1 The normal tuition of the Euler number e and natural logarithmic function $\ln x$

Following our usual tuition system, we introduce the number e as the definition

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty}\frac{1}{n!}$$

and define the functions

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\ln x = u \iff x = e^u$$

which is, as 'defined' by some natural force or by some certainty. It is also natural to write

$$e^{ax} = \lim_{n\to\infty}(1+\frac{ax}{n})^n = \lim_{n\to\infty}(1+\frac{x}{n})^{an}$$

which is just like some transposition of variables. But one must ask: Why is this number defined in this fashion? Does the limit really exist? Is it unique? And the most important thing, why is the log function called 'natural'?

To answer these questions, we must use some higher level techniques, and I would like to inherit the content of the book 'Elementary Mathematics from a Higher Standpoint' written by F. Klein, so that the insight become more fruitful.

2 Existence and uniqueness of e

We first tackle the problem of its existence and uniqueness, so that we have interest in discussing the meaning of the number e. This time we could follow the definition on limit.

Theorem. *The infinite sum* $\sum_{n=0}^{\infty}\frac{1}{n!}$ *converges.*

Proof. By ratio test, we have

$$\frac{1}{n+1}\cdot\frac{n}{1} =$$

□

Theorem. *The limit* $\lim_{n\to\infty}(1+\frac{1}{n})^n$ *exists, and is unique.*

Proof. To prove the limit exists, we consider its boundedness and monotonicity. □

3 The Origination: Natural growth rate yields the natural estimation

4 The art of abstraction: Summation form