



12

VECTORS AND THE GEOMETRY OF SPACE

OVERVIEW To apply calculus in many real-world situations and in higher mathematics, we need a mathematical description of three-dimensional space. In this chapter we introduce three-dimensional coordinate systems and vectors. Building on what we already know about coordinates in the xy -plane, we establish coordinates in space by adding a third axis that measures distance above and below the xy -plane. Vectors are used to study the analytic geometry of space, where they give simple ways to describe lines, planes, surfaces, and curves in space. We use these geometric ideas later in the book to study motion in space and the calculus of functions of several variables, with their many important applications in science, engineering, economics, and higher mathematics.

12.1 | Three-Dimensional Coordinate Systems

To locate a point in space, we use three mutually perpendicular coordinate axes, arranged as in Figure 12.1. The axes shown there make a *right-handed* coordinate frame. When you hold your right hand so that the fingers curl from the positive x -axis toward the positive y -axis, your thumb points along the positive z -axis. So when you look down on the xy -plane from the positive direction of the z -axis, positive angles in the plane are measured counterclockwise from the positive x -axis and around the positive z -axis. (In a *left-handed* coordinate frame, the z -axis would point downward in Figure 12.1 and angles in the plane would be positive when measured clockwise from the positive x -axis. Right-handed and left-handed coordinate frames are not equivalent.)

The Cartesian coordinates (x, y, z) of a point P in space are the values at which the planes through P perpendicular to the axes cut the axes. Cartesian coordinates for space are also called **rectangular coordinates** because the axes that define them meet at right angles. Points on the x -axis have y - and z -coordinates equal to zero. That is, they have coordinates of the form $(x, 0, 0)$. Similarly, points on the y -axis have coordinates of the form $(0, y, 0)$, and points on the z -axis have coordinates of the form $(0, 0, z)$.

The planes determined by the coordinate axes are the **xy -plane**, whose standard equation is $z = 0$; the **yz -plane**, whose standard equation is $x = 0$; and the **xz -plane**, whose standard equation is $y = 0$. They meet at the **origin** $(0, 0, 0)$ (Figure 12.2). The origin is also identified by simply 0 or sometimes the letter O .

The three **coordinate planes** $x = 0$, $y = 0$, and $z = 0$ divide space into eight cells called **octants**. The octant in which the point coordinates are all positive is called the **first octant**; there is no convention for numbering the other seven octants.

The points in a plane perpendicular to the x -axis all have the same x -coordinate, this being the number at which that plane cuts the x -axis. The y - and z -coordinates can be any numbers. Similarly, the points in a plane perpendicular to the y -axis have a common y -coordinate and the points in a plane perpendicular to the z -axis have a common z -coordinate. To write equations for these planes, we name the common coordinate's value. The plane $x = 2$ is the plane perpendicular to the x -axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the y -axis

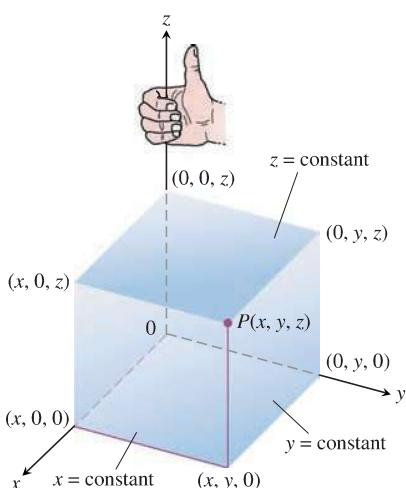


FIGURE 12.1 The Cartesian coordinate system is right-handed.

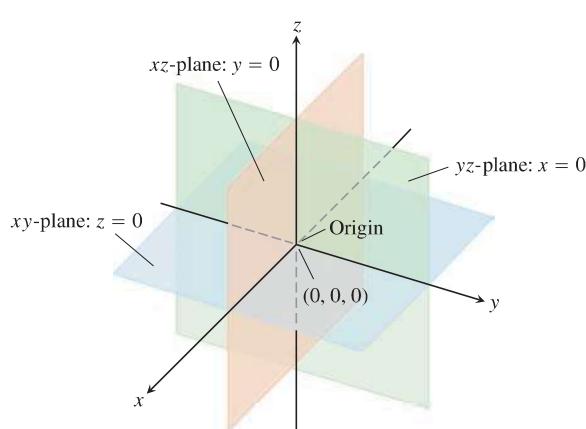


FIGURE 12.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

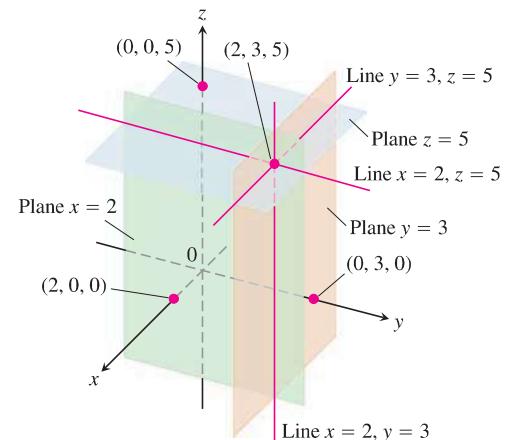


FIGURE 12.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

at $y = 3$. The plane $z = 5$ is the plane perpendicular to the z -axis at $z = 5$. Figure 12.3 shows the planes $x = 2$, $y = 3$, and $z = 5$, together with their intersection point $(2, 3, 5)$.

The planes $x = 2$ and $y = 3$ in Figure 12.3 intersect in a line parallel to the z -axis. This line is described by the pair of equations $x = 2, y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$. Similarly, the line of intersection of the planes $y = 3$ and $z = 5$ is described by the equation pair $y = 3, z = 5$. This line runs parallel to the x -axis. The line of intersection of the planes $x = 2$ and $z = 5$, parallel to the y -axis, is described by the equation pair $x = 2, z = 5$.

In the following examples, we match coordinate equations and inequalities with the sets of points they define in space.

EXAMPLE 1 We interpret these equations and inequalities geometrically.

(a) $z \geq 0$

The half-space consisting of the points on and above the xy -plane.

(b) $x = -3$

The plane perpendicular to the x -axis at $x = -3$. This plane lies parallel to the yz -plane and 3 units behind it.

(c) $z = 0, x \leq 0, y \geq 0$

The second quadrant of the xy -plane.

(d) $x \geq 0, y \geq 0, z \geq 0$

The first octant.

(e) $-1 \leq y \leq 1$

The slab between the planes $y = -1$ and $y = 1$ (planes included).

(f) $y = -2, z = 2$

The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x -axis. ■

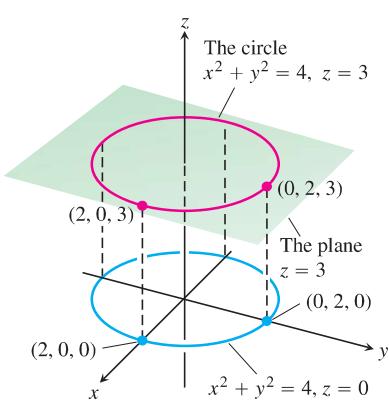


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

EXAMPLE 2 What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad z = 3?$$

Solution The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points “the circle $x^2 + y^2 = 4$ in the plane $z = 3$ ” or, more simply, “the circle $x^2 + y^2 = 4, z = 3$ ” (Figure 12.4). ■

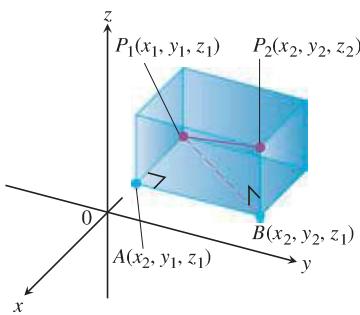


FIGURE 12.5 We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2 .

Distance and Spheres in Space

The formula for the distance between two points in the xy -plane extends to points in space.

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof We construct a rectangular box with faces parallel to the coordinate planes and the points P_1 and P_2 at opposite corners of the box (Figure 12.5). If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then the three box edges P_1A , AB , and BP_2 have lengths

$$|P_1A| = |x_2 - x_1|, \quad |AB| = |y_2 - y_1|, \quad |BP_2| = |z_2 - z_1|.$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 \quad \text{and} \quad |P_1B|^2 = |P_1A|^2 + |AB|^2$$

(see Figure 12.5).

So

$$\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\ &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Substitute
 $|P_1B|^2 = |P_1A|^2 + |AB|^2$.

Therefore

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 3 The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} \approx 6.708. \end{aligned}$$

We can use the distance formula to write equations for spheres in space (Figure 12.6). A point $P(x, y, z)$ lies on the sphere of radius a centered at $P_0(x_0, y_0, z_0)$ precisely when $|P_0P| = a$ or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

EXAMPLE 4 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x -, y -, and z -terms as necessary and write each

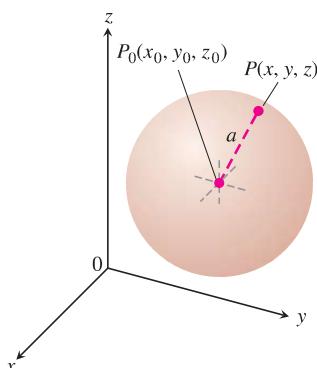


FIGURE 12.6 The sphere of radius a centered at the point (x_0, y_0, z_0) .

quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$\begin{aligned}x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\(x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\ \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) &= -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2 \\ \left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 &= -1 + \frac{9}{4} + 4 = \frac{21}{4}.\end{aligned}$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$. ■

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

- (a) $x^2 + y^2 + z^2 < 4$
- (b) $x^2 + y^2 + z^2 \leq 4$
- (c) $x^2 + y^2 + z^2 > 4$
- (d) $x^2 + y^2 + z^2 = 4, z \leq 0$

- The interior of the sphere $x^2 + y^2 + z^2 = 4$.
- The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.
- The exterior of the sphere $x^2 + y^2 + z^2 = 4$.
- The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy -plane (the plane $z = 0$). ■

Just as polar coordinates give another way to locate points in the xy -plane (Section 11.3), alternative coordinate systems, different from the Cartesian coordinate system developed here, exist for three-dimensional space. We examine two of these coordinate systems in Section 15.7.

Exercises 12.1

Geometric Interpretations of Equations

In Exercises 1–16, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

- 1. $x = 2, y = 3$
- 2. $x = -1, z = 0$
- 3. $y = 0, z = 0$
- 4. $x = 1, y = 0$
- 5. $x^2 + y^2 = 4, z = 0$
- 6. $x^2 + y^2 = 4, z = -2$
- 7. $x^2 + z^2 = 4, y = 0$
- 8. $y^2 + z^2 = 1, x = 0$
- 9. $x^2 + y^2 + z^2 = 1, x = 0$
- 10. $x^2 + y^2 + z^2 = 25, y = -4$
- 11. $x^2 + y^2 + (z + 3)^2 = 25, z = 0$
- 12. $x^2 + (y - 1)^2 + z^2 = 4, y = 0$
- 13. $x^2 + y^2 = 4, z = y$
- 14. $x^2 + y^2 + z^2 = 4, y = x$
- 15. $y = x^2, z = 0$
- 16. $z = y^2, x = 1$

Geometric Interpretations of Inequalities and Equations

In Exercises 17–24, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

- 17. a. $x \geq 0, y \geq 0, z = 0$ b. $x \geq 0, y \leq 0, z = 0$
- 18. a. $0 \leq x \leq 1$ b. $0 \leq x \leq 1, 0 \leq y \leq 1$
c. $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$
- 19. a. $x^2 + y^2 + z^2 \leq 1$ b. $x^2 + y^2 + z^2 > 1$
- 20. a. $x^2 + y^2 \leq 1, z = 0$ b. $x^2 + y^2 \leq 1, z = 3$
c. $x^2 + y^2 \leq 1$, no restriction on z
- 21. a. $1 \leq x^2 + y^2 + z^2 \leq 4$
b. $x^2 + y^2 + z^2 \leq 1, z \geq 0$
- 22. a. $x = y, z = 0$ b. $x = y$, no restriction on z
- 23. a. $y \geq x^2, z \geq 0$ b. $x \leq y^2, 0 \leq z \leq 2$
- 24. a. $z = 1 - y$, no restriction on x
b. $z = y^3, x = 2$

In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

25. The plane perpendicular to the
- x -axis at $(3, 0, 0)$
 - y -axis at $(0, -1, 0)$
 - z -axis at $(0, 0, -2)$
26. The plane through the point $(3, -1, 2)$ perpendicular to the
- x -axis
 - y -axis
 - z -axis
27. The plane through the point $(3, -1, 1)$ parallel to the
- xy -plane
 - yz -plane
 - xz -plane
28. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the
- xy -plane
 - yz -plane
 - xz -plane
29. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the
- xy -plane
 - yz -plane
 - plane $y = 2$
30. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the
- xy -plane
 - yz -plane
 - xz -plane
31. The line through the point $(1, 3, -1)$ parallel to the
- x -axis
 - y -axis
 - z -axis
32. The set of points in space equidistant from the origin and the point $(0, 2, 0)$
33. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z -axis meets the sphere of radius 5 centered at the origin
34. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

Inequalities to Describe Sets of Points

Write inequalities to describe the sets in Exercises 35–40.

35. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included)
36. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$
37. The half-space consisting of the points on and below the xy -plane
38. The upper hemisphere of the sphere of radius 1 centered at the origin
39. The (a) interior and (b) exterior of the sphere of radius 1 centered at the point $(1, 1, 1)$
40. The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin. (*Closed* means the spheres are to be included. Had we wanted the spheres left out, we would have asked for the *open* region bounded by the spheres. This is analogous to the way we use *closed* and *open* to describe intervals: *closed* means endpoints included, *open* means endpoints left out. Closed sets include boundaries; open sets leave them out.)

Distance

In Exercises 41–46, find the distance between points P_1 and P_2 .

41. $P_1(1, 1, 1)$, $P_2(3, 3, 0)$
 42. $P_1(-1, 1, 5)$, $P_2(2, 5, 0)$
 43. $P_1(1, 4, 5)$, $P_2(4, -2, 7)$

44. $P_1(3, 4, 5)$, $P_2(2, 3, 4)$
 45. $P_1(0, 0, 0)$, $P_2(2, -2, -2)$
 46. $P_1(5, 3, -2)$, $P_2(0, 0, 0)$

Spheres

Find the centers and radii of the spheres in Exercises 47–50.

47. $(x + 2)^2 + y^2 + (z - 2)^2 = 8$
 48. $(x - 1)^2 + \left(y + \frac{1}{2}\right)^2 + (z + 3)^2 = 25$
 49. $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 + (z + \sqrt{2})^2 = 2$
 50. $x^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \frac{16}{9}$

Find equations for the spheres whose centers and radii are given in Exercises 51–54.

Center	Radius
51. $(1, 2, 3)$	$\sqrt{14}$
52. $(0, -1, 5)$	2
53. $\left(-1, \frac{1}{2}, -\frac{2}{3}\right)$	$\frac{4}{9}$
54. $(0, -7, 0)$	7

Find the centers and radii of the spheres in Exercises 55–58.

55. $x^2 + y^2 + z^2 + 4x - 4z = 0$
 56. $x^2 + y^2 + z^2 - 6y + 8z = 0$
 57. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$
 58. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$

Theory and Examples

59. Find a formula for the distance from the point $P(x, y, z)$ to the
- x -axis
 - y -axis
 - z -axis
60. Find a formula for the distance from the point $P(x, y, z)$ to the
- xy -plane
 - yz -plane
 - xz -plane
61. Find the perimeter of the triangle with vertices $A(-1, 2, 1)$, $B(1, -1, 3)$, and $C(3, 4, 5)$.
62. Show that the point $P(3, 1, 2)$ is equidistant from the points $A(2, -1, 3)$ and $B(4, 3, 1)$.
63. Find an equation for the set of all points equidistant from the planes $y = 3$ and $y = -1$.
64. Find an equation for the set of all points equidistant from the point $(0, 0, 2)$ and the xy -plane.
65. Find the point on the sphere $x^2 + (y - 3)^2 + (z + 5)^2 = 4$ nearest
- the xy -plane.
 - the point $(0, 7, -5)$.
66. Find the point equidistant from the points $(0, 0, 0)$, $(0, 4, 0)$, $(3, 0, 0)$, and $(2, 2, -3)$.

12.2 | Vectors

Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going. In this section we show how to represent things that have both magnitude and direction in the plane or in space.

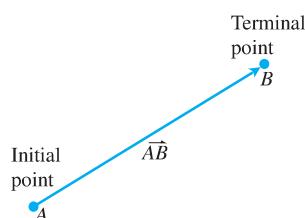


FIGURE 12.7 The directed line segment \overrightarrow{AB} is called a vector.

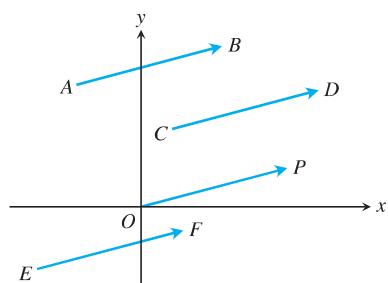


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

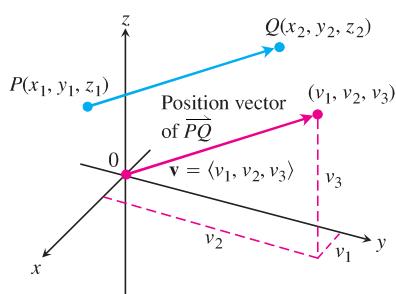


FIGURE 12.10 A vector \overrightarrow{PQ} in standard position has its initial point at the origin. The directed line segments \overrightarrow{PQ} and v are parallel and have the same length.

Component Form

A quantity such as force, displacement, or velocity is called a **vector** and is represented by a **directed line segment** (Figure 12.7). The arrow points in the direction of the action and its length gives the magnitude of the action in terms of a suitably chosen unit. For example, a force vector points in the direction in which the force acts and its length is a measure of the force's strength; a velocity vector points in the direction of motion and its length is the speed of the moving object. Figure 12.8 displays the velocity vector \mathbf{v} at a specific location for a particle moving along a path in the plane or in space. (This application of vectors is studied in Chapter 13.)

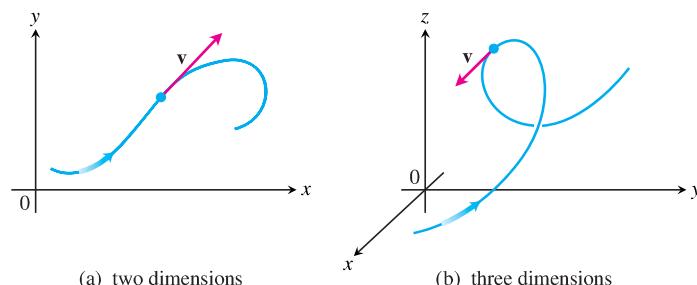


FIGURE 12.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

DEFINITIONS The vector represented by the directed line segment \overrightarrow{AB} has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

The arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction (Figure 12.9) regardless of the initial point.

In textbooks, vectors are usually written in lowercase, boldface letters, for example \mathbf{u} , \mathbf{v} , and \mathbf{w} . Sometimes we use uppercase boldface letters, such as \mathbf{F} , to denote a force vector. In handwritten form, it is customary to draw small arrows above the letters, for example \vec{u} , \vec{v} , and \vec{F} .

We need a way to represent vectors algebraically so that we can be more precise about the direction of a vector. Let $\mathbf{v} = \overrightarrow{PQ}$. There is one directed line segment equal to \overrightarrow{PQ} whose initial point is the origin (Figure 12.10). It is the representative of \mathbf{v} in **standard position** and is the vector we normally use to represent \mathbf{v} . We can specify \mathbf{v} by writing the

coordinates of its terminal point (v_1, v_2, v_3) when \mathbf{v} is in standard position. If \mathbf{v} is a vector in the plane its terminal point (v_1, v_2) has two coordinates.

DEFINITION

If \mathbf{v} is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If \mathbf{v} is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

So a two-dimensional vector is an ordered pair $\mathbf{v} = \langle v_1, v_2 \rangle$ of real numbers, and a three-dimensional vector is an ordered triple $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ of real numbers. The numbers v_1 , v_2 , and v_3 are the **components** of \mathbf{v} .

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is represented by the directed line segment \overrightarrow{PQ} , where the initial point is $P(x_1, y_1, z_1)$ and the terminal point is $Q(x_2, y_2, z_2)$, then $x_1 + v_1 = x_2$, $y_1 + v_2 = y_2$, and $z_1 + v_3 = z_2$ (see Figure 12.10). Thus, $v_1 = x_2 - x_1$, $v_2 = y_2 - y_1$, and $v_3 = z_2 - z_1$ are the components of \overrightarrow{PQ} .

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to \overrightarrow{PQ} is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

If \mathbf{v} is two-dimensional with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as points in the plane, then $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. There is no third component for planar vectors. With this understanding, we will develop the algebra of three-dimensional vectors and simply drop the third component when the vector is two-dimensional (a planar vector).

Two vectors are equal if and only if their standard position vectors are identical. Thus $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are **equal** if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

The **magnitude** or **length** of the vector \overrightarrow{PQ} is the length of any of its equivalent directed line segment representations. In particular, if $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the standard position vector for \overrightarrow{PQ} , then the distance formula gives the magnitude or length of \mathbf{v} , denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(see Figure 12.10).

The only vector with length 0 is the **zero vector** $\mathbf{0} = \langle 0, 0 \rangle$ or $\mathbf{0} = \langle 0, 0, 0 \rangle$. This vector is also the only vector with no specific direction.

EXAMPLE 1 Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector \mathbf{v} representing \overrightarrow{PQ} has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \quad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of \vec{PQ} is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

- (b) The length or magnitude of $\mathbf{v} = \vec{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

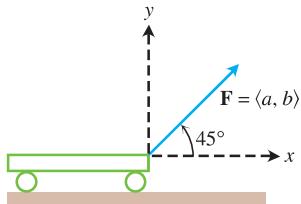


FIGURE 12.11 The force pulling the cart forward is represented by the vector \mathbf{F} whose horizontal component is the effective force (Example 2).

EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force \mathbf{F} making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45^\circ = (20) \left(\frac{\sqrt{2}}{2} \right) \approx 14.14 \text{ lb.}$$

Notice that \mathbf{F} is a two-dimensional vector.

Vector Algebra Operations

Two principal operations involving vectors are *vector addition* and *scalar multiplication*. A **scalar** is simply a real number, and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero and are used to “scale” a vector by multiplication.

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

We add vectors by adding the corresponding components of the vectors. We multiply a vector by a scalar by multiplying each component by the scalar. The definitions apply to planar vectors except there are only two components, $\langle u_1, u_2 \rangle$ and $\langle v_1, v_2 \rangle$.

The definition of vector addition is illustrated geometrically for planar vectors in Figure 12.12a, where the initial point of one vector is placed at the terminal point of the other. Another interpretation is shown in Figure 12.12b (called the **parallelogram law** of

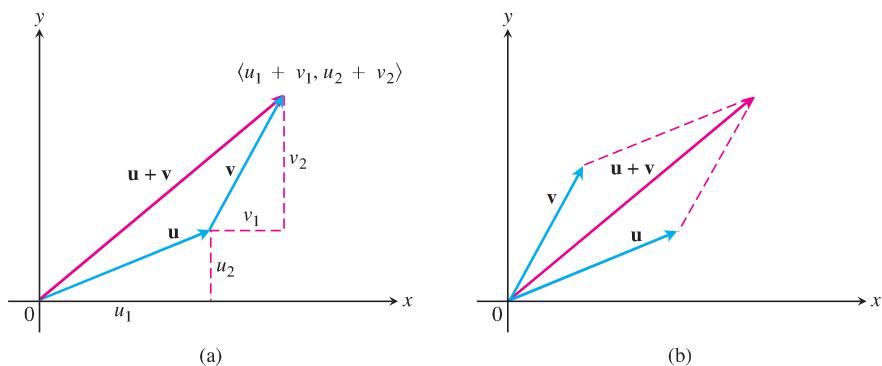
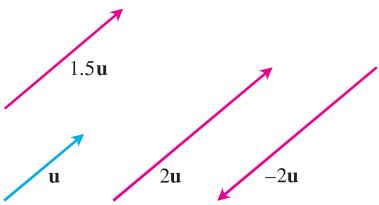


FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

FIGURE 12.13 Scalar multiples of \mathbf{u} .

addition), where the sum, called the **resultant vector**, is the diagonal of the parallelogram. In physics, forces add vectorially as do velocities, accelerations, and so on. So the force acting on a particle subject to two gravitational forces, for example, is obtained by adding the two force vectors.

Figure 12.13 displays a geometric interpretation of the product $k\mathbf{u}$ of the scalar k and vector \mathbf{u} . If $k > 0$, then $k\mathbf{u}$ has the same direction as \mathbf{u} ; if $k < 0$, then the direction of $k\mathbf{u}$ is opposite to that of \mathbf{u} . Comparing the lengths of \mathbf{u} and $k\mathbf{u}$, we see that

$$\begin{aligned} |k\mathbf{u}| &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2}\sqrt{u_1^2 + u_2^2 + u_3^2} = |k||\mathbf{u}|. \end{aligned}$$

The length of $k\mathbf{u}$ is the absolute value of the scalar k times the length of \mathbf{u} . The vector $(-1)\mathbf{u} = -\mathbf{u}$ has the same length as \mathbf{u} but points in the opposite direction.

The **difference** $\mathbf{u} - \mathbf{v}$ of two vectors is defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Note that $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}$, so adding the vector $(\mathbf{u} - \mathbf{v})$ to \mathbf{v} gives \mathbf{u} (Figure 12.14a). Figure 12.14b shows the difference $\mathbf{u} - \mathbf{v}$ as the sum $\mathbf{u} + (-\mathbf{v})$.

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

- (a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$ ■

Vector operations have many of the properties of ordinary arithmetic.

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars.

- | | |
|--|--|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ |
| 5. $0\mathbf{u} = \mathbf{0}$ | 6. $1\mathbf{u} = \mathbf{u}$ |
| 7. $a(b\mathbf{u}) = (ab)\mathbf{u}$ | 8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ |
| 9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ | |

These properties are readily verified using the definitions of vector addition and multiplication by a scalar. For instance, to establish Property 1, we have

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle \\ &= \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\ &= \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \\ &= \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \\ &= \mathbf{v} + \mathbf{u}. \end{aligned}$$

When three or more space vectors lie in the same plane, we say they are **coplanar** vectors. For example, the vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} + \mathbf{v}$ are always coplanar.

Unit Vectors

A vector \mathbf{v} of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.\end{aligned}$$

We call the scalar (or number) v_1 the **i-component** of the vector \mathbf{v} , v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

(Figure 12.15).

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of \mathbf{v} , called the **direction** of the nonzero vector \mathbf{v} .

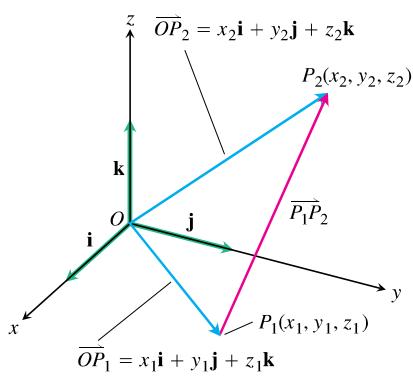


FIGURE 12.15 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

EXAMPLE 4 Find a unit vector \mathbf{u} in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector \mathbf{u} is the direction of $\overrightarrow{P_1P_2}$. ■

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express \mathbf{v} as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of \mathbf{v} :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as \mathbf{v} :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

HISTORICAL BIOGRAPHY

Hermann Grassmann
(1809–1877)

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\substack{\text{Length} \\ \text{Direction of motion} \\ (\text{speed})}} \right).$$

■

In summary, we can express any nonzero vector \mathbf{v} in terms of its two important features, length and direction, by writing $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$.

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force \mathbf{F} as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned} \mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right). \end{aligned}$$

■

Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

To see why, observe (Figure 12.16) that

$$\begin{aligned} \overrightarrow{OM} &= \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{P}_1\overrightarrow{P}_2) = \overrightarrow{OP}_1 + \frac{1}{2}(\overrightarrow{OP}_2 - \overrightarrow{OP}_1) \\ &= \frac{1}{2}(\overrightarrow{OP}_1 + \overrightarrow{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}. \end{aligned}$$

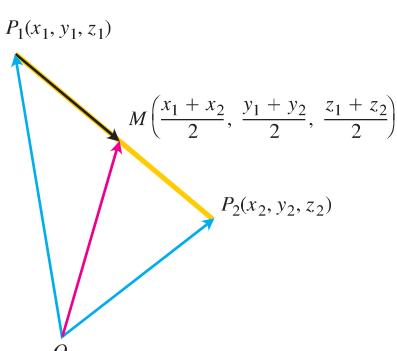


FIGURE 12.16 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2 .

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5, 1, 2).$$

■

Applications

An important application of vectors occurs in navigation.

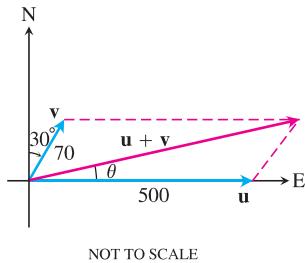


FIGURE 12.17 Vectors representing the velocities of the airplane \mathbf{u} and tailwind \mathbf{v} in Example 8.

EXAMPLE 8 A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution If \mathbf{u} = the velocity of the airplane alone and \mathbf{v} = the velocity of the tailwind, then $|\mathbf{u}| = 500$ and $|\mathbf{v}| = 70$ (Figure 12.17). The velocity of the airplane with respect to the ground is given by the magnitude and direction of the resultant vector $\mathbf{u} + \mathbf{v}$. If we let the positive x -axis represent east and the positive y -axis represent north, then the component forms of \mathbf{u} and \mathbf{v} are

$$\mathbf{u} = \langle 500, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\mathbf{j}$$

$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ. \quad \text{Figure 12.17}$$

The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east. ■

Another important application occurs in physics and engineering when several forces are acting on a single object.

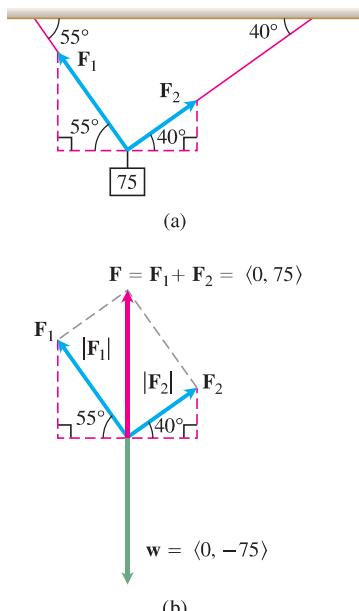


FIGURE 12.18 The suspended weight in Example 9.

EXAMPLE 9 A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces \mathbf{F}_1 and \mathbf{F}_2 acting in both wires.

Solution The force vectors \mathbf{F}_1 and \mathbf{F}_2 have magnitudes $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$ and components that are measured in Newtons. The resultant force is the sum $\mathbf{F}_1 + \mathbf{F}_2$ and must be equal in magnitude and acting in the opposite (or upward) direction to the weight vector \mathbf{w} (see Figure 12.18b). It follows from the figure that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75 \rangle \quad \text{and} \quad \mathbf{F}_2 = \langle |\mathbf{F}_2| \cos 40^\circ, |\mathbf{F}_2| \sin 40^\circ \rangle.$$

Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75 \rangle$, the resultant vector leads to the system of equations

$$-|\mathbf{F}_1| \cos 55^\circ + |\mathbf{F}_2| \cos 40^\circ = 0$$

$$|\mathbf{F}_1| \sin 55^\circ + |\mathbf{F}_2| \sin 40^\circ = 75.$$

Solving for $|\mathbf{F}_2|$ in the first equation and substituting the result into the second equation, we get

$$|\mathbf{F}_2| = \frac{|\mathbf{F}_1| \cos 55^\circ}{\cos 40^\circ} \quad \text{and} \quad |\mathbf{F}_1| \sin 55^\circ + \frac{|\mathbf{F}_1| \cos 55^\circ}{\cos 40^\circ} \sin 40^\circ = 75.$$

It follows that

$$|\mathbf{F}_1| = \frac{75}{\sin 55^\circ + \cos 55^\circ \tan 40^\circ} \approx 57.67 \text{ N},$$

and

$$\begin{aligned} |\mathbf{F}_2| &= \frac{75 \cos 55^\circ}{\sin 55^\circ \cos 40^\circ + \cos 55^\circ \sin 40^\circ} \\ &= \frac{75 \cos 55^\circ}{\sin(55^\circ + 40^\circ)} \approx 43.18 \text{ N}. \end{aligned}$$

The force vectors are then $\mathbf{F}_1 = \langle -33.08, 47.24 \rangle$ and $\mathbf{F}_2 = \langle 33.08, 27.76 \rangle$. ■

Exercises 12.2

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

- | | |
|--|--|
| 1. $3\mathbf{u}$ | 2. $-2\mathbf{v}$ |
| 3. $\mathbf{u} + \mathbf{v}$ | 4. $\mathbf{u} - \mathbf{v}$ |
| 5. $2\mathbf{u} - 3\mathbf{v}$ | 6. $-2\mathbf{u} + 5\mathbf{v}$ |
| 7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$ | 8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$ |

In Exercises 9–16, find the component form of the vector.

9. The vector \overrightarrow{PQ} , where $P = (1, 3)$ and $Q = (2, -1)$
10. The vector \overrightarrow{OP} where O is the origin and P is the midpoint of segment RS , where $R = (2, -1)$ and $S = (-4, 3)$
11. The vector from the point $A = (2, 3)$ to the origin
12. The sum of \overrightarrow{AB} and \overrightarrow{CD} , where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$
13. The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x -axis
14. The unit vector that makes an angle $\theta = -3\pi/4$ with the positive x -axis
15. The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ 120° counterclockwise about the origin
16. The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ 135° counterclockwise about the origin

Vectors in Space

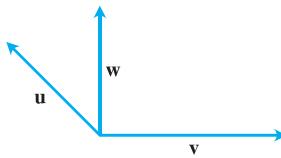
In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

17. $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$
18. $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$
19. \overrightarrow{AB} if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$
20. \overrightarrow{AB} if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$
21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$
22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

Geometric Representations

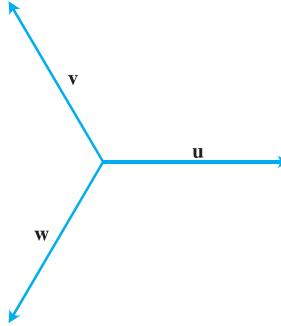
In Exercises 23 and 24, copy vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} head to tail as needed to sketch the indicated vector.

23.



- a. $\mathbf{u} + \mathbf{v}$
b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
c. $\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} - \mathbf{w}$

24.



- a. $\mathbf{u} - \mathbf{v}$
b. $\mathbf{u} - \mathbf{v} + \mathbf{w}$
c. $2\mathbf{u} - \mathbf{v}$
d. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

Length and Direction

In Exercises 25–30, express each vector as a product of its length and direction.

25. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
26. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
27. $5\mathbf{k}$
28. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$
29. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$
30. $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

31. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 2	\mathbf{i}
b. $\sqrt{3}$	$-\mathbf{k}$
c. $\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$
d. 7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

32. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 7	$-\mathbf{j}$
b. $\sqrt{2}$	$-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
c. $\frac{13}{12}$	$\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
d. $a > 0$	$\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

33. Find a vector of magnitude 7 in the direction of $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$.

34. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.

Direction and Midpoints

In Exercises 35–38, find

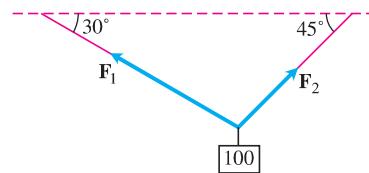
- a. the direction of $\overrightarrow{P_1P_2}$ and
 - b. the midpoint of line segment P_1P_2 .
35. $P_1(-1, 1, 5)$ $P_2(2, 5, 0)$
36. $P_1(1, 4, 5)$ $P_2(4, -2, 7)$
37. $P_1(3, 4, 5)$ $P_2(2, 3, 4)$
38. $P_1(0, 0, 0)$ $P_2(2, -2, -2)$

39. If $\overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and B is the point $(5, 1, 3)$, find A .

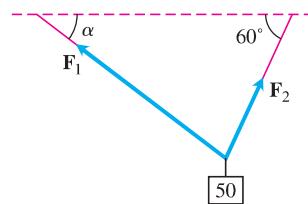
40. If $\overrightarrow{AB} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$ and A is the point $(-2, -3, 6)$, find B .

Theory and Applications

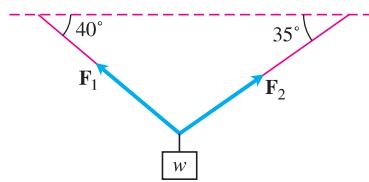
41. **Linear combination** Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. Find scalars a and b such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$.
42. **Linear combination** Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where \mathbf{u}_1 is parallel to \mathbf{v} and \mathbf{u}_2 is parallel to \mathbf{w} . (See Exercise 41.)
43. **Velocity** An airplane is flying in the direction 25° west of north at 800 km/h. Find the component form of the velocity of the airplane, assuming that the positive x -axis represents due east and the positive y -axis represents due north.
44. (Continuation of Example 8.) What speed and direction should the jetliner in Example 8 have in order for the resultant vector to be 500 mph due east?
45. Consider a 100-N weight suspended by two wires as shown in the accompanying figure. Find the magnitudes and components of the force vectors \mathbf{F}_1 and \mathbf{F}_2 .



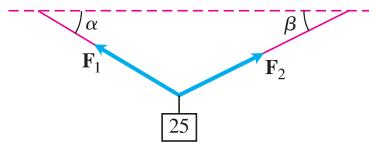
46. Consider a 50-N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector \mathbf{F}_1 is 35 N, find angle α and the magnitude of vector \mathbf{F}_2 .



47. Consider a w -N weight suspended by two wires as shown in the accompanying figure. If the magnitude of vector \mathbf{F}_2 is 100 N, find w and the magnitude of vector \mathbf{F}_1 .



48. Consider a 25-N weight suspended by two wires as shown in the accompanying figure. If the magnitudes of vectors \mathbf{F}_1 and \mathbf{F}_2 are both 75 N, then angles α and β are equal. Find α .



49. **Location** A bird flies from its nest 5 km in the direction 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy -coordinate system so that the origin is the bird's nest, the x -axis points east, and the y -axis points north.

- a. At what point is the tree located?

- b. At what point is the telephone pole?

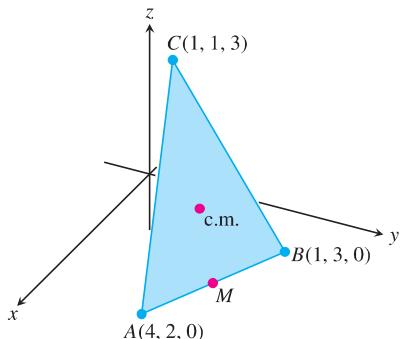
50. Use similar triangles to find the coordinates of the point Q that divides the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ into two lengths whose ratio is $p/q = r$.

51. **Medians of a triangle** Suppose that A , B , and C are the corner points of the thin triangular plate of constant density shown here.

- a. Find the vector from C to the midpoint M of side AB .

- b. Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM .

- c. Find the coordinates of the point in which the medians of $\triangle ABC$ intersect. According to Exercise 17, Section 6.6, this point is the plate's center of mass.



52. Find the vector from the origin to the point of intersection of the medians of the triangle whose vertices are

$$A(1, -1, 2), \quad B(2, 1, 3), \quad \text{and} \quad C(-1, 2, -1).$$

53. Let $ABCD$ be a general, not necessarily planar, quadrilateral in space. Show that the two segments joining the midpoints of opposite sides of $ABCD$ bisect each other. (*Hint:* Show that the segments have the same midpoint.)

54. Vectors are drawn from the center of a regular n -sided polygon in the plane to the vertices of the polygon. Show that the sum of the vectors is zero. (*Hint:* What happens to the sum if you rotate the polygon about its center?)

55. Suppose that A , B , and C are vertices of a triangle and that a , b , and c are, respectively, the midpoints of the opposite sides. Show that $\overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = 0$.

56. **Unit vectors in the plane** Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$, obtained by rotating \mathbf{i} through an angle θ in the counterclockwise direction. Explain why this form gives every unit vector in the plane.

12.3 | The Dot Product

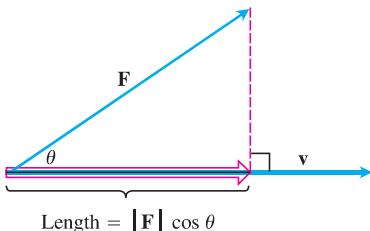


FIGURE 12.19 The magnitude of the force \mathbf{F} in the direction of vector \mathbf{v} is the length $|\mathbf{F}| \cos \theta$ of the projection of \mathbf{F} onto \mathbf{v} .

If a force \mathbf{F} is applied to a particle moving along a path, we often need to know the magnitude of the force in the direction of motion. If \mathbf{v} is parallel to the tangent line to the path at the point where \mathbf{F} is applied, then we want the magnitude of \mathbf{F} in the direction of \mathbf{v} . Figure 12.19 shows that the scalar quantity we seek is the length $|\mathbf{F}| \cos \theta$, where θ is the angle between the two vectors \mathbf{F} and \mathbf{v} .

In this section we show how to calculate easily the angle between two vectors directly from their components. A key part of the calculation is an expression called the *dot product*. Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another (as displayed in Figure 12.19) and to finding the work done by a constant force acting through a displacement.

Angle Between Vectors

When two nonzero vectors \mathbf{u} and \mathbf{v} are placed so their initial points coincide, they form an angle θ of measure $0 \leq \theta \leq \pi$ (Figure 12.20). If the vectors do not lie along the same line, the angle θ is measured in the plane containing both of them. If they do lie along the same line, the angle between them is 0 if they point in the same direction and π if they point in opposite directions. The angle θ is the **angle between \mathbf{u} and \mathbf{v}** . Theorem 1 gives a formula to determine this angle.

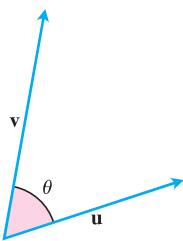


FIGURE 12.20 The angle between \mathbf{u} and \mathbf{v} .

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

Before proving Theorem 1, we focus attention on the expression $u_1 v_1 + u_2 v_2 + u_3 v_3$ in the calculation for θ . This expression is the sum of the products of the corresponding components for the vectors \mathbf{u} and \mathbf{v} .

DEFINITION The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“ \mathbf{u} dot \mathbf{v} ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE 1

(a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$
 $= -6 - 4 + 3 = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$ ■

The dot product of a pair of two-dimensional vectors is defined in a similar fashion:

$$\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2.$$

We will see throughout the remainder of the book that the dot product is a key tool for many important geometric and physical calculations in space (and the plane), not just for finding the angle between two vectors.

Proof of Theorem 1 Applying the law of cosines (Equation (8), Section 1.3) to the triangle in Figure 12.21, we find that

$$\begin{aligned} |\mathbf{w}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta && \text{Law of cosines} \\ 2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2. \end{aligned}$$

Because $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the component form of \mathbf{w} is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$\begin{aligned} |\mathbf{u}|^2 &= (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = u_1^2 + u_2^2 + u_3^2 \\ |\mathbf{v}|^2 &= (\sqrt{v_1^2 + v_2^2 + v_3^2})^2 = v_1^2 + v_2^2 + v_3^2 \\ |\mathbf{w}|^2 &= (\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2})^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \end{aligned}$$

and

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1 v_1 + u_2 v_2 + u_3 v_3).$$

Therefore,

$$\begin{aligned} 2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ |\mathbf{u}||\mathbf{v}|\cos\theta &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ \cos\theta &= \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}||\mathbf{v}|}. \end{aligned}$$

Since $0 \leq \theta < \pi$, we have

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}||\mathbf{v}|} \right). ■$$

In the notation of the dot product, the angle between two vectors \mathbf{u} and \mathbf{v} is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \right).$$

EXAMPLE 2 Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{-4}{(3)(7)} \right) \approx 1.76 \text{ radians.} \quad \blacksquare$$

The angle formula applies to two-dimensional vectors as well.

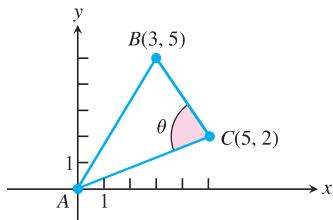


FIGURE 12.22 The triangle in Example 3.

EXAMPLE 3 Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (Figure 12.22).

Solution The angle θ is the angle between the vectors \overrightarrow{CA} and \overrightarrow{CB} . The component forms of these two vectors are

$$\overrightarrow{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}| |\overrightarrow{CB}|} \right) \\ &= \cos^{-1} \left(\frac{4}{(\sqrt{29})(\sqrt{13})} \right) \\ &\approx 78.1^\circ \quad \text{or} \quad 1.36 \text{ radians.} \end{aligned} \quad \blacksquare$$

Perpendicular (Orthogonal) Vectors

Two nonzero vectors \mathbf{u} and \mathbf{v} are perpendicular or **orthogonal** if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u} \cdot \mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If \mathbf{u} and \mathbf{v} are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 0$, then $\cos \theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

DEFINITION Vectors \mathbf{u} and \mathbf{v} are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 To determine if two vectors are orthogonal, calculate their dot product.

- (a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.
- (b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector \mathbf{u} since

$$\begin{aligned}\mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0.\end{aligned}$$

Dot Product Properties and Vector Projections

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

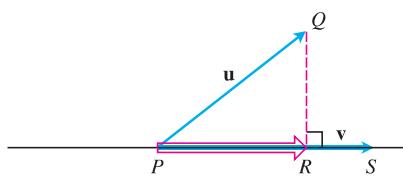
Properties of the Dot Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

- | | |
|---|---|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | 2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | 4. $\mathbf{u} \cdot \mathbf{u} = \mathbf{u} ^2$ |
| 5. $\mathbf{0} \cdot \mathbf{u} = 0.$ | |

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)



Proofs of Properties 1 and 3 The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

$$1. \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$$

$$3. \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

$$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$$

$$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3$$

$$= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3)$$

$$= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

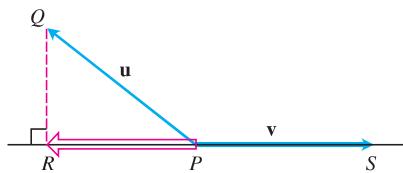


FIGURE 12.23 The vector projection of \mathbf{u} onto \mathbf{v} .

We now return to the problem of projecting one vector onto another, posed in the opening to this section. The **vector projection** of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ (Figure 12.23) is the vector \overrightarrow{PR} determined by dropping a perpendicular from Q to the line PS . The notation for this vector is

$\text{proj}_{\mathbf{v}} \mathbf{u}$ (“the vector projection of \mathbf{u} onto \mathbf{v} ”).

If \mathbf{u} represents a force, then $\text{proj}_{\mathbf{v}} \mathbf{u}$ represents the effective force in the direction of \mathbf{v} (Figure 12.24).

If the angle θ between \mathbf{u} and \mathbf{v} is acute, $\text{proj}_{\mathbf{v}} \mathbf{u}$ has length $|\mathbf{u}| \cos \theta$ and direction $\mathbf{v}/|\mathbf{v}|$ (Figure 12.25). If θ is obtuse, $\cos \theta < 0$ and $\text{proj}_{\mathbf{v}} \mathbf{u}$ has length $-|\mathbf{u}| \cos \theta$ and direction $-\mathbf{v}/|\mathbf{v}|$. In both cases,

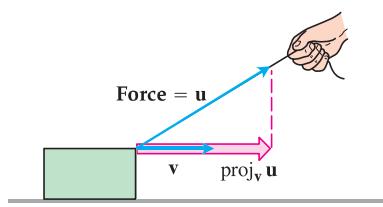


FIGURE 12.24 If we pull on the box with force \mathbf{u} , the effective force moving the box forward in the direction \mathbf{v} is the projection of \mathbf{u} onto \mathbf{v} .

$$\text{proj}_{\mathbf{v}} \mathbf{u} = (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} \quad |\mathbf{u}| \cos \theta = \frac{|\mathbf{u}| |\mathbf{v}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$

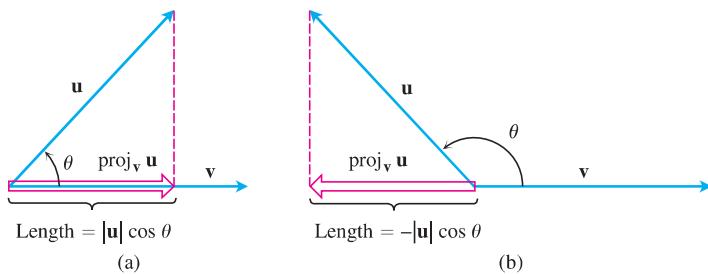


FIGURE 12.25 The length of $\text{proj}_v \mathbf{u}$ is (a) $|\mathbf{u}| \cos \theta$ if $\cos \theta \geq 0$ and (b) $-|\mathbf{u}| \cos \theta$ if $\cos \theta < 0$.

The number $|\mathbf{u}| \cos \theta$ is called the **scalar component of \mathbf{u} in the direction of \mathbf{v}** (or of \mathbf{u} onto \mathbf{v}). To summarize,

The vector projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_v \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}. \quad (1)$$

The scalar component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (2)$$

Note that both the vector projection of \mathbf{u} onto \mathbf{v} and the scalar component of \mathbf{u} onto \mathbf{v} depend only on the direction of the vector \mathbf{v} and not its length (because we dot \mathbf{u} with $\mathbf{v}/|\mathbf{v}|$, which is the direction of \mathbf{v}).

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

Solution We find $\text{proj}_v \mathbf{u}$ from Equation (1):

$$\begin{aligned} \text{proj}_v \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}. \end{aligned}$$

We find the scalar component of \mathbf{u} in the direction of \mathbf{v} from Equation (2):

$$\begin{aligned} |\mathbf{u}| \cos \theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}. \end{aligned}$$

Equations (1) and (2) also apply to two-dimensional vectors. We demonstrate this in the next example.

EXAMPLE 6 Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of \mathbf{F} in the direction of \mathbf{v} .

Solution The vector projection is

$$\begin{aligned}\text{proj}_v \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \\ &= \frac{5 - 6}{1 + 9} (\mathbf{i} - 3\mathbf{j}) = -\frac{1}{10} (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10} \mathbf{i} + \frac{3}{10} \mathbf{j}.\end{aligned}$$

The scalar component of \mathbf{F} in the direction of \mathbf{v} is

$$|\mathbf{F}| \cos \theta = \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{5 - 6}{\sqrt{1 + 9}} = -\frac{1}{\sqrt{10}}.$$

A routine calculation (see Exercise 29) verifies that the vector $\mathbf{u} - \text{proj}_v \mathbf{u}$ is orthogonal to the projection vector $\text{proj}_v \mathbf{u}$ (which has the same direction as \mathbf{v}). So the equation

$$\mathbf{u} = \text{proj}_v \mathbf{u} + (\mathbf{u} - \text{proj}_v \mathbf{u}) = \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right)}_{\text{Orthogonal to } \mathbf{v}}$$

expresses \mathbf{u} as a sum of orthogonal vectors.

Work

In Chapter 6, we calculated the work done by a constant force of magnitude F in moving an object through a distance d as $W = Fd$. That formula holds only if the force is directed along the line of motion. If a force \mathbf{F} moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of \mathbf{F} in the direction of \mathbf{D} . If θ is the angle between \mathbf{F} and \mathbf{D} (Figure 12.26), then

$$\begin{aligned}\text{Work} &= \left(\text{scalar component of } \mathbf{F} \text{ in the direction of } \mathbf{D} \right) (\text{length of } \mathbf{D}) \\ &= (|\mathbf{F}| \cos \theta) |\mathbf{D}| \\ &= \mathbf{F} \cdot \mathbf{D}.\end{aligned}$$

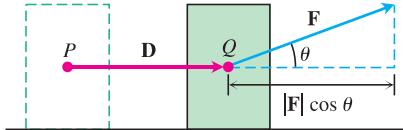


FIGURE 12.26 The work done by a constant force \mathbf{F} during a displacement \mathbf{D} is $(|\mathbf{F}| \cos \theta)|\mathbf{D}|$, which is the dot product $\mathbf{F} \cdot \mathbf{D}$.

DEFINITION The **work** done by a constant force \mathbf{F} acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

EXAMPLE 7 If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60^\circ$, the work done by \mathbf{F} in acting from P to Q is

$$\begin{aligned}\text{Work} &= \mathbf{F} \cdot \mathbf{D} && \text{Definition} \\ &= |\mathbf{F}| |\mathbf{D}| \cos \theta \\ &= (40)(3) \cos 60^\circ && \text{Given values} \\ &= (120)(1/2) = 60 \text{ J (joules).} && \blacksquare\end{aligned}$$

We encounter more challenging work problems in Chapter 16 when we learn to find the work done by a variable force along a *path* in space.

Exercises 12.3

Dot Product and Projections

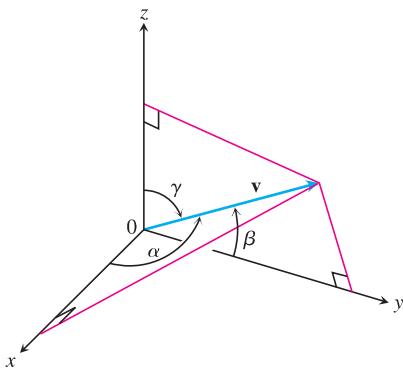
In Exercises 1–8, find

- $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$
 - the cosine of the angle between \mathbf{v} and \mathbf{u}
 - the scalar component of \mathbf{u} in the direction of \mathbf{v}
 - the vector $\text{proj}_{\mathbf{v}} \mathbf{u}$.
- $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$, $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
 - $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$, $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$
 - $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$
 - $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
 - $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
 - $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$
 - $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{u} = 2\mathbf{i} + \sqrt{17}\mathbf{j}$
 - $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

Angle Between Vectors

Find the angles between the vectors in Exercises 9–12 to the nearest hundredth of a radian.

- $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$
- $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
- Triangle** Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.
- Rectangle** Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$, and $D = (4, 1)$.
- Direction angles and direction cosines** The *direction angles* α , β , and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:
 α is the angle between \mathbf{v} and the positive x -axis ($0 \leq \alpha \leq \pi$)
 β is the angle between \mathbf{v} and the positive y -axis ($0 \leq \beta \leq \pi$)
 γ is the angle between \mathbf{v} and the positive z -axis ($0 \leq \gamma \leq \pi$).



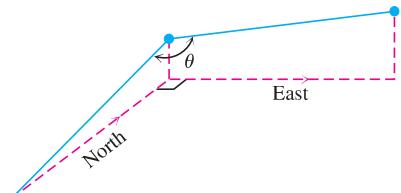
- a. Show that

$$\cos \alpha = \frac{a}{|\mathbf{v}|}, \quad \cos \beta = \frac{b}{|\mathbf{v}|}, \quad \cos \gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. These cosines are called the *direction cosines* of \mathbf{v} .

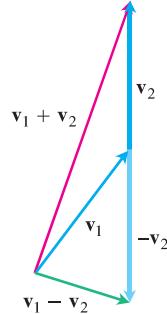
- b. **Unit vectors are built from direction cosines** Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a , b , and c are the direction cosines of \mathbf{v} .

16. **Water main construction** A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.

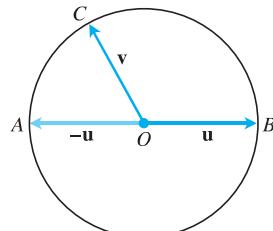


Theory and Examples

17. **Sums and differences** In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.

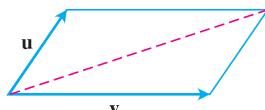


18. **Orthogonality on a circle** Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B . Show that \overrightarrow{CA} and \overrightarrow{CB} are orthogonal.

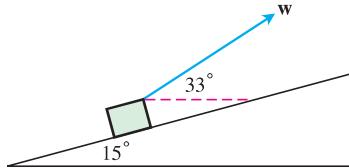


19. **Diagonals of a rhombus** Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.

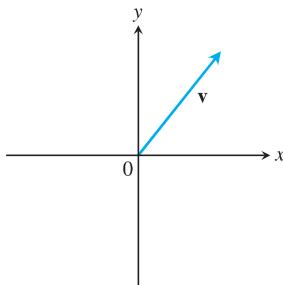
- 20. Perpendicular diagonals** Show that squares are the only rectangles with perpendicular diagonals.
- 21. When parallelograms are rectangles** Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. (This fact is often exploited by carpenters.)
- 22. Diagonal of parallelogram** Show that the indicated diagonal of the parallelogram determined by vectors \mathbf{u} and \mathbf{v} bisects the angle between \mathbf{u} and \mathbf{v} if $|\mathbf{u}| = |\mathbf{v}|$.



- 23. Projectile motion** A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.
- 24. Inclined plane** Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force \mathbf{w} needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.



- 25. a. Cauchy-Schwartz inequality** Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$, show that the inequality $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}|$ holds for any vectors \mathbf{u} and \mathbf{v} .
- b.** Under what circumstances, if any, does $|\mathbf{u} \cdot \mathbf{v}|$ equal $|\mathbf{u}| |\mathbf{v}|$? Give reasons for your answer.
- 26.** Copy the axes and vector shown here. Then shade in the points (x, y) for which $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} \leq 0$. Justify your answer.



- 27. Orthogonal unit vectors** If \mathbf{u}_1 and \mathbf{u}_2 are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v} \cdot \mathbf{u}_1$.
- 28. Cancellation in dot products** In real-number multiplication, if $uv_1 = uv_2$ and $u \neq 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product? That is, if $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$ and $\mathbf{u} \neq \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.
- 29.** Using the definition of the projection of \mathbf{u} onto \mathbf{v} , show by direct calculation that $(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \text{proj}_{\mathbf{v}} \mathbf{u} = 0$.
- 30.** A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express \mathbf{F} as a sum of a vector parallel to \mathbf{v} and a vector orthogonal to \mathbf{v} .

Equations for Lines in the Plane

- 31. Line perpendicular to a vector** Show that $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of the vector \mathbf{v} is the negative reciprocal of the slope of the given line.
- 32. Line parallel to a vector** Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing \mathbf{v} is the same as the slope of the given line.

In Exercises 33–36, use the result of Exercise 31 to find an equation for the line through P perpendicular to \mathbf{v} . Then sketch the line. Include \mathbf{v} in your sketch as a vector starting at the origin.

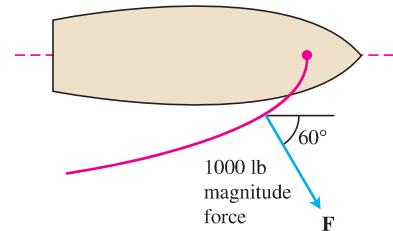
33. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ 34. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$
 35. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ 36. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 37–40, use the result of Exercise 32 to find an equation for the line through P parallel to \mathbf{v} . Then sketch the line. Include \mathbf{v} in your sketch as a vector starting at the origin.

37. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ 38. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
 39. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ 40. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

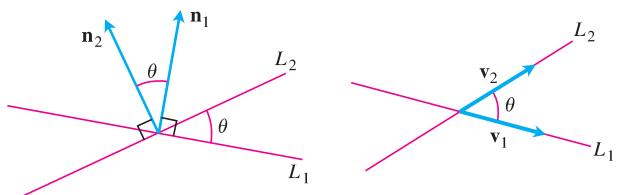
Work

- 41. Work along a line** Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point $(1, 1)$ (distance in meters).
- 42. Locomotive** The Union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?
- 43. Inclined plane** How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?
- 44. Sailboat** The wind passing over a boat's sail exerted a 1000-lb magnitude force \mathbf{F} as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The acute angle between intersecting lines that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



Use this fact and the results of Exercise 31 or 32 to find the acute angles between the lines in Exercises 45–50.

45. $3x + y = 5$, $2x - y = 4$
 46. $y = \sqrt{3}x - 1$, $y = -\sqrt{3}x + 2$

47. $\sqrt{3}x - y = -2$, $x - \sqrt{3}y = 1$
 48. $x + \sqrt{3}y = 1$, $(1 - \sqrt{3})x + (1 + \sqrt{3})y = 8$
 49. $3x - 4y = 3$, $x - y = 7$
 50. $12x + 5y = 1$, $2x - 2y = 3$

12.4 | The Cross Product

In studying lines in the plane, when we needed to describe how a line was tilting, we used the notions of slope and angle of inclination. In space, we want a way to describe how a *plane* is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane. The direction of this third vector tells us the “inclination” of the plane. The product we use to multiply the vectors together is the *vector* or *cross product*, the second of the two vector multiplication methods. We study the cross product in this section.

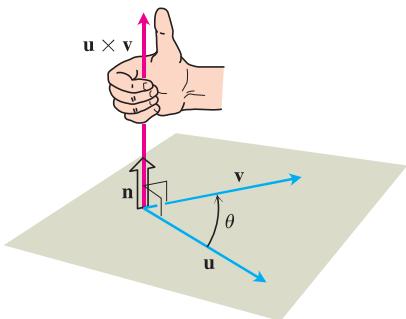


FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

The Cross Product of Two Vectors in Space

We start with two nonzero vectors \mathbf{u} and \mathbf{v} in space. If \mathbf{u} and \mathbf{v} are not parallel, they determine a plane. We select a unit vector \mathbf{n} perpendicular to the plane by the **right-hand rule**. This means that we choose \mathbf{n} to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from \mathbf{u} to \mathbf{v} (Figure 12.27). Then the **cross product** $\mathbf{u} \times \mathbf{v}$ (“ \mathbf{u} cross \mathbf{v} ”) is the *vector* defined as follows.

DEFINITION

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}| |\mathbf{v}| \sin \theta) \mathbf{n}$$

Unlike the dot product, the cross product is a vector. For this reason it's also called the **vector product** of \mathbf{u} and \mathbf{v} , and applies *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} because it is a scalar multiple of \mathbf{n} .

There is a straightforward way to calculate the cross product of two vectors from their components. The method does not require that we know the angle between them (as suggested by the definition), but we postpone that calculation momentarily so we can focus first on the properties of the cross product.

Since the sines of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\mathbf{0}$. If one or both of \mathbf{u} and \mathbf{v} are zero, we also define $\mathbf{u} \times \mathbf{v}$ to be zero. This way, the cross product of two vectors \mathbf{u} and \mathbf{v} is zero if and only if \mathbf{u} and \mathbf{v} are parallel or one or both of them are zero.

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The cross product obeys the following laws.

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r , s are scalars, then

- | | |
|--|---|
| 1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$ | 2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ |
| 3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ | 4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$ |
| 5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$ | 6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ |

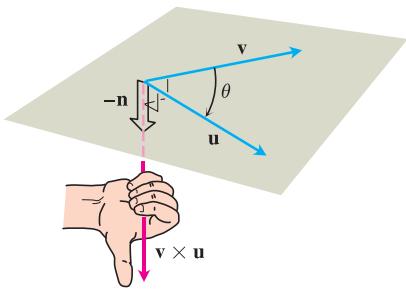


FIGURE 12.28 The construction of $\mathbf{v} \times \mathbf{u}$.

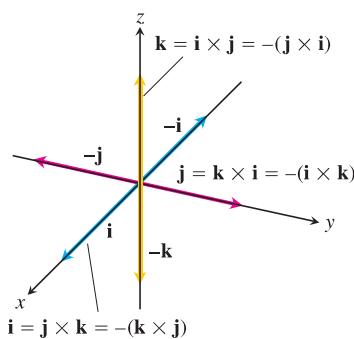


FIGURE 12.29 The pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} .

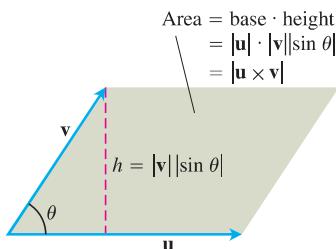


FIGURE 12.30 The parallelogram determined by \mathbf{u} and \mathbf{v} .

To visualize Property 3, for example, notice that when the fingers of your right hand curl through the angle θ from \mathbf{v} to \mathbf{u} , your thumb points the opposite way; the unit vector we choose in forming $\mathbf{v} \times \mathbf{u}$ is the negative of the one we choose in forming $\mathbf{u} \times \mathbf{v}$ (Figure 12.28).

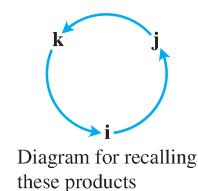
Property 1 can be verified by applying the definition of cross product to both sides of the equation and comparing the results. Property 2 is proved in Appendix 8. Property 4 follows by multiplying both sides of the equation in Property 2 by -1 and reversing the order of the products using Property 3. Property 5 is a definition. As a rule, cross product multiplication is *not associative* so $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ does not generally equal $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. (See Additional Exercise 17.)

When we apply the definition to calculate the pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} , we find (Figure 12.29)

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$



and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because \mathbf{n} is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

This is the area of the parallelogram determined by \mathbf{u} and \mathbf{v} (Figure 12.30), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}| |\sin \theta|$ the height.

Determinant Formula for $\mathbf{u} \times \mathbf{v}$

Our next objective is to calculate $\mathbf{u} \times \mathbf{v}$ from the components of \mathbf{u} and \mathbf{v} relative to a Cartesian coordinate system.

Suppose that

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

Then the distributive laws and the rules for multiplying \mathbf{i} , \mathbf{j} , and \mathbf{k} tell us that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\ &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\ &\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\ &\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}. \end{aligned}$$

The component terms in the last line are hard to remember, but they are the same as the terms in the expansion of the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Determinants

2×2 and 3×3 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (1)(-4) = 6 + 4 = 10$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{aligned} \text{EXAMPLE} \quad & \begin{vmatrix} -5 & 3 & 1 \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ & - (3) \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \\ & = -5(1 - 3) - 3(2 + 4) + 1(6 + 4) \\ & = 10 - 18 + 10 = 2 \end{aligned}$$

(For more information, see the Web site at www.aw.com/thomas.)

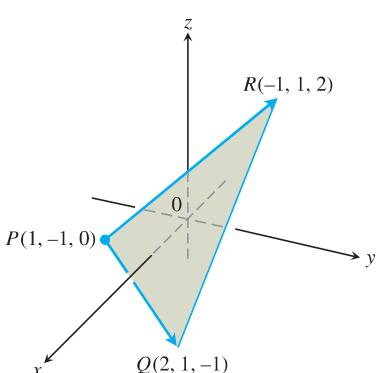


FIGURE 12.31 The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\vec{PQ} \times \vec{PR}|$ (Example 3).

So we restate the calculation in this easy-to-remember form.

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1 Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \\ \mathbf{v} \times \mathbf{u} &= -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \end{aligned}$$

EXAMPLE 2 Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Solution The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\begin{aligned} \vec{PQ} &= (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ \vec{PR} &= (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}. \end{aligned}$$

EXAMPLE 3 Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Solution The area of the parallelogram determined by P , Q , and R is

$$\begin{aligned} |\vec{PQ} \times \vec{PR}| &= |6\mathbf{i} + 6\mathbf{k}| \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}. \end{aligned}$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE 4 Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane, its direction \mathbf{n} is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Torque

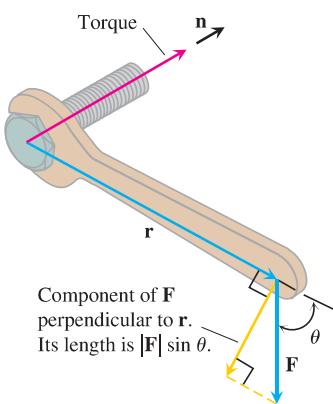


FIGURE 12.32 The torque vector describes the tendency of the force \mathbf{F} to drive the bolt forward.

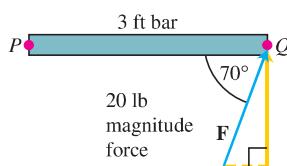


FIGURE 12.33 The magnitude of the torque exerted by \mathbf{F} at P is about 56.4 ft-lb (Example 5). The bar rotates counter-clockwise around P .

When we turn a bolt by applying a force \mathbf{F} to a wrench (Figure 12.32), we produce a torque that causes the bolt to rotate. The **torque vector** points in the direction of the axis of the bolt according to the right-hand rule (so the rotation is counterclockwise when viewed from the *tip* of the vector). The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm \mathbf{r} and the scalar component of \mathbf{F} perpendicular to \mathbf{r} . In the notation of Figure 12.32,

$$\text{Magnitude of torque vector} = |\mathbf{r}| |\mathbf{F}| \sin \theta,$$

or $|\mathbf{r} \times \mathbf{F}|$. If we let \mathbf{n} be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r} \times \mathbf{F}$, or

$$\text{Torque vector} = (|\mathbf{r}| |\mathbf{F}| \sin \theta) \mathbf{n}.$$

Recall that we defined $\mathbf{u} \times \mathbf{v}$ to be $\mathbf{0}$ when \mathbf{u} and \mathbf{v} are parallel. This is consistent with the torque interpretation as well. If the force \mathbf{F} in Figure 12.32 is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

EXAMPLE 5 The magnitude of the torque generated by force \mathbf{F} at the pivot point P in Figure 12.33 is

$$\begin{aligned} |\vec{PQ} \times \mathbf{F}| &= |\vec{PQ}| |\mathbf{F}| \sin 70^\circ \\ &\approx (3)(20)(0.94) \\ &\approx 56.4 \text{ ft-lb}. \end{aligned}$$

In this example the torque vector is pointing out of the page toward you. ■

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} (Figure 12.34). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base

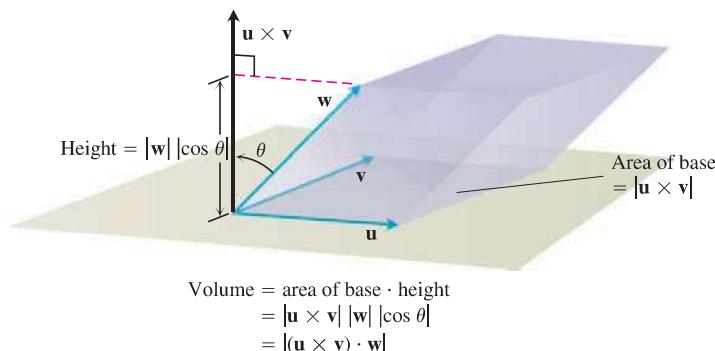


FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

parallelogram. The number $|\mathbf{w}||\cos \theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} .

By treating the planes of \mathbf{v} and \mathbf{w} and of \mathbf{w} and \mathbf{u} as the base planes of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} , we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w} \\ &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■

Exercises 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

1. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$

2. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$

3. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

4. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$

5. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$

6. $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$

7. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

8. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$

10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$

11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

Triangles in Space

In Exercises 15–18,

- Find the area of the triangle determined by the points P , Q , and R .
 - Find a unit vector perpendicular to plane PQR .
15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$
 16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$
 17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
 18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Triple Scalar Products

In Exercises 19–22, verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped (box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} .

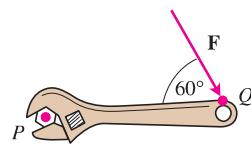
\mathbf{u}	\mathbf{v}	\mathbf{w}
19. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
20. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
21. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
22. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Theory and Examples

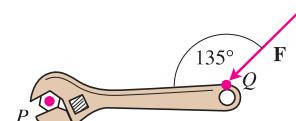
23. **Parallel and perpendicular vectors** Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.
24. **Parallel and perpendicular vectors** Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

In Exercises 25 and 26, find the magnitude of the torque exerted by \mathbf{F} on the bolt at P if $|\overrightarrow{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25.



26.



27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$
- $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$
- $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

c. $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$

d. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number c)

e. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number c)

f. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

g. $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$

h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

29. Given nonzero vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} , use dot product and cross product notation, as appropriate, to describe the following.

- a. The vector projection of \mathbf{u} onto \mathbf{v}

- b. A vector orthogonal to \mathbf{u} and \mathbf{v}

- c. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and \mathbf{w}

- d. The volume of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w}

- e. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$

- f. A vector of length $|\mathbf{u}|$ in the direction of \mathbf{v}

30. Compute $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. What can you conclude about the associativity of the cross product?

31. Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors. Which of the following make sense, and which do not? Give reasons for your answers.

- a. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

- b. $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

- c. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

- d. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

32. **Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of \mathbf{u} and \mathbf{v} , whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of \mathbf{v} and \mathbf{w} . What are the degenerate cases?

33. **Cancellation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

34. **Double cancellation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$

36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$

37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$

38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$

39. $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$

40. $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

41. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$

42. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$

43. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$

44. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$

45. $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$

46. $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$

47. $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$

48. Find the volume of a parallelepiped if four of its eight vertices are $A(0, 0, 0)$, $B(1, 2, 0)$, $C(0, -3, 2)$, and $D(3, -4, 5)$.
49. **Triangle area** Find a formula for the area of the triangle in the xy -plane with vertices at $(0, 0)$, (a_1, a_2) , and (b_1, b_2) . Explain your work.
50. **Triangle area** Find a concise formula for the area of a triangle in the xy -plane with vertices (a_1, a_2) , (b_1, b_2) , and (c_1, c_2) .

12.5 | Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space. We will use these representations throughout the rest of the book.

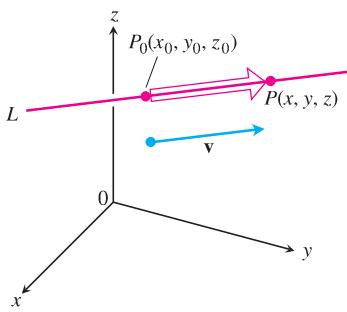


FIGURE 12.35 A point P lies on L through P_0 parallel to \mathbf{v} if and only if $\overrightarrow{P_0P}$ is a scalar multiple of \mathbf{v} .

Lines and Line Segments in Space

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a *vector* giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to \mathbf{v} (Figure 12.35). Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter t . The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \quad (1)$$

If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and \mathbf{r}_0 is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty, \quad (2)$$

where \mathbf{r} is the position vector of a point $P(x, y, z)$ on L and \mathbf{r}_0 is the position vector of $P_0(x_0, y_0, z_0)$.

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter t :

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \quad (3)$$

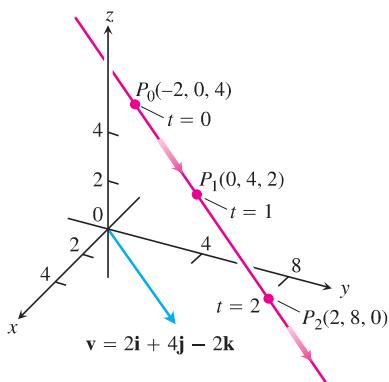


FIGURE 12.36 Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing t .

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.36).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t. \quad \blacksquare$$

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the “base point” and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t . \blacksquare

Notice that parametrizations are not unique. Not only can the “base point” change, but so can the parameter. The equations $x = -3 + 4t^3$, $y = 2 - 3t^3$, and $z = -3 + 7t^3$ also parametrize the line in Example 2.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the t -values for the endpoints and restrict t to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.

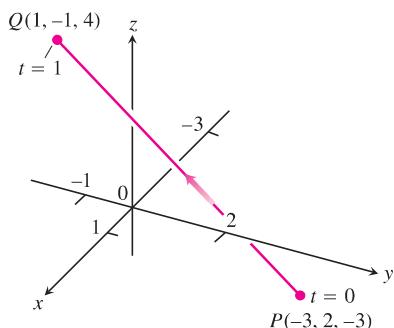


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ . The arrow shows the direction of increasing t .

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).

Solution We begin with equations for the line through P and Q , taking them, in this case, from Example 2:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1. \quad \blacksquare$$

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector \mathbf{v} . Rewriting Equation (2), we have

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \mathbf{r}_0 + t|\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}.\end{aligned} \tag{4}$$

Initial position Time Speed Direction

In other words, the position of the particle at time t is its initial position plus its distance moved (speed \times time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60 \text{ ft/sec})(10 \text{ sec}) = 600 \text{ ft}$, which is the length of the vector $\mathbf{r}(10)$. ■

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector \mathbf{v} , we find the absolute value of the scalar component of \vec{PS} in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is $|\vec{PS} \times \mathbf{v}|$, which is $\frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

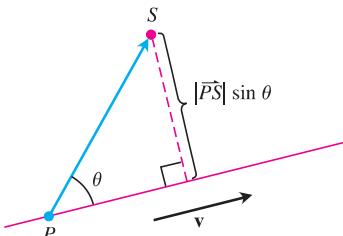


FIGURE 12.38 The distance from S to the line through P parallel to \mathbf{v} is $|\vec{PS}| \sin \theta$, where θ is the angle between \vec{PS} and \mathbf{v} .

Distance from a Point S to a Line Through P Parallel to \mathbf{v}

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\vec{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}. \blacksquare$$

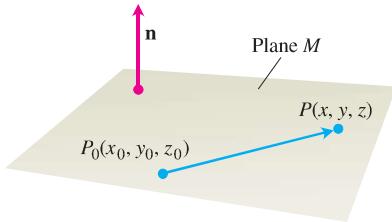


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to \mathbf{n} if and only if $\mathbf{n} \cdot \vec{P_0P} = 0$.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\vec{P_0P}$ is orthogonal to \mathbf{n} (Figure 12.39). Thus, the dot product $\mathbf{n} \cdot \vec{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \vec{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where

$$D = Ax_0 + By_0 + Cz_0$$

EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22. \blacksquare$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x , y , and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6. \quad \blacksquare$$

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k . Two planes that are not parallel intersect in a line.

EXAMPLE 8 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to both planes' normal vectors \mathbf{n}_1 and \mathbf{n}_2 (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■

EXAMPLE 9 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z . The different choices would simply give different parametrizations of the same line. ■

Sometimes we want to know where a line and a plane intersect. For example, if we are looking at a flat plate and a line segment passes through it, we may be interested in knowing what portion of the line segment is hidden from our view by the plate. This application is used in computer graphics (Exercise 74).

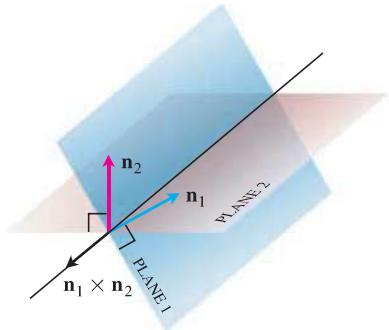


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

EXAMPLE 10 Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t \right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1 \right) = \left(\frac{2}{3}, 2, 0 \right). \quad \blacksquare$$

The Distance from a Point to a Plane

If P is a point on a plane with normal \mathbf{n} , then the distance from any point S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} . That is, the distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of \overrightarrow{PS} onto a vector \mathbf{n} normal to the plane (Figure 12.41). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

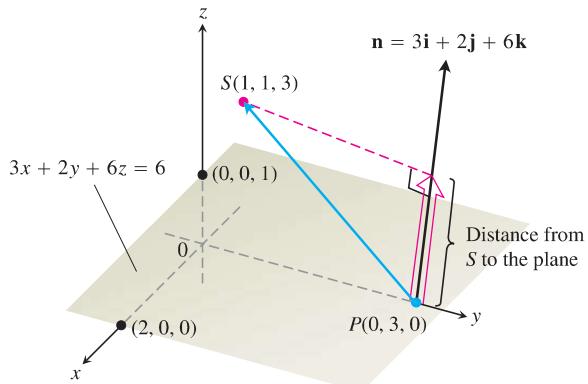


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} (Example 11).

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y -intercept $(0, 3, 0)$, then

$$\begin{aligned}\vec{PS} &= (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.\end{aligned}$$

The distance from S to the plane is

$$\begin{aligned}d &= \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| && \text{length of proj}_{\mathbf{n}} \vec{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}.\end{aligned}$$
■

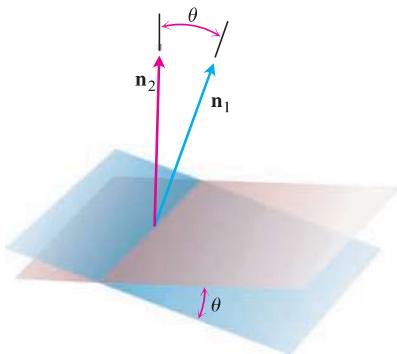


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{4}{21} \right) \\ &\approx 1.38 \text{ radians.} && \text{About } 79 \text{ deg}\end{aligned}$$
■

Exercises 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z -axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$

10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

11. The x -axis
12. The z -axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0), (1, 1, 3/2)$
14. $(0, 0, 0), (1, 0, 0)$
15. $(1, 0, 0), (1, 1, 0)$
16. $(1, 1, 0), (1, 1, 1)$
17. $(0, 1, 1), (0, -1, 1)$
18. $(0, 2, 0), (3, 0, 0)$
19. $(2, 0, 2), (0, 2, 0)$
20. $(1, 0, -1), (0, 3, 0)$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
 22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1), (2, 0, 2)$, and $(0, -2, 1)$
 24. The plane through $(2, 4, 5), (1, 5, 7)$, and $(-1, 6, 8)$
 25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
 27. Find the point of intersection of the lines $x = 2t + 1, y = 3t + 2, z = 4t + 3$, and $x = s + 2, y = 2s + 4, z = -4s - 1$, and then find the plane determined by these lines.
 28. Find the point of intersection of the lines $x = t, y = -t + 2, z = t + 1$, and $x = 2s + 2, y = s + 3, z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane determined by the intersecting lines.

29. L1: $x = -1 + t, y = 2 + t, z = 1 - t; -\infty < t < \infty$
 L2: $x = 1 - 4s, y = 1 + 2s, z = 2 - 2s; -\infty < s < \infty$
 30. L1: $x = t, y = 3 - 3t, z = -2 - t; -\infty < t < \infty$
 L2: $x = 1 + s, y = 4 + s, z = -1 + s; -\infty < s < \infty$
 31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3, x + 2y + z = 2$.
 32. Find a plane through the points $P_1(1, 2, 3), P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12); x = 4t, y = -2t, z = 2t$
 34. $(0, 0, 0); x = 5 + 3t, y = 5 + 4t, z = -3 - 5t$
 35. $(2, 1, 3); x = 2 + 2t, y = 1 + 6t, z = 3$
 36. $(2, 1, -1); x = 2t, y = 1 + 2t, z = 2t$
 37. $(3, -1, 4); x = 4 - t, y = 3 + 2t, z = -5 + 3t$
 38. $(-1, 4, 3); x = 10 + 4t, y = -3, z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4), x + 2y + 2z = 13$
 40. $(0, 0, 0), 3x + 2y + 6z = 6$
 41. $(0, 1, 1), 4y + 3z = -12$
 42. $(2, 2, 3), 2x + y + 2z = 4$
 43. $(0, -1, 0), 2x + y + 2z = 4$
 44. $(1, 0, -1), -4x + y + z = 4$
 45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.
 46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, 2x + y - 2z = 2$
 48. $5x + y - z = 10, x - 2y + 3z = -1$

T Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, 2x - 2y - z = 5$
 50. $x + y + z = 1, z = 0$ (the xy -plane)
 51. $2x + 2y - z = 3, x + 2y + z = 2$
 52. $4y + 3z = -12, 3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, y = 3t, z = 1 + t; 2x - y + 3z = 6$
 54. $x = 2, y = 3 + 2t, z = -2 - 2t; 6x + 3y - 4z = -12$
 55. $x = 1 + 2t, y = 1 + 5t, z = 3t; x + y + z = 2$
 56. $x = -1 + 3t, y = -2, z = 5t; 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, x + y = 2$
 58. $3x - 6y - 2z = 3, 2x + y - 2z = 2$
 59. $x - 2y + 4z = 2, x + y - 2z = 5$
 60. $5x - 2y = 11, 4y - 5z = -17$

Given two lines in space, either they are parallel, or they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky). Exercises 61 and 62 each give three lines. In each exercise, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection.

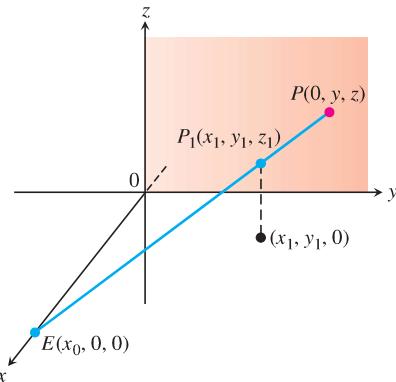
61. L1: $x = 3 + 2t, y = -1 + 4t, z = 2 - t; -\infty < t < \infty$
 L2: $x = 1 + 4s, y = 1 + 2s, z = -3 + 4s; -\infty < s < \infty$
 L3: $x = 3 + 2r, y = 2 + r, z = -2 + 2r; -\infty < r < \infty$
 62. L1: $x = 1 + 2t, y = -1 - t, z = 3t; -\infty < t < \infty$
 L2: $x = 2 - s, y = 3s, z = 1 + s; -\infty < s < \infty$
 L3: $x = 5 + 2r, y = 1 - r, z = 8 + 3r; -\infty < r < \infty$

Theory and Examples

63. Use Equations (3) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.
 64. Use the component form to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$.
 65. Find the points in which the line $x = 1 + 2t, y = -1 - t, z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.
 66. Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with \mathbf{i} and an angle of $\pi/3$ rad with \mathbf{j} . Describe the reasoning behind your answer.
 67. Is the line $x = 1 - 2t, y = 2 + 5t, z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.

68. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Give reasons for your answer.
69. Find two different planes whose intersection is the line $x = 1 + t, y = 2 - t, z = 3 + 2t$. Write equations for each plane in the form $Ax + By + Cz = D$.
70. Find a plane through the origin that is perpendicular to the plane $M: 2x + 3y + z = 12$ in a right angle. How do you know that your plane is perpendicular to M ?
71. The graph of $(x/a) + (y/b) + (z/c) = 1$ is a plane for any nonzero numbers a, b , and c . Which planes have an equation of this form?
72. Suppose L_1 and L_2 are disjoint (nonintersecting) nonparallel lines. Is it possible for a nonzero vector to be perpendicular to both L_1 and L_2 ? Give reasons for your answer.
73. **Perspective in computer graphics** In computer graphics and perspective drawing, we need to represent objects seen by the eye in space as images on a two-dimensional plane. Suppose that the eye is at $E(x_0, 0, 0)$ as shown here and that we want to represent a point $P_1(x_1, y_1, z_1)$ as a point on the yz -plane. We do this by projecting P_1 onto the plane with a ray from E . The point P_1 will be portrayed as the point $P(0, y, z)$. The problem for us as graphics designers is to find y and z given E and P_1 .
- a. Write a vector equation that holds between \vec{EP} and \vec{EP}_1 . Use the equation to express y and z in terms of x_0, x_1, y_1 , and z_1 .

- b. Test the formulas obtained for y and z in part (a) by investigating their behavior at $x_1 = 0$ and $x_1 = x_0$ and by seeing what happens as $x_0 \rightarrow \infty$. What do you find?



74. **Hidden lines in computer graphics** Here is another typical problem in computer graphics. Your eye is at $(4, 0, 0)$. You are looking at a triangular plate whose vertices are at $(1, 0, 1)$, $(1, 1, 0)$, and $(-2, 2, 2)$. The line segment from $(1, 0, 0)$ to $(0, 2, 2)$ passes through the plate. What portion of the line segment is hidden from your view by the plate? (This is an exercise in finding intersections of lines and planes.)

12.6 | Cylinders and Quadric Surfaces

Up to now, we have studied two special types of surfaces: spheres and planes. In this section, we extend our inventory to include a variety of cylinders and quadric surfaces. Quadric surfaces are surfaces defined by second-degree equations in x , y , and z . Spheres are quadric surfaces, but there are others of equal interest which will be needed in Chapters 14–16.

Cylinders

A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder (Figure 12.43). In solid geometry, where *cylinder* means *circular cylinder*, the generating curves are circles, but now we allow generating curves of any kind. The cylinder in our first example is generated by a parabola.

EXAMPLE 1 Find an equation for the cylinder made by the lines parallel to the z -axis that pass through the parabola $y = x^2, z = 0$ (Figure 12.44).

Solution The point $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy -plane. Then, for any value of z , the point $Q(x_0, x_0^2, z)$ lies on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z -axis. Conversely, any point $Q(x_0, x_0^2, z)$ whose y -coordinate is the square of its x -coordinate lies on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z -axis (Figure 12.44).

Regardless of the value of z , therefore, the points on the surface are the points whose coordinates satisfy the equation $y = x^2$. This makes $y = x^2$ an equation for the cylinder. Because of this, we call the cylinder “the cylinder $y = x^2$.” ■

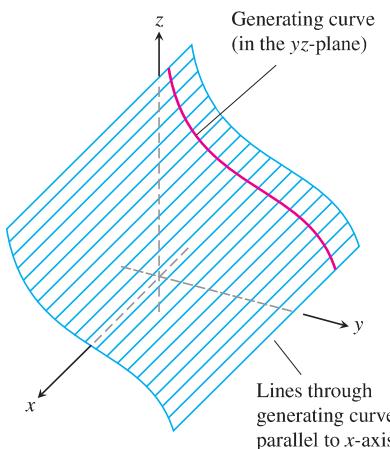


FIGURE 12.43 A cylinder and generating curve.

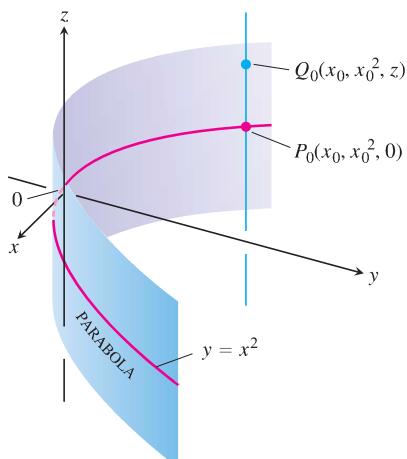


FIGURE 12.44 Every point of the cylinder in Example 1 has coordinates of the form (x_0, x_0^2, z) . We call it “the cylinder $y = x^2$.”

As Example 1 suggests, any curve $f(x, y) = c$ in the xy -plane defines a cylinder parallel to the z -axis whose equation is also $f(x, y) = c$. For instance, the equation $x^2 + y^2 = 1$ defines the circular cylinder made by the lines parallel to the z -axis that pass through the circle $x^2 + y^2 = 1$ in the xy -plane.

In a similar way, any curve $g(x, z) = c$ in the xz -plane defines a cylinder parallel to the y -axis whose space equation is also $g(x, z) = c$. Any curve $h(y, z) = c$ defines a cylinder parallel to the x -axis whose space equation is also $h(y, z) = c$. The axis of a cylinder need not be parallel to a coordinate axis, however.

Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in x , y , and z . We focus on the special equation

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where A , B , C , D , and E are constants. The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**. Spheres are special cases of ellipsoids. We present a few examples illustrating how to sketch a quadric surface, and then give a summary table of graphs of the basic types.

EXAMPLE 2 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Figure 12.45) cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$. It lies within the rectangular box defined by the inequalities $|x| \leq a$, $|y| \leq b$, and $|z| \leq c$. The surface is symmetric with respect to each of the coordinate planes because each variable in the defining equation is squared.

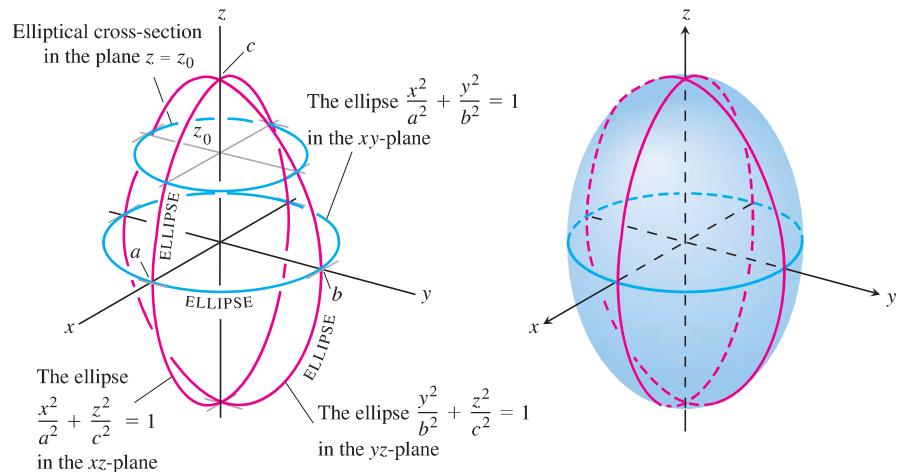


FIGURE 12.45 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

The curves in which the three coordinate planes cut the surface are ellipses. For example,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{when} \quad z = 0.$$

The curve cut from the surface by the plane $z = z_0$, $|z_0| < c$, is the ellipse

$$\frac{x^2}{a^2(1 - (z_0/c)^2)} + \frac{y^2}{b^2(1 - (z_0/c)^2)} = 1.$$

If any two of the semiaxes a , b , and c are equal, the surface is an **ellipsoid of revolution**. If all three are equal, the surface is a sphere. ■

EXAMPLE 3 The **hyperbolic paraboloid**

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$

has symmetry with respect to the planes $x = 0$ and $y = 0$ (Figure 12.46). The cross-sections in these planes are

$$x = 0: \text{ the parabola } z = \frac{c}{b^2}y^2. \quad (1)$$

$$y = 0: \text{ the parabola } z = -\frac{c}{a^2}x^2. \quad (2)$$

In the plane $x = 0$, the parabola opens upward from the origin. The parabola in the plane $y = 0$ opens downward.

If we cut the surface by a plane $z = z_0 > 0$, the cross-section is a hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c},$$

with its focal axis parallel to the y -axis and its vertices on the parabola in Equation (1). If z_0 is negative, the focal axis is parallel to the x -axis and the vertices lie on the parabola in Equation (2).

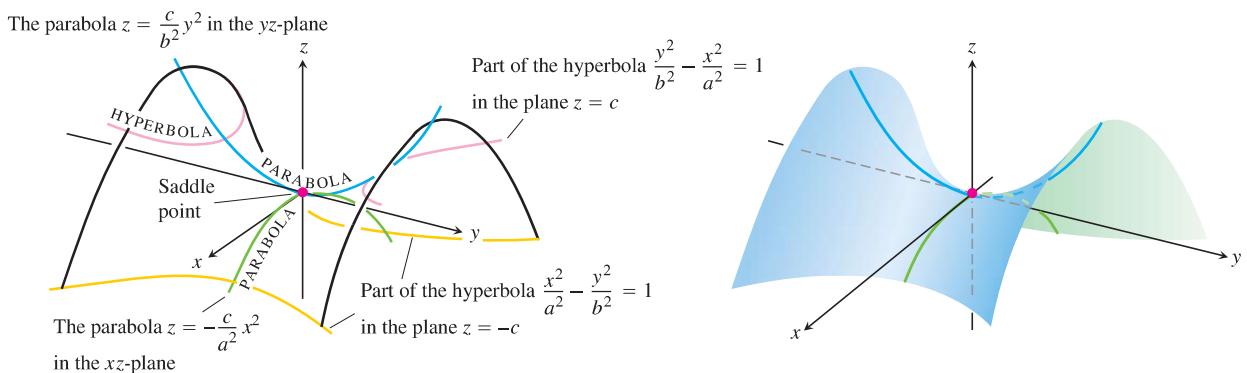
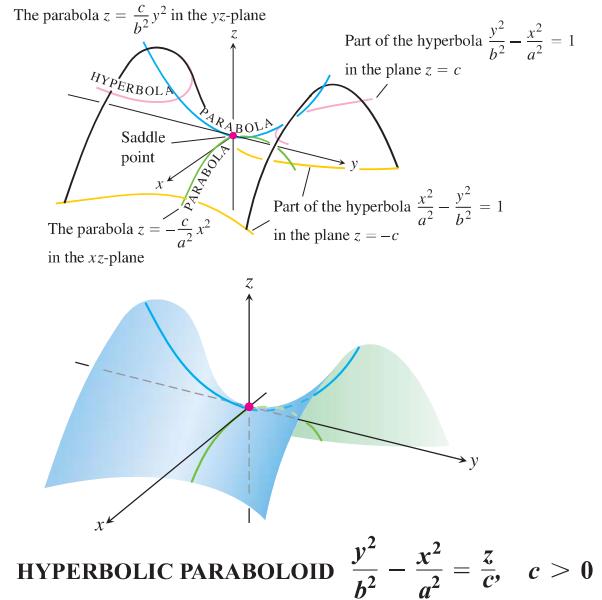
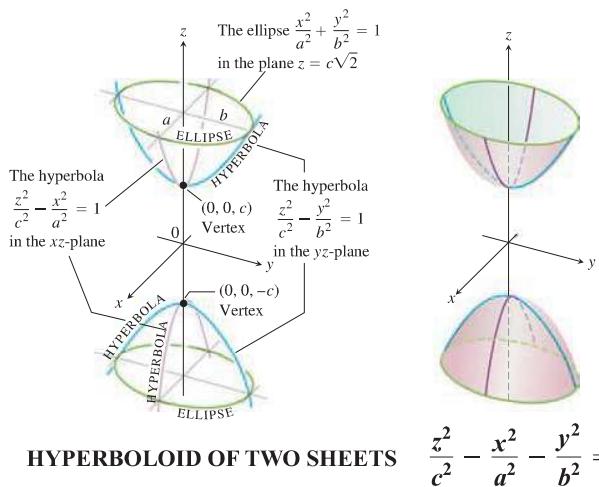
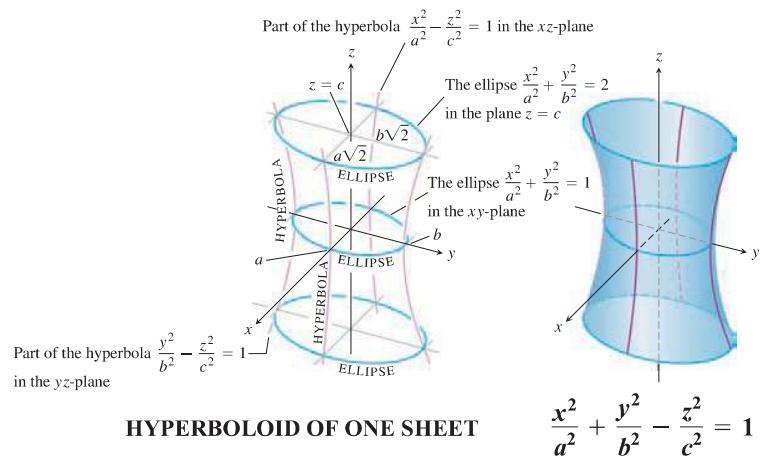
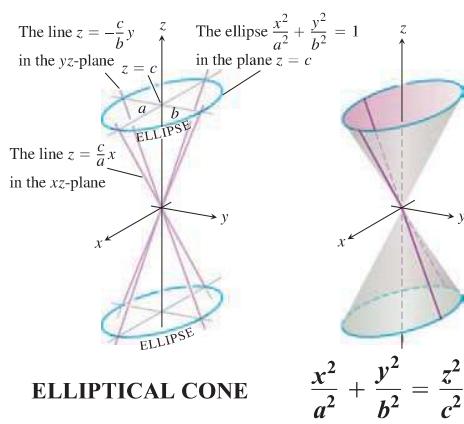
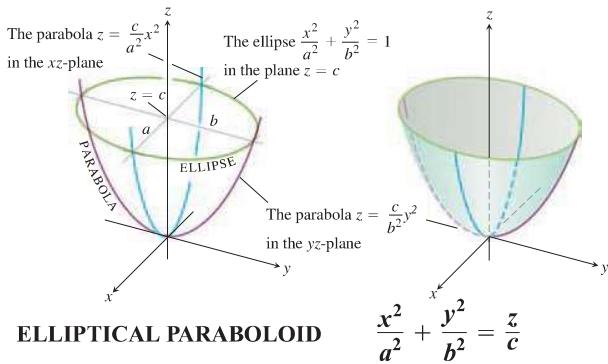
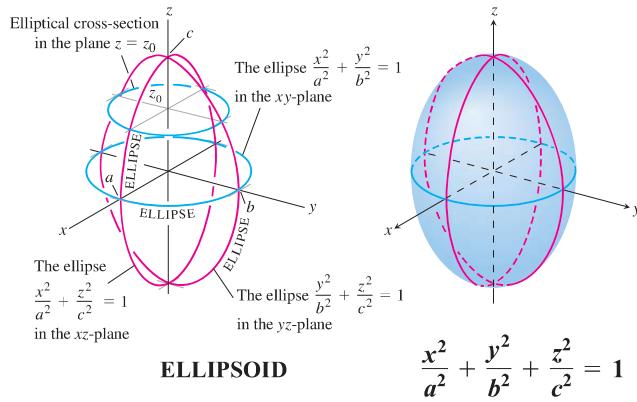


FIGURE 12.46 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z -axis above and below the xy -plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

Near the origin, the surface is shaped like a saddle or mountain pass. To a person traveling along the surface in the yz -plane the origin looks like a minimum. To a person traveling the xz -plane the origin looks like a maximum. Such a point is called a **saddle point** of a surface. We will say more about saddle points in Section 14.7. ■

Table 12.1 shows graphs of the six basic types of quadric surfaces. Each surface shown is symmetric with respect to the z -axis, but other coordinate axes can serve as well (with appropriate changes to the equation).

TABLE 12.1 Graphs of Quadric Surfaces

Exercises 12.6

Matching Equations with Surfaces

In Exercises 1–12, match the equation with the surface it defines. Also, identify each surface by type (paraboloid, ellipsoid, etc.) The surfaces are labeled (a)–(l).

1. $x^2 + y^2 + 4z^2 = 10$

3. $9y^2 + z^2 = 16$

5. $x = y^2 - z^2$

7. $x^2 + 2z^2 = 8$

9. $x = z^2 - y^2$

11. $x^2 + 4z^2 = y^2$

2. $z^2 + 4y^2 - 4x^2 = 4$

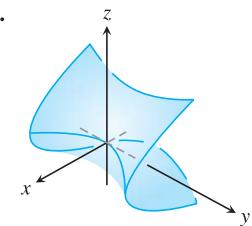
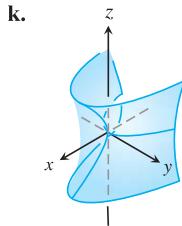
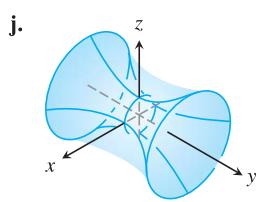
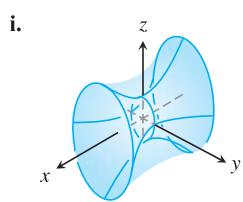
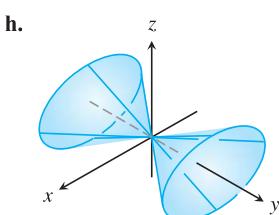
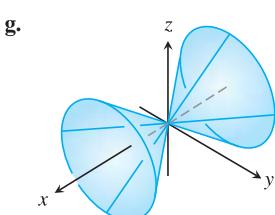
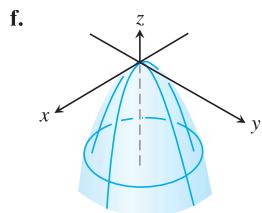
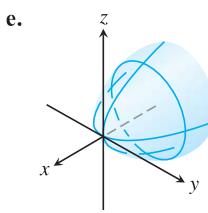
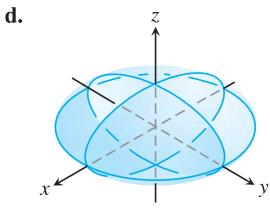
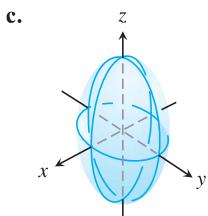
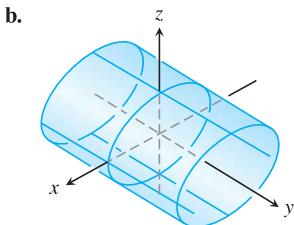
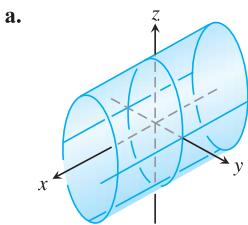
4. $y^2 + z^2 = x^2$

6. $x = -y^2 - z^2$

8. $z^2 + x^2 - y^2 = 1$

10. $z = -4x^2 - y^2$

12. $9x^2 + 4y^2 + 2z^2 = 36$



Drawing

Sketch the surfaces in Exercises 13–44.

CYLINDERS

13. $x^2 + y^2 = 4$

15. $x^2 + 4z^2 = 16$

14. $z = y^2 - 1$

16. $4x^2 + y^2 = 36$

ELLIPSOIDS

17. $9x^2 + y^2 + z^2 = 9$

19. $4x^2 + 9y^2 + 4z^2 = 36$

18. $4x^2 + 4y^2 + z^2 = 16$

20. $9x^2 + 4y^2 + 36z^2 = 36$

PARABOLOIDS AND CONES

21. $z = x^2 + 4y^2$

23. $x = 4 - 4y^2 - z^2$

25. $x^2 + y^2 = z^2$

22. $z = 8 - x^2 - y^2$

24. $y = 1 - x^2 - z^2$

26. $4x^2 + 9z^2 = 9y^2$

HYPERBOLOIDS

27. $x^2 + y^2 - z^2 = 1$

29. $z^2 - x^2 - y^2 = 1$

28. $y^2 + z^2 - x^2 = 1$

30. $(y^2/4) - (x^2/4) - z^2 = 1$

HYPERBOLIC PARABOLOIDS

31. $y^2 - x^2 = z$

32. $x^2 - y^2 = z$

ASSORTED

33. $z = 1 + y^2 - x^2$

34. $4x^2 + 4y^2 = z^2$

35. $y = -(x^2 + z^2)$

36. $16x^2 + 4y^2 = 1$

37. $x^2 + y^2 - z^2 = 4$

38. $x^2 + z^2 = y$

39. $x^2 + z^2 = 1$

40. $16y^2 + 9z^2 = 4x^2$

41. $z = -(x^2 + y^2)$

42. $y^2 - x^2 - z^2 = 1$

43. $4y^2 + z^2 - 4x^2 = 4$

44. $x^2 + y^2 = z$

Theory and Examples

45. a. Express the area A of the cross-section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by the plane $z = c$ as a function of c . (The area of an ellipse with semiaxes a and b is πab .)

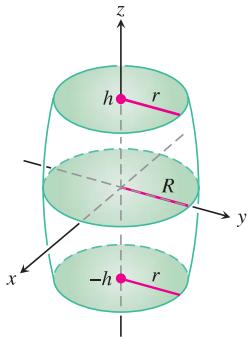
- b. Use slices perpendicular to the z -axis to find the volume of the ellipsoid in part (a).

- c. Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Does your formula give the volume of a sphere of radius a if $a = b = c$?

46. The barrel shown here is shaped like an ellipsoid with equal pieces cut from the ends by planes perpendicular to the z -axis. The cross-sections perpendicular to the z -axis are circular. The barrel is $2h$ units high, its midsection radius is R , and its end radii are both r . Find a formula for the barrel's volume. Then check two things. First, suppose the sides of the barrel are straightened to turn the barrel into a cylinder of radius R and height $2h$. Does your formula give the cylinder's volume? Second, suppose $r = 0$ and $h = R$ so the barrel is a sphere. Does your formula give the sphere's volume?



47. Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude.

48. a. Find the volume of the solid bounded by the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the planes $z = 0$ and $z = h$, $h > 0$.

- b. Express your answer in part (a) in terms of h and the areas A_0 and A_h of the regions cut by the hyperboloid from the planes $z = 0$ and $z = h$.

- c. Show that the volume in part (a) is also given by the formula

$$V = \frac{h}{6} (A_0 + 4A_m + A_h),$$

where A_m is the area of the region cut by the hyperboloid from the plane $z = h/2$.

Viewing Surfaces

- T Plot the surfaces in Exercises 49–52 over the indicated domains. If you can, rotate the surface into different viewing positions.

49. $z = y^2$, $-2 \leq x \leq 2$, $-0.5 \leq y \leq 2$

50. $z = 1 - y^2$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$

51. $z = x^2 + y^2$, $-3 \leq x \leq 3$, $-3 \leq y \leq 3$

52. $z = x^2 + 2y^2$ over

a. $-3 \leq x \leq 3$, $-3 \leq y \leq 3$

b. $-1 \leq x \leq 1$, $-2 \leq y \leq 3$

c. $-2 \leq x \leq 2$, $-2 \leq y \leq 2$

d. $-2 \leq x \leq 2$, $-1 \leq y \leq 1$

COMPUTER EXPLORATIONS

Use a CAS to plot the surfaces in Exercises 53–58. Identify the type of quadric surface from your graph.

53. $\frac{x^2}{9} + \frac{y^2}{36} = 1 - \frac{z^2}{25}$ 54. $\frac{x^2}{9} - \frac{z^2}{9} = 1 - \frac{y^2}{16}$

55. $5x^2 = z^2 - 3y^2$ 56. $\frac{y^2}{16} = 1 - \frac{x^2}{9} + z$

57. $\frac{x^2}{9} - 1 = \frac{y^2}{16} + \frac{z^2}{2}$ 58. $y = \sqrt{4 - z^2} = 0$

Chapter 12 Questions to Guide Your Review

- When do directed line segments in the plane represent the same vector?
- How are vectors added and subtracted geometrically? Algebraically?
- How do you find a vector's magnitude and direction?
- If a vector is multiplied by a positive scalar, how is the result related to the original vector? What if the scalar is zero? Negative?
- Define the *dot product (scalar product)* of two vectors. Which algebraic laws are satisfied by dot products? Give examples. When is the dot product of two vectors equal to zero?
- What geometric interpretation does the dot product have? Give examples.
- What is the vector projection of a vector \mathbf{u} onto a vector \mathbf{v} ? Give an example of a useful application of a vector projection.
- Define the *cross product (vector product)* of two vectors. Which algebraic laws are satisfied by cross products, and which are not? Give examples. When is the cross product of two vectors equal to zero?
- What geometric or physical interpretations do cross products have? Give examples.
- What is the determinant formula for calculating the cross product of two vectors relative to the Cartesian \mathbf{i} , \mathbf{j} , \mathbf{k} -coordinate system? Use it in an example.
- How do you find equations for lines, line segments, and planes in space? Give examples. Can you express a line in space by a single equation? A plane?
- How do you find the distance from a point to a line in space? From a point to a plane? Give examples.
- What are box products? What significance do they have? How are they evaluated? Give an example.
- How do you find equations for spheres in space? Give examples.
- How do you find the intersection of two lines in space? A line and a plane? Two planes? Give examples.
- What is a cylinder? Give examples of equations that define cylinders in Cartesian coordinates.
- What are quadric surfaces? Give examples of different kinds of ellipsoids, paraboloids, cones, and hyperboloids (equations and sketches).

Chapter 12 Practice Exercises

Vector Calculations in Two Dimensions

In Exercises 1–4, let $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 2, -5 \rangle$. Find (a) the component form of the vector and (b) its magnitude.

1. $3\mathbf{u} - 4\mathbf{v}$ 2. $\mathbf{u} + \mathbf{v}$
 3. $-2\mathbf{u}$ 4. $5\mathbf{v}$

In Exercises 5–8, find the component form of the vector.

5. The vector obtained by rotating $\langle 0, 1 \rangle$ through an angle of $2\pi/3$ radians
 6. The unit vector that makes an angle of $\pi/6$ radian with the positive x -axis
 7. The vector 2 units long in the direction $4\mathbf{i} - \mathbf{j}$
 8. The vector 5 units long in the direction opposite to the direction of $(3/5)\mathbf{i} + (4/5)\mathbf{j}$

Express the vectors in Exercises 9–12 in terms of their lengths and directions.

9. $\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$ 10. $-\mathbf{i} - \mathbf{j}$
 11. Velocity vector $\mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}$ when $t = \pi/2$.
 12. Velocity vector $\mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$ when $t = \ln 2$.

Vector Calculations in Three Dimensions

Express the vectors in Exercises 13 and 14 in terms of their lengths and directions.

13. $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ 14. $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

15. Find a vector 2 units long in the direction of $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.
 16. Find a vector 5 units long in the direction opposite to the direction of $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$.

In Exercises 17 and 18, find $|\mathbf{v}|$, $|\mathbf{u}|$, $\mathbf{v} \cdot \mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, $\mathbf{u} \times \mathbf{v}$, $|\mathbf{v} \times \mathbf{u}|$, the angle between \mathbf{v} and \mathbf{u} , the scalar component of \mathbf{u} in the direction of \mathbf{v} , and the vector projection of \mathbf{u} onto \mathbf{v} .

17. $\mathbf{v} = \mathbf{i} + \mathbf{j}$ 18. $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$
 $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ $\mathbf{u} = -\mathbf{i} - \mathbf{k}$

In Exercises 19 and 20, find $\text{proj}_{\mathbf{v}} \mathbf{u}$.

19. $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ 20. $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$
 $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

In Exercises 21 and 22, draw coordinate axes and then sketch \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ as vectors at the origin.

21. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$ 22. $\mathbf{u} = \mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$
 23. If $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$, and the angle between \mathbf{v} and \mathbf{w} is $\pi/3$, find $|\mathbf{v} - 2\mathbf{w}|$.
 24. For what value or values of a will the vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} - 8\mathbf{j} + a\mathbf{k}$ be parallel?

In Exercises 25 and 26, find (a) the area of the parallelogram determined by vectors \mathbf{u} and \mathbf{v} and (b) the volume of the parallelepiped determined by the vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} .

25. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
 26. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Lines, Planes, and Distances

27. Suppose that \mathbf{n} is normal to a plane and that \mathbf{v} is parallel to the plane. Describe how you would find a vector \mathbf{n} that is both perpendicular to \mathbf{v} and parallel to the plane.
 28. Find a vector in the plane parallel to the line $ax + by = c$.

In Exercises 29 and 30, find the distance from the point to the line.

29. $(2, 2, 0)$; $x = -t$, $y = t$, $z = -1 + t$
 30. $(0, 4, 1)$; $x = 2 + t$, $y = 2 + t$, $z = t$

31. Parametrize the line that passes through the point $(1, 2, 3)$ parallel to the vector $\mathbf{v} = -3\mathbf{i} + 7\mathbf{k}$.
 32. Parametrize the line segment joining the points $P(1, 2, 0)$ and $Q(1, 3, -1)$.

In Exercises 33 and 34, find the distance from the point to the plane.

33. $(6, 0, -6)$, $x - y = 4$
 34. $(3, 0, 10)$, $2x + 3y + z = 2$

35. Find an equation for the plane that passes through the point $(3, -2, 1)$ normal to the vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.
 36. Find an equation for the plane that passes through the point $(-1, 6, 0)$ perpendicular to the line $x = -1 + t$, $y = 6 - 2t$, $z = 3t$.

In Exercises 37 and 38, find an equation for the plane through points P , Q , and R .

37. $P(1, -1, 2)$, $Q(2, 1, 3)$, $R(-1, 2, -1)$
 38. $P(1, 0, 0)$, $Q(0, 1, 0)$, $R(0, 0, 1)$

39. Find the points in which the line $x = 1 + 2t$, $y = -1 - t$, $z = 3t$ meets the three coordinate planes.
 40. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.
 41. Find the acute angle between the planes $x = 7$ and $x + y + \sqrt{2}z = -3$.
 42. Find the acute angle between the planes $x + y = 1$ and $y + z = 1$.
 43. Find parametric equations for the line in which the planes $x + 2y + z = 1$ and $x - y + 2z = -8$ intersect.
 44. Show that the line in which the planes

$$x + 2y - 2z = 5 \quad \text{and} \quad 5x - 2y - z = 0$$

intersect is parallel to the line

$$x = -3 + 2t, \quad y = 3t, \quad z = 1 + 4t.$$

45. The planes $3x + 6z = 1$ and $2x + 2y - z = 3$ intersect in a line.
- Show that the planes are orthogonal.
 - Find equations for the line of intersection.
46. Find an equation for the plane that passes through the point $(1, 2, 3)$ parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
47. Is $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ related in any special way to the plane $2x + y = 5$? Give reasons for your answer.
48. The equation $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ represents the plane through P_0 normal to \mathbf{n} . What set does the inequality $\mathbf{n} \cdot \overrightarrow{P_0P} > 0$ represent?
49. Find the distance from the point $P(1, 4, 0)$ to the plane through $A(0, 0, 0)$, $B(2, 0, -1)$, and $C(2, -1, 0)$.
50. Find the distance from the point $(2, 2, 3)$ to the plane $2x + 3y + 5z = 0$.
51. Find a vector parallel to the plane $2x - y - z = 4$ and orthogonal to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
52. Find a unit vector orthogonal to \mathbf{A} in the plane of \mathbf{B} and \mathbf{C} if $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
53. Find a vector of magnitude 2 parallel to the line of intersection of the planes $x + 2y + z - 1 = 0$ and $x - y + 2z + 7 = 0$.
54. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.
55. Find the point in which the line through $P(3, 2, 1)$ normal to the plane $2x - y + 2z = -2$ meets the plane.
56. What angle does the line of intersection of the planes $2x + y - z = 0$ and $x + y + 2z = 0$ make with the positive x -axis?
57. The line
- $$L: x = 3 + 2t, \quad y = 2t, \quad z = t$$
- intersects the plane $x + 3y - z = -4$ in a point P . Find the coordinates of P and find equations for the line in the plane through P perpendicular to L .
58. Show that for every real number k the plane
- $$x - 2y + z + 3 + k(2x - y - z + 1) = 0$$
- contains the line of intersection of the planes
- $$x - 2y + z + 3 = 0 \quad \text{and} \quad 2x - y - z + 1 = 0.$$
59. Find an equation for the plane through $A(-2, 0, -3)$ and $B(1, -2, 1)$ that lies parallel to the line through $C(-2, -13/5, 26/5)$ and $D(16/5, -13/5, 0)$.
60. Is the line $x = 1 + 2t$, $y = -2 + 3t$, $z = -5t$ related in any way to the plane $-4x - 6y + 10z = 9$? Give reasons for your answer.
61. Which of the following are equations for the plane through the points $P(1, 1, -1)$, $Q(3, 0, 2)$, and $R(-2, 1, 0)$?
62. The parallelogram shown here has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$, and D . Find
-
- a. the coordinates of D ,
- b. the cosine of the interior angle at B ,
- c. the vector projection of \overrightarrow{BA} onto \overrightarrow{BC} ,
- d. the area of the parallelogram,
- e. an equation for the plane of the parallelogram,
- f. the areas of the orthogonal projections of the parallelogram on the three coordinate planes.
63. **Distance between lines** Find the distance between the line L_1 through the points $A(1, 0, -1)$ and $B(-1, 1, 0)$ and the line L_2 through the points $C(3, 1, -1)$ and $D(4, 5, -2)$. The distance is to be measured along the line perpendicular to the two lines. First find a vector \mathbf{n} perpendicular to both lines. Then project \overline{AC} onto \mathbf{n} .
64. (Continuation of Exercise 63.) Find the distance between the line through $A(4, 0, 2)$ and $B(2, 4, 1)$ and the line through $C(1, 3, 2)$ and $D(2, 2, 4)$.

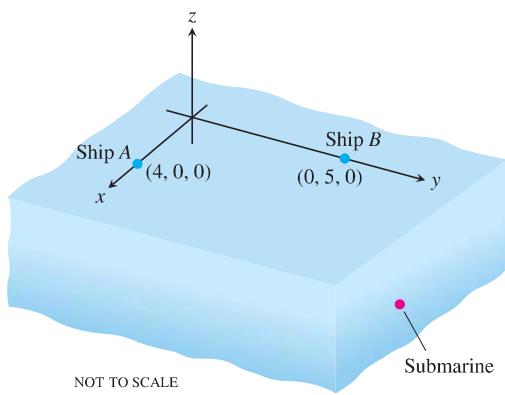
Quadratic Surfaces

Identify and sketch the surfaces in Exercises 65–76.

65. $x^2 + y^2 + z^2 = 4$ 66. $x^2 + (y - 1)^2 + z^2 = 1$
 67. $4x^2 + 4y^2 + z^2 = 4$ 68. $36x^2 + 9y^2 + 4z^2 = 36$
 69. $z = -(x^2 + y^2)$ 70. $y = -(x^2 + z^2)$
 71. $x^2 + y^2 = z^2$ 72. $x^2 + z^2 = y^2$
 73. $x^2 + y^2 - z^2 = 4$ 74. $4y^2 + z^2 - 4x^2 = 4$
 75. $y^2 - x^2 - z^2 = 1$ 76. $z^2 - x^2 - y^2 = 1$

Chapter 12 Additional and Advanced Exercises

- 1. Submarine hunting** Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. As shown here, ship A is located at $(4, 0, 0)$, whereas ship B is located at $(0, 5, 0)$. All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - (1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i} - 6\mathbf{j} - \mathbf{k}$. Four minutes ago, the submarine was located at $(2, -1, -1/3)$. The aircraft is due in 20 min. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?

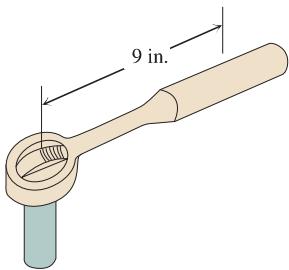


- 2. A helicopter rescue** Two helicopters, H_1 and H_2 , are traveling together. At time $t = 0$, they separate and follow different straight-line paths given by

$$\begin{aligned} H_1: \quad &x = 6 + 40t, \quad y = -3 + 10t, \quad z = -3 + 2t \\ H_2: \quad &x = 6 + 110t, \quad y = -3 + 4t, \quad z = -3 + t. \end{aligned}$$

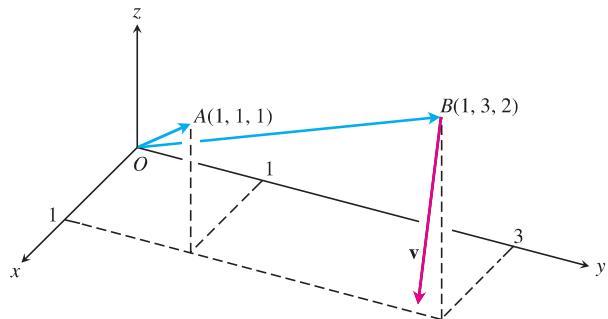
Time t is measured in hours and all coordinates are measured in miles. Due to system malfunctions, H_2 stops its flight at $(446, 13, 1)$ and, in a negligible amount of time, lands at $(446, 13, 0)$. Two hours later, H_1 is advised of this fact and heads toward H_2 at 150 mph. How long will it take H_1 to reach H_2 ?

- 3. Torque** The operator's manual for the Toro® 21 in. lawnmower says "tighten the spark plug to 15 ft-lb (20.4 N·m)." If you are installing the plug with a 10.5-in. socket wrench that places the center of your hand 9 in. from the axis of the spark plug, about how hard should you pull? Answer in pounds.

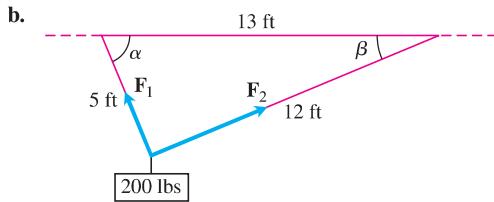
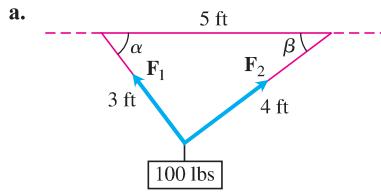


- 4. Rotating body** The line through the origin and the point $A(1, 1, 1)$ is the axis of rotation of a right body rotating with a constant angular speed of $3/2$ rad/sec. The rotation appears to be

clockwise when we look toward the origin from A . Find the velocity \mathbf{v} of the point of the body that is at the position $B(1, 3, 2)$.

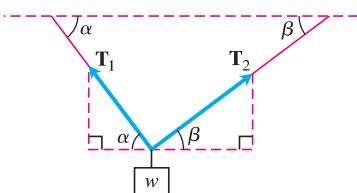


- 5.** Consider the weight suspended by two wires in each diagram. Find the magnitudes and components of vectors \mathbf{F}_1 and \mathbf{F}_2 , and angles α and β .



(Hint: This triangle is a right triangle.)

- 6.** Consider a weight of w N suspended by two wires in the diagram, where \mathbf{T}_1 and \mathbf{T}_2 are force vectors directed along the wires.



- a. Find the vectors \mathbf{T}_1 and \mathbf{T}_2 and show that their magnitudes are

$$|\mathbf{T}_1| = \frac{w \cos \beta}{\sin(\alpha + \beta)}$$

and

$$|\mathbf{T}_2| = \frac{w \cos \alpha}{\sin(\alpha + \beta)}$$

- b. For a fixed β determine the value of α which minimizes the magnitude $|\mathbf{T}_1|$.
c. For a fixed α determine the value of β which minimizes the magnitude $|\mathbf{T}_2|$.

7. Determinants and planes

- a. Show that

$$\begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$$

is an equation for the plane through the three noncollinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

- b. What set of points in space is described by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0?$$

8. Determinants and lines

Show that the lines

$$x = a_1s + b_1, \quad y = a_2s + b_2, \quad z = a_3s + b_3, \quad -\infty < s < \infty$$

and

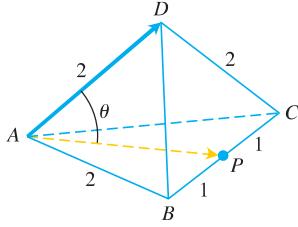
$$x = c_1t + d_1, \quad y = c_2t + d_2, \quad z = c_3t + d_3, \quad -\infty < t < \infty,$$

intersect or are parallel if and only if

$$\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0.$$

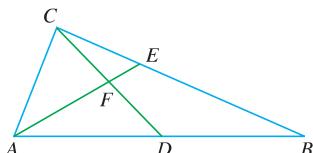
9. Consider a regular tetrahedron of side length 2.

- a. Use vectors to find the angle θ formed by the base of the tetrahedron and any one of its other edges.



- b. Use vectors to find the angle θ formed by any two adjacent faces of the tetrahedron. This angle is commonly referred to as a dihedral angle.

10. In the figure here, D is the midpoint of side AB of triangle ABC , and E is one-third of the way between C and B . Use vectors to prove that F is the midpoint of line segment CD .



11. Use vectors to show that the distance from $P_1(x_1, y_1)$ to the line $ax + by = c$ is

$$d = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

12. a. Use vectors to show that the distance from $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz = D$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- b. Find an equation for the sphere that is tangent to the planes $x + y + z = 3$ and $x + y + z = 9$ if the planes $2x - y = 0$ and $3x - z = 0$ pass through the center of the sphere.

13. a. Show that the distance between the parallel planes $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$ is

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.$$

- b. Find the distance between the planes $2x + 3y - z = 6$ and $2x + 3y - z = 12$.

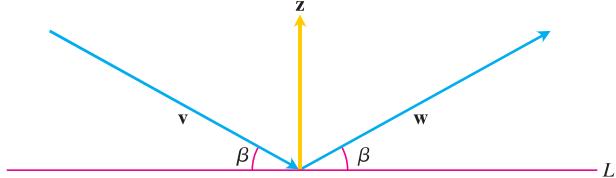
- c. Find an equation for the plane parallel to the plane $2x - y + 2z = -4$ if the point $(3, 2, -1)$ is equidistant from the two planes.

- d. Write equations for the planes that lie parallel to and 5 units away from the plane $x - 2y + z = 3$.

14. Prove that four points A , B , C , and D are coplanar (lie in a common plane) if and only if $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

15. **The projection of a vector on a plane** Let P be a plane in space and let \mathbf{v} be a vector. The vector projection of \mathbf{v} onto the plane P , $\text{proj}_P \mathbf{v}$, can be defined informally as follows. Suppose the sun is shining so that its rays are normal to the plane P . Then $\text{proj}_P \mathbf{v}$ is the “shadow” of \mathbf{v} onto P . If P is the plane $x + 2y + 6z = 6$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find $\text{proj}_P \mathbf{v}$.

16. The accompanying figure shows nonzero vectors \mathbf{v} , \mathbf{w} , and \mathbf{z} , with \mathbf{z} orthogonal to the line L , and \mathbf{v} and \mathbf{w} making equal angles β with L . Assuming $|\mathbf{v}| = |\mathbf{w}|$, find \mathbf{w} in terms of \mathbf{v} and \mathbf{z} .



17. **Triple vector products** The triple vector products $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are usually not equal, although the formulas for evaluating them from components are similar:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}.$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Verify each formula for the following vectors by evaluating its two sides and comparing the results.

\mathbf{u}	\mathbf{v}	\mathbf{w}
a. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
b. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
c. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
d. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

- 18. Cross and dot products** Show that if \mathbf{u} , \mathbf{v} , \mathbf{w} , and \mathbf{r} are any vectors, then

a. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

b. $\mathbf{u} \times \mathbf{v} = (\mathbf{u} \cdot \mathbf{v} \times \mathbf{i})\mathbf{i} + (\mathbf{u} \cdot \mathbf{v} \times \mathbf{j})\mathbf{j} + (\mathbf{u} \cdot \mathbf{v} \times \mathbf{k})\mathbf{k}$

c. $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}.$

- 19. Cross and dot products** Prove or disprove the formula

$$\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v})) \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}.$$

- 20. By forming the cross product of two appropriate vectors, derive the trigonometric identity**

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

- 21. Use vectors to prove that**

$$(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

for any four numbers a , b , c , and d . (*Hint:* Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$.)

- 22. Dot multiplication is positive definite** Show that dot multiplication of vectors is *positive definite*; that is, show that $\mathbf{u} \cdot \mathbf{u} \geq 0$ for every vector \mathbf{u} and that $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

- 23.** Show that $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ for any vectors \mathbf{u} and \mathbf{v} .

- 24.** Show that $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ bisects the angle between \mathbf{u} and \mathbf{v} .

- 25.** Show that $|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ and $|\mathbf{v}|\mathbf{u} - |\mathbf{u}|\mathbf{v}$ are orthogonal.

Chapter 12 Technology Application Projects

Mathematica/Maple Module:

Using Vectors to Represent Lines and Find Distances

Parts I and II: Learn the advantages of interpreting lines as vectors.

Part III: Use vectors to find the distance from a point to a line.

Putting a Scene in Three Dimensions onto a Two-Dimensional Canvas

Use the concept of planes in space to obtain a two-dimensional image.

Getting Started in Plotting in 3D

Part I: Use the vector definition of lines and planes to generate graphs and equations, and to compare different forms for the equations of a single line.

Part II: Plot functions that are defined implicitly.