 6).

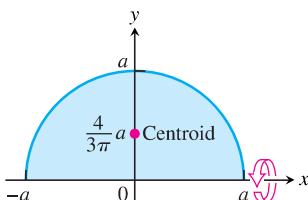


FIGURE 6.58 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 7).

THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \quad (9)$$

Proof We draw the axis of revolution as the x -axis with the region R in the first quadrant (Figure 6.56). We let $L(y)$ denote the length of the cross-section of R perpendicular to the y -axis at y . We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x -axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height}) dy = 2\pi \int_c^d y L(y) dy. \quad (10)$$

The y -coordinate of R 's centroid is

$$\bar{y} = \frac{\int_c^d \bar{y} dA}{A} = \frac{\int_c^d y L(y) dy}{A}, \quad \bar{y} = y, dA = L(y) dy$$

so that

$$\int_c^d y L(y) dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With ρ equal to \bar{y} , we have $V = 2\pi\rho A$. ■

EXAMPLE 6 Find the volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.57).

Solution We apply Pappus's Theorem for volumes. The centroid of a disk is located at its center, the area is $A = \pi a^2$, and $\rho = b$ is the distance from the centroid to the axis of revolution (see Figure 6.57). Substituting these values into Equation (9), we find the volume of the torus to be

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2. \quad \blacksquare$$

The next example shows how we can use Equation (9) in Pappus's Theorem to find one of the coordinates of the centroid of a plane region of known area A when we also know the volume V of the solid generated by revolving the region about the other coordinate axis. That is, if \bar{y} is the coordinate we want to find, we revolve the region around the x -axis so that $\bar{y} = \rho$ is the distance from the centroid to the axis of revolution. The idea is that the rotation generates a solid of revolution whose volume V is an already known quantity. Then we can solve Equation (9) for ρ , which is the value of the centroid's coordinate \bar{y} .

EXAMPLE 7 Locate the centroid of a semicircular region of radius a .

Solution We consider the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.58) and the x -axis and imagine revolving the region about the x -axis to generate a solid sphere. By symmetry, the x -coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a. \quad \blacksquare$$

THEOREM 2 Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \quad (11)$$

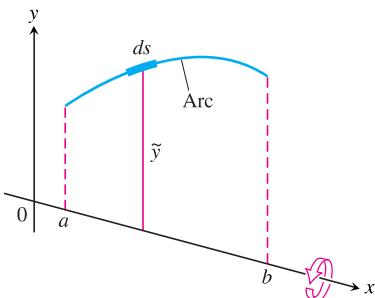


FIGURE 6.59 Figure for proving Pappus's Theorem for surface area. The arc length differential ds is given by Equation (6) in Section 6.3.

The proof we give assumes that we can model the axis of revolution as the x -axis and the arc as the graph of a continuously differentiable function of x .

Proof We draw the axis of revolution as the x -axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.59). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y \, ds = 2\pi \int_{x=a}^{x=b} y \, ds. \quad (12)$$

The y -coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y} \, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y \, ds}{L}. \quad L = \int ds \text{ is the arc's length and } \tilde{y} = y.$$

Hence

$$\int_{x=a}^{x=b} y \, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to \bar{y} , we have $S = 2\pi\rho L$. ■

EXAMPLE 8 Use Pappus's area theorem to find the surface area of the torus in Example 6.

Solution From Figure 6.57, the surface of the torus is generated by revolving a circle of radius a about the z -axis, and $b \geq a$ is the distance from the centroid to the axis of revolution. The arc length of the smooth curve generating this surface of revolution is the circumference of the circle, so $L = 2\pi a$. Substituting these values into Equation (11), we find the surface area of the torus to be

$$S = 2\pi(b)(2\pi a) = 4\pi^2ba.$$

Exercises 6.6

Thin Plates with Constant Density

In Exercises 1–14, find the center of mass of a thin plate of constant density δ covering the given region.

1. The region bounded by the parabola $y = x^2$ and the line $y = 4$
2. The region bounded by the parabola $y = 25 - x^2$ and the x -axis
3. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
4. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$

5. The region bounded by the y -axis and the curve $x = y - y^3$, $0 \leq y \leq 1$
6. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$
7. The region bounded by the x -axis and the curve $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$
8. The region between the curve $y = \sec^2 x$, $-\pi/4 \leq x \leq \pi/4$ and the x -axis

- T** 9. The region between the curve $y = 1/x$ and the x -axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

10. a. The region cut from the first quadrant by the circle $x^2 + y^2 = 9$
 b. The region bounded by the x -axis and the semicircle $y = \sqrt{9 - x^2}$

Compare your answer in part (b) with the answer in part (a).

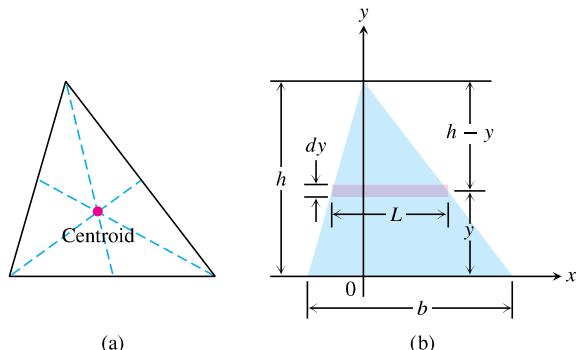
11. The region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$
 12. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$
 13. The region between the curve $y = 1/\sqrt{x}$ and the x -axis from $x = 1$ to $x = 16$
 14. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a \rightarrow \infty} \bar{x}$.

Thin Plates with Varying Density

15. Find the center of mass of a thin plate covering the region between the x -axis and the curve $y = 2/x^2$, $1 \leq x \leq 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.
 16. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.
 17. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y -axis to generate a solid.
 a. Find the volume of the solid.
 b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.
 c. Sketch the plate and show the center of mass in your sketch.
 18. The region between the curve $y = 2/x$ and the x -axis from $x = 1$ to $x = 4$ is revolved about the x -axis to generate a solid.
 a. Find the volume of the solid.
 b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.
 c. Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

19. **The centroid of a triangle lies at the intersection of the triangle's medians** You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.
 i) Stand one side of the triangle on the x -axis as in part (b) of the accompanying figure. Express dm in terms of L and dy .
 ii) Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm .
 iii) Show that $\bar{y} = h/3$.
 iv) Extend the argument to the other sides.



Use the result in Exercise 19 to find the centroids of the triangles whose vertices appear in Exercises 20–24. Assume $a, b > 0$.

20. $(-1, 0), (1, 0), (0, 3)$ 21. $(0, 0), (1, 0), (0, 1)$
 22. $(0, 0), (a, 0), (0, a)$ 23. $(0, 0), (a, 0), (0, b)$
 24. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

25. **Constant density** Find the moment about the x -axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.
 26. **Constant density** Find the moment about the x -axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.
 27. **Variable density** Suppose that the density of the wire in Example 4 is $\delta = k \sin \theta$ (k constant). Find the center of mass.
 28. **Variable density** Suppose that the density of the wire in Example 4 is $\delta = 1 + k|\cos \theta|$ (k constant). Find the center of mass.

Plates Bounded by Two Curves

In Exercises 29–32, find the centroid of the thin plate bounded by the graphs of the given functions. Use Equations (6) and (7) with $\delta = 1$ and $M = \text{area of the region covered by the plate}$.

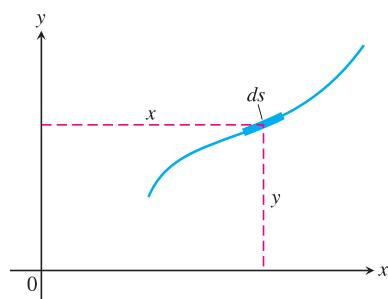
29. $g(x) = x^2$ and $f(x) = x + 6$
 30. $g(x) = x^2(x + 1)$, $f(x) = 2$, and $x = 0$
 31. $g(x) = x^2(x - 1)$ and $f(x) = x^2$
 32. $g(x) = 0$, $f(x) = 2 + \sin x$, $x = 0$, and $x = 2\pi$
 (Hint: $\int x \sin x \, dx = \sin x - x \cos x + C$.)

Theory and Examples

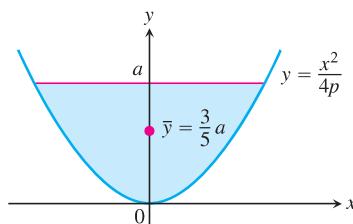
Verify the statements and formulas in Exercises 33 and 34.

33. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x \, ds}{\text{length}}, \quad \bar{y} = \frac{\int y \, ds}{\text{length}}.$$



34. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y -coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



The Theorems of Pappus

35. The square region with vertices $(0, 2)$, $(2, 0)$, $(4, 2)$, and $(2, 4)$ is revolved about the x -axis to generate a solid. Find the volume and surface area of the solid.
36. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$ (see Exercise 19).
37. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y -axis.
38. Use the theorems of Pappus to find the lateral surface area and the volume of a right-circular cone.
39. Use Pappus's Theorem for surface area and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

40. As found in Exercise 39, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

41. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x -axis is $(1/2)\pi ab$, and the volume of the ellipsoid generated by revolving R about the x -axis is $(4/3)\pi ab^2$. Find the centroid of R . Notice that the location is independent of a .

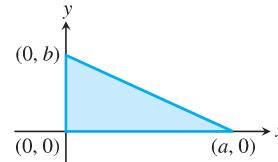
42. As found in Example 7, the centroid of the region enclosed by the x -axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

43. The region of Exercise 42 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

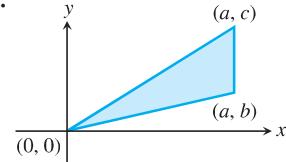
44. As found in Exercise 39, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

In Exercises 45 and 46, use a theorem of Pappus to find the centroid of the given triangle. Use the fact that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

45.



46.



Chapter 6

Questions to Guide Your Review

- How do you define and calculate the volumes of solids by the method of slicing? Give an example.
- How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
- Describe the method of cylindrical shells. Give an example.
- How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
- How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \leq x \leq b$, about the x -axis? Give an example.
- How do you define and calculate the work done by a variable force directed along a portion of the x -axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
- How do you calculate the force exerted by a liquid against a portion of a flat vertical wall? Give an example.
- What is a center of mass? a centroid?
- How do you locate the center of mass of a thin flat plate of material? Give an example.
- How do you locate the center of mass of a thin plate bounded by two curves $y = f(x)$ and $y = g(x)$ over $a \leq x \leq b$?

Chapter 6

Practice Exercises

Volumes

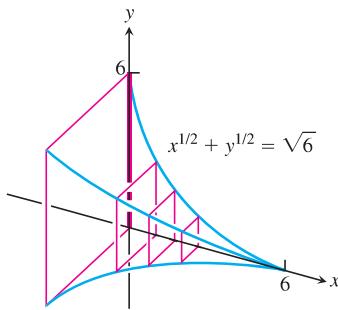
Find the volumes of the solids in Exercises 1–16.

- The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

- The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the x -axis are equilateral triangles whose bases stretch from the line to the curve.
- The solid lies between planes perpendicular to the x -axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular

disks whose diameters run from the curve $y = 2 \cos x$ to the curve $y = 2 \sin x$.

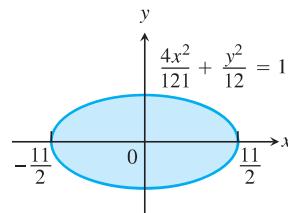
4. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x -axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



5. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the x -axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.
6. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy -plane. Each cross-section perpendicular to the x -axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)
7. Find the volume of the solid generated by revolving the region bounded by the x -axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 1$; (d) the line $y = 3$.
8. Find the volume of the solid generated by revolving the “triangular” region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 2$; (d) the line $y = 4$.
9. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 5$.
10. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 4$; (d) the line $y = 4$.
11. Find the volume of the solid generated by revolving the “triangular” region bounded by the x -axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x -axis.
12. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.
13. Find the volume of the solid generated by revolving the region bounded by the curve $x = e^{y^2}$ and the lines $y = 0$, $x = 0$, and $y = 1$ about the x -axis.
14. Find the volume of the solid generated by revolving about the x -axis the region bounded by $y = 2 \tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)

15. **Volume of a solid sphere hole** A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

16. **Volume of a football** The profile of a football resembles the ellipse shown here. Find the football’s volume to the nearest cubic inch.



Lengths of Curves

Find the lengths of the curves in Exercises 17–20.

17. $y = x^{1/2} - (1/3)x^{3/2}$, $1 \leq x \leq 4$
 18. $x = y^{2/3}$, $1 \leq y \leq 8$
 19. $y = x^2 - (\ln x)/8$, $1 \leq x \leq 2$
 20. $x = (y^3/12) + (1/y)$, $1 \leq y \leq 2$

Areas of Surfaces of Revolution

In Exercises 21–24, find the areas of the surfaces generated by revolving the curves about the given axes.

21. $y = \sqrt{2x + 1}$, $0 \leq x \leq 3$; x -axis
 22. $y = x^3/3$, $0 \leq x \leq 1$; x -axis
 23. $x = \sqrt{4y - y^2}$, $1 \leq y \leq 2$; y -axis
 24. $x = \sqrt{y}$, $2 \leq y \leq 6$; y -axis

Work

25. **Lifting equipment** A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (Hint: Solve for the rope and equipment separately, then add.)
26. **Leaky tank truck** You drove an 800-gal tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 lb/U.S. gal.
27. **Stretching a spring** If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? An additional foot?
28. **Garage door spring** A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?
29. **Pumping a reservoir** A reservoir shaped like a right-circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

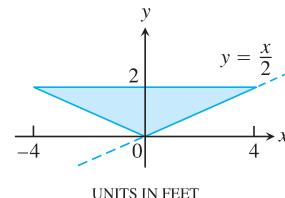
- 30. Pumping a reservoir** (Continuation of Exercise 29.) The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?
- 31. Pumping a conical tank** A right-circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3 . How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft-lb/sec ($1/2$ hp), how long will it take to empty the tank?
- 32. Pumping a cylindrical tank** A storage tank is a right-circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3 , find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

Centers of Mass and Centroids

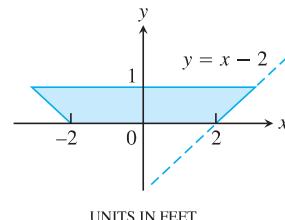
- 33.** Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.
- 34.** Find the centroid of a thin, flat plate covering the region enclosed by the x -axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.
- 35.** Find the centroid of a thin, flat plate covering the “triangular” region in the first quadrant bounded by the y -axis, the parabola $y = x^2/4$, and the line $y = 4$.
- 36.** Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.
- 37.** Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)
- 38. a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x -axis from $x = 1$ to $x = 9$.
- b.** Find the plate’s center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Fluid Force

- 39. Trough of water** The vertical triangular plate shown here is the end plate of a trough full of water ($w = 62.4$). What is the fluid force against the plate?



- 40. Trough of maple syrup** The vertical trapezoidal plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3 . What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?



- 41. Force on a parabolic gate** A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate ($w = 62.4$).
- T 42.** You plan to store mercury ($w = 849 \text{ lb/ft}^3$) in a vertical rectangular tank with a 1 ft square base side whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

Chapter 6

Additional and Advanced Exercises

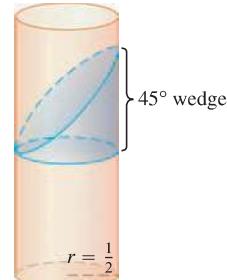
Volume and Length

- A solid is generated by revolving about the x -axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x -axis, and the fixed line $x = a$ and the variable line $x = b$, $b > a$. Its volume, for all b , is $b^2 - ab$. Find $f(x)$.
- A solid is generated by revolving about the x -axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x -axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.
- Suppose that the increasing function $f(x)$ is smooth for $x \geq 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C . What are the allowable values for C ?
- a. Show that for $0 < \alpha \leq \pi/2$,

$$\int_0^\alpha \sqrt{1 + \cos^2 \theta} d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}.$$

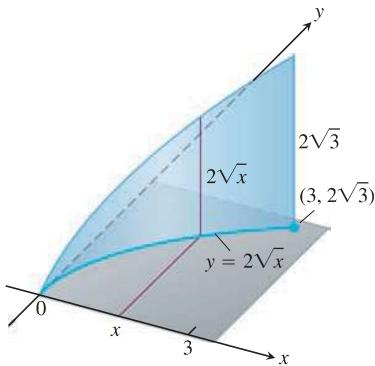
- b. Generalize the result in part (a).

- Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x$ and $y = x^2$ about the line $y = x$.
- Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder (see accompanying diagram). Find the volume of the wedge.

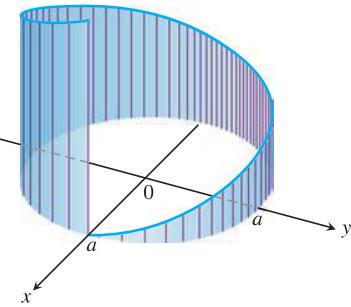


Surface Area

7. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy -plane. (See accompanying figure.) Find the area of the surface formed by these perpendiculars from $(0, 0)$ to $(3, 2\sqrt{3})$.



8. At points on a circle of radius a , line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks , where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.

**Work**

9. A particle of mass m starts from rest at time $t = 0$ and is moved along the x -axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.

10. **Work and kinetic energy** Suppose a 1.6-oz golf ball is placed on a vertical spring with force constant $k = 2$ lb/in. The spring is compressed 6 in. and released. About how high does the ball go (measured from the spring's rest position)?

Centers of Mass

11. Find the centroid of the region bounded below by the x -axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \rightarrow \infty$?
12. If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be 3 ft or so behind the pole's center of mass to provide an adequate "tongue" weight. The 40-ft wooden telephone poles used by Verizon have a 27-in. circumference at the top and a 43.5-in. circumference at the base. About how far from the top is the center of mass?
13. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy -plane, and let M_y be the plate's moment about the y -axis. Show that the plate's moment about the line $x = b$ is
- $M_y - b\delta A$ if the plate lies to the right of the line, and
 - $b\delta A - M_y$ if the plate lies to the left of the line.
14. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, a = positive constant, if the density at (x, y) is directly proportional to (a) x , (b) $|y|$.
15.
 - Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b , $0 < a < b$, and their centers are at the origin.
 - Find the limits of the coordinates of the centroid as a approaches b and discuss the meaning of the result.
16. A triangular corner is cut from a square 1 ft on a side. The area of the triangle removed is 36 in^2 . If the centroid of the remaining region is 7 in. from one side of the original square, how far is it from the remaining sides?

Fluid Force

17. A triangular plate ABC is submerged in water with its plane vertical. The side AB , 4 ft long, is 6 ft below the surface of the water, while the vertex C is 2 ft below the surface. Find the force exerted by the water on one side of the plate.
18. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

Chapter 6**Technology Application Projects****Mathematica/Maple Modules:****Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves**

Visualize and approximate areas and volumes in **Part I** and **Part II**: Volumes of Revolution; and **Part III**: Lengths of Curves.

Modeling a Bungee Cord Jump

Collect data (or use data previously collected) to build and refine a model for the force exerted by a jumper's bungee cord. Use the work-energy theorem to compute the distance fallen for a given jumper and a given length of bungee cord.