

1. **Solution.** Let $P(n)$ be the proposition

$$\sum_{k=1}^n T_k = n[(n+1)!]$$

where $T_n = (n^2 + 1)(n!)$ for any positive integer n .

For $P(1)$:

$$\begin{aligned}\text{L.H.S.} &= \sum_{k=1}^1 T_k \\ &= T_1 \\ &= (1^2 + 1)(1!) \\ &= 2 \\ \text{R.H.S.} &= 1[(1+1)!] \\ &= 2\end{aligned}$$

$\therefore \text{L.H.S.} = \text{R.H.S.}$

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m , i.e.

$$\sum_{k=1}^m T_k = m[(m+1)!]$$

Then for $P(m+1)$:

$$\begin{aligned}\sum_{k=1}^{m+1} T_k &= m[(m+1)!] + [(m+1)^2 + 1][(m+1)!] \\ &= [m^2 + 2m + 2 + m][(m+1)!] \\ &= (m+1)(m+2)[(m+1)!] \\ &= (m+1)[(m+2)!]\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n . ■

2. (a) **Solution.** Let $P(n)$ be the proposition

$$A_n = (-1)^{n-1} B_n$$

where $A_n = \sum_{k=1}^n (-1)^{k-1} k^2$ and $B_n = \frac{n(n+1)}{2}$ for any positive integer n .

For $P(1)$:

$$\begin{aligned}\text{L.H.S.} &= A_1 \\ &= \sum_{k=1}^1 (-1)^{k-1} k^2 \\ &= (-1)^0 1^2 \\ &= (-1)^0 1 \\ &= (-1)^{1-1} \frac{1(1+1)}{2} \\ &= \text{R.H.S.}\end{aligned}$$

$\therefore \text{L.H.S.} = \text{R.H.S.}$

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m , i.e.

$$A_m = (-1)^{m-1} B_m$$

Then for $P(m+1)$:

$$\begin{aligned}A_{m+1} &= \sum_{k=1}^{m+1} (-1)^{k-1} k^2 \\ &= (-1)^{m-1} \frac{m(m+1)}{2} + (-1)^m (m+1)^2 \\ &= (-1)^m (m+1) \left[(m+1) - \frac{m}{2} \right] \\ &= (-1)^m (m+1) \frac{m+2}{2} \\ &= (-1)^m B_{m+1}\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n . ■

(b) **Solution.**

$$\begin{aligned}\sum_{n=1}^{2m} A_n &= \sum_{n=1}^{2m} (-1)^{n-1} B_n \\ &= \sum_{n=1}^m (B_{2n-1} - B_{2n}) \\ &= \sum_{n=1}^m (-2n) \\ &= -2 \sum_{n=1}^m n \\ &= -2B_m \\ &= -m(m+1) \\ \sum_{n=1}^{2m+1} A_n &= -m(m+1) + A_{2m+1} \\ &= -m(m+1) + B_{2m+1} \\ &= -m(m+1) + (m+1)(2m+1) \\ &= (m+1)^2\end{aligned}$$

■

3. **Solution.** By considering the corresponding coefficients, we have

$$\begin{aligned}\lambda_1 &= C_1^8 a = 8a \\ \lambda_2 &= C_2^8 a^2 = 28a^2 \\ \mu_7 &= C_7^9 b^2 = 36b^2 \\ \mu_8 &= C_8^9 b = 9b\end{aligned}$$

That means we have to solve the following system:

$$\begin{cases} \frac{28a^2}{36b^2} = \frac{7}{4} \implies 4a^2 = 9b^2 & \implies 2a = \pm 3b \\ 8a + 9b + 6 = 0 \end{cases}$$

If $2a = 3b$, we have

$$\begin{aligned}12b + 9b + 6 &= 0 \\ b &= -\frac{2}{7}, a = -\frac{3}{7}\end{aligned}$$

If $2a = -3b$, we have

$$-12b + 9b + 6 = 0$$

$$b = -2, a = 3$$

Then $a = -\frac{3}{7}$ or $a = 3$. ■

4. (a) **Solution.** We have the fundamental identities:

$$\sin 2x = 2 \sin x \cos x$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

Then

$$\sin x - \sin 2x + \sin 3x = 0$$

$$\sin x(1 - 2 \cos x + 3 - 4 \sin^2 x) = 0$$

One solution is $\sin x = 0$ which means $x = \pi$. Otherwise,

$$1 - 2 \cos x + 3 - 4 \sin^2 x = 0$$

$$-2 \cos x + 4 \cos^2 x = 0$$

$$-2 \cos x(1 - 2 \cos x) = 0$$

Then, either $\cos x = 0 \implies x = \pi/2, 3\pi/2$ or $\cos x = 1/2 \implies x = \pi/3, 5\pi/3$.

In conclusion, the set of solution to the equation is $\{\pi/3, \pi/2, \pi, 3\pi/2, 5\pi/3\}$. ■

(b) i. **Solution.**

$$f(\theta) = \sin 2\theta + \sin \theta + \cos \theta$$

$$= 2 \sin \theta \cos \theta + \sin \theta + \cos \theta$$

$$= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta + \sin \theta + \cos \theta - 1$$

$$= p^2 + p - 1$$
■

- ii. **Solution.** $f(\theta) = (p + 1/2)^2 - 5/4$. So $f(\theta)$ has the minimum value $-5/4$. Such θ can be computed as follows:

$$\begin{aligned} p + \frac{1}{2} &= 0 \\ p &= -\frac{1}{2} \\ \sin 2\theta - \frac{1}{2} &= -\frac{5}{4} \\ \sin 2\theta &= -\frac{3}{4} \end{aligned}$$

Case 1:

$$\begin{aligned} \tan 2\theta &= \frac{3}{\sqrt{7}} \\ \frac{2 \tan \theta}{1 - \tan^2 \theta} &= \frac{3}{\sqrt{7}} \\ 2\sqrt{7} \tan \theta &= 3 - 3 \tan^2 \theta \\ 3 \tan^2 \theta + 2\sqrt{7} \tan \theta - 3 &= 0 \\ \tan \theta &= \frac{-2\sqrt{7} \pm \sqrt{28 + 36}}{6} \\ &= \frac{-\sqrt{7} \pm 4}{3} \end{aligned}$$

By $\tan \theta < 0$, $\theta = \tan^{-1}(-\frac{4+\sqrt{7}}{3})$.

Case 2:

$$\begin{aligned} \tan 2\theta &= -\frac{3}{\sqrt{7}} \\ \frac{2 \tan \theta}{1 - \tan^2 \theta} &= -\frac{3}{\sqrt{7}} \\ -2\sqrt{7} \tan \theta &= 3 - 3 \tan^2 \theta \\ 3 \tan^2 \theta - 2\sqrt{7} \tan \theta - 3 &= 0 \\ \tan \theta &= \frac{2\sqrt{7} \pm \sqrt{28 + 36}}{6} \\ &= \frac{\sqrt{7} \pm 4}{3} \end{aligned}$$

By $\tan \theta < 0$, $\theta = \tan^{-1}(\frac{\sqrt{7}-4}{3})$.

Hence the conclusion is f attains minimum at $\theta = \tan^{-1}(\frac{-4 \pm \sqrt{7}}{3})$. ■

5. (a) **Solution.**

$$\begin{aligned}
 \tan 4\theta &= \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} \\
 &= \frac{2 \frac{2 \tan \theta}{1 - \tan^2 \theta}}{1 - \left(\frac{2 \tan \theta}{1 - \tan^2 \theta}\right)^2} \\
 &= \frac{4 \tan \theta (1 - \tan^2 \theta)}{(1 - \tan^2 \theta)^2 - 4 \tan^2 \theta} \\
 &= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}
 \end{aligned}$$

Then

$$\begin{aligned}
 \cot 4\theta &= \frac{1}{\tan 4\theta} \\
 &= \frac{1 - 6 \tan^2 \theta + \tan^4 \theta}{4 \tan \theta - 4 \tan^3 \theta} \\
 &= \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}
 \end{aligned}$$

Hence $\cot \theta$ solve the equation $x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$ when $\cot 4\theta = 1$, which is

$$4\theta = n\pi + \pi/4$$

$$\theta = n\pi/4 + \pi/16$$

$$x = \cot(\pi/16), \cot(5\pi/16), \cot(9\pi/16), \cot(13\pi/16)$$

restricting to $[0, 2\pi]$. ■

(b) i. **Solution.**

$$\begin{aligned}
 \frac{1}{2}(2 - a^2 - b^2) &= \frac{1}{2}(2 - \cos^2 \theta + 2 \cos \theta \cos \phi - \cos^2 \phi - \sin^2 \theta + 2 \sin \theta \sin \phi - \sin^2 \phi) \\
 &= \cos \theta \cos \phi + \sin \theta \sin \phi \\
 &= \cos(\theta - \phi)
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{-a}{b} &= \frac{\cos \phi - \cos \theta}{\sin \theta - \sin \phi} \\
 &= \frac{-2 \sin \frac{\phi+\theta}{2} \sin \frac{\phi-\theta}{2}}{2 \cos \frac{\theta+\phi}{2} \sin \frac{\theta-\phi}{2}} \\
 &= \tan \frac{\theta + \phi}{2}
 \end{aligned}$$
■

ii. **Solution.** By i.

$$\begin{aligned}a &= 1, b = \sqrt{3} \\ \implies &\begin{cases} \cos(\theta - \phi) = -1 \\ \tan \frac{\theta + \phi}{2} = -\frac{1}{\sqrt{3}} \end{cases} \\ \implies &\begin{cases} \theta - \phi = 2n\pi + \pi \\ \theta + \phi = 2n\pi - \pi/3 \end{cases} \\ \implies &\theta = \pi/3, \phi = 4\pi/3\end{aligned}$$

■

6. **Solution.** Let $f(x) = (x^2 - 1)e^x$. Then

$$\begin{aligned}f(1+h) &= [(1+h)^2 - 1]e^{1+h} = h(2+h)e^{1+h} \\ f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2+h)e^{1+h}}{h} \\ &= \lim_{h \rightarrow 0} (2+h)e^{1+h} \\ &= 2e\end{aligned}$$

■

7. (a) **Solution.**

$$\begin{aligned}f'(x) &= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) \\ &= 2e^x \cos x \\ f''(x) &= 2e^x \cos x - 2e^x \sin x \\ &= 2e^x(\cos x - \sin x)\end{aligned}$$

■

(b) **Solution.**

$$\begin{aligned}f''(x) - f'(x) + f(x) &= 0 \\ 2e^x(\cos x - \sin x) - 2e^x \cos x + e^x(\sin x + \cos x) &= 0 \\ \cos x - \sin x &= 0 \\ \tan x &= 1 \\ x &= n\pi + \pi/4\end{aligned}$$

where $n \in \mathbb{Z}$ is integer. ■

8. **Solution.**

$$\begin{aligned}\frac{dy}{dx} &= \frac{-6}{(x+1)^2} = -\frac{1}{6} \\ (x+1)^2 &= 36 \\ x &= 5, -7\end{aligned}$$

Then the equation of tangent to C at $(5, 1)$ is

$$\begin{aligned}y - 1 &= -\frac{1}{6}(x - 5) \\ x + 6y - 11 &= 0\end{aligned}$$

Then the equation of tangent to C at $(-7, -1)$ is

$$\begin{aligned}y + 1 &= -\frac{1}{6}(x + 7) \\ x + 6y + 13 &= 0\end{aligned}$$
■

9. (a) i. **Solution.** $x = 4 \sin \theta$. ■

ii. **Solution.** Since $\frac{dx}{dt} = \frac{1}{2}$, we have

$$\begin{aligned}\frac{dx}{dt} &= 4 \cos \theta \frac{d\theta}{dt} \\ \frac{d\theta}{dt} &= \frac{1}{8 \cos \theta}\end{aligned}$$
■

(b) i. **Solution.** Write $y = 4 \cos \theta$ and $z = \sqrt{5^2 - x^2} = \sqrt{25 - 16 \sin^2 \theta}$.

Then

$$\begin{aligned}\frac{dy}{dt} &= -4 \sin \theta \frac{d\theta}{dt} \\ &= -4 \sin \theta \frac{1}{8 \cos \theta} \\ &= -\frac{1}{2} \tan \theta \\ \frac{dz}{dt} &= \frac{-32 \sin \theta \cos \theta}{2\sqrt{25 - 16 \sin^2 \theta}} \frac{1}{8 \cos \theta} \\ &= \frac{-2 \sin \theta}{\sqrt{25 - 16 \sin^2 \theta}}\end{aligned}$$
■

ii. **Solution.**

$$\begin{aligned}
 \frac{dPQ}{dt}\bigg|_{\theta=\pi/6} &= \frac{dy}{dt}\bigg|_{\theta=\pi/6} + \frac{dz}{dt}\bigg|_{\theta=\pi/6} \\
 &= -\frac{1}{2} \tan \pi/3 - \frac{2 \sin \pi/6}{\sqrt{25 - 16 \sin^2 \pi/6}} \\
 &= -\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{25 - 4}} \\
 &= -\frac{\sqrt{3}}{2} - \frac{\sqrt{21}}{21}
 \end{aligned}$$

■

(c) i. **Solution.** Recall area of $\triangle OPR$ is $\frac{xy}{2}$,

$$\begin{aligned}
 \frac{d}{dt} \frac{xy}{2} &= \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right) \\
 &= \frac{1}{2} (-2 \sin \theta \tan \theta + 2 \cos \theta) \\
 &= \cos \theta - \sin \theta \tan \theta
 \end{aligned}$$

Set $\frac{d}{dt} \frac{xy}{2} = 0$, then we have

$$\begin{aligned}
 \tan^2 \theta &= 1 \\
 \tan \theta &= \pm 1 \\
 \theta &= n\pi/2 + \pi/4
 \end{aligned}$$

Since $0 < \theta < \pi/2$, $\theta = \pi/4$.

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ii. **Solution.** Let $\angle OQR = \phi$, then

$$4 \sin \theta = 5 \sin \phi$$

Similar to i., we have to have $\phi = \pi/4$ so that $\triangle ORQ$ is having maximum area. Then

$$\begin{aligned}
 \theta &= \sin^{-1} \left(\frac{5 \sin \pi/4}{4} \right) \\
 &= \sin^{-1} \left(\frac{5\sqrt{2}}{8} \right) \\
 &= 1.08 \text{ rad}
 \end{aligned}$$

■