Abstract

To understand the concept of calculus well, it is indeed not a compulsory to learn the theory of limit. However, the consolidation and confirmation of the knowledge in both differentiation and integration depends on limit application heavily, so it worths learning the sense behind. We all know that Newton found caculus a.k.a. dynamic flow without limit thoery, but the mystery of calculus makes sense until the explanation by limit.

1 Sense of approximation

What comes first is the sense of approximation. We need to approximate *almost everything* in the real world, such as taking values on a ruler, the water level in a beaker, making a 3-point shot in a basketball game. Not everything can be controlled exactly, even that writing a figure or an alphabet can be so different in every trial. That means we are living in a world of approximation.

Now, imagine if we have two slices of thin bread and a thin slice of cheese, we placed one thin bread on the plate first, then the thin slice of cheese on top of the placed bread, then the other thin bread on top of the placed cheese. For instance, we have no information about the coordinates of the objects. How should we describe the places of the slice of **thin cheese**?

It is natural to answer 'between the slices of bread'. The step to understand limit is to think more about the case when the slices are thin enough to say 'of zero length'. In such cases, are the places of cheese and bread 'the same'?

It turns out that they are nearly *the same* place.

2 Limit definition

Therefore, we ought to write down the meaning of 'close enough' in a logical way. Define a function to represent distance:

Definition 2.1 (Absolute value function)**.** *The function* $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *is defined by*

$$x \mapsto \sqrt{x^2}$$

or in separation form

$$x \mapsto \begin{cases} x & , \textit{if } x \geq 0, \\ -x & , \textit{if } x < 0 \end{cases}$$

Example. *1.* $|1| = |-1| = 1$.

2. $|2| = |-2| = 2$.

3. $|3| = |-3| = 3$.

In order to facilitate the usage of absolute value function, we ought to develop some properties on its computation.

Proposition. *The absolute value function is positive-definite:*

1. *For every* $x \in \mathbb{R}, |x| \geq 0$*;*

2. $|x| = 0$ *if and only if* $x = 0$.

Proof.

Both statement are trivially following from the definition of absolute value function. The checking will be left as an intuition for students. $\square$

Proposition. *Given* $x \in \mathbb{R}$*,*

$$|x| \geq x.$$

Proof.

If $x \geq 0$, then $|x| = x$; if $x < 0$, then $|x| > 0 > x$. $\square$

Proposition. *Given* $x, y \in \mathbb{R}$*,*

$$|xy| = |x||y|.$$

Proof.

It is easy to check all cases for it, i.e. $(x, y) \in \{(+,+), (+,-), (-,+), (-,-)\}$. I will put the analytic way of proof to facilitate understanding.

Notice $|x| = \sqrt{x^2}$, therefore $|xy| = \sqrt{(xy)^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|$. $\square$

Proposition. *Triangle inequality holds for absolute value function: For every real pair* x *and* y*,*

$$|x + y| \leq |x| + |y|.$$

Proof.

Consider the definition of absolute value function that $|x| = \sqrt{x^2}$, we will pay attention to prove

$$|x + y|^2 \leq (|x| + |y|)^2$$

instead of the one stated. The statement in the proposition follows from positivity of absolute value function.

Notice that

$$\begin{aligned}(|x|+|y|)^2-|x+y|^2 &= |x|^2+2|x||y|+|y|^2-x^2-2xy-y^2\\ &= 2(|x||y|-xy)\\ &\geq 0\end{aligned}$$

Therefore, by rearrangement of terms, the proof is done. □

For the distance between any two numbers x and y, where they are not necessary be distinct, can be defined as

$$\operatorname{dist}(x,y)=|x-y|.$$

In addition to the meaning of distance, we can observe the following property:

Proposition. *The distance function is symmetric:*

$$\operatorname{dist}(x,y)=\operatorname{dist}(y,x).$$

Proof.
Due to $|-x|=|-1||x|=|x|$, we have

$$|x-y|=|-(y-x)|=|y-x|.$$

Hence, $\operatorname{dist}(x,y)=\operatorname{dist}(y,x)$. □

From this definition, we can now write the following:

Definition 2.2 (Formal definition of limit of a function)**.** *Suppose $f:\mathbb{R}\to\mathbb{R}$ be a function defined on real number domain, and not necessary be defined on $x=x_0$. Let $\varepsilon>0$ be an arbitrary number. If for such ε there always exists a number $\delta>0$ depends on it, write $\delta:=\delta(\varepsilon)$ as a function depends on ε, such that whenever $0<|x-x_0|<\delta$, there is always a number L such that $|f(x)-L|<\varepsilon$, then we will say L to be the* ***limit of*** *f* ***at*** *x_0, written as*

$$\lim_{x\to x_0} f(x)=L.$$

For a limit, we can consider without knowing the value of the function at the limit point. It is because every discussion of limit is the discussion of approximated value. Let us analyze the writing in the definition.

- A function should be defined to discuss limit of a function, but *not necessary* be defined on the point we want to have limit. The point is that approximation need no information about the actual value, as we did for looking at a ruler, we could never say any words about the exact length. For who couldn't understand the meaning of it, let me ask a question: do you know the exact length of a pencil if it is measured by a centimetre-ruler?

- The number ε is set to be arbitrary to keep the variation of boundary. This variable acts as an upper limit for the distance between the limit L of f at x_0.

- The number $\delta = \delta(\varepsilon)$ defines an upper bound for the distance between x and x_0. The function-like presentation is to clarify that δ is dependent on ε.

The question is whether the number L unique. We will take care of it in short and dive into the application and theorems of limit.

Proof of the uniqueness of limit of a function.

Suppose the limit situation holds for two numbers L and L':

$$\forall \varepsilon > 0, \exists \delta > 0, (0 < |x - x_0| < \delta) \implies (|f(x) - L| < \varepsilon)$$

and

$$\forall \varepsilon' > 0, \exists \delta' > 0, (0 < |x - x_0| < \delta') \implies (|f(x) - L'| < \varepsilon')$$

Pick for an arbitrary $\varepsilon'' > 0$, then we may choose $\delta'' := \min\{\delta, \delta'\}$ such that

$$(0 < |x - x_0| < \delta'') \implies (|f(x) - L| < \varepsilon''), (|f(x) - L'| < \varepsilon'')$$

Then

$$\begin{aligned} |L - L'| &= |L - f(x) + f(x) - L'| \\ &\leq |L - f(x)| + |f(x) - L'| \\ &< 2\varepsilon'' \end{aligned}$$

Since ε is arbitrarily chosen, it turns out only $L = L'$ makes sense in any situation. $\square$

Remark. *Since limit of a function is unique, we will call $\lim_{x\to x_0} f(x)$ to be **the limit of** f **at** x_0.*

3 Evaluation of simple limits

We will see the techniques of evaluating a limit in this section, provided with some useful theorems.

Theorem. *If $f(x)$ is well-defined at x_0, and continuous near x_0, then*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Example. *The following examples shows continuous function's limit.*

1. $\lim_{x\to 0} x = 0$, $\lim_{x\to 1} x = 1$, $\lim_{x\to 2} x = 2$, $\dots$;
2. $\lim_{x\to 0} x^2 = 0$, $\lim_{x\to 1} x^2 = 1$, $\lim_{x\to 2} x^2 = 4$, $\dots$.

In order to manage polynomial functions, we have some rules of arithmetic of limit.

Theorem (Arithmetic on limit)**.** *Let f, g be functions having limit at x_0. Then*

1. $\lim\limits_{x\to x_0}(f \pm g) = \lim\limits_{x\to x_0} f \pm \lim\limits_{x\to x_0} g$;
2. $\lim\limits_{x\to x_0} fg = (\lim\limits_{x\to x_0} f)(\lim\limits_{x\to x_0} g)$;
3. *If* $\lim\limits_{x\to x_0} g(x) \neq 0$, *then* $\lim\limits_{x\to x_0} \dfrac{f}{g} = \dfrac{\lim_{x\to x_0} f}{\lim_{x\to x_0} g}$.

Proof.

The proofs for the three statement will be done by the definition of limit. Let $\varepsilon > 0$ be an arbitrary number, with $\delta := \delta(\varepsilon)$ be chosen according to that in definition, such that $0 < |x - x_0| < \delta$ implies both $|f(x) - F| < \varepsilon$ and $|g(x) - G| < \varepsilon$. It is clear that $\lim\limits_{x\to x_0} f(x) = F$ and $\lim\limits_{x\to x_0} g(x) = G$ in this construction. Each deduction part will be given as follows.

1. We need to show $\big|[f(x) \pm g(x)] - [F \pm G]\big| < \varepsilon$:

$$\begin{aligned}\big|[f(x) \pm g(x)] - [F \pm G]\big| &= \big|[f(x) - F] \pm [g(x) - G]\big| \\ &\leq |f(x) - F| + |g(x) - G| \\ &< 2\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, the result follows.

2. We need to show $|f(x)g(x) - FG| < \varepsilon$:

$$\begin{aligned}|f(x)g(x) - FG| &= |f(x)g(x) - Fg(x) + Fg(x) - FG| \\ &\leq |f(x)g(x) - Fg(x)| + |Fg(x) - FG| \\ &= |g(x)||f(x) - F| + |F||g(x) - G|\end{aligned}$$

The odd part in the deduction is the value of $|g(x)|$, but remember the properties of absolute value function leads us to the conclusion

$$|g(x) - G| < \varepsilon \implies G - \varepsilon < g(x) < G + \varepsilon.$$

Then the finiteness of G and ε tells there exists another finite number G_0 such that

$$|g(x)| < G_0$$

and that

$$\begin{aligned}|f(x)g(x) - FG| &\leq |g(x)||f(x) - F| + |F||g(x) - G| \\ &< (G_0 + |F|)\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, the result follows.

3. Similar to previous proofs, we aim to show that $|\frac{f(x)}{g(x)} - \frac{F}{G}| < \varepsilon$:

$$\begin{aligned}|\frac{f(x)}{g(x)} - \frac{F}{G}| &= |\frac{Gf(x) - Fg(x)}{Gg(x)}| \\ &= |\frac{Gf(x) - FG + FG - Fg(x)}{Gg(x)}| \\ &\leq \frac{1}{|G||g(x)|}(|G||f(x) - F| + |F||G - g(x)|) \\ &< \frac{|G| + |F|}{|G||g(x)|}\varepsilon\end{aligned}$$

For this time, the tricky part is to bound the value of $\frac{1}{|g(x)|}$. In fact, from previous proof we know that

$$G - \varepsilon < g(x) < G + \varepsilon \implies \frac{1}{G + \varepsilon} < \frac{1}{g(x)} < \frac{1}{G - \varepsilon}$$

which, by the arbitrariness of ε, there is always a $g_0 > 0$ such that $\frac{1}{|g(x)|} < g_0$. The result then follows from writing

$$|\frac{f(x)}{g(x)} - \frac{F}{G}| < \frac{|G| + |F|}{|G|}g_0\varepsilon.$$

$\square$

We will examine the rules by checking some basic examples using polynomial functions.

Example. *Check the following limits:*

1. $\lim_{x\to 0}(x^2 + 2x - 1) = 0^2 + 0 - 1 = -1$;
2. $\lim_{x\to 3}[4(x^2 - 2)(x + 3)] = 4(3^2 - 2)(3 + 3) = 168$;
3. $\lim_{x\to -1} \frac{x-1}{x-2} = \frac{-1-1}{-1-2} = \frac{2}{3}$.

Essential Practice 3.1. *Compute the following limits with the help of rules provided:*

1. $\lim_{x\to 1}(3x^2 - 2x + 4)$;
2. $\lim_{x\to 2}[x(x + 1)(x + 2)]$;
3. $\lim_{x\to 0} \frac{x+2}{(x+1)^2}$.

More than direct substitution, one may bump into the situation of $\frac{0}{0}$, which is undefined in usual sense. For if we are doing limits, we know that $x \to x_0$ means x is approaching x_0, but they can never be equal. We hence introduce a concept of identification on limit results.

Proposition. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. If $f(x) = (x - a)g(x)$, and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then*

$$\lim_{x\to a} \frac{f(x)}{x-a} = \lim_{x\to a} g(x).$$

Proof.

By definition of limit, when $0 < |x - a| < \delta$,

$$|\frac{f(x)}{x-a} - L| = |\frac{(x-a)g(x)}{x-a} - L| = |g(x) - L|.$$

Therefore, the result follows. □

Corollary. *Let $f, g, p, q : \mathbb{R} \to \mathbb{R}$ be functions. If $f(x) = (x - a)p(x)$ and $g(x) = (x - a)q(x)$, and all limit exists with $q(x) \neq 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{p(x)}{q(x)}.$$

Remark. *The above proposition can be named as the* ***removable discontinuity****. It's application can be seen in extending functions with finite discontinuities to continuous functions.*

Here are some applications of the properties when evaluating limits.

Example. *Check the following limit:*

1. $\lim_{x\to 0} \frac{x(x+1)}{x} = \lim_{x\to 0}(x+1) = 1$;
2. $\lim_{x\to 1} \frac{x(x-1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x}{x-2} = -1$;
3. $\lim_{x\to -1} \frac{x^2-3x-4}{x^2+3x+2} = \lim_{x\to -1} \frac{(x+1)(x-4)}{(x+1)(x+2)} = \lim_{x\to -1} \frac{x-4}{x+2} = -5$;

Essential Practice 3.2. *Compute the following limit:*

1. $\lim_{x\to -3} \frac{x(x+3)}{x+3}$;
2. $\lim_{x\to 0} \frac{x(x-1)}{x(x-2)}$;
3. $\lim_{x\to 1} \frac{x^2+3x-4}{x^2-2x+1}$.

4 Limits to infinity

When the limit tends to a finite number, it is easy to show the convergence when discontinuities are removed. However, we have less idea on how it works when limit goes nowhere finite.

We observe one divengence to extend the logic of limit. Consider the counting on integers increased from $1, 2, 3, \ldots$ to nowhere terminate. If we let any integer M as a virtual boundary of the counting, one could see $M+1$ can again be an integer such that $M+1 > M$ and M will no longer be an upper bound. This facilitate a sense of 'unlimited' and we use the symbol ∞ to denote the meaning.

Definition 4.1 (Infinity)**.** *Define a symbol ∞ such that for any integer N, N is always less than ∞.*

Proposition. *Any real number $x \in \mathbb{R}$ is less than ∞.*

Proof.
For all $x \in \mathbb{R}$, there is always some integer M such that

$$M \leq x < M+1.$$

Since $M+1 \in \mathbb{N}$, it follows $x < M+1 < \infty$. □

Remark. *In the proof, such M is called the* ***integral part of*** *x, and we can define the* ***floor function*** $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{N}$ *such that*

$$\lfloor x \rfloor = M.$$

In addition, the ***fractional part****, follows the meaning in mixed fraction ($a\frac{b}{c}$), can be defined by* $[\cdot] : \mathbb{R} \to (0,1)$ *as*

$$[x] = x - \lfloor x \rfloor,$$

and the ***ceiling function*** $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{N}$ *to be*

$$\lceil x \rceil = \lfloor x \rfloor + 1.$$

Observe the function $f(x) := \frac{1}{x}$, we may check that

1. $\lim_{x\to 0^+} f(x) = \infty$;
2. $\lim_{x\to 0^-} f(x) = -\infty$;
3. $\lim_{x\to\infty} f(x) = 0$.

The above results provides a foundation on defining a bounded version of real numbers, call it the **extended real number set**.

Definition 4.2 (Extended real number set)**.** *The set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is called an* ***extended real number set****. We usually simply write $\mathbb{R}$ to mean the extended one.*

Such definition transform the usage of ∞ from a simple concept to a 'number', though it is not rigorous to say it is a number, but it founds the properties of $\mathbb{R}$ useful.

Theorem. *For any finite real numbers $a > 0, b < 0$, the following holds:*

1. $a + \infty = b + \infty = \infty + a = \infty + b = \infty$;
2. $a - \infty = b - \infty = -\infty$;
3. $a\infty = \infty$;
4. $b\infty = -\infty$;
5. $a/\infty = b/\infty = 0$.

5 Limits with special functions

We will see many ideas floats from the definition of limit. The popular one will be the exponential function exp.

