1. (a)

$$\begin{aligned}\left(u+\frac{1}{u}\right)^4 &= u^4+4u^3\left(\frac{1}{u}\right)+6u^2\left(\frac{1}{u}\right)^2+4u\left(\frac{1}{u}\right)^3+\left(\frac{1}{u}\right)^4\\ &= u^4+4u^2+6+\frac{4}{u^2}+\frac{1}{u^4}\end{aligned}$$

(b)

$$\begin{aligned}(e^{ax}+e^{-ax})^4 &= e^{4ax}+4e^{2ax}+6+4e^{-2ax}+e^{-4ax}\\ &= \left[1+\frac{4ax}{1!}+\frac{(4ax)^2}{2!}+\cdots\right]+4\left[1+\frac{2ax}{1!}+\frac{(2ax)^2}{2!}+\cdots\right]+6\\ &+4\left[1+\frac{-2ax}{1!}+\frac{(-2ax)^2}{2!}+\cdots\right]+\left[1+\frac{-4ax}{1!}+\frac{(-4ax)^2}{2!}+\cdots\right]\\ &= 1+4ax+8a^2x^2+4+8ax+8a^2x^2+6\\ &+4-8ax+8a^2x^2+1-4ax+8a^2x^2+\cdots\\ &= 16+32a^2x^2+\cdots\end{aligned}$$

(c)

$$\begin{aligned}32a^2 &= 2\\ a^2 &= \frac{1}{16}\\ a &= \pm\frac{1}{4}\end{aligned}$$

2.

$$\begin{aligned}(1+e^{3x})^2 &= 1+2e^{3x}+e^{6x}\\ &= 1+2\left[1+\frac{3ax}{1!}+\frac{(3ax)^2}{2!}+\cdots\right]+\left[1+\frac{6ax}{1!}+\frac{(6ax)^2}{2!}+\cdots\right]\\ &= 4+12x+27x^2+\cdots\end{aligned}$$

3.

$$\begin{aligned}(5-x)^4 &= 5^4-C^4_1(5^3)x+C^4_2(5^2)x^2+C^4_3(5)x^3+x^4\\ &= 625-500x+150x^2-20x^3+x^4\end{aligned}$$

所求係數:

$$\begin{aligned}&= (625)(27)+(-500)(12)+(150)(4)\\ &= 11475\end{aligned}$$

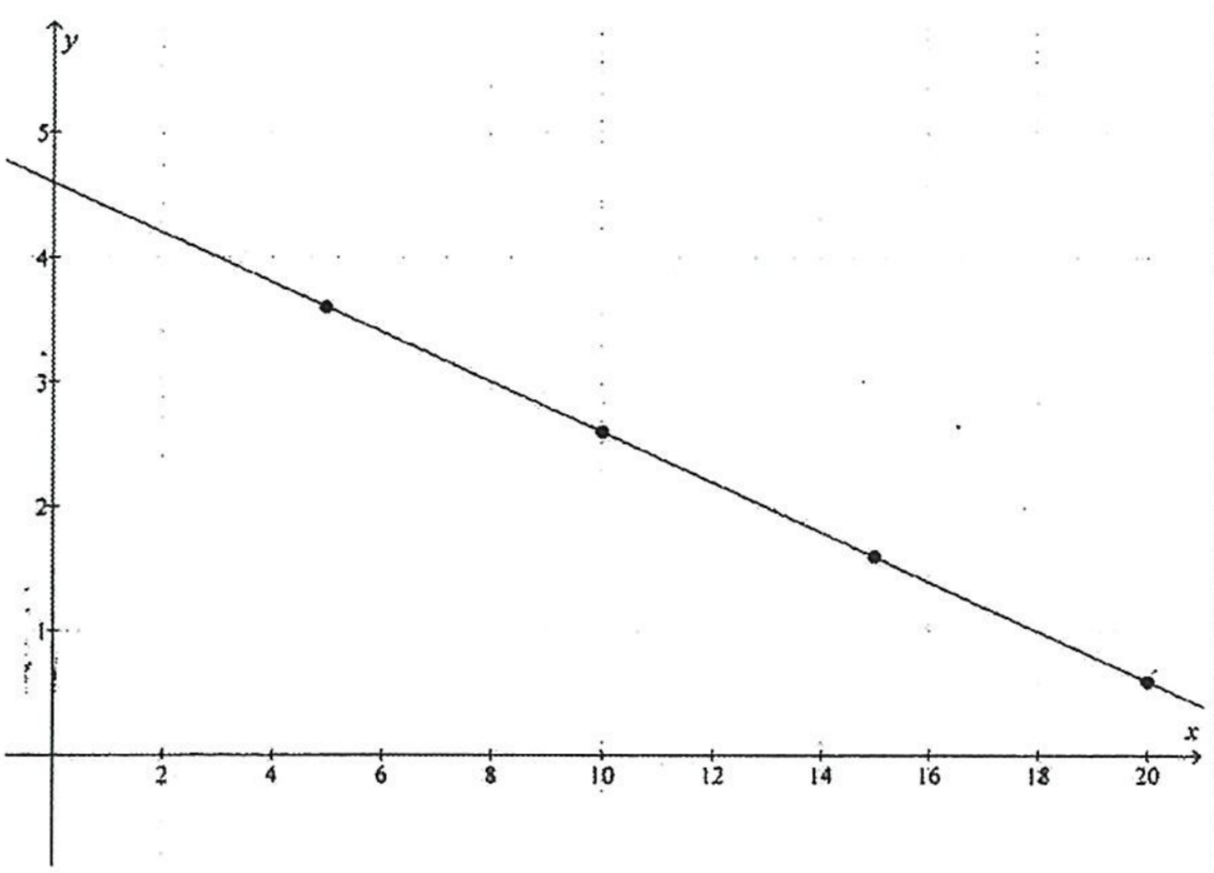
4. (a)

$$\begin{aligned} N(t) &= \frac{500}{1+ae^{-kt}} \\ \ln\left(\frac{500}{N(t)}-1\right) &= \ln\left(ae^{-kt}\right) \\ &= -kt+\ln a \end{aligned}$$

(b) 先求所需數值：

t	5	10	15	20
$\ln\left(\frac{500}{N(t)}-1\right)$	3.6	2.6	1.6	0.6

圖像如下：



從圖像可得

$$\begin{aligned}\ln a &\approx 4.6\\ a &\approx 99.48431564\\ &\approx 99.5\\ -k &= \frac{0.6-3.6}{20-5}\\ k &= 0.2\end{aligned}$$

(c)

$$\begin{aligned}270 &= \frac{500}{1+99.48431564e^{-0.2t}}\\ t &= 23.80171325\end{aligned}$$

因此，疾病爆發後的第24天，受感染的魚的數量會到達270條。

挑戰題. (a)

$$\begin{aligned}\sum_{k=0}^{n} a_k &= \sum_{k=0}^{n} C_k^n p^k (1-p)^{n-k}\\ &= (1-p+p)^n\\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}k \cdot C_k^n &= k \cdot \frac{n!}{k!(n-k)!}\\ &= k \cdot \frac{n(n-1)!}{k(k-1)!(n-k)!}\\ &= n \cdot \frac{(n-1)!}{(k-1)!(n-k)!}\\ &= n \cdot \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!}\\ &= n \cdot C_{k-1}^{n-1}\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} k a_k &= \sum_{k=1}^{n} k \cdot C_k^n p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} n \cdot C_{k-1}^{n-1} p^k (1-p)^{n-k} \\
&= n \sum_{k=0}^{n-1} \cdot C_k^{n-1} p^{k+1} (1-p)^{n-k-1} \\
&= np \sum_{k=0}^{n-1} \cdot C_k^{n-1} p^k (1-p)^{n-k-1} \\
&= np(1) \\
&= np
\end{aligned}$$