1. (a)

$$P(1): \sum_{r=1}^{1} r^3 = \frac{1}{4}(1^2)(1+1)^2$$

$$\begin{aligned} P(n) \implies P(n+1):& \\ \sum_{r=1}^{n+1} r^3 &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ &= \frac{1}{4}(n+1)^2(n^2+4n+4) \\ &= \frac{1}{4}(n+1)^2(n+2)^2 \end{aligned}$$

(b)

$$\begin{aligned} &1^3 - 2^3 + 3^3 - 4^3 + \cdots + (-1)^{r+1}r^3 + \cdots - (2n)^3 \\ &= \sum_{r=1}^{2n} r^3 - 2\sum_{r=1}^{n}(2r)^3 \\ &= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{16}{4}n^2(n+1)^2 \\ &= n^2[(2n+1)^2 - 4(n+1)^2] \\ &= -n^2(4n+3) \end{aligned}$$

2.

$$\begin{aligned} 4n + 16C_2^n &= 180 \\ 16n^2 - 8n - 360 &= 0 \\ n &= 5 \\ C_3^5 &= 10 \end{aligned}$$

3. (a)

$$\begin{aligned} n - 8 &= 1 \\ n &= 9 \end{aligned}$$

(b)

$$16 - 8 \cdot 9 + 36 = -20$$

4. (a)

$$\begin{aligned}\sin^2 x\cos^2 x &= \frac{1}{4}(\sin 2x)^2\\ &= \frac{1}{4}\frac{1-\cos 4x}{2}\\ &= \frac{1-\cos 4x}{8}\end{aligned}$$

(b) i.

$$\begin{aligned}f(x) &= \cos^4 x + \sin^4 x\\ &= 1-2\sin^2 x\cos^2 x\\ &= 1-2\frac{1-\cos 4x}{8}\\ &= \frac{3}{4}+\frac{1}{4}\cos 4x\end{aligned}$$

ii.

$$\begin{aligned}8f(x) &= 7\\ \frac{3}{4}+\frac{1}{4}\cos 4x &= \frac{7}{8}\\ \cos 4x &= \frac{1}{2}\\ 4x &= \pi/3, -\pi/3\\ x &= \pi/12\end{aligned}$$

5. (a)

$$P(1) : \sin\frac{x}{2}\cos x$$

$$P(n) \implies P(n+1) :$$

$$\begin{aligned}\sin\frac{x}{2}\sum_{k=1}^{n+1}\cos kx &= \sin\frac{nx}{2}\cos\frac{(n+1)x}{2}+\sin\frac{x}{2}\cos[(n+1)x]\\ &= \frac{1}{2}[\sin\left(nx+\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)+\sin\left(nx+\frac{3x}{2}\right)-\sin nx+\frac{x}{2}]\\ &= \sin\frac{x}{2}\cos\frac{(n+2)x}{2}\end{aligned}$$

(b)

$$\begin{aligned}\sin\frac{\pi}{14}\sum_{k=1}^{567}\cos\frac{k\pi}{7} &= \sin\frac{567\pi}{14}\cos\frac{568\pi}{14}\\ &= \cos\frac{4\pi}{7}\\ &= -\sin\frac{\pi}{14}\sum_{k=1}^{567}\cos\frac{k\pi}{7}\\ &= -1\end{aligned}$$

6.

$$\begin{aligned}\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+h}} &= \frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}\sqrt{x}}\\ &= \frac{h}{\sqrt{x+h}\sqrt{x}(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}\\ \frac{dy}{dx}\sqrt{\frac{3}{x}} &= \lim_{h\to 0}\frac{\sqrt{3/(x+h)}-\sqrt{3/x}}{h}\\ &= \sqrt{3}\lim_{h\to 0}\frac{1}{h}\frac{h}{(x+h)\sqrt{x}+x\sqrt{x+h}}\\ &= \sqrt{3}\lim_{h\to 0}\frac{1}{(x+h)\sqrt{x}+x\sqrt{x+h}}\\ &= \frac{\sqrt{3}}{2x^{3/2}}\end{aligned}$$

7. (a)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{x^2}\sec^2\frac{1}{x}\\ x^2\frac{dy}{dx} &= -(1+\tan^2\frac{1}{x})\end{aligned}$$

$$x^2\frac{dy}{dx}+(y^2+1)=0$$

(b)

$$\begin{aligned}x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y\frac{dy}{dx} &= 0\\ \frac{d^2y}{dx^2}+\frac{2(x+y)}{x^2}\frac{dy}{dx} &= 0\end{aligned}$$

8. (a)

$$\begin{aligned} y' &= 3x^2 \\ y - b &= 3a^2(x-a) \\ 2 - b &= 3a^2(-a) \\ b &= 3a^3 + 2 \end{aligned}$$

(b)

$$\begin{aligned} a^3 &= 3a^3 + 2 \\ a &= -1 \\ b &= -1 \end{aligned}$$

9. (a)

$$\begin{aligned} y^2 + 3 &= 4 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

So the points are $\{(0,-1),(0,1)\}$.

(b)

$$\begin{aligned} (y^2+3) + 2y\frac{dy}{dx}(x-2) &= 0 \\ \frac{dy}{dx}|_{(0,\pm 1)} &= \pm 1 \end{aligned}$$