1. **Solution**. Let $P(n)$ be the proposition

$$\sum_{k=1}^{n} T_k = n[(n+1)!]$$

where $T_n = (n^2+1)(n!)$ for any positive integer n.

For $P(1)$:

$$\begin{aligned}\text{L.H.S.} &= \sum_{k=1}^{1} T_k \\ &= T_1 \\ &= (1^2+1)(1!) \\ &= 2 \\ \text{R.H.S.} &= 1[(1+1)!] \\ &= 2\end{aligned}$$

$\therefore$ L.H.S=R.H.S.

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m, i.e.

$$\sum_{k=1}^{m} T_k = m[(m+1)!]$$

Then for $P(m+1)$:

$$\begin{aligned}\sum_{k=1}^{m+1} T_k &= m[(m+1)!] + [(m+1)^2+1][(m+1)!] \\ &= [m^2+2m+2+m][(m+1)!] \\ &= (m+1)(m+2)[(m+1)!] \\ &= (m+1)[(m+2)!]\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n. ■

2. (a) **Solution**. Let $P(n)$ be the proposition

$$A_n = (-1)^{n-1} B_n$$

where $A_n = \sum_{k=1}^{n}(-1)^{k-1}k^2$ and $B_n = \dfrac{n(n+1)}{2}$ for any positive integer n.

For $P(1)$:

$$\begin{aligned}
\text{L.H.S.} &= A_1 \\
&= \sum_{k=1}^{1}(-1)^{k-1}k^2 \\
&= (-1)^0 1^2 \\
&= (-1)^0 1 \\
&= (-1)^{1-1}\frac{1(1+1)}{2} \\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore$ L.H.S=R.H.S.

$\therefore P(1)$ is true.

Assume $P(m)$ is true for some positive integer m, i.e.

$$A_m = (-1)^{m-1}B_m$$

Then for $P(m+1)$:

$$\begin{aligned}
A_{m+1} &= \sum_{k=1}^{m+1}(-1)^{k-1}k^2 \\
&= (-1)^{m-1}\frac{m(m+1)}{2} + (-1)^m(m+1)^2 \\
&= (-1)^m(m+1)[(m+1) - \frac{m}{2}] \\
&= (-1)^m(m+1)\frac{m+2}{2} \\
&= (-1)^m B_{m+1}
\end{aligned}$$

$\therefore P(m+1)$ is true when $P(m)$ is true.

Thus, by the principle of Mathematical Induction, $P(n)$ is true for all positive integer n. ■

(b) **Solution**.

$$\begin{aligned}
\sum_{n=1}^{2m} A_n &= \sum_{n=1}^{2m}(-1)^{n-1}B_n \\
&= \sum_{n=1}^{m}(B_{2n-1}-B_{2n}) \\
&= \sum_{n=1}^{m}(-2n) \\
&= -2\sum_{n=1}^{m} n \\
&= -2B_m \\
&= -m(m+1) \\
\sum_{n=1}^{2m+1} A_n &= -m(m+1)+A_{2m+1} \\
&= -m(m+1)+B_{2m+1} \\
&= -m(m+1)+(m+1)(2m+1) \\
&= (m+1)^2
\end{aligned}$$

■

3. **Solution**. By considering the corresponding coefficients, we have

$$\begin{aligned}
\lambda_1 &= C_1^8 a = 8a \\
\lambda_2 &= C_2^8 a^2 = 28a^2 \\
\mu_7 &= C_7^9 b^2 = 36b^2 \\
\mu_8 &= C_8^9 b = 9b
\end{aligned}$$

That means we have to solve the folloing system:

$$\begin{cases} \frac{28a^2}{36b^2} = \frac{7}{4} \implies 4a^2 = 9b^2 \quad \implies 2a = \pm 3b \\ 8a+9b+6=0 \end{cases}$$

If $2a = 3b$, we have

$$\begin{aligned}
12b+9b+6 &= 0 \\
b = -\frac{2}{7}&, a = -\frac{3}{7}
\end{aligned}$$

If $2a = -3b$, we have

$$\begin{aligned} -12b + 9b + 6 &= 0 \\ b &= -2, a = 3 \end{aligned}$$

Then $a = -\dfrac{3}{7}$ or $a = 3$. ■

4. (a) **Solution**. We have the fundamental identities:

$$\begin{aligned} \sin 2x &= 2\sin x \cos x \\ \sin 3x &= 3\sin x - 4\sin^3 x \end{aligned}$$

Then

$$\begin{aligned} \sin x - \sin 2x + \sin 3x &= 0 \\ \sin x(1 - 2\cos x + 3 - 4\sin^2 x) &= 0 \end{aligned}$$

One solution is $\sin x = 0$ which means $x = \pi$. Otherwise,

$$\begin{aligned} 1 - 2\cos x + 3 - 4\sin^2 x &= 0 \\ -2\cos x + 4\cos^2 x &= 0 \\ -2\cos x(1 - 2\cos x) &= 0 \end{aligned}$$

Then, either $\cos x = 0 \implies x = \pi/2, 3\pi/2$ or $\cos x = 1/2 \implies x = \pi/3, 5\pi/3$.

In conclusion, the set of solution to the equation is $\{\pi/3, \pi/2, \pi, 3\pi/2, 5\pi/3\}$. ■

(b) i. **Solution**.

$$\begin{aligned} f(\theta) &= \sin 2\theta + \sin\theta + \cos\theta \\ &= 2\sin\theta\cos\theta + \sin\theta + \cos\theta \\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \sin\theta + \cos\theta - 1 \\ &= p^2 + p - 1 \end{aligned}$$

■

ii. **Solution**. $f(\theta) = (p+1/2)^2 - 5/4$. So $f(\theta)$ has the minimum value $-5/4$. Such θ can be computed as follows:

Method 1(Subsidiary angle):

$$\begin{aligned}
\sin\theta + \cos\theta &= -\frac{1}{2} \\
\sin\theta\cos\frac{5\pi}{4} + \cos\theta\sin\frac{5\pi}{4} &= \frac{\sqrt{2}}{4} \\
\sin\left(\theta + \frac{5\pi}{4}\right) &= \frac{\sqrt{2}}{4} \\
\theta &= \sin^{-1}(\frac{\sqrt{2}}{4}) + \frac{3\pi}{4}
\end{aligned}$$

or

Method 2(Realisation of triangle):

$$\begin{aligned}
p + \frac{1}{2} &= 0 \\
p &= -\frac{1}{2} \\
\sin 2\theta - \frac{1}{2} &= -\frac{5}{4} \\
\sin 2\theta &= -\frac{3}{4}
\end{aligned}$$

Case 1:

$$\begin{aligned}
\tan 2\theta &= \frac{3}{\sqrt{7}} \\
\frac{2\tan\theta}{1-\tan^2\theta} &= \frac{3}{\sqrt{7}} \\
2\sqrt{7}\tan\theta &= 3 - 3\tan^2\theta \\
3\tan^2\theta + 2\sqrt{7}\tan\theta - 3 &= 0 \\
\tan\theta &= \frac{-2\sqrt{7} \pm \sqrt{28+36}}{6} \\
&= \frac{-\sqrt{7} \pm 4}{3}
\end{aligned}$$

By $\tan\theta < 0$, $\theta = \tan^{-1}(-\frac{4+\sqrt{7}}{3})$.

Case 2:

$$\begin{aligned}
\tan 2\theta &= -\frac{3}{\sqrt{7}} \\
\frac{2\tan\theta}{1-\tan^2\theta} &= -\frac{3}{\sqrt{7}} \\
-2\sqrt{7}\tan\theta &= 3 - 3\tan^2\theta \\
3\tan^2\theta - 2\sqrt{7}\tan\theta - 3 &= 0 \\
\tan\theta &= \frac{2\sqrt{7} \pm \sqrt{28+36}}{6} \\
&= \frac{\sqrt{7} \pm 4}{3}
\end{aligned}$$

By $\tan\theta < 0$, $\theta = \tan^{-1}(\frac{\sqrt{7}-4}{3})$.

Hence the conclusion is f attains minimum at $\theta = \tan^{-1}(\frac{-4\pm\sqrt{7}}{3})$. ■

5. (a) **Solution**.

$$\begin{aligned}
\tan 4\theta &= \frac{2\tan 2\theta}{1-\tan^2 2\theta} \\
&= \frac{2\frac{2\tan\theta}{1-\tan^2\theta}}{1-(\frac{2\tan\theta}{1-\tan^2\theta})^2} \\
&= \frac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2 - 4\tan^2\theta} \\
&= \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}
\end{aligned}$$

Then

$$\begin{aligned}
\cot 4\theta &= \frac{1}{\tan 4\theta} \\
&= \frac{1-6\tan^2\theta+\tan^4\theta}{4\tan\theta-4\tan^3\theta} \\
&= \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}
\end{aligned}$$

Hence $\cot\theta$ solve the equation $x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$ when $\cot 4\theta = 1$, which is

$$\begin{aligned}
4\theta &= n\pi + \pi/4 \\
\theta &= n\pi/4 + \pi/16 \\
x &= \cot(\pi/16), \cot(5\pi/16), \cot(9\pi/16), \cot(13\pi/16)
\end{aligned}$$

restricting to $[0, 2\pi]$. ■

(b) i. **Solution**.

$$\begin{aligned}\frac{1}{2}(2-a^2-b^2) &= \frac{1}{2}(2-\cos^2\theta+2\cos\theta\cos\phi-\cos^2\phi-\sin^2\theta+2\sin\theta\sin\phi-\sin^2\phi)\\ &= \cos\theta\cos\phi+\sin\theta\sin\phi\\ &= \cos(\theta-\phi)\end{aligned}$$

and

$$\begin{aligned}\frac{-a}{b} &= \frac{\cos\phi-\cos\theta}{\sin\theta-\sin\phi}\\ &= \frac{-2\sin\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}}{2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}}\\ &= \tan\frac{\theta+\phi}{2}\end{aligned}$$

■

ii. **Solution**. By i.

$$\begin{aligned}& a=1, b=\sqrt{3}\\ \Longrightarrow & \begin{cases}\cos(\theta-\phi)=-1\\ \tan\frac{\theta+\phi}{2}=-\frac{1}{\sqrt{3}}\end{cases}\\ \Longrightarrow & \begin{cases}\theta-\phi=2n\pi+\pi\\ \theta+\phi=2n\pi-\pi/3\end{cases}\\ \Longrightarrow & \theta=\pi/3, \phi=4\pi/3\end{aligned}$$

■

6. **Solution**. Let $f(x)=(x^2-1)e^x$. Then

$$\begin{aligned}f(1+h) &= [(1+h)^2-1]e^{1+h} = h(2+h)e^{1+h}\\ f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\ &= \lim_{h\to 0}\frac{h(2+h)e^{1+h}}{h}\\ &= \lim_{h\to 0}(2+h)e^{1+h}\\ &= 2e\end{aligned}$$

■

7. (a) **Solution**.

$$\begin{aligned}
f'(x) &= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) \\
&= 2e^x \cos x \\
f''(x) &= 2e^x \cos x - 2e^x \sin x \\
&= 2e^x(\cos x - \sin x)
\end{aligned}$$

■

(b) **Solution**.

$$\begin{aligned}
f''(x) - f'(x) + f(x) &= 0 \\
2e^x(\cos x - \sin x) - 2e^x \cos x + e^x(\sin x + \cos x) &= 0 \\
\cos x - \sin x &= 0 \\
\tan x &= 1 \\
x &= n\pi + \pi/4
\end{aligned}$$

where $n \in \mathbb{Z}$ is integer. ■

8. **Solution**.

$$\begin{aligned}
\frac{dy}{dx} = \frac{-6}{(x+1)^2} &= -\frac{1}{6} \\
(x+1)^2 &= 36 \\
x &= 5, -7
\end{aligned}$$

Then the equation of tangent to C at $(5, 1)$ is

$$\begin{aligned}
y - 1 &= -\frac{1}{6}(x - 5) \\
x + 6y - 11 &= 0
\end{aligned}$$

Then the equation of tangent to C at $(-7, -1)$ is

$$\begin{aligned}
y + 1 &= -\frac{1}{6}(x + 7) \\
x + 6y + 13 &= 0
\end{aligned}$$

■

9. (a) i. **Solution**. $x = 4\sin\theta$. ■

ii. **Solution**. Since $\dfrac{dx}{dt} = \dfrac{1}{2}$, we have

$$\begin{aligned}\frac{dx}{dt} &= 4\cos\theta\frac{d\theta}{dt}\\ \frac{d\theta}{dt} &= \frac{1}{8\cos\theta}\end{aligned}$$

■

(b) i. **Solution**. Write $y = 4\cos\theta$ and $z = \sqrt{5^2 - x^2} = \sqrt{25 - 16\sin^2\theta}$. Then

$$\begin{aligned}\frac{dy}{dt} &= -4\sin\theta\frac{d\theta}{dt}\\ &= -4\sin\theta\frac{1}{8\cos\theta}\\ &= -\frac{1}{2}\tan\theta\\ \frac{dz}{dt} &= \frac{-32\sin\theta\cos\theta}{2\sqrt{25-16\sin^2\theta}}\frac{1}{8\cos\theta}\\ &= \frac{-2\sin\theta}{\sqrt{25-16\sin^2\theta}}\end{aligned}$$

■

ii. **Solution**.

$$\begin{aligned}\frac{dPQ}{dt}|_{\theta=\pi/6} &= \frac{dy}{dt}|_{\theta=\pi/6} + \frac{dz}{dt}|_{\theta=\pi/6}\\ &= -\frac{1}{2}\tan\pi/3 - \frac{2\sin\pi/6}{\sqrt{25-16\sin^2\pi/6}}\\ &= -\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{25-4}}\\ &= -\frac{\sqrt{3}}{2} - \frac{\sqrt{21}}{21}\end{aligned}$$

■

(c) i. **Solution**. Recall area of $\triangle OPR$ is $\dfrac{xy}{2}$,

$$\begin{aligned}\frac{d}{dt}\frac{xy}{2} &= \frac{1}{2}(x\frac{dy}{dt} + y\frac{dx}{dt})\\ &= \frac{1}{2}(-2\sin\theta\tan\theta + 2\cos\theta)\\ &= \cos\theta - \sin\theta\tan\theta\end{aligned}$$

Set $\dfrac{d}{dt}\dfrac{xy}{2} = 0$, then we have

$$\begin{aligned}\tan^2\theta &= 1\\ \tan\theta &= \pm 1\\ \theta &= n\pi/2 + \pi/4\end{aligned}$$

Since $0 < \theta < \pi/2$, $\theta = \pi/4$. ■

ii. **Solution**. Let $\angle OQR = \phi$, then

$$4\sin\theta = 5\sin\phi$$

Similar to i., we have to have $\phi = \pi/4$ so that $\triangle ORQ$ is having maximum area. Then

$$\begin{aligned}\theta &= \sin^{-1}(\frac{5\sin\pi/4}{4})\\ &= \sin^{-1}(\frac{5\sqrt{2}}{8})\\ &= 1.08rad\end{aligned}$$

■