1 Quaternion

Definition (Quaternion). *Let $a, b, c, d \in \mathbb{R}$ and unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be pointing positively along 3 spatial axes such that a* ***quaternion*** *q can be written in the form of*

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

The scalar part of q is denoted by $\mathrm{Re}\{q\} := a$ and the vector part of q is denoted by $\mathrm{Im}\{q\} := b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. The space of quaternion is denoted by $\mathbb{H}$, called the ***Hamiltonian****.*

To type it simple, I may sometimes denote $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ as (a, b, c, d). For readability, it may sometimes be a column vector $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

Definition (Quaternion Arithmetic). *Let $p := (p_0, p_1, p_2, p_3), q := (q_0, q_1, q_2, q_3) \in \mathbb{H}$. Then the arithmetic of quaternion is defined by*

- *Addition: $p + q := (p_0 + q_0, p_1 + q_1, p_2 + q_2, p_3 + q_3)$;*
- *Scalar multiplication: $\lambda p := (\lambda p_0, \lambda p_1, \lambda p_2, \lambda p_3)$ for $\lambda \in \mathbb{R}$.*
- $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, $\mathbf{ijk} = -1$.

Proposition (Identity element). *$1 \in \mathbb{H}$ is the only identity element.*

Proof. Suppose the identity element $e \in \mathbb{H}$ is in the form of (e_0, e_1, e_2, e_3). Then

$$(q_0, q_1, q_2, q_3)(e_0, e_1, e_2, e_3) = \begin{pmatrix} q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3 \\ q_0e_1 + q_1e_0 + q_2e_3 - q_3e_2 \\ q_0e_2 + q_2e_0 + q_3e_1 - q_1e_3 \\ q_0e_3 + q_3e_0 + q_1e_2 - q_2e_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Solving equation yields $e_0 = 1$ and $e_1 = e_2 = e_3 = 0$. □

Definition (Norm). *The norm of a quaternion $q = (q_0, q_1, q_2, q_3) \in \mathbb{H}$ is defined as*

$$\|q\| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Proposition (Conjugation)**.** *For $q = (q_0, q_1, q_2, q_3)$, the algebraic conjugation $\bar{q}$ is*

$$(q_0, -q_1, -q_2, -q_3)$$

Proof. Consider $\bar{q} := (\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$, we have to have

$$(q_0, q_1, q_2, q_3)(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3) = \begin{pmatrix} q_0\bar{q}_0 - q_1\bar{q}_1 - q_2\bar{q}_2 - q_3\bar{q}_3 \\ q_0\bar{q}_1 + q_1\bar{q}_0 + q_2\bar{q}_3 - q_3\bar{q}_2 \\ q_0\bar{q}_2 + q_2\bar{q}_0 + q_3\bar{q}_1 - q_1\bar{q}_3 \\ q_0\bar{q}_3 + q_3\bar{q}_0 + q_1\bar{q}_2 - q_2\bar{q}_1 \end{pmatrix} = \begin{pmatrix} q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which yields $\bar{q_0} = q_0$, $\bar{q_1} = -q_1$, $\bar{q_2} = -q_2$, $\bar{q_3} = -q_3$. □

Corollary. *By seeing a quaternion as a scalar adjoin with a vector, we can rewrite the above in another way. Suppose $q = r + \vec{v}$, where $r \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$.*

- *Norm:* $\|q\| = \sqrt{r^2 + \langle\vec{v}|\vec{v}\rangle}$.
- *Conjugation:* $\bar{q} = r - \vec{v}$.

2 Rubik's cube with quaternion