1. (a)

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(b) 欲證明$\mathbf{y}_k = A\mathbf{x}_k$，則證明

$$\begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{nk} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{bmatrix}$$

將之簡化，則需證明$\forall i, \forall k$，

$$y_{ik} = \sum_{j=1}^{n} a_{ij} x_{jk}$$

此為矩陣乘法定義，因此證畢。

(c) 此矩陣按固定變換方式將$\mathbf{x}_k$映射至$\mathbf{y}_k$。視$\mathbf{x}_k$與$\mathbf{y_k}$為n維空間內的兩組坐標，則是n組坐標按同一方式進行坐標變換，可視作幾何圖形的綫性變換。

2. (a) $f(x) = g^{-1}(d(g(x))) = P^{-1}DPx = Ax$。$f$為以矩陣$A$對坐標$x$進行綫性變換。變換過程：先從$X$的基底變換至$Y$的基底，按$Y$的基底利用矩陣$D$放大縮小，再變換回$X$基底。

(b) 若X的維數為m，Y的維數為n，設$x \in X, y \in Y$，則

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

定$y = Bx$。若$m < n$，則$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} & 0 & \cdots & b_{1n} = 0 \\ b_{21} & b_{22} & \cdots & b_{2m} & 0 & \cdots & b_{2n} = 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} & 0 & \cdots & b_{nn} = 0 \end{bmatrix}$使

得$|B|=0$，則B不可逆；若$m>n$，則$B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}=0 & b_{m2}=0 & \cdots & b_{mm}=0 \end{bmatrix}$使得$|B|=0$，則$B$不可逆。

(c) 不可被對角化的幾何含義為綫性變換的維數不同。

3. (a) i.

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\lim_{h\to 0}\frac{A-A}{h}=0$$

ii.

$$\begin{aligned}\frac{\mathrm{d}(At^n)}{\mathrm{d}t}&=\lim_{h\to 0}\frac{A(t+h)^n-At^n}{h}\\&=A\lim_{h\to 0}\frac{(t+h)^n-t^n}{h}\\&=A\frac{\mathrm{d}t^n}{\mathrm{d}t}\\&=A(nt^{n-1})\\&=nAt^{n-1}\end{aligned}$$

iii.

$$\begin{aligned}\frac{\mathrm{d}e^{At}}{\mathrm{d}t}&=\lim_{h\to 0}\frac{e^{A(t+h)}-e^{At}}{h}\\&=e^{At}\lim_{h\to 0}\frac{e^{Ah}-I}{h}\end{aligned}$$

考慮

$$\begin{aligned}
\frac{e^{Ah}-I}{h} &= \frac{1}{h}(\sum_{n=0}^{\infty}\frac{(Ah)^n}{n!}-I) \\
&= \frac{1}{h}(\sum_{n=1}^{\infty}\frac{(Ah)^n}{n!}) \\
&= \sum_{n=1}^{\infty}\frac{A^n h^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty}\frac{A^{n+1}h^n}{(n+1)!} \\
&= A+\sum_{n=1}^{\infty}\frac{A^{n+1}h^n}{(n+1)!}
\end{aligned}$$

因此

$$\begin{aligned}
\lim_{h\to 0}\frac{e^{Ah}-I}{h} &= A \\
\frac{\mathrm{d}e^{At}}{\mathrm{d}t} &= e^{At}A \\
&= \sum_{n=0}^{\infty}\frac{A^n t^n}{n!}A \\
&= \sum_{n=0}^{\infty}\frac{A^{n+1}t^n}{n!} \\
&= A\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} \\
&= Ae^{At}
\end{aligned}$$

(b) i. 設$F(t) := \begin{bmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}(t) & f_{m2}(t) & \cdots & f_{mn}(t) \end{bmatrix}$，$F'(t) = \begin{bmatrix} f'_{11}(t) & f'_{12}(t) & \cdots & f'_{1n}(t) \\ f'_{21}(t) & f'_{22}(t) & \cdots & f'_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f'_{m1}(t) & f'_{m2}(t) & \cdots & f'_{mn}(t) \end{bmatrix}$,

$G(t) := \begin{bmatrix} g_{11}(t) & g_{12}(t) & \cdots & g_{1\ell}(t) \\ g_{21}(t) & g_{22}(t) & \cdots & g_{2\ell}(t) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(t) & g_{n2}(t) & \cdots & g_{n\ell}(t) \end{bmatrix}$，$G'(t) = \begin{bmatrix} g'_{11}(t) & g'_{12}(t) & \cdots & g'_{1\ell}(t) \\ g'_{21}(t) & g'_{22}(t) & \cdots & g'_{2\ell}(t) \\ \vdots & \vdots & \ddots & \vdots \\ g'_{n1}(t) & g'_{n2}(t) & \cdots & g'_{n\ell}(t) \end{bmatrix}$

則

$$\begin{aligned}(FG)_{ij} &= \sum_{k=1}^{n} f_{ik}(t)g_{kj}(t) \\ (FG)'_{ij} &= \sum_{k=1}^{n}[f'_{ik}(t)g_{kj}(t) + f_{ik}(t)g'_{kj}(t)] \\ &= \sum_{k=1}^{n}[f'_{ik}(t)g_{kj}(t)] + \sum_{k=1}^{n}[f_{ik}(t)g'_{kj}(t)] \\ &= (F'G + FG')_{ij}\end{aligned}$$

ii. 不適用。

(c) At^{A-I}

4. (a) 出錯。

(b) 設$X = e^A = P^{-1}e^K P$，則$A = P^{-1}KP = P^{-1}\log DP$。

(c) 設$A(t) = \log Xt$，則$e^{A(t)}A'(t) = X \implies A'(t) = (Xt)^{-1}X = t^{-1}I$