Abstract

In this pieces of notes, we will go through the concepts related to functions and the meaning of graphs of the functions.

Contents

1 Function and its graph			and its graph	5
	1.1	What	is a Function?	5
		1.1.3	Function as an input-output pair	7
		1.1.4	Defining a function with variables	7
		1.1.5	Domain, Codomain and Range	7
	1.2	Graph	of a function	7
		1.2.1	Blobs-and-arrows diagram for discrete functions	7
		1.2.2	The pairing table and xy-coordination	7
	1.3	Funda	mental properties of a graph	7
		1.3.1	points lying on the graph	7
		1.3.2	x-intercept	7
		1.3.3	y-intercept	7
		1.3.4	general intersection	7
	1.4	Solving	g equations using graphs of functions	7
	1.5	Transf	ormation of functions	7
		1.5.1	Translation	7
		1.5.2	Dilation	7
	1.6	Challe	nging questions	7
2	Line	ear fun	actions	8
	2.1	Funda	mental concepts of points	8
		2.1.1	Distance between two points	8
		2.1.2	Mid-point and Division points	8
	2.2	Differe	ent forms of a linear function	8
		2.2.1	Slope of a line	8
		2.2.2	Two-point form	8
		2.2.3	Point-slope form	8
		2.2.4	Slope-intercept form	8
		2.2.5	General form	8

		2.2.6	Special: Intercept-form	8
	2.3	Paralle	el lines and Perpendicular lines	8
		2.3.1	Point of intersection	8
		2.3.2	Parallel lines	8
		2.3.3	Perpendicular lines	8
		2.3.4	Number of intersections	8
	2.4	Angle	of elevation and depression	8
	2.5	Additi	ional content: Point-line distance	8
	2.6	Linear	rinequalities	9
		2.6.1	One variable inequality	9
		2.6.2	Two variable inequality	9
		2.6.3	Linear programming	9
	2.7	Challe	enging questions	9
3	Qua	adratic	functions	10
	3.1	Solvin	g Quadratic Equations	10
		3.1.1	The square-root method	10
		3.1.2	General form and Completing the square method	10
		3.1.3	The quadratic formula	10
	3.2	Solvab	oility of Quadratic Equations	10
		3.2.1	Discriminant: the factor affecting solvability	10
		3.2.2	Application of solvability	10
	3.3	Relation	on between coefficients and roots	10
		3.3.1	What does it mean by a root?	10
		3.3.2	The Vieta formula	10
		3.3.3	Forming quadratic equations with given roots	10
	3.4	Vertex	form of a quadratic function	10
		3.4.1	Obtaining vertex form using Transformation	10
		3.4.2	Relation between vertex form and general form	10
		3.4.3	Opening direction	10
		3.4.4	The axis of symmetry	10
		3.4.5	The extremum of quadratic functions	10
	3.5	Quadr	ratic inequalities	10
		3.5.1	Boolean algebra	10
		3.5.2	Solving quadratic inequalities	10

	3.6	Challenging questions	. 10		
4	Pol	olynomial functions			
	4.1	Arithmetic rules for polynomials	. 11		
		4.1.1 Addition, Subtraction and Multiplication	. 11		
		4.1.2 Division	. 11		
	4.2	Divisibility of polynomials	. 11		
		4.2.1 Division algorithm	. 11		
		4.2.2 Remainder Theorem	. 11		
		4.2.3 Factor Theorem and General Vieta formula	. 11		
	4.3	G.C.D. and L.C.M	. 11		
		4.3.1 Greatest common divisor	. 11		
		4.3.2 Least common multiple	. 11		
	4.4	Positional notation	. 11		
		4.4.1 Binary, Octal, Decimal and Hexadecimal	. 11		
		4.4.2 Conversion between different bases	. 11		
	4.5	Challenging questions	. 11		
5	Exp	ponential and Logarithmic functions	12		
5	Exp 5.1	ponential and Logarithmic functions Exponential functions			
5	-	•	. 12		
5	-	Exponential functions	. 12		
5	-	Exponential functions	. 12 . 12		
5	-	Exponential functions	. 12 . 12 . 12		
5	5.1	Exponential functions	. 12 . 12 . 12 . 12		
5	5.1	Exponential functions	. 12 . 12 . 12 . 12 . 12		
5	5.1	Exponential functions	. 12 . 12 . 12 . 12 . 12 . 12		
5	5.1	Exponential functions 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents 5.1.3 Solving equations using exponetiation method Logarithmic functions 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra	. 12 . 12 . 12 . 12 . 12 . 12 . 12		
	5.1 5.2 5.3	Exponential functions. 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents. 5.1.3 Solving equations using exponetiation method Logarithmic functions. 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra 5.2.3 Solving equations using Logarithmic method	. 12 . 12 . 12 . 12 . 12 . 12 . 12		
5	5.1 5.2 5.3	Exponential functions. 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents. 5.1.3 Solving equations using exponetiation method Logarithmic functions. 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra 5.2.3 Solving equations using Logarithmic method Challenging questions.	12 12 12 12 12 12 12 12 12 13		
	5.1 5.2 5.3 Seq	Exponential functions 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents 5.1.3 Solving equations using exponetiation method Logarithmic functions 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra 5.2.3 Solving equations using Logarithmic method Challenging questions quence as a function of natural numbers	12 12 12 12 12 12 12 12 12 12 12 13		
	5.1 5.2 5.3 Seq 6.1	Exponential functions 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents 5.1.3 Solving equations using exponetiation method Logarithmic functions 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra 5.2.3 Solving equations using Logarithmic method Challenging questions quence as a function of natural numbers What is a sequence?	12 12 12 12 12 12 12 12 12 12 13 13		
	5.1 5.2 5.3 Seq 6.1	Exponential functions 5.1.1 Law of Exponential algebra 5.1.2 Seeing surds as fractional exponents 5.1.3 Solving equations using exponetiation method Logarithmic functions 5.2.1 What is Logarithm? 5.2.2 Law of Logarithmic algebra 5.2.3 Solving equations using Logarithmic method Challenging questions quence as a function of natural numbers What is a sequence? Arithmetic sequence	12 12 12 12 12 12 12 12 12 12 13 13 13		

Function and graphs

	6.3.1	General form	13
	6.3.2	Summation on Geometric sequence	13
	6.3.3	Infinite sum of Geometric sequence	13
6.4	Additi	onal content: Arithmetic-Geometric sequence	13
6.5	Challe	enging questions	13

1 Function and its graph

'Functions describes the world!', one Professor in Mathematics of Massachusetts Institute of Technology (a.k.a. MIT) said that. His speech was greatly influential, as I have never heard such conclusive thinking about functions. In fact, in the past few years, whenever I was studying in schools, my thought about functions is always only about projecting elements from oen set to another set, but what he said had a big impact to my knowledge about functions.

What function talks about, is a subjection of one elements to one another. It can be thought of as a pointing action started from element A to element B, which not so far away, if we could think of subjecting a lot of, a bunch of, or a list of, whatever, objects from one collection to another collection, and they can all be matched under this pointing action, then it is a so-called function.

For example, in an Indian factory, the production of food undergoes many different process. Those can all be called functions. Let say a raw material comes to the factory first, it then be put to a machine to chop into many small pieces. It is the chopping function inside the factory. Next, the chopped material will be put into a pool of yellowish-brownish liquid and be stirred by dirty hands. It is the Mixing function in the factory. After that, the liquid will be drained on the dirty floor and be stepped on by Indian workers so that they can be smell freshed. It is the flavouring function in the factory. Finally, it will be sold to stores, which is the selling function.

Another example is what our body does. We eat and drink, going down the digestive system, and we sit on a toilet. Although we stupid human knows nothing about how the digestive system works, we could still name the conversion from food to poops a digestion, which means the digestive function representing the process in our body.

So we know that function as an english word represents the naming of a process of conversion, it is the time to explore how Math functions works.

1.1 What is a Function?

A function is defined as follow:

Definition (Function). Given an input x and an output y, a **function** is a relation between x and y so that we can write y = f(x) to represent the relationship.

Essential Practice 1.1.1. Write down functions for the following input-output variables:

- 1. u as input and v as output;
- 2. b as input and a as output;

- 3. n as input and 1 as output (which we call it a constant function);
- 4. x^2 as input and y as output;
- 5. xy as input and z as output;
- 6. 2^x as input and k as output;
- 7. \sqrt{p} as input and q as output;

Remark. It is notable that we may write functions as y = g(x), y = h(x), y = d(x), ... as we want. The 'naming' of a function is always definitive and up to user's construction.

We can also apply functions after functions. To do so, we have to talk about the following:

Definition (Composite functions). Let f and g be functions such that f takes x as input and y as output, and g takes y as input and z as output. Then we can say there is a function h that takes x as input and z as output. In other words, h is a **composite function** such that z = h(x) = g(f(x)).

Essential Practice 1.1.2. Let f and g be functions such that b = f(a), c = g(b). Write a function for a as input and c as output.

- 1.1.3 Function as an input-output pair
- 1.1.4 Defining a function with variables
- 1.1.5 Domain, Codomain and Range
- 1.2 Graph of a function
- 1.2.1 Blobs-and-arrows diagram for discrete functions
- 1.2.2 The pairing table and xy-coordination
- 1.3 Fundamental properties of a graph
- 1.3.1 points lying on the graph
- 1.3.2 x-intercept
- 1.3.3 y-intercept
- 1.3.4 general intersection
- 1.4 Solving equations using graphs of functions
- 1.5 Transformation of functions
- 1.5.1 Translation
- 1.5.2 Dilation
- 1.6 Challenging questions

2 Linear functions

2.1 Fundamental concepts of points

- 2.1.1 Distance between two points
- 2.1.2 Mid-point and Division points

2.2 Different forms of a linear function

- 2.2.1 Slope of a line
- 2.2.2 Two-point form
- 2.2.3 Point-slope form
- 2.2.4 Slope-intercept form
- 2.2.5 General form
- 2.2.6 Special: Intercept-form

2.3 Parallel lines and Perpendicular lines

- 2.3.1 Point of intersection
- 2.3.2 Parallel lines
- 2.3.3 Perpendicular lines
- 2.3.4 Number of intersections

2.4 Angle of elevation and depression

2.5 Additional content: Point-line distance

By definition, the distance between two points on a \mathbb{R}^2 plane is

Definition (Distance between two points). Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on \mathbb{R}^2 -plane. The distance between A and B is computed by the formula

$$dist(A, B) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Now let L: ax + by + c = 0 be a straight line and $P(x_0, y_0)$ be a point on \mathbb{R}^2 -plane. Note that with the following axiom

Axiom. The distance between a point P and a line L is defined by the shortest distance between P and a point on L

$$dist(P, L) := \inf\{dist(P, Q) : Q \in L\}$$

we can choose the perpendicular displacement of P from L to define the distance. Using the line Γ perpendicular to L passing through P, we can find such Q by following computation:

$$\begin{cases} L : ax + by + c = 0 \\ \Gamma : bx - ay + (ay_0 - bx_0) = 0 \end{cases} \implies Q(\frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2})$$

Therefore, the distance between P an Q is

$$\operatorname{dist}(P,Q) = \frac{1}{a^2 + b^2} \sqrt{[(a^2 + b^2)x_0 - (b^2x_0 - aby_0 - ac)]^2 + [(a^2 + b^2)y_0 - (a^2y_0 - abx_0 - bc)]^2}$$

$$= \frac{1}{a^2 + b^2} \sqrt{(a^2x_0 + aby_0 + ac)^2 + (b^2y_0 + abx_0 + bc)^2}$$

$$= \frac{\sqrt{a^2 + b^2} \sqrt{(ax_0 + by_0 + c)^2}}{a^2 + b^2}$$

$$= \frac{\sqrt{(ax_0 + by_0 + c)^2}}{\sqrt{a^2 + b^2}}$$

$$= \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \right|$$

2.6 Linear inequalities

- 2.6.1 One variable inequality
- 2.6.2 Two variable inequality
- 2.6.3 Linear programming

2.7 Challenging questions

3 Quadratic functions

3.1	Solving Quadratic Equations
3.1.1	The square-root method
3.1.2	General form and Completing the square method
3.1.3	The quadratic formula
3.2	Solvability of Quadratic Equations
3.2.1	Discriminant: the factor affecting solvability
3.2.2	Application of solvability
3.3	Relation between coefficients and roots
3.3.1	What does it mean by a root?
3.3.2	The Vieta formula
3.3.3	Forming quadratic equations with given roots
3.4	Vertex form of a quadratic function
3.4.1	Obtaining vertex form using Transformation
3.4.2	Relation between vertex form and general form
3.4.3	Opening direction
3.4.4	The axis of symmetry
3.4.5	The extremum of quadratic functions
3.5	Quadratic inequalities
3.5.1	Boolean algebra
3.5.2	Solving quadratic inequalities

Challenging questions

3.6

4 Polynomial functions

- 4.1 Arithmetic rules for polynomials
- 4.1.1 Addition, Subtraction and Multiplication
- 4.1.2 Division
- 4.2 Divisibility of polynomials
- 4.2.1 Division algorithm
- 4.2.2 Remainder Theorem
- 4.2.3 Factor Theorem and General Vieta formula
- 4.3 G.C.D. and L.C.M.
- 4.3.1 Greatest common divisor
- 4.3.2 Least common multiple
- 4.4 Positional notation
- 4.4.1 Binary, Octal, Decimal and Hexadecimal
- 4.4.2 Conversion between different bases
- 4.5 Challenging questions

5 Exponential and Logarithmic functions

5.1 Exponential functions

- 5.1.1 Law of Exponential algebra
- 5.1.2 Seeing surds as fractional exponents
- 5.1.3 Solving equations using exponetiation method
- 5.2 Logarithmic functions
- 5.2.1 What is Logarithm?
- 5.2.2 Law of Logarithmic algebra
- 5.2.3 Solving equations using Logarithmic method
- 5.3 Challenging questions

6 Sequence as a function of natural numbers

- 6.1 What is a sequence?
- 6.2 Arithmetic sequence
- 6.2.1 General form
- 6.2.2 Summation of arithmetic sequence
- 6.3 Geometric sequence
- 6.3.1 General form
- 6.3.2 Summation on Geometric sequence
- 6.3.3 Infinite sum of Geometric sequence
- 6.4 Additional content: Arithmetic-Geometric sequence
- 6.5 Challenging questions