1. **Solution**. Let $P(n)$ be the proposition

$$\sum_{k=1}^{n} T_k = n[(n+1)!]$$

where $T_n = (n^2+1)(n!)$ for any positive integer n.

For $P(1)$:

$$\begin{aligned}
\text{L.H.S.} &= \sum_{k=1}^{1} T_k \\
&= T_1 \\
&= (1^2+1)(1!) \\
&= 2 \\
\text{R.H.S.} &= 1[(1+1)!] \\
&= 2
\end{aligned}$$

$\therefore$ L.H.S=R.H.S.

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m, i.e.

$$\sum_{k=1}^{m} T_k = m[(m+1)!]$$

Then for $P(m+1)$:

$$\begin{aligned}
\sum_{k=1}^{m+1} T_k &= m[(m+1)!] + [(m+1)^2+1][(m+1)!] \\
&= [m^2+2m+2+m][(m+1)!] \\
&= (m+1)(m+2)[(m+1)!] \\
&= (m+1)[(m+2)!]
\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n. ∎

2. (a) **Solution**. Let $P(n)$ be the proposition

$$A_n = (-1)^{n-1} B_n$$

where $A_n = \sum_{k=1}^{n}(-1)^{k-1}k^2$ and $B_n = \dfrac{n(n+1)}{2}$ for any positive integer n.

For $P(1)$:

$$\begin{aligned}\text{L.H.S.} &= A_1 \\ &= \sum_{k=1}^{1}(-1)^{k-1}k^2 \\ &= (-1)^0 1^2 \\ &= (-1)^0 1 \\ &= (-1)^{1-1}\frac{1(1+1)}{2} \\ &= \text{R.H.S.}\end{aligned}$$

$\therefore$ L.H.S=R.H.S.

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m, i.e.

$$A_m = (-1)^{m-1}B_m$$

Then for $P(m+1)$:

$$\begin{aligned}A_{m+1} &= \sum_{k=1}^{m+1}(-1)^{k-1}k^2 \\ &= (-1)^{m-1}\frac{m(m+1)}{2} + (-1)^m(m+1)^2 \\ &= (-1)^m(m+1)[(m+1) - \frac{m}{2}] \\ &= (-1)^m(m+1)\frac{m+2}{2} \\ &= (-1)^m B_{m+1}\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n. ■

(b) **Solution**.

$$\begin{aligned}\sum_{n=1}^{2m} A_n &= \sum_{n=1}^{2m}(-1)^{n-1}B_n \\ &= \sum_{n=1}^{m}(B_{2n-1}-B_{2n}) \\ &= \sum_{n=1}^{m}(-2n) \\ &= -2\sum_{n=1}^{m} n \\ &= -2B_m \\ &= -m(m+1) \\ \sum_{n=1}^{2m+1} A_n &= -m(m+1)+A_{2m+1} \\ &= -m(m+1)+B_{2m+1} \\ &= -m(m+1)+(m+1)(2m+1) \\ &= (m+1)^2\end{aligned}$$

■

3. **Solution**. By considering the corresponding coefficients, we have

$$\begin{aligned}\lambda_1 &= C_1^8 a = 8a \\ \lambda_2 &= C_2^8 a^2 = 28a^2 \\ \mu_7 &= C_7^9 b^2 = 36b^2 \\ \mu_8 &= C_8^9 b = 9b\end{aligned}$$

That means we have to solve the folloing system:

$$\begin{cases} \frac{28a^2}{36b^2} = \frac{7}{4} \implies 4a^2 = 9b^2 \quad \implies 2a = \pm 3b \\ 8a + 9b + 6 = 0 \end{cases}$$

If $2a = 3b$, we have

$$\begin{aligned}12b + 9b + 6 = 0 \\ b = -\frac{2}{7} a = -\frac{3}{7}\end{aligned}$$

If $2a = -3b$, we have

$$\begin{aligned} -12b + 9b + 6 &= 0 \\ b = -2a &= 3 \end{aligned}$$

Then $a = -\dfrac{3}{7}$ or $a = 3$. ■

4. (a) **Solution**. We have the fundamental identities:

$$\begin{aligned} \sin 2x &= 2\sin x \cos x \\ \sin 3x &= 3\sin x - 4\sin^3 x \end{aligned}$$

Then

$$\begin{aligned} \sin x - \sin 2x + \sin 3x &= 0 \\ \sin x(1 - 2\cos x + 3 - 4\sin^2 x) &= 0 \end{aligned}$$

One solution is $\sin x = 0$ which means $x = \pi$. Otherwise,

$$\begin{aligned} 1 - 2\cos x + 3 - 4\sin^2 x &= 0 \\ -2\cos x + 4\cos^2 x &= 0 \\ -2\cos x(1 - 2\cos x) &= 0 \end{aligned}$$

Then, either $\cos x = 0 \implies x = \pi/2, 3\pi/2$ or $\cos x = 1/2 \implies x = \pi/3, 5\pi/3$.

In conclusion, the set of solution to the equation is $\{\pi/3, \pi/2, \pi, 3\pi/2, 5\pi/3\}$. ■

(b) i.

$$\begin{aligned} f(\theta) &= \sin 2\theta + \sin\theta + \cos\theta \\ &= 2\sin\theta\cos\theta + \sin\theta + \cos\theta \\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \sin\theta + \cos\theta - 1 \\ &= p^2 + p - 1 \end{aligned}$$

ii. $f(\theta) = (p + 1/2)^2 - 5/4$. So $f(\theta)$ has the minimum value $-5/4$. Such θ can be computed as follows: