8

Techniques of Integration

OVERVIEW The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. Table 8.1 summarizes the forms of antiderivatives for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this chapter we study a number of other important techniques for finding antiderivatives (or indefinite integrals) for many combinations of functions whose antiderivatives cannot be found using the methods presented before.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)	**12.** $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	**13.** $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
3. $\int \dfrac{dx}{x} = \ln\lvert x\rvert + C$	**14.** $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	**15.** $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$	**16.** $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	**17.** $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	**18.** $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
8. $\int \sec^2 x\,dx = \tan x + C$	**19.** $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
9. $\int \csc^2 x\,dx = -\cot x + C$	**20.** $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left\lvert\dfrac{x}{a}\right\rvert + C$
10. $\int \sec x\tan x\,dx = \sec x + C$	**21.** $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \quad (a > 0)$
11. $\int \csc x\cot x\,dx = -\csc x + C$	**22.** $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \quad (x > a > 0)$

8.1 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx,$$

leading to the **integration by parts** formula

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du and v, and finally we apply the formula in Equation (2).

EXAMPLE 1 Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\displaystyle\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= \cos x\,dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$

Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C. \qquad \blacksquare$$

There are four choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x\cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x\cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3 leads to the integral

$$\int (x\cos x - x^2\sin x)\,dx.$$

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. When finding v from dv, any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in v because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \dfrac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C. \qquad \blacksquare$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 3 Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned} \qquad \blacksquare$$

The technique of Example 3 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x\cos x\,dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x\cos x\,dx &= e^x\sin x - \left(-e^x\cos x - \int(-\cos x)(e^x\,dx)\right) \\ &= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.
\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx. \qquad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}
\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\
&= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.
\end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

In applying Equation (3), we normally use the u and v notation from Equation (2) because it is easier to remember. Here is an example.

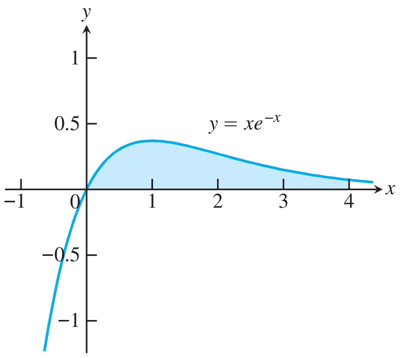


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}
\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\
&= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx \\
&= -4e^{-4} - e^{-x}\Big]_0^4 \\
&= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.
\end{aligned}$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a way to organize the calculations that prevents these pitfalls and makes the work much easier. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Evaluate

$$\int x^2e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3. ■

EXAMPLE 8 Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$ ■

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Exercises 8.1

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2} \, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

4. $\int x^2 \sin x \, dx$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int xe^x \, dx$

8. $\int xe^{3x} \, dx$

9. $\int x^2 e^{-x} \, dx$

10. $\int (x^2 - 2x + 1) e^{2x} \, dx$

11. $\int \tan^{-1} y \, dy$

12. $\int \sin^{-1} y \, dy$

13. $\int x \sec^2 x \, dx$

14. $\int 4x \sec^2 2x \, dx$

15. $\int x^3 e^x \, dx$

16. $\int p^4 e^{-p} \, dp$

17. $\int (x^2 - 5x)e^x \, dx$

18. $\int (r^2 + r + 1)e^r \, dr$

19. $\int x^5 e^x \, dx$

20. $\int t^2 e^{4t} \, dt$

21. $\int e^\theta \sin \theta \, d\theta$

22. $\int e^{-y} \cos y \, dy$

23. $\int e^{2x} \cos 3x \, dx$

24. $\int e^{-2x} \sin 2x \, dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}} \, ds$

26. $\int_0^1 x\sqrt{1 - x} \, dx$

27. $\int_0^{\pi/3} x \tan^2 x \, dx$

28. $\int \ln (x + x^2) \, dx$

29. $\int \sin (\ln x) \, dx$

30. $\int z(\ln z)^2 \, dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–50. Some integrals do not require integration by parts.

31. $\int x \sec x^2 \, dx$

32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

33. $\int x (\ln x)^2 \, dx$

34. $\int \frac{1}{x (\ln x)^2} \, dx$

35. $\int \frac{\ln x}{x^2} \, dx$

36. $\int \frac{(\ln x)^3}{x} \, dx$

37. $\int x^3 e^{x^4} \, dx$

38. $\int x^5 e^{x^3} \, dx$

39. $\int x^3 \sqrt{x^2 + 1} \, dx$

40. $\int x^2 \sin x^3 \, dx$

41. $\int \sin 3x \cos 2x \, dx$

42. $\int \sin 2x \cos 4x \, dx$

43. $\int e^x \sin e^x \, dx$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

45. $\int \cos \sqrt{x} \, dx$

46. $\int \sqrt{x} \, e^{\sqrt{x}} \, dx$

47. $\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta$

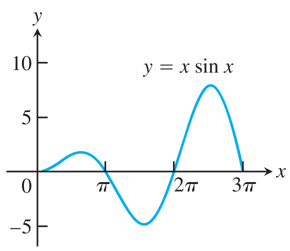
48. $\int_0^{\pi/2} x^3 \cos 2x \, dx$

49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt$

50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2) \, dx$

Theory and Examples

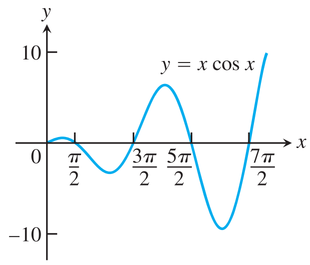
51. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for
   a. $0 \le x \le \pi$.
   b. $\pi \le x \le 2\pi$.
   c. $2\pi \le x \le 3\pi$.
   d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n + 1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



52. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for
   a. $\pi/2 \le x \le 3\pi/2$.
   b. $3\pi/2 \le x \le 5\pi/2$.
   c. $5\pi/2 \le x \le 7\pi/2$.
   d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n - 1}{2}\right)\pi \le x \le \left(\frac{2n + 1}{2}\right)\pi,$$

   n an arbitrary positive integer? Give reasons for your answer.



53. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

54. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
   a. about the y-axis.
   b. about the line $x = 1$.

55. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
   a. the y-axis.
   b. the line $x = \pi/2$.

56. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
   a. the y-axis.
   b. the line $x = \pi$.
   (See Exercise 51 for a graph.)

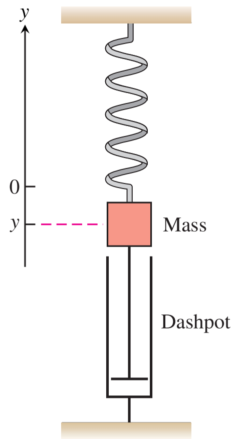
57. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.
   a. Find the area of the region.
   b. Find the volume of the solid formed by revolving this region about the x-axis.
   c. Find the volume of the solid formed by revolving this region about the line $x = -2$.
   d. Find the centroid of the region.

58. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.
   a. Find the area of the region.
   b. Find the volume of the solid formed by revolving this region about the y-axis.

59. **Average value** A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



60. Average value In a mass-spring-dashpot system like the one in Exercise 59, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 61–64, use integration by parts to establish the reduction formula.

61. $\displaystyle\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

62. $\displaystyle\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

63. $\displaystyle\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx, \quad a \ne 0$

64. $\displaystyle\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

65. Show that

$$\int_a^b \left(\int_x^b f(t)\, dt\right) dx = \int_a^b (x - a) f(x)\, dx.$$

66. Use integration by parts to obtain the formula

$$\int \sqrt{1 - x^2}\, dx = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1 - x^2}}\, dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\, dx &= \int y f'(y)\, dy && y = f^{-1}(x),\quad x = f(y) \\ & && dx = f'(y)\, dy \\ &= y f(y) - \int f(y)\, dy && \text{Integration by parts with} \\ & && u = y,\ dv = f'(y)\, dy \\ &= x f^{-1}(x) - \int f(y)\, dy\end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned}\int \ln x\, dx &= \int y e^y\, dy && y = \ln x,\quad x = e^y \\ & && dx = e^y\, dy \\ &= y e^y - e^y + C \\ &= x \ln x - x + C.\end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned}\int \cos^{-1} x\, dx &= x \cos^{-1} x - \int \cos y\, dy && y = \cos^{-1} x \\ &= x \cos^{-1} x - \sin y + C \\ &= x \cos^{-1} x - \sin(\cos^{-1} x) + C.\end{aligned}$$

Use the formula

$$\int f^{-1}(x)\, dx = x f^{-1}(x) - \int f(y)\, dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 67–70. Express your answers in terms of x.

67. $\displaystyle\int \sin^{-1} x\, dx$

68. $\displaystyle\int \tan^{-1} x\, dx$

69. $\displaystyle\int \sec^{-1} x\, dx$

70. $\displaystyle\int \log_2 x\, dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\, dx = x f^{-1}(x) - \int x\left(\frac{d}{dx} f^{-1}(x)\right) dx. \tag{5}$$

Exercises 71 and 72 compare the results of using Equations (4) and (5).

71. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\displaystyle\int \cos^{-1} x\, dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\displaystyle\int \cos^{-1} x\, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

72. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\displaystyle\int \tan^{-1} x\, dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C$ Eq. (4)

b. $\displaystyle\int \tan^{-1} x\, dx = x \tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 73 and 74 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

73. $\displaystyle\int \sinh^{-1} x\, dx$

74. $\displaystyle\int \tanh^{-1} x\, dx$

8.2 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\,dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\,dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x\,dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\,dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\,dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x\,dx.$$

Solution This is an example of Case 1.

$$
\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d\,(\cos x)) && \sin x \, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2)\, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C. && \blacksquare
\end{aligned}
$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$
\begin{aligned}
\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx &= \int (1 - \sin^2 x)^2 \, d(\sin x) && \cos x \, dx = d(\sin x) \\
&= \int (1 - u^2)^2 \, du && u = \sin x \\
&= \int (1 - 2u^2 + u^4)\, du && \text{Square } 1 - u^2. \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C. && \blacksquare
\end{aligned}
$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$
\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx && m \text{ and } n \text{ both even} \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].
\end{aligned}
$$

For the term involving $\cos^2 2x$, we use

$$
\begin{aligned}
\int \cos^2 2x \, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx \\
&= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). && \text{Omitting the constant of integration until the final result}
\end{aligned}
$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x)\cos 2x\, dx$$

$u = \sin 2x,$
$du = 2\cos 2x\, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$\cos 2x \ge 0$ on $[0, \pi/4]$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

■

Integrals of Powers of tan *x* and sec *x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C. \qquad \blacksquare$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx)\\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx \qquad \tan^2 x = \sec^2 x - 1\\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \qquad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m - n)x + \sin(m + n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]. \tag{5}$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Exercises 8.2

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$
3. $\int \cos^3 x \sin x \, dx$
4. $\int \sin^4 2x \cos 2x \, dx$
5. $\int \sin^3 x \, dx$
6. $\int \cos^3 4x \, dx$
7. $\int \sin^5 x \, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$
9. $\int \cos^3 x \, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$
11. $\int \sin^3 x \cos^3 x \, dx$
12. $\int \cos^3 2x \sin^5 2x \, dx$
13. $\int \cos^2 x \, dx$
14. $\int_0^{\pi/2} \sin^2 x \, dx$
15. $\int_0^{\pi/2} \sin^7 y \, dy$
16. $\int 7 \cos^7 t \, dt$
17. $\int_0^{\pi} 8 \sin^4 x \, dx$
18. $\int 8 \cos^4 2\pi x \, dx$
19. $\int 16 \sin^2 x \cos^2 x \, dx$
20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$
21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$
22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx$
24. $\int_0^{\pi} \sqrt{1 - \cos 2x} \, dx$
25. $\int_0^{\pi} \sqrt{1 - \sin^2 t} \, dt$
26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta} \, d\theta$
27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} \, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx$

    $\left(\textit{Hint}: \text{Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}.\right)$
29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}} \, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} \, dx$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x \, dx$
34. $\int \sec x \tan^2 x \, dx$
35. $\int \sec^3 x \tan x \, dx$
36. $\int \sec^3 x \tan^3 x \, dx$
37. $\int \sec^2 x \tan^2 x \, dx$
38. $\int \sec^4 x \tan^2 x \, dx$
39. $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx$
40. $\int e^x \sec^3 e^x \, dx$
41. $\int \sec^4 \theta \, d\theta$
42. $\int 3 \sec^4 3x \, dx$
43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$
44. $\int \sec^6 x \, dx$
45. $\int 4 \tan^3 x \, dx$
46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx$
47. $\int \tan^5 x \, dx$
48. $\int \cot^6 2x \, dx$
49. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx$
50. $\int 8 \cot^4 t \, dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 51–56.

51. $\int \sin 3x \cos 2x \, dx$

52. $\int \sin 2x \cos 3x \, dx$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

54. $\int_{0}^{\pi/2} \sin x \cos x \, dx$

55. $\int \cos 3x \cos 4x \, dx$

56. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Exercises 57–62 require the use of various trigonometric identities before you evaluate the integrals.

57. $\int \sin^2 \theta \cos 3\theta \, d\theta$

58. $\int \cos^2 2\theta \sin \theta \, d\theta$

59. $\int \cos^3 \theta \sin 2\theta \, d\theta$

60. $\int \sin^3 \theta \cos 2\theta \, d\theta$

61. $\int \sin \theta \cos \theta \cos 3\theta \, d\theta$

62. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 63–68.

63. $\int \frac{\sec^3 x}{\tan x} \, dx$

64. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

65. $\int \frac{\tan^2 x}{\csc x} \, dx$

66. $\int \frac{\cot x}{\cos^2 x} \, dx$

67. $\int x \sin^2 x \, dx$

68. $\int x \cos^3 x \, dx$

Applications

69. Arc length Find the length of the curve

$$y = \ln(\sec x), \quad 0 \le x \le \pi/4.$$

70. Center of gravity Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

71. Volume Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

72. Area Find the area between the x-axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \le x \le \pi$.

73. Centroid Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \le x \le 2\pi$.

74. Volume Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x-axis.

8.3 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 8.2.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

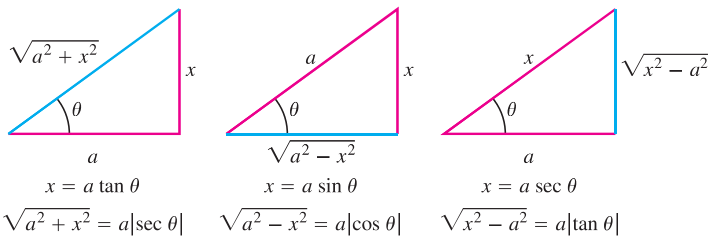


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan \theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a \sin \theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a \sec \theta$.

As we know from Section 1.6, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a \tan \theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\[2mm] \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

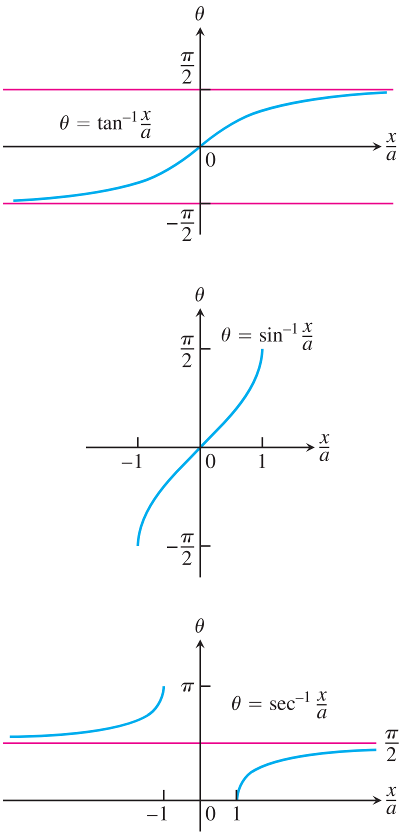


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

To simplify calculations with the substitution $x = a \sec \theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan \theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta| = a \tan \theta$, free of absolute values, provided $a > 0$.

Procedure For a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

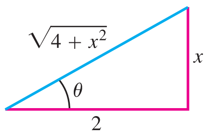


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\
&= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. && \text{From Fig. 8.4}
\end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Evaluate

$$\int \frac{x^2\, dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\begin{aligned}
\int \frac{x^2\, dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\, d\theta && \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= 9\int \frac{1-\cos 2\theta}{2}\, d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

■

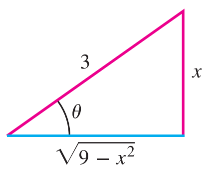


FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 3 Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned}
\sqrt{25x^2-4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\
&= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}
\end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

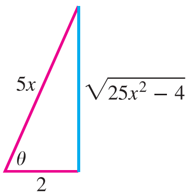


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\,d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{Fig. 8.6} \quad ■$$

EXERCISES 8.3

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{dx}{\sqrt{9 + x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1 + 9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4 + x^2}$

4. $\displaystyle\int_0^2 \frac{dx}{8 + 2x^2}$

5. $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9 - x^2}}$

6. $\displaystyle\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1 - 4x^2}}$

7. $\displaystyle\int \sqrt{25 - t^2}\,dt$

8. $\displaystyle\int \sqrt{1 - 9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2 - 49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2 - 9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2 - 49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2 - 25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2 - 1}}, \quad x > 1$

19. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4 - w^2}}$

20. $\displaystyle\int \frac{\sqrt{9 - w^2}}{w^2}\,dw$

21. $\displaystyle\int \frac{100}{36 + 25x^2}\,dx$

22. $\displaystyle\int x\sqrt{x^2 - 4}\,dx$

23. $\displaystyle\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1 - x^2)^{3/2}}$

24. $\displaystyle\int_0^1 \frac{dx}{(4 - x^2)^{3/2}}$

25. $\displaystyle\int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$

26. $\displaystyle\int \frac{x^2\,dx}{(x^2 - 1)^{5/2}}, \quad x > 1$

27. $\displaystyle\int \frac{(1 - x^2)^{3/2}}{x^6}\,dx$

28. $\displaystyle\int \frac{(1 - x^2)^{1/2}}{x^4}\,dx$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\displaystyle\int \frac{x}{\sqrt{9 - x^2}}\,dx$

16. $\displaystyle\int \frac{x^2}{4 + x^2}\,dx$

17. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2 + 4}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 1}}$

29. $\displaystyle\int \frac{8\,dx}{(4x^2 + 1)^2}$

30. $\displaystyle\int \frac{6\,dt}{(9t^2 + 1)^2}$

31. $\displaystyle\int \frac{x^3\,dx}{x^2 - 1}$

32. $\displaystyle\int \frac{x\,dx}{25 + 4x^2}$

33. $\displaystyle\int \frac{v^2\,dv}{(1 - v^2)^{5/2}}$

34. $\displaystyle\int \frac{(1 - r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\displaystyle\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t} + 9}}$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1 + e^{2t})^{3/2}}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}}$

38. $\displaystyle\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

39. $\int \frac{dx}{x\sqrt{x^2 - 1}}$

40. $\int \frac{dx}{1 + x^2}$

41. $\int \frac{x\,dx}{\sqrt{x^2 - 1}}$

42. $\int \frac{dx}{\sqrt{1 - x^2}}$

43. $\int \frac{x\,dx}{\sqrt{1 + x^4}}$

44. $\int \frac{\sqrt{1 - (\ln x)^2}}{x \ln x}\,dx$

45. $\int \sqrt{\frac{4 - x}{x}}\,dx$
(*Hint*: Let $x = u^2$.)

46. $\int \sqrt{\frac{x}{1 - x^3}}\,dx$
(*Hint*: Let $u = x^{3/2}$.)

47. $\int \sqrt{x}\sqrt{1 - x}\,dx$

48. $\int \frac{\sqrt{x - 2}}{\sqrt{x - 1}}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 49–52 for y as a function of x.

49. $x\frac{dy}{dx} = \sqrt{x^2 - 4}, \quad x \geq 2, \quad y(2) = 0$

50. $\sqrt{x^2 - 9}\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

51. $(x^2 + 4)\frac{dy}{dx} = 3, \quad y(2) = 0$

52. $(x^2 + 1)^2\frac{dy}{dx} = \sqrt{x^2 + 1}, \quad y(0) = 1$

Applications and Examples

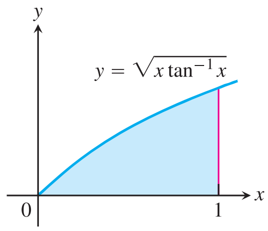
53. **Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2}/3$.

54. **Area** Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

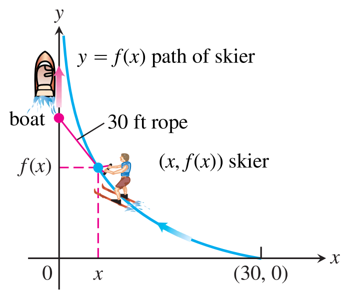
55. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.
    a. Find the area of the region.
    b. Find the centroid of the region.

56. Consider the region bounded by the graphs of $y = \sqrt{x}\tan^{-1} x$ and $y = 0$ for $0 \leq x \leq 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



57. Evaluate $\int x^3\sqrt{1 - x^2}\,dx$ using
    a. integration by parts.
    b. a u-substitution.
    c. a trigonometric substitution.

58. **Path of a water skier** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.
    a. Show that $f'(x) = \frac{-\sqrt{900 - x^2}}{x}$.
    (*Hint*: Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)
    b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



NOT TO SCALE

8.4 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function

$(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x - 3}{(x + 1)(x - 3)}\,dx = \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx$$
$$= 2 \ln |x + 1| + 3 \ln |x - 3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \qquad (1)$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

continued

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate
$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\, dx.$$

Solution The partial fraction decomposition has the form
$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$
To find the values of the undetermined coefficients A, B, and C, we clear fractions and get
$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$
The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining
$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$
There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have
$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\, dx &= \int \left[\frac{3}{4}\frac{1}{x - 1} + \frac{1}{2}\frac{1}{x + 1} - \frac{1}{4}\frac{1}{x + 3}\right] dx \\ &= \frac{3}{4}\ln|x - 1| + \frac{1}{2}\ln|x + 1| - \frac{1}{4}\ln|x + 3| + K, \end{aligned}$$
where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 Use partial fractions to evaluate
$$\int \frac{6x + 7}{(x + 2)^2}\, dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned}\frac{6x+7}{(x+2)^2} &= \frac{A}{x+2} + \frac{B}{(x+2)^2}\\ 6x+7 &= A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.\\ &= Ax + (2A+B)\end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned}\int \frac{6x+7}{(x+2)^2}\,dx &= \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx\\ &= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx\\ &= 6\ln|x+2| + 5(x+2)^{-1} + C.\end{aligned}$$

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \phantom{{}-4x^2-x-3}\\ x^2-2x-3\,\overline{)\,2x^3-4x^2-x-3}\\ \underline{2x^3-4x^2-6x}\phantom{{}-3}\\ 5x-3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x-3}{x^2-2x-3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned}\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx\\ &= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx\\ &= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.\end{aligned}$$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} &\text{Coefficients of } x^3: & 0 &= A + C \\ &\text{Coefficients of } x^2: & 0 &= -2A + B - C + D \\ &\text{Coefficients of } x^1: & -2 &= A - 2B + C \\ &\text{Coefficients of } x^0: & 4 &= B - C + D \end{aligned}$$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$\begin{aligned} -4 &= -2A, \quad A = 2 && \text{Subtract fourth equation from second.} \\ C &= -A = -2 && \text{From the first equation} \\ B &= (A + C + 2)/2 = 1 && \text{From the third equation and } C = -A \\ D &= 4 - B + C = 1. && \text{From the fourth equation} \end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned} \int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C. \end{aligned}$$ ■

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \qquad C = 0, \qquad 2A + B + D = 0, \qquad C + E = 0, \qquad A = 1.$$

Solving this system gives $A = 1, B = -1, C = 0, D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{array}{l} u = x^2+1, \\ du = 2x\,dx \end{array} \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

■

HISTORICAL BIOGRAPHY

Oliver Heaviside (1850–1925)

The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0,$$

$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} \tag{4}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2+1}{\underbrace{\boxed{(x-1)}}_{\Uparrow\ \text{Cover}}(1-2)(1-3)} = \frac{2}{(-1)(-2)} = 1.$$

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Expression (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2 + 1}{(2 - 1)\underbrace{\boxed{(x - 2)}}_{\text{Cover}}(2 - 3)} = \frac{5}{(1)(-1)} = -5.$$

Finally, C is found by covering the $(x - 3)$ in Expression (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3 - 1)(3 - 2)\underbrace{\boxed{(x - 3)}}_{\text{Cover}}} = \frac{10}{(2)(1)} = 5.$$

■

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 7 Use the Heaviside Method to evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of the cubic polynomial $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0+4}{\underset{\substack{\Uparrow\\ \text{Cover}}}{\boxed{x}}\,(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$$A_2 = \frac{2+4}{2\,\underset{\substack{\Uparrow\\ \text{Cover}}}{\boxed{(x-2)}}\,(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$$A_3 = \frac{-5+4}{(-5)(-5-2)\,\underset{\substack{\Uparrow\\ \text{Cover}}}{\boxed{(x+5)}}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C. \qquad \blacksquare$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}. \qquad \blacksquare$$

In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution Clear fractions to get

$$x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0)\\
2 &= 2A\\
A &= 1\\
x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0)\\
5 &= -B\\
B &= -5\\
x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1)\\
10 &= 2C\\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}.$$

Exercises 8.4

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{x^2 - x + 2}{x^3 - 1}\,dx$

28. $\displaystyle\int \frac{1}{x^4 + x}\,dx$

29. $\displaystyle\int \frac{x^2}{x^4 - 1}\,dx$

30. $\displaystyle\int \frac{x^2 + x}{x^4 - 3x^2 - 4}\,dx$

31. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

32. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

34. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

35. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

36. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

37. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

38. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–50.

39. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

40. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

41. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

42. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

43. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

44. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

45. $\displaystyle\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

46. $\displaystyle\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$

(*Hint*: Let $x = u^6$.)

47. $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

(*Hint*: Let $x + 1 = u^2$.)

48. $\displaystyle\int \frac{1}{x\sqrt{x+9}}\,dx$

49. $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$

$\left(\text{\textit{Hint}: Multiply by } \dfrac{x^3}{x^3}.\right)$

50. $\displaystyle\int \frac{1}{x^6(x^5+4)}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 51–54 for x as a function of t.

51. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

52. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

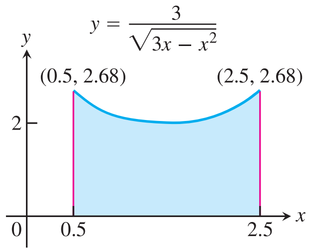
53. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

54. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

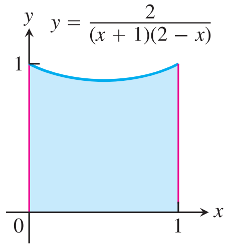
Applications and Examples

In Exercises 55 and 56, find the volume of the solid generated by revolving the shaded region about the indicated axis.

55. The x-axis

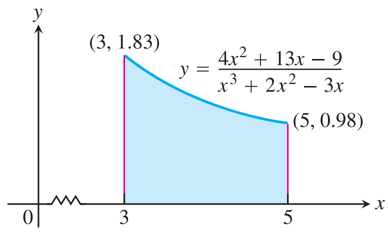


56. The y-axis



T 57. Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

T 58. Find the x-coordinate of the centroid of this region to two decimal places.



T 59. Social diffusion Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

T 60. Second-order chemical reactions Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

8.5 Integral Tables and Computer Algebra Systems

In this section we discuss how to use tables and computer algebra systems to evaluate integrals.

Integral Tables

A Brief Table of Integrals is provided at the back of the book, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 21 requires $n \neq -1$, for example, and Formula 27 requires $n \neq -2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 24 assumes that $a \neq 0$, and Formulas 29a and 29b cannot be used unless b is positive.

EXAMPLE 1 Find

$$\int x(2x+5)^{-1}\,dx.$$

Solution We use Formula 24 at the back of the book (not 22, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C.$$ ■

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 29b:

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{2x-4}{4}} + C = \tan^{-1}\sqrt{\frac{x-2}{2}} + C.$$ ■

EXAMPLE 3 Find

$$\int x\sin^{-1}x\,dx.$$

Solution We begin by using Formula 106:

$$\int x^n \sin^{-1}ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 \, dx}{\sqrt{1 - x^2}}.$$

Next we use Formula 49 to find the integral on the right:

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C.$$

The combined result is

$$\begin{aligned} \int x \sin^{-1} x \, dx &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C\right) \\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right) \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C'. \end{aligned}$$

■

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x \, dx = \frac{1}{n - 1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \tag{1}$$

$$\int (\ln x)^n \, dx = x (\ln x)^n - n \int (\ln x)^{n-1} \, dx \tag{2}$$

$$\int \sin^n x \cos^m x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m + n} + \frac{n - 1}{m + n} \int \sin^{n-2} x \cos^m x \, dx \qquad (n \neq -m). \tag{3}$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 4 Find

$$\int \tan^5 x \, dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + C.$$

The combined result is

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C'.$$

■

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 8.1.)

Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 5 Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f\!:= x\hat{\ }2 * \text{sqrt}\,(a\hat{\ }2 + x\hat{\ }2);$$

Then you use the integrate command on f, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see if the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4\ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple will return the expression

$$\frac{1}{64}\pi(4a^2 + \pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2 + \pi^2} + \frac{1}{8}a^4\ln(2)$$
$$- \frac{1}{8}a^4\ln\left(\pi + \sqrt{4a^2 + \pi^2}\right) + \frac{1}{16}a^4\ln(a^2).$$

You can also find the definite integral for a particular value of the constant a:

$$> a\!:= 1;$$
$$> \text{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2} - 1\right).$$ ■

EXAMPLE 6 Use a CAS to find

$$\int \sin^2 x \cos^3 x\,dx.$$

Solution With Maple, we have the entry

$$> \text{int}\,((\sin\hat{\ }2)(x) * (\cos\hat{\ }3)(x), x);$$

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$ ■

Computer algebra systems vary in how they process integrations. We used Maple in Examples 5 and 6. Mathematica would have returned somewhat different results:

1. In Example 5, given

$$\textit{In [1]:=}\ \text{Integrate}\,[x\verb|^|2 * \text{Sqrt}\,[a\verb|^|2\,+\,x\verb|^|2], x]$$

Mathematica returns

$$\textit{Out [1]=}\ \sqrt{a^2+x^2}\left(\frac{a^2 x}{8}+\frac{x^3}{4}\right)-\frac{1}{8}a^4\,\text{Log}\left[x+\sqrt{a^2+x^2}\right]$$

without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

2. The Mathematica answer to the integral

$$\textit{In [2]:=}\ \text{Integrate}\,[\text{Sin}\,[x]\verb|^|2 * \text{Cos}\,[x]\verb|^|3, x]$$

in Example 6 is

$$\textit{Out [2]=}\ \frac{\text{Sin}\,[x]}{8}-\frac{1}{48}\text{Sin}\,[3\,x]-\frac{1}{80}\text{Sin}\,[5\,x]$$

differing from the Maple answer. Both answers are correct.

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica returns an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 67.

Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 10) or numerical methods for their evaluation, which give only an approximation. Examples of nonelementary integrals include the error function (which measures the probability of random errors)

$$\text{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1+x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1-k^2\sin^2 x}\,dx, \qquad 0<k<1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives, as a consequence of the Fundamental Theorem of Calculus, Part 1, because they are continuous. However, none of the antiderivatives are elementary.

None of the integrals you are asked to evaluate in the present chapter fall into this category, but you may encounter nonelementary integrals in your other work.

Exercises 8.5

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–26.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$
2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$
3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$
4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$
5. $\displaystyle\int x\sqrt{2x-3}\,dx$
6. $\displaystyle\int x(7x+5)^{3/2}\,dx$
7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$
8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$
9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$
10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$
11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$
12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$
13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$
14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$
15. $\displaystyle\int e^{2t}\cos 3t\,dt$
16. $\displaystyle\int e^{-3t}\sin 4t\,dt$
17. $\displaystyle\int x\cos^{-1}x\,dx$
18. $\displaystyle\int x\tan^{-1}x\,dx$
19. $\displaystyle\int x^2\tan^{-1}x\,dx$
20. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$
21. $\displaystyle\int \sin 3x\cos 2x\,dx$
22. $\displaystyle\int \sin 2x\cos 3x\,dx$
23. $\displaystyle\int 8\sin 4t\sin\frac{t}{2}\,dt$
24. $\displaystyle\int \sin\frac{t}{3}\sin\frac{t}{6}\,dt$
25. $\displaystyle\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta$
26. $\displaystyle\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta$

Substitution and Integral Tables

In Exercises 27–40, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

27. $\displaystyle\int \frac{x^3+x+1}{(x^2+1)^2}\,dx$
28. $\displaystyle\int \frac{x^2+6x}{(x^2+3)^2}\,dx$
29. $\displaystyle\int \sin^{-1}\sqrt{x}\,dx$
30. $\displaystyle\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx$
31. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$
32. $\displaystyle\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx$
33. $\displaystyle\int \cot t\sqrt{1-\sin^2 t}\,dt,\quad 0<t<\pi/2$
34. $\displaystyle\int \frac{dt}{\tan t\sqrt{4-\sin^2 t}}$
35. $\displaystyle\int \frac{dy}{y\sqrt{3+(\ln y)^2}}$
36. $\displaystyle\int \tan^{-1}\sqrt{y}\,dy$
37. $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$ (*Hint*: Complete the square.)
38. $\displaystyle\int \frac{x^2}{\sqrt{x^2-4x+5}}\,dx$
39. $\displaystyle\int \sqrt{5-4x-x^2}\,dx$
40. $\displaystyle\int x^2\sqrt{2x-x^2}\,dx$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 41–50.

41. $\displaystyle\int \sin^5 2x\,dx$
42. $\displaystyle\int 8\cos^4 2\pi t\,dt$
43. $\displaystyle\int \sin^2 2\theta\cos^3 2\theta\,d\theta$
44. $\displaystyle\int 2\sin^2 t\sec^4 t\,dt$
45. $\displaystyle\int 4\tan^3 2x\,dx$
46. $\displaystyle\int 8\cot^4 t\,dt$
47. $\displaystyle\int 2\sec^3 \pi x\,dx$
48. $\displaystyle\int 3\sec^4 3x\,dx$
49. $\displaystyle\int \csc^5 x\,dx$
50. $\displaystyle\int 16x^3(\ln x)^2\,dx$

Evaluate the integrals in Exercises 51–56 by making a substitution (possibly trigonometric) and then applying a reduction formula.

51. $\displaystyle\int e^t\sec^3(e^t-1)\,dt$
52. $\displaystyle\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$
53. $\displaystyle\int_0^1 2\sqrt{x^2+1}\,dx$
54. $\displaystyle\int_0^{\sqrt{3}/2} \frac{dy}{(1-y^2)^{5/2}}$
55. $\displaystyle\int_1^2 \frac{(r^2-1)^{3/2}}{r}\,dr$
56. $\displaystyle\int_0^{1/\sqrt{3}} \frac{dt}{(t^2+1)^{7/2}}$

Applications

57. **Surface area** Find the area of the surface generated by revolving the curve $y=\sqrt{x^2+2}$, $0\le x\le\sqrt{2}$, about the x-axis.

58. **Arc length** Find the length of the curve $y=x^2$, $0\le x\le\sqrt{3}/2$.

59. **Centroid** Find the centroid of the region cut from the first quadrant by the curve $y=1/\sqrt{x+1}$ and the line $x=3$.

60. **Moment about *y*-axis** A thin plate of constant density $\delta=1$ occupies the region enclosed by the curve $y=36/(2x+3)$ and the line $x=3$ in the first quadrant. Find the moment of the plate about the y-axis.

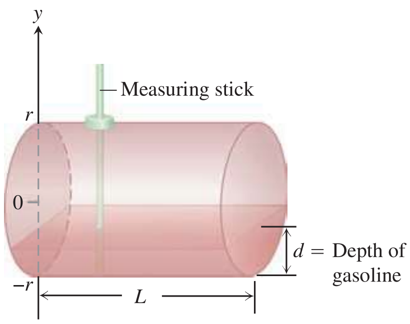
T 61. Use the integral table and a calculator to find to two decimal places the area of the surface generated by revolving the curve $y=x^2$, $-1\le x\le 1$, about the x-axis.

62. **Volume** The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L, mounted horizontally, as shown in the accompanying figure. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a. Show, in the notation of the figure, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

b. Evaluate the integral.



63. What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

64. What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 65 and 66, use a CAS to perform the integrations.

65. Evaluate the integrals

a. $\int x \ln x\, dx$ **b.** $\int x^2 \ln x\, dx$ **c.** $\int x^3 \ln x\, dx$.

d. What pattern do you see? Predict the formula for $\int x^4 \ln x\, dx$ and then see if you are correct by evaluating it with a CAS.

e. What is the formula for $\int x^n \ln x\, dx$, $n \geq 1$? Check your answer using a CAS.

66. Evaluate the integrals

a. $\int \frac{\ln x}{x^2}\, dx$ **b.** $\int \frac{\ln x}{x^3}\, dx$ **c.** $\int \frac{\ln x}{x^4}\, dx$.

d. What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\, dx$$

and then see if you are correct by evaluating it with a CAS.

e. What is the formula for

$$\int \frac{\ln x}{x^n}\, dx, \quad n \geq 2?$$

Check your answer using a CAS.

67. a. Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

where n is an arbitrary positive integer. Does your CAS find the result?

b. In succession, find the integral when $n = 1, 2, 3, 5$, and 7. Comment on the complexity of the results.

c. Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

8.6 Numerical Integration

The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1 + x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f. This procedure is an example of numerical integration. In this section we study two such methods, the *Trapezoidal Rule* and *Simpson's Rule*. In our presentation we assume that f is positive, but the only requirement is for it to be continuous over the interval of integration $[a, b]$.

Trapezoidal Approximations

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in

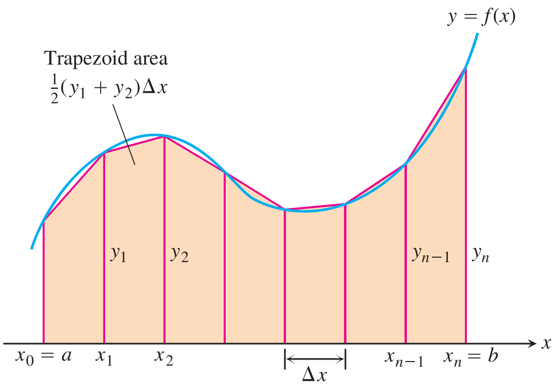


FIGURE 8.7 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the areas of the trapezoids made by joining the ends of the segments to the x-axis.

Figure 8.7. It is not necessary for the subdivision points $x_0, x_1, x_2, \ldots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1} + y_i),$$

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. This area is the length Δx of the trapezoid's horizontal "altitude" times the average of its two vertical "bases." (See Figure 8.7.) The area below the curve $y = f(x)$ and above the x-axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \qquad y_1 = f(x_1), \qquad \ldots, \qquad y_{n-1} = f(x_{n-1}), \qquad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b.

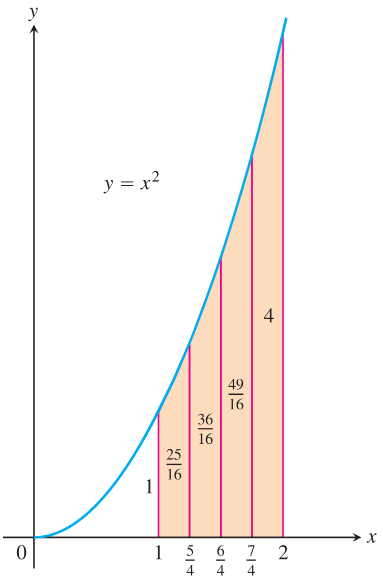


FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution Partition [1, 2] into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).

Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

Simpson's Rule: Approximations Using Parabolas

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight line segments that produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola, as shown in Figure 8.9. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.

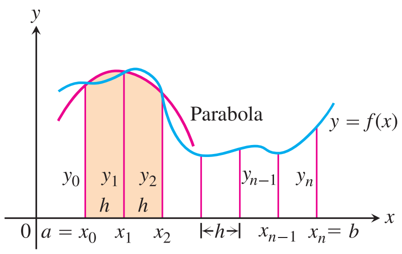


FIGURE 8.9 Simpson's Rule approximates short stretches of the curve with parabolas.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and

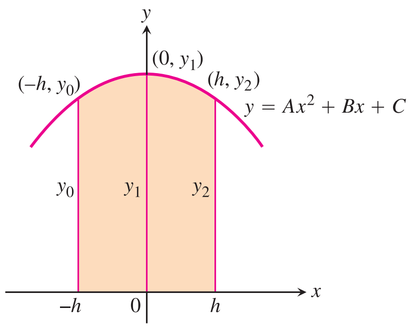


FIGURE 8.10 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

$x_2 = h$ (Figure 8.10), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

so the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned} A_p &= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\ &= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \bigg]_{-h}^{h} \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C). \end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$\begin{aligned} C &= y_1, \\ Ah^2 - Bh &= y_0 - y_1, \\ Ah^2 + Bh &= y_2 - y_1, \\ 2Ah^2 &= y_0 + y_2 - 2y_1. \end{aligned}$$

Hence, expressing the area A_p in terms of the ordinates y_0, y_1, and y_2, we have

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.9 does not change the area under it. Thus the area under the parabola through (x_0, y_0), (x_1, y_1), and (x_2, y_2) in Figure 8.9 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned} \int_a^b f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

HISTORICAL BIOGRAPHY

Thomas Simpson
(1720–1761)

The result is known as Simpson's Rule. The function need not be positive, as in our derivation, but the number n of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: $1, 4, 2, 4, 2, 4, 2, \ldots, 4, 1$.

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

Error Analysis

Whenever we use an approximation technique, the issue arises as to how accurate the approximation might be. The following theorem gives formulas for estimating the errors when using the Trapezoidal Rule and Simpson's Rule. The **error** is the difference between the approximation obtained by the rule and the actual value of the definite integral $\int_a^b f(x)\,dx$.

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

To see why Theorem 1 is true in the case of the Trapezoidal Rule, we begin with a result from advanced calculus, which says that if f'' is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12}\cdot f''(c)(\Delta x)^2$$

for some number c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero as the *square* of Δx.

The inequality

$$|E_T| \le \frac{b-a}{12} \max |f''(x)| (\Delta x)^2$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max|f''(x)|$ and have to estimate an upper bound or "worst case" value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \le M$ on $[a, b]$, then

$$|E_T| \le \frac{b-a}{12} M(\Delta x)^2.$$

If we substitute $(b-a)/n$ for Δx, we get

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

To estimate the error in Simpson's rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point c between a and b. Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4,$$

approaches zero as the *fourth power* of Δx. (This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \le \frac{b-a}{180} \max |f^{(4)}(x)|\,(\Delta x)^4,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max|f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max|f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \le \frac{b-a}{180} M(\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

EXAMPLE 3 Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \le x \le 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

This estimate is consistent with the result of Example 2. ■

Theorem 1 can also be used to estimate the number of subintervals required when using the Trapezoidal or Simpson's Rules if we specify a certain tolerance for the error.

EXAMPLE 4 Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4}.

Solution Using the inequality in Theorem 1, if we choose the number of subintervals n to satisfy

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

then the error E_S in Simpson's Rule satisfies $|E_S| < 10^{-4}$ as required.

From the solution in Example 3, we have $M = 120$ and $b - a = 2$, so we want n to satisfy

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4}$$

or, equivalently,

$$n^4 > \frac{64 \cdot 10^4}{3}.$$

It follows that

$$n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5.$$

Since n must be even in Simpson's Rule, we estimate the minimum number of subintervals required for the error tolerance to be $n = 22$. ■

EXAMPLE 5 As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

Table 8.4 shows T and S values for approximations of $\int_1^2 (1/x)\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule.

TABLE 8.4 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x)\,dx$

n	T_n	|Error| less than . . .	S_n	|Error| less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f. In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero, and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f. This is no surprise, for the trapezoids fit the graph perfectly.

Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice. When Δx is very small, say $\Delta x = 10^{-5}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. Although this is not an issue in this book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

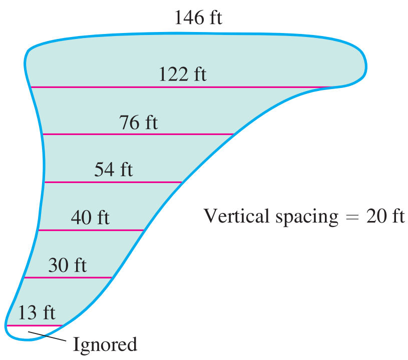


FIGURE 8.11 The dimensions of the swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Figure 8.11). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the y's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$\begin{aligned} S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100 \end{aligned}$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3. ■

Exercises 8.6

Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x - 1)\,dx$

3. $\int_{-1}^{1}(x^2+1)\,dx$

4. $\int_{-2}^{0}(x^2-1)\,dx$

5. $\int_{0}^{2}(t^3+t)\,dt$

6. $\int_{-1}^{1}(t^3+1)\,dt$

7. $\int_{1}^{2}\frac{1}{s^2}\,ds$

8. $\int_{2}^{4}\frac{1}{(s-1)^2}\,ds$

9. $\int_{0}^{\pi}\sin t\,dt$

10. $\int_{0}^{1}\sin \pi t\,dt$

Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

11. $\int_{1}^{2}x\,dx$

12. $\int_{1}^{3}(2x-1)\,dx$

13. $\int_{-1}^{1}(x^2+1)\,dx$

14. $\int_{-2}^{0}(x^2-1)\,dx$

15. $\int_{0}^{2}(t^3+t)\,dt$

16. $\int_{-1}^{1}(t^3+1)\,dt$

17. $\int_{1}^{2}\frac{1}{s^2}\,ds$

18. $\int_{2}^{4}\frac{1}{(s-1)^2}\,ds$

19. $\int_{0}^{3}\sqrt{x+1}\,dx$

20. $\int_{0}^{3}\frac{1}{\sqrt{x+1}}\,dx$

21. $\int_{0}^{2}\sin(x+1)\,dx$

22. $\int_{-1}^{1}\cos(x+\pi)\,dx$

Estimates with Numerical Data

23. **Volume of water in a swimming pool** A rectangular swimming pool is 30 ft wide and 50 ft long. The accompanying table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

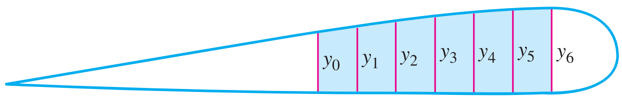
$$V = \int_{0}^{50} 30 \cdot h(x)\,dx.$$

Position (ft) x	**Depth (ft)** $h(x)$	**Position (ft)** x	**Depth (ft)** $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

24. **Distance traveled** The accompanying table shows time-to-speed data for a sports car accelerating from rest to 130 mph. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	**Time (sec)**
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

25. **Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank by Simpson's Rule.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

26. **Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	**Oil consumption rate (liters/h)**
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

27. **Usable values of the sine-integral function** *The sine-integral function,*

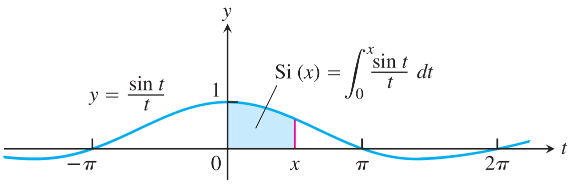
$$\mathrm{Si}(x) = \int_{0}^{x}\frac{\sin t}{t}\,dt, \quad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



a. Use the fact that $|f^{(4)}| \le 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

b. Estimate $\mathrm{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).

28. The error function *The error function*,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a. Use Simpson's Rule with $n = 10$ to estimate $\mathrm{erf}(1)$.

b. In $[0, 1]$,

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \le 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

29. Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

30. Prove that the sum S in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 29.)

T **31. Elliptic integrals** The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t}\,dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

Applications

T **32.** The length of one arch of the curve $y = \sin x$ is given by

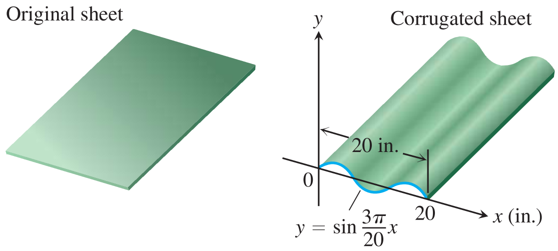
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx.$$

Estimate L by Simpson's Rule with $n = 8$.

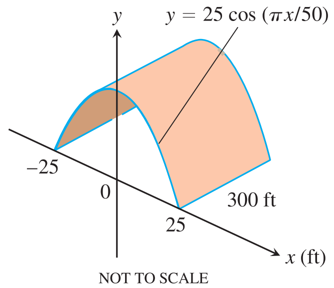
T **33.** Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20} x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



T **34.** Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25 \cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the x-axis.

35. $y = \sin x, \quad 0 \le x \le \pi$

36. $y = x^2/4, \quad 0 \le x \le 2$

37. Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

38. Use numerical integration to estimate the value of

$$\pi = 4 \int_0^1 \frac{1}{1 + x^2}\,dx.$$

8.7 Improper Integrals

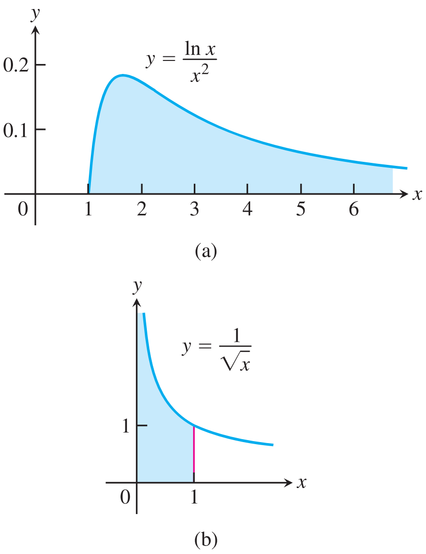


FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

Up to now, we have required definite integrals to have two properties. First, that the domain of integration $[a, b]$ be finite. Second, that the range of the integrand be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.12a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.12b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see in Chapter 10 that improper integrals play an important role when investigating the convergence of certain infinite series.

Infinite Limits of Integration

Consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.13a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.13b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

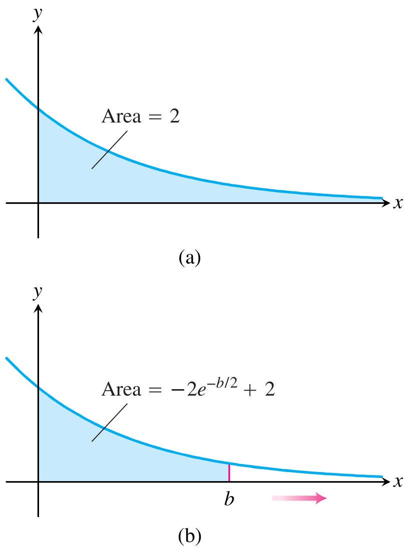


FIGURE 8.13 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ (b) The area is an improper integral of the first type.

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$
where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

It can be shown that the choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.13 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

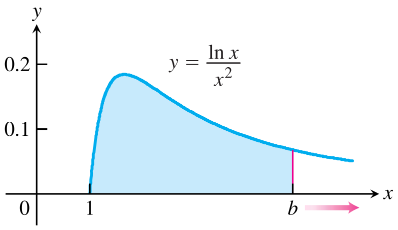


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.14). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x,\ dv = dx/x^2,\ du = dx/x,\ v = -1/x$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

l'Hôpital's Rule

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

HISTORICAL BIOGRAPHY

Lejeune Dirichlet
(1805–1859)

Next we evaluate each improper integral on the right side of the equation above.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0$$

$$= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\begin{aligned}\int_0^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} \\ &= \lim_{b\to\infty} \tan^{-1} x \Big]_0^b \\ &= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.15). ■

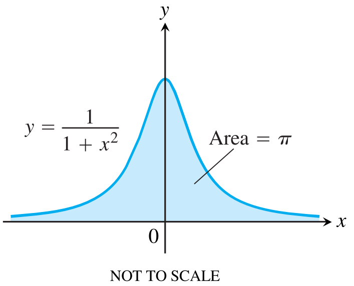


FIGURE 8.15 The area under this curve is finite (Example 2).

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}\end{aligned}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \int_1^\infty \frac{dx}{x} \\ &= \lim_{b\to\infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \ln x\Big]_1^b \\ &= \lim_{b\to\infty} (\ln b - \ln 1) = \infty .\end{aligned}$$

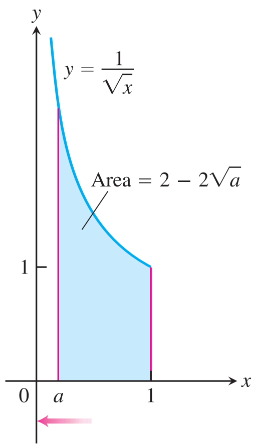


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.12b). First we find the area of the portion from a to 1 (Figure 8.16).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

Therefore the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

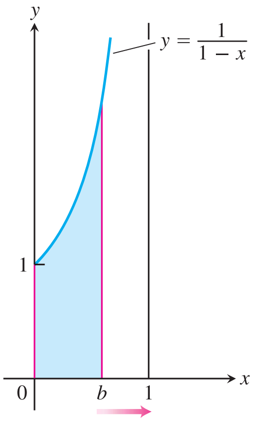


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.17). We evaluate the integral as

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx &= \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b\\ &= \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

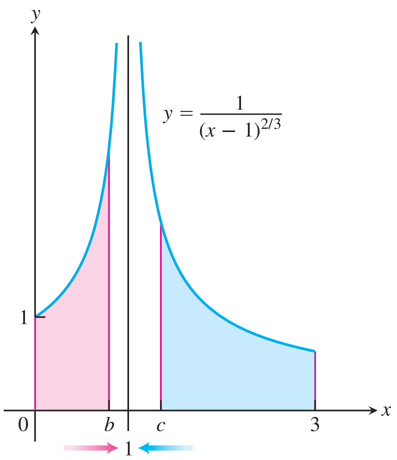


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}}\\ &= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b\\ &= \lim_{b\to 1^-}[3(b-1)^{1/3}+3] = 3\\ \int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}}\\ &= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3\\ &= \lim_{c\to 1^+}\Big[3(3-1)^{1/3} - 3(c-1)^{1/3}\Big] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

Improper Integrals with a CAS

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx$$

(which converges) using Maple, enter

$$> f := (x + 3)/((x - 1) * (x\^{}2 + 1));$$

Then use the integration command

$$> \text{int}(f, x = 2..\text{infinity});$$

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$\textit{In [1]:=} \text{ Integrate } [(x + 3)/((x - 1)(x\^{}2 + 1)), \{x, 2, \text{Infinity}\}]$$

returns

$$\textit{Out [1]=} \frac{-\pi}{2} + \text{ArcTan}[2] + \text{Log}[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

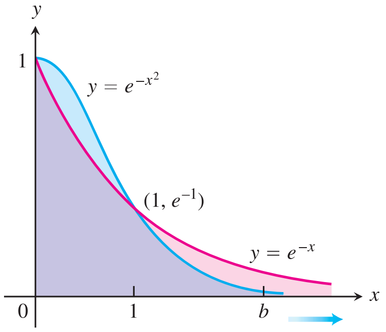


FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

EXAMPLE 6 Does the integral $\int_1^{\infty} e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^{\infty} e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x^2}\,dx.$$

We cannot evaluate this integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$ (Figure 8.19) so that

$$\int_1^{b} e^{-x^2}\,dx \leq \int_1^{b} e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence,

$$\int_1^{\infty} e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x^2}\,dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 6. ■

The comparison of e^{-x^2} and e^{-x} in Example 6 is a special case of the following test.

HISTORICAL BIOGRAPHY

Karl Weierstrass
(1815–1897)

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Proof The reasoning behind the argument establishing Theorem 2 is similar to that in Example 6. If $0 \le f(x) \le g(x)$ for $x \ge a$, then from Rule 7 in Theorem 2 of Section 5.3 we have

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 6, that

$$\int_a^\infty f(x)\,dx \quad \text{converges if} \quad \int_a^\infty g(x)\,dx \quad \text{converges.}$$

Turning this around says that

$$\int_a^\infty g(x)\,dx \quad \text{diverges if} \quad \int_a^\infty f(x)\,dx \quad \text{diverges.}$$

■

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$

Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$

Example 3 ■

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

We omit the more advanced proof of Theorem 3.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

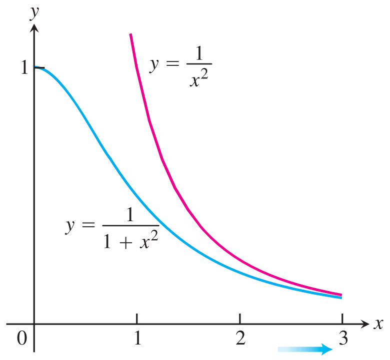


FIGURE 8.20 The functions in Example 8.

EXAMPLE 8 Show that

$$\int_1^\infty \frac{dx}{1 + x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1 + x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1 + x^2)} = \lim_{x\to\infty} \frac{1 + x^2}{x^2}$$
$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1 + x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2 - 1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1 + x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1 + x^2}$$
$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

■

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1 - e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$. ■

TABLE 8.5

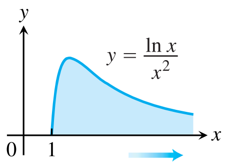
b	$\int_1^b \frac{1 - e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

Types of Improper Integrals Discussed in This Section

Infinite Limits of Integration: Type I

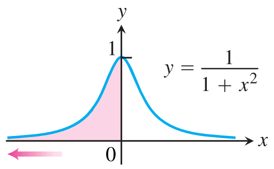
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



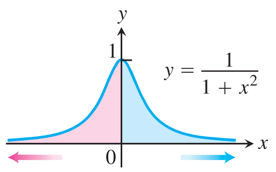
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$



3. Both limits

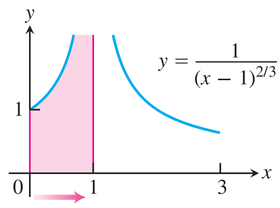
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$



Integrand Becomes Infinite: Type II

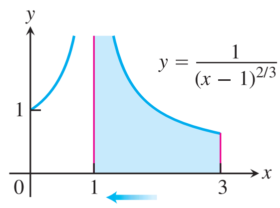
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



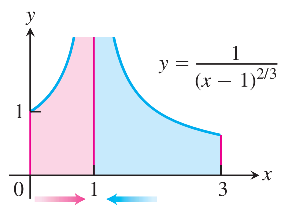
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Exercises 8.7

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^{\infty} \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^{1} \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^{1} \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^{1} \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^{\infty} \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^{\infty} \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\displaystyle\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$

22. $\displaystyle\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^{0} e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^{1} x\ln x\,dx$

26. $\displaystyle\int_0^{1} (-\ln x)\,dx$

27. $\displaystyle\int_0^{2} \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^{1} \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^{2} \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^{4} \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^{2} \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\displaystyle\int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$

39. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\displaystyle\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

42. $\displaystyle\int_0^{1} \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\displaystyle\int_0^{2} \frac{dx}{1-x^2}$

44. $\displaystyle\int_0^{2} \frac{dx}{1-x}$

45. $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

46. $\displaystyle\int_{-1}^{1} -x\ln|x|\,dx$

47. $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$

48. $\displaystyle\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$

49. $\displaystyle\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

50. $\displaystyle\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$

51. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$

52. $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

53. $\displaystyle\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\displaystyle\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

56. $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$

57. $\displaystyle\int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$

58. $\displaystyle\int_2^{\infty} \frac{1}{\ln x}\,dx$

59. $\displaystyle\int_1^{\infty} \frac{e^x}{x}\,dx$

60. $\displaystyle\int_{e^e}^{\infty} \ln(\ln x)\,dx$

61. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\displaystyle\int_1^{\infty} \frac{1}{e^x-2^x}\,dx$

63. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$

64. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^{2} \frac{dx}{x(\ln x)^p}$ **b.** $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

66. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

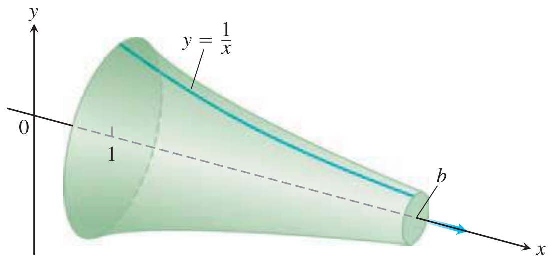
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

74. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx > 2\pi \int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

a. Calculate it.

b. This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

75. Sine-integral function The integral

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the *sine-integral function*, has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\operatorname{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\operatorname{Si}(x)$ for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

76. Error function The function

$$\operatorname{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.4, Exercise 41.

77. Normal probability distribution The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^n f(x)\,dx$$

for $n = 1$, 2, and 3.

c. Give a convincing argument that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_b^\infty e^{-x/2}\,dx \to 0 \quad \text{as} \quad b \to \infty.)$$

78. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both converge if and only if $\int_{-\infty}^b f(x)\,dx$ and $\int_b^\infty f(x)\,dx$ both converge.

b. $\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$ when the integrals involved converge.

COMPUTER EXPLORATIONS

In Exercises 79–82, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

79. $\displaystyle\int_0^e x^p \ln x\,dx$

80. $\displaystyle\int_e^\infty x^p \ln x\,dx$

81. $\displaystyle\int_0^\infty x^p \ln x\,dx$

82. $\displaystyle\int_{-\infty}^\infty x^p \ln |x|\,dx$

Chapter 8 Questions to Guide Your Review

1. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
2. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
3. If an integrand is a product of the form $\sin^n x \cos^m x$, where m and n are nonnegative integers, how do you evaluate the integral? Give a specific example of each case.
4. What substitutions are made to evaluate integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$? Give an example of each case.
5. What substitutions are sometimes used to transform integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals that can be evaluated directly? Give an example of each case.
6. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?
7. What is the goal of the method of partial fractions?
8. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
   - **a.** is a product of distinct linear factors?
   - **b.** consists of a repeated linear factor?
   - **c.** contains an irreducible quadratic factor?

   What do you do if the degree of f is *not* less than the degree of g?
9. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
10. What is a reduction formula? How are reduction formulas used? Give an example.
11. You are collaborating to produce a short "how-to" manual for numerical integration, and you are writing about the Trapezoidal Rule. **(a)** What would you say about the rule itself and how to use it? How to achieve accuracy? **(b)** What would you say if you were writing about Simpson's Rule instead?
12. How would you compare the relative merits of Simpson's Rule and the Trapezoidal Rule?
13. What is an improper integral of Type I? Type II? How are the values of various types of improper integrals defined? Give examples.
14. What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.

Chapter 8 Practice Exercises

Integration by Parts

Evaluate the integrals in Exercises 1–8 using integration by parts.

1. $\displaystyle\int \ln(x+1)\,dx$

2. $\displaystyle\int x^2 \ln x\,dx$

3. $\displaystyle\int \tan^{-1} 3x\,dx$

4. $\displaystyle\int \cos^{-1}\left(\frac{x}{2}\right)dx$

5. $\displaystyle\int (x+1)^2 e^x\,dx$

6. $\displaystyle\int x^2 \sin(1-x)\,dx$

7. $\displaystyle\int e^x \cos 2x\,dx$

8. $\displaystyle\int e^{-2x} \sin 3x\,dx$

Partial Fractions

Evaluate the integrals in Exercises 9–28. It may be necessary to use a substitution first.

9. $\displaystyle\int \frac{x\,dx}{x^2 - 3x + 2}$

10. $\displaystyle\int \frac{x\,dx}{x^2 + 4x + 3}$

11. $\displaystyle\int \frac{dx}{x(x+1)^2}$

12. $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

13. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

14. $\displaystyle\int \frac{\cos\theta\,d\theta}{\sin^2\theta + \sin\theta - 6}$

15. $\displaystyle\int \frac{3x^2 + 4x + 4}{x^3 + x}\,dx$

16. $\displaystyle\int \frac{4x\,dx}{x^3 + 4x}$

17. $\displaystyle\int \frac{v+3}{2v^3 - 8v}\,dv$

18. $\displaystyle\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)}$

19. $\displaystyle\int \frac{dt}{t^4 + 4t^2 + 3}$

20. $\displaystyle\int \frac{t\,dt}{t^4 - t^2 - 2}$

21. $\displaystyle\int \frac{x^3 + x^2}{x^2 + x - 2}\,dx$

22. $\displaystyle\int \frac{x^3 + 1}{x^3 - x}\,dx$

23. $\displaystyle\int \frac{x^3 + 4x^2}{x^2 + 4x + 3}\,dx$

24. $\displaystyle\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8}\,dx$

25. $\displaystyle\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$

26. $\displaystyle\int \frac{dx}{x\left(1 + \sqrt[3]{x}\right)}$

27. $\displaystyle\int \frac{ds}{e^s - 1}$

28. $\displaystyle\int \frac{ds}{\sqrt{e^s + 1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 29–32 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

29. $\displaystyle\int \frac{y\,dy}{\sqrt{16 - y^2}}$

30. $\displaystyle\int \frac{x\,dx}{\sqrt{4 + x^2}}$

31. $\displaystyle\int \frac{x\,dx}{4 - x^2}$

32. $\displaystyle\int \frac{t\,dt}{\sqrt{4t^2 - 1}}$

Evaluate the integrals in Exercises 33–36.

33. $\displaystyle\int \frac{x\,dx}{9 - x^2}$

34. $\displaystyle\int \frac{dx}{x(9 - x^2)}$

35. $\displaystyle\int \frac{dx}{9 - x^2}$

36. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 37–44.

37. $\displaystyle\int \sin^3 x \cos^4 x\,dx$

38. $\displaystyle\int \cos^5 x \sin^5 x\,dx$

39. $\displaystyle\int \tan^4 x \sec^2 x\,dx$

40. $\displaystyle\int \tan^3 x \sec^3 x\,dx$

41. $\displaystyle\int \sin 5\theta \cos 6\theta\,d\theta$

42. $\displaystyle\int \cos 3\theta \cos 3\theta\,d\theta$

43. $\displaystyle\int \sqrt{1 + \cos(t/2)}\,dt$

44. $\displaystyle\int e^t\sqrt{\tan^2 e^t + 1}\,dt$

Numerical Integration

45. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x}\,dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)

46. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

47. A direct calculation shows that

$$\int_0^\pi 2\sin^2 x\,dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

48. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x)\,dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's Rule the number has to be even.)

T **49. Mean temperature** Use Simpson's Rule to approximate the average value of the temperature function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

50. Heat capacity of a gas Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Use Simpson's Rule to find the average value of C_v and the temperature at which it is attained for $20° \le T \le 675°$C.

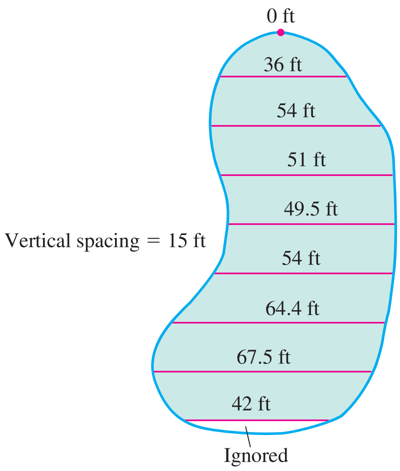
51. Fuel efficiency An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	Gal/h	Time	Gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

b. If the automobile covered 60 mi in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

52. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Use Simpson's Rule to find out if the job can be done for \$11,000.



Improper Integrals

Evaluate the improper integrals in Exercises 53–62.

53. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9 - x^2}}$

54. $\displaystyle\int_0^1 \ln x\, dx$

55. $\displaystyle\int_0^2 \frac{dy}{(y - 1)^{2/3}}$

56. $\displaystyle\int_{-2}^0 \frac{d\theta}{(\theta + 1)^{3/5}}$

57. $\displaystyle\int_3^\infty \frac{2\, du}{u^2 - 2u}$

58. $\displaystyle\int_1^\infty \frac{3v - 1}{4v^3 - v^2}\, dv$

59. $\displaystyle\int_0^\infty x^2 e^{-x}\, dx$

60. $\displaystyle\int_{-\infty}^0 x e^{3x}\, dx$

61. $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2 + 9}$

62. $\displaystyle\int_{-\infty}^\infty \frac{4\, dx}{x^2 + 16}$

Which of the improper integrals in Exercises 63–68 converge and which diverge?

63. $\displaystyle\int_6^\infty \frac{d\theta}{\sqrt{\theta^2 + 1}}$

64. $\displaystyle\int_0^\infty e^{-u} \cos u\, du$

65. $\displaystyle\int_1^\infty \frac{\ln z}{z}\, dz$

66. $\displaystyle\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\, dt$

67. $\displaystyle\int_{-\infty}^\infty \frac{2\, dx}{e^x + e^{-x}}$

68. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2(1 + e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 69–116. The integrals are listed in random order.

69. $\displaystyle\int \frac{x\, dx}{1 + \sqrt{x}}$

70. $\displaystyle\int \frac{x^3 + 2}{4 - x^2}\, dx$

71. $\displaystyle\int \frac{dx}{x(x^2 + 1)^2}$

72. $\displaystyle\int \frac{dx}{\sqrt{-2x - x^2}}$

73. $\displaystyle\int \frac{2 - \cos x + \sin x}{\sin^2 x}\, dx$

74. $\displaystyle\int \frac{\sin^2 \theta}{\cos^2 \theta}\, d\theta$

75. $\displaystyle\int \frac{9\, dv}{81 - v^4}$

76. $\displaystyle\int_2^\infty \frac{dx}{(x - 1)^2}$

77. $\displaystyle\int \theta \cos (2\theta + 1)\, d\theta$

78. $\displaystyle\int \frac{x^3\, dx}{x^2 - 2x + 1}$

79. $\displaystyle\int \frac{\sin 2\theta\, d\theta}{(1 + \cos 2\theta)^2}$

80. $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x}\, dx$

81. $\displaystyle\int \frac{x\, dx}{\sqrt{2 - x}}$

82. $\displaystyle\int \frac{\sqrt{1 - v^2}}{v^2}\, dv$

83. $\displaystyle\int \frac{dy}{y^2 - 2y + 2}$

84. $\displaystyle\int \frac{x\, dx}{\sqrt{8 - 2x^2 - x^4}}$

85. $\displaystyle\int \frac{z + 1}{z^2(z^2 + 4)}\, dz$

86. $\displaystyle\int x^3 e^{(x^2)}\, dx$

87. $\displaystyle\int \frac{t\, dt}{\sqrt{9 - 4t^2}}$

88. $\displaystyle\int \frac{\tan^{-1} x}{x^2}\, dx$

89. $\displaystyle\int \frac{e^t\, dt}{e^{2t} + 3e^t + 2}$

90. $\displaystyle\int \tan^3 t\, dt$

91. $\displaystyle\int_1^\infty \frac{\ln y}{y^3}\, dy$

92. $\displaystyle\int \frac{\cot v\, dv}{\ln \sin v}$

93. $\displaystyle\int e^{\ln \sqrt{x}}\, dx$

94. $\displaystyle\int e^\theta \sqrt{3 + 4e^\theta}\, d\theta$

95. $\displaystyle\int \frac{\sin 5t\, dt}{1 + (\cos 5t)^2}$

96. $\displaystyle\int \frac{dv}{\sqrt{e^{2v} - 1}}$

97. $\displaystyle\int \frac{dr}{1 + \sqrt{r}}$

98. $\displaystyle\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9}\, dx$

99. $\displaystyle\int \frac{x^3}{1 + x^2}\, dx$

100. $\displaystyle\int \frac{x^2}{1 + x^3}\, dx$

101. $\displaystyle\int \frac{1 + x^2}{1 + x^3}\, dx$

102. $\displaystyle\int \frac{1 + x^2}{(1 + x)^3}\, dx$

103. $\displaystyle\int \sqrt{x} \cdot \sqrt{1 + \sqrt{x}}\, dx$

104. $\displaystyle\int \sqrt{1 + \sqrt{1 + x}}\, dx$

105. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\, dx$

106. $\displaystyle\int_0^{1/2} \sqrt{1 + \sqrt{1 - x^2}}\, dx$

107. $\displaystyle\int \frac{\ln x}{x + x \ln x}\, dx$

108. $\displaystyle\int \frac{1}{x \cdot \ln x \cdot \ln (\ln x)}\, dx$

109. $\displaystyle\int \frac{x^{\ln x} \ln x}{x}\, dx$

110. $\displaystyle\int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln (\ln x)}{x}\right] dx$

111. $\displaystyle\int \frac{1}{x\sqrt{1 - x^4}}\, dx$

112. $\displaystyle\int \frac{\sqrt{1 - x}}{x}\, dx$

113. **a.** Show that $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$.

b. Use part (a) to evaluate

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx.$$

114. $\displaystyle\int \frac{\sin x}{\sin x + \cos x}\, dx$

115. $\displaystyle\int \frac{\sin^2 x}{1 + \sin^2 x}\, dx$

116. $\displaystyle\int \frac{1 - \cos x}{1 + \cos x}\, dx$

Chapter 8 Additional and Advanced Exercises

Evaluating Integrals

Evaluate the integrals in Exercises 1–6.

1. $\displaystyle\int (\sin^{-1} x)^2\, dx$

2. $\displaystyle\int \frac{dx}{x(x + 1)(x + 2)\cdots(x + m)}$

3. $\displaystyle\int x \sin^{-1} x\, dx$

4. $\displaystyle\int \sin^{-1} \sqrt{y}\, dy$

5. $\displaystyle\int \frac{dt}{t - \sqrt{1 - t^2}}$

6. $\displaystyle\int \frac{dx}{x^4 + 4}$

Evaluate the limits in Exercises 7 and 8.

7. $\displaystyle\lim_{x\to\infty} \int_{-x}^{x} \sin t\, dt$

8. $\displaystyle\lim_{x\to 0^+} x \int_{x}^{1} \frac{\cos t}{t^2}\, dt$

Evaluate the limits in Exercises 9 and 10 by identifying them with definite integrals and evaluating the integrals.

9. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + \frac{k}{n}}$

10. $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

Applications

11. Finding arc length Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

12. Finding arc length Find the length of the graph of the function $y = \ln(1 - x^2)$, $0 \le x \le 1/2$.

13. Finding volume The region in the first quadrant that is enclosed by the x-axis and the curve $y = 3x\sqrt{1 - x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

14. Finding volume The region in the first quadrant that is enclosed by the x-axis, the curve $y = 5/\left(x\sqrt{5 - x}\right)$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

15. Finding volume The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

16. Finding volume The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

17. Finding volume Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

a. the x-axis. **b.** the line $y = 1$.

18. Finding volume (*Continuation of Exercise 17.*) Find the volume of the solid generated by revolving the region R about

a. the y-axis. **b.** the line $x = 1$.

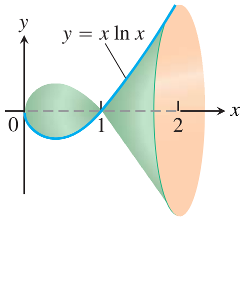
19. Finding volume The region between the x-axis and the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a. Show that f is continuous at $x = 0$.

b. Find the volume of the solid.



20. Finding volume The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

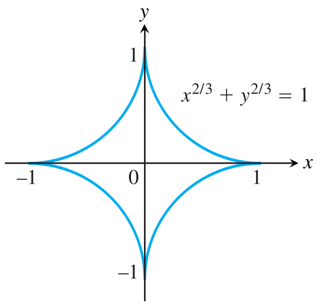
21. Centroid of a region Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

22. Centroid of a region Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

23. Length of a curve Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

24. Finding surface area Find the area of the surface generated by revolving the curve in Exercise 23 about the y-axis.

25. The surface generated by an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is an *astroid* (see accompanying figure). Find the area of the surface generated by revolving the curve about the x-axis.



26. Length of a curve Find the length of the curve

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt, \quad 1 \le x \le 16.$$

27. For what value or values of a does

$$\int_1^\infty \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

28. For each $x > 0$, let $G(x) = \int_0^\infty e^{-xt}\, dt$. Prove that $xG(x) = 1$ for each $x > 0$.

29. Infinite area and finite volume What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

30. Infinite area and finite volume What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval [0, 1] on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 8.21 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 23 in Chapter 14.

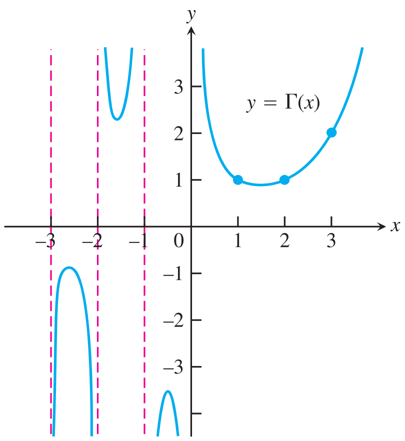


FIGURE 8.21 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x+1))/x$, which is the subject of Exercise 31.

31. If n is a nonnegative integer, $\Gamma(n+1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives

$$\begin{aligned} \Gamma(2) &= 1\Gamma(1) = 1 \\ \Gamma(3) &= 2\Gamma(2) = 2 \\ \Gamma(4) &= 3\Gamma(3) = 6 \\ &\vdots \\ \Gamma(n+1) &= n\,\Gamma(n) = n! \end{aligned} \tag{1}$$

c. Use mathematical induction to verify Equation (1) for every nonnegative integer n.

32. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty} \left(\frac{e}{x}\right)^x \sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so, for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\,(1 + \epsilon(x)), \qquad \epsilon(x) \to 0 \text{ as } x \to \infty. \tag{2}$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \qquad \textbf{(Stirling's formula)}. \tag{3}$$

a. Stirling's approximation for $n!$ Use Equation (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \qquad \textbf{(Stirling's approximation)}. \tag{4}$$

As you will see if you do Exercise 104 in Section 10.1, Equation (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \tag{5}$$

T **b.** Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots$, as far as your calculator can go.

T **c.** A refinement of Equation (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)}(1 + \epsilon(x))$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)},$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/(12n)}. \tag{6}$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Equation (6).

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)g(x)\,dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x \, dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} and its derivatives		**$\cos x$ and its integrals**
e^{2x}	$(+)$	$\cos x$
$2e^{2x}$	$(-)$	$\sin x$
$4e^{2x}$	$(+)$	$-\cos x$ ← *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, −1 on the right).

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x}\cos x\,dx = +(e^{2x}\sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x)\,dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x$$

or

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}\sin x + 2e^{2x}\cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 33–40.

33. $\int e^{2x}\cos 3x\,dx$ **34.** $\int e^{3x}\sin 4x\,dx$

35. $\int \sin 3x \sin x\,dx$ **36.** $\int \cos 5x \sin 4x\,dx$

37. $\int e^{ax}\sin bx\,dx$ **38.** $\int e^{ax}\cos bx\,dx$

39. $\int \ln(ax)\,dx$ **40.** $\int x^2\ln(ax)\,dx$

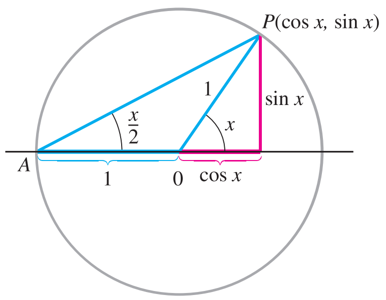
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{7}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1+\tan^2(x/2)} - 1 = \frac{2}{1+z^2} - 1$$

$$\cos x = \frac{1-z^2}{1+z^2}, \tag{8}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)}\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$$

$$\sin x = \frac{2z}{1+z^2}. \tag{9}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1+z^2}. \tag{10}$$

Examples

a.
$$\int \frac{1}{1+\cos x}dx = \int \frac{1+z^2}{2}\frac{2\,dz}{1+z^2}$$
$$= \int dz = z + C$$
$$= \tan\left(\frac{x}{2}\right) + C$$

b.
$$\int \frac{1}{2+\sin x}dx = \int \frac{1+z^2}{2+2z+2z^2}\frac{2\,dz}{1+z^2}$$
$$= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+(1/2))^2+3/4}$$
$$= \int \frac{du}{u^2+a^2}$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1+2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (7)–(10) to evaluate the integrals in Exercises 41–48. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

41. $\int \frac{dx}{1-\sin x}$ **42.** $\int \frac{dx}{1+\sin x+\cos x}$

43. $\int_0^{\pi/2} \frac{dx}{1+\sin x}$ **44.** $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x}$

45. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$ **46.** $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta}$

47. $\int \frac{dt}{\sin t-\cos t}$ **48.** $\int \frac{\cos t\,dt}{1-\cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 49 and 50.

49. $\int \sec\theta\,d\theta$ **50.** $\int \csc\theta\,d\theta$

Chapter 8 Technology Application Projects

Mathematica/Maple Modules:

Riemann, Trapezoidal, and Simpson Approximations

Part I: Visualize the error involved in using Riemann sums to approximate the area under a curve.

Part II: Build a table of values and compute the relative magnitude of the error as a function of the step size Δx.

Part III: Investigate the effect of the derivative function on the error.

Parts IV and **V**: Trapezoidal Rule approximations.

Part VI: Simpson's Rule approximations.

Games of Chance: Exploring the Monte Carlo Probabilistic Technique for Numerical Integration
Graphically explore the Monte Carlo method for approximating definite integrals.

Computing Probabilities with Improper Integrals
More explorations of the Monte Carlo method for approximating definite integrals.

9

FIRST-ORDER DIFFERENTIAL EQUATIONS

OVERVIEW In Section 4.8 we introduced differential equations of the form $dy/dx = f(x)$, where f is given and y is an unknown function of x. When f is continuous over some interval, we found the general solution $y(x)$ by integration, $y = \int f(x)\,dx$. In Section 7.2 we solved separable differential equations. Such equations arise when investigating exponential growth or decay, for example. In this chapter we study some other types of *first-order* differential equations. They involve only first derivatives of the unknown function.

9.1 Solutions, Slope Fields, and Euler's Method

We begin this section by defining general differential equations involving first derivatives. We then look at slope fields, which give a geometric picture of the solutions to such equations. Many differential equations cannot be solved by obtaining an explicit formula for the solution. However, we can often find numerical approximations to solutions. We present one such method here, called *Euler's method*, upon which many other numerical methods are based.

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

We need to show that the differential equation is satisfied when we substitute into it the expressions $(C/x) + 2$ for y, and $-C/x^2$ for dy/dx. That is, we need to verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right-hand side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left side of the equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

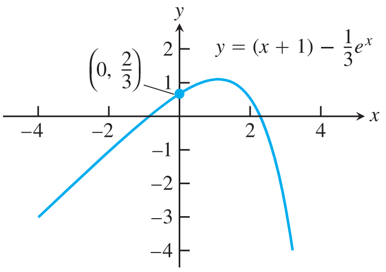


FIGURE 9.1 Graph of the solution to the initial value problem in Example 2.

On the right side of the equation:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1. ■

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

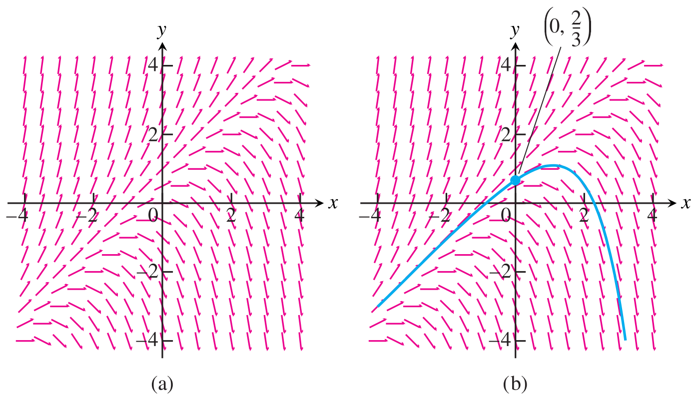


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

Figure 9.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields. Slope fields are useful because they display the overall behavior of the family of solution curves for a given differential equation. For instance, the slope field in Figure 9.3b reveals that every solution $y(x)$ to the differential equation specified in the figure satisfies $\lim_{x\to\pm\infty} y(x) = 0$. We will see that knowing the overall behavior of the solution curves is often critical to understanding and predicting outcomes in a real-world system modeled by a differential equation.

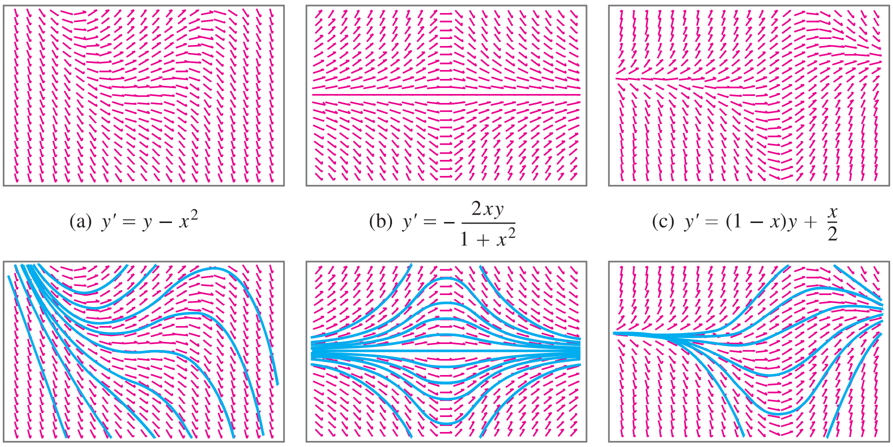


FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

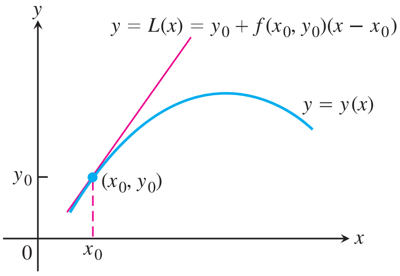


FIGURE 9.4 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer.

Euler's Method

If we do not require or cannot immediately find an *exact* solution giving an explicit formula for an initial value problem $y' = f(x, y), y(x_0) = y_0$, we can often use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem, and the method by which we generate the table is called a **numerical method**.

Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \qquad \text{or} \qquad L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 (Figure 9.4). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

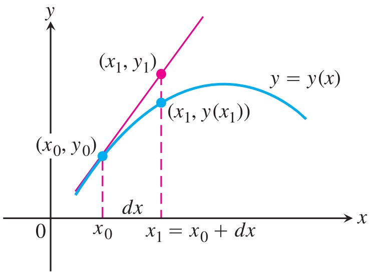


FIGURE 9.5 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. (Recall that $dx = \Delta x$ in the definition of differentials.) If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve (Figure 9.5).

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) to calculate

$$y_2 = y_1 + f(x_1, y_1)\,dx.$$

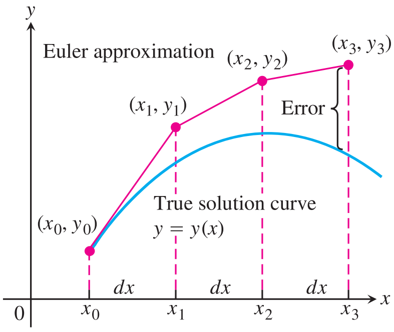


FIGURE 9.6 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Figure 9.6). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation

$$y_3 = y_2 + f(x_2, y_2)\,dx,$$

and so on. We are literally building an approximation to one of the solutions by following the direction of the slope field of the differential equation.

The steps in Figure 9.6 are drawn large to illustrate the construction process, so the approximation looks crude. In practice, dx would be small enough to make the red curve hug the blue one and give a good approximation throughout.

EXAMPLE 3 Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ with $dx = 0.1$.

Solution We have the starting values $x_0 = 0$ and $y_0 = 1$. Next we determine the values of x at which the Euler approximations will take place: $x_1 = x_0 + dx = 0.1$, $x_2 = x_0 + 2\,dx = 0.2$, and $x_3 = x_0 + 3\,dx = 0.3$. Then we find

$$\begin{aligned}
\textit{First:} \quad y_1 &= y_0 + f(x_0, y_0)\,dx \\
&= y_0 + (1 + y_0)\,dx \\
&= 1 + (1 + 1)(0.1) = 1.2 \\
\textit{Second:} \quad y_2 &= y_1 + f(x_1, y_1)\,dx \\
&= y_1 + (1 + y_1)\,dx \\
&= 1.2 + (1 + 1.2)(0.1) = 1.42 \\
\textit{Third:} \quad y_3 &= y_2 + f(x_2, y_2)\,dx \\
&= y_2 + (1 + y_2)\,dx \\
&= 1.42 + (1 + 1.42)(0.1) = 1.662
\end{aligned}$$

■

The step-by-step process used in Example 3 can be continued easily. Using equally spaced values for the independent variable in the table for the numerical solution, and generating n of them, set

$$\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}$$

Then calculate the approximations to the solution,

$$\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\,dx \\
y_2 &= y_1 + f(x_1, y_1)\,dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\,dx.
\end{aligned}$$

The number of steps n can be as large as we like, but errors can accumulate if n is too large.

Euler's method is easy to implement on a computer or calculator. A computer program generates a table of numerical solutions to an initial value problem, allowing us to input x_0 and y_0, the number of steps n, and the step size dx. It then calculates the approximate solution values $y_1, y_2, \ldots, y_n$ in iterative fashion, as just described.

HISTORICAL BIOGRAPHY

Leonhard Euler
(1703–1783)

Solving the separable equation in Example 3, we find that the exact solution to the initial value problem is $y = 2e^x - 1$. We use this information in Example 4.

EXAMPLE 4 Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking **(a)** $dx = 0.1$ and **(b)** $dx = 0.05$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution

(a) We used a computer to generate the approximate values in Table 9.1. The "error" column is obtained by subtracting the unrounded Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

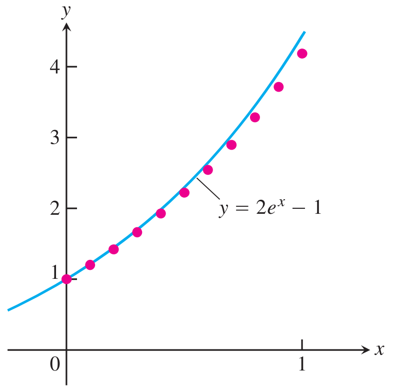


FIGURE 9.7 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 4).

By the time we reach $x = 1$ (after 10 steps), the error is about 5.6% of the exact solution. A plot of the exact solution curve with the scatterplot of Euler solution points from Table 9.1 is shown in Figure 9.7.

(b) One way to try to reduce the error is to decrease the step size. Table 9.2 shows the results and their comparisons with the exact solutions when we decrease the step size to 0.05, doubling the number of steps to 20. As in Table 9.1, all computations are performed before rounding. This time when we reach $x = 1$, the relative error is only about 2.9%. ■

It might be tempting to reduce the step size even further in Example 4 to obtain greater accuracy. Each additional calculation, however, not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers inside the computer.

The analysis of error and the investigation of methods to reduce it when making numerical calculations are important but are appropriate for a more advanced course. There are numerical methods more accurate than Euler's method, usually presented in a further study of differential equations.

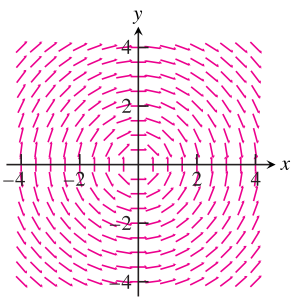
TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

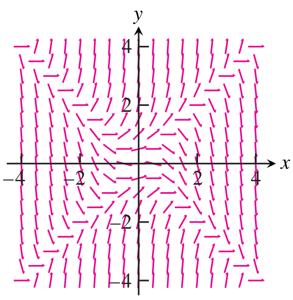
Exercises 9.1

Slope Fields

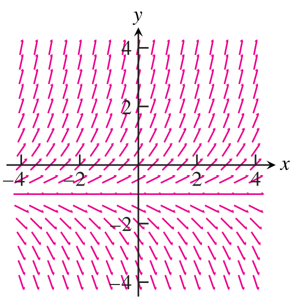
In Exercises 1–4, match the differential equations with their slope fields, graphed here.



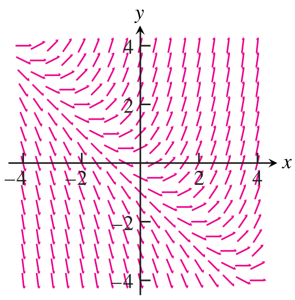
(a)



(b)



(c)



(d)

1. $y' = x + y$

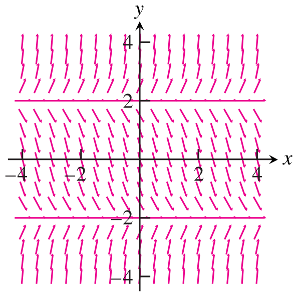
2. $y' = y + 1$

3. $y' = -\frac{x}{y}$

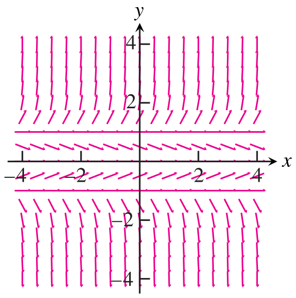
4. $y' = y^2 - x^2$

In Exercises 5 and 6, copy the slope fields and sketch in some of the solution curves.

5. $y' = (y + 2)(y - 2)$



6. $y' = y(y + 1)(y - 1)$



Integral Equations

In Exercises 7–10, write an equivalent first-order differential equation and initial condition for y.

7. $y = -1 + \int_1^x (t - y(t))\, dt$

8. $y = \int_1^x \frac{1}{t}\, dt$

9. $y = 2 - \int_0^x (1 + y(t)) \sin t\, dt$

10. $y = 1 + \int_0^x y(t)\, dt$

Using Euler's Method

In Exercises 11–16, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to four decimal places.

11. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

12. $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$

13. $y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$

14. $y' = y^2(1 + 2x), \quad y(-1) = 1, \quad dx = 0.5$

T **15.** $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1$

T **16.** $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

17. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?

18. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?

19. Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?

20. Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = x \sin y$ and $y(0) = 1$. What is the exact value of $y(2)$?

21. Show that the solution of the initial value problem

$$y' = x + y, \quad y(x_0) = y_0$$

is

$$y = -1 - x + (1 + x_0 + y_0)\, e^{x - x_0}.$$

22. What integral equation is equivalent to the initial value problem $y' = f(x),\ y(x_0) = y_0$?

COMPUTER EXPLORATIONS

In Exercises 23–28, obtain a slope field and add to it graphs of the solution curves passing through the given points.

23. $y' = y$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$

24. $y' = 2(y - 4)$ with

a. $(0, 1)$ **b.** $(0, 4)$ **c.** $(0, 5)$

25. $y' = y(x + y)$ with

a. $(0, 1)$ **b.** $(0, -2)$ **c.** $(0, 1/4)$ **d.** $(-1, -1)$

26. $y' = y^2$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$ **d.** $(0, 0)$

27. $y' = (y - 1)(x + 2)$ with

a. $(0, -1)$ **b.** $(0, 1)$ **c.** $(0, 3)$ **d.** $(1, -1)$

28. $y' = \dfrac{xy}{x^2 + 4}$ with

a. $(0, 2)$ **b.** $(0, -6)$ **c.** $\left(-2\sqrt{3}, -4\right)$

In Exercises 29 and 30, obtain a slope field and graph the particular solution over the specified interval. Use your CAS DE solver to find the general solution of the differential equation.

29. **A logistic equation** $y' = y(2 - y), \quad y(0) = 1/2;$ $0 \le x \le 4, \quad 0 \le y \le 3$

30. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6$

Exercises 31 and 32 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations.

31. $y' = \cos(2x - y), \quad y(0) = 2; \quad 0 \le x \le 5, \quad 0 \le y \le 5$

32. **A Gompertz equation** $y' = y(1/2 - \ln y), \quad y(0) = 1/3;$ $0 \le x \le 4, \quad 0 \le y \le 3$

33. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a. $2x$ **b.** $\sin 2x$ **c.** $3e^{x/2}$ **d.** $2e^{-x/2} \cos 2x$.

Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

34. **a.** Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

b. Separate the variables and use a CAS integrator to find the general solution in implicit form.

c. Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d. Find and graph the solution that satisfies the initial condition $y(0) = -1$.

In Exercises 35–38, use Euler's method with the specified step size to estimate the value of the solution at the given point x^*. Find the value of the exact solution at x^*.

35. $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1, \quad x^* = 1$

36. $y' = 2y^2(x - 1), \quad y(2) = -1/2, \quad dx = 0.1, \quad x^* = 3$

37. $y' = \sqrt{x}/y, \quad y > 0, \quad y(0) = 1, \quad dx = 0.1, \quad x^* = 1$

38. $y' = 1 + y^2, \quad y(0) = 0, \quad dx = 0.1, \quad x^* = 1$

Use a CAS to explore graphically each of the differential equations in Exercises 39–42. Perform the following steps to help with your explorations.

a. Plot a slope field for the differential equation in the given xy-window.

b. Find the general solution of the differential equation using your CAS DE solver.

c. Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

d. Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

e. Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the x-interval and plot the Euler approximation superimposed on the graph produced in part (d).

f. Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

g. Find the error $(y(\text{exact}) - y(\text{Euler}))$ at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

39. $y' = x + y, \quad y(0) = -7/10; \quad -4 \le x \le 4, \quad -4 \le y \le 4; \quad b = 1$

40. $y' = -x/y, \quad y(0) = 2; \quad -3 \le x \le 3, \quad -3 \le y \le 3; \quad b = 2$

41. $y' = y(2 - y), \quad y(0) = 1/2; \quad 0 \le x \le 4, \quad 0 \le y \le 3; \quad b = 3$

42. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6; \quad b = 3\pi/2$

9.2 First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 7.2) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y,\ e^y, \text{ or } \sqrt{dy/dx}\right)$.

EXAMPLE 1 Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by positive } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y).$$

$$v(x) \cdot y = \int v(x)Q(x)\,dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \tag{2}$$

Equation (2) expresses the solution of Equation (1) in terms of the functions $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (1) because its presence makes the equation integrable.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Derivative Product Rule}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated, } v > 0$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0 \text{, we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{3}$$

Thus a formula for the general solution to Equation (1) is given by Equation (2), where $v(x)$ is given by Equation (3). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified. Any antiderivative of P works for Equation (3).

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

When you integrate the product on the left-hand side in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

HISTORICAL BIOGRAPHY

Adrien Marie Legendre
(1752–1833)

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) &= e^{\int P(x)\,dx} = e^{\int(-3/x)\,dx} \\ &= e^{-3\ln|x|} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left-hand side is } \frac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int\frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$ ■

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left-hand side is } vy.$$

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

■

In solving the linear equation in Example 2, we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example 3, by remembering that the left-hand side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation (2) this means that

$$v(x)y = \int v(x)Q(x)\,dx. \tag{4}$$

We need only integrate the product of the integrating factor $v(x)$ with $Q(x)$ on the right-hand side of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example 2.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable and can be solved by the method of Section 7.2:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) \equiv 0$$

$$\frac{dy}{y} = -P(x)\,dx \qquad \text{Separating the variables}$$

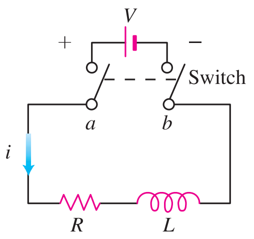


FIGURE 9.8 The *RL* circuit in Example 4.

RL Circuits

The diagram in Figure 9.8 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = RI$, has to be augmented for such a circuit. The correct equation accounting for both resistance and inductance is

$$L\frac{di}{dt} + Ri = V, \tag{5}$$

where i is the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

EXAMPLE 4 The switch in the *RL* circuit in Figure 9.8 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (5) is a first-order linear differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{6}$$

and the corresponding solution, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t}. \tag{7}$$

(We leave the calculation of the solution for you to do in Exercise 28.) Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes, the current approaches the **steady-state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Figure 9.9).

Equation (7) expresses the solution of Equation (6) as the sum of two terms: a steady-state solution V/R and a transient solution $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ■

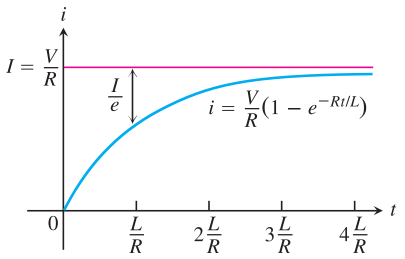


FIGURE 9.9 The growth of the current in the *RL* circuit in Example 4. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 27).

Exercises 9.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

1. $x\frac{dy}{dx} + y = e^x, \quad x > 0$

2. $e^x\frac{dy}{dx} + 2e^x y = 1$

3. $xy' + 3y = \frac{\sin x}{x^2}, \quad x > 0$

4. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

5. $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$

6. $(1 + x)y' + y = \sqrt{x}$

7. $2y' = e^{x/2} + y$

8. $e^{2x}y' + 2e^{2x}y = 2x$

9. $xy' - y = 2x\ln x$

10. $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y, \quad x > 0$

11. $(t-1)^3 \dfrac{ds}{dt} + 4(t-1)^2 s = t + 1, \quad t > 1$

12. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

13. $\sin\theta \dfrac{dr}{d\theta} + (\cos\theta) r = \tan\theta, \quad 0 < \theta < \pi/2$

14. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

17. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

18. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3 \sec\theta \tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

19. $(x+1)\dfrac{dy}{dx} - 2(x^2 + x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \quad y(0) = 5$

20. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

21. Solve the exponential growth/decay initial value problem for y as a function of t by thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

22. Solve the following initial value problem for u as a function of t:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

a. as a first-order linear equation.

b. as a separable equation.

Theory and Examples

23. Is either of the following equations correct? Give reasons for your answers.

a. $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$ **b.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

24. Is either of the following equations correct? Give reasons for your answers.

a. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

25. Current in a closed *RL* circuit How many seconds after the switch in an RL circuit is closed will it take the current i to reach half of its steady-state value? Notice that the time depends on R and L and not on how much voltage is applied.

26. Current in an open *RL* circuit If the switch is thrown open after the current in an RL circuit has built up to its steady-state value $I = V/R$, the decaying current (see accompanying figure) obeys the equation

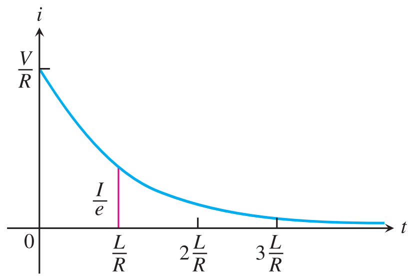
$$L\frac{di}{dt} + Ri = 0,$$

which is Equation (5) with $V = 0$.

a. Solve the equation to express i as a function of t.

b. How long after the switch is thrown will it take the current to fall to half its original value?

c. Show that the value of the current when $t = L/R$ is I/e. (The significance of this time is explained in the next exercise.)



27. Time constants Engineers call the number L/R the *time constant* of the RL circuit in Figure 9.9. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Figure 9.9). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

a. Find the value of i in Equation (7) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady-state value $I = V/R$.

b. Approximately what percentage of the steady-state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

28. Derivation of Equation (7) in Example 4

a. Show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

b. Then use the initial condition $i(0) = 0$ to determine the value of C. This will complete the derivation of Equation (7).

c. Show that $i = V/R$ is a solution of Equation (6) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

HISTORICAL BIOGRAPHY

James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the Bernoulli equations in Exercises 29–32.

29. $y' - y = -y^2$

30. $y' - y = xy^2$

31. $xy' + y = y^{-2}$

32. $x^2y' + 2xy = y^3$

9.3 Applications

We now look at four applications of first-order differential equations. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth. The third application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles). The final application analyzes chemical concentrations entering and leaving a container. The various models involve separable or linear first-order equations.

Motion with Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. Picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.2)

$$v = v_0e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t}). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: if m is large, the body will coast a long way.

In the English system, where weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

EXAMPLE 1 For a 192-lb ice skater, the k in Equation (1) is about $1/3$ slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Equation (1) for t:

$$\begin{aligned} 11e^{-t/18} &= 1 && \text{Eq. (1) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1 \\ e^{-t/18} &= 1/11 \\ -t/18 &= \ln(1/11) = -\ln 11 \\ t &= 18 \ln 11 \approx 43 \text{ sec}. \end{aligned}$$

We answer the second question with Equation (3):

$$\begin{aligned}\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3} \\ &= 198 \text{ ft}.\end{aligned}$$

■

Inaccuracy of the Exponential Population Growth Model

In Section 7.2 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 7.2 we found the solution $P = P_0 e^{kt}$ to this model.

To assess the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

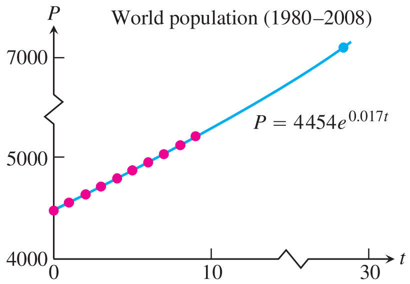


FIGURE 9.10 Notice that the value of the solution $P = 4454e^{0.017t}$ is 7169 when $t = 28$, which is nearly 7% more than the actual population in 2008.

TABLE 9.3 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 2007): www.census .gov/ipc/www/idb.

is constant. This rate is called the **relative growth rate**. Now, Table 9.3 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by the initial value problem,

$$\frac{dP}{dt} = 0.017P, \qquad P(0) = 4454.$$

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 2008 (so $t = 28$), the solution predicts the world population in midyear to be about 7169 million, or 7.2 billion (Figure 9.10), which is more than the actual population of 6707 million from the U.S. Bureau of the Census. A more realistic model would consider environmental and other factors affecting the growth rate, which has been steadily declining to about 0.012 since 1987. We consider one such model in Section 9.4.

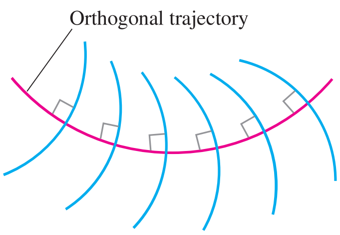


FIGURE 9.11 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.11). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 9.12). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to strength of an electric field and those in the other family correspond to constant electric potential. They also occur in hydrodynamics and heat-flow problems.

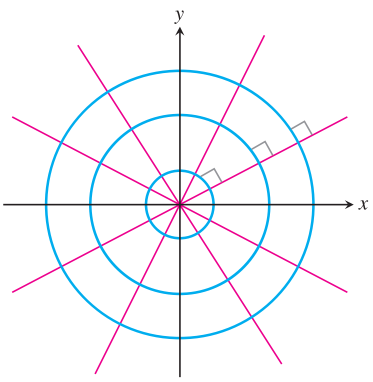


FIGURE 9.12 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 2 Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas having the coordinate axes as asymptotes. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

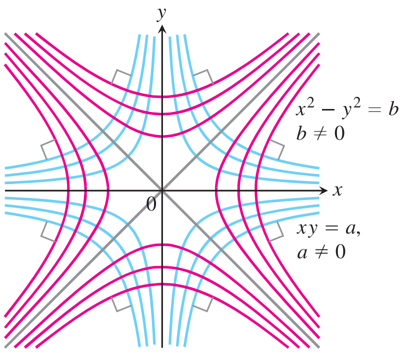


FIGURE 9.13 Each curve is orthogonal to every curve it meets in the other family (Example 2).

This differential equation is separable and we solve it as in Section 7.2:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{5}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (5) and sketched in Figure 9.13. ■

Mixture Problems

Suppose a chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{array}{c}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{array} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix} \tag{6}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate})\\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate}).\end{aligned} \tag{7}$$

Accordingly, Equation (6) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{8}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Equation (8) are

$$\frac{\text{pounds}}{\text{minutes}} = \frac{\text{pounds}}{\text{minutes}} - \frac{\text{pounds}}{\text{gallons}} \cdot \frac{\text{gallons}}{\text{minutes}}.$$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min.

The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.14)?

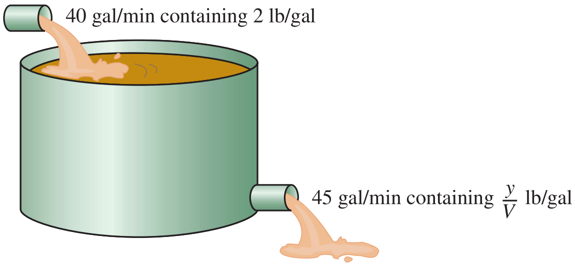


FIGURE 9.14 The storage tank in Example 3 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (7)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min and } v = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t} \qquad \text{Eq. (8)}$$

in pounds per minute.

To solve this differential equation, we first write it in standard linear form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t} y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) = e^{\int P\, dt} &= e^{\int \frac{45}{2000 - 5t} dt} \\ &= e^{-9 \ln (2000 - 5t)} && 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$(2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}\, y\right) = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9} \frac{dy}{dt} + 45(2000 - 5t)^{-10}\, y = 80(2000 - 5t)^{-9}$$

$$\frac{d}{dt}\left[(2000 - 5t)^{-9} y\right] = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9} y = \int 80(2000 - 5t)^{-9}\, dt$$

$$(2000 - 5t)^{-9} y = 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C.$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^9.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$100 = 2(2000 - 0) + C(2000 - 0)^9$$

$$C = -\frac{3900}{(2000)^9}.$$

The particular solution of the initial value problem is

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$

■

Exercises 9.3

Motion Along a Line

1. **Coasting bicycle** A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.
   a. About how far will the cyclist coast before reaching a complete stop?
   b. How long will it take the cyclist's speed to drop to 1 m/sec?
2. **Coasting battleship** Suppose that an Iowa class battleship has mass around 51,000 metric tons (51,000,000 kg) and a k value in Equation (1) of about 59,000 kg/sec. Assume that the ship loses power when it is moving at a speed of 9 m/sec.
   a. About how far will the ship coast before it is dead in the water?
   b. About how long will it take the ship's speed to drop to 1 m/sec?
3. The data in Table 9.4 were collected with a motion detector and a CBL™ by Valerie Sharritts, a mathematics teacher at St. Francis DeSales High School in Columbus, Ohio. The table shows the distance s (meters) coasted on in-line skates in t sec by her daughter Ashley when she was 10 years old. Find a model for Ashley's

position given by the data in Table 9.4 in the form of Equation (2). Her initial velocity was $v_0 = 2.75$ m/sec, her mass $m = 39.92$ kg (she weighed 88 lb), and her total coasting distance was 4.91 m.

TABLE 9.4 Ashley Sharritts skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	2.24	3.05	4.48	4.77
0.16	0.31	2.40	3.22	4.64	4.82
0.32	0.57	2.56	3.38	4.80	4.84
0.48	0.80	2.72	3.52	4.96	4.86
0.64	1.05	2.88	3.67	5.12	4.88
0.80	1.28	3.04	3.82	5.28	4.89
0.96	1.50	3.20	3.96	5.44	4.90
1.12	1.72	3.36	4.08	5.60	4.90
1.28	1.93	3.52	4.18	5.76	4.91
1.44	2.09	3.68	4.31	5.92	4.90
1.60	2.30	3.84	4.41	6.08	4.91
1.76	2.53	4.00	4.52	6.24	4.90
1.92	2.73	4.16	4.63	6.40	4.91
2.08	2.89	4.32	4.69	6.56	4.91

4. **Coasting to a stop** Table 9.5 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form of Equation (2). Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.

TABLE 9.5 Kelly Schmitzer skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

Orthogonal Trajectories

In Exercises 5–10, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

5. $y = mx$
6. $y = cx^2$
7. $kx^2 + y^2 = 1$
8. $2x^2 + y^2 = c^2$
9. $y = ce^{-x}$
10. $y = e^{kx}$
11. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.
12. Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.
    - **a.** $x\,dx + y\,dy = 0$
    - **b.** $x\,dy - 2y\,dx = 0$

Mixture Problems

13. **Salt mixture** A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.
    - **a.** At what rate (pounds per minute) does salt enter the tank at time t?
    - **b.** What is the volume of brine in the tank at time t?
    - **c.** At what rate (pounds per minute) does salt leave the tank at time t?
    - **d.** Write down and solve the initial value problem describing the mixing process.
    - **e.** Find the concentration of salt in the tank 25 min after the process starts.
14. **Mixture problem** A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.
    - **a.** At what time will the tank be full?
    - **b.** At the time the tank is full, how many pounds of concentrate will it contain?
15. **Fertilizer mixture** A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
16. **Carbon monoxide pollution** An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

9.4 Graphical Solutions of Autonomous Equations

In Chapter 4 we learned that the sign of the first derivative tells where the graph of a function is increasing and where it is decreasing. The sign of the second derivative tells the concavity of the graph. We can build on our knowledge of how derivatives determine the shape of a graph to solve differential equations graphically. We will see that the ability to discern physical behavior from graphs is a powerful tool in understanding real-world systems. The starting ideas for a graphical solution are the notions of *phase line* and *equilibrium value*. We arrive at these notions by investigating, from a point of view quite different from that studied in Chapter 4, what happens when the derivative of a differentiable function is zero.

Equilibrium Values and Phase Lines

When we differentiate implicitly the equation

$$\frac{1}{5}\ln(5y - 15) = x + 1,$$

we obtain

$$\frac{1}{5}\left(\frac{5}{5y - 15}\right)\frac{dy}{dx} = 1.$$

Solving for $y' = dy/dx$ we find $y' = 5y - 15 = 5(y - 3)$. In this case the derivative y' is a function of y only (the dependent variable) and is zero when $y = 3$.

A differential equation for which dy/dx is a function of y only is called an **autonomous** differential equation. Let's investigate what happens when the derivative in an autonomous equation equals zero. We assume any derivatives are continuous.

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

Thus, equilibrium values are those at which no change occurs in the dependent variable, so y is at *rest*. The emphasis is on the value of y where $dy/dx = 0$, not the value of x, as we studied in Chapter 4. For example, the equilibrium values for the autonomous differential equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

are $y = -1$ and $y = 2$.

To construct a graphical solution to an autonomous differential equation, we first make a **phase line** for the equation, a plot on the y-axis that shows the equation's equilibrium values along with the intervals where dy/dx and d^2y/dx^2 are positive and negative. Then we know where the solutions are increasing and decreasing, and the concavity of the solution curves. These are the essential features we found in Section 4.4, so we can determine the shapes of the solution curves without having to find formulas for them.

EXAMPLE 1 Draw a phase line for the equation

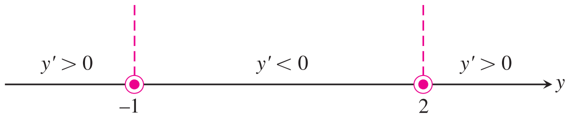
$$\frac{dy}{dx} = (y + 1)(y - 2)$$

and use it to sketch solutions to the equation.

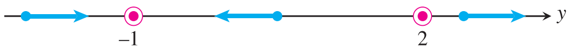
Solution

1. *Draw a number line for y and mark the equilibrium values $y = -1$ and $y = 2$, where $dy/dx = 0$.*

2. *Identify and label the intervals where $y' > 0$ and $y' < 0$.* This step resembles what we did in Section 4.3, only now we are marking the y-axis instead of the x-axis.



We can encapsulate the information about the sign of y' on the phase line itself. Since $y' > 0$ on the interval to the left of $y = -1$, a solution of the differential equation with a y-value less than -1 will increase from there toward $y = -1$. We display this information by drawing an arrow on the interval pointing to -1.



Similarly, $y' < 0$ between $y = -1$ and $y = 2$, so any solution with a value in this interval will decrease toward $y = -1$.

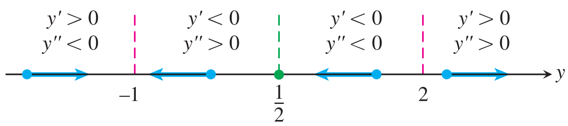
For $y > 2$, we have $y' > 0$, so a solution with a y-value greater than 2 will increase from there without bound.

In short, solution curves below the horizontal line $y = -1$ in the xy-plane rise toward $y = -1$. Solution curves between the lines $y = -1$ and $y = 2$ fall away from $y = 2$ toward $y = -1$. Solution curves above $y = 2$ rise away from $y = 2$ and keep going up.

3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.* To find y'', we differentiate y' *with respect to* x, using implicit differentiation.

$$
\begin{aligned}
y' &= (y+1)(y-2) = y^2 - y - 2 && \text{Formula for } y' \ldots \\
y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - y - 2) \\
&= 2yy' - y' && \text{differentiated implicitly with respect to } x \\
&= (2y-1)y' \\
&= (2y-1)(y+1)(y-2).
\end{aligned}
$$

From this formula, we see that y'' changes sign at $y = -1$, $y = 1/2$, and $y = 2$. We add the sign information to the phase line.



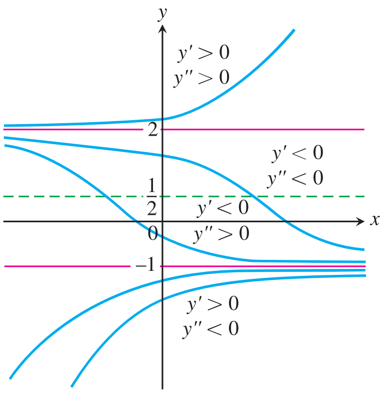


FIGURE 9.15 Graphical solutions from Example 1 include the horizontal lines $y = -1$ and $y = 2$ through the equilibrium values. No two solution curves can ever cross or touch each other.

4. *Sketch an assortment of solution curves in the xy-plane.* The horizontal lines $y = -1$, $y = 1/2$, and $y = 2$ partition the plane into horizontal bands in which we know the signs of y' and y''. In each band, this information tells us whether the solution curves rise or fall and how they bend as x increases (Figure 9.15).

The "equilibrium lines" $y = -1$ and $y = 2$ are also solution curves. (The constant functions $y = -1$ and $y = 2$ satisfy the differential equation.) Solution curves that cross the line $y = 1/2$ have an inflection point there. The concavity changes from concave down (above the line) to concave up (below the line).

As predicted in Step 2, solutions in the middle and lower bands approach the equilibrium value $y = -1$ as x increases. Solutions in the upper band rise steadily away from the value $y = 2$. ■

Stable and Unstable Equilibria

Look at Figure 9.15 once more, in particular at the behavior of the solution curves near the equilibrium values. Once a solution curve has a value near $y = -1$, it tends steadily toward that value; $y = -1$ is a **stable equilibrium**. The behavior near $y = 2$ is just the opposite: all solutions except the equilibrium solution $y = 2$ itself move *away* from it as x increases. We call $y = 2$ an **unstable equilibrium**. If the solution is *at* that value, it stays, but if it is off by any amount, no matter how small, it moves away. (Sometimes an equilibrium value is unstable because a solution moves away from it only on one side of the point.)

Now that we know what to look for, we can already see this behavior on the initial phase line (the second diagram in Step 2 of Example 1). The arrows lead away from $y = 2$ and, once to the left of $y = 2$, toward $y = -1$.

We now present several applied examples for which we can sketch a family of solution curves to the differential equation models using the method in Example 1.

Newton's Law of Cooling

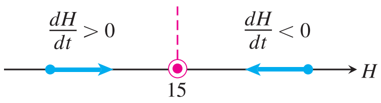


FIGURE 9.16 First step in constructing the phase line for Newton's law of cooling. The temperature tends towards the equilibrium (surrounding-medium) value in the long run.

In Section 7.2 we solved analytically the differential equation

$$\frac{dH}{dt} = -k(H - H_S), \qquad k > 0$$

modeling Newton's law of cooling. Here H is the temperature of an object at time t and H_S is the constant temperature of the surrounding medium.

Suppose that the surrounding medium (say a room in a house) has a constant Celsius temperature of 15°C. We can then express the difference in temperature as $H(t) - 15$. Assuming H is a differentiable function of time t, by Newton's law of cooling, there is a constant of proportionality $k > 0$ such that

$$\frac{dH}{dt} = -k(H - 15) \tag{1}$$

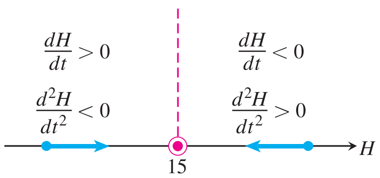


FIGURE 9.17 The complete phase line for Newton's law of cooling.

(*minus* k to give a negative derivative when $H > 15$).

Since $dH/dt = 0$ at $H = 15$, the temperature 15°C is an equilibrium value. If $H > 15$, Equation (1) tells us that $(H - 15) > 0$ and $dH/dt < 0$. If the object is hotter than the room, it will get cooler. Similarly, if $H < 15$, then $(H - 15) < 0$ and $dH/dt > 0$. An object cooler than the room will warm up. Thus, the behavior described by Equation (1) agrees with our intuition of how temperature should behave. These observations are captured in the initial phase line diagram in Figure 9.16. The value $H = 15$ is a stable equilibrium.

We determine the concavity of the solution curves by differentiating both sides of Equation (1) with respect to t:

$$\frac{d}{dt}\left(\frac{dH}{dt}\right) = \frac{d}{dt}(-k(H - 15))$$

$$\frac{d^2H}{dt^2} = -k\frac{dH}{dt}.$$

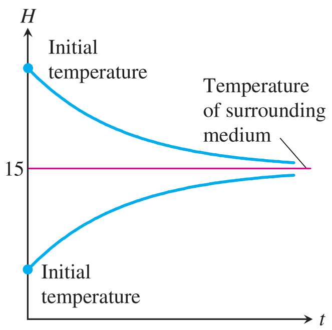


FIGURE 9.18 Temperature versus time. Regardless of initial temperature, the object's temperature $H(t)$ tends toward 15°C, the temperature of the surrounding medium.

Since $-k$ is negative, we see that d^2H/dt^2 is positive when $dH/dt < 0$ and negative when $dH/dt > 0$. Figure 9.17 adds this information to the phase line.

The completed phase line shows that if the temperature of the object is above the equilibrium value of 15°C, the graph of $H(t)$ will be decreasing and concave upward. If the temperature is below 15°C (the temperature of the surrounding medium), the graph of $H(t)$ will be increasing and concave downward. We use this information to sketch typical solution curves (Figure 9.18).

From the upper solution curve in Figure 9.18, we see that as the object cools down, the rate at which it cools slows down because dH/dt approaches zero. This observation is implicit in Newton's law of cooling and contained in the differential equation, but the flattening of the graph as time advances gives an immediate visual representation of the phenomenon.

A Falling Body Encountering Resistance

Newton observed that the rate of change in momentum encountered by a moving object is equal to the net force applied to it. In mathematical terms,

$$F = \frac{d}{dt}(mv), \tag{2}$$

where F is the net force acting on the object, and m and v are the object's mass and velocity. If m varies with time, as it will if the object is a rocket burning fuel, the right-hand side of Equation (2) expands to

$$m\frac{dv}{dt} + v\frac{dm}{dt}$$

using the Derivative Product Rule. In many situations, however, m is constant, $dm/dt = 0$, and Equation (2) takes the simpler form

$$F = m\frac{dv}{dt} \qquad \text{or} \qquad F = ma, \tag{3}$$

known as *Newton's second law of motion* (see Section 9.3).

In free fall, the constant acceleration due to gravity is denoted by g and the one force acting downward on the falling body is

$$F_p = mg,$$

the force due to gravity. If, however, we think of a real body falling through the air—say, a penny from a great height or a parachutist from an even greater height—we know that at some point air resistance is a factor in the speed of the fall. A more realistic model of free fall would include air resistance, shown as a force F_r in the schematic diagram in Figure 9.19.

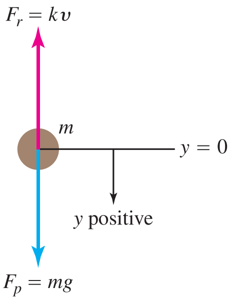


FIGURE 9.19 An object falling under the influence of gravity with a resistive force assumed to be proportional to the velocity.

For low speeds well below the speed of sound, physical experiments have shown that F_r is approximately proportional to the body's velocity. The net force on the falling body is therefore

$$F = F_p - F_r,$$

giving

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v. \tag{4}$$

We can use a phase line to analyze the velocity functions that solve this differential equation.

The equilibrium point, obtained by setting the right-hand side of Equation (4) equal to zero, is

$$v = \frac{mg}{k}.$$

If the body is initially moving faster than this, dv/dt is negative and the body slows down. If the body is moving at a velocity below mg/k, then $dv/dt > 0$ and the body speeds up. These observations are captured in the initial phase line diagram in Figure 9.20.

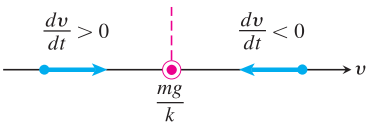


FIGURE 9.20 Initial phase line for the falling body encountering resistance.

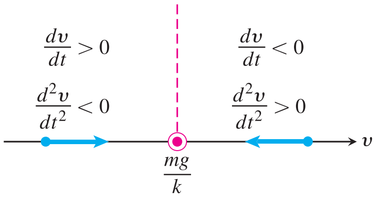


FIGURE 9.21 The completed phase line for the falling body.

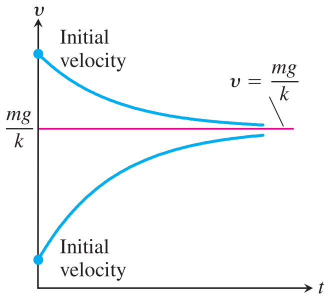


FIGURE 9.22 Typical velocity curves for a falling body encountering resistance. The value $v = mg/k$ is the terminal velocity.

We determine the concavity of the solution curves by differentiating both sides of Equation (4) with respect to t:

$$\frac{d^2v}{dt^2} = \frac{d}{dt}\left(g - \frac{k}{m}v\right) = -\frac{k}{m}\frac{dv}{dt}.$$

We see that $d^2v/dt^2 < 0$ when $v < mg/k$ and $d^2v/dt^2 > 0$ when $v > mg/k$. Figure 9.21 adds this information to the phase line. Notice the similarity to the phase line for Newton's law of cooling (Figure 9.17). The solution curves are similar as well (Figure 9.22).

Figure 9.22 shows two typical solution curves. Regardless of the initial velocity, we see the body's velocity tending toward the limiting value $v = mg/k$. This value, a stable equilibrium point, is called the body's **terminal velocity**. Skydivers can vary their terminal velocity from 95 mph to 180 mph by changing the amount of body area opposing the fall, which affects the value of k.

Logistic Population Growth

In Section 9.3 we examined population growth using the model of exponential change. That is, if P represents the number of individuals and we neglect departures and arrivals, then

$$\frac{dP}{dt} = kP, \tag{5}$$

where $k > 0$ is the birth rate minus the death rate per individual per unit time.

Because the natural environment has only a limited number of resources to sustain life, it is reasonable to assume that only a maximum population M can be accommodated. As the population approaches this **limiting population** or **carrying capacity**, resources become less abundant and the growth rate k decreases. A simple relationship exhibiting this behavior is

$$k = r(M - P),$$

where $r > 0$ is a constant. Notice that k decreases as P increases toward M and that k is negative if P is greater than M. Substituting $r(M - P)$ for k in Equation (5) gives the differential equation

$$\frac{dP}{dt} = r(M - P)P = rMP - rP^2. \tag{6}$$

The model given by Equation (6) is referred to as **logistic growth**.

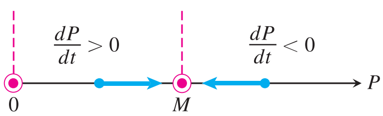


FIGURE 9.23 The initial phase line for logistic growth (Equation 6).

We can forecast the behavior of the population over time by analyzing the phase line for Equation (6). The equilibrium values are $P = M$ and $P = 0$, and we can see that $dP/dt > 0$ if $0 < P < M$ and $dP/dt < 0$ if $P > M$. These observations are recorded on the phase line in Figure 9.23.

We determine the concavity of the population curves by differentiating both sides of Equation (6) with respect to t:

$$\begin{aligned}\frac{d^2P}{dt^2} &= \frac{d}{dt}(rMP - rP^2)\\ &= rM\frac{dP}{dt} - 2rP\frac{dP}{dt}\\ &= r(M - 2P)\frac{dP}{dt}.\end{aligned} \tag{7}$$

If $P = M/2$, then $d^2P/dt^2 = 0$. If $P < M/2$, then $(M - 2P)$ and dP/dt are positive and $d^2P/dt^2 > 0$. If $M/2 < P < M$, then $(M - 2P) < 0$, $dP/dt > 0$, and $d^2P/dt^2 < 0$.

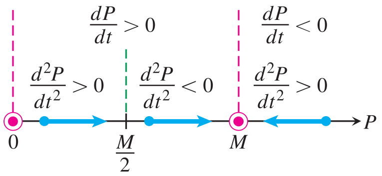


FIGURE 9.24 The completed phase line for logistic growth (Equation 6).

If $P > M$, then $(M - 2P)$ and dP/dt are both negative and $d^2P/dt^2 > 0$. We add this information to the phase line (Figure 9.24).

The lines $P = M/2$ and $P = M$ divide the first quadrant of the tP-plane into horizontal bands in which we know the signs of both dP/dt and d^2P/dt^2. In each band, we know how the solution curves rise and fall, and how they bend as time passes. The equilibrium lines $P = 0$ and $P = M$ are both population curves. Population curves crossing the line $P = M/2$ have an inflection point there, giving them a **sigmoid** shape (curved in two directions like a letter S). Figure 9.25 displays typical population curves. Notice that each population curve approaches the limiting population M as $t \to \infty$.

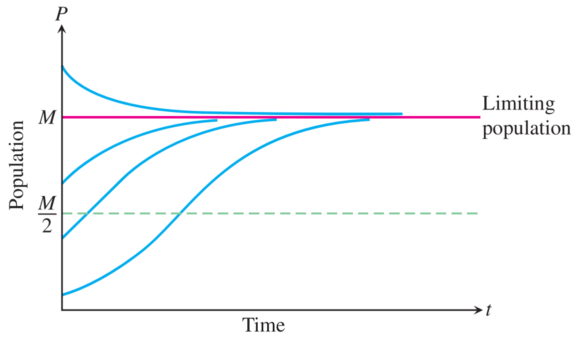


FIGURE 9.25 Population curves for logistic growth.

Exercises 9.4

Phase Lines and Solution Curves

In Exercises 1–8,

- **a.** Identify the equilibrium values. Which are stable and which are unstable?
- **b.** Construct a phase line. Identify the signs of y' and y''.
- **c.** Sketch several solution curves.

1. $\frac{dy}{dx} = (y + 2)(y - 3)$

2. $\frac{dy}{dx} = y^2 - 4$

3. $\frac{dy}{dx} = y^3 - y$

4. $\frac{dy}{dx} = y^2 - 2y$

5. $y' = \sqrt{y}, \quad y > 0$

6. $y' = y - \sqrt{y}, \quad y > 0$

7. $y' = (y - 1)(y - 2)(y - 3)$

8. $y' = y^3 - y^2$

Models of Population Growth

The autonomous differential equations in Exercises 9–12 represent models for population growth. For each exercise, use a phase line analysis to sketch solution curves for $P(t)$, selecting different starting values $P(0)$. Which equilibria are stable, and which are unstable?

9. $\frac{dP}{dt} = 1 - 2P$

10. $\frac{dP}{dt} = P(1 - 2P)$

11. $\frac{dP}{dt} = 2P(P - 3)$

12. $\frac{dP}{dt} = 3P(1 - P)\left(P - \frac{1}{2}\right)$

13. Catastrophic change in logistic growth Suppose that a healthy population of some species is growing in a limited environment and that the current population P_0 is fairly close to the carrying capacity M_0. You might imagine a population of fish living in a freshwater lake in a wilderness area. Suddenly a catastrophe such as the Mount St. Helens volcanic eruption contaminates the lake and destroys a significant part of the food and oxygen on which the fish depend. The result is a new environment with a carrying capacity M_1 considerably less than M_0 and, in fact, less than the current population P_0. Starting at some time before the catastrophe, sketch a "before-and-after" curve that shows how the fish population responds to the change in environment.

14. Controlling a population The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M - P)(P - m),$$

where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

b. Explain how this model differs from the logistic model $dP/dt = rP(M - P)$. Is it better or worse than the logistic model?

c. Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$.

d. What happens if $P < m$ for all t?

e. Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P. About how many permits should be issued?

Applications and Examples

15. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of velocity, then the body's velocity t seconds into the fall satisfies the equation

$$m\frac{dv}{dt} = mg - kv^2, \qquad k > 0$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is too short to be affected by changes in the air's density.)

a. Draw a phase line for the equation.

b. Sketch a typical velocity curve.

c. For a 110-lb skydiver ($mg = 110$) and with time in seconds and distance in feet, a typical value of k is 0.005. What is the diver's terminal velocity? Repeat for a 200-lb skydiver.

16. Resistance proportional to $\sqrt{v}$ A body of mass m is projected vertically downward with initial velocity v_0. Assume that the resisting force is proportional to the square root of the velocity and find the terminal velocity from a graphical analysis.

17. Sailing A sailboat is running along a straight course with the wind providing a constant forward force of 50 lb. The only other force acting on the boat is resistance as the boat moves through the water. The resisting force is numerically equal to five times the boat's speed, and the initial velocity is 1 ft/sec. What is the maximum velocity in feet per second of the boat under this wind?

18. The spread of information Sociologists recognize a phenomenon called *social diffusion*, which is the spreading of a piece of information, technological innovation, or cultural fad among a population. The members of the population can be divided into two classes: those who have the information and those who do not. In a fixed population whose size is known, it is reasonable to assume that the rate of diffusion is proportional to the number who have the information times the number yet to receive it. If X denotes the number of individuals who have the information in a population of N people, then a mathematical model for social diffusion is given by

$$\frac{dX}{dt} = kX(N - X),$$

where t represents time in days and k is a positive constant.

a. Discuss the reasonableness of the model.

b. Construct a phase line identifying the signs of X' and X''.

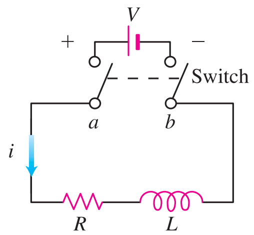
c. Sketch representative solution curves.

d. Predict the value of X for which the information is spreading most rapidly. How many people eventually receive the information?

19. Current in an *RL*-circuit The accompanying diagram represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts. From Section 9.2, we have

$$L\frac{di}{dt} + Ri = V,$$

where i is the current in amperes and t is the time in seconds.



Use a phase line analysis to sketch the solution curve assuming that the switch in the *RL*-circuit is closed at time $t = 0$. What happens to the current as $t \to \infty$? This value is called the *steady-state solution.*

20. A pearl in shampoo Suppose that a pearl is sinking in a thick fluid, like shampoo, subject to a frictional force opposing its fall and proportional to its velocity. Suppose that there is also a resistive buoyant force exerted by the shampoo. According to *Archimedes' principle*, the buoyant force equals the weight of the fluid displaced by the pearl. Using m for the mass of the pearl and P for the mass of the shampoo displaced by the pearl as it descends, complete the following steps.

a. Draw a schematic diagram showing the forces acting on the pearl as it sinks, as in Figure 9.19.

b. Using $v(t)$ for the pearl's velocity as a function of time t, write a differential equation modeling the velocity of the pearl as a falling body.

c. Construct a phase line displaying the signs of v' and v''.

d. Sketch typical solution curves.

e. What is the terminal velocity of the pearl?

9.5 Systems of Equations and Phase Planes

In some situations we are led to consider not one, but several first-order differential equations. Such a collection is called a **system** of differential equations. In this section we present an approach to understanding systems through a graphical procedure known as a *phase-plane analysis*. We present this analysis in the context of modeling the populations of trout and bass living in a common pond.

Phase Planes

A general system of two first-order differential equations may take the form

$$\frac{dx}{dt} = F(x, y),$$

$$\frac{dy}{dt} = G(x, y).$$

Such a system of equations is called **autonomous** because dx/dt and dy/dt do not depend on the independent variable time t, but only on the dependent variables x and y. A **solution** of such a system consists of a pair of functions $x(t)$ and $y(t)$ that satisfies both of the differential equations simultaneously for every t over some time interval (finite or infinite).

We cannot look at just one of these equations in isolation to find solutions $x(t)$ or $y(t)$ since each derivative depends on both x and y. To gain insight into the solutions, we look at both dependent variables together by plotting the points $(x(t), y(t))$ in the xy-plane starting at some specified point. Therefore the solution functions define a solution curve through the specified point, called a **trajectory** of the system. The xy-plane itself, in which these trajectories reside, is referred to as the **phase plane**. Thus we consider both solutions together and study the behavior of all the solution trajectories in the phase plane. It can be proved that two trajectories can never cross or touch each other. (Solution trajectories are examples of *parametric curves*, which are studied in detail in Chapter 11.)

A Competitive-Hunter Model

Imagine two species of fish, say trout and bass, competing for the same limited resources (such as food and oxygen) in a certain pond. We let $x(t)$ represent the number of trout and $y(t)$ the number of bass living in the pond at time t. In reality $x(t)$ and $y(t)$ are always integer valued, but we will approximate them with real-valued differentiable functions. This allows us to apply the methods of differential equations.

Several factors affect the rates of change of these populations. As time passes, each species breeds, so we assume its population increases proportionally to its size. Taken by itself, this would lead to exponential growth in each of the two populations. However, there is a countervailing effect from the fact that the two species are in competition. A large number of bass tends to cause a decrease in the number of trout, and vice-versa. Our model takes the size of this effect to be proportional to the frequency with which the two species interact, which in turn is proportional to xy, the product of the two populations. These considerations lead to the following model for the growth of the trout and bass in the pond:

$$\frac{dx}{dt} = (a - by)x, \tag{1a}$$

$$\frac{dy}{dt} = (m - nx)y. \tag{1b}$$

Here $x(t)$ represents the trout population, $y(t)$ the bass population, and a, b, m, n are positive constants. A solution of this system then consists of a pair of functions $x(t)$ and $y(t)$ that gives the population of each fish species at time t. Each equation in (1) contains both of the unknown functions x and y, so we are unable to solve them individually. Instead, we will use a graphical analysis to study the solution trajectories of this **competitive-hunter model**.

We now examine the nature of the phase plane in the trout-bass population model. We will be interested in the 1st quadrant of the xy-plane, where $x \geq 0$ and $y \geq 0$, since populations cannot be negative. First, we determine where the bass and trout populations are both constant. Noting that the $(x(t), y(t))$ values remain unchanged when $dx/dt = 0$ and $dy/dt = 0$, Equations (1a and 1b) then become

$$(a - by)x = 0,$$

$$(m - nx)y = 0.$$

This pair of simultaneous equations has two solutions: $(x, y) = (0, 0)$ and $(x, y) = (m/n, a/b)$. At these (x, y) values, called **equilibrium** or **rest points**, the two populations remain at constant values over all time. The point (0, 0) represents a pond containing no members of either fish species; the point $(m/n, a/b)$ corresponds to a pond with an unchanging number of each fish species.

Next, we note that if $y = a/b$, then Equation (1a) implies $dx/dt = 0$, so the trout population $x(t)$ is constant. Similarly, if $x = m/n$, then Equation (1b) implies $dy/dt = 0$, and the bass population $y(t)$ is constant. This information is recorded in Figure 9.26.

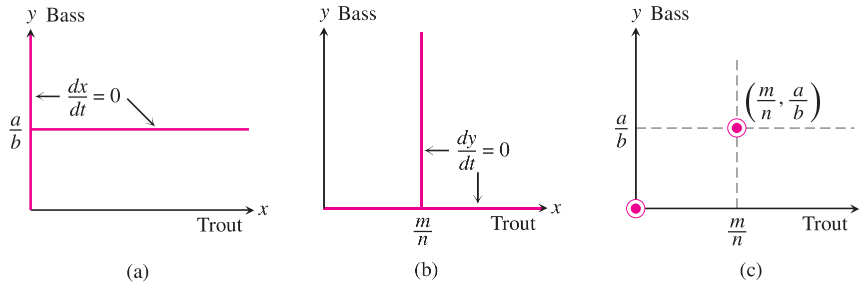


FIGURE 9.26 Rest points in the competitive-hunter model given by Equations (1a) and (1b).

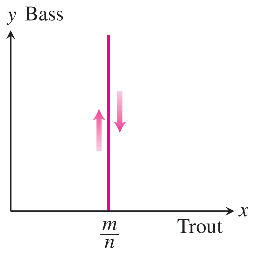


FIGURE 9.27 To the left of the line $x = m/n$ the trajectories move upward, and to the right they move downward.

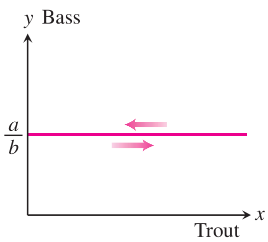


FIGURE 9.28 Above the line $y = a/b$ the trajectories move to the left, and below it they move to the right.

In setting up our competitive-hunter model, precise values of the constants a, b, m, n will not generally be known. Nonetheless, we can analyze the system of Equations (1) to learn the nature of its solution trajectories. We begin by determining the signs of dx/dt and dy/dt throughout the phase plane. Although $x(t)$ represents the number of trout and $y(t)$ the number of bass at time t, we are thinking of the pair of values $(x(t), y(t))$ as a point tracing out a trajectory curve in the phase plane. When dx/dt is positive, $x(t)$ is increasing and the point is moving to the right in the phase plane. If dx/dt is negative, the point is moving to the left. Likewise, the point is moving upward where dy/dt is positive and downward where dy/dt is negative.

We saw that $dy/dt = 0$ along the vertical line $x = m/n$. To the left of this line, dy/dt is positive since $dy/dt = (m - nx)y$ and $x < m/n$. So the trajectories on this side of the line are directed upward. To the right of this line, dy/dt is negative and the trajectories point downward. The directions of the associated trajectories are indicated in Figure 9.27. Similarly, above the horizontal line $y = a/b$, we have $dx/dt < 0$ and the trajectories head leftward; below this line they head rightward, as shown in Figure 9.28. Combining this information gives four distinct regions in the plane A, B, C, D, with their respective trajectory directions shown in Figure 9.29.

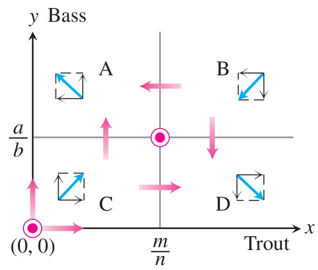


FIGURE 9.29 Composite graphical analysis of the trajectory directions in the four regions determined by $x = m/n$ and $y = a/b$.

Next, we examine what happens near the two equilibrium points. The trajectories near (0, 0) point away from it, upward and to the right. The behavior near the equilibrium point $(m/n, a/b)$ depends on the region in which a trajectory begins. If it starts in region B, for instance, then it will move downward and leftward towards the equilibrium point. Depending on where the trajectory begins, it may move downward into region D, leftward into region A,

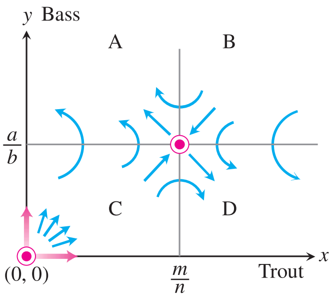


FIGURE 9.30 Motion along the trajectories near the rest points (0, 0) and $(m/n, a/b)$.

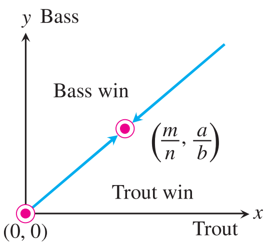


FIGURE 9.31 Qualitative results of analyzing the competitive-hunter model. There are exactly two trajectories approaching the point $(m/n, a/b)$.

or perhaps straight into the equilibrium point. If it enters into regions A or D, then it will continue to move away from the rest point. We say that both rest points are **unstable**, meaning (in this setting) there are trajectories near each point that head away from them. These features are indicated in Figure 9.30.

It turns out that in each of the half-planes above and below the line $y = a/b$, there is exactly one trajectory approaching the equilibrium point $(m/n, a/b)$ (see Exercise 7). Above these two trajectories the bass population increases and below them it decreases. The two trajectories approaching the equilibrium point are suggested in Figure 9.31.

Our graphical analysis leads us to conclude that, under the assumptions of the competitive-hunter model, it is unlikely that both species will reach equilibrium levels. This is because it would be almost impossible for the fish populations to move exactly along one of the two approaching trajectories for all time. Furthermore, the initial populations point (x_0, y_0) determines which of the two species is likely to survive over time, and mutual coexistence of the species is highly improbable.

Limitations of the Phase-Plane Analysis Method

Unlike the situation for the competitive-hunter model, it is not always possible to determine the behavior of trajectories near a rest point. For example, suppose we know that the trajectories near a rest point, chosen here to be the origin (0, 0), behave as in Figure 9.32. The information provided by Figure 9.32 is not sufficient to distinguish between the three possible trajectories shown in Figure 9.33. Even if we could determine that a trajectory near an equilibrium point resembles that of Figure 9.33c, we would still not know how the other trajectories behave. It could happen that a trajectory closer to the origin behaves like the motions displayed in Figure 9.33a or 9.33b. The spiraling trajectory in Figure 9.33b can never actually reach the rest point in a finite time period.

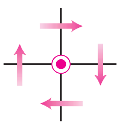


FIGURE 9.32 Trajectory direction near the rest point (0, 0).

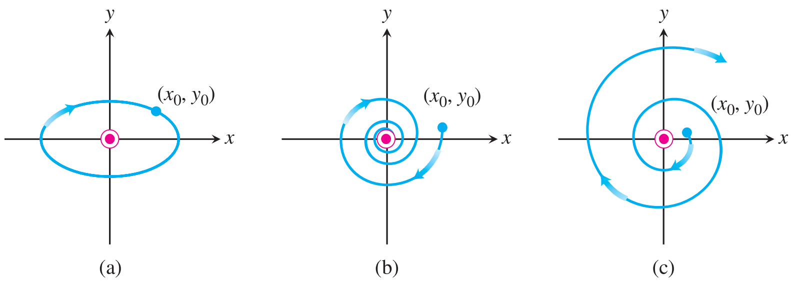


FIGURE 9.33 Three possible trajectory motions: (a) periodic motion, (b) motion toward an asymptotically stable rest point, and (c) motion near an unstable rest point.

Another Type of Behavior

The system

$$\frac{dx}{dt} = y + x - x(x^2 + y^2), \tag{2a}$$

$$\frac{dy}{dt} = -x + y - y(x^2 + y^2) \tag{2b}$$

can be shown to have only one equilibrium point at (0, 0). Yet any trajectory starting on the unit circle traverses it clockwise because, when $x^2 + y^2 = 1$, we have $dy/dx = -x/y$ (see Exercise 2). If a trajectory starts inside the unit circle, it spirals outward, asymptotically approaching the circle as $t \to \infty$. If a trajectory starts outside the unit circle, it spirals inward, again asymptotically approaching the circle as $t \to \infty$. The circle $x^2 + y^2 = 1$ is called a **limit cycle** of the system (Figure 9.34). In this system, the values of x and y eventually become periodic.

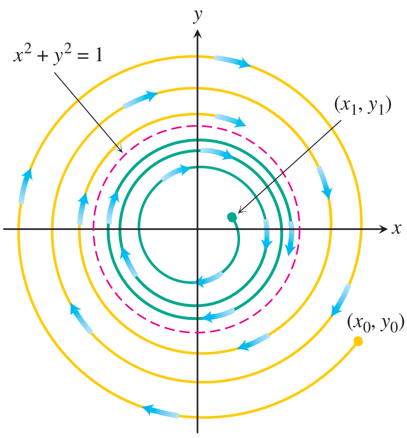


FIGURE 9.34 The solution $x^2 + y^2 = 1$ is a limit cycle.

Exercises 9.5

1. List three important considerations that are ignored in the competitive-hunter model as presented in the text.
2. For the system (2a) and (2b), show that any trajectory starting on the unit circle $x^2 + y^2 = 1$ will traverse the unit circle in a periodic solution. First introduce polar coordinates and rewrite the system as $dr/dt = r(1 - r^2)$ and $-d\theta/dt = -1$.
3. Develop a model for the growth of trout and bass, assuming that in isolation trout demonstrate exponential decay [so that $a < 0$ in Equations (1a) and (1b)] and that the bass population grows logistically with a population limit M. Analyze graphically the motion in the vicinity of the rest points in your model. Is coexistence possible?
4. How might the competitive-hunter model be validated? Include a discussion of how the various constants a, b, m, and n might be estimated. How could state conservation authorities use the model to ensure the survival of both species?
5. Consider another competitive-hunter model defined by

$$\frac{dx}{dt} = a\left(1 - \frac{x}{k_1}\right)x - bxy,$$
$$\frac{dy}{dt} = m\left(1 - \frac{y}{k_2}\right)y - nxy,$$

where x and y represent trout and bass populations, respectively.

a. What assumptions are implicitly being made about the growth of trout and bass in the absence of competition?

b. Interpret the constants a, b, m, n, k_1, and k_2 in terms of the physical problem.

c. Perform a graphical analysis:

i) Find the possible equilibrium levels.

ii) Determine whether coexistence is possible.

iii) Pick several typical starting points and sketch typical trajectories in the phase plane.

iv) Interpret the outcomes predicted by your graphical analysis in terms of the constants a, b, m, n, k_1, and k_2.

Note: When you get to part (iii), you should realize that five cases exist. You will need to analyze all five cases.

6. **An economic model** Consider the following economic model. Let P be the price of a single item on the market. Let Q be the quantity of the item available on the market. Both P and Q are functions of time. If one considers price and quantity as two interacting species, the following model might be proposed:

$$\frac{dP}{dt} = aP\left(\frac{b}{Q} - P\right),$$
$$\frac{dQ}{dt} = cQ(fP - Q),$$

where a, b, c, and f are positive constants. Justify and discuss the adequacy of the model.

a. If $a = 1$, $b = 20{,}000$, $c = 1$, and $f = 30$, find the equilibrium points of this system. If possible, classify each equilibrium point with respect to its stability. If a point cannot be readily classified, give some explanation.

b. Perform a graphical stability analysis to determine what will happen to the levels of P and Q as time increases.

c. Give an economic interpretation of the curves that determine the equilibrium points.

7. **Two trajectories approach equilibrium** Show that the two trajectories leading to $(m/n, a/b)$ shown in Figure 9.31 are unique by carrying out the following steps.

a. From system (1a) and (1b) apply the Chain Rule to derive the following equation:

$$\frac{dy}{dx} = \frac{(m - nx)y}{(a - by)x}.$$

b. Separate the variables, integrate, and exponentiate to obtain

$$y^a e^{-by} = Kx^m e^{-nx},$$

where K is a constant of integration.

c. Let $f(y) = y^a/e^{by}$ and $g(x) = x^m/e^{nx}$. Show that $f(y)$ has a unique maximum of $M_y = (a/eb)^a$ when $y = a/b$ as shown in Figure 9.35. Similarly, show that $g(x)$ has a unique maximum $M_x = (m/en)^m$ when $x = m/n$, also shown in Figure 9.35.

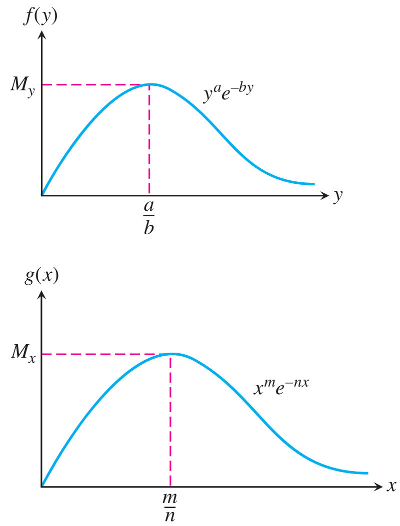


FIGURE 9.35 Graphs of the functions $f(y) = y^a/e^{by}$ and $g(x) = x^m/e^{nx}$.

d. Consider what happens as (x, y) approaches $(m/n, a/b)$. Take limits in part (b) as $x \to m/n$ and $y \to a/b$ to show that either

$$\lim_{\substack{x \to m/n \\ y \to a/b}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = K$$

or $M_y/M_x = K$. Thus any solution trajectory that approaches $(m/n, a/b)$ must satisfy

$$\frac{y^a}{e^{by}} = \left(\frac{M_y}{M_x}\right)\left(\frac{x^m}{e^{nx}}\right).$$

e. Show that only one trajectory can approach $(m/n, a/b)$ from below the line $y = a/b$. Pick $y_0 < a/b$. From Figure 9.35 you can see that $f(y_0) < M_y$, which implies that

$$\frac{M_y}{M_x}\left(\frac{x^m}{e^{nx}}\right) = {y_0}^a/e^{by_0} < M_y.$$

This in turn implies that

$$\frac{x^m}{e^{nx}} < M_x.$$

Figure 9.35 tells you that for $g(x)$ there is a unique value $x_0 < m/n$ satisfying this last inequality. That is, for each $y < a/b$ there is a unique value of x satisfying the equation in part (d). Thus there can exist only one trajectory solution approaching $(m/n, a/b)$ from below, as shown in Figure 9.36.

f. Use a similar argument to show that the solution trajectory leading to $(m/n, a/b)$ is unique if $y_0 > a/b$.

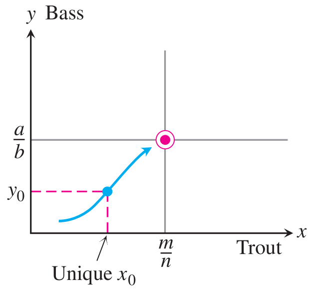


FIGURE 9.36 For any $y < a/b$ only one solution trajectory leads to the rest point $(m/n, a/b)$.

8. Show that the second-order differential equation $y'' = F(x, y, y')$ can be reduced to a system of two first-order differential equations

$$\frac{dy}{dx} = z,$$

$$\frac{dz}{dx} = F(x, y, z).$$

Can something similar be done to the nth-order differential equation $y^{(n)} = F\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$?

Lotka-Volterra Equations for a Predator-Prey Model

In 1925 Lotka and Volterra introduced the *predator-prey* equations, a system of equations that models the populations of two species, one of which preys on the other. Let $x(t)$ represent the number of rabbits living in a region at time t, and $y(t)$ the number of foxes in the same region. As time passes, the number of rabbits increases at a rate proportional to their population, and decreases at a rate proportional to the number of encounters between rabbits and foxes. The foxes, which compete for food, increase in number at a rate proportional to the number of encounters with rabbits but decrease at a rate proportional to the number of foxes. The number of encounters between rabbits and foxes is assumed to be proportional to the product of the two populations. These assumptions lead to the autonomous system

$$\frac{dx}{dt} = (a - by)x$$

$$\frac{dy}{dt} = (-c + dx)y$$

where a, b, c, d are positive constants. The values of these constants vary according to the specific situation being modeled. We can study the nature of the population changes without setting these constants to specific values.

9. What happens to the rabbit population if there are no foxes present?

10. What happens to the fox population if there are no rabbits present?

11. Show that $(0, 0)$ and $(c/d, a/b)$ are equilibrium points. Explain the meaning of each of these points.

12. Show, by differentiating, that the function

$$C(t) = a \ln y(t) - by(t) - dx(t) + c \ln x(t)$$

is constant when $x(t)$ and $y(t)$ are positive and satisfy the predator-prey equations.

While x and y may change over time, $C(t)$ does not. Thus, C is a *conserved quantity* and its existence gives a *conservation law*. A trajectory that begins at a point (x, y) at time $t = 0$ gives a value of C that remains unchanged at future times. Each value of the constant C gives a trajectory for the autonomous system, and these trajectories close up, rather than spiraling inwards or outwards. The rabbit and fox populations oscillate through repeated cycles along a fixed trajectory. Figure 9.37 shows several trajectories for the predator-prey system.

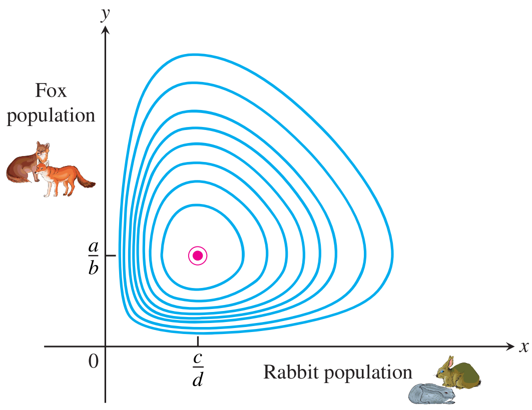


FIGURE 9.37 Some trajectories along which C is conserved.

13. Using a procedure similar to that in the text for the competitive-hunter model, show that each trajectory is traversed in a counterclockwise direction as time t increases.

Along each trajectory, both the rabbit and fox populations fluctuate between their maximum and minimum levels. The maximum and minimum levels for the rabbit population occur where the trajectory intersects the horizontal line $y = a/b$. For the fox population, they occur where the trajectory intersects the vertical line $x = c/d$. When the rabbit population is at its maximum, the fox population is below its maximum value. As the rabbit population declines from this point in time, we move counterclockwise around the trajectory, and the fox population grows until it reaches its maximum value. At this point the rabbit population has declined to $x = c/d$ and is no longer at its peak value. We see that the fox population reaches its maximum value at a later time than the rabbits. The predator population *lags behind* that of the prey in achieving its maximum values. This lag effect is shown in Figure 9.38, which graphs both $x(t)$ and $y(t)$.

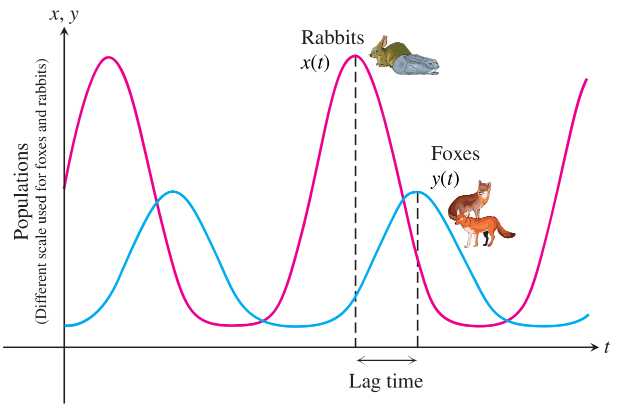


FIGURE 9.38 The fox and rabbit populations oscillate periodically, with the maximum fox population lagging the maximum rabbit population.

14. At some time during a trajectory cycle, a wolf invades the rabbit-fox territory, eats some rabbits, and then leaves. Does this mean that the fox population will from then on have a lower maximum value? Explain your answer.

Chapter 9 Questions to Guide Your Review

1. What is a first-order differential equation? When is a function a solution of such an equation?
2. What is a general solution? A particular solution?
3. What is the slope field of a differential equation $y' = f(x, y)$? What can we learn from such fields?
4. Describe Euler's method for solving the initial value problem $y' = f(x, y), y(x_0) = y_0$ numerically. Give an example. Comment on the method's accuracy. Why might you want to solve an initial value problem numerically?
5. How do you solve linear first-order differential equations?
6. What is an orthogonal trajectory of a family of curves? Describe how one is found for a given family of curves.
7. What is an autonomous differential equation? What are its equilibrium values? How do they differ from critical points? What is a stable equilibrium value? Unstable?
8. How do you construct the phase line for an autonomous differential equation? How does the phase line help you produce a graph which qualitatively depicts a solution to the differential equation?
9. Why is the exponential model unrealistic for predicting long-term population growth? How does the logistic model correct for the deficiency in the exponential model for population growth? What is the logistic differential equation? What is the form of its solution? Describe the graph of the logistic solution.
10. What is an autonomous system of differential equations? What is a solution to such a system? What is a trajectory of the system?

Chapter 9 Practice Exercises

In Exercises 1–16 solve the differential equation.

1. $y' = xe^y\sqrt{x-2}$

2. $y' = xye^{x^2}$

3. $\sec x\, dy + x\cos^2 y\, dx = 0$

4. $2x^2\, dx - 3\sqrt{y}\csc x\, dy = 0$

5. $y' = \dfrac{e^y}{xy}$

6. $y' = xe^{x-y}\csc y$

7. $x(x-1)\, dy - y\, dx = 0$

8. $y' = (y^2 - 1)x^{-1}$

9. $2y' - y = xe^{x/2}$

10. $\dfrac{y'}{2} + y = e^{-x}\sin x$

11. $xy' + 2y = 1 - x^{-1}$

12. $xy' - y = 2x\ln x$

13. $(1 + e^x)\, dy + (ye^x + e^{-x})\, dx = 0$

14. $e^{-x}\, dy + (e^{-x}y - 4x)\, dx = 0$

15. $(x + 3y^2)\, dy + y\, dx = 0$ (*Hint:* $d(xy) = y\, dx + x\, dy$)

16. $x\, dy + (3y - x^{-2}\cos x)\, dx = 0, \quad x > 0$

Initial Value Problems

In Exercises 17–22 solve the initial value problem.

17. $(x+1)\dfrac{dy}{dx} + 2y = x, \quad x > -1, \quad y(0) = 1$

18. $x\dfrac{dy}{dx} + 2y = x^2 + 1, \quad x > 0, \quad y(1) = 1$

19. $\dfrac{dy}{dx} + 3x^2y = x^2, \quad y(0) = -1$

20. $x\, dy + (y - \cos x)\, dx = 0, \quad y\left(\dfrac{\pi}{2}\right) = 0$

21. $xy' + (x-2)y = 3x^3e^{-x}, \quad y(1) = 0$

22. $y\, dx + (3x - xy + 2)\, dy = 0, \quad y(2) = -1, \quad y < 0$

Euler's Method

In Exercises 23 and 24, use Euler's method to solve the initial value problem on the given interval starting at x_0 with $dx = 0.1$.

T **23.** $y' = y + \cos x, \quad y(0) = 0; \quad 0 \le x \le 2; \quad x_0 = 0$

T **24.** $y' = (2 - y)(2x + 3), \quad y(-3) = 1;$
$-3 \le x \le -1; \quad x_0 = -3$

In Exercises 25 and 26, use Euler's method with $dx = 0.05$ to estimate $y(c)$ where y is the solution to the given initial value problem.

T **25.** $c = 3; \quad \dfrac{dy}{dx} = \dfrac{x - 2y}{x + 1}, \quad y(0) = 1$

T **26.** $c = 4; \quad \dfrac{dy}{dx} = \dfrac{x^2 - 2y + 1}{x}, \quad y(1) = 1$

In Exercises 27 and 28, use Euler's method to solve the initial value problem graphically, starting at $x_0 = 0$ with

a. $dx = 0.1$. **b.** $dx = -0.1$.

T **27.** $\dfrac{dy}{dx} = \dfrac{1}{e^{x+y+2}}, \quad y(0) = -2$

T **28.** $\dfrac{dy}{dx} = -\dfrac{x^2 + y}{e^y + x}, \quad y(0) = 0$

Slope Fields

In Exercises 29–32, sketch part of the equation's slope field. Then add to your sketch the solution curve that passes through the point $P(1, -1)$. Use Euler's method with $x_0 = 1$ and $dx = 0.2$ to estimate $y(2)$. Round your answers to four decimal places. Find the exact value of $y(2)$ for comparison.

29. $y' = x$ **30.** $y' = 1/x$

31. $y' = xy$ **32.** $y' = 1/y$

Autonomous Differential Equations and Phase Lines

In Exercises 33 and 34:

a. Identify the equilibrium values. Which are stable and which are unstable?

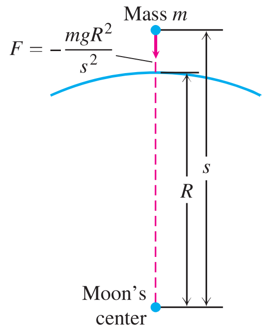
b. Construct a phase line. Identify the signs of y' and y''.

c. Sketch a representative selection of solution curves.

33. $\dfrac{dy}{dx} = y^2 - 1$ **34.** $\dfrac{dy}{dx} = y - y^2$

Applications

35. Escape velocity The gravitational attraction F exerted by an airless moon on a body of mass m at a distance s from the moon's center is given by the equation $F = -mg\,R^2 s^{-2}$, where g is the acceleration of gravity at the moon's surface and R is the moon's radius (see accompanying figure). The force F is negative because it acts in the direction of decreasing s.



a. If the body is projected vertically upward from the moon's surface with an initial velocity v_0 at time $t = 0$, use Newton's second law, $F = ma$, to show that the body's velocity at position s is given by the equation

$$v^2 = \frac{2gR^2}{s} + v_0^{\,2} - 2gR.$$

Thus, the velocity remains positive as long as $v_0 \ge \sqrt{2gR}$. The velocity $v_0 = \sqrt{2gR}$ is the moon's **escape velocity**. A body projected upward with this velocity or a greater one will escape from the moon's gravitational pull.

b. Show that if $v_0 = \sqrt{2gR}$, then

$$s = R\left(1 + \frac{3v_0}{2R}t\right)^{2/3}.$$

36. Coasting to a stop Table 9.6 shows the distance s (meters) coasted on in-line skates in t sec by Johnathon Krueger. Find a model for his position in the form of Equation (2) of Section 9.3. His initial velocity was $v_0 = 0.86$ m/sec, his mass $m = 30.84$ kg (he weighed 68 lb), and his total coasting distance 0.97 m.

TABLE 9.6 Johnathon Krueger skating data

t (sec)	*s* (m)	*t* (sec)	*s* (m)	*t* (sec)	*s* (m)
0	0	0.93	0.61	1.86	0.93
0.13	0.08	1.06	0.68	2.00	0.94
0.27	0.19	1.20	0.74	2.13	0.95
0.40	0.28	1.33	0.79	2.26	0.96
0.53	0.36	1.46	0.83	2.39	0.96
0.67	0.45	1.60	0.87	2.53	0.97
0.80	0.53	1.73	0.90	2.66	0.97

Chapter 9 Additional and Advanced Exercises

Theory and Applications

1. Transport through a cell membrane Under some conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\frac{A}{V}(c - y).$$

In this equation, y is the concentration of the substance inside the cell and dy/dt is the rate at which y changes over time. The letters

k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

a. Solve the equation for $y(t)$, using y_0 to denote $y(0)$.

b. Find the steady-state concentration, $\lim_{t\to\infty} y(t)$.

2. Height of a rocket If an external force F acts upon a system whose mass varies with time, Newton's law of motion is

$$\frac{d(mv)}{dt} = F + (v + u)\frac{dm}{dt}.$$

In this equation, m is the mass of the system at time t, v is its velocity, and $v + u$ is the velocity of the mass that is entering (or leaving) the system at the rate dm/dt. Suppose that a rocket of initial mass m_0 starts from rest, but is driven upward by firing some of its mass directly backward at the constant rate of $dm/dt = -b$ units per second and at constant speed relative to the rocket $u = -c$. The only external force acting on the rocket is $F = -mg$ due to gravity. Under these assumptions, show that the height of the rocket above the ground at the end of t seconds (t small compared to m_0/b) is

$$y = c\left[t + \frac{m_0 - bt}{b}\ln\frac{m_0 - bt}{m_0}\right] - \frac{1}{2}gt^2.$$

3. a. Assume that $P(x)$ and $Q(x)$ are continuous over the interval $[a, b]$. Use the Fundamental Theorem of Calculus, Part 1 to show that any function y satisfying the equation

$$v(x)y = \int v(x)Q(x)\,dx + C$$

for $v(x) = e^{\int P(x)\,dx}$ is a solution to the first-order linear equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

b. If $C = y_0 v(x_0) - \int_{x_0}^{x} v(t)Q(t)\,dt$, then show that any solution y in part (a) satisfies the initial condition $y(x_0) = y_0$.

4. (*Continuation of Exercise 3.*) Assume the hypotheses of Exercise 3, and assume that $y_1(x)$ and $y_2(x)$ are both solutions to the first-order linear equation satisfying the initial condition $y(x_0) = y_0$.

a. Verify that $y(x) = y_1(x) - y_2(x)$ satisfies the initial value problem

$$y' + P(x)y = 0, \quad y(x_0) = 0.$$

b. For the integrating factor $v(x) = e^{\int P(x)\,dx}$, show that

$$\frac{d}{dx}\left(v(x)[y_1(x) - y_2(x)]\right) = 0.$$

Conclude that $v(x)[y_1(x) - y_2(x)] \equiv \text{constant}$.

c. From part (a), we have $y_1(x_0) - y_2(x_0) = 0$. Since $v(x) > 0$ for $a < x < b$, use part (b) to establish that $y_1(x) - y_2(x) \equiv 0$ on the interval (a, b). Thus $y_1(x) = y_2(x)$ for all $a < x < b$.

Homogeneous Equations

A first-order differential equation of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is called *homogeneous*. It can be transformed into an equation whose variables are separable by defining the new variable $v = y/x$. Then, $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution into the original differential equation and collecting terms with like variables then gives the separable equation

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0.$$

After solving this separable equation, the solution of the original equation is obtained when we replace v by y/x.

Solve the homogeneous equations in Exercises 5–10. First put the equation in the form of a homogeneous equation.

5. $(x^2 + y^2)\,dx + xy\,dy = 0$

6. $x^2\,dy + (y^2 - xy)\,dx = 0$

7. $(xe^{y/x} + y)\,dx - x\,dy = 0$

8. $(x + y)\,dy + (x - y)\,dx = 0$

9. $y' = \dfrac{y}{x} + \cos\dfrac{y - x}{x}$

10. $\left(x\sin\dfrac{y}{x} - y\cos\dfrac{y}{x}\right)dx + x\cos\dfrac{y}{x}\,dy = 0$

Chapter 9 Technology Application Projects

Mathematica/Maple Modules:

Drug Dosages: Are They Effective? Are They Safe?

Formulate and solve an initial value model for the absorption of a drug in the bloodstream.

First-Order Differential Equations and Slope Fields

Plot slope fields and solution curves for various initial conditions to selected first-order differential equations.