Abstract

In this pieces of notes, we will go through the concepts related to functions and the meaning of graphs of the functions.

Contents

1 Function and its graph

'Functions describes the world!', one Professor in Mathematics of Massachusetts Institute of Technology (a.k.a. MIT) said that. His speech was greatly influential, as I have never heard such conclusive thinking about functions. In fact, in the past few years, whenever I was studying in schools, my thought about functions is always only about projecting elements from oen set to another set, but what he said had a big impact to my knowledge about functions.

What function talks about, is a subjection of one elements to one another. It can be thought of as a pointing action started from element A to element B, which not so far away, if we could think of subjecting a lot of, a bunch of, or a list of, whatever, objects from one collection to another collection, and they can all be matched under this pointing action, then it is a so-called function.

For example, in an Indian factory, the production of food undergoes many different process. Those can all be called functions. Let say a raw material comes to the factory first, it then be put to a machine to chop into many small pieces. It is the chopping function inside the factory. Next, the chopped material will be put into a pool of yellowish-brownish liquid and be stirred by dirty hands. It is the Mixing function in the factory. After that, the liquid will be drained on the dirty floor and be stepped on by Indian workers so that they can be smell freshed. It is the flavouring function in the factory. Finally, it will be sold to stores, which is the selling function.

Another example is what our body does. We eat and drink, going down the digestive system, and we sit on a toilet. Although we stupid human knows nothing about how the digestive system works, we could still name the conversion from food to poops a digestion, which means the digestive function representing the process in our body.

So we know that function as an english word represents the naming of a process of conversion, it is the time to explore how Math functions works.

1.1 What is a Function?

A function is defined as follow:

Definition 1.1 (Function)**.** *Given an input x and an output y, a **function** is a relation between x and y so that we can write $y = f(x)$ to represent the relationship.*

Essential Practice 1.1.0.1. *Write down functions for the following input-output variables:*

1. *u as input and v as output;*

2. *b as input and a as output;*

3. *n as input and 1 as output (which we call it a constant function);*

4. *x^2 as input and y as output;*

5. *xy as input and z as output;*

6. *2^x as input and k as output;*

7. *$\sqrt{p}$ as input and q as output;*

Remark. *It is notable that we may write functions as $y = g(x)$, $y = h(x)$, $y = d(x)$, $\ldots$ as we want. The 'naming' of a function is always definitive and up to user's construction.*

We can also apply functions after functions. To do so, we have to talk about the following:

Definition 1.2 (Composite functions)**.** *Let f and g be functions such that f takes x as input and y as output, and g takes y as input and z as output. Then we can say there is a function h that takes x as input and z as output. In other words, h is a* ***composite function*** *such that $z = h(x) = g(f(x))$.*

Essential Practice 1.1.0.2. *Let f and g be functions such that $b = f(a), c = g(b)$. Write a function for a as input and c as output.*

1.1.1 Function as an input-output pair

We may now consider how functions carry things to things using arrow notations. We may use a stroked arrow $\mapsto$ to emphasis the carrying process. Let's take a look at the following examples.

Example. *Given a function that undergoes the process of adding one to the given element. We can say*

$$\ldots, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, \ldots$$

In other words, we may write

$$x \mapsto x + 1$$

to generalize the process of adding one to the given element, which is how we usually write to describe any functions.

The example shows that we can generalize the process using algebraic notation, in which it is usually in the form of

$$x \mapsto f(x)$$

with the function f. It is equivalent to say that

$$\ldots, 1 \mapsto f(1), 2 \mapsto f(2), \ldots$$

but what's important is how explicit the mapping process is done. More examples could be viewed to familiarize with it.

Example. *To describe a function undergoes the process 'multiplying the element by two and add one to it', we may examine that*

$$\ldots, 0 \mapsto 1, 1 \mapsto 3, 2 \mapsto 5, \ldots$$

so that it is equivalent to write

$$x \mapsto 2x + 1$$

as a generalization of the function. It is equivalent to write $f(x) := 2x + 1$ to emphasize that we shall call the function f as a naming for the given process, as long as we can write

$$x \mapsto f(x), f(x) := 2x + 1$$

Essential Practice 1.1.1.1. *Examine the following functions from 0 to 3, and generalize the function using algebraic notation. You may either choose writing $x \mapsto \square$ directly or $x \mapsto f(x), f(x) := \square$.*

1. *Multiply the element by 3 and then add 2 to it.*
2. *Divide the element by 10 and then Subtract 7 from it.*
3. *Multiply the element by a and then add b to it.*
4. *Squaring the element.*
5. *Multiply 3 to the square of the element, an then add 5 to it.*
6. *Add 1 to the element first, then multiply the square of the result by 6, and then add 1 to it.*
7. *Subtract 8 from the element first, then divide the square of the result by 2, and then add 4 to it.*
8. *Subtract h from the element first, then multiply the square of the result by a, and then add k to it.*

It is notable that a function can only give one output for each input, which makes the next page a fruitful discussion.

1.1.2 Defining a function with variables

So far, we have learned how writing a generalization of a function is, and have we used algebraic notation to shorten the examination, one suggest we can always write algebraic notation for a function definition. In addition, we also find that a function cannot have more than one output, as we see functions as a pointing process from one element to another element. So we have the following definition:

Definition 1.3 (Function)**.** *A **function** f **of** x defines the pointing process $x \mapsto f(x)$ is a one-to-one pointing process, which can have only one output.*

It is important to note that $f(x)$ is a function of x if and only if one x produce one $f(x)$. The x is called a *dummy variable*, which is a variable that can be changed all the time. For example, writing $f(y)$ or $f(z)$ still makes sense to say a function f, but not of x.

From now on, we can determine whether a given relation is a function or not.

Example. *Given the relation $y = mx + c$. Since one x can produce only one y, y is a function of x; on the other hand, we also see one y produces only one x, so x is also a function of y.*

Example. *Given the relation $y = x^2$. Since one x can produce only one y, y is a function of x; however, one y may produce more than one x, say if $y = 4$ then x can be 2 or -2, so x is not a function of y.*

Essential Practice 1.1.2.1. *Determine whether the following given relation between x and y is a function of one another or not. Provide counterexample if it is not a function.*

1. $y = -x$;
2. $y = 4x + 3$;
3. $y = \frac{1}{x}$;
4. $y = \frac{x}{6}$;
5. $y^2 = x$;
6. $y^2 = x^2$;
7. $y^3 = 4x^2 - 3$.

1.1.3 Domain, Co-domain and Range

For explicit definition of a function, we need the following concepts to help with: *domain*, *co-domain* and *range*.

A **domain** is where the input comes from, which is usually half-customized and half-restricted. For example, $f(x) = \frac{1}{x}$ can have input of negative real numbers, positive real numbers, any complex numbers except 0. This means that 0 is naturally restricted by the operation of $\frac{1}{x}$, but other than 0, we can choose freely our input from all complex numbers. Thus, the largest domain of $f(x) = \frac{1}{x}$ is all complex numbers except 0. However, it is not saying that the domain of $f(x) = \frac{1}{x}$ must be all complex numbers except 0, we can still put restrictions on our own, what means by customize, like all real numbers except 0 or all positive real numbers except 0 as its domain, is still a possible choice. Hence, we shall usually talk about the *greatest possible domain of a function* if we need to find the natural restrictions, and the *domain of a function* if we are going to define our source of input.

Essential Practice 1.1.3.1. *Find the greatest possible domain of the following functions if (i) the output is restricted to complex numbers and (ii) the output is restricted to be real numbers:*

1. $f(x) := x$;
2. $f(x) := ax + b$;
3. $f(x) := \frac{a}{x}$;
4. $f(x) := x^2$;
5. $f(x) := a(x - h)^2 + k$;
6. $f(x) := \sqrt{x}$;
7. $f(x) := \sqrt[3]{x}$;
8. $f(x) := \frac{1}{\sqrt{x}}$;

A **co-domain** is where the output can go to. It is more likely a limitation of the output of the function so that we know where our target is. Similarly, we shall usually talk about the *greatest possible co-domain of the function* if we need to find the natural restrictions, and the *domain of the function* if we are going to define our target output.

Essential Practice 1.1.3.2. *Find the greatest possible co-domain of the following functions if the input is unrestricted:*

1. *$f(x) := x$;*
2. *$f(x) := ax + b$;*
3. *$f(x) := \frac{a}{x}$;*
4. *$f(x) := x^2$;*
5. *$f(x) := a(x-h)^2 + k$;*
6. *$f(x) := \sqrt{x}$;*
7. *$f(x) := \sqrt[3]{x}$;*
8. *$f(x) := \frac{1}{\sqrt{x}}$;*

With domain and co-domain, we can now define a function in a more explicit manner. Writing a function with where the inputs are in and where the outputs to go, we have a nice notation - an arrow $\rightarrow$ emphasizing the direction from domain to co-domain. In general, we will write

$$f : D \rightarrow R$$

to specify the function f is a function goes from domain D to co-domain R. The formal way to define a function is like below:

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)$$

which reads 'a function f sending a real number x to a real number $f(x)$'.

Example. *For a function sending a natural number n to a natural number 2^n, we may write its definition as*

$$f : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto 2^n$$

Example. *For a function sending an integer n to a rational number 2^n, we may write its definition as*

$$f : \mathbb{Z} \rightarrow \mathbb{Q}, n \mapsto 2^n$$

We shall see both example shows the same function process $f(n) := 2^n$ but different domain and codomain. Therefore, we acknowledge that although both functions are having the same process, they are indeed representing different things, which yields they are in fact different functions.

From this point of view, we can further examine the so-called **range** of a function, which is the exact target region of the function under the codomain. We shall define some set notation to present its meaning.

Definition 1.4 (Union and Intersection)**.** *Let A and B be sets. The* ***union*** *of A and B is defined as*

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

while the ***intersection*** *of A and B is defined as*

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

In fact, we say union is the collection of the objects which are either in set A or set B, or simply say it is the joined set of two sets. We can take a look at the following examples:

Example. *Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5\}$, then*

$$A \cup B = \{1, 2, 3, 4, 5\}$$

Example. *Let $A = \{1, 3, 5, 7, 9, \dots\}$ be the set of all positive odd numbers, $B = \{2, 4, 6, 8, 10, \dots\}$ be the set of positive even numbers, then*

$$A \cup B = \{1, 2, 3, 4, 5, 6 \dots\}$$

which is the set of all positive numbers. Sometimes, we may write the set by specifying the property of the elements as following:

$$A \cup B = \{x : x \text{ is a positive integer}\}$$

For intersection, it is generally speaking the collection of repeated elements in both sets, or we can say the sharing elements. We can take a look at the following examples:

Example. *Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5\}$, then*

$$A \cap B = \{3, 4\}$$

Example. *Let $A = \{1, 3, 5, 7, 9, \dots\}$ be the set of all positive odd numbers, $B = \{2, 4, 6, 8, 10, \dots\}$ be the set of positive even numbers, then*

$$A \cap B = \emptyset$$

which is the set with no element, an empty set.

We may represent the two definitions by shading regions in a Venn-diagram. Suppose we are calling a set by enclosing the region by a circle, then we have the following representation.

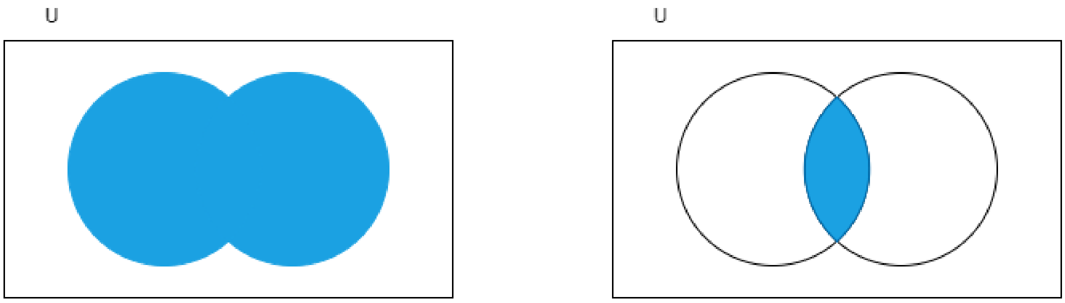


Figure 1: Union(Left) and Intersection(Right) of two sets

Essential Practice 1.1.3.3. *Let* $A = \{1, 3, 4, 6, 7\}, B = \{3, 4, 5, 7, 8\}$*, then find* $A \cup B$ *and* $A \cap B$*.*

Essential Practice 1.1.3.4. *Prove the following identities:*

1. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$;
2. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$.

Definition 1.5 (Range)**.** *The* ***range*** *of a function* $f : D \to R$*, denoted by* $\mathbf{Ran}(f)$*, is the set* $f(D) \cap R$*, where* $f(D) := \{f(x) : x \in D\}$*.*

Usually, a teacher in high school aims to discuss the range of a function rather than the co-domain of a function, as what he shall teach is the size of the output, but not where the output shall be in. However, the difference between the co-domain of a function and the range of a function is we can further define the concept of a well-defined function with the concept of range.

Example. *For a function* f *defined as*

$$f : \mathbb{N} \to \mathbb{N}, n \mapsto 2^n$$

The range of f*, denoted by* $\mathrm{Ran}(f)$*, is the set of all possible outcomes of* $f(n)$ *with natural numbers* n*. That is, the set*

$$\{2^n : n \in \mathbb{N}\} = \{2, 4, 8, 16, 32, \dots\}$$

Example. *For a function* f *defined as*

$$f : \mathbb{Z} \to \mathbb{Q}, n \mapsto 2^n$$

$\mathrm{Ran}(f)$ *is the set of all possible outcomes of* $f(n)$ *with integers* n*. That is, the set*

$$\{2^n : n \in \mathbb{Z}\} = \{\dots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, \dots\}$$

We must observe that the two ranges of the same function process f are different when their co-domains are different. This shows the importance of discussion of co-domain when we are defining ranges of functions.

Essential Practice 1.1.3.5. *Find the range of the following functions:*

1. $f : \mathbb{N} \to \mathbb{N}, n \mapsto n$;
2. $f : \mathbb{N} \to \mathbb{N}, n \mapsto 3^n$;
3. $f : \mathbb{N} \to \mathbb{N}, n \mapsto 3^n - n^3$;
4. $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto n$;
5. $f : \mathbb{Z} \to \mathbb{Z}, n \mapsto n^2$;
6. $f : \mathbb{Z} \to \mathbb{Q}, n \mapsto 5^n$;
7. $f : \mathbb{Q} \to \mathbb{Q}, n \mapsto n$;
8. $f : \mathbb{Z} \to \mathbb{Q}, n \mapsto n/2$;
9. $f : \mathbb{N} \to \mathbb{Q}, n \mapsto n/10$;
10. $f : \mathbb{Z} \to \mathbb{R}, n \mapsto \pi n^2$;

Essential Practice 1.1.3.6. *Are the functions in previous practice all well-defined? Which of them are not?*

1.2 Graph of a function

Graphing has been an essential skill in understanding mathematical objects. We usually say it is a mathematical modeling technique. Through graphing, we can see the relationship between the input and output clearly, whether they are related, increasing or decreasing. It is also easier to draw conclusion to estimations by valid graphs. This section aims to build the concept of representing functions by graphs.

1.2.1 Blobs-and-arrows diagram for discrete functions

We shall first take a step backward to a simpler function - a *discrete function* with finite inputs. This will help the construction very much.

Definition 1.6 (Discrete functions)**.** *A* ***discrete function*** *is a function with direct indication of the function process for each element in domain. That is, for each element $x \in D$, the output $f(x)$ is assigned to co-domain directly.*

A discrete function is usually with a discrete domain, and random assignment.

Example. *Let $D := \{1, 2, 3, 4, 5\}$ be the domain of a function f. Suppose f is defined by*

$$f(x) := \begin{cases} 1 & , x = 1 \\ 4 & , x = 2 \\ 3 & , x = 3 \\ 2 & , x = 4 \\ 5 & , x = 5 \end{cases}$$

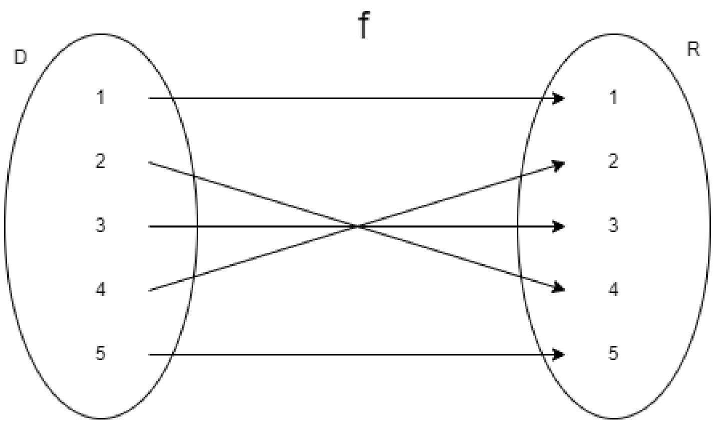
In this case, f is a discrete function.

To represent the above function using a graph, it is recommended to use a *blobs-and-arrows* diagram.

Definition 1.7 (Blobs-and-arrows diagram)**.** *A* ***blobs-and-arrows*** *diagram is a diagram representing a function by denoting the domain and co-domain by two circles, and for each element in the circle of domain, a pointer arrow over-set with f pointing to one of the element in the co-domain, to emphasize the relation between the two elements are input and output pair.*

It is undesired to read through such complicated definition of the diagram. Let's see the following example.

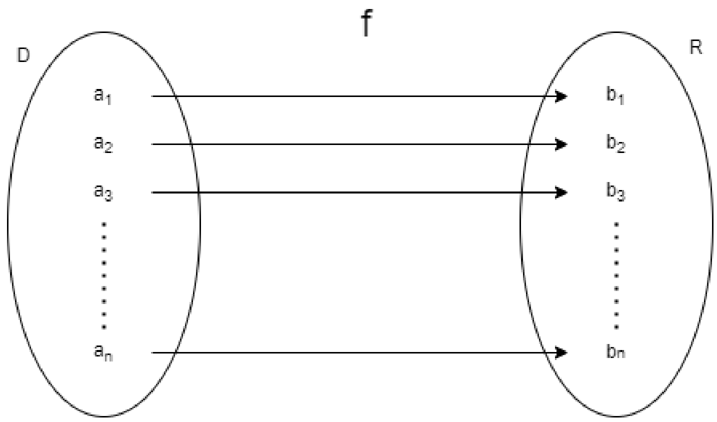
Example. *Recall the function in previous example, we could draw the blobs-and-arrows diagram as shown.*



In the figure above, the name of the function is hung over the main diagram, while the left ellipse denote the set of elements of domain D and the right ellipse denote the range of elements of codomain R. The letter R could also be interpreted as the range of f.

For a discrete function with finitely many inputs, where we will take them as a sequence, it can also be used to represent it efficiently. We just need to have some modification.

Axiom. *Let $f : D \to R$ be a function. Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of numbers in D and $b_1, b_2, b_3, \ldots, b_n$ be a sequence of numbers in R. Suppose $b_1 = f(a_1), b_2 = f(a_2), \ldots, b_n = f(a_n)$ defines the one-to-one correspodence from D to R. Then the following blobs-and-arrows diagram describes f formally:*



Essential Practice 1.2.1.1. *Draw the blobs-and-arrows diagram for the following discrete functions:*

1. $f(1) = 3, f(2) = 6, f(3) = 7, f(4) = 2, f(5) = 3.$
2. $f(x) := \begin{cases} 1 & , x = 1,2,3,4,5 \\ 2 & , x = 6,7 \\ 3 & , x = 8,9,10 \\ 4 & , x = 11,12,13,14,15 \\ 5 & , x = 16,17,18,19 \\ 6 & , x = 20 \end{cases}$

1.2.2 The pairing table and xy-coordination

To extend our concept of function from discrete version to continuous version, that is, infinitely many numbers can be plugged into the function for calculation, we may need to 'fill up' the holes of the domain.

Let say we have 1 and 2 in the domain D, then all the real numbers within 1 and 2 also in D. This is the process of **continuation** or **extension**, and the whole set of real numbers within 1 and 2 is called the **interval** between 1 and 2. We can denote the interval by $[1, 2]$ if 1 and 2 are included in the meaning, or $(1, 2)$ if 1 and 2 are excluded from the meaning. To better distinct the difference between $[1, 2]$ and $(1, 2)$, we call the previous one the **closed interval** between 1 and 2, while the latter one the **open interval** between 1 and 2.

From this point of view, a real number line is used to represent the concept of any intervals. Let's define the real number line and discuss intervals using a number line.

Definition 1.8 (Real number line)**.** *A* ***real number line*** *is a straight line with ordered property so that one of its direction is increasing, and the other direction is decreasing.*

We usually draw a horizontal real number line for convenience.

For the sake of simplicity and connected-understanding of open and closed intervals, we may put the desired parenthesis in the real number line to denote the meaning. Following that a closed interval is in a block-typed while open interval is in a round-typed, we can directly place them on the real number line.

Example. *The open interval* $(0, 1)$.

Example. *The open interval* $(-1.5, 5.5)$.

Example. *The closed interval* $[0.5, 1]$.

Example. *The closed interval* $[-100, -67]$.

Essential Practice 1.2.2.1. *Draw the mentioned interval on the provided real number line with suitable parenthesis.*

1. $[0, 1]$.
2. $[\alpha, \beta]$.
3. $(0, 1)$.
4. (α, β).

It is of course we could draw it vertically with the same meaning, by indicating upward as positive direction and downward as negative direction. Consider a function $y = f(x)$ defines the relation between x and y coordinate, we can combine two real number lines in an orthogonal way, like a cross, to present the coordination in a sensible way. This is called the **continuous graph of a function**, or we simply call it the *graph of a function* if there's no ambiguity of what we are discussing.

In order to draw the graph of a function, we need to figure out the valid pairs of x and y first, so that the points could be addressed correctly for the function. We will make use of a **pairing table** so that each value of x in the domain is paired with a unique value of y in the co-domain. We will show its use with some examples.

Example. *Let G be the graph of $y = x + 1$. Then we have the table of testing xy-pair*

x	0	1	2	3	4
$y = x + 1$	1	2	3	4	5

and thus the graph of G be like

by linking up the points with straight lines.

Example. *Let G be the graph of $y = x^2$. Then we have the table of testing xy-pair*

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4

and thus the graph of G be like

by linking up the points with straight lines. Moreover, by increasing the points of testing value to, say, 100 or 1000, a see-smoothing curve can be drawn by computer.

It is the continuation of the curve under testing value control.

Even higher power polynomial functions can be drawn using this strategy.

Example. *Let G be the graph of $y = x^3 + 2x^2 + x - 1$. Then we have the table of testing xy-pair*

x	0	1	2	3	4
$y = x^3 + 2x^2 + x - 1$	-1	3	17	47	99

and thus the graph of G be like, with many enough testing points,

with the continuation.

Essential Practice 1.2.2.2. *Configure the xy-table with testing values in between -3 to 3 and draw a suitable graph for the following functions.*

1. *$y = x - 3$;*

2. *$y = x^2 + x + 1$;*

3. *$y = (x - 3)^3 + 2$.*

You may also check the graph using WolframAlpha desktop to convince yourself with the plot. Try to conclude the difference between large amount and small amount of points of testing.

With the xy-coordination of a graph, many features and prediction could be done. We will try to analyze the properties that we could make use of in the next subsection.

1.3 Fundamental properties of a graph

A graph shows the relationship between two variables, namely the vertical component (in this case, it is the value of y) and the horizontal component (respectively, the value of x). We will make use of this relationship and consider different situations.

1.3.1 points lying on the graph

Let us reverse the sight of plotting a graph of a function. We plot the graph by linking up the points we get from inputting x into the function to get the corresponding value of y, so that each point (x, y) describes when the function has the input x. We could now see that if a point (x, y) is passed through by a curve then it is the same as the point lies on that curve. We shall say the point (x, y) **satisfies** the equation $y = f(x)$, since this is the definition of a graph.

Theorem. *For any real-valued continuous functions f, any coordinates (x, y) lies on the graph $y = f(x)$ if and only if the xy-pair satisfies the equation $y = f(x)$.*

Example. *For a given function $f(x) = 3x + 1$, the following are true:*

1. *The point $(0, 1)$ lies on $y = f(x)$;*

2. *The point $(3, 10)$ lies on $y = f(x)$;*

3. *The point $(-2, 1)$ does not lay on $y = f(x)$;*

4. *The point $(-10, 2)$ does not lay on $y = f(x)$.*

Example. *For a given function $f(x) = 3x^3 + 2x^2 + x + 1$, the following are true:*

1. *The point $(0, 1)$ lies on $y = f(x)$;*

2. *The point* $(-1,-1)$ *lies on* $y = f(x)$*;*

3. *The point* $(-2,0)$ *does not lay on* $y = f(x)$*;*

4. *The point* $(-10,2)$ *does not lay on* $y = f(x)$*.*

Essential Practice 1.3.1.1. *For a given function* $f(x) = x^2 - 1$*, show whether the following points are lying on the graph of* $y = f(x)$*:*

1. *The point* $(0,1)$*;*

2. *The point* $(-1,0)$*;*

3. *The point* $(-2,3)$*;*

4. *The set of points* $(-t, t^2 - 1)$ *parametrized by the variable* t*.*

We may now examine another feature of a graph, namely **the region of inequality**. In the previous paragraph we find a point satisfying the equation $y = f(x)$ has the same meaning as the point lies on the curve. If we make a small translation of the y coordinate to the upper region separate by the curve, it becomes that all the points have a larger value of y than we could calculate by the function. For that region we will call it the region for $y > f(x)$; Similarly, if we make a small translation of the y coordinate to the lower region, it becomes that all the points in that region have a smaller value of y than we could calculate by the function. For that region, we will call it the region for $y < f(x)$.

Example. *For a given function* $f(x) = 3x + 1$*, the following are true:*

1. *The point* $(0,2)$ *lies above* $y = f(x)$*;*

2. *The point* $(3,9)$ *lies below* $y = f(x)$*;*

3. *The point* $(-2,1)$ *lies above* $y = f(x)$*;*

4. *The point* $(-10,-120)$ *lies below* $y = f(x)$*.*

Example. *For a given function* $f(x) = 3x^3 + 2x^2 + x + 1$*, the following are true:*

1. *The point* $(0,3)$ *lies above* $y = f(x)$*;*

2. *The point* $(-1,-2)$ *lies below* $y = f(x)$*;*

3. *The point* $(-2,0)$ *lies above* $y = f(x)$*;*

4. The point $(10, 2)$ *lies below* $y = f(x)$.

Essential Practice 1.3.1.2. *For a given function* $f(x) = x^2 - 1$*, determine the position of the following points with respect to the graph of* $y = f(x)$*:*

1. The point $(0, 9)$*;*

2. The point $(-1, -1)$*;*

3. The point $(-2, 20)$*;*

4. The point $(10, 10)$.

1.3.2 Special intersections: axis intercepts

Usually the most important element of a graph is whether it cuts the coordinating axis, namely the y-axis and the x-axis. For practical reason we value them so much and we may give them special names: the y-intercept and the x-intercept.

Definition 1.9 (y-intercept of a graph)**.** *Given a function* f*, the function has its* ***y-intercept*** *when its cuts the y-axis. In other words, the y-intercept is defined as*

$$y_0 := f(0)$$

whenever the function is well-defined at $x = 0$.

Similarly, we may want the same definition for x-intercept of a graph, however, we do not know whether the inverse function exists or not. This comes to some kind of difficulties to say directly the x-intercept. But still, we can give the possible condition for the discussion.

Definition 1.10 (x-intercept of a graph)**.** *Given a function* f*, the function has its* ***x-intercept*** *when its cuts the x-axis. In other words, the x-intercept is defined as the roots of*

$$0 = f(x)$$

whenever the given equation is solvable.

We will be familiar with them with some valid examples.

Example. *For the graph of the function* $f(x) := x + 1$*, we have:*

- *y-intercept:* $y_0 = 0 + 1 = 1$*;*

- *x-intercept: Set $0 = x + 1$, then $x_0 = -1$ is the root to the given equation, which is the x-intercept of the function.*

Example. *For the graph of the function $f(x) := x^2 - 1$, we have:*

- *y-intercept: $y_0 = 0^2 - 1 = -1$;*
- *x-intercept: Set $0 = x^2 - 1$, then $x_1 = -1$ or $x_2 = 1$ are the roots to the given equation, which are the x-intercepts of the function.*

Example. *For the graph of the function $f(x) := x^2 + 1$, we have:*

- *y-intercept: $y_0 = 0^2 + 1 = 1$;*
- *x-intercept: Set $0 = x^2 + 1$, which has no real solution. Then the graph of the function has no x-intercepts.*

We see a function has its y-intercept most of the time, but not the same case for x-intercepts. It could have either no x-intercepts, only 1 x-intercept, or more than one x-intercept. It is a consequence of the fundamental theorem of algebra, but we will dig deeper in the future to discuss simple cases we could.

As long as we could plot not only the xy-coordinate but we could pair up something weird, where they could be different functions say $v(y)$ and $u(x)$, the graphs could no longer talked about y-intercept or x-intercept - they are not exactly about x and y! Hence, we turned the naming into something general: we plot the graph by vertical axis and horizontal axis, so let us call them the **vertical intercept** and the **horizontal intercept** respectively.

1.4 Solving equations using graphs of functions

1.4.1 Homogeneous equations

1.4.2 Non-homogeneous equations

1.4.3 Simultaneous equations

1.5 Transformation of functions

1.5.1 Translation

1.5.2 Dilation

1.6 Challenging questions

2 Linear functions

In this section, we focus on a specific form of function, which is called **Linear functions**, a type of functions that generate straight lines in the Euclidean space, or we may call it the xy-plane, denoted by $\mathbb{R}^2$. It is the simplest form of function, and has many properties we are interested in.

2.1 Fundamental concepts of points

Revisiting the xy-plane, we may define some useful tools for measurement and ratio properties, since we shall always be equipped with the sense of measuring and performing comparisons.

2.1.1 Distance between two points

One important practical measurement in the xy-plane is the so-called **distance between two points**. We recall that given a right-angled triangle $\triangle ABC$ with $\angle ABC = 90°$, the following relation on the side lengths holds:

$$|\overline{AB}|^2 + |\overline{BC}|^2 = |\overline{AC}|^2$$

where the stroke sign $|\cdot|$ denotes the length of a certain line and each line is mentioned explicitly using the over-lined notation $\overline{XY}$ to indicate the line connecting point X and point Y. As we learned the relation is mentioned by *Pythagoras Theorem* or *Pythagorean Theorem.*

Following from the Pythagoras Theorem, we shall examine the distance between any two points on the xy-plane by the length of the straight line connecting them. With the help of the following figure, let us carry out our definition of distance between two points.

Definition 2.1 (Distance between two points)**.** *Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on $\mathbb{R}^2$-plane. The distance between A and B is computed by the formula*

$$\operatorname{dist}(A, B) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We may think of the following construction for distance between two points:

Taking the right-angled triangle as distance reference, the vertical displacement becomes $|y_1 - y_2|$ and the horizontal displacement becomes $|x_1 - x_2|$. Putting the oblique displacement $\operatorname{dist}(A, B)$ gives the Pythagorean relation, and we could get the desired formula.

Example. *Let $A(1, 2)$ and $B(4, -2)$ be two points on the rectangular coordinate plane, i.e. the xy-plane. Then,*

$$\operatorname{dist}(A, B) = \sqrt{(1-4)^2 + (2-(-2))^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Example. *Let $A(1,2)$ and $B(k,k-1)$ be two points on the rectangular coordinate plane such that* $\operatorname{dist}(A,B)=2$. *Then,*

$$\begin{aligned}
\sqrt{(k-1)^2+(k-3)^2} &= 2\\
(k-1)^2+(k-3)^2 &= 4\\
k^2-2k+1+k^2-6k+9 &= 4\\
2k^2-8k+6 &= 0\\
k^2-4k+3 &= 0\\
(k-1)(k-3) &= 0
\end{aligned}$$

Then $k=1$ or $k=3$, i.e. the possible coordinates of B are $(1,0)$ or $(3,2)$.

Example. *Given two trajectories $T_1:(x_1(t),y_1(t))$ and $T_2:(x_2(t),y_2(t))$, we say the distance between T_1 and T_2 is*

$$\operatorname{dist}(T_1,T_2) := \min_{t>0}\sqrt{(x_1(t)-x_2(t))^2+(y_1(t)-y_2(t))^2}$$

It is quite abstract to talk about the distance between two trajectories in this stage. Just keep in mind that this is a truth, or say you may believe in it.

Essential Practice 2.1.1.1. *Compute the distance between following pair of points:*

1. $(1,0)$ *and* $(0,1)$;
2. $(1,-1)$ *and* $(-1,1)$;
3. $(3,4)$ *and* $(12,-13)$;
4. $(7,-8)$ *and* $(-3,0)$;

Essential Practice 2.1.1.2. *Given the distance between two points, find the possible value(s) of the unknowns:*

1. $A(0,0), B(3,k)$; $\operatorname{dist}(A,B)=3$.
2. $A(2,0), B(3,k)$; $\operatorname{dist}(A,B)=1$.
3. $A(k,0), B(0,k)$; $\operatorname{dist}(A,B)=5$.
4. $A(3+k,1-k), B(3,-3)$; $\operatorname{dist}(A,B)=8$.

2.1.2 Mid-point and Division points

Definition 2.2 (Mid-way of a trajectory)**.** *Given a trajectory defined by $(x(t), y(t))$ where $t \in [0, r]$ with $r > 0$ a real number, the **midway** (or **half-way**) of the trajectory is given by the point M such that*

$$\text{arclength}((x(0), y(0)), M) = \text{arclength}(M, (x(r), y(r)))$$

In particular, if we are talking about a mid-point, we usually pick the shortest trajectory between the two points. This is due to the uniqueness of the shortest trajectory if our distance is well-defined. In the Euclidean space, the shortest trajectory is usually the straight line between the two points.

Theorem (Mid-point of two points)**.** *Given $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the Euclidean space, the **mid-point** M of A and B is defined by*

$$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Proof. Let L be the straight line connecting A and B. Assume that $M := (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$, then

$$\begin{aligned}
\text{dist}(A, M) &= \sqrt{(x_1 - \frac{x_1 + x_2}{2})^2 + (y_1 - \frac{y_1 + y_2}{2})^2} \\
&= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \frac{1}{2}\text{dist}(A, B) \\
\text{dist}(M, B) &= \sqrt{(\frac{x_1 + x_2}{2} - x_2)^2 + (\frac{y_1 + y_2}{2} - x_2)^2} \\
&= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \frac{1}{2}\text{dist}(A, B)
\end{aligned}$$

In particular,

$$\text{dist}(A, M) + \text{dist}(M, B) = \text{dist}(A, B)$$

Thus M preserve a point on the straight line connecting A and B and in fact the mid-point of A and B. □

Example. *Given $A(100, 20)$ and $B(30, 70)$, their mid-point is the point $M(65, 45)$.*

Essential Practice 2.1.2.1. *For the following given pairs of points, find their mid-points:*

1. $(2,2)$ *and* $(0,0)$*;*

2. $(2,1)$ *and* $(-2,-1)$*.*

Other than the mid-point, we also consider the general version of line separation, which is the point of division.

Definition 2.3 (Point of division)**.** *Given two points A and B, any point P lies in between A and B is called **the point of division with ratio** $\operatorname{dist}(A,P) : \operatorname{dist}(P,B)$.*

Theorem (Formula for point of division)**.** *Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be a point on the Euclidean space. Let P be the point of division of A and B with ratio $r : s$. Then the coordinates of P is given by*

$$\frac{sA + rB}{r+s} = (\frac{sx_1 + rx_2}{r+s}, \frac{sy_1 + ry_2}{r+s})$$

In particular, the mid-point of A and B is given by setting $s = r = 1$.

Proof. Similar to the proof of mid-point. Let $P := (\frac{sx_1 + rx_2}{r+s}, \frac{sy_1 + ry_2}{r+s})$, then

$$\begin{aligned}
\operatorname{dist}(A,P) &= \sqrt{(x_1 - \frac{sx_1 + rx_2}{r+s})^2 + (y_1 - \frac{sy_1 + ry_2}{r+s})^2} \\
&= \frac{1}{r+s}\sqrt{(rx_1 - rx_2)^2 + (ry_1 - ry_2)^2} \\
&= \frac{r}{r+s}\operatorname{dist}(A,B) \\
\operatorname{dist}(P,B) &= \sqrt{(\frac{sx_1 + rx_2}{r+s} - x_2)^2 + (\frac{sy_1 + ry_2}{r+s} - x_2)^2} \\
&= \frac{1}{r+s}\sqrt{(sx_1 - sx_2)^2 + (sy_1 - sy_2)^2} \\
&= \frac{s}{r+s}\operatorname{dist}(A,B)
\end{aligned}$$

In particular,

$$\operatorname{dist}(A,P) + \operatorname{dist}(P,B) = \operatorname{dist}(A,B)$$

Thus P preserve a point on the straight line connecting A and B and satisfies $\operatorname{dist}(A,P) : \operatorname{dist}(P,B) = \frac{r}{r+s} : \frac{s}{r+s} = r : s$. ☐

Example. *Let $A(2,4)$ and $B(-4,7)$. To trisect the line connecting A and B, we need two points on the line where they divide the straight line in the ratio of $1:1:1$. In particular, let the point* *closer to* A *be* X *and the other be* Y*, we have* X *the division point of ratio* $1:2$ *and* Y *the division point of ratio* $2:1$*. Therefore,*

$$X := (\frac{2(2)+1(-4)}{1+2}, \frac{2(4)+1(7)}{1+2}) = (0,5)$$
$$Y := (\frac{1(2)+2(-4)}{1+2}, \frac{1(4)+2(7)}{1+2}) = (-2,6)$$

Essential Practice 2.1.2.2. *Given* $A(-100,-10)$ *and* $B(100,100)$*. Find the point of division for the following given ratio* $r:s$ *:*

1. $r:s = 1:1$;
2. $r:s = 1:3$;
3. $r:s = 3:7$;
4. $r:s = 19:91$.

2.1.3 Slope of two points

For two points, we want to know not only how far away they are, but also how steep they see each other, i.e. the direction of proceeding. We call it the slope of the two points.

To be professional, we say a slope of the two points is how line goes from the left point to the right point; that is, whether y increase with x or y decrease with x.

Definition 2.4 (Slope)**.** *Let* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *be two points on the Euclidean space. Then the* ***slope*** *of* A *and* B*, denoted by* $\mathfrak{M}_{AB}$*, is given by*

$$\mathfrak{M}_{AB} := \frac{y_2 - y_1}{x_2 - x_1}$$

Example. *The slope of* $A(1,2)$ *and* $B(4,1)$ *is* $\mathfrak{M}_{AB} = \frac{1-2}{4-1} = -\frac{1}{3}$.

We could observe the following fact. For if two points are aligned horizontally on the plane, then their slope is computed equal to 0; if the right point is higher than the left, then the slope is positive, meaning the corresponding line is increasing with x; otherwise the slope is negative, meaning the corresponding line is decreasing with x. For if two points are aligned vertically on the plane, then the slope is undefined, since we could not explicitly tell whether the corresponding line is increasing or decreasing - it could be both. Due to the uncertainty, we identify it as undefined slope. However, we can still define the slope of vertical lines for consistent usage, but that's another page of discussion.

Essential Practice 2.1.3.1. *Compute the slope of the following pairs of points:*

1. $(1, 0)$ *and* $(0, 1)$*;*

2. $(1, -1)$ *and* $(-1, 1)$*;*

3. $(3, 4)$ *and* $(12, -13)$*;*

4. $(7, -8)$ *and* $(-3, 0)$*;*

Definition 2.5 (Collinear)**.** *3 points are called* ***collinear*** *if there is a straight line passes through 3 points at the same time.*

In fact, collinearity is somehow the converse of computing points of division, by the definition of point of division. We will make use of the equivalence in proving the following theorem:

Theorem. *3 points on the plane are collinear if and only if the slope of any two of the three points are the same.*

Proof. For the direction from collinear to slope equality, we have if 3 points are collinear, then they can be written as

$$A(x_1, x_2), B(\frac{sx_1 + rx_2}{r + s}, \frac{sy_1 + ry_2}{r + s}), C(x_2, y_2)$$

We then have

$$\begin{aligned}
\mathfrak{M}_{AB} &= \frac{\frac{sy_1+ry_2}{r+s} - y_1}{\frac{sx_1+rx_2}{r+s} - x_1} \\
&= \frac{sy_1 + ry_2 - (r + s)y_1}{sx_1 + rx_2 - (r + s)x_1} \\
&= \frac{ry_2 - ry_1}{rx_2 - rx_1} \\
&= \frac{y_2 - y_1}{x_2 - x_1} \\
\mathfrak{M}_{BC} &= \frac{y_2 - \frac{sy_1+ry_2}{r+s}}{x_2 - \frac{sx_1+rx_2}{r+s}} \\
&= \frac{(r + s)y_2 - sy_1 + ry_2}{(r + s)x_2 - sx_1 + rx_2} \\
&= \frac{sy_2 - sy_1}{sx_2 - sx_1} \\
&= \frac{y_2 - y_1}{x_2 - x_1} \\
\mathfrak{M}_{AC} &= \frac{y_2 - y_1}{x_2 - x_1}
\end{aligned}$$

Then collinear implies the equality of the slopes.

Conversely, suppose 3 points A, B, C such that $\mathfrak{M}_{AB} = \mathfrak{M}_{BC} = \mathfrak{M}_{AC} = m$. We may compute that

$$\begin{aligned}\operatorname{dist}(A, B) &= \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} \\ &= |x_A - x_B|\sqrt{1 + \mathfrak{M}_{AB}^2}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\operatorname{dist}(B, C) &= |x_B - x_C|\sqrt{1 + \mathfrak{M}_{BC}^2} \\ \operatorname{dist}(A, C) &= |x_A - x_C|\sqrt{1 + \mathfrak{M}_{AC}^2}\end{aligned}$$

Without loss of generality, we may assume their x-coordinates are in fixed order, where I will set $x_A < x_B < x_C$. Then

$$\begin{aligned}\operatorname{dist}(A, B) + \operatorname{dist}(B, C) &= (|x_A - x_B| + |x_B - x_C|)\sqrt{1 + m^2} \\ &= (x_B - x_A + x_C - x_B)\sqrt{1 + m^2} \\ &= |x_A - x_C|\sqrt{1 + \mathfrak{M}_{AC}^2} \\ &= \operatorname{dist}(A, C)\end{aligned}$$

This shows that B is a point of division of A and C, and hence A, B, C are collinear. □

This measurement provides another way of line construction. In fact, by one of its powerful property, there is a generalized version of slope with various application in higher mathematics and other fields including computer science.

Theorem (Invariant properties of slope of two points)**.** *Let A and B be two points on the Euclidean space, and $\mathfrak{M}_{AB}$ be the slope of A and B. Under any simultaneous translation to A and B, $\mathfrak{M}_{AB}$ remains unchanged.*

Proof. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points. Suppose A' and B' be the resulting coordinates after simultaneously translating A and B, we may write $A'(x_1 + \Delta x, y_1 + \Delta y)$ and $B'(x_2 + \Delta x, y_2 + \Delta y)$. Then the slope after simultaneous translation is

$$\mathfrak{M}_{A'B'} = \frac{(y_2 + \Delta y) - (y_1 + \Delta y)}{(x_2 + \Delta x) - (x_1 + \Delta x)} = \frac{y_2 - y_1}{x_2 - x_1} = \mathfrak{M}_{AB}$$

□

2.2 Different forms of a linear function

2.2.1 Fundamental construction of a line

2.2.2 Two-point form

2.2.3 Point-slope form

2.2.4 Slope-intercept form

2.2.5 General form

2.2.6 Special: Intercept-form

2.3 Parallel lines and Perpendicular lines

2.3.1 Point of intersection

2.3.2 Parallel lines

2.3.3 Perpendicular lines

2.3.4 Number of intersections

2.4 Angle of elevation and depression

2.5 Additional content: Point-line distance

By definition, the distance between two points on a $\mathbb{R}^2$ plane is

Definition 2.6 (Distance between two points). *Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on $\mathbb{R}^2$-plane. The distance between A and B is computed by the formula*

$$\operatorname{dist}(A, B) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Now let $L : ax + by + c = 0$ be a straight line and $P(x_0, y_0)$ be a point on $\mathbb{R}^2$-plane. Note that with the following axiom

Axiom. *The distance between a point P and a line L is defined by the shortest distance between P and a point on L*

$$\operatorname{dist}(P, L) := \inf\{\operatorname{dist}(P, Q) : Q \in L\}$$

we can choose the perpendicular displacement of P from L to define the distance. Using the line Γ perpendicular to L passing through P, we can find such Q by following computation:

$$\begin{cases} L : ax + by + c = 0 \\ \Gamma : bx - ay + (ay_0 - bx_0) = 0 \end{cases} \implies Q(\frac{b^2x_0 - aby_0 - ac}{a^2 + b^2}, \frac{a^2y_0 - abx_0 - bc}{a^2 + b^2})$$

Therefore, the distance between P an Q is

$$\begin{aligned}
\mathrm{dist}(P, Q) &= \frac{1}{a^2 + b^2}\sqrt{[(a^2 + b^2)x_0 - (b^2x_0 - aby_0 - ac)]^2 + [(a^2 + b^2)y_0 - (a^2y_0 - abx_0 - bc)]^2} \\
&= \frac{1}{a^2 + b^2}\sqrt{(a^2x_0 + aby_0 + ac)^2 + (b^2y_0 + abx_0 + bc)^2} \\
&= \frac{\sqrt{a^2 + b^2}\sqrt{(ax_0 + by_0 + c)^2}}{a^2 + b^2} \\
&= \frac{\sqrt{(ax_0 + by_0 + c)^2}}{\sqrt{a^2 + b^2}} \\
&= \left|\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\right|
\end{aligned}$$

2.6 Linear inequalities

2.6.1 One variable inequality

2.6.2 Two variable inequality

2.6.3 Linear programming

2.7 Challenging questions

3 Quadratic functions

3.1 Solving Quadratic Equations

3.1.1 The square-root method

3.1.2 General form and Completing the square method

3.1.3 The quadratic formula

3.2 Solvability of Quadratic Equations

3.2.1 Discriminant: the factor affecting solvability

3.2.2 Application of solvability

3.3 Relation between coefficients and roots

3.3.1 What does it mean by a root?

3.3.2 The Vieta formula

3.3.3 Forming quadratic equations with given roots

3.4 Vertex form of a quadratic function

3.4.1 Obtaining vertex form using Transformation

3.4.2 Relation between vertex form and general form

3.4.3 Opening direction

3.4.4 The axis of symmetry

3.4.5 The extremum of quadratic functions

3.5 Quadratic inequalities

3.5.1 Boolean algebra

3.5.2 Solving quadratic inequalities

3.6 Challenging questions

4 Polynomial functions

4.1 Arithmetic rules for polynomials

4.1.1 Addition, Subtraction and Multiplication

4.1.2 Division

4.2 Divisibility of polynomials

4.2.1 Division algorithm

4.2.2 Remainder Theorem

4.2.3 Factor Theorem and General Vieta formula

4.3 G.C.D. and L.C.M.

4.3.1 Greatest Common Divisor

4.3.2 Least Common Multiple

4.4 Rational Function

4.4.1 Simplification of Rational Functions

4.4.2 Extended: Partial Fractions

4.5 Positional notation

4.5.1 Binary, Octal, Decimal and Hexadecimal

4.5.2 Conversion between different bases

4.6 Challenging questions

5 Exponential and Logarithmic functions

5.1 Exponential functions

5.1.1 Law of Exponential algebra

5.1.2 Seeing surds as fractional exponents

5.1.3 Solving equations using exponetiation method

5.2 Logarithmic functions

5.2.1 What is Logarithm?

5.2.2 Law of Logarithmic algebra

5.2.3 Solving equations using Logarithmic method

5.3 Challenging questions

6 Sequence as a function of natural numbers

6.1 What is a sequence?

6.2 Arithmetic sequence

6.2.1 General form

6.2.2 Summation of arithmetic sequence

6.3 Geometric sequence

6.3.1 General form

6.3.2 Summation on Geometric sequence

6.3.3 Infinite sum of Geometric sequence

6.4 Additional content: Arithmetic-Geometric sequence

6.5 Challenging questions