Learning objectives

By studying this unit, we will achieve the following goals:

1. Solving quadratic equations by the factor methods.
2. Forming quadratic equations from given roots.
3. Solving the equation $ax^2 + bx + c = 0$ by plotting the graph of the parabola $y = ax^2 + bx + c$ and reading the x-intercepts.
4. Solving quadratic equations by the quadratic formula.
5. Understanding the relations between the discriminant of a quadratic equation and the nature of its roots.
6. Solving problems involving quadratic equations.
7. Understanding the relations between roots and coefficients and form quadratic equations using those relations.
8. Appreciating the development of the number systems including the system of complex numbers.
9. Performing addition, subtraction, multiplication and division of complex numbers.

Background

It was a long time ago that we may not determine when did humanity started to examine quadratic equation, but it was already known to ancient Egyptians. Here, we recall some basic algebraic operations and think of a question.

We recall the algebraic expressions we learned in junior secondary, and read the following problem.

'Given a rectangle of perimeter p and area a, how long should its side lengths be?'

In order to solve the above problem, we may need to let x and y to be its side lengths. Converting the words into algebraic expressions gives us the following system of simultaneous equations:

$$\begin{cases} x + y = p \\ xy = a \end{cases}$$

To be simpler, we may fix some number for p and a, so that the problem looks easier to solve - we could literally focus on the unknowns, where we may adopt the similar deduction from the simplified case to general case.

Let's pick $p = 1$ and $a = 1$ for instance, which looks simpler. And now the system becomes

$$\begin{cases} x + y = 1 \\ xy = 1 \end{cases}$$

The genius from history thought of the solution using substitution, through transforming the system of two unknown into one first, then solve for the other using the deduced relations. Now, we may take a look at the second equation in the system, which can be modified as

$$xy = 1 \implies y = \frac{1}{x}$$

as long as $x \neq 0$ and $y \neq 0$. In fact, it must not be happening since if $x = 0$ or $y = 0$, $xy = 0$ in which it falsified the system argument. We then substitute $y = \frac{1}{x}$ into the first equation to form the new argument in one unknown:

$$\begin{aligned} x + (\frac{1}{x}) &= 1 && (\text{sub } y = \frac{1}{x}) \\ x(x + \frac{1}{x}) &= x(1) && (\text{multiply both sides by } x) \\ x^2 + 1 &= x \\ x^2 - x + 1 &= 0 \end{aligned}$$

where it comes to the stopping point. This is where we starts the discussion of quadratic equation.

Factor method

One observed that, to solve some quadratic equations (some but not all, you will figure out why later), we have the knowledge to construct the following:

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$

which, in fact, satisfy the form of a quadratic equation. We may also notice the fact that solving $(x-\alpha)(x-\beta) = 0$ is much easier than seeing the original quadratic form. We then exhibit the following:

Theorem. *If $ab = 0$, then $a = 0$ or $b = 0$.*

The meaning of this theorem is that, we must have either one of the factor being 0 if the product of them is zero. You may think it naturally that 0 dominant other numbers.

Therefore, we can apply this theorem to the problem $(x - \alpha)(x - \beta) = 0$, and we may deduce that

$$\begin{aligned} x - \alpha &= 0 & \text{or} & & x - \beta &= 0 \\ x &= \alpha & \text{or} & & x &= \beta \end{aligned}$$

The above result is cool, but the difficulties are still existing. We are still looking for the way to perform factorization. One suggested the usage of cross method.

The cross method is literally a testing by exhausting every possible results and choose the suitable one as the answer. For example, we may look at $x^2 - 5x + 6 = 0$. We shall see $6 = 1 \times 6 = 2 \times 3 = -1 \times -6 = -2 \times -3$. As we see from previous page, we need the middle term to be $-5x$, which means $\alpha + \beta = 5$. The pair that satisfies this result is 2 and 3, so the factor form becomes $(x - 2)(x - 3) = 0$, and solved by $x = 2$ or $x = 3$.

But this turns out that we are in fact not seeing the quadratic equation itself but eventually returning the situation

$$\begin{cases} \alpha + \beta = 5 \\ \alpha\beta = 6 \end{cases}$$

with the method of testing, which is not a good practice, and not a reasonable system we want. We must see the factor method is not effective enough to draw out solutions for all cases.

Forming quadratic equations from given roots

Alternatively, and for the sake of seeing why factor method is not effective enough, we may acquire the concepts of forming quadratic equations using given roots.

Recalling the relation from above, we may see that

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

gives us the concrete formula to construct a quadratic equation using given roots. We may simply plug the roots in α and β respectively. For instance, if we choose to pick complicated numbers as roots, we will see why not every quadratic equation could be solved using cross method, as we are not able to directly guess such monsters. Let's check the following:

1. $\alpha = \frac{2}{7}, \beta = \frac{7}{2} \implies 14x^2 - 53x + 14 = 0$
2. $\alpha = 1+\sqrt{2}, \beta = 1-\sqrt{2} \implies x^2 - 2x - 1 = 0$

We have no capability of these kinds of monster numbers, at least we could not directly get to the solutions. It proves the limitation of factor method.

Graphical solution to quadratic equation

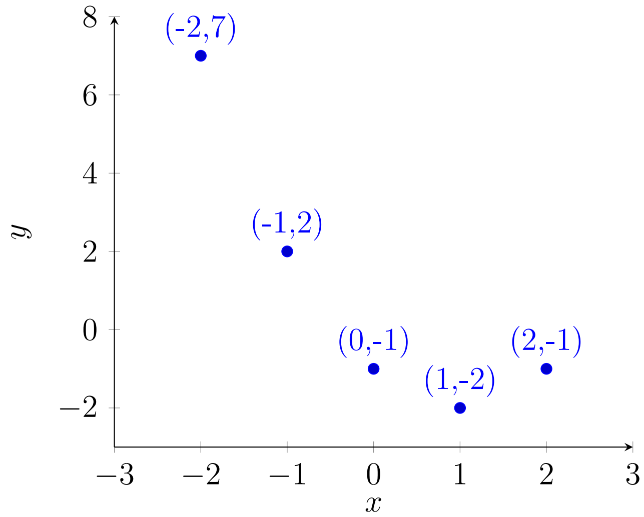
One suggested the thought of graphing, in which we should have learned the technique of plotting a graph in junior secondary. Let's recall the memory of it first.

Consider we have a coordinate plane, and our coordinate are presented in the form of an ordered pair (x, y). We set the relation $y = ax^2 + bx + c$, for instance, $y = x^2 - 2x - 1$. Observe some of the values of this equation, and we may use a table to write them down:

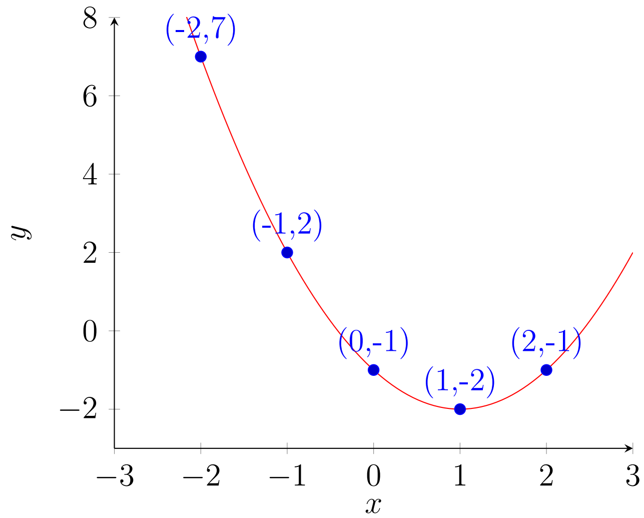
x	-2	-1	0	1	2
y	7	2	-1	-2	-1

We could observe that $y = 2 > 0$ when $x = -1$, and $y = -1 < 0$ when $x = 0$. So there must be some x between -1 and 0 so that $y = 0$, ensured by a solid theorem called Mean Value Theorem. It turns out the solution to the quadratic equation exists. Why is this so? As we consider $y = ax^2 + bx + c$, we are observing a more general case of the problem. In particular, the scenario for $y = 0$ returns to the quadratic equation, and our work before provided information for the existence of $y = 0$, so there is some x satisfying the condition, which indeed is concluding the existence of solution.

May we complete the graphical method in following way. We first plot the points calculated before. They are $(-2, 7), (-1, 2), (0, -1), (1, -2), (2, -1)$ according to the table. We then have the following graph:



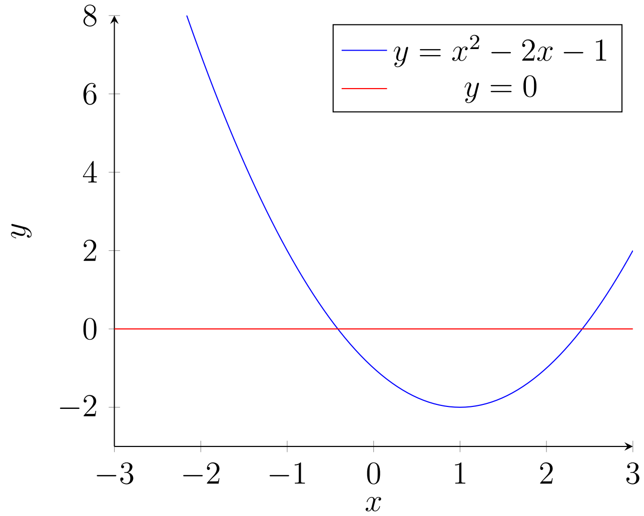
We shall link the points up, and with some smoothness, we may draw the graph like:



And we observe that solving $x^2 - 2x - 1 = 0$ is equivalent to solve the system

$$\begin{cases} y = x^2 - 2x - 1 \\ y = 0 \end{cases}$$

which, we can put the line $y = 0$ on the graph to search for the intersections, as we knew the solution to the system of equations are the intersections of the equations. Hence, we have



Here, we can observe the intersections lie in between -1 to 0 and 2 to 3. They are the desired roots, a.k.a. solutions, to the quadratic equation. This is the visualisation to quadratic equation in some sense. But one could still not easily determine the roots of the quadratic equation explicitly from the graph. We still need other effective thinking to achieve the goal.

The quadratic formula

Modern society equipped with modern tools. The explicit solution usually exists in algebraic ways, because algebraic expressions are meaningful in expressing thoughts. We could now study two of the genius thinking to solve **any** quadratic equation effectively.

Completing square

One should see if $x^2 = c$ with $c \geq 0$ constant, then $x = \pm\sqrt{c}$. We adopt the meaning of $\pm$ as 'plus or minus'. The historical genius thought that if they could complete a square from the original equation, then they would be able to resolve the problems. Recall the identity

$$(a+b)^2 \equiv a^2 + 2ab + b^2$$

such that we are able to see $x = a$ and remain to find what b substitutes. Let's check a simpler case (monic case) of a quadratic equation:

$$x^2 + px + q = 0$$

We shall compare the like terms and retrieve the condition $p = 2b$ such that there is a complete square to simplify the variables. We have the following trick:

$$\begin{aligned}
0 = x^2 + px + q &= x^2 + 2(\frac{p}{2})x + q \\
&= x^2 + 2(\frac{p}{2})x + (\frac{p}{2})^2 - (\frac{p}{2})^2 + q \\
&= (x + \frac{p}{2})^2 + q - \frac{p^2}{4}
\end{aligned}$$

Then,

$$\begin{aligned}
(x + \frac{p}{2})^2 &= \frac{p^2}{4} - q \\
(x + \frac{p}{2})^2 &= \frac{p^2 - 4q}{4} \\
x + \frac{p}{2} &= \pm\sqrt{\frac{p^2 - 4q}{4}} \\
x &= \frac{-p \pm \sqrt{p^2 - 4q}}{2}
\end{aligned}$$

We can next consider the more general case of quadratic equation

$$ax^2 + bx + c = 0$$

with $a \neq 0$ and b, c be constants. With the previous work, the simplification is simple:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
\implies x &= \frac{-b/a \pm \sqrt{(b/a)^2 - 4(c/a)}}{2} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Hence, the solutions could be deduced using coeficients of the quadratic equation. This is called the quadratic formula. The formula is useful and we will emphasis the content later.

Lagrange resolvent

Another method to find the quadratic equation is by the Lagrange resolvent, which is quite abstract but powerful. We recall the relation

$$(a + b)^2 - (a - b)^2 = 4ab \implies (a - b)^2 = (a + b)^2 - 4ab$$

Also, from the part of forming equations, we knew that if

$$x^2 + px + q = (x - \alpha)(x - \beta)$$

then

$$p = -(\alpha + \beta), q = \alpha\beta$$

by comparing like terms. More about coefficients and roots will be emphasis in later sections.

We may consider the following:

$$\begin{cases} r_1 = \alpha + \beta \\ r_2 = \alpha - \beta \end{cases}$$

in which we can easily see $\alpha = \frac{r_1+r_2}{2}$ and $\beta = \frac{r_1-r_2}{2}$. It suffices to resolve r_1 and r_2.

It is easy to see $r_1 = -p$, and for r_2, we use the identity $r_2^2 = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = p^2 - 4q$. Hence, $r_2 = \pm\sqrt{p^2 - 4q}$.

Finally, we conclude $\frac{-p\pm\sqrt{p^2-4q}}{2}$ the quadratic formula of monic case. The general case is similar to previous method.

Discriminant

Throughout the process, we have the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as the solution to the quadratic equation $ax^2 + bx + c = 0$. Observe the formula has a square root operation, which is not defining for negative numbers.

It is interesting that if we apply the above thinking to the quadratic formula, we will have 3 cases: (1) if $b^2 - 4ac > 0$, then there are two distinct roots; (2) if $b^2 - 4ac = 0$, there will be double roots, or we call it only one root; (3) if $b^2 - 4ac < 0$, the square root broke, and no solution can be found. In fact, we focus on real solution and results follow.

We extensively define $\Delta = b^2 - 4ac$ for quadratic equation usage, and this will be convenient to deal with when there is more than one quadratic function to be considered.

As a consequence, we could now write the components separately as: for $ax^2 + bx + c = 0$, we have

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

where $\Delta = b^2 - 4ac$ is the discriminant of the quadratic equation.

Extending to complex numbers