Abstract

Usually we discuss about trigonometry with geometry sense, however, it would be quite interesting to discuss trigonometry with series and calculus. We shall call it mathematical analysis.

Defining trigonometric functions

Let $\mathbb{C}$ be the set of complex numbers, and $V = (\mathbb{C}^n, +, \cdot)$ be the n-dimensional complex vector space, where $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are component-wise addition and multiplication respectively. Define a metric on V by $d(z, w) := \|z - w\|$ implicitly as a metric induced by norm, so that a metric (open) ball of radius $\varepsilon \geq 0$ centered at z_0, in the universal meaning, is the set

$$B(z_0, \varepsilon) := \{z \in V \mid d(z, z_0) < \varepsilon\}.$$

A metric sphere of radius $\varepsilon \geq 0$ centered at z_0 is the set

$$S(z_0, \varepsilon) := \{z \in V \mid d(z, z_0) = \varepsilon\}.$$

A metric (closed) ball of radius $\varepsilon \geq 0$ centered at z_0 is the set

$$\overline{B}(z_0, \varepsilon) := \{z \in V \mid d(z, z_0) \leq \varepsilon\}.$$

Recall some of the meaning:

Definition (Projection)**.** *A* ***projection*** *is a function $P : \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^{n-m}$, where $m \leq n$, such that $P \circ P = P$. We will denote P^k to be the k-th composition $P^k = P \circ P^{k-1}$. In particular, $P^k = P$ for $k \in \mathbb{N}$.*

Definition (The Pseudo-sine Function)**.** *Define the* ***pseudo-sine function*** *$PS_m : \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^{n-m}$ be a projection from $\mathbb{C}^n$ to $\mathbb{C}^m \times \{0\}^{n-m}$. In particular, for $z = (z_1, z_2, \ldots, z_m, z_{m+1}, \ldots, z_n) \in V$,*

$$PS_m(z) := (z_1, z_2, \ldots, z_m, 0, \ldots, 0).$$

It is not hard to check the pseudo-sine function is indeed a projection on $\mathbb{C}^m$, and we may observe that:

Proposition. *The pseudo-sine function satisfies the following properties:*

- *Given the component-wise $PS_m(z + w) = PS_m(z) + PS_m(w)$.*
- *$PS_m(zw) = PS_m(z)PS_m(w)$.*

Proof.
Both equality follows from the component-wise operation on V. □

The inner product on V can be constructed from the usual sense of complex group. We define $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ by the mapping

$$\langle x, y\rangle = \sum x_i \overline{y}_i$$

where x_i denotes the i-th component of x and $\overline{y}_i$ denotes the complex conjugate of y_i. The inner product is

- Positive definite: $\langle x, x\rangle \geq 0$, equality holds if and only if $x = 0$.
- Conjugate symmetric: $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
- Sesquilinearity: $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$, $\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle$.

The inner porduct induced a norm $\|\cdot\| = \|\cdot\|_V := (\langle p, p\rangle)^{1/2}$ on V satisfies

- Absolute homogeneity: For $\alpha \in \mathbb{C}$, $\|\alpha x\| = |\alpha|\|x\|$.
- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Hence, we define the argument as the following:

Definition (Argument of a point)**.** *Let $p \in V$. Define the argument of p, denoted by* $\arg p$*, in the following manner:*

$$\arg_m p := \frac{\|PS_m(p)\|}{\|p - PS_m(p)\|}.$$

Analogous to the real-valued case, $\arg_m p$ here acts like the tangent function, but not really the same. The argument takes ratio against orthogonal space, still, not a meaningful value to use.

Definition (Exact argument of a point)**.** *Let $p \in V$. The exact argument of a point p is the function*

$$\arg p := (\arg_1 p, \arg_2 p, \ldots, \arg_{n-1} p)$$

which is a mapping $\mathbb{C}^n \to \mathbb{R}^{n-1}$.