Chapter II

APPLICATIONS TO GEOMETRY

15. *Introduction.* This chapter contains a treatment by vector methods of various elementary topics in geometry. This treatment is included in the present volume for two reasons. First, it indicates the ease and power which vector methods lend to studies in geometry. Secondly, it affords the student an opportunity to gain additional skill in the use of the vector operations introduced in the previous chapter.

Proofs of some well-known theorems of plane geometry will first be given. Then a fairly broad treatment of solid analytic geometry will be presented, in which some of the more familiar formulas will be deduced in vector form and by vector methods. Finally, the differential geometry of curves in space will be considered briefly. For more complete treatments of analytic geometry and differential geometry by vector methods, the reader is referred to the excellent books by F. D. Murnaghan,[1] W. C. Graustein[2] and C. E. Weatherburn.[3]

16. *Some theorems of plane geometry.* In this section we shall consider proofs by vector methods of two well-known theorems of plane geometry.

Theorem 1. The diagonals of a parallelogram bisect each other.

Proof. Let us consider the parallelogram $OABC$ in Figure 21. The diagonals cut each other at the point D. We must prove that D bisects both of the line segments OB and AC.

For convenience we denote the vectors drawn from O to the points

[1] F. D. Murnaghan, Analytic geometry, Prentice-Hall, New York, 1946.

[2] W. C. Graustein, Differential geometry, The Macmillan Company, New York, 1935.

[3] C. E. Weatherburn, Differential geometry of three dimensions, Cambridge University Press, Cambridge, England. Vol. 1, 1927. Vol. 2, 1930.

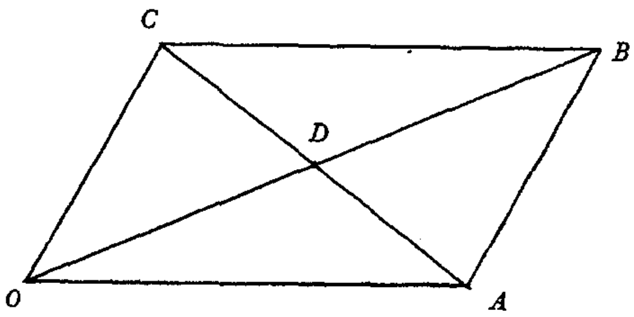

Figure 21

A, B, C and D by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, respectively. Since D lies on the line OB, there exists a scalar u such that

$$\mathbf{d} = u\mathbf{b}. \tag{16.1}$$

Also, from the figure we see that

$$\mathbf{d} = \overline{OA} + \overline{AD}. \tag{16.2}$$

Since D lies on the line AC there exists a scalar v such that

$$\overline{AD} = v\overline{AC}.$$

Hence we can write (16.2) in the form

$$\mathbf{d} = \mathbf{a} + v\overline{AC}. \tag{16.3}$$

We now equate the above two expressions given for $\mathbf{d}$ in (16.1) and (16.3), obtaining

$$\mathbf{a} + v\overline{AC} = u\,\mathbf{b}. \tag{16.4}$$

The next step is to express all vectors in this equation as linear functions of any two vectors in the plane, say $\mathbf{a}$ and $\mathbf{c}$. From the figure we see that

$$\overline{AC} = -\mathbf{a} + \mathbf{c}, \qquad \mathbf{b} = \mathbf{a} + \mathbf{c},$$

whence (16.4) becomes

$$\mathbf{a} + v\,(-\mathbf{a} + \mathbf{c}) = u\,(\mathbf{a} + \mathbf{c}),$$

or

$$(1 - u - v)\,\mathbf{a} = (u - v)\,\mathbf{c}.$$

Since $\mathbf{a}$ and $\mathbf{c}$ do not have the same line of action it then follows that

$$1-u-v=0, \qquad u-v=0.$$

We solve these equations for u and v, obtaining $u=v=\frac{1}{2}$. Thus Equation (16.1) becomes $\mathbf{d}=\frac{1}{2}\,\mathbf{b}$, from which it follows that D is the middle point of OB.

We have now proved that the point of intersection D of the diagonals is the middle point of one of these diagonals. From symmetry, D must also be the middle point of the other diagonal.

Theorem 2. The medians of a triangle meet in a single point which trisects each of them.

Proof. Let us consider the triangle OAB in Figure 22. The points

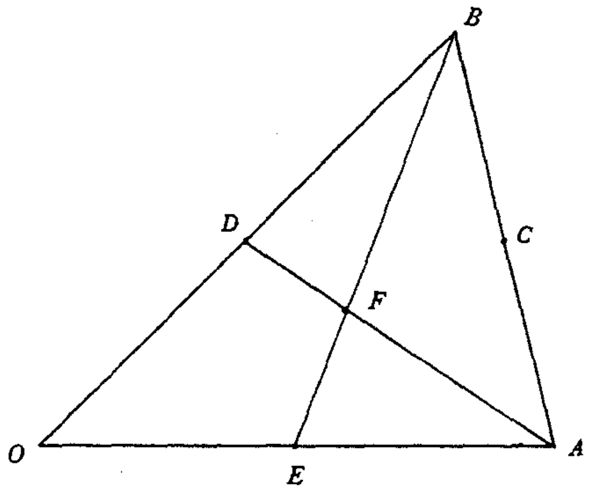

Figure 22

C, D and E are the middle points of the sides, and F is the point of intersection of the medians AD and BE. We must prove that F is a point of trisection of each of the three medians AD, BE and OC.

For convenience we denote the vectors drawn from O to the points A, B, C, D, E and F by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$, respectively. Now F lies on the median AD. Thus there exists a scalar u such that

$$\overline{DF}=u\overline{DA} \tag{16.5}$$

and we then have

$$\mathbf{f}=\overline{OD}+u\overline{DA}.$$

Similarly, since F lies on the median BE, we have

$$\mathbf{f} = \overline{OE} + v\overline{EB},$$

where v is some scalar. We now equate these two expressions for $\mathbf{f}$, obtaining

$$\overline{OD} + u\overline{DA} = \overline{OE} + v\overline{EB}. \tag{16.6}$$

The next step is to express all vectors in this equation as linear functions of any two vectors in the plane, say $\mathbf{a}$ and $\mathbf{b}$. From Figure 22 we see that

$$\begin{aligned} \overline{OD} &= \tfrac{1}{2}\,\mathbf{b}, & \overline{OE} &= \tfrac{1}{2}\,\mathbf{a}, \\ \overline{DA} &= -\tfrac{1}{2}\,\mathbf{b}+\mathbf{a}, & \overline{EB} &= -\tfrac{1}{2}\,\mathbf{a}+\mathbf{b}. \end{aligned} \tag{16.7}$$

Thus (16.6) becomes, after substitution from (16.7) and collection of like terms,

$$(\tfrac{1}{2} - u - \tfrac{1}{2}\,v)\mathbf{a} = (\tfrac{1}{2} - \tfrac{1}{2}\,u - v)\mathbf{b}.$$

Since $\mathbf{a}$ and $\mathbf{b}$ do not have the same line of action, it follows that

$$\tfrac{1}{2} - u - \tfrac{1}{2}v = 0, \qquad \tfrac{1}{2} - \tfrac{1}{2}u - v = 0.$$

We solve these equations for u and v, obtaining $u = v = \frac{1}{3}$. Thus Equation (16.5) becomes $\overline{DF} = \frac{1}{3}\,\overline{DA}$, from which it follows that F is a point of trisection of the median DA.

We have now proved that the point of intersection F of two medians is a point of trisection of one of these medians. From symmetry, F must also be a point of trisection of the other of these medians. From symmetry F must also be a point of trisection of the third median.

Solid Analytic Geometry

17. *Notation.* We shall now consider the analytic geometry of points in space. A right-handed set of rectangular cartesian coordinates is introduced, with origin at a point 0. Just as in Chapter I, we denote these coordinates by the symbols x_1, x_2 and x_3, as shown in Figure 23. Unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ pointing in the directions of the positive coordinate axes are also introduced, as shown.

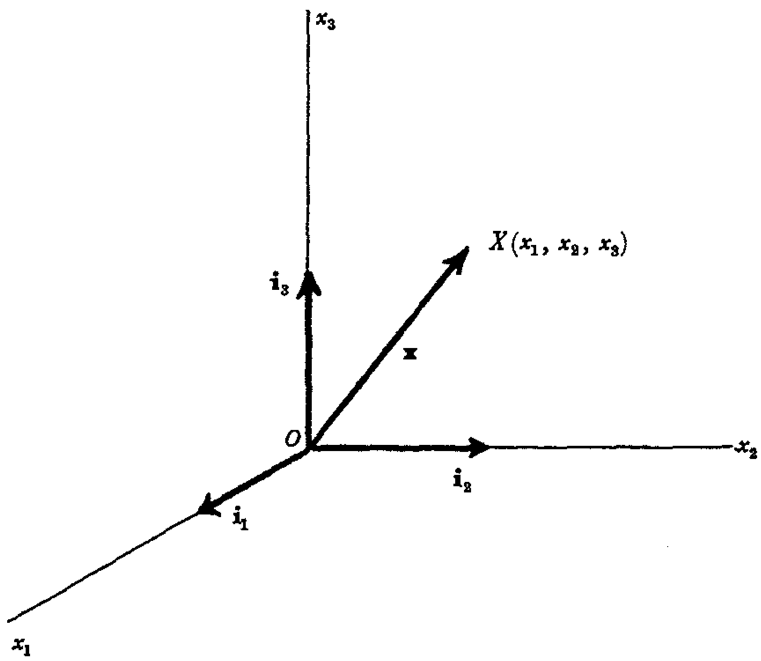

Figure 23

To denote a general point in space we shall use the letter X, and we shall call the vector $\overline{OX}$ the *position-vector* of X. For convenience we shall denote this vector also by the symbol $\mathbf{x}$, and its components by the symbols (x_1, x_2, x_3). We then have

$$\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3.$$

The quantities x_1, x_2 and x_3 are also the coordinates of the point X.

We shall use the letters $A, B, C, \cdots$ to denote specific points in space, and shall denote the position-vectors of these points by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$. The component of these vectors will be denoted in the usual way by the symbols (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), $\cdots$. We note that these quantities are also the coordinates of the points $A, B, C, \cdots$.

18. *Division of a line segment in a given ratio.* Let us suppose that we are given two points A and B, and that it is desired to find a third point C which divides the line segment AB in the given ratio m to n. Figure 24 illustrates the problem. If C lies between A and B, then $0 < m/n < +\infty$; if C lies beyond B then $-\infty < m/n < -1$; if C lies beyond A, then $-1 < m/n < 0$. In any event we have

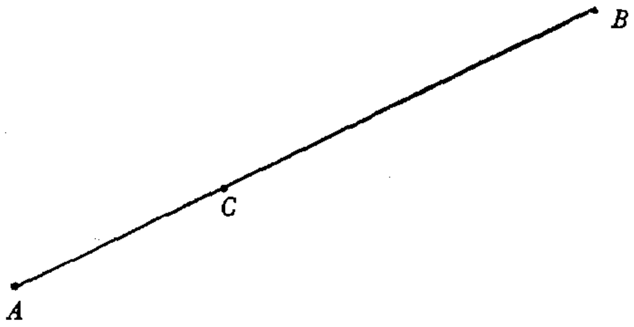

Figure 24

$$\frac{\overline{AC}}{m} = \frac{\overline{CB}}{n}. \tag{18.1}$$

If we now denote the position-vectors of A, B and C by **a**, **b** and **c**, respectively, then

$$\overline{AC} = \mathbf{c} - \mathbf{a}, \qquad \overline{CB} = \mathbf{b} - \mathbf{c},$$

and (18.1) can then be written in the form

$$n(\mathbf{c} - \mathbf{a}) = m(\mathbf{b} - \mathbf{c}).$$

Solving this equation for **c**, we obtain

$$\mathbf{c} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}. \tag{18.2}$$

This formula expresses the position-vector **c** of the desired point C in terms of the known quantities **a**, **b**, m and n.

In books on Analytic Geometry, formulas are usually given which express the coordinates of C in terms of m, n and the coordinates of A and B. It should be noted that (18.2) is entirely equivalent to these formulas, for these formulas can be deduced from (18.2) simply by equating the components of the left side of (18.2) to the components of the right side of (18.2).

19. *The distance between two points.* Let us suppose that A and B are two given points, and that it is desired to find the distance d between A and B in terms of the position-vectors **a** and **b** of A and B. Figure 25 illustrates the problem. Now

$$d = |\overline{AB}| \, .$$

But $\overline{AB} = \mathbf{b} - \mathbf{a}$. Thus

$$d^2 = |\overline{AB}|^2 = \overline{AB} \cdot \overline{AB} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

whence

$$d = \sqrt{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})}.$$

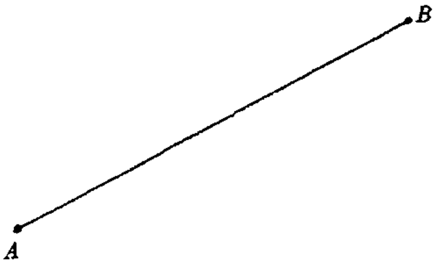

Figure 25

20. *The area of a triangle.* Let us suppose that A, B and C are three given points, and that it is desired to find the area Δ_{abc} of the triangle ABC in terms of the position-vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of A, B and C. Figure 26 illustraties the problem.

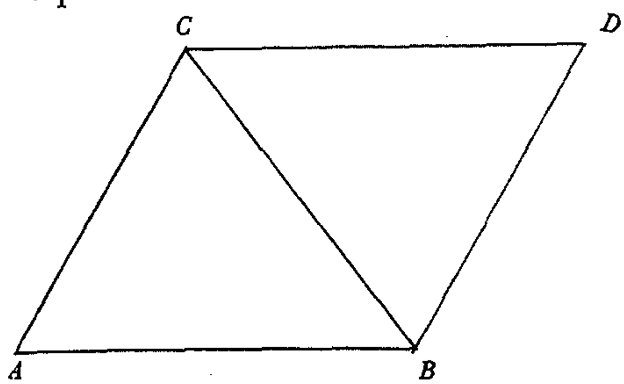

Figure 26

We first construct the parallelogram of which AB and AC form two adjacent edges, as shown. Then Δ_{abc} is equal to one half the area of this parallelogram. But, by Theorem 1 of § 8, the area of this parallelogram is $|\overline{AB} \times \overline{AC}|$. Hence we can write

(20.1) $$\Delta_{abc} = \tfrac{1}{2}\varphi,$$

where

$$\varphi = \overline{AB} \times \overline{AC}. \tag{20.2}$$

Now $\overline{AB} = \mathbf{b} - \mathbf{a}$, $\overline{AC} = \mathbf{c} - \mathbf{a}$. Thus

$$\boldsymbol{\varphi} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}).$$

This simplifies to

$$\boldsymbol{\varphi} = \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}. \tag{20.3}$$

The required area of the triangle is thus given by (20.1), φ being the magnitude of the vector given by (20.3).

A property of the vector $\boldsymbol{\varphi}$ will now be recorded, for future use. Since this vector is equal to $\overline{AB} \times \overline{AC}$ we conclude that *the vector* $\boldsymbol{\varphi}$ *is perpendicular to the plane of the triangle ABC, and its direction is that indicated by the thumb of the right hand when the fingers are placed to indicate the direction of the passage around the triangle from A to B to C.*

21. *The equation of a plane.* There are several ways in which a plane can be specified. For example, three points which are on the plane and do not lie on a single straight line can be given, or a line in the plane and a point on the plane but not on the line can be given. In each of several such cases we shall now deduce the equation which must be satisfied by the position-vector $\mathbf{x}$ of every point X on the plane. This equation will be referred to simply as the *equation of the plane.* In books on analytic geometry the equation of a plane usually appears as an equation which involves scalars only, and is satisfied only by the coordinates of points on the plane. We shall refer to this latter equation as the *cartesian form of the equation of the plane.*

(i) *To find the equation of the plane through a given point and perpendicular to a given vector.* Let A be the given point and $\mathbf{b}$ be the given vector. Figure 27 illustrates the problem, the plane P being the plane in question.

Let X be a general point on P, and let $\mathbf{a}$ and $\mathbf{x}$ denote the position-vectors of A and X, respectively. Now $\overline{AX}$ is perpendicular to $\mathbf{b}$. Thus

$$\overline{AX} \cdot \mathbf{b} = 0.$$

But $\overline{AX} = \mathbf{x} - \mathbf{a}$, whence it follows that

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0. \tag{21.1}$$

This is the desired equation of the plane P.

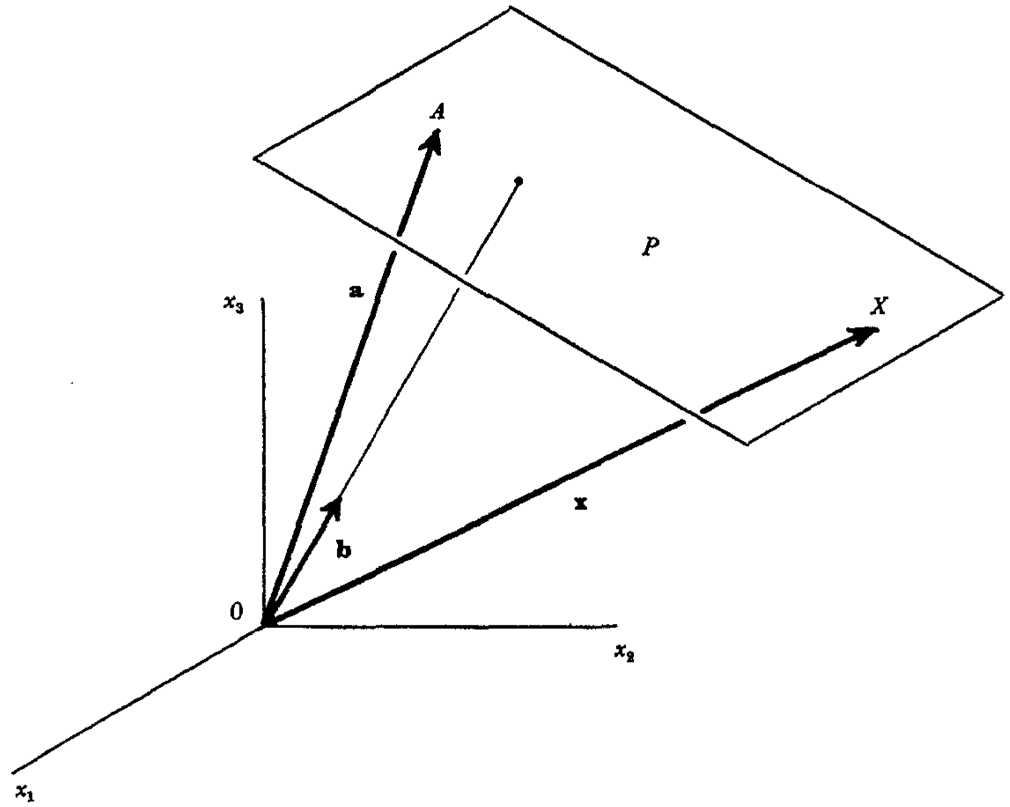

Figure 27

The cartesian form of the equation of the plane P can be obtained readily from (21.1). It is only necessary to express (21.1) in terms of the components of the vectors involved. In this way we obtain the equation

$$(x_1 - a_1)\, b_1 + (x_2 - a_2)\, b_2 + (x_3 - a_3)\, b_3 = 0.$$

(ii) *To find the equation of the plane through three given points.* Let A, B and C be three given points. It is desired to find the equation of the plane P containing these three points. Figure 28 illustrates the problem.

Let X be a general point on the plane P, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$ denote the position-vectors of the points A, B, C and X, respectively. In § 20 we saw that the vector $\boldsymbol{\varphi}$ given by the relation

$$\boldsymbol{\varphi} = \mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$$

is perpendicular to the plane P. Hence, by Problem (i) above the equation of P is

$$(\mathbf{x}-\mathbf{a})\cdot\boldsymbol{\varphi} = 0. \tag{21.2}$$

Since $\mathbf{a}\cdot\boldsymbol{\varphi} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$,

Equation (21.2) can be written in the equivalent form

$$\mathbf{x}\cdot\boldsymbol{\varphi} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}). \tag{21.3}$$

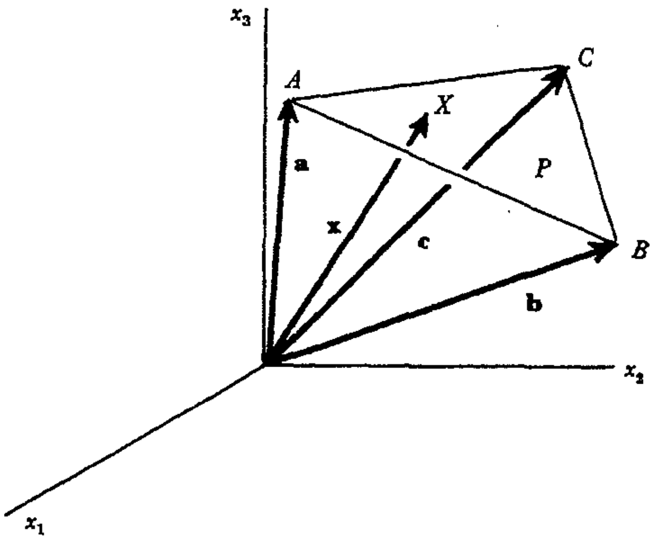

Figure 28

22. *The vector-perpendicular from a point to a plane.* The vector-perpendicular from a point D to a plane P is the vector with origin at D and terminus at the point on P nearest D.

(i) *To find the vector-perpendicular from a point D to a plane P through a given point and perpendicular to a given vector.* Let A be the given point and let $\mathbf{b}$ be the given vector. Figure 29 illustrates the problem. We denote the position-vectors of the points A and D by $\mathbf{a}$ and $\mathbf{d}$, respectively. If the point E is the foot of the perpendicular from the point D to the plane P, then $\overline{DE}$ is the vector-perpendicular from the point D to the plane P. We shall denote it by the symbol $\mathbf{p}$. It is required to find $\mathbf{p}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

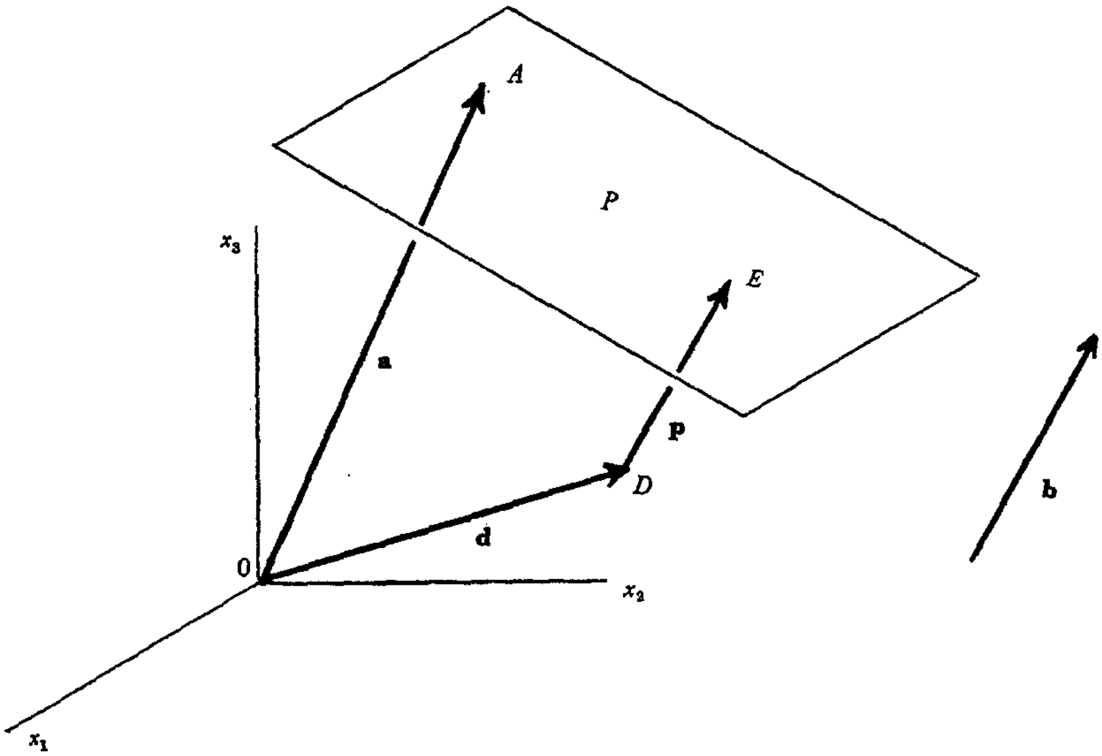

Figure 29

Now $\mathbf{p}$ and $\mathbf{b}$ are parallel. Thus there exists a scalar constant K such that

$$\mathbf{p} = K\mathbf{b}. \tag{22.1}$$

From the figure we see that

$$\overline{OD}+\overline{DE}+\overline{EA}+\overline{AO} = 0,$$

or

$$\mathbf{d}+K\mathbf{b}+\overline{EA}-\mathbf{a} = 0. \tag{22.2}$$

Now $\mathbf{b}$ is perpendicular to $\overline{EA}$. Hence $\mathbf{b}\cdot\overline{EA} = 0$, and so scalar multiplication of (22.2) by $\mathbf{b}$ yields

$$(\mathbf{d}-\mathbf{a})\cdot\mathbf{b}+Kb^2 = 0. \tag{22.3}$$

Thus

$$K = \frac{(\mathbf{a}-\mathbf{d})\cdot\mathbf{b}}{b^2},$$

and substitution for K in Equation (22.1) then yields

$$\mathbf{p} = \frac{(\mathbf{a}-\mathbf{d})\cdot\mathbf{b}}{b^2}\,\mathbf{b}\,. \tag{22.4}$$

In particular, if the point D is at the origin, then $\mathbf{d} = 0$ and

$$\mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{b^2}\,\mathbf{b}\,. \tag{22.5}$$

(ii) *To find the vector-perpendicular from a point D to a plane P through three given points.* Let A, B and C be the three given points, with position-vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. Figure 30 illustrates the problem. It is desired to find the vector-perpendicular $\mathbf{p}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

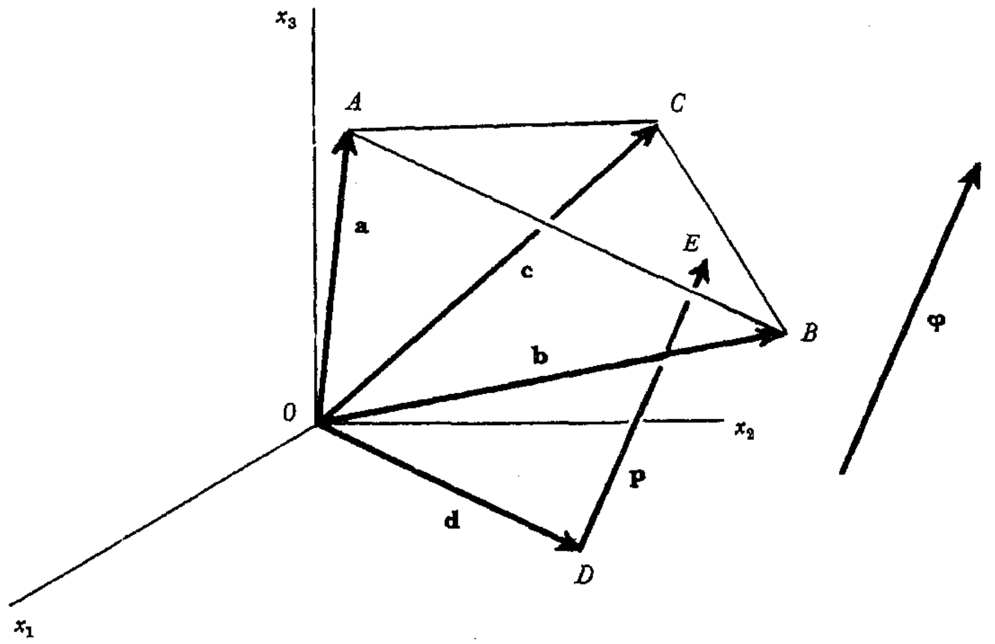

Figure 30

In § 20 we saw that the vector $\boldsymbol{\varphi}$ given by the relation

$$\boldsymbol{\varphi} = \mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$$

is perpendicular to the plane P. Hence we may regard P as the plane through the given point A and perpendicular to the given vector $\boldsymbol{\varphi}$. Thus, from Equation (22.4) it follows that the required vector-perpendicular is given by the relation

$$\mathbf{p} = \frac{(\mathbf{a}-\mathbf{d})\cdot\boldsymbol{\varphi}}{\varphi^2}\,\boldsymbol{\varphi}.$$

Since $\mathbf{a}\cdot\boldsymbol{\varphi} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ it follows that

$$\mathbf{p} = \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) - \mathbf{d}\cdot\boldsymbol{\varphi}}{\varphi^2}\,\boldsymbol{\varphi}. \tag{22.6}$$

23. *The equation of a line.* There are several ways in which a line in space can be specified. For example, two points on the line can be given, or two planes through the line can be given. In each of several such cases we shall now deduce the equation which must be satisfied by the position-vector $\mathbf{x}$ of every point X on the line. This equation will be referred to simply as the *equation of the line.*

(i) *To find the equation of the line through a given point and parallel to a given vector.* Let A be the given point, with position-vector $\mathbf{a}$, and let $\mathbf{b}$ be the given vector. Figure 31 illustrates the problem, L being the line in question.

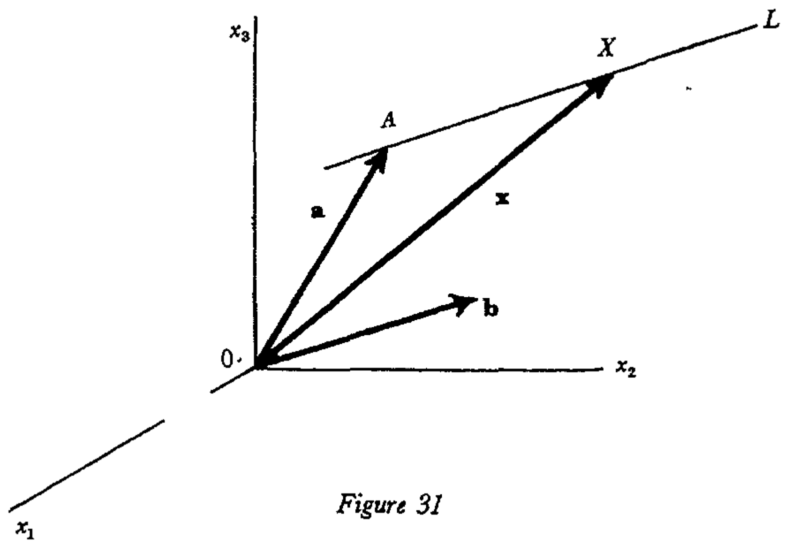

Figure 31

Let X be a general point on L, and let $\mathbf{x}$ denote the position-vector of X. Now $\overline{AX}$ is parallel to $\mathbf{b}$. Thus

$$\overline{AX} \times \mathbf{b} = 0.$$

But $\overline{AX} = \mathbf{x} - \mathbf{a}$, whence it follows that

$$(23.1) \qquad (\mathbf{x} - \mathbf{a}) \times \mathbf{b} = 0\,.$$

This is the desired equation of the line L.

(ii) *To find the equation of the line through two given points.* Let A and B be the given points, with position-vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Figure 32

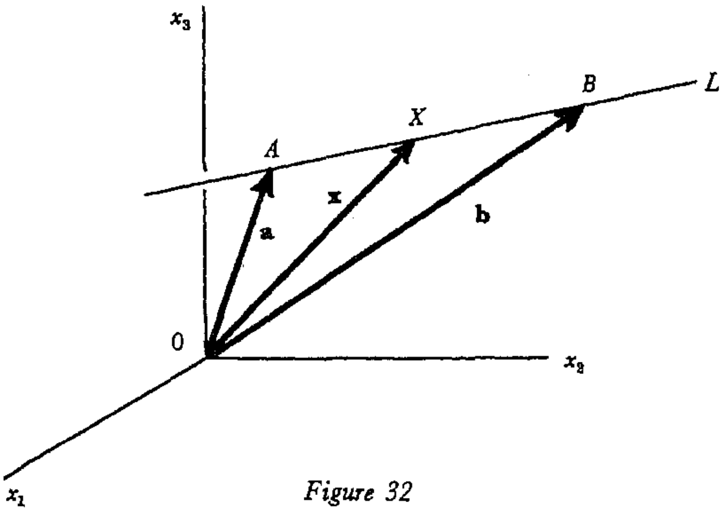

Figure 32

illustrates the problem, L being the line in question. Now L is parallel to the vector $\overline{AB}$, and $\overline{AB} = \mathbf{b} - \mathbf{a}$. Thus, by Problem (i) above, the desired equation of L is

$$(23.2) \qquad (\mathbf{x} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = 0\,.$$

(iii) *To find the equation of the line through a given point and perpendicular to two given vectors.* Let A be the given point with position-vector $\mathbf{a}$, and let $\mathbf{b}$ and $\mathbf{c}$ be the given vectors. Figure 33 illustrates the problem, L being the line in question. Now L is parallel to the vector $\mathbf{b} \times \mathbf{c}$. Hence, by Problem (i) above, the desired equation of L is

$$(23.3) \qquad (\mathbf{x} - \mathbf{a}) \times (\mathbf{b} \times \mathbf{c}) = 0\,.$$

(iv) *To find the equation of the line through a given point and perpendicular to the plane through three given points.* Let A be the given point on the line, and let B, C and D be the given points on the plane. Figure 34 illustrates

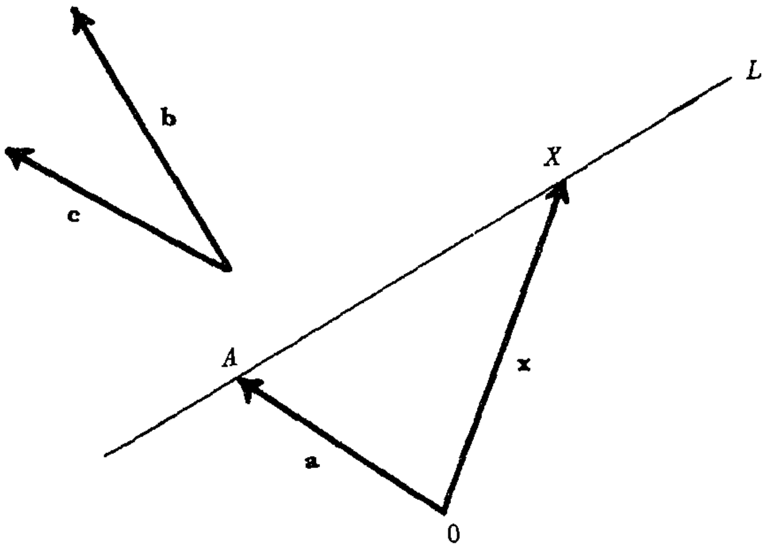

Figure 33

the problem, L and P being the line and plane in question. We denote the position-vectors of A, B, C and D in the usual manner. Let us consider the vector $\boldsymbol{\varphi}$ given by the relation

$$\boldsymbol{\varphi} = \mathbf{c}\times\mathbf{d}+\mathbf{d}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}.$$

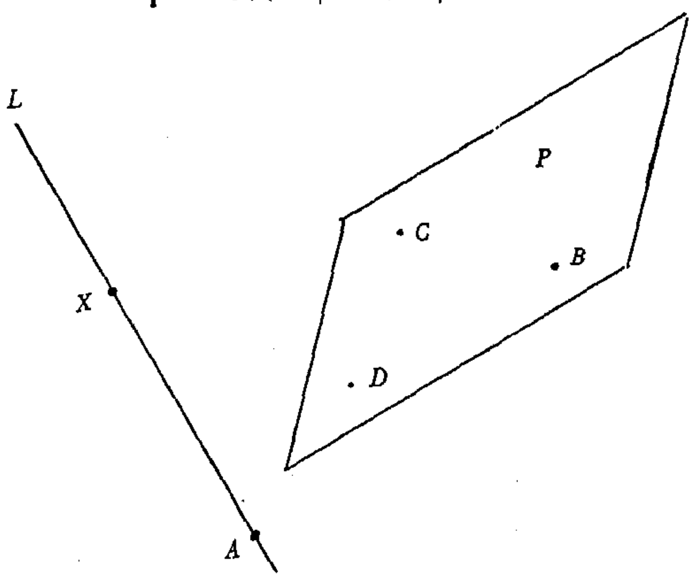

Figure 34

According to § 20, this vector is perpendicular to the plane P. Thus L is the line through A and parallel to $\boldsymbol{\varphi}$, and hence by Problem (i) above the desired equation of L is

(23.4) $$(\mathbf{x}-\mathbf{a})\times\boldsymbol{\varphi}=0.$$

24. *The equation of a sphere.* Let S be a sphere of radius a with center at a point C, as shown in Figure 35. If X is general point on the sphere S, then

$$\overline{CX}\cdot\overline{CX}=|\overline{CX}|^2=a^2.$$

But $\overline{CX}=\mathbf{x}-\mathbf{c}$. Thus

(24.1) $$(\mathbf{x}-\mathbf{c})\cdot(\mathbf{x}-\mathbf{c})=\mathrm{a}^2.$$

This is an equation satisfied by the position-vector of a general point X on the sphere. It is thus the equation of the sphere.

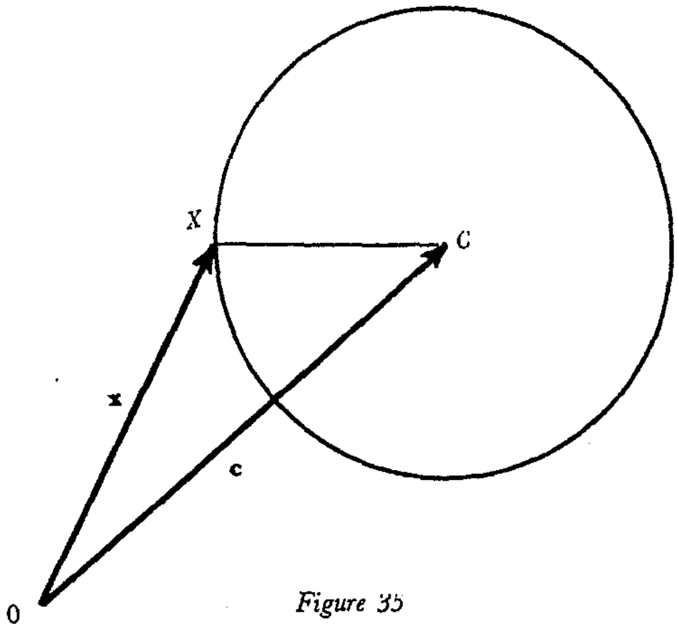

Figure 35

We shall now prove the following well-known property of a sphere: the angle at the surface of a sphere subtended by a diameter of the sphere is a right angle. For convenience, the origin of the coordinate system is chosen at the center of the sphere, as shown in Figure 36. Let D and E be points at the ends of a diameter, and let X be a general point on the sphere. We denote the position-vectors of these points in the usual manner. From the figure

$$\overline{DX}=\mathbf{x}-\mathbf{d},\qquad \overline{EX}=\mathbf{x}-\mathbf{e}.$$

Thus

$$\overline{DX}\cdot\overline{EX} = (\mathbf{x}-\mathbf{d})\cdot(\mathbf{x}-\mathbf{e}) = \mathbf{x}\cdot\mathbf{x}-\mathbf{x}\cdot(\mathbf{d}+\mathbf{e})+\mathbf{d}\cdot\mathbf{e}.$$

But $\mathbf{d}+\mathbf{e}=0$, and if a denotes the radius of the sphere then $\mathbf{x}\cdot\mathbf{x} = a^2$, $\mathbf{d}\cdot\mathbf{e} = -a^2$. Hence it follows that

$$\overline{DX}\cdot\overline{EX} = 0\,,$$

and so $\overline{DX}$ is perpendicular to $\overline{EX}$.

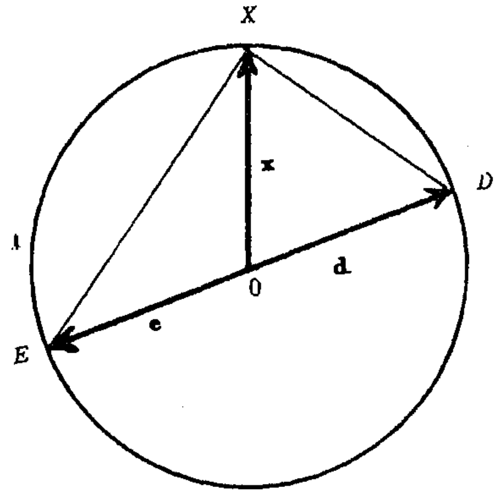

Figure 36

25. *The tangent plane to a sphere.* Let S be a sphere of radius a with center at a point C, as shown in Figure 37. We shall now find

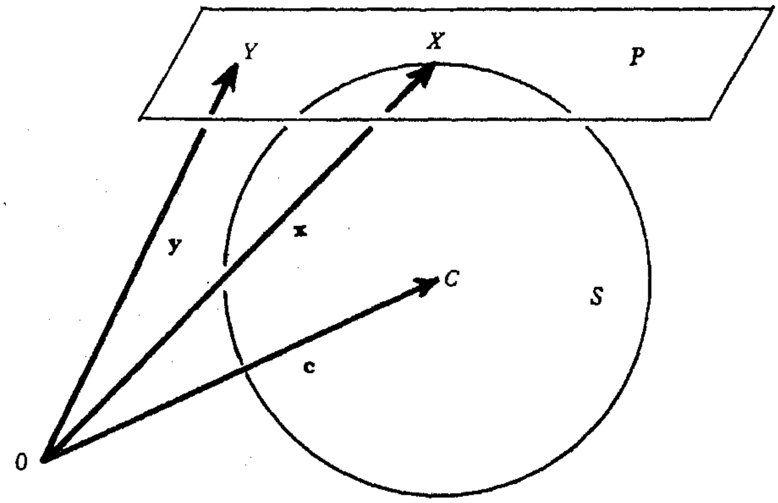

Figure 37

the equation of the plane P which touches S at a given point X.

Let Y be a general point on P. We denote the position-vectors of the various points in the usual manner. From the figure it follows that

$$\overline{CY} = \overline{CX} + \overline{XY},$$

or

$$\mathbf{y} - \mathbf{c} = \overline{CX} + \overline{XY}.$$

We now multiply this equation scalarly by $\overline{CX}$. Now $\overline{CX} \cdot \overline{CX} = a^2$, and $\overline{CX} \cdot \overline{XY} = 0$ because $\overline{CX}$ is perpendicular to $\overline{XY}$. Thus we have

$$\overline{CX} \cdot (\mathbf{y} - \mathbf{c}) = a^2. \tag{25.1}$$

But $\overline{CX} = \mathbf{x} - \mathbf{c}$. Thus Equation (25.1) becomes

$$(\mathbf{x} - \mathbf{c}) \cdot (\mathbf{y} - \mathbf{c}) = a^2.$$

This is the desired equation of the plane P.

Differential Geometry

26. *Introduction.* We shall consider here only a small portion of the differential geometry of curves in space. Rectangular cartesian co-ordinates x_1, x_2 and x_3 are introduced, with origin at a point O. The quantities x_1, x_2 and x_3 denote the coordinates of a general point X with position-vector $\mathbf{x}$. If $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are unit vectors in the directions of the positive coordinate axes, then as before,

$$\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3. \tag{26.1}$$

A curve consists of the set of points the position-vectors of which satisfy the relation

$$\mathbf{x} = \mathbf{x}(u),$$

where $\mathbf{x}(u)$ is a function of a scalar parameter u. We shall consider only those parts of the curve which are free of singularities of all kinds.

If the set of points comprising a curve all lie in a single plane, the curve is said to be a *plane curve*. If this set of points does not lie in a single plane, the curve is said to be a *skew curve*.

It is convenient to choose as the scalar parameter the length s of the arc of the curve measured from some fixed point A. The quantity s is positive for points on one side of A, and negative for points on the other side of A. The equation of the curve may then take the form

$$\mathbf{x} = \mathbf{x}(s).$$

The derivatives with respect to s of the function $\mathbf{x}(s)$ will be denoted by $\mathbf{x}'$, $\mathbf{x}''$, $\mathbf{x}'''$, etc.

27. *The principal triad.* Let us consider a general point X on a curve C. The position-vector of X is $\mathbf{x}$. We shall now define a set of three orthogonal unit vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ at X. They are functions of the parameter s, and their derivatives respect to s will be denoted in the usual

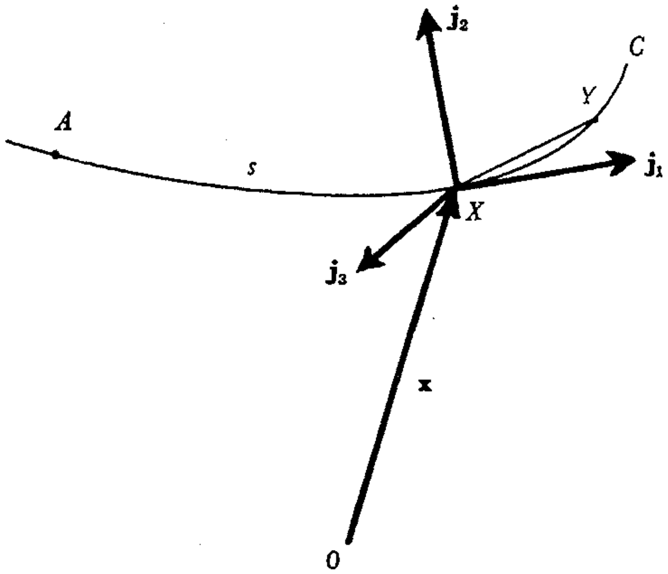

Figure 38

way by the symbols $\mathbf{j}_1'$, $\mathbf{j}_2'$ and $\mathbf{j}_3'$. They are shown in Figure 38, and are defined by the conditions:

(i) $\mathbf{j}_1$ is tangent to the curve C, and points in the direction of s increasing;
(ii) $\mathbf{j}_2$ lies in the plane of the vectors $\mathbf{j}_1$ and $\mathbf{j}_1'$, and makes an acute angle with $\mathbf{j}_1'$;
(iii) $\mathbf{j}_3$ is such that the vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ form a right-handed triad[1].

[1] At points on the curve where $\mathbf{j}_1'$ is equal to zero, these conditions are not sufficient for a unique determination of $\mathbf{j}_2$ and $\mathbf{j}_3$. We exclude such points from consideration here.

The straight line through the point X and parallel to $\mathbf{j}_2$ is called the *principal normal* to the curve. The straight line through X and parallel to $\mathbf{j}_3$ is called the *binormal* to the curve. The vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ are called the *unit tangent vector*, *unit normal vector*, and *unit binormal vector*, respectively. The triad formed by these vectors is called the *principal triad*. The plane through X and perpendicular to $\mathbf{j}_1$ is called the *normal plane*. The plane through X and perpendicular to $\mathbf{j}_3$ is called the *osculating plane*.

28. *The Serret-Frenet formulas*. Let Y be a point on the curve C and near the point X, as shown in Figure 38. We denote the length of the arc XY by the symbol Δs, and the vector $\overline{XY}$ by the symbol $\Delta\mathbf{x}$. Let us now consider the vector

$$\mathbf{x}' = \operatorname*{limit}_{\Delta s \to 0} \frac{\Delta\mathbf{x}}{\Delta s}.$$

Now

$$\operatorname*{limit}_{\Delta s \to 0} \frac{|\Delta\mathbf{x}|}{\Delta s} = 1.$$

Thus $\mathbf{x}'$ is a unit vector. Further, the vector $\Delta\mathbf{x}/\Delta s$ lies along $\overline{XY}$, and its direction then tends to that of $\mathbf{j}_1$ as Δs tends to zero. Since $\mathbf{j}_1$ is a unit vector, we can then write

$$\mathbf{j}_1 = \mathbf{x}'. \tag{28.1}$$

The vectors $\mathbf{j}_1'$, $\mathbf{j}_2'$ and $\mathbf{j}_3'$ can each be expressed as a linear function of any three non-coplaner vectors. In particular, they can be expressed as linear functions of the vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$, and we then have relations of the form

$$\begin{aligned} \mathbf{j}_1' &= a_{11}\mathbf{j}_1 + a_{12}\mathbf{j}_2 + a_{13}\mathbf{j}_3, \\ \mathbf{j}_2' &= a_{21}\mathbf{j}_1 + a_{22}\mathbf{j}_2 + a_{23}\mathbf{j}_3, \\ \mathbf{j}_3' &= a_{31}\mathbf{j}_1 + a_{32}\mathbf{j}_2 + a_{33}\mathbf{j}_3, \end{aligned} \tag{28.2}$$

where the scalar coefficients are functions of the parameter s. Since the vectors $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ are orthogonal unit vectors, they satisfy the relations

$$
\begin{array}{llll}
(28.3) & \mathbf{j}_1\cdot\mathbf{j}_1=1, & \mathbf{j}_1\cdot\mathbf{j}_2=0, & \mathbf{j}_1\cdot\mathbf{j}_3=0, \\
 & \mathbf{j}_2\cdot\mathbf{j}_1=0, & \mathbf{j}_2\cdot\mathbf{j}_2=1, & \mathbf{j}_2\cdot\mathbf{j}_3=0, \\
 & \mathbf{j}_3\cdot\mathbf{j}_1=0, & \mathbf{j}_3\cdot\mathbf{j}_2=0, & \mathbf{j}_3\cdot\mathbf{j}_3=1.
\end{array}
$$

We differentiate with respect to s the first equation in the first line of (28.3). This yields the relation

$$\mathbf{j}_1\cdot\mathbf{j}_1'+\mathbf{j}_1'\cdot\mathbf{j}_1=0.$$

Since in a scalar product the order in which the vectors appear is immaterial, we can interchange the vectors in the second scalar product. It then follows that

$$\mathbf{j}_1\cdot\mathbf{j}_1'=0.$$

If we substitute here for $\mathbf{j}_1'$ from the first equation in (28.2), and then make use of Equations (28.3), we find that $a_{11}=0$. Similarly $a_{22}=a_{33}=0$, and we may write

$$a_{11}=a_{22}=a_{33}=0. \tag{28.4}$$

We now differentiate with respect to s the second relation in the first line of (28.3). This yields

$$\mathbf{j}_1\cdot\mathbf{j}_2'+\mathbf{j}_1'\cdot\mathbf{j}_2=0.$$

If we substitute here for $\mathbf{j}_1'$ and $\mathbf{j}_2'$ from the first two equations in (28.2), and then make use of Equations (28.3), we find that $a_{12}+a_{21}=0$. Similarly we can find two similar relations, and we have altogether

$$a_{12}+a_{21}=0, \qquad a_{23}+a_{32}=0, \qquad a_{31}+a_{13}=0. \tag{28.5}$$

So far, only conditions (i) and (iii) above have been used. By condition (ii) the vector $\mathbf{j}_1'$ is to be in the plane of $\mathbf{j}_1$ and $\mathbf{j}_2$. This can be true only if

$$a_{13}=0. \tag{28.6}$$

By condition (ii), the vector $\mathbf{j}_1'$ is to make an acute angle with $\mathbf{j}_2$. If this angle is denoted by α, then cos α must be positive. But

$$|\mathbf{j}_1'|\cos\alpha=\mathbf{j}_2\cdot\mathbf{j}_1'.$$

If we substitute here for $\mathbf{j}_1'$ from Equations (28.2) and then use Equations (28.3) we find that

$$|\mathbf{j}_1'| \cos \alpha = a_{12}.$$

Thus

(28.7) $$a_{12} > 0.$$

We now define two quantities $\varkappa$ and τ by the relations

(28.8) $$\varkappa = a_{12}, \qquad \tau = a_{23}.$$

Then, by (28.7) it follows that

(28.9) $$\varkappa > 0,$$

and because of Equations (28.4), (28.5), (28.6) and (28.8), we can now express Equations (28.2) in the form

(28.10) $$\begin{aligned} \dot{\mathbf{j}}_1' &= \varkappa\dot{\mathbf{j}}_2, \\ \dot{\mathbf{j}}_2' &= \tau\dot{\mathbf{j}}_3 - \varkappa\dot{\mathbf{j}}_1, \\ \dot{\mathbf{j}}_3' &= -\tau\dot{\mathbf{j}}_2. \end{aligned}$$

These are the Serret-Frenet formulas. They were given originally by Serret (1851) and Frenet (1852) in an equivalent form which did not involve vectors. The quantities $\varkappa$ and τ, which are functions of the arc length s of the curve C, will be considered in some detail in the next section.

29. *Curvature and torsion.* The quantity $\varkappa$ appearing in Equations (28.10) is called the *curvature* of the curve. It can be shown that

$$\varkappa = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s},$$

where $\Delta\theta$ is the angle between the tangents to the curve C at the points X and Y in Figure 38. Thus $\varkappa$ is the rate at which the tangent at the point X rotates as X moves along the curve. The reciprocal of $\varkappa$ is called the *radius of curvature*, and will be denoted by the symbol ρ.

The quantity τ appearing in Equations (28.10) is called the *torsion* of the curve. It can be shown that

$$\tau = \lim_{\Delta s \to 0} \frac{\Delta\Phi}{\Delta s},$$

where $\Delta\Phi$ is the angle between the binormals to the curve C at the points X and Y in Figure 38. Thus τ is the rate at which the unit binormal at the point X rotates as X moves along the curve. The reciprocal of τ is called the *radius of torsion*, and will be denoted by σ.

To find $\varkappa$, we note from Equation (28.1) that

$$\mathbf{j}_1 = \mathbf{x}', \qquad \mathbf{j}_1' = \mathbf{x}''. \tag{29.1}$$

Substitution from the second of these relations for $\mathbf{j}_1'$ in the first of the Serret-Frenet formulas then yields the equation

$$\mathbf{x}'' = \varkappa \mathbf{j}_2. \tag{29.2}$$

We now multiply each side of this equation scalarly by itself, obtaining

$$\varkappa^2 = \mathbf{x}'' \cdot \mathbf{x}''.$$

Because of (28.9), $\varkappa$ is positive, and so

$$\varkappa = \sqrt{\mathbf{x}'' \cdot \mathbf{x}''}. \tag{29.3}$$

To find the torsion τ we differentiate Equation (29.2) with respect to s. This yields

$$\mathbf{x}''' = \varkappa \mathbf{j}_2' + \varkappa' \mathbf{j}_2.$$

We now substitute for $\mathbf{j}_2'$ from the second Serret-Frenet formula, obtaining

$$\mathbf{x}''' = \varkappa\,(\tau \mathbf{j}_3 - \varkappa \mathbf{j}_1) + \varkappa' \mathbf{j}_2. \tag{29.4}$$

From Equations (29.1), (29.2) and (29.4) it now follows that

$$\begin{aligned} \mathbf{x}' \cdot (\mathbf{x}'' \times \mathbf{x}''') &= \mathbf{j}_1 \cdot [\varkappa \mathbf{j}_2 \times (\varkappa\tau \mathbf{j}_3 - \varkappa^2 \mathbf{j}_1 + \varkappa' \mathbf{j}_2)] \\ &= \mathbf{j}_1 \cdot [\varkappa^2 \tau \mathbf{j}_1 + \varkappa^3 \mathbf{j}_3] \\ &= \varkappa^2 \tau. \end{aligned}$$

Thus

$$\tau = \frac{1}{\varkappa^2}\, \mathbf{x}' \cdot (\mathbf{x}'' \times \mathbf{x}'''). \tag{29.5}$$

The curvature and torsion can be computed from Equations (29.3) and (29.5). We can express these equations in different forms. Now

$$\begin{aligned} \mathbf{x} &= x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3, \\ \mathbf{x}' &= x_1' \mathbf{i}_1 + x_2' \mathbf{i}_2 + x_3' \mathbf{i}_3, \\ \mathbf{x}'' &= x_1'' \mathbf{i}_1 + x_2'' \mathbf{i}_2 + x_3'' \mathbf{i}_3, \\ \mathbf{x}''' &= x_1''' \mathbf{i}_1 + x_2''' \mathbf{i}_2 + x_3''' \mathbf{i}_3. \end{aligned}$$

Substitution from these relations in Equations (29.3) and (29.4) then yields

$$\varkappa = \sqrt{x_1''^2 + x_2''^2 + x_3''^2}, \tag{29.6}$$

$$\tau = \frac{1}{\varkappa^2} \begin{vmatrix} x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \\ x_1''' & x_2''' & x_3''' \end{vmatrix}. \tag{29.7}$$

Since $\varkappa$ can now be found, we can obtain the unit tangent vector $\mathbf{j}_1$ and the unit normal vector $\mathbf{j}_2$ by use of Equations (29.1) and (29.2). The unit binormal vector $\mathbf{j}_3$ can then be found easily, since it is equal to $\mathbf{j}_1 \times \mathbf{j}_2$. We have the collected results

$$\mathbf{j}_1 = \mathbf{x}', \quad \mathbf{j}_2 = \frac{1}{\varkappa}\mathbf{x}'', \quad \mathbf{j}_3 = \frac{1}{\varkappa}\mathbf{x}' \times \mathbf{x}''. \tag{29.8}$$

Let us now find the equation of the tangent to the curve at the point X. If Y is a general point on this tangent, the desired equation is easily seen to be

$$(\mathbf{y} - \mathbf{x}) \times \mathbf{j}_1 = 0.$$

Because of Equation (29.8), this can be written in the form

$$(\mathbf{y} - \mathbf{x}) \times \mathbf{x}' = 0. \tag{29.9}$$

In the same way, the equations of the principal normal and binormal can be found in the forms

$$(\mathbf{y} - \mathbf{x}) \times \mathbf{x}'' = 0, \tag{29.10}$$

$$(\mathbf{y} - \mathbf{x}) \times (\mathbf{x}' \times \mathbf{x}'') = 0. \tag{29.11}$$

The equation of the normal plane at the point X is easily seen to be

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{j}_1 = 0.$$

Because of Equation (29.8), this can be written in the form

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{x}' = 0. \tag{29.12}$$

In the same way we can find the equation of the osculating plane in the form

$$(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{x}' \times \mathbf{x}'') = 0. \tag{29.13}$$

1. Prove that the line joining the middle points of any two sides of a triangle is parallel to the third side, and is equal in length to one half the length of the third side.

2. Prove that the lines joining the middle points of the sides of a quadrilateral form a parallelogram.

3. If O is a point in space, ABC is a triangle, and D, E and F are the middle points of the sides, prove that

$$\overline{OA}+\overline{OB}+\overline{OC} = \overline{OD}+\overline{OE}+\overline{OF}.$$

4. If O is a point in space, $ABCD$ is a parallelogram, and E is the point of intersection of the diagonals, prove that

$$\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD} = 4\overline{OE}.$$

5. Prove that the line joining one vertex of a parallelogram to the middle point of an opposite side trisects a diagonal of the parallelogram.

6. Prove that it is possible to construct a triangle with sides equal and parallel to the medians of a given triangle.

7. Prove that the lines joining the middle points of opposite sides of a skew quadrilateral bisect each other. Prove also that the point where these lines cross is the middle point of the line joining the middle points of the diagonals of the quadrilateral.

8. Prove that the three perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

9. Prove that the bisectors of the angles of a triangle meet in a point. Hint: the sum of unit vectors along two sides lies along the bisector of the contained angle.

10. Prove that the perpendicular bisectors of the sides of a triangle meet in a point.

11. If O is a point in space and ABC is a triangle with sides of lengths l, m and n, then

$$l\,\overline{OA}+m\,\overline{OB}+n\,\overline{OC} = (l+m+n)\,\overline{OD},$$

where D is the center of the inscribed circle.

12. If ABC is a given triangle, the middle points of the sides BC, CA

and AB are denoted by D, E and F respectively, G is the point of intersection of the perpendiculars from the vertices to the opposite sides, and H is the center of the circumscribed circle, prove that

$$\overline{AG} = 2\,\overline{HD}, \quad \overline{BG} = 2\,\overline{HE}, \quad \overline{CG} = 2\,\overline{HF}.$$

Hence prove that

$$\overline{GA}+\overline{GB}+\overline{GC} = 2\,\overline{GH}.$$

In problems 13–22 the points A, B, C and D have the following coordinates: $A(-1, 2, 3)$, $B(2, 5, -3)$, $C(4, 1, -1)$, $D(1, 3, -3)$.

13. Find the position-vectors of the points of trisection of the line segment AB.

14. Find the distance between the points A and B.

15. Find the area of the triangle ABC.

16. Find the cartesian form of the equation of the plane through A and perpendicular to $\overline{OB}$.

17. Find the cartesian form of the equation of the plane through (i) the origin and the points A and B, (ii) the points A, B and C.

18. Find the distance from the point D to the plane through A and perpendicular to $\overline{OB}$.

19. Find the distance from D to the plane ABC.

20. Find the cartesian form of the equation of the line through A and parallel to $\overline{BC}$.

21. Find the cartesian form of the equation of the line through A and B.

22. Find the cartesian form of the equation of the line through D and (i) perpendicular to the plane through A, B and C, (ii) perpendicular to $\overline{BC}$ and $\overline{OC}$.

23. A plane passes through a given point A with position-vector $\mathbf{a}$, and is parallel to each of two given vectors $\mathbf{b}$ and $\mathbf{c}$. Derive the equation of this plane in the form

$$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) = 0.$$

24. A straight line L passes through a point A with position-vector $\mathbf{a}$, and is parallel to a vector $\mathbf{b}$. A vector $\mathbf{p}$ has its origin at a point C

with position-vector $\mathbf{c}$, its line of action along the perpendicular from C to L, and its terminus on L. Show that

$$\mathbf{p} = \mathbf{a} - \mathbf{c} - \frac{(\mathbf{a}-\mathbf{c})\cdot\mathbf{b}}{b^2}\,\mathbf{b}\,.$$

25. A straight line L passes through a point A with position-vector $\mathbf{a}$, and is parallel to a vector $\mathbf{b}$. A second straight line L' passes through a point A' with position-vector $\mathbf{a}'$, and is parallel to a vector $\mathbf{b}'$. The vector $\mathbf{p}$ runs from L to L' along the common perpendicular. Show that

$$\mathbf{p} = \frac{(\mathbf{a}'-\mathbf{a})\cdot\mathbf{c}}{c^2}\,\mathbf{c}\,,$$

where $\mathbf{c} = \mathbf{b}\times\mathbf{b}'$.

26. Prove that if the torsion of a curve is equal to zero, the curve is a plane curve.

27. Prove that

$$\mathbf{x}'''' = -3\varkappa\,\varkappa'\mathbf{j}_1 + (\varkappa'' - \varkappa^3 - \varkappa\tau^2)\mathbf{j}_2 + (2\,\varkappa'\tau + \varkappa\tau')\mathbf{j}_3\,.$$

28. If the position-vector $\mathbf{x}$ of a general point on a curve is given as a function of a parameter t, and if primes denote differentiations with respect to t prove that

$$\varkappa = \frac{1}{s'^2}\sqrt{\mathbf{x}''\cdot\mathbf{x}'' - s''^2}\,, \qquad \tau = \frac{1}{\varkappa^2 s'^6}\,\mathbf{x}'\cdot(\mathbf{x}''\times\mathbf{x}''')\,,$$

$$\mathbf{j}_1 = \frac{\mathbf{x}'}{s'}\,, \quad \mathbf{j}_2 = \frac{1}{\varkappa s'^2}\left(\mathbf{x}'' - \frac{s''}{s'}\,\mathbf{x}'\right), \quad \mathbf{j}_3 = \frac{\mathbf{x}'\times\mathbf{x}''}{\varkappa s'^3}\,.$$

Also, derive the equations of the tangent, principal normal, binormal, normal plane and osculating plane in the following forms:

tangent,	$(\mathbf{y}-\mathbf{x})\times\mathbf{x}' = 0;$
principal normal,	$(\mathbf{y}-\mathbf{x})\times\left(\mathbf{x}'' - \frac{s''}{s'}\,\mathbf{x}'\right) = 0;$
binormal,	$(\mathbf{y}-\mathbf{x})\times(\mathbf{x}'\times\mathbf{x}'') = 0;$
normal plane,	$(\mathbf{y}-\mathbf{x})\cdot\mathbf{x}' = 0;$
osculating plane,	$(\mathbf{y}-\mathbf{x})\cdot(\mathbf{x}'\times\mathbf{x}'') = 0.$

29. The position-vector $\mathbf{x}$ of a general point on a circular helix is given by the relation

$$\mathbf{x} = a \cos t\, \mathbf{i}_1 + a \sin t\, \mathbf{i}_2 + at \cot \alpha\, \mathbf{i}_3,$$

where a and α are constants, and t is a parameter. Find ρ, σ and the principal triad. Answer: $\rho = a \operatorname{cosec}^2 \alpha$, $\sigma = 2a \operatorname{cosec} 2\alpha$, $\mathbf{j}_1 = \sin \alpha\, (-\mathbf{i}_1 \sin t + \mathbf{i}_2 \cos t + \mathbf{i}_3 \cot \alpha)$, $\mathbf{j}_2 = -\mathbf{i}_1 \cos t - \mathbf{i}_2 \sin t$, $\mathbf{j}_3 = \cos \alpha\, (\mathbf{i}_1 \sin t - \mathbf{i}_2 \cos t + \mathbf{i}_3 \tan \alpha)$.

30. The position-vector $\mathbf{x}$ of a general point on a curve is given by the relation

$$\mathbf{x} = a(3t - t^3)\mathbf{i}_1 + 3at^2\mathbf{i}_2 + a(3t + t^3)\mathbf{i}_3,$$

where a is a constant and t is a parameter. Find ρ, σ and the principal triad. Answer: $\rho = \sigma = 3a\, \gamma^2$, $\sqrt{2}\, \mathbf{j}_1 = \gamma^{-1}(\alpha\, \mathbf{i}_1 + \beta \mathbf{i}_2 + \gamma \mathbf{i}_3)$, $\mathbf{j}_2 = \gamma^{-1}(-\beta \mathbf{i}_1 + \alpha\, \mathbf{i}_2)$, $\sqrt{2}\, \mathbf{j}_3 = \gamma^{-1}(-\alpha\, \mathbf{i}_1 - \beta\, \mathbf{i}_2 + \gamma\, \mathbf{i}_3)$, where $\alpha = 1 - t^2$, $\beta = 2t$, $\gamma = 1 + t^2$.

31. The position-vector $\mathbf{x}$ of a general point on a curve is given by the relation

$$\mathbf{x} = a[(t - \sin t)\mathbf{i}_1 + (1 - \cos t)\mathbf{i}_2 + t\, \mathbf{i}_3],$$

where a is a constant and t is a parameter. Find ρ and σ. Answer: $\rho = a\, \alpha^{3/2}, \beta^{-1/2}$, $\sigma = -a\, \beta$, where $\alpha = 3 - 2 \cos t$, $\beta = 2 - 2 \cos t + \cos^2 t$.