Abstract

Physics, as a newly introduced subject apart from junior secondary, its concepts are relatively complicated when there is a lost of mathematical tools and insufficient explanations. In this set of notes, we will go through the concepts in Highschool physics with more mathematical tools.

Contents

1 Preliminaries

Differentiation

The first tool for us to use is a powerful mathematical tool, namely **Differentiation**. The principle concept of differentiation is the approximation of the slope at a point of a curve. We then call such slope, if dependent of time, the **rate of change** of some function $f(t)$. It will be useful in our discussion.

Definition 1.1 (Limit)**.** *By a limit of a function $f(t)$ as t approaches some fixed value a, we denote and define the writing by*

$$\lim_{t\to a} f(t) := \begin{cases} f(a) & \text{if } f(a) \text{ is well-defined at } t=a \\ g(a) & \text{if } f(x)=g(x) \text{ near } t=a \text{ but } f(a) \text{ is undefined} \end{cases}$$

Such concept on limit is a basis for differentiation. Now, we recall that for any straight line L if $A(x_a, y_a)$ and $B(x_2, y_2)$ are points on L then the slope of L can be computed as

$$m_L = m_{AB} := \frac{y_2 - y_1}{x_2 - x_1}.$$

It is reasonable to assume that every function depends on t is continuously differentiable, so we are able to discuss differentiation on such functions.

Definition 1.2 (Derivative of a function)**.** *Let $f(t)$ be a function of t and be continuously differentiable. The **derivative of** f **at time** t, denoted by either $\frac{\mathrm{d}f}{\mathrm{d}t}$, $f'(t)$ or $\dot{f}$, is defined by the **first principle***

$$\frac{\mathrm{d}f}{\mathrm{d}t} := \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}.$$

Proposition. *The following are fundamental results from the first principle of differentiation:*

1. $\frac{\mathrm{d}}{\mathrm{d}t}(C) = 0$.
2. $\frac{\mathrm{d}}{\mathrm{d}t}(t^\alpha) = \alpha t^{\alpha-1}$ *if* $\alpha \neq 0$.
3. $\frac{\mathrm{d}}{\mathrm{d}t}(\ln t) = \dfrac{1}{t}$ *for* $t > 0$.
4. $\frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha t}) = \alpha e^{\alpha t}$ *for* $\alpha \in \mathbb{R}$.
5. $\frac{\mathrm{d}}{\mathrm{d}t}(\sin \alpha t) = \alpha \cos \alpha t$ *if* $\alpha \neq 0$.
6. $\frac{\mathrm{d}}{\mathrm{d}t}(\cos \alpha t) = -\alpha \sin \alpha t$ *if* $\alpha \neq 0$.

7. $\frac{\mathrm{d}}{\mathrm{d}t}(\tan \alpha t) = \alpha \sec^2 \alpha t$ *if* $\alpha \neq 0$.

Some important rules are needed to fasten the implementation of differentiation.

Theorem. *The following rules are sufficient to handle all types of one-variable differentiation. Let f, g be continuous functions of t,*

1. *Addition rule:* $\frac{\mathrm{d}}{\mathrm{d}t}(f \pm g) = \dot{f} \pm \dot{g}$.
2. *Product rule:* $\frac{\mathrm{d}}{\mathrm{d}t}(fg) = \dot{f}g + f\dot{g}$.
3. *Quotient rule:* $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f}{g}\right) = \frac{\dot{f}g - f\dot{g}}{g^2}$ *if* $g \neq 0$.
4. *Chain rule:* $\frac{\mathrm{d}}{\mathrm{d}t}(f(g(t))) = \dot{f}(g(t)) \cdot \dot{g}(t)$.

We also subscribe the partial derivative to enclose this session.

Definition 1.3 (Partial derivative)**.** *If f is a two-variable function such that $f : (x, t) \mapsto f(x, t)$, then the* ***partial derivatives*** *writes*

- $\frac{\partial f}{\partial x}$ *to be differentiating only with respect to x;*
- $\frac{\partial f}{\partial t}$ *to be differentiating only with respect to t.*

In particular, we mat assume x and t are variables independent to each other, and immediately define $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial t}{\partial x} = 0$. All operations on partial derivatives are nearly the same as the derivative introduced before.

Integration

The purpose of integration deals with infinite sums, and we shall quickly introduce Riemann Sum to get closed to its concept.

Definition 1.4 (Sigma notation / Summation)**.** *The* ***Sigma notation*** Σ *refers to summing over a certain quantity. Let $a_1, a_2, \ldots, a_n$ to be n values, either equal or not, and we define the sum over all these values to be*

$$\sum_{k=1}^{n} a_k$$

By a graph, we can visualize the idea of integration and write down a clear definition.

For intuition, we should understand an integration as calculating the area under the curve generated by $y = f(t)$ in a certain interval $[a, b]$, as shown in the graph. To approximate such area, we may use rectangles or trapeziums. Such origination is proposed by Riemann, a great mathematician, and publicized by Darboux, another great mathematician.

Definition 1.5 (Riemann Sum / Definite Integration). *Define the **Riemann Sum** of n-th partition by*

$$S_n(f) := \sum_{k=0}^{n-1} \frac{f(t_k) + f(t_{k+1})}{2}(t_{k+1} - t_k)$$

*The **Riemann Integral** is an infinite partitioning approximation progress*

$$\int_a^b f(t)dt := \lim_{n\to\infty} S_n(f)$$

of the area under the curve $f(t)$ on the interval $[a, b]$.

It is an appreciated result that Darboux and Steiljes fixed Riemann's thought a lot and we honour them by calling the integration result a **Riemann-Darboux Integration** or **Riemann-Steiljes Integral**. Of course, we don't care who made it.

On behalf of Riemann Integral, which we named it a definite integral by its functionality, the **Indefinite Integral** can be defined as

$$I(f) := \int_k^t f(t)dt$$

by abusing the variable t so as to demonstrate a general result. The value at k is then a constant so that

$$I(f) = \int f(t)dt + C$$

is an acceptable result.

As f can be understood as some other function's derivatives, i.e. We are summing up rate of changes, the result of Indefinite integral can be thought of as an **anti-derivative**.

Proposition. *The following are fundamental results anti-derivatives:*

1. $\int 0dx = C$.
2. $\int t^\alpha dt = \frac{t^{\alpha+1}}{\alpha+1} + C$ *if* $\alpha \neq -1$.
3. $\int \frac{1}{t}dt = \ln t + C$ *for* $t > 0$.
4. $\int e^{\alpha t}dt = \frac{1}{\alpha}e^{\alpha t} + C$ *for* $\alpha \in \mathbb{R}$.
5. $\int \cos \alpha t dt = \frac{1}{\alpha}\sin \alpha t + C$.
6. $\int \sin \alpha t dt = -\frac{1}{\alpha}\cos \alpha t + C$.
7. $\int \sec^2 \alpha t dt = \frac{1}{\alpha}\tan \alpha t + C$.

And some advanced results with further insight:

- $\int \tan t dt = \ln|\sec t| + C$.
- $\int \sec t dt = \ln|\sec t + \tan t| + C$.

Theorem. *Let f, g be continuous functions of t. Then*

- $\int f(t) \pm g(t) dt = \int f(t) dt \pm \int g(t) dt$.
- $\int f(g(t)) g'(t) dt = \int f(u) du$.
- $\int f(t) g'(t) dt = f(t) g(t) - \int g(t) f'(t) dt$.

And the final relation combining differentiation and integration is the **Fundamental theorem of Calculus** (FTC).

Theorem (Fundamental theorem of Calculus(FTC))**.** *Given a continuous function f of t on $[a, b]$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t) dt\right) = f(x)$$

2 Heat and Gases

3 Newtonian Mechanics

An apple fell and gravity discovered. Newton was not hit by any apple, but his revolutionary ideas.

3.1 Considering quantities in Newtonian mechanics

Scalar quantity

A **scalar quantity** is a quantity representing rate and ratios. It contains no information about directions.

Vector quantity

A **vector quantity** is a quantity representing rate equipped with directions. The equality about vector quantities preserved directional equivalence as long as quantitative equivalence.

3.2 Newton's law of motion

Momentum

Newton concerns about the movement of any objects, placing a high value on the properties of any individuals. Let us consider these kinds of thinking intrinsic, depends on the object itself highly. One individual has a physical quantity called **mass**, which can be thought of as a measure of inertia of one body when there is no external force acting on the body. It immediately takes up a question: What does it mean to have higher inertia? Newton provided a natural thinking on this question, that if one body with higher mass is moving with a certain velocity, the higher tendancy it continues to move in a certain manner. Similarly, for some certain objects with same mass, we may admit that one with higher velocity is much likely to continue its run, or we consider it is harder to be stopped. From this view point, we concluded one measure to combine these assumption, which is called **momentum**, a vector-valued quantity

$$\mathbf{p} = m\mathbf{v}$$

given by the product of its mass m and its velocity vector $\mathbf{v}$. The words originated from Latin word '*pellere*' means 'push', showing its connection with the tendency of continuing its movement.

From this point of view, we may also understand the mass as the scaling factor for a given velocity to calculate momentum. For an object A with mass m_A and an object B with mass m_B, if $m_A > m_B$, then A has a higher tendency than B to continue its movement after any collision.

Newton's first law of motion: Law of inertia

The first law stated as follows:

A body remains at rest or in uniform motion unless acted upon by a force.

For example, a ball staying on a ground without movement continues to stay at rest on the ground unless there are winds or humans picking it up. Of course if it is rolling then it will keep rolling unless there is something changes it.

The statement includes the word *force*, which acts as the changer of the momentum. Therefore, in mathematical representation, it is like

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

so that the amount of force given to the object affects the scale and the direction of change in momentum.

Newton's second law of motion: Equation of motion

The second law stated as follows:

A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

It is the wordy version of the formula

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

and if we see $\mathbf{F}$ a instantaneous measure of force and the mass m of an object is fixed under the inertial frame, we have

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a}$$

The formulation tells us the relation between acceleration and force: if we apply a greater force to an object, it will much likely to change its motion with the direction of given force, which mathes our daily life experience very much. For example, the slow-motion of a boxer puching others' faces deals a great impact on one's face and destroy the facial structure in a short period. May the force be with you!

Newton's third law of motion: Action and reaction pair

The third law stated as follows:

If two bodies exert forces on each other these forces are equal in magnitude and opposite in direction.

That is, if one object acts force on some other object, then a force of same magnitude will be reacted immediately back to the original force actor.

We may formulate the statement in the following form: suppose two objects are colliding with each other, one object is called A and another is called B. Denote the force acting on B from A by $\mathbf{F}_{A\to B}$ and similarly for the force acting on A from B we denote it by $\mathbf{F}_{B\to A}$. Then it is natural to write

$$\mathbf{F}_{A\to B} = \mathbf{F}_{B\to A}$$

Example. *For a person with mass m, by believing the gravitational acceleration is actually g ms^{-2}, we could write the force acting on the floor by the person is*

$$\mathbf{F}_{\downarrow} = mg.$$

By the Newton's third law, the magnitude of the reacting force from the ground to the person will be as same as the person gives, that is

$$\mathbf{F}_{\uparrow} = -\mathbf{F}_{\downarrow} = -mg$$

for if we take downward as positive direction.

4 Electricity and Magnetism

5 Waves

6 Nuclear Power

7 Extended Part I: Astronomy

8 Extended Part II: Atomic World

9 Extended Part III: Domestic Electrical Appliances

10 Extended Part IV: Medical Physics