## Chapter III

# APPLICATION OF VECTORS TO MECHANICS

### *Motion of a particle*

30. *Kinematics of a particle.* The phrase „kinematics of a particle" refers to that portion of the study of the motion of a particle which is not concerned with the forces producing the motion, but is concerned rather with the mathematical concepts useful in describing the motion.

Let us consider a moving particle. It is necessary to introduce a „frame of reference" relative to which the motion of the particle can be measured. For a frame of reference we take a rigid body. Such a body is one having the property that the distances between all pairs of particles in it do not vary with the time. We then introduce a set of rectangular cartesian coordinate axes fixed in the frame of reference. Figure 39 shows these axes and the associated unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$.

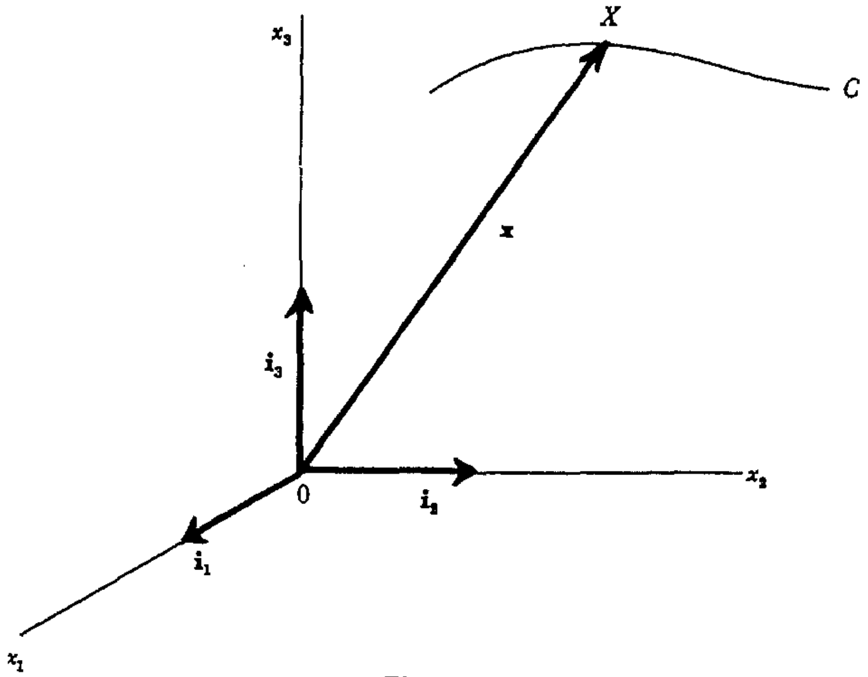


*Figure 39*

Let the curve $C$ in this figure be the path of the particle, and let the point $X$ denote the position of the particle at time $t$. The vector $\overline{OX}$ is the *position-vector* of the particle. We denote this vector also by $\mathbf{x}$. It is a function of the time $t$.

The velocity $\mathbf{v}$ of the particle relative to the frame of reference, and the acceleration $\mathbf{a}$ of the particle relative to the frame of reference, are defined by the relations

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}. \tag{30.1}$$

The magnitude $v$ of the velocity $\mathbf{v}$ is called the *speed* of the particle. Thus, velocity is a vector and speed is a scalar. We shall now compute various sets of components of the vectors $\mathbf{v}$ and $\mathbf{a}$.

(i) *The components of the velocity and acceleration in the directions of rectangular cartesian coordinate axes.* Let $x_1$, $x_2$, $x_3$ denote the rectangular cartesian coordinates of the point $X$ in Figure 39. Then

$$\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3.$$

If we now adopt the convention that a single superimposed dot denotes a first time derivative, and a pair of superimposed dots denotes a second time derivative, then

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot{x}_1\mathbf{i}_1 + \dot{x}_2\mathbf{i}_2 + \dot{x}_3\mathbf{i}_3,$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}_1\mathbf{i}_1 + \ddot{x}_2\mathbf{i}_2 + \ddot{x}_3\mathbf{i}_3.$$

Thus *the desired components of* $\mathbf{v}$ *and* $\mathbf{a}$ *are*

$$\dot{x}_1, \dot{x}_2, \dot{x}_3; \qquad \ddot{x}_1, \ddot{x}_2, \ddot{x}_3. \tag{30.2}$$

(ii) *The components of the velocity and acceleration in the directions of the principal triad of the curve traced out by the particle.* The curve $C$ in Figure 39 is the path of the particle. Let $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$ denote the principal triad at the general point $X$ on $C$. The principal triad was discussed in § 27. If $s$ denotes the arc length of $C$, then from Equations (28.1) and (28.10) we have

$$\mathbf{j}_1 = \frac{d\mathbf{x}}{ds}, \qquad \frac{d\mathbf{j}_1}{ds} = \varkappa\mathbf{j}_2, \tag{30.3}$$

where $\varkappa$ is the curvature of $C$. Now

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\dot{s},$$

and because of the first equation in (30.3) we then have

$$\mathbf{v} = \dot{s}\,\mathbf{j}_1. \tag{30.4}$$

Thus *the velocity of the particle is directed along the tangent to its path, and the speed is* $v = \dot{s}$.

Because of (30.4) we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{s}\,\mathbf{j}_1 + \dot{s}\,\frac{d\mathbf{j}_1}{dt}.$$

But because of the second equation in (30.3) we have

$$\frac{d\mathbf{j}_1}{dt} = \frac{d\mathbf{j}_1}{ds}\dot{s} = \varkappa\dot{s}\mathbf{j}_2,$$

and hence

$$\mathbf{a} = \ddot{s}\mathbf{j}_1 + \varkappa\dot{s}^2\mathbf{j}_2. \tag{30.5}$$

Thus *the acceleration* $\mathbf{a}$ *lies in the osculating plane of* $C$. Also, *the components of* $\mathbf{a}$ *in the directions of the tangent, normal and binormal are*

$$\ddot{s} = \dot{v} = v\frac{dv}{ds}, \qquad \varkappa\dot{s}^2 = \varkappa v^2 = \frac{v^2}{\rho}, \qquad 0, \tag{30.6}$$

*where* $\rho$ *is the radius of curvature of* $C$.

(iii) *The components of the velocity and acceleration in the directions of the parametric lines of cylindrical coordinates.* Let $r, \theta, x_3$ be cylindrical coordinates of a general point $X$ on the path $C$ of the particle. Figure 40 shows these coordinates. We introduce a triad of unit vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ at $O$ as shown; $\mathbf{k}_1$ points toward the point $X'$ which is the projection of $X$ on the $x_1x_2$ plane, $\mathbf{k}_3$ is equal to $\mathbf{i}_3$, and $\mathbf{k}_2$ is such that the triad is right-handed. It will be noted that $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ point in the directions of the parametric lines[1] of the cylindrical coordinates $r$, $\theta$, $x_3$ at $X$.

[1] It will be recalled that the directions of the parametric lines of a coordinate system at a point $X$ are those directions in which one of the coordinates increases while the other coordinates do not vary.

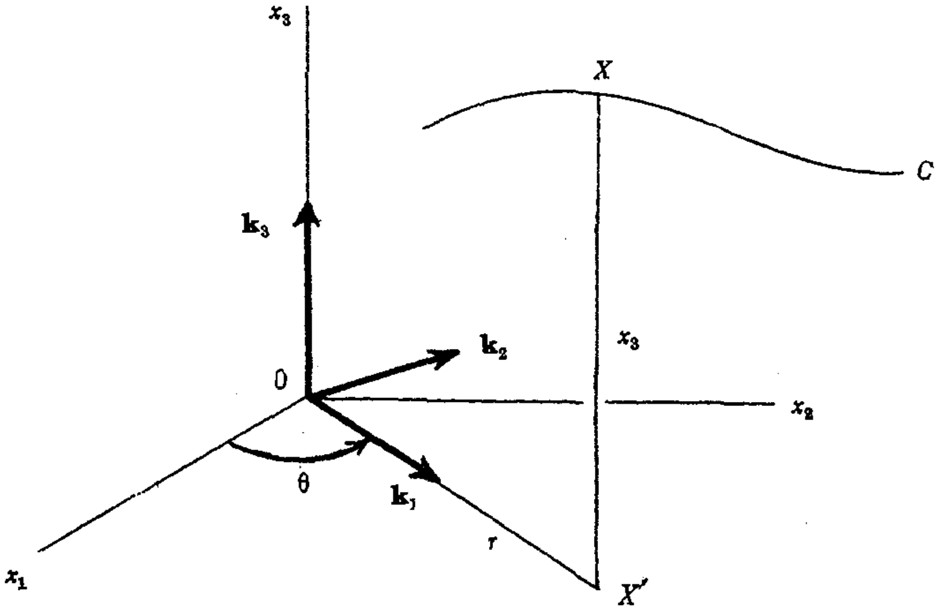


*Figure 40*

If $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the usual unit vectors associated with the rectangular cartesian coordinate axes in Figure 40, then

$$\begin{aligned}\mathbf{k}_1 &= \mathbf{i}_1 \cos\theta + \mathbf{i}_2 \sin\theta\,,\\ \mathbf{k}_2 &= -\mathbf{i}_1 \sin\theta + \mathbf{i}_2 \cos\theta\,,\\ \mathbf{k}_3 &= \mathbf{i}_3\,,\end{aligned}$$

and so

$$\begin{aligned}&\frac{d\mathbf{k}_1}{dt} = \frac{d\mathbf{k}_1}{d\theta}\,\dot\theta = (-\mathbf{i}_1 \sin\theta + \mathbf{i}_2\cos\theta)\,\dot\theta = \mathbf{k}_2\dot\theta\,,\\ (30.7)\qquad &\frac{d\mathbf{k}_2}{dt} = \frac{d\mathbf{k}_2}{d\theta}\,\dot\theta = (-\mathbf{i}_1 \cos\theta - \mathbf{i}_2\sin\theta)\,\dot\theta = -\mathbf{k}_1\dot\theta\,,\\ &\frac{d\mathbf{k}_3}{dt} = 0\,.\end{aligned}$$

From Figure 40 it follows that the position-vector $\mathbf{x}$ of the particle is given by the relation $\mathbf{x} = r\mathbf{k}_1 + x_3\mathbf{k}_3$. Thus

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot r\mathbf{k}_1 + \dot x_3\mathbf{k}_3 + r\,\frac{d\mathbf{k}_1}{dt} + x_3\,\frac{d\mathbf{k}_3}{dt}\,.$$

Because of (30.7) we then obtain

$$(30.8) \qquad \mathbf{v} = \dot{r}\mathbf{k}_1 + r\dot{\theta}\mathbf{k}_2 + \dot{x}_3\mathbf{k}_3 .$$

Thus *the desired components of* $\mathbf{v}$ *are*

$$(30.9) \qquad \dot{r}, \ r\dot{\theta}, \ \dot{x}_3 .$$

From (30.8) we find that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{k}_1 + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{k}_2 + \ddot{x}_3\mathbf{k}_3$$

$$+\dot{r}\frac{d\mathbf{k}_1}{dt} + r\dot{\theta}\frac{d\mathbf{k}_2}{dt} + \dot{x}_3\frac{d\mathbf{k}_3}{dt} .$$

Substitution in this equation from (30.7) then yields

$$(30.10) \qquad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{k}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{k}_2 + \ddot{x}_3\mathbf{k}_3 .$$

Hence *the desired components of* $\mathbf{a}$ *are*

$$(30.11) \qquad \ddot{r} - r\dot{\theta}^2, \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}), \quad \ddot{x}_3 .$$

Of course, if the particle is confined to the $x_1x_2$ plane then $x_3 = 0$, $r = x$, and we find from (30.9) and (30.11) that the components of $\mathbf{v}$ and $\mathbf{a}$ in the directions of the parametric lines of the plane polar coordinates $x$, $\theta$ are

$$(30.12) \qquad x, \quad x\dot{\theta}; \quad \ddot{x} - x\dot{\theta}^2, \frac{1}{x}\frac{d}{dt}(x^2\dot{\theta}) .$$

31. *Newton's laws.* The concept of force is intuitive. We can define a unit force as that force which produces a standard deflection of a standard spring. Hence we can assign a numerical value to the magnitude of any force.

We know that forces have magnitude and direction. It has been verified experimentally to within the limits of experimental error that forces obey the law of vector addition. Hence we shall assume that *forces are vectors*. The sum of two or more forces is sometimes called the resultant of the forces.

The term "mass of a body" refers to the quantity of matter present in the body. We can define a unit mass as that mass which, when suspended from a standard spring at a standard place in the earth's gravitational field, produces a standard deflection of the spring. Hence we can assign a numerical value to the mass of any body.

We now introduce the laws governing the motion of a particle. These laws, which were first stated by Isaac Newton and are called Newton's laws, are as follows:

(i) Every particle continues in a state of rest or uniform motion in a straight line unless compelled by some external force to change that state.

(ii) The product of the mass and acceleration of a particle is proportional to the force applied to the particle, and the acceleration is in the same direction as the force.

(iii) When two particles exert forces on each other, the forces have the same magnitudes and act in opposite directions along the line joining the two particles.

In the second law, the acceleration of the particle enters. This acceleration depends on the frame of reference employed. It thus appears that Newton's second law cannot apply in all frames of reference. Those frames of reference in which this law does apply are called *Newtonian frames of reference*. A frame of reference fixed with respect to the stars is Newtonian, and in making an accurate study of any motion such a frame of reference should be used. However, for many problems we may consider the earth as a Newtonian frame of reference, when effects due to the motion of the earth are negligible.

Let us now consider a particle of mass $m$ acted upon by a force $\mathbf{F}$. Let $\mathbf{a}$ denote the acceleration of the particle relative to a Newtonian frame of reference. Then according to Newton's second law

$$\mathbf{F} = k\, m\, \mathbf{a},$$

where $k$ is a constant of proportionality. It is customary to choose units of length, mass, time and force so that $k$ is equal to unity. We then have

$$\mathbf{F} = m\, \mathbf{a}. \tag{31.1}$$

There are three such systems of units in general use. These are indicated in Table 1, together with abbreviations commonly used for these units. Thus, for example, when a force of one pdl. acts on a particle with a mass of one lb., the acceleration of the particle is one ft./sec.$^2$. The systems of units in the second and third columns of Table 1 are called foot-pound-second sytems, or simply f.p.s. systems. The system of units in the fourth column is called the centimeter–gram–second system, or simply the c.g.s. system.

| | f.p.s. | | c.g.s. |
|---|---|---|---|
| Unit of length | foot (ft.) | foot (ft.) | centimeter (cm.) |
| Unit of mass | pound (lb.) | slug | gram (gm.) |
| Unit of time | second (sec.) | second (sec.) | second (sec.) |
| Unit of force | poundal (pdl.) | pound-weight (lb.wt.) | dyne |

TABLE 1. Systems of units used in mechanics.

The lb. wt. is the force exerted on a mass of one lb. by the earth's gravitational field. If $G$ denotes the acceleration due to gravity, expressed in ft./sec.$^2$, then

$$\begin{aligned} 1 \text{ lb. wt.} &= G \text{ pdl.}, \\ 1 \text{ slug} &= G \text{ lb.} \end{aligned}$$

At points near the surface of the earth, $G$ is approximately equal to 32.

Equation (31.1), which governs the motion of a particle, may also be written in the equivalent forms

$$\text{(31.2)} \qquad \mathbf{F} = m\frac{d\mathbf{v}}{dt}, \qquad \mathbf{F} = m\frac{d^2\mathbf{x}}{dt^2}.$$

32. *Motion of a particle acted upon by a force which is a given function of the time.* When the force $\mathbf{F}$ acting on a particle is a given function of the time, Equations (31.2) can be solved by integration for the velocity $\mathbf{v}$ and position-vector $\mathbf{x}$ of the particle.

As an example, let us suppose that

$$\mathbf{F} = 12\,\mathbf{p} + \mathbf{q} \cos t,$$

where $\mathbf{p}$ and $\mathbf{q}$ are given constant vectors. Because of the first equation in (31.2) we then have

$$m\frac{d\mathbf{v}}{dt} = 12\mathbf{p}+\mathbf{q}\cos t,$$

whence

$$m\mathbf{v} = \int(12\mathbf{p}+\mathbf{q}\cos t)\,\mathrm{dt}.$$

We now carry out this integration in the manner outlined in § 13, obtaining

$$m\mathbf{v} = 12\mathbf{p}t+\mathbf{q}\sin t+\mathbf{r}, \tag{32.1}$$

where $\mathbf{r}$ is an arbitrary constant vector.

Now $\mathbf{v} = d\mathbf{x}/dt$. Thus from (32.1) we have

$$\begin{aligned} m\mathbf{x} &= \int(12\mathbf{p}t+\mathbf{q}\sin t+\mathbf{r})\,dt \\ &= 6\mathbf{p}t^2 - \mathbf{q}\cos t+\mathbf{r}t+\mathbf{s}, \end{aligned} \tag{32.2}$$

where $\mathbf{s}$ is an arbitrary constant vector. The arbitrary constant vectors $\mathbf{r}$ and $\mathbf{s}$ can be found if the initial values of $\mathbf{x}$ and $\mathbf{v}$ are known. If these initial values are $\mathbf{x}_0$ and $\mathbf{v}_0$, it is readily found that

$$\mathbf{r} = m\,\mathbf{v}_0\,, \qquad \mathbf{s} = m\mathbf{x}_0+\mathbf{q}.$$

33. *Simple harmonic motion.* Let $O$ be a point fixed in a Newtonian frame of reference. Let us consider a particle moving under the action of a force directed toward $O$, the force having a magnitude proportional to the distance from the particle to $O$. If $\mathbf{x}$ denotes the position-vector of the particle relative to $O$, then the force $\mathbf{F}$ acting on the particle satisfies the relation

$$\mathbf{F} = -k\mathbf{x},$$

where $k$ is a constant. From Equation (31.2) we then have

$$-k\mathbf{x} = m\frac{d^2\mathbf{x}}{dt^2},$$

or

$$\frac{d^2\mathbf{x}}{dt^2} + \frac{k}{m}\mathbf{x} = 0.$$

This is a differential equation of the type considered in § 14. According to the procedure demonstrated there, the general solution of this differential equation is

$$\mathbf{x} = \mathbf{c}_1 \cos \sqrt{\frac{k}{m}}\, t + \mathbf{c}_2 \sin \sqrt{\frac{k}{m}}\, t\,,$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are arbitrary constant vectors. These arbitrary constant vectors can be found if the initial values of $\mathbf{x}$ and $\mathbf{v}$ are known. If these initial values are $\mathbf{x}_0$ and $\mathbf{v}_0$, it is readily found that

$$\mathbf{c}_1 = \mathbf{x}_0\,, \qquad \mathbf{c}_2 = \sqrt{\frac{m}{k}}\, \mathbf{v}_0,$$

whence we have

$$\mathbf{x} = \mathbf{x}_0 \cos \sqrt{\frac{k}{m}}\, t + \mathbf{v}_0 \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}}\, t\,.$$

It will be noted that $\mathbf{x}$ is a linear function of $\mathbf{x}_0$ and $\mathbf{v}_0$; hence it follows that the motion of the particle is confined to the plane $P$ containing the given vectors $\mathbf{x}_0$ and $\mathbf{v}_0$. This result could have been anticipated, for the force $\mathbf{F}$ acting on the particle has no component perpendicular to the plane $P$.

34. *Central orbits.* Let $O$ be a point fixed in a Newtonian frame of reference. Let us consider a particle acted upon by a force $\mathbf{F}$ directed toward $O$, the magnitude of $\mathbf{F}$ being a function of the distance from $O$ to the particle. The path of the particle is called a *central orbit.* It will be noted that the problem considered in § 33 dealt with a special type of central orbit.

Denoting the vector $\overline{OX}$ by $\mathbf{x}$, we then have

$$F = F(x)\,.$$

The equation of motion is

$$\mathbf{F} = m\mathbf{a}\,.$$

Let $\mathbf{x}_0$ and $\mathbf{v}_0$ be the initial values of $\mathbf{x}$ and the velocity $\mathbf{v}$. The entire path of the particle will be in the plane $P$ containing the vectors $\mathbf{x}_0$ and $\mathbf{v}_0$. Let $x$ and $\theta$ be polar coordinates in this plane. The components of $\mathbf{F}$ in the directions of the parametric lines of these coordinates are $-F$, 0. Also, the components of **a** in these directions are given in Equation (30.12). Hence we have

$$-F = m\,(\ddot{x} - x\dot{\theta}^2), \tag{34.1}$$

$$0 = \frac{m}{x}\frac{d}{dt}\,(x^2\dot{\theta})\,. \tag{34.2}$$

These are two equations from which $x$ and $\theta$ can be determined as functions of the time $t$. It is more convenient, however, to determine from (34.1) and (34.2) a single equation by elimination of the time variable. This single equation will now be deduced.

We first introduce a variable $y$ defined by the relation

$$y = 1/x. \tag{34.3}$$

Then from (34.2) we have

$$\dot{\theta}y^{-2} = \text{const.} = h$$

whence

$$\dot{\theta} = h\,y^2. \tag{34.4}$$

Then

$$\dot{x} = -y^{-2}\dot{y} = -\,y^{-2}\frac{dy}{d\theta}\,\dot{\theta} = -\,h\,\frac{dy}{d\theta},$$

$$\ddot{x} = -\,h\,\frac{d^2y}{d\theta^2}\,\dot{\theta} = -\,h^2y^2\,\frac{d^2y}{d\theta^2}. \tag{34.5}$$

By substitution in (34.1) for $x$, $\dot{\theta}$ and $\ddot{x}$ from (34.3), (34.4) and (34.5), we finally obtain

$$\frac{d^2y}{d\theta^2} + y = \frac{F}{mh^2y^2}. \tag{34.6}$$

Now $F$ is a function of $y$ alone. Once the form of this function has been assigned, we can find the path of the particle by solving Equation (34.6).

Let us now consider the special case when $F$ varies inversely as the square of $x$. Then we can write

$$F = \gamma m y^2,$$

where $\gamma$ is a constant, and Equation (34.6) becomes

$$\frac{d^2y}{d\theta^2} + y = \frac{\Upsilon}{h^2}.$$

The general solution of this equation, expressed in terms of $x$, is

$$\frac{1}{x} = \frac{\Upsilon}{h^2} + c_1 \cos\theta + c_2 \sin\theta, \tag{34.7}$$

where $c_1$ and $c_2$ are arbitrary constants. These constants can be found if the initial values $\mathbf{x}_0$ and $\mathbf{v}_0$ are known. It can be shown that (34.7) represents either an ellipse, parabola or hyperbola, depending on the values of $x_0$ and $v_0$.

### *Motion of a system of particles*

35. *The center of mass of a system of particles.* Let us consider a system of $N$ particles. We denote their masses by the symbols $m_1$, $m_2$, $m_3$, $\cdots$, $m_N$. The total mass $m$ of the system is then given by the relation

$$m = \sum_{j=1}^{N} m_j. \tag{35.1}$$

We denote the coordinates of the particle of mass $m_j$ by the symbols $(x_{j1}, x_{j2}, x_{j3})$. The position-vector $\mathbf{x}_j$ of this particle then satisfies the relation

$$\mathbf{x}_j = x_{j1}\,\mathbf{i}_1 + x_{j2}\,\mathbf{i}_2 + x_{j3}\,\mathbf{i}_3 \quad (j = 1, 2, \cdots, N).$$

We have then a set of $3N$ scalars $x_{jk}$ $(j = 1, 1, 3, \cdots, N;\ k = 1, 2, 3)$ which denotes the coordinates of the particles.

The center of mass of the system of particles is defined to be the point $C$ with position-vector $\mathbf{x}_C$ determined by the equation

$$m\,\mathbf{x}_C = \sum_{j=1}^{N} m_j\,\mathbf{x}_j. \tag{35.2}$$

The center of mass is sometimes called the mass center or centroid or center of gravity.

If the distances between the individual particles in a system remain unaltered, as mentioned previously the system is called a *rigid body*. A rigid body often consists of a continuous distribution of matter, and in this case the summations in Equations (35.1) and (35.2) above must then be replaced by integrations. Thus, if $\rho$ is the density of matter in the body, $V$ is the region occupied by the body, $dV$ is the volume of an element of the body and $\mathbf{x}$ is the position-vector of a point in $dV$, then

$$m = \int_V \rho \, dV, \tag{35.3}$$

$$m\mathbf{x}_C = \int_V \rho \, \mathbf{x} \, dV. \tag{35.4}$$

36. *The moments and products of inertia of a system of particles.* Let us first consider a single particle of mass $m$. Let $l$ denote the length of the perpendicular from the particle to a line $L$, as shown in Figure 41.

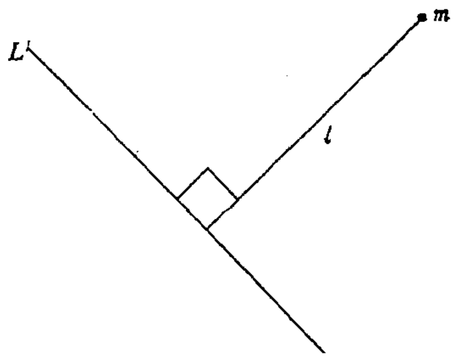


*Figure 41*

The moment of inertia of the particle about the line $L$ is defined to be the scalar $I$ given by the relation

$$I = m\, l^2.$$

Let $p$ and $q$ denote the lengths of the perpendiculars from the particle to two perpendicular planes $P$ and $Q$, as shown in Figure 42. The product of inertia of the particle with respect to these two planes is defined to be the scalar $K$ given by the relation

$$K = m\, p\, q.$$

The moment of inertia about a line of a system of particles is defined to be the sum of the moments of inertia about the line of the individual particles. Also, the product of inertia with respect to two planes of

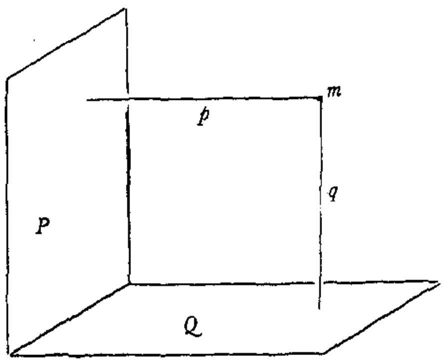


*Figure 42*

a system of particles is defined to be the sum of the products of inertia with respect to the two planes of the individual particles.

We now consider a set of $N$ particles and introduce a set of rectangular cartesian coordinate axes with origin at a point 0. As in § 35 we denote the masses of these particles by $m_j$ ($j = 1, 2, 3, \cdots, N$) and their coordinates by the $3N$ symbols $x_{jk}$ ($j = 1, 2, \cdots, N$; $k = 1, 2, 3$). The moments of inertia of this system of particles about the three coordinate axes are denoted by $I_1$, $I_2$, and $I_3$. It is easily seen that

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{N} m_j \, (x^2_{j2}+x^2_{j3}) \, , \\
I_2 &= \sum_{j=1}^{N} m_j \, (x^2_{j3}+x^2_{j1}) \, , \qquad (36.1) \\
I_3 &= \sum_{j=1}^{N} m_j \, (x^2_{j1}+x^2_{j2}) \, .
\end{aligned}
$$

The products of inertia of this system of particles with respect to the three coordinate planes, taken in pairs, are denoted by $K_1$, $K_2$ and $K_3$. It is easily seen that

$$
\begin{aligned}
K_1 &= \sum_{j=1}^{N} m_j \, x_{j2} \, x_{j3} \, , \\
K_2 &= \sum_{j=1}^{N} m_j \, x_{j3} \, x_{j1} \, , \qquad (36.2) \\
K_3 &= \sum_{j=1}^{N} m_j \, x_{j1} \, x_{j2} \, .
\end{aligned}
$$

Of course, if the system of particles forms a rigid body, the summations in Equations (36.1) and (36.2) must be replaced by integrations.

If $I_1$, $I_2$, $I_3$, $K_1$, $K_2$ and $K_3$ are known, the moment of inertia $I$ of the system about any line $L$ through the origin can be found easily.

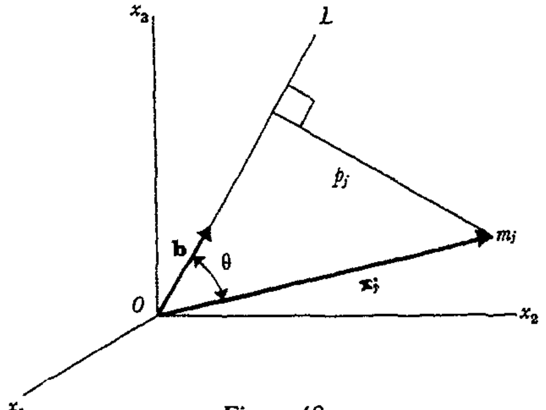


*Figure 43*

In order to prove this we let $p_j$ denote the length of the perpendicular from the particle of mass $m_j$ to the line $L$, as shown in Figure 43. Then

$$I = \sum_{j=1}^{N} m_j p_j^2.$$

But from the figure we see that

$$p_j = x_j \sin\theta = |\mathbf{b} \times \mathbf{x}_j|\ ,$$

where $\mathbf{x}_j$ is the position-vector of the particle of mass $m_j$, $x_j$ is the magnitude of $\mathbf{x}_j$, $\mathbf{b}$ is a unit vector on $L$, and $\theta$ is the angle between $\mathbf{x}_j$ and $\mathbf{b}$. The components of the vector $\mathbf{b} \times \mathbf{x}_j$ are

$$b_2\, x_{j3} - b_3\, x_{j2}, \quad b_3\, x_{j1} - b_1\, x_{j3}\ , \quad b_1\, x_{j2} - b_2\, x_{j1}\ ,$$

and hence $p_j^2$ is equal to the sum of the squares of these three components. Thus

$$\begin{aligned} I &= \sum_{j=1}^{N} m_j \left[(b_2 x_{j3} - b_3\, x_{j2})^2 + (b_3\, x_{j1} - b_1\, x_{j3})^2 + (b_1\, x_{j2} - b_2\, x_{j1})^2\right] \\ &= b_1^2 \sum_{j=1}^{N} m_j\, (x_{j2}^2 + x_{j3}^2) + b_2^2 \sum_{j=1}^{N} m_j\, (x_{j3}^2 + x_{j1}^2)\ + \end{aligned}$$

$$+ b_3^2 \sum_{j=1}^{N} m_j (x_{j1}^2+x_{j2}^2) - 2\, b_2 b_3 \sum_{j=1}^{N} m_j\, x_{j2}\, x_{j3}$$

$$- 2\, b_3\, b_1 \sum_{j=1}^{N} m_j\, x_{j3}\, x_{j1} - 2\, b_1\, b_2 \sum_{j=1}^{N} m_j\, x_{j1}\, x_{j2}\,.$$

Because of (36.1) and (36.2) we then have

$$I = I_1\, b_1^2+I_2\, b_2^2+I_3\, b_3^2 - 2\, K_1\, b_2\, b_3 - 2\, K_2\, b_3\, b_1 - 2\, K_3\, b_1\, b_2\,. \tag{36.3}$$

It will be noted that $b_1$, $b_2$ and $b_3$, which are the components of the unit vector $\mathbf{b}$ on the line $L$, are also the direction cosines of $L$. Equation (36.3) is the desired equation which permits a simple determination of the moment of inertia $I$ of a system about any line $L$ through the origin, once $I_1$, $I_2$, $I_3$, $K_1$, $K_2$ and $K_3$ have been found.

If it happens that $K_1 = K_2 = K_3 = O$, the coordinate axes are said to be *principal axes of inertia* at the point $O$. It can be proved that at every point there is at least one set of principal axes of inertia.[1] In many cases, principal axes of inertia can be deduced readily by considerations of symmetry of the system of particles. For example, at the center of a rectangular parallelepiped the principal axes of inertia are parallel to the edges of the body.

We shall now state without proof two theorems the proofs of which are very simple and may be found in almost any text book on calculus.

*The theorem of perpendicular axes.* If a system of particles lies entirely in a plane $P$, the moment of inertia of the system with respect to a line $L$ perpendicular to the plane $P$ is equal to the sum of the moments of inertia of the system with respect to any two perpendicular lines intersecting $L$ and lying in $P$.

*The theorem of parallel axes.* The moment of inertia $I$ of a system of particles about a line $L$ satisfies the relation

$$I = I'+m\, l^2\,,$$

where $I'$ is the moment of inertia of the system about a line $L'$ parallel to $L$ and through the center of mass of the system, $m$ is the total mass

[1] See J. L. Synge and B. A. Griffith, *Principles of Mechanics*, McGraw-Hill Book Co., 1942, pp. 311–321.

of the system, and $l$ is the perpendicular distance between $L$ and $L'$.

In most books of mathematical tables there are listed the moments of inertia of many bodies with respect to certain axes associated with these bodies. By the use of such tables together with Equation (36.3) and the above two theorems, it is frequently possible to determine rapidly the moment of inertia of a body with respect to any given line.

37. *Kinematics of a rigid body.* Let us consider a rigid body which is rotating about a line $L$ at the rate of $\omega$ radians per unit time. The body is said to have an angular velocity. We can represent this angular velocity completely by an arrow defined as follows: its length represents the scalar to some convenient scale; its origin is an arbitrary point on $L$; its line of action coincides with $L$; it points in the direction indicated by the thumb of the right hand when the fingers are placed to indicate the sense of the rotation about $L$. To prove that angular velocity is a vector, it is then only necessary to prove that it obeys the law of vector addition. This will be done presently. We shall anticipate this result, and denote angular velocities by symbols in bold-faced type.

We shall first determine the velocity $\mathbf{v}$ of a general point $X$ in a body which has an angular velocity $\boldsymbol{\omega}$. Let 0 denote a point on the line of action of $\boldsymbol{\omega}$, and let $\mathbf{x}$ denote the vector $\overline{OX}$, as shown in Figure 44.

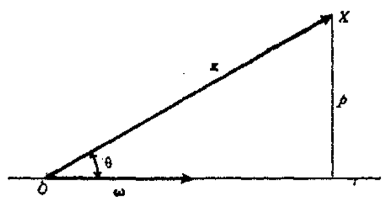


*Figure 44*

Let $\theta$ denote the angle between $\boldsymbol{\omega}$ and $\mathbf{x}$, and let $p$ denote the length of the perpendicular from $X$ to the line of action of $\boldsymbol{\omega}$. The displacement $d\mathbf{x}$ of the point $X$ in time $dt$ has the following properties:

(i) its direction is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{x}$;
(ii) its direction is that indicated by the thumb of the right hand

when the fingers are placed to indicate the sense of the rotation $\theta$ from $\boldsymbol{\omega}$ to $\mathbf{x}$;

(iii) its magnitude is $p\omega dt$, which is equal to $x\omega dt \sin\theta$, $x$ being the magnitude of the vector $\mathbf{x}$.

In view of the definition in § 8 of the vector product of two vectors, it then appears that $d\mathbf{x} = \boldsymbol{\omega}\times\mathbf{x}\, dt$. Thus

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\omega}\times\mathbf{x}.$$

If the point $O$ is fixed in a frame of reference $S$, the velocity $\mathbf{v}$ of the point $X$ relative to $S$ is then

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x}. \tag{37.1}$$

On the other hand, if the point $O$ has a velocity $\mathbf{u}$ relative to a frame of reference $S$, the velocity of the point $X$ relative to $S$ is

$$\mathbf{v} = \mathbf{u}+\boldsymbol{\omega}\times\mathbf{x}. \tag{37.2}$$

We shall now prove that angular velocity obeys the law of vector addition. Let us consider a body which is rotating simultaneously about two lines $L$ and $L'$ which intersect at a point $O$ fixed in a frame of reference $S$. These angular velocities can be represented by the arrows $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ in Figure 45. Let $X$ be a general point in the body,

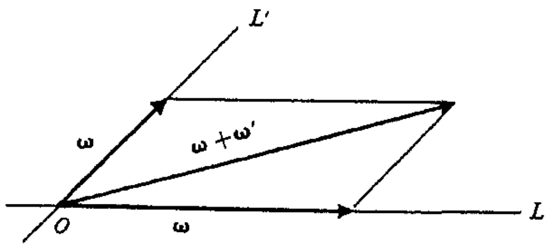


*Figure 45*

with position-vector $\mathbf{x}$ relative to $O$. The two angular velocities impart to $X$ the two velocities $\boldsymbol{\omega}\times\mathbf{x}$ and $\boldsymbol{\omega}'\times\mathbf{x}$ which, being vectors, can be added to yield the resultant velocity

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x}+\boldsymbol{\omega}'\times\mathbf{x}. \tag{37.3}$$

To complete the proof we must show that (37.3) can be written in the form $\mathbf{v} = \boldsymbol{\omega}''\times\mathbf{x}$, where $\boldsymbol{\omega}''$ is an arrow obtained by the application of the law of vector addition to the arrows $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$. Even though angular velocity has not been assumed to satisfy the law of vector addition, Equation (8.5) may be applied to the two products in (37.3) to yield

$$\begin{aligned} v_1 &= \omega_2 x_3 - \omega_3 x_2 + \omega'_2 x_3 - \omega'_3 x_2 \\ &= (\omega_2 + \omega'_2) x_3 - (\omega_3 + \omega'_3) x_2, \end{aligned}$$

and two similar expressions for $v_2$ and $v_3$. Hence we can write

$$\mathbf{v} = \boldsymbol{\omega}'' \times \mathbf{x},$$

where $\boldsymbol{\omega}''$ is an arrow having components $\omega_1 + \omega_1'$, $\omega_2 + \omega'_2$, $\omega_3 + \omega'_3$. But these are the components of the vector obtained by application of the law of vector addition to the arrows $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$. Hence $\boldsymbol{\omega}''$ is equal to the vector sum of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$, and so angular velocity is a vector.

It will be noted that two angular velocities can be added only when their lines of action have a point of intersection, and that the line of action of the sum passes through this point of intersection.

38. *The time derivative of a vector.* Let us consider a set of rectangular cartesian coordinate axes with origin $O$ fixed in a frame of reference $S$, and with axes rotating relative to $S$ with angular velocity $\boldsymbol{\omega}$. Then the line of action of $\boldsymbol{\omega}$ passes though $O$.

If $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the usual unit vectors associated with these coordinate axes, then the velocities relative to $S$ of the terminuses of these vectors are

$$\frac{d\mathbf{i}_1}{dt}, \qquad \frac{d\mathbf{i}_2}{dt}, \qquad \frac{d\mathbf{i}_3}{dt}.$$

But by the previous section these velocities are

$$\boldsymbol{\omega} \times \mathbf{i}_1, \quad \boldsymbol{\omega} \times \mathbf{i}_2, \quad \boldsymbol{\omega} \times \mathbf{i}_3.$$

Hence

$$\frac{d\mathbf{i}_1}{dt} = \boldsymbol{\omega} \times \mathbf{i}_1, \quad \frac{d\mathbf{i}_2}{dt} = \boldsymbol{\omega} \times \mathbf{i}_2, \quad \frac{d\mathbf{i}_3}{dt} = \boldsymbol{\omega} \times \mathbf{i}_3. \tag{38.1}$$

Let $\mathbf{a}$ be a vector with components $a_1$, $a_2$, $a_3$ relative to the rotating coordinate axes. Then

$$\mathbf{a} = a_1\,\mathbf{i}_1 + a_2\,\mathbf{i}_2 + a_3\,\mathbf{i}_3\,,$$

and the time derivative of $\mathbf{a}$, relative to $S$, is then

$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i}_1 + \frac{da_2}{dt}\mathbf{i}_2 + \frac{da_3}{dt}\mathbf{i}_3 + a_1\frac{d\mathbf{i}_1}{dt} + a_2\frac{d\mathbf{i}_2}{dt} + a_3\frac{d\mathbf{i}_3}{dt}\,.$$

Because of Equations (38.1) we can write the last three terms in the form

$$a_1\boldsymbol{\omega}\times\mathbf{i}_1 + a_2\boldsymbol{\omega}\times\mathbf{i}_2 + a_3\boldsymbol{\omega}\times\mathbf{i}_3,$$

which reduces to $\boldsymbol{\omega}\times\mathbf{a}$. Hence we can write

(38.2) $$\frac{d\mathbf{a}}{dt} = \frac{\delta\mathbf{a}}{\delta t} + \boldsymbol{\omega}\times\mathbf{a}\,,$$

where

(38.3) $$\frac{\delta\mathbf{a}}{\delta t} = \frac{da_1}{dt}\mathbf{i}_1 + \frac{da_2}{dt}\mathbf{i}_2 + \frac{da_3}{dt}\mathbf{i}_3\,.$$

Equation (38.2) expresses $d\mathbf{a}/dt$ as the sum of two parts. The part $\delta\mathbf{a}/\delta t$ is the time derivative of $\mathbf{a}$ relative to the moving coordinate system. The part $\boldsymbol{\omega}\times\mathbf{a}$ is the time derivative of $\mathbf{a}$ relative to $S$ when $\mathbf{a}$ is fixed relative to the moving coordinate system.

When the origin of the coordinate system is not at rest relative to $S$ but has a velocity $\mathbf{u}$, Equations (38.1) still hold, and hence also does Equation (38.2).

39. *Linear and angular momentum.* Let us consider a particle of mass $m$, with a position-vector $\mathbf{x}$ relative to a point $O$ fixed in a frame of

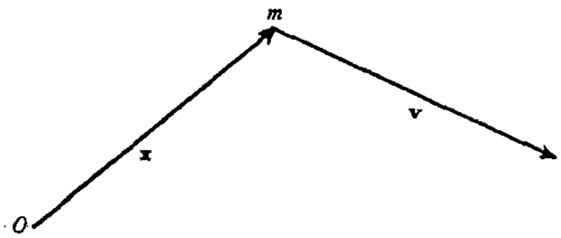


*Figure 46*

reference $S$. Let $\mathbf{v}$ denote the velocity of the particle, as shown in Figure 46. The linear momentum of the particle is a vector $\mathbf{M}$ defined by the relation

$$\mathbf{M} = m\,\mathbf{v}.$$

The angular momentum of the particle about the point $O$ is by definition the moment of $\mathbf{M}$ about $O$. We shall denote it by the symbol $\mathbf{h}$. Hence, by § 10 where the moment of a vector about a point is considered, we have

$$\mathbf{h} = \mathbf{x}\times\mathbf{M} = \mathbf{x}\times m\mathbf{v} = m\mathbf{x}\times\mathbf{v}.$$

Let us now consider a system of $N$ particles. As before, we denote the mass and position-vector of the *j-th* particle by $m_j$ and $\mathbf{x}_j$, respectively. Also, we denote the velocity of this particle relative to $S$ by $\mathbf{v}_j$. Then for this system the linear momentum $\mathbf{M}$ and the angular momentum $\mathbf{h}$ about $O$ are defined by the relations

$$(39.1) \qquad \mathbf{M} = \sum_{j=1}^{N} m_j\mathbf{v}_j, \qquad \mathbf{h} = \sum_{j=1}^{N} m_j\mathbf{x}_j\times\mathbf{v}_j.$$

*Theorem*. The linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of the center of mass of the system.

Proof. The position-vector $\mathbf{x}_C$ of the center of mass of the system is given by Equation (35.2). We differentiate this equation with respect to the time $t$, obtaining

$$m\frac{d\mathbf{x}_C}{dt} = \sum_{j=1}^{N} m_j\frac{d\mathbf{x}_j}{dt}.$$

But

$$\frac{d\mathbf{x}_C}{dt} = \mathbf{v}_C, \qquad \frac{d\mathbf{x}_j}{dt} = \mathbf{v}_j,$$

where $\mathbf{v}_C$ is the velocity of the center of mass $C$ relative to $S$. Hence

$$(39.2) \qquad m\mathbf{v}_C = \sum_{j=1}^{N} m_j\mathbf{v}_j = \mathbf{M}.$$

This completes the proof.

Let us now suppose that the system of particles constitutes a rigid body, and that the body is rotating about the point $O$ which is fixed in the frame of reference $S$. The body then has an angular velocity $\boldsymbol{\omega}$

with a line of action which passes through $O$. The velocity relative to $S$ of the $j$-*th* particle in the body is then

$$\mathbf{v}_j = \boldsymbol{\omega} \times \mathbf{x}_j,$$

and by Equation (39.1) the angular momentum of the system about $O$ is then

$$\mathbf{h} = \sum_{j=1}^{N} m_j \, \mathbf{x}_j \times (\boldsymbol{\omega} \times \mathbf{x}_j) \,. \tag{39.3}$$

Because of the identity (9.3), we can then write

$$\mathbf{h} = \sum_{j=1}^{N} m_j \left[\boldsymbol{\omega} \, x_j^2 - \mathbf{x}_j(\mathbf{x}_j \cdot \boldsymbol{\omega})\right] .$$

Now let us introduce coordinate axes with origin at the point $O$ fixed in $S$. The directions of these coordinate axes need not be fixed in $S$. As before we denote the coordinates of the $j$-*th* particle by $(x_{j1}, x_{j2}, x_{j3})$. The component $h_1$ of $\mathbf{h}$ then has the value

$$\begin{aligned} \mathbf{h}_1 &= \sum_{j=1}^{N} m_j \left[\omega_1 \,(x_{j1}^2 + x_{j2}^2 + x_{j3}^2) - x_{j1}\,(x_{j1}\omega_1 + x_{j2}\omega_2 + x_{j3}\omega_3)\right] \\ &= \omega_1 \sum_{j=1}^{N} m_j \,(x_{j2}^2 + x_{j3}^2) - \omega_2 \sum_{j=1}^{N} m_j \, x_{j1} \, x_{j2} - \omega_3 \sum_{j=1}^{N} m_j \, x_{j3} \, x_{j1} \\ &= I_1\omega_1 - K_3\omega_2 - K_2\omega_3 \,, \end{aligned}$$

where $I_1$, $K_2$ and $K_3$ are moments and products of inertia defined in § 36. There are similar expressions for $h_2$ and $h_3$. We have finally

$$\begin{aligned} h_1 &= I_1\omega_1 - K_3\omega_2 - K_2\omega_3 \,, \\ h_2 &= -K_3\omega_1 + I_2\omega_2 - K_1\omega_3 \,, \\ h_3 &= -K_2\omega_1 - K_1\omega_2 + I_3\omega_3 \,. \end{aligned} \tag{39.4}$$

Let us now consider a rigid body which is moving in a general fashion relative to a frame of reference $S$. Let us introduce coordinate axes with origin at the center of mass $C$ of the body. The directions of the coordinate axes need not be fixed in the body, however. We may consider the body as having a velocity of translation $\mathbf{v}_C$ plus an angular velocity about a line through $C$. Then, as seen in § 37, the velocity of the $j$-*th* particle can be expressed in the form

$$\mathbf{v}_j = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{x}_j \,.$$

Hence the angular momentum $\mathbf{h}$ of the system about the center of mass $C$ has the value

$$\begin{aligned}\mathbf{h} &= \sum_{j=1}^{N} m_j \, \mathbf{x}_j \times (\mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{x}_j) \\ &= \left( \sum_{j=1}^{N} m_j \, \mathbf{x}_j \right) \times \mathbf{v}_C + \sum_{j=1}^{N} m_j \, \mathbf{x}_j \times (\boldsymbol{\omega} \times \mathbf{x}_j) \, .\end{aligned}$$

By Equation (35.2) the first sum is equal to $m\mathbf{x}_C$. Since the origin of the coordinate system and the center of mass $C$ of the body coincide, $\mathbf{x}_C = 0$. Thus

$$\mathbf{h} = \sum_{j=1}^{N} m_j \, \mathbf{x}_j \times (\boldsymbol{\omega} \times \mathbf{x}_j) \, .$$

The right side of this equation is the same as the right side of Equation (39.3). Hence in the present case the components of $\mathbf{h}$ are also given by Equations (39.4).

We have then the important result: *Equations* (39.4) *may be used for the determination of the components of the angular momentum* $\mathbf{h}$ *of a rigid body about either a fixed point O in the body or the center of mass C of the body. In the two cases the origin of the coordinates is at O and C, respectively, the directions of the coordinate axes being quite general.* Equations (39.4) cannot be used in the case of the angular momentum of a rigid body about a moving point which is not the center of mass of the body.

40. *The motion of a system of particles.* Let us consider a general system of $N$ particles. Let $m_j$ denote the mass of the *j-th* particle, and let $\mathbf{v}_j$ denote its velocity relative to a Newtonian system. The forces acting on the *j-th* particle can be divided into two groups called *internal forces* and *external forces.* Internal forces are those due to other particles in the system. External forces include all other forces. Let $\mathbf{F}_{jk}$ denote the internal force exerted on the *j-th* particle by the *k-th* particle, and let $\mathbf{F}_j$ denote the total external force exerted on the *j-th* particle.

*Theorem* 1. The rate of change of the linear momentum of the system is equal to the sum the *external* forces acting on the system.

Proof. Applying to the *j-th* particle Newton's Second Law as stated in § 31, we have

$$m_j \frac{d\mathbf{v}_j}{dt} = \mathbf{F}_j + \sum_{k=1}^{N} \mathbf{F}_{jk} . \tag{40.1}$$

We now sum the $N$ equations in (40.1), obtaining

$$\sum_{j=1}^{N} m_j \frac{d\mathbf{v}_j}{dt} = \sum_{j=1}^{N} \mathbf{F}_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbf{F}_{jk} . \tag{40.2}$$

Because of Newton's Third Law, as stated in § 31, $\mathbf{F}_{jk} = -\mathbf{F}_{jk}$. Thus the double sum in Equation (40.2) vanishes, and we can then write (40.2) in the form

$$\frac{d\mathbf{M}}{dt} = \mathbf{F} , \tag{40.3}$$

where $\mathbf{M}$ is the linear momentum of the system and $\mathbf{F}$ is the sum of the external forces acting on the system.

*Theorem* 2. The center of mass of a system of particles moves like a particle with a mass equal to the total mass of the system acted upon by a force equal to the sum of the external forces acting on the system.

Proof. In § 39 we saw that $\mathbf{M} = m\mathbf{v}_C$, where $m$ is the total mass of the system, and $\mathbf{v}_C$ is the velocity of the center of mass of the system. Thus Equation (40.3) can be written in the form

$$m \frac{d\mathbf{v}_C}{dt} = \mathbf{F} . \tag{40.4}$$

This completes the proof.

41. *The motion of a rigid body with a fixed point.* Let us now consider a system of particles which constitutes a rigid body with a point $O$ fixed relative to a Newtonian frame of reference.

*Theorem* 1. The rate of change of the angular momentum of the body about $O$ is equal to the total moment about $O$ of the external forces.

Proof. Let us introduce coordinates with origin at $O$. Then

$$\mathbf{v}_j = \frac{d\mathbf{x}_j}{dt} , \tag{41.1}$$

where $\mathbf{v}_j$, $\mathbf{x}_j$ and $t$ have the usual meanings. By Equation (39.1), the angular momentum $\mathbf{h}$ of the body about the fixed point $O$ is

$$\mathbf{h} = \sum_{j=1}^{N} m_j\, \mathbf{x}_j \times \mathbf{v}_j\,,$$

and so

$$\frac{d\mathbf{h}}{dt} = \mathbf{A}+\mathbf{B}\,, \tag{41.2}$$

where

$$\mathbf{A} = \sum_{j=1}^{N} m_j \frac{d\mathbf{x}_j}{dt} \times \mathbf{v}_j\,, \quad \mathbf{B} = \sum_{j=1}^{N} m_j\, \mathbf{x}_j \times \frac{d\mathbf{v}_j}{dt}\,.$$

Because of Equation (41.1) we have

$$\mathbf{A} = \sum_{j=1}^{N} m_j\, \mathbf{v}_j \times \mathbf{v}_j = 0\,.$$

Equation (40.1) gives an expression for $m_j d\mathbf{v}_j/dt$. Because of this we have

$$\mathbf{B} = \sum_{j=1}^{N} \mathbf{x}_j \times (\mathbf{F}_j + \sum_{k=1}^{N} \mathbf{F}_{jk}) = \mathbf{G}+\mathbf{H},$$

where

$$\mathbf{G} = \sum_{j=1}^{N} \mathbf{x}_j \times \mathbf{F}_j, \tag{41.3}$$

$$\mathbf{H} = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbf{x}_j \times \mathbf{F}_{jk}\,.$$

It will be recalled that $\mathbf{F}_j$ is the external force acting on the *j-th* particle and $\mathbf{F}_{jk}$ is the internal force exerted on the *j-th* particle by *k-th* particle. We note that $\mathbf{G}$ is the sum of the moments about $O$ of the external forces. Now

$$\mathbf{H} = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbf{x}_j \times \mathbf{F}_{jk} = \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbf{x}_k \times \mathbf{F}_{kj}.$$

Thus

$$2\,\mathbf{H} = \sum_{j=1}^{N} \sum_{k=1}^{N} (\mathbf{x}_j \times \mathbf{F}_{jk} + \mathbf{x}_k \times \mathbf{F}_{kj})\,. \tag{41.4}$$

But $\mathbf{F}_{kj} = -\mathbf{F}_{jk}$. Thus (41.4) becomes

$$2\,\mathbf{H} = \sum_{j=1}^{N} \sum_{k=1}^{N} (\mathbf{x}_j - \mathbf{x}_k) \times \mathbf{F}_{jk}.$$

Since the lines of action of the vectors $\mathbf{x}_j - \mathbf{x}_k$ and $\mathbf{F}_{jk}$ coincide, their vector product vanishes. Hence $\mathbf{H} = O$, and $\mathbf{B} = \mathbf{G}$, so Equation (41.2) reduces to the form

$$\frac{d\mathbf{h}}{dt} = \mathbf{G}\,, \tag{41.5}$$

where $\mathbf{h}$ is the angular momentum of the system about the fixed point $O$, and $\mathbf{G}$ is the total moment about $O$ of the external forces. This completes the proof.

We have placed the origin of the coordinate system at the fixed point $O$. Let us now choose as coordinate axes a set of principal axes of inertia of the body at $O$. (Principal axes of inertia are defined in § 36.) Then the products of inertia $K_1$, $K_2$, $K_3$ all vanish, and from Equations (39.4) we obtain for the components of the angular momentum $\mathbf{h}$ of the body about 0 the expressions

$$h_1 = I_1\omega_1\,, \quad h_2 = I_2\omega_2\,, \quad h_3 = I_3\omega_3\,, \tag{41.6}$$

where $I_1$, $I_2$, $I_3$ are the moments of inertia of the body about the coordinate axes, and $\omega_1$, $\omega_2$, $\omega_3$ are the components of the angular velocity $\boldsymbol{\omega}$ of the body about $O$.

In most cases the coordinate axes will be fixed in the body and will hence have an angular velocity $\boldsymbol{\omega}$ about $O$. However, in a few special cases when the body has a certain symmetry it will be found possible and desirable to choose coordinate axes not fixed in the body. To include such special cases we denote the angular velocity of the axes about $O$ by $\boldsymbol{\Omega}$, which may or may not differ from $\boldsymbol{\omega}$. According to Equation (38.3) we then have

$$\frac{d\mathbf{h}}{dt} = \frac{\delta\mathbf{h}}{\delta t} + \boldsymbol{\Omega}\times\mathbf{h}$$

or

$$\frac{d\mathbf{h}}{dt} = \dot{h}_1\mathbf{i}_1 + \dot{h}_2\mathbf{i}_2 + \dot{h}_3\mathbf{i}_3 + (\Omega_2 h_3 - \Omega_3 h_2)\,\mathbf{i}_1 + (\Omega_3 h_1 - \Omega_1 h_3)\,\mathbf{i}_2 + (\Omega_1 h_2 - \Omega_2 h_1)\,\mathbf{i}_3\,. \tag{41.7}$$

From this equation we can read off the components of the vector $d\mathbf{h}/dt$. According to Equation (41.5) these components are equal to the components of $\mathbf{G}$. Hence we have the equations

$$\begin{aligned} \dot{h}_1+\Omega_2 h_3-\Omega_3 h_2 &= G_1, \\ \dot{h}_2+\Omega_3 h_1-\Omega_1 h_3 &= G_2, \\ \dot{h}_3+\Omega_1 h_2-\Omega_2 h_1 &= G_3, \end{aligned}$$

where $G_1$, $G_2$, $G_3$ are the components of $\mathbf{G}$. Because of Equations (41.6) these relations can be written in the form

$$\begin{aligned} I_1\dot{\omega}_1-I_2\omega_2\Omega_3+I_3\omega_3\Omega_2 &= G_1, \\ I_2\dot{\omega}_2-I_3\omega_3\Omega_1+I_1\omega_1\Omega_3 &= G_2, \\ I_3\dot{\omega}_3-I_1\omega_1\Omega_2+I_2\omega_2\Omega_1 &= G_3. \end{aligned} \tag{41.8}$$

In the case when the coordinate axes are fixed in the rigid body, then $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and so (12.8) reduce to the form

$$\begin{aligned} I_1\dot{\omega}_1-(I_2-I_3)\ \omega_2\omega_3 &= G_1, \\ I_2\dot{\omega}_2-(I_3-I_1)\ \omega_3\omega_1 &= G_2, \\ I_3\dot{\omega}_3-(I_1-I_2)\ \omega_1\omega_2 &= G_3. \end{aligned} \tag{41.9}$$

These equations are called Euler's equations of motion.

*Theorem* 3. The total moment about $O$ of the gravity forces acting on a system of particles is equal to the moment about $O$ of a single force equal to the resultant of the gravity forces and acting at the center of mass of the system.

Proof. Let us introduce a coordinate system with origin at the point $O$. Let $\mathbf{k}$ be a unit vector in the direction of the gravity forces. Then the gravity force acting on the *j-th* particle is $m_j g\mathbf{k}$, and the total moment about $O$ of the gravity forces is

$$\mathbf{G}' = \sum_{j=1}^{N} \mathbf{x}_j \times m_j g\mathbf{k} = \left(\sum_{j=1}^{N} m_j\,\mathbf{x}_j\right)\times g\mathbf{k}.$$

But by Equation (35.2) we have

$$\sum_{j=1}^{N} m_j\,\mathbf{x}_j = m\,\mathbf{x}_C$$

where $m$ is the total mass and $\mathbf{x}_C$ is the position-vector of the center of mass. Thus

$$\mathbf{G}' = m\mathbf{x}_C \times g\mathbf{k} = \mathbf{x}_C \times (mg\mathbf{k}) .$$

But $\mathbf{x}_C \times (mg\mathbf{k})$ is the moment about $O$ of a single force $mg\mathbf{k}$ equal to the resultant of the gravity forces and acting at the center of mass $C$ of the system. This completes the proof.

*Example* 1. *A sphere of radius a is placed on a rough plane which makes an angle α with the horizontal, and is then released. Find the distance the sphere moves down the plane in time t.*

Figure 47 shows the configuration of the system at a general time $t$.

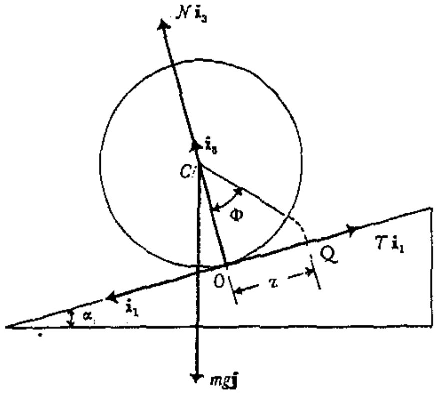


*Figure 47*

The center of mass of the sphere is at its geometrical center. The point Q is the initial point of contact of the sphere and plane, and in the time $t$ the sphere has rolled through an angle $\Phi$, as shown. The point on the sphere which is in contact with the plane is at rest. Hence the sphere has a fixed point, and we select this point as the origin $O$ of the coordinate system. We must select the coordinate axes to coincide with principal directions of inertia of the sphere at $O$. This requirement is satisfied if the unit vectors $\mathbf{i}_1$ and $\mathbf{i}_3$ are chosen as shown in the figure. The unit vector $\mathbf{i}_2$ is then perpendicular to $\mathbf{i}_1$ and $\mathbf{i}_3$, and points up from the page. We note that the coordinate axes are not fixed in the body, and that consequently Equations (41.8) apply.

The external forces acting on the sphere consist of gravity and the reaction of the plane. Because of Theorem 3 above, the forces exerted by gravity on all the particles of the sphere may be replaced by a single force $mg\mathbf{j}$ acting at $C$, as shown, where $\mathbf{j}$ is a unit vector. The reaction of the plane is a force which may be resolved into a force $N\,\mathbf{i}_3$ normal to the plane and a force $-T\,\mathbf{i}_1$ along the plane, as shown. The moment $\mathbf{G}$ of the external forces about $O$ is given by the relation

$$\mathbf{G} = \overline{OC}\times mg\mathbf{j},$$

since the moments of $\mathbf{N}$ and $\mathbf{T}$ about $O$ are equal to zero. But

$$\overline{OC} = a\mathbf{i}_3, \quad \mathbf{j} = \mathbf{i}_1 \sin\alpha - \mathbf{i}_3 \cos\alpha.$$

Thus

$$\begin{aligned} \mathbf{G} &= a\mathbf{i}_3\times mg\,(\mathbf{i}_1 \sin\alpha - \mathbf{i}_3\cos\alpha) \\ &= mga\mathbf{i}_2 \sin\alpha, \end{aligned}$$

so

(41.10) $$G_1 = 0, \quad G_2 = mga\sin\alpha, \quad G_3 = 0.$$

The coordinate axes have no angular velocity, so

(41.11) $$\Omega_1 = \Omega_2 = \Omega_3 = 0.$$

The angular velocity $\boldsymbol{\omega}$ of the sphere about 0 is

$$\boldsymbol{\omega} = \dot{\Phi}\,\mathbf{i}_2,$$

where the superimposed dot denotes differentiation with respect to $t$. Thus

(41.12) $$\omega_1 = 0, \quad \omega_2 = \dot{\Phi}, \quad \omega_3 = 0.$$

The moments of inertia of the sphere about the coordinate axes are

(41.13) $$I_1 = I_2 = \tfrac{7}{5}ma^2, \quad I_3 = \tfrac{2}{5}ma^2.$$

We now substitute in Euler's equations (41.8) from Equations (41.10), (41.11), (41.12) and (41.13) to obtain the relation

$$\tfrac{7}{5}ma^2\ddot{\Phi}\mathbf{i}_2 = mga\mathbf{i}_2\sin\alpha.$$

Thus

$$\ddot{\Phi} = \frac{5g\sin\alpha}{7a},$$

and two integrations then yield

$$\Phi = \frac{5g \sin \alpha}{14a} t^2,$$

since $\Phi = \dot{\Phi} = 0$ when $t = 0$.

If $z$ is the distance the sphere has rolled down the plane in time $t$, then $z = a\Phi$ and we have

(41.14) $$z = \tfrac{5}{14} g t^2 \sin \alpha.$$

*Example* 2. *Two shafts are attached to the corners A and C of a rectangular plate ABCD, in such a way that the axes of the shafts are continuations of the diagonal AC. The shafts turn in two bearings each at a distance c from the center of the plate. The system is made to rotate at a constant rate of* $\omega$ *radians per unit time. Find the forces exerted on the bearings.*

The system is shown in Figure 48. We choose the center $O$ of the

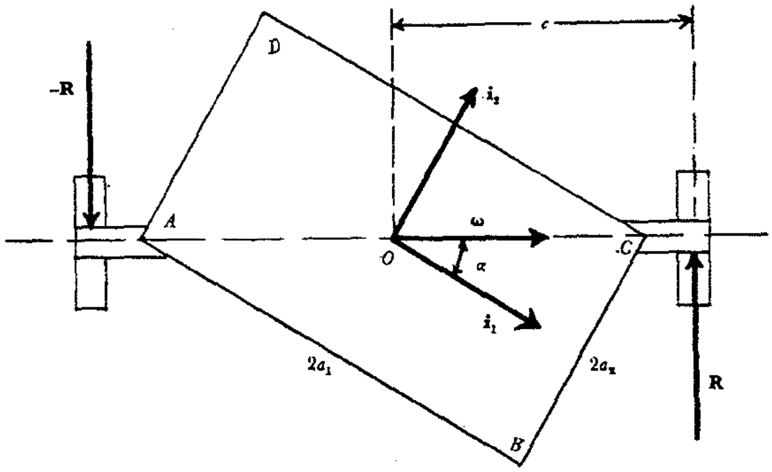


*Figure 48*

plate as the origin of the coordinate system. We also choose the unit vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ parallel to edges of the plate, as shown. The unit vector $\mathbf{i}_3$ is then perpendicular to the plane of the plate. The directions of these three vectors are principal directions of inertia of the plate at $O$. The moments of inertia of the plate about the coordinate axes are

(41.15) $$I_1 = \tfrac{1}{3} m a_2^2, \quad I_2 = \tfrac{1}{3} m a_1^2, \quad I_3 = \tfrac{1}{3} m (a_1^2 + a_2^2),$$

where $m$ is the mass of the plate, and $2a_1$ and $2a_2$ are the lengths of the edges.

The plate has an angular velocity $\boldsymbol{\omega}$ the line of action of which is the diagonal $AC$ and the magnitude of which is the given constant $\omega$. If $\alpha$ is the angle between $\boldsymbol{\omega}$ and $\mathbf{i}_1$, then

$$\tan \alpha = \frac{a_2}{a_1} \tag{41.16}$$

and

$$\boldsymbol{\omega} = \omega \mathbf{i}_1 \cos \alpha + \omega \mathbf{i}_2 \sin \alpha .$$

Thus

$$\omega_1 = \omega \cos \alpha, \quad \omega_2 = \omega \sin \alpha, \quad \omega_3 = 0. \tag{41.17}$$

Since the coordinate axes are fixed in the body, Euler's Equations (41.9) apply. We substitute in these equations from (41.15) and (41.17), obtaining the relations

$$G_1 = 0, \quad G_2 = 0, \quad G_3 = \tfrac{1}{3} m (a_1^2 - a_2^2)\, \omega^2 \sin \alpha \cos \alpha .$$

Thus the moment $\mathbf{G}$ about $O$ of the external forces is normal to the plate and rotates with it. Hence the forces exerted on the shaft by the bearings must be in the plane of the plate. Let us denote these forces by $\mathbf{R}$ and $-\mathbf{R}$, as shown in Figure 48. We must then have

$$2cR = G_3$$

whence we find that

$$\begin{aligned} R &= \tfrac{1}{6c} m \,(a_1^2 - a_2^2)\, \omega^2 \sin \alpha \cos \alpha \\ &= \tfrac{1}{6} m\omega^2 \frac{a_1 a_2 (a_1^2 - a_2^2)}{c(a_1^2 + a_2^2)} . \end{aligned} \tag{41.18}$$

By Newton's third law (§ 31), the forces exerted on the right and left bearings are $-\mathbf{R}$ and $\mathbf{R}$, with magnitudes $R$ given in Equation (41.18) above.

*Example* 3. *A gyroscope is mounted so that one point on its axis is fixed. Investigate those motions under gravity in which the axis of the gyroscope makes a constant angle with the vertical.*

A gyroscope is a body with an axis of symmetry, the shape of the body being such that the moment of inertia of the body about its axis of symmetry is large. For example, the disc and shaft in Figure 49 constitute a gyroscope.

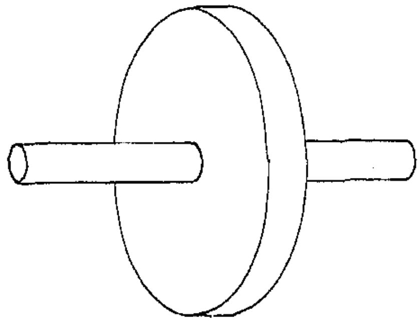

*Figure 49*

In Figure 50 the line $OA$ is the axis of the gyroscope, the fixed point being at $O$ and the center of mass being at $C$. We introduce a fixed unit vector $\mathbf{j}$ pointing up from $O$, and a set of moving orthogonal

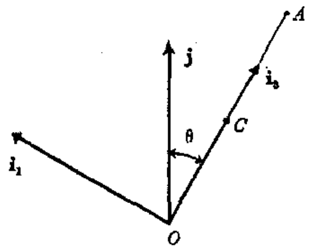


*Figure 50*

unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ defined as follows: $\mathbf{i}_3$ is along $OA$; $\mathbf{i}_1$ is in the plane of $\mathbf{j}$ and $\mathbf{i}_3$, as shown; $\mathbf{i}_2$ completes the triad. We note that $\mathbf{i}_2$ is horizontal.

The angle between $OA$ and $\mathbf{j}$ is denoted by $\theta$; it is constant. The plane of $\mathbf{j}$ and $\mathbf{i}_3$ rotates about $\mathbf{j}$ at a rate of $p$ radians per unit time; $p$ is called the precession. The gyroscope spins about its axis at the rate of $s$ radians per unit time; $s$ is called the spin. We denote the mass of the gyroscope by $m$ and the distance $OC$ by $l$.

The only external forces are the reaction of the pivot support at $O$ and the gravity forces. The former force has no moment about $O$. The latter forces may be replaced by a single force $-mg\mathbf{j}$ at $C$. Hence

$$\mathbf{G} = (l\,\mathbf{i}_3) \times (-mg\mathbf{j}).$$

But

$$(41.19) \qquad \mathbf{j} = \mathbf{i}_1 \sin\theta + \mathbf{i}_3 \cos\theta,$$

whence we get

$$\mathbf{G} = -lmg \sin\theta\, \mathbf{i}_2.$$

Thus

$$(41.20) \qquad G_1 = 0, \quad G_2 = -lmg \sin\theta, \quad G_3 = 0.$$

The angular velocity $\boldsymbol{\omega}$ of the gyroscope is

$$\boldsymbol{\omega} = s\mathbf{i}_3 + p\mathbf{j}.$$

Because of Equation (41.19) we then have

$$\boldsymbol{\omega} = p \sin\theta \mathbf{i}_1 + (s + p \cos\theta)\, \mathbf{i}_3,$$

whence

$$(41.21) \qquad \omega_1 = p \sin\theta, \quad \omega_2 = 0, \quad \omega_3 = s + p\cos\theta.$$

The angular velocity $\boldsymbol{\Omega}$ of the coordinate axes is given by the relation

$$\begin{aligned} \boldsymbol{\Omega} &= p\mathbf{j} \\ &= p\,(\mathbf{i}_1 \sin\theta + \mathbf{i}_3 \cos\theta). \end{aligned}$$

Thus

$$(41.22) \qquad \Omega_1 = p \sin\theta, \quad \Omega_2 = 0, \quad \Omega_3 = p\cos\theta.$$

The coordinate axes associated with $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are principal axes of inertia at $O$, and we have

$$(41.23) \qquad I_1 = I_2, \qquad K_1 = K_2 = K_3 = 0.$$

We now substitute in Euler's equations (41.8) from Equations (41.20), (41.21), (41.22) and (41.23) to obtain the relations

$$(41.24) \qquad I_1 \dot{p} \sin\theta = 0,$$

$$(41.25) \qquad [I_3 s + (I_3 - I_1)\, p \cos\theta]\, p \sin\theta = lmg \sin\theta,$$

$$(41.26) \qquad I_3\,(\dot{s} - \dot{p} \cos\theta) = 0.$$

One solution of these equations is $\theta = 0$. In this case the axis of the gyroscope is vertical, and the gyroscope is said to be "sleeping". If $\theta$ is not equal to zero, then Equations (41.24) and (41.26) yield

$$p = \text{constant}, \qquad s = \text{constant},$$

and Equation (41.25) takes the form

$$(I_3 - I_1) \cos \theta \, p^2 + I_3 sp - lmg = 0. \tag{41.27}$$

This is a relation among the three constants $p$, $s$ and $\theta$. Hence it appears that we may assign arbitrarily values for two of these constants and there will exist a corresponding motion of the top with $\theta$ a constant, provided of course the value of third constant, as obtained from Equation (41.27), is real.

We note from (41.27) that

$$s = \frac{lmg}{I_3 p} - \frac{(I_3 - I_1) \, p \cos \theta}{I_3}. \tag{41.28}$$

The quantities $p$ and $\theta$ may be observed readily. The corresponding spin $s$ may be computed by means of this relation. If the precession is small, we note from Equation (41.28) that the spin is large and has the approximate value

$$s = \frac{lmg}{I_3}.$$

42. *The general motion of a rigid body.* We now consider a rigid body moving in a general manner. It may or may not have a fixed point. The motion of its mass center can be determined from Theorem 2 of § 40, which applies to the motion of any system of particles. This theorem yields

$$m \frac{d\mathbf{v}_C}{dt} = \mathbf{F}, \tag{42.1}$$

where $m$ is the total mass of the body, $\mathbf{v}_C$ is the velocity of its center of mass, and $\mathbf{F}$ is the sum of the external forces acting on the body. Integration of (42.1) gives the position-vector $\mathbf{x}_C$ of the center of mass $C$ of the body as a function of the time $t$.

To determine the complete motion of the body it is then only necessary to find the angular velocity of the body about its center of mass.

To do this, we choose the origin $O$ of the coordinate system at the center of mass $C$ of the body. We then consider the body as having a velocity of translation $\mathbf{v}_C$ plus an angular velocity $\boldsymbol{\omega}$ with a line of action through $C$. The velocity $\mathbf{v}_j$ of the *j-th* particle is then given by the relations

(42.2) $$\mathbf{v}_j = \mathbf{v}_C + \frac{d\mathbf{x}_j}{dt} = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{x}_j \, .$$

But by definition the angular momentum $\mathbf{h}$ of the body about the point $O$ is

$$\mathbf{h} = \sum_{j=1}^{N} m_j \mathbf{x}_j \times \mathbf{v}_j ,$$

and we then have, just as in § 41,

(42.3) $$\frac{d\mathbf{h}}{dt} = \mathbf{A} + \mathbf{B} \, ,$$

where

(42.4) $$\mathbf{A} = \sum_{j=1}^{N} m_j \frac{d\mathbf{x}_j}{dt} \times \mathbf{v}_j \, , \qquad \mathbf{B} = \sum_{j=1}^{N} m_j \mathbf{x}_j \times \frac{d\mathbf{v}_j}{dt} \, .$$

From Equation (42.2) we then have

$$\begin{aligned} \mathbf{A} &= \sum_{j=1}^{N} m_j (\mathbf{v}_j - \mathbf{v}_C) \times \mathbf{v}_j \\ &= \sum_{j=1}^{N} m_j \mathbf{v}_j \times \mathbf{v}_j - \mathbf{v}_C \times \sum_{j=1}^{N} m_j \, \mathbf{v}_j \\ &= 0 - \mathbf{v}_C \times \mathbf{M}, \end{aligned}$$

where $\mathbf{M}$ is the linear momentum of the body. But $\mathbf{M} = m\mathbf{v}_C$ by Equation (39.2). Thus $\mathbf{A} = O$. Just as in § 41 we find that $\mathbf{B} = \mathbf{G}$, where $\mathbf{G}$ denotes the total moment about $O$ of all the external forces. Thus Equation (42.3) takes the form

(42.5) $$\frac{d\mathbf{h}}{dt} = \mathbf{G} \, .$$

We have thus the result: *the rate of change of the angular momentum of a body about its center of mass is equal to the total moment of the external forces about the center of mass.*

We have placed the origin of the coordinate system at the center of mass $C$ of the body. If we choose the coordinate axes to coincide with principal axes of inertia of the body at $C$, then just as in § 41 we obtain Equations (41.6) and finally Euler's Equations (41.8) from which we can find the unknown quantities $\omega_1$, $\omega_2$ and $\omega_3$ which characterize the rotation of the body about its center of mass.

In conclusion, it should be noted particularly that the equation $d\mathbf{h}/dt = \mathbf{G}$ can be used only in the two following cases: (i) the body has a fixed point and the origin is at this fixed point; (ii) the origin is at the center of mass of the body.

*Example* 1. *A gyroscope with a constant spin is carried along a horizontal circular path at a constant speed, with its axis tangent to the path of its center of mass. Find the forces exerted on the axle of the gyroscope by the bearings in which the axle turns, neglecting gravity.*

We choose the center of mass of the gyroscope as the origin $O$ of the coordinate system. The path of $O$ is shown in Figure 51; it is a circle

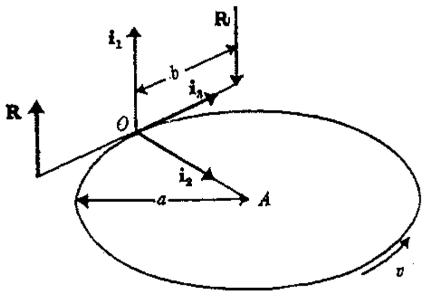


*Figure 51*

with center $A$ and radius $a$. We introduce an orthogonal right triad of unit vectors at $O$, as shown; $\mathbf{i}_1$ points vertically up; $\mathbf{i}_2$ points towards $A$; $\mathbf{i}_3$ is tangent to the circle and hence lies along the axis of the gyroscope.

Let us suppose that $O$ travels at a speed $v$ in the direction opposite to $\mathbf{i}_3$, and that $s$ is the rate at which the gyroscope spins about its axis, $s$ being positive when it is in the sense of the 90° rotation from $\mathbf{i}_1$ to $\mathbf{i}_2$.

The time required for $O$ to go around the circle is $2\pi a/v$. In this time the gyroscope has turned about $\mathbf{i}_1$ through an angle of $2\pi$ radians. Hence the angular velocity $\boldsymbol{\omega}$ of the gyroscope is

$$\boldsymbol{\omega} = s\mathbf{i}_3 + \frac{2\pi}{2\pi\, a/v}\mathbf{i}_1$$

$$= \frac{v}{a}\mathbf{i}_1 + s\mathbf{i}_3 .$$

Thus

(42.6) $$\omega_1 = \frac{v}{a}, \qquad \omega_2 = 0, \qquad \omega_3 = s .$$

The angular velocity $\boldsymbol{\Omega}$ of the coordinate axes. is

$$\boldsymbol{\Omega} = \frac{v}{a}\mathbf{i}_1$$

whence

(42.7) $$\Omega_1 = \frac{v}{a}, \qquad \Omega_2 = 0, \qquad \Omega_3 = 0 .$$

Also

(42.8) $$I_1 = I_2, \qquad K_1 = K_2 = K_3 = 0 .$$

We now substitute in Euler's equations (41.8) from Equations (42.6), (42.7) and (42.8) to obtain the relations

$$G_1 = 0, \qquad G_2 = -I_3\frac{sv}{a}, \qquad G_3 = 0 .$$

Hence the forces exerted on the gyroscope by the bearings must be in the $x_3x_1$ plane. If $\mathbf{R}$ and $-\mathbf{R}$ are these forces, and $2b$ is the distance between the bearings, then

$$2bR = I_3\frac{vs}{a}$$

whence

$$R = I_3\frac{vs}{2ab} .$$

*Problems*

1. A particle moves on the curve $x_2 = h \tan kx_1$, $x_3 = 0$, where $h$ and $k$ are constants. The $x_2$ component of the velocity is constant. Find the acceleration.

2. A particle moves with constant speed. Prove that its acceleration is perpendicular to its velocity.

3. A particle moves on an elliptical path with constant speed. At what points is the magnitude of its acceleration (i) a maximum, (ii) a minimum?

4. A particle moves in space. Find the components of its velocity and acceleration along the parametric lines of spherical polar coordinates.

5. A particle moves in space. Its position-vector $\mathbf{x}$ relative to the origin of a fixed set of rectangular cartesian coordinate axes is given in terms of the time $t$ by the relation

$$\mathbf{x} = h(\mathbf{i}_1 \cos t + \mathbf{i}_2 \sin t + \mathbf{i}_3 t),$$

where $h$ is a constant and $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the usual unit vectors in the directions of the coordinate axes. Find the components of the velocity and acceleration in the directions of (i) the coordinate axes mentioned above, (ii) the principal triad of the path of the particle, (iii) the parametric lines of spherical polar coordinates. Find the speed and the magnitude of the acceleration.

6. A particle describes a rhumb line on a sphere in such a way that its longitude increases uniformly. Prove that the resultant acceleration varies as the cosine of the latitude, and that its direction makes with the inner normal an angle equal to the latitude.

7. Two forces $\mathbf{A}$ and $\mathbf{B}$ act at a point. If $\alpha$ is the angle between their lines of action, prove that the magnitude of the resultant $\mathbf{R}$ is given by the relation

$$R^2 = A^2 + B^2 + 2AB \cos \alpha.$$

8. Four forces $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ act at a point $O$ and are in equilibrium, the forces $\mathbf{C}$ and $\mathbf{D}$ being perpendicular and having equal magnitudes. Find $C$ in terms of $A$, $B$ and the angle $\alpha$ between $\mathbf{A}$ and $\mathbf{B}$.

9. Forces with magnitudes 1, 4, 4 and $2\sqrt{3}$ lb. wt. act at a point. The directions of the first three forces are respectively the directions of the positive axes of $x_1$, $x_2$ and $x_3$. The direction of the fourth force makes equal acute angles with these axes. Find the magnitude and direction of the resultant.

10. A force $\mathbf{F}$ acts on a particle of mass $m$. Find the magnitude of the acceleration, given that (i) $F = 6$ poundals, $m = 3$ lb., (ii) $F =$ 6 lb. wt., $m = 2$ slugs, (iii) $F = 6$ poundals, $m = 2$ slugs, (iv) $F =$ 6 lb. wt., $m = 3$ lb., (v) $F = 5$ dynes, $m = 10$ gm.

11. A particle of mass $m$ is acted upon by a force $\mathbf{F}$ given by the relation

$$\mathbf{F} = 16\mathbf{p} \sin 2t + \mathbf{q}e^{-t},$$

where $\mathbf{p}$ and $\mathbf{q}$ are constant vectors and $t$ is the time. Find the velocity $\mathbf{v}$ and position-vector $\mathbf{x}$ of the particle in terms of $t$, given that $\mathbf{v} = 0$ and $\mathbf{x} = 0$ when $t = 0$.

12. A particle of mass $m$ is acted upon by two forces $\mathbf{P}$ and $\mathbf{Q}$. The force $\mathbf{P}$ acts in the direction of the $x_1$ axis. The force $\mathbf{Q}$ makes angles of 45° with the axes of $x_2$ and $x_3$. Also $P = p \sin kt$ and $Q = q \cos kt$, where $p$, $q$ and $k$ are constants and $t$ is the time. At time $t = 0$ the particle has coordinates $(b, 0, 0)$ and is moving towards the origin with a speed $p/mk$. Find the position-vector $\mathbf{x}$ of the particle. Prove that the particle moves on an ellipse, and find the center and lengths of the axes of the ellipse.

13. A particle of mass $m$ moves under the action of a force $\mathbf{p}e^{-qt}$ and a resistance $-l\mathbf{v}$, where $\mathbf{p}$ is a constant vector, $q$ and $l$ are positive constants, $t$ is the time, and $\mathbf{v}$ is the velocity of the particle. Prove that

$$\mathbf{x}_\infty - \mathbf{x}_0 = \frac{1}{lq}(\mathbf{p} + m\, q\, \mathbf{u})$$

where $\mathbf{u}$ is the velocity when $t = 0$, and $\mathbf{x}_0$ and $\mathbf{x}_\infty$ are respectively the position-vectors of the particle when $t = 0$ and when $t$ becomes infinite. Is the above result true when $l = mq$?

14. A particle of mass $m$ moves under the action of a force $\mathbf{p} \cos qt - k\mathbf{x}$, where $\mathbf{p}$ is a constant vector, $q$ and $k$ are positive constants, $t$ is the time, and $\mathbf{x}$ is the position-vector of the particle relative to a fixed point $O$. When $t = 0$, the particle is at $O$ and has a velocity $\mathbf{u}$. Find $\mathbf{x}$ in terms of $t$ when (i) $k \neq mq^2$, (ii) $k = mq^2$.

15. A particle of mass $m$ is acted upon by a single force $\gamma\, m/x^2$ directed towards a fixed point $O$, where $\gamma$ is a constant and $x$ is the

distance from $O$ to the particle. At time $t = 0$ the particle is at a point $B$ and has a velocity of magnitude $u$ in a direction perpendicular to the line $OB$. Prove that the orbit is (i) an ellipse if $bu^2 < 2\gamma$, (ii) a parabola if $bu^2 = 2\gamma$, (iii) an hyperbola if $bu^2 > 2\gamma$, where $b = OB$.

16. Find the moment of inertia of a circular disk of mass $m$ and radius $a$ about (i) the axis of the disk, (ii) a diameter of the disk. [Answer: (i) $\frac{1}{2}ma^2$; (ii) $\frac{1}{4}ma^2$.]

17. Using the result of Problem 16, find the moment of inertia of a circular cylinder of mass $m$, length $2l$ and radius $a$ about (i) the axis of the cylinder, (ii) a generator of the cylinder, (iii) a line through the center of the cylinder perpendicular to its axis, (iv) a diameter of one end of the cylinder. [Answer: (i) $\frac{1}{2}ma^2$; (ii) $\frac{3}{2}ma^2$; (iii) $\frac{1}{12} m\,(4l^2+3a^2)$; (iv) $\frac{1}{12} m\,(16l^2+3a^2)$.]

18. A circular cylinder has a mass $m$, length $2l$ and radius $a$. Rectangular cartesian coordinates are introduced, with origin $O$ at the center of the cylinder, and the $x_3$ axis coinciding with the axis of the cylinder. Two particles each of mass $m'$ are attached to the cylinder at the points $(0, a, l)$ and $(0, -a, -l)$. Find the moments and products of inertia $I_1$, $I_2$, $I_3$, $K_1$, $K_2$, and $K_3$ for the system consisting of the cylinder and the two particles.

19. A circular disk of mass $m$ and radius $a$ spins with angular speed $\omega$ about a line through its center $O$, making an angle $\alpha$ with its axis. Find the angular momentum of the disk about $O$.

20. For the system of masses in Problem 18, find the angular momentum about the point $O$ when the system has an angular speed $\omega$ about (i) the $x_1$ axis, (ii) the $x_2$ axis, (iii) the $x_3$ axis.

21. A circular cylinder of mass $m$, length $2l$ and radius $a$ turns freely about its axis which is horizontal. A light inextensible cord is wrapped around the cylinder several times. A constant force $F$ is applied to the end of the cord. If the cylinder starts from rest at time $t = 0$, show that at time $t$ it has turned through the angle $Ft^2/ma$.

22. The circular cylinder of Problem 21 is again mounted with its axis horizontal, and has a light inextensible cord wrapped around it. A body with a mass $m'$ is attached to the end of the cord. If the cylinder starts from rest at time $t = 0$ show that at time $t$ the cylinder has

turned through the angle $\frac{mgt^2}{a(m+2m')}$, where $g$ is the acceleration due to gravity.

23. A circular cylinder of mass $m$, length $2l$ and radius $a$ is placed on a rough plane which makes an angle $\alpha$ with the horizontal, and is then released. Find the distance the cylinder moves down the plane in time $t$.

24. In Problem 18 there was introduced a system consisting of a circular cylinder of mass $m$ with two attached particles each of mass $m'$. This system is mounted so it can turn about the axis of the cylinder in two smooth bearings each at a distance $c$ from the center of the cylinder. The system is made to rotate with constant angular speed $\omega$. Find the reactions of the bearings.

25. A circular disk of mass $m$ and radius a turns with constant angular speed $\omega$ about an axis through the center $O$ of the disk and making a constant angle $\alpha$ with the axis of the disk. The disk turns in two smooth bearings each at a distance $c$ from the point $O$. Find the reactions of the bearings.

26. A uniform rod of length $2l$ is free to turn about an axis $L$ perpendicular to it and through its center. The center of the rod moves at constant speed $v$ around a circular track of radius $a$, the axis $L$ being always tangent to the track. Deduce the equations of motion of system.