Henglein and Steets/Getty Images

One of the crowning achievements of calculus is its ability to capture continuous motion mathematically, allowing that motion to be analyzed instant by instant.

2

THE DERIVATIVE

Many real-world phenomena involve changing quantities—the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal, and so forth. In this chapter we will develop the concept of a "derivative," which is the mathematical tool for studying the rate at which one quantity changes relative to another. The study of rates of change is closely related to the geometric concept of a tangent line to a curve, so we will also be discussing the general definition of a tangent line and methods for finding its slope and equation.

2.1 TANGENT LINES AND RATES OF CHANGE

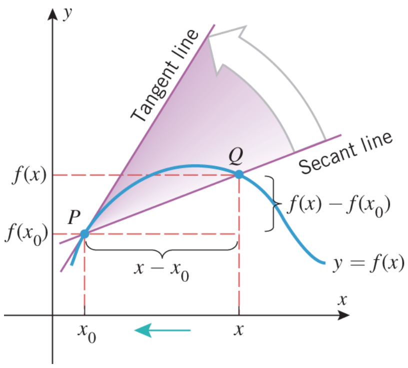
In this section we will discuss three ideas: tangent lines to curves, the velocity of an object moving along a line, and the rate at which one variable changes relative to another. Our goal is to show how these seemingly unrelated ideas are, in actuality, closely linked.

TANGENT LINES

In Example 1 of Section 1.1 we used an informal argument to find the equation of a tangent line to a curve. However, at that stage in the text we did not have a precise definition of a tangent line. Now that limits have been defined precisely we can give a mathematical definition of the tangent line to a curve $y = f(x)$ at a point $P(x_0, f(x_0))$ on the curve. As illustrated in Figure 2.1.1, the slope m_{PQ} of the secant line through P and a second point $Q(x, f(x))$ on the graph of f is

$$m_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}$$

If we let x approach x_0, then the point Q will move along the curve and approach the point P. Suppose the slope m_{PQ} of the secant line through P and Q approaches a limit as $x \to x_0$. In that case we can take the value of the limit to be the slope $m_{\tan}$ of the tangent line at P. Thus, we make the following definition.



▶ **Figure 2.1.1**

2.1.1 DEFINITION Suppose that x_0 is in the domain of the function f. The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.

Example 1 Use Definition 2.1.1 to find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$, and confirm the result agrees with that obtained in Example 1 of Section 1.1.

Solution. Applying Formula (1) with $f(x) = x^2$ and $x_0 = 1$, we have

$$\begin{aligned} m_{\tan} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2 \end{aligned}$$

Thus, the tangent line to $y = x^2$ at $(1, 1)$ has equation

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

which agrees with Example 1 of Section 1.1. ◀

There is an alternative way of expressing Formula (1) that is commonly used. If we let h denote the difference

$$h = x - x_0$$

then the statement that $x \to x_0$ is equivalent to the statement $h \to 0$, so we can rewrite (1) in terms of x_0 and h as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

Formulas (1) and (2) for $m_{\tan}$ usually lead to indeterminate forms of type $0/0$, so you will generally need to perform algebraic simplifications or use other methods to determine limits of such indeterminate forms.

Figure 2.1.2 shows how Formula (2) expresses the slope of the tangent line as a limit of slopes of secant lines.

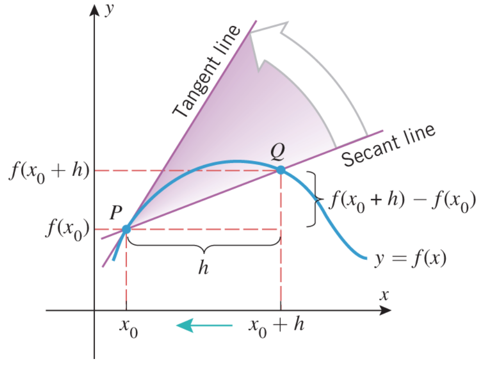
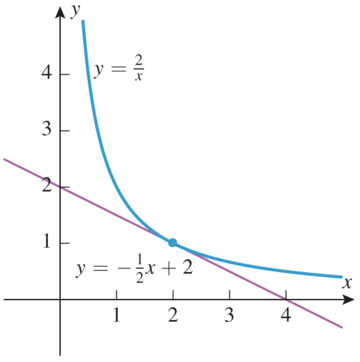


Figure 2.1.2

Example 2 Find an equation for the tangent line to the curve $y = 2/x$ at the point $(2, 1)$ on this curve.

Solution. First, we will find the slope of the tangent line by applying Formula (2) with $f(x) = 2/x$ and $x_0 = 2$. This yields

$$m_{\tan} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{2}{2+h} - 1}{h} = \lim_{h\to 0} \frac{\left(\dfrac{2-(2+h)}{2+h}\right)}{h}$$
$$= \lim_{h\to 0} \frac{-h}{h(2+h)} = -\left(\lim_{h\to 0} \frac{1}{2+h}\right) = -\frac{1}{2}$$



▲ **Figure 2.1.3**

Thus, an equation of the tangent line at (2, 1) is

$$y - 1 = -\tfrac{1}{2}(x - 2) \quad \text{or equivalently} \quad y = -\tfrac{1}{2}x + 2$$

(see Figure 2.1.3). ◀

▶ **Example 3** Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $x_0 = 1$, $x_0 = 4$, and $x_0 = 9$.

Solution. We could compute each of these slopes separately, but it will be more efficient to find the slope for a general value of x_0 and then substitute the specific numerical values. Proceeding in this way we obtain

$$m_{\tan} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0+h} + \sqrt{x_0}}{\sqrt{x_0+h} + \sqrt{x_0}}$$

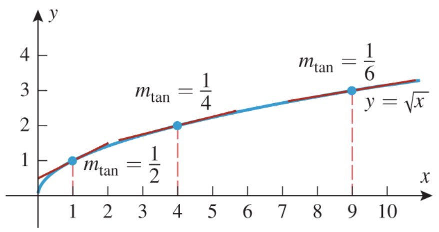
Rationalize the numerator to help eliminate the indeterminate form of the limit.

$$= \lim_{h\to 0} \frac{x_0 + h - x_0}{h(\sqrt{x_0+h} + \sqrt{x_0})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x_0+h} + \sqrt{x_0})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x_0+h} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

The slopes at $x_0 = 1, 4$, and 9 can now be obtained by substituting these values into our general formula for $m_{\tan}$. Thus,

$$\text{slope at } x_0 = 1: \ \frac{1}{2\sqrt{1}} = \frac{1}{2}$$
$$\text{slope at } x_0 = 4: \ \frac{1}{2\sqrt{4}} = \frac{1}{4}$$
$$\text{slope at } x_0 = 9: \ \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

(see Figure 2.1.4). ◀



▶ **Figure 2.1.4**

Carlos E. Santa Maria/iStockphoto

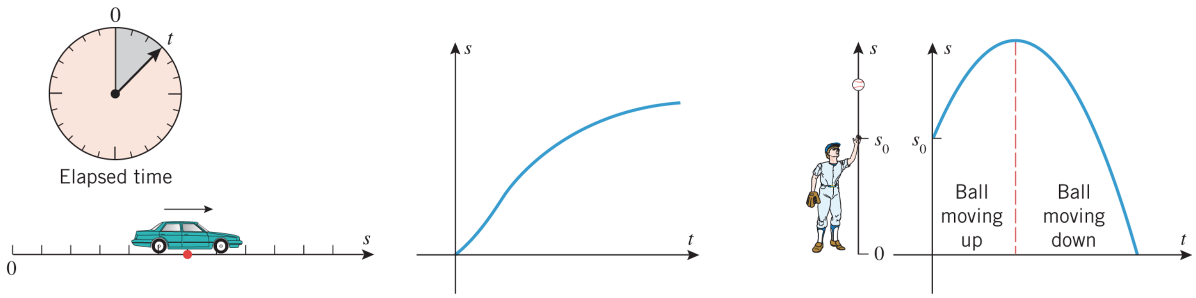
The velocity of an airplane describes its speed and direction.

VELOCITY

One of the important themes in calculus is the study of motion. To describe the motion of an object completely, one must specify its ***speed*** (how fast it is going) and the direction in which it is moving. The speed and the direction of motion together comprise what is called the ***velocity*** of the object. For example, knowing that the speed of an aircraft is 500 mi/h tells us how fast it is going, but not which way it is moving. In contrast, knowing that the velocity of the aircraft is 500 mi/h *due south* pins down the speed and the direction of motion.

Later, we will study the motion of objects that move along curves in two- or three-dimensional space, but for now we will only consider motion along a line; this is called ***rectilinear motion***. Some examples are a piston moving up and down in a cylinder, a race car moving along a straight track, an object dropped from the top of a building and falling straight down, a ball thrown straight up and then falling down along the same line, and so forth.

For computational purposes we will assume that a particle in rectilinear motion moves along a coordinate line that we will call the s-axis. A plot of the s-coordinate of the particle versus the elapsed time t is called the ***position versus time curve*** for the particle. Figure 2.1.5 shows two typical position versus time curves. The first is for a car that moves in the positive direction along the s-axis and the second is for a ball that is thrown straight up in the positive direction of an s-axis from some initial height s_0.



Car moves only in the positive direction. Position versus time curve Position versus time curve

▲ **Figure 2.1.5**

If a particle in rectilinear motion moves along an s-axis so that its position coordinate function of the elapsed time t is

$$s = f(t) \tag{3}$$

then f is called the ***position function of the particle***; the graph of (3) is the position versus time curve. The ***average velocity*** of the particle over a time interval $[t_0, t_0 + h]$, $h > 0$, is defined to be

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h} \tag{4}$$

Show that (4) is also correct for a time interval $[t_0 + h, t_0]$, $h < 0$.

▶ **Example 4** Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.

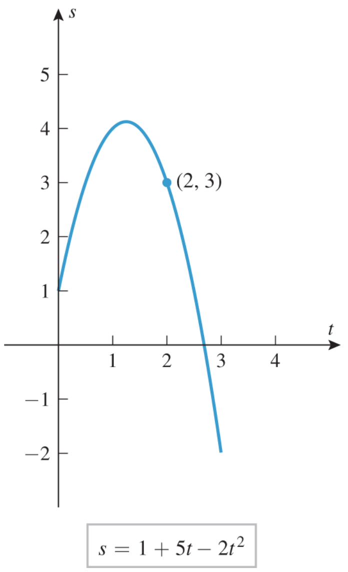
The change in position

$$f(t_0 + h) - f(t_0)$$

is also called the ***displacement*** of the particle over the time interval between t_0 and $t_0 + h$.

Solution (a). Applying (4) with $t_0 = 0$ and $h = 2$, we see that the average velocity is

$$v_{\text{ave}} = \frac{f(t_0 + h) - f(t_0)}{h} = \frac{f(2) - f(0)}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1 \text{ m/s}$$



▲ **Figure 2.1.6**

Solution (b). Applying (4) with $t_0 = 2$ and $h = 1$, we see that the average velocity is

$$v_{\text{ave}} = \frac{f(t_0+h) - f(t_0)}{h} = \frac{f(3) - f(2)}{1} = \frac{-2-3}{1} = \frac{-5}{1} = -5 \text{ m/s} \blacktriangleleft$$

For a particle in rectilinear motion, average velocity describes its behavior over an *interval* of time. We are also interested in the particle's "instantaneous velocity," its speed and direction at a specific *instant* $t = t_0$ in time. Average velocities over small time intervals between $t = t_0$ and $t = t_0 + h$ can be viewed as approximations to this instantaneous velocity. If these average velocities have a limit as h approaches 0, then we can take that limit to be the instantaneous velocity of the particle at time t_0. Here is an example.

▶ **Example 5** Consider the particle in Example 4, whose position function is

$$s = f(t) = 1 + 5t - 2t^2$$

The position of the particle at time $t = 2$ s is $s = 3$ m (Figure 2.1.6). Find the particle's instantaneous velocity at time $t = 2$ s.

Table 2.1.1

TIME INTERVAL	AVERAGE VELOCITY (m/s)
$2.0 \le t \le 3.0$	−5
$2.0 \le t \le 2.1$	−3.2
$2.0 \le t \le 2.01$	−3.02
$2.0 \le t \le 2.001$	−3.002
$2.0 \le t \le 2.0001$	−3.0002

Solution. Table 2.1.1 displays the average velocity of the particle over a succession of smaller and smaller time intervals. The table suggests that the instantaneous velocity at time $t = 2$ s is -3 m/s. To confirm this analytically, we start by computing the object's average velocity over a general time interval between $t = 2$ and $t = 2 + h$ using Formula (4):

$$v_{\text{ave}} = \frac{f(2+h) - f(2)}{h} = \frac{[1 + 5(2+h) - 2(2+h)^2] - 3}{h}$$

The object's instantaneous velocity at time $t = 2$ is calculated as a limit as $h \to 0$:

$$\begin{aligned}
\text{instantaneous velocity} &= \lim_{h\to 0} \frac{[1 + 5(2+h) - 2(2+h)^2] - 3}{h} \\
&= \lim_{h\to 0} \frac{-2 + (10 + 5h) - (8 + 8h + 2h^2)}{h} \\
&= \lim_{h\to 0} \frac{-3h - 2h^2}{h} = \lim_{h\to 0} (-3 - 2h) = -3
\end{aligned}$$

This confirms our numerical conjecture that the instantaneous velocity after 2 s is -3 m/s. ◀

Note the negative values for the velocities in Example 5. This is consistent with the fact that the object is moving in the negative direction along the s-axis.

Consider a particle in rectilinear motion with position function $s = f(t)$. Motivated by Example 5, we define the ***instantaneous velocity*** v_{inst} of the particle at time t_0 to be the limit as $h \to 0$ of the average velocities v_{ave} in Equation (4):

$$v_{\text{inst}} = \lim_{h\to 0} v_{\text{ave}} = \lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{5}$$

Geometrically, the average velocity v_{ave} between $t = t_0$ and $t = t_0 + h$ is the slope of the secant line through points $P(t_0, f(t_0))$ and $Q(t_0 + h, f(t_0 + h))$ on the position versus time curve, and the instantaneous velocity v_{inst} at time t_0 is the slope of the tangent line to the position versus time curve at the point $P(t_0, f(t_0))$ (Figure 2.1.7).

Confirm the solution to Example 4(b) by computing the slope of an appropriate secant line.

SLOPES AND RATES OF CHANGE

Velocity can be viewed as *rate of change*—the rate of change of position with respect to time. Rates of change occur in other applications as well. For example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.

- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.

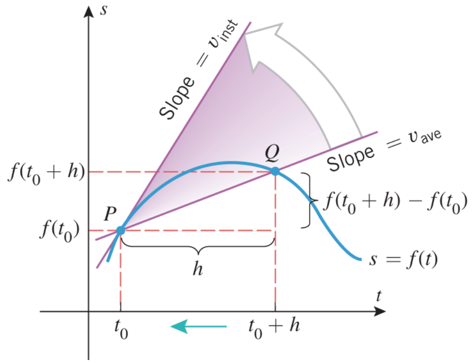
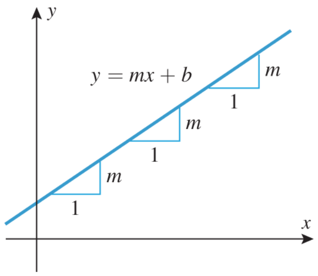
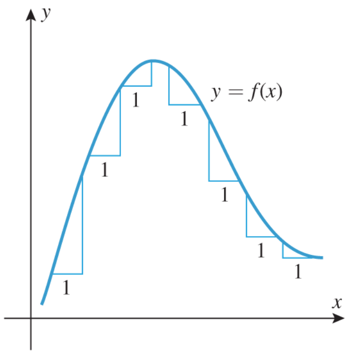


Figure 2.1.7



A 1-unit increase in x always produces an m-unit change in y.

Figure 2.1.8



A 1-unit increase in x produces changes in y that vary in size.

Figure 2.1.9

When y is a linear function of x, say $y = mx + b$, the slope m is the natural measure of the rate of change of y with respect to x. As illustrated in Figure 2.1.8, each 1-unit increase in x anywhere along the line produces an m-unit change in y, so we see that y changes at a constant rate with respect to x along the line and that m measures this rate of change.

Although the rate of change of y with respect to x is constant along a nonvertical line $y = mx + b$, this is not true for a general curve $y = f(x)$ (Figure 2.1.9). As with velocity, we need to distinguish between the average rate of change over an interval and the instantaneous rate of change at a specific point.

If $y = f(x)$, then we define the ***average rate of change of y with respect to x over the interval*** $[x_0, x_1]$ to be

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{6}$$

and we define the ***instantaneous rate of change of y with respect to x at*** x_0 to be

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{7}$$

Geometrically, the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope of the secant line through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ (Figure 2.1.10), and the instantaneous rate of change of y with respect to x at x_0 is the slope of the tangent line at the point $P(x_0, f(x_0))$ (since it is the limit of the slopes of the secant lines through P).

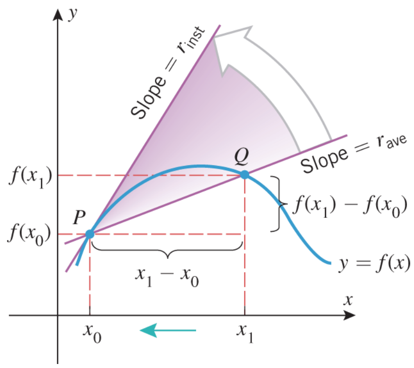


Figure 2.1.10

If desired, we can let $h = x_1 - x_0$, and rewrite (6) and (7) as

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h} \tag{8}$$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{9}$$

▶ **Example 6** Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval $[3, 5]$.

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution (a). We will apply Formula (6) with $f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$. This yields

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, y increases an average of 8 units per unit increase in x over the interval $[3, 5]$.

Solution (b). We will apply Formula (7) with $f(x) = x^2 + 1$ and $x_0 = -4$. This yields

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Thus, a small increase in x from $x = -4$ will produce approximately an 8-fold decrease in y. ◀

Perform the calculations in Example 6 using Formulas (8) and (9).

RATES OF CHANGE IN APPLICATIONS

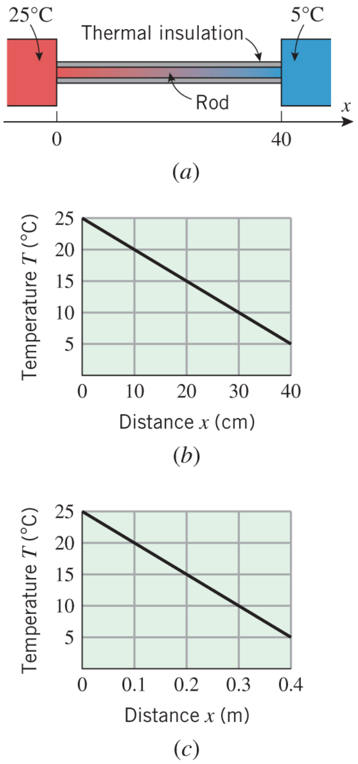
In applied problems, changing the units of measurement can change the slope of a line, so it is essential to include the units when calculating the slope and describing rates of change. The following example illustrates this.

▶ **Example 7** Suppose that a uniform rod of length 40 cm (= 0.4 m) is thermally insulated around the lateral surface and that the exposed ends of the rod are held at constant temperatures of 25°C and 5°C, respectively (Figure 2.1.11*a*). It is shown in physics that under appropriate conditions the graph of the temperature T versus the distance x from the left-hand end of the rod will be a straight line. Parts (*b*) and (*c*) of Figure 2.1.11 show two such graphs: one in which x is measured in centimeters and one in which it is measured in meters. The slopes in the two cases are

$$m = \frac{5 - 25}{40 - 0} = \frac{-20}{40} = -0.5 \tag{10}$$

$$m = \frac{5 - 25}{0.4 - 0} = \frac{-20}{0.4} = -50 \tag{11}$$

The slope in (10) implies that the temperature *decreases* at a rate of 0.5°C per centimeter of distance from the left end of the rod, and the slope in (11) implies that the temperature decreases at a rate of 50°C per meter of distance from the left end of the rod. The two statements are equivalent physically, even though the slopes differ. ◀

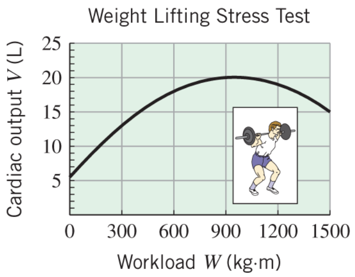


▲ **Figure 2.1.11**

In general, the units for a rate of change of y with respect to x are obtained by "dividing" the units of y by the units of x and then simplifying according to the standard rules of algebra. Here are some examples:

- If y is in degrees Fahrenheit (°F) and x is in inches (in), then a rate of change of y with respect to x has units of degrees Fahrenheit per inch (°F/in).
- If y is in feet per second (ft/s) and x is in seconds (s), then a rate of change of y with respect to x has units of feet per second per second (ft/s/s), which would usually be written as ft/s^2.

- If y is in newton-meters (N·m) and x is in meters (m), then a rate of change of y with respect to x has units of newtons (N), since N·m/m = N.
- If y is in foot-pounds (ft·lb) and x is in hours (h), then a rate of change of y with respect to x has units of foot-pounds per hour (ft·lb/h).



▲ **Figure 2.1.12**

▶ **Example 8** The limiting factor in athletic endurance is cardiac output, that is, the volume of blood that the heart can pump per unit of time during an athletic competition. Figure 2.1.12 shows a stress-test graph of cardiac output V in liters (L) of blood versus workload W in kilogram-meters (kg·m) for 1 minute of weight lifting. This graph illustrates the known medical fact that cardiac output increases with the workload, but after reaching a peak value begins to decrease.

(a) Use the secant line shown in Figure 2.1.13*a* to estimate the average rate of change of cardiac output with respect to workload as the workload increases from 300 to 1200 kg·m.

(b) Use the line segment shown in Figure 2.1.13*b* to estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 300 kg·m.

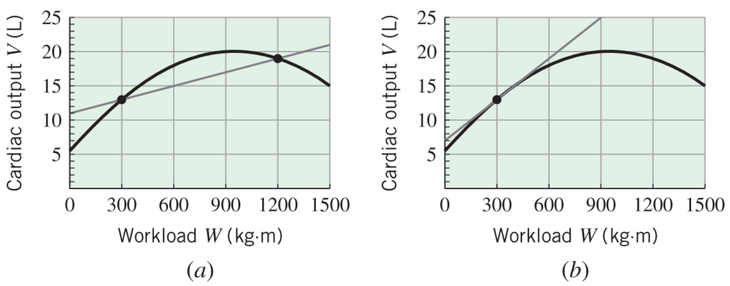
Solution (a). Using the estimated points (300, 13) and (1200, 19) to find the slope of the secant line, we obtain

$$r_{\text{ave}} \approx \frac{19-13}{1200-300} \approx 0.0067 \frac{\text{L}}{\text{kg}\cdot\text{m}}$$

This means that on average a 1-unit increase in workload produced a 0.0067 L increase in cardiac output over the interval.

Solution (b). We estimate the slope of the cardiac output curve at $W = 300$ by sketching a line that appears to meet the curve at $W = 300$ with slope equal to that of the curve (Figure 2.1.13*b*). Estimating points (0, 7) and (900, 25) on this line, we obtain

$$r_{\text{inst}} \approx \frac{25-7}{900-0} = 0.02 \frac{\text{L}}{\text{kg}\cdot\text{m}} \quad ◀$$



▶ **Figure 2.1.13**

✓ QUICK CHECK EXERCISES 2.1 (See page 89 for answers.)

1. The slope $m_{\tan}$ of the tangent line to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is given by

$$m_{\tan} = \lim_{x \to x_0} \underline{\qquad\qquad} = \lim_{h \to 0} \underline{\qquad\qquad}$$

2. The tangent line to the curve $y = (x-1)^2$ at the point $(-1, 4)$ has equation $4x + y = 0$. Thus, the value of the limit

$$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1}$$

is ________.

3. A particle is moving along an s-axis, where s is in feet. During the first 5 seconds of motion, the position of the particle is given by

$$s = 10 - (3-t)^2, \quad 0 \le t \le 5$$

Use this position function to complete each part.

(a) Initially, the particle moves a distance of ________ ft in the (positive/negative) ________ direction; then it reverses direction, traveling a distance of ________ ft during the remainder of the 5-second period.

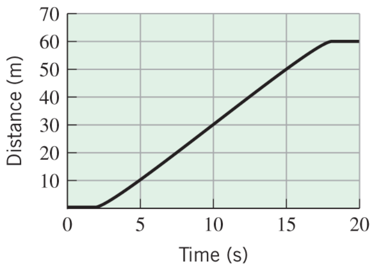
(b) The average velocity of the particle over the 5-second period is ________.

4. Let $s = f(t)$ be the equation of a position versus time curve for a particle in rectilinear motion, where s is in meters and t is in seconds. Assume that $s = -1$ when $t = 2$ and that the instantaneous velocity of the particle at this instant is $3\ \text{m/s}$. The equation of the tangent line to the position versus time curve at time $t = 2$ is ________.

5. Suppose that $y = x^2 + x$.
   (a) The average rate of change of y with respect to x over the interval $2 \le x \le 5$ is ________.
   (b) The instantaneous rate of change of y with respect to x at $x = 2$, r_{inst}, is given by the limit ________.

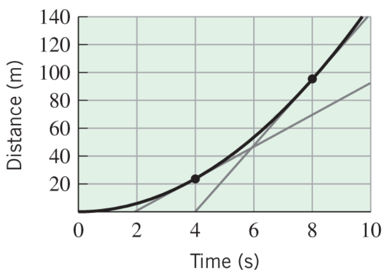
EXERCISE SET 2.1

1. The accompanying figure shows the position versus time curve for an elevator that moves upward a distance of 60 m and then discharges its passengers.
   (a) Estimate the instantaneous velocity of the elevator at $t = 10$ s.
   (b) Sketch a velocity versus time curve for the motion of the elevator for $0 \le t \le 20$.



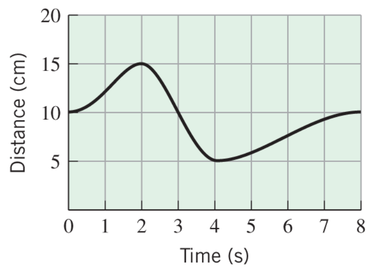
◀ **Figure Ex-1**

2. The accompanying figure shows the position versus time curve for an automobile over a period of time of 10 s. Use the line segments shown in the figure to estimate the instantaneous velocity of the automobile at time $t = 4$ s and again at time $t = 8$ s.



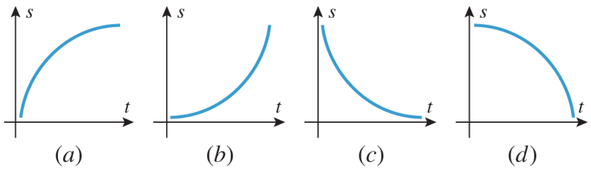
◀ **Figure Ex-2**

3. The accompanying figure shows the position versus time curve for a certain particle moving along a straight line. Estimate each of the following from the graph:
   (a) the average velocity over the interval $0 \le t \le 3$
   (b) the values of t at which the instantaneous velocity is zero
   (c) the values of t at which the instantaneous velocity is either a maximum or a minimum
   (d) the instantaneous velocity when $t = 3$ s.



◀ **Figure Ex-3**

4. The accompanying figure shows the position versus time curves of four different particles moving on a straight line. For each particle, determine whether its instantaneous velocity is increasing or decreasing with time.



▲ **Figure Ex-4**

FOCUS ON CONCEPTS

5. If a particle moves at constant velocity, what can you say about its position versus time curve?

6. An automobile, initially at rest, begins to move along a straight track. The velocity increases steadily until suddenly the driver sees a concrete barrier in the road and applies the brakes sharply at time t_0. The car decelerates rapidly, but it is too late—the car crashes into the barrier at time t_1 and instantaneously comes to rest. Sketch a position versus time curve that might represent the motion of the car. Indicate how characteristics of your curve correspond to the events of this scenario.

7–10 For each exercise, sketch a curve and a line L satisfying the stated conditions. ■

7. L is tangent to the curve and intersects the curve in at least two points.

8. L intersects the curve in exactly one point, but L is not tangent to the curve.

9. L is tangent to the curve at two different points.

10. L is tangent to the curve at two different points and intersects the curve at a third point.

11–14 A function $y = f(x)$ and values of x_0 and x_1 are given.

(a) Find the average rate of change of y with respect to x over the interval $[x_0, x_1]$.

(b) Find the instantaneous rate of change of y with respect to x at the specified value of x_0.

(c) Find the instantaneous rate of change of y with respect to x at an arbitrary value of x_0.

(d) The average rate of change in part (a) is the slope of a certain secant line, and the instantaneous rate of change in part (b) is the slope of a certain tangent line. Sketch the graph of $y = f(x)$ together with those two lines.

11. $y = 2x^2;\ x_0 = 0,\ x_1 = 1$ **12.** $y = x^3;\ x_0 = 1,\ x_1 = 2$

13. $y = 1/x;\ x_0 = 2,\ x_1 = 3$ **14.** $y = 1/x^2;\ x_0 = 1,\ x_1 = 2$

15–18 A function $y = f(x)$ and an x-value x_0 are given.

(a) Find a formula for the slope of the tangent line to the graph of f at a general point $x = x_0$.

(b) Use the formula obtained in part (a) to find the slope of the tangent line for the given value of x_0.

15. $f(x) = x^2 - 1;\ x_0 = -1$

16. $f(x) = x^2 + 3x + 2;\ x_0 = 2$

17. $f(x) = x + \sqrt{x};\ x_0 = 1$

18. $f(x) = 1/\sqrt{x};\ x_0 = 4$

19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. If $\lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = 3$, then $\lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = 3$.

20. A tangent line to a curve $y = f(x)$ is a particular kind of secant line to the curve.

21. The velocity of an object represents a change in the object's position.

22. A 50-foot horizontal metal beam is supported on either end by concrete pillars and a weight is placed on the middle of the beam. If $f(x)$ models how many inches the center of the beam sags when the weight measures x tons, then the units of the rate of change of $y = f(x)$ with respect to x are inches/ton.

23. Suppose that the outside temperature versus time curve over a 24-hour period is as shown in the accompanying figure.

(a) Estimate the maximum temperature and the time at which it occurs.

(b) The temperature rise is fairly linear from 8 A.M. to 2 P.M. Estimate the rate at which the temperature is increasing during this time period.

(c) Estimate the time at which the temperature is decreasing most rapidly. Estimate the instantaneous rate of change of temperature with respect to time at this instant.

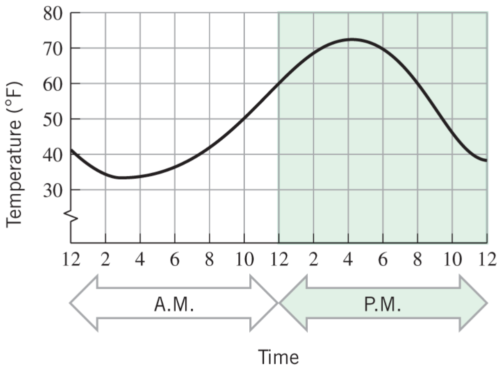


Figure Ex-23

24. The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in liters (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the line segments shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.

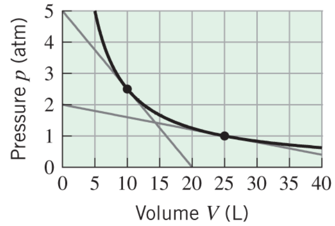


Figure Ex-24

25. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.

(a) When is the growth rate greatest?

(b) Estimate the growth rate at age 5.

(c) At approximately what age between 10 and 20 is the growth rate greatest? Estimate the growth rate at this age.

(d) Draw a rough graph of the growth rate versus age.

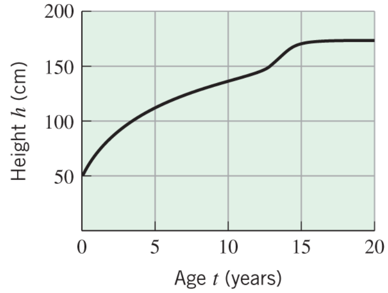


Figure Ex-25

26. An object is released from rest (its initial velocity is zero) from the Empire State Building at a height of 1250 ft above street level (Figure Ex-26 on the next page). The height of the object can be modeled by the position function $s = f(t) = 1250 - 16t^2$.

(a) Verify that the object is still falling at $t = 5$ s.

(b) Find the average velocity of the object over the time interval from $t = 5$ to $t = 6$ s.

(c) Find the object's instantaneous velocity at time $t = 5$ s.

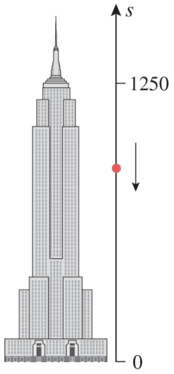


Figure Ex-26

27. During the first 40 s of a rocket flight, the rocket is propelled straight up so that in t seconds it reaches a height of $s = 0.3t^3$ ft.
(a) How high does the rocket travel in 40 s?
(b) What is the average velocity of the rocket during the first 40 s?
(c) What is the average velocity of the rocket during the first 1000 ft of its flight?
(d) What is the instantaneous velocity of the rocket at the end of 40 s?

28. An automobile is driven down a straight highway such that after $0 \leq t \leq 12$ seconds it is $s = 4.5t^2$ feet from its initial position.
(a) Find the average velocity of the car over the interval [0, 12].
(b) Find the instantaneous velocity of the car at $t = 6$.

29. A robot moves in the positive direction along a straight line so that after t minutes its distance is $s = 6t^4$ feet from the origin.
(a) Find the average velocity of the robot over the interval [2, 4].
(b) Find the instantaneous velocity at $t = 2$.

30. **Writing** Discuss how the tangent line to the graph of a function $y = f(x)$ at a point $P(x_0, f(x_0))$ is defined in terms of secant lines to the graph through point P.

31. **Writing** A particle is in rectilinear motion during the time interval $0 \leq t \leq 2$. Explain the connection between the instantaneous velocity of the particle at time $t = 1$ and the average velocities of the particle during portions of the interval $0 \leq t \leq 2$.

✓ QUICK CHECK ANSWERS 2.1 1. $\dfrac{f(x)-f(x_0)}{x-x_0}$; $\dfrac{f(x_0+h)-f(x_0)}{h}$ 2. -4 3. (a) 9; positive; 4 (b) 1 ft/s 4. $s = 3t - 7$
5. (a) 8 (b) $\lim\limits_{x\to 2}\dfrac{(x^2+x)-6}{x-2}$ or $\lim\limits_{h\to 0}\dfrac{[(2+h)^2+(2+h)]-6}{h}$

2.2 THE DERIVATIVE FUNCTION

In this section we will discuss the concept of a "derivative," which is the primary mathematical tool that is used to calculate and study rates of change.

DEFINITION OF THE DERIVATIVE FUNCTION

In the last section we showed that if the limit

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

exists, then it can be interpreted either as the slope of the tangent line to the curve $y = f(x)$ at $x = x_0$ or as the instantaneous rate of change of y with respect to x at $x = x_0$ [see Formulas (2) and (9) of that section]. This limit is so important that it has a special notation:

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \tag{1}$$

You can think of f' (read "f prime") as a function whose input is x_0 and whose output is the number $f'(x_0)$ that represents either the slope of the tangent line to $y = f(x)$ at $x = x_0$ or the instantaneous rate of change of y with respect to x at $x = x_0$. To emphasize this function point of view, we will replace x_0 by x in (1) and make the following definition.

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

The expression

$$\frac{f(x+h) - f(x)}{h}$$

that appears in (2) is commonly called the *difference quotient*.

The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

Example 1 Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.

Solution. It follows from (2) that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x \end{aligned}$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$ (Figure 2.2.1). ◀

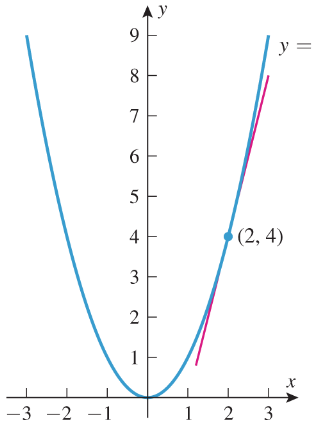


Figure 2.2.1

The function f' is a "slope-producing function" since the value of $f'(x)$ at $x = x_0$ is the slope of the tangent line to the graph of f at $x = x_0$. This aspect of the derivative is illustrated in Figure 2.2.2, which shows the graphs of $f(x) = x^2$ and its derivative $f'(x) = 2x$ (obtained in Example 1). The figure illustrates that the values of $f'(x) = 2x$ at $x = -2, 0,$ and 2 correspond to the slopes of the tangent lines to the graph of $f(x) = x^2$ at those values of x.

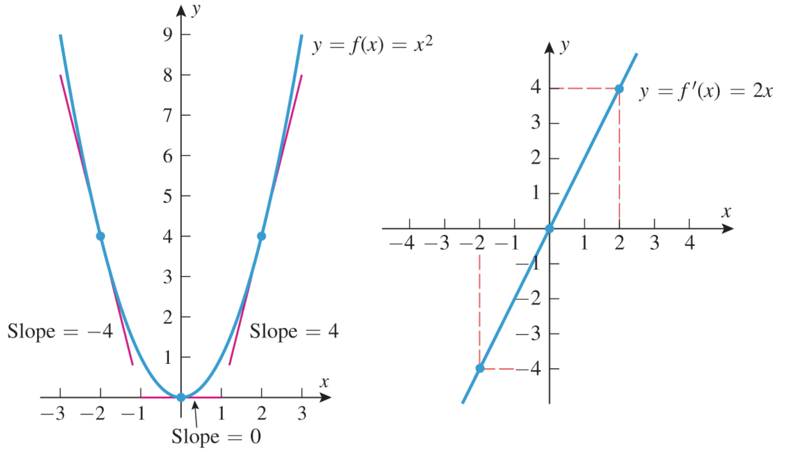


Figure 2.2.2

In general, if $f'(x)$ is defined at $x = x_0$, then the point-slope form of the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ may be found using the following steps.

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{3}$$

▶ Example 2

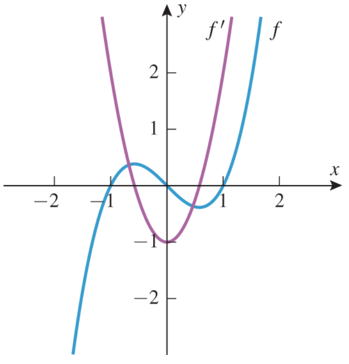
In Solution (a), the binomial formula is used to expand $(x+h)^3$. This formula may be found on the front endpaper.

(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

(b) Graph f and f' together, and discuss the relationship between the two graphs.

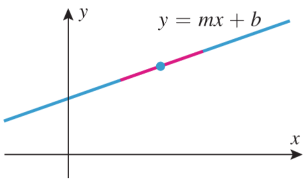
Solution (a).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h\to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1 \end{aligned}$$



▲ **Figure 2.2.3**

Solution (b). Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope, is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal. We leave it for you to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 2.2.3. ◀



At each value of x the tangent line has slope m.

▲ **Figure 2.2.4**

▶ Example 3 At each value of x, the tangent line to a line $y = mx + b$ coincides with the line itself (Figure 2.2.4), and hence all tangent lines have slope m. This suggests geometrically that if $f(x) = mx + b$, then $f'(x) = m$ for all x. This is confirmed by the following computations:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[m(x+h) + b] - [mx + b]}{h} \\ &= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m \end{aligned}$$ ◀

The result in Example 3 is consistent with our earlier observation that the rate of change of y with respect to x along a line $y = mx + b$ is constant and that constant is m.

▶ Example 4

(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$.

(c) Find the limits of $f'(x)$ as $x \to 0^+$ and as $x \to +\infty$, and explain what those limits say about the graph of f.

Solution (a). Recall from Example 3 of Section 2.1 that the slope of the tangent line to $y = \sqrt{x}$ at $x = x_0$ is given by $m_{\tan} = 1/(2\sqrt{x_0})$. Thus, $f'(x) = 1/(2\sqrt{x})$.

Anton, H., Bivens, I. C., & Davis, S. (2016). Calculus early transcendentals, 11th edition. Wiley.
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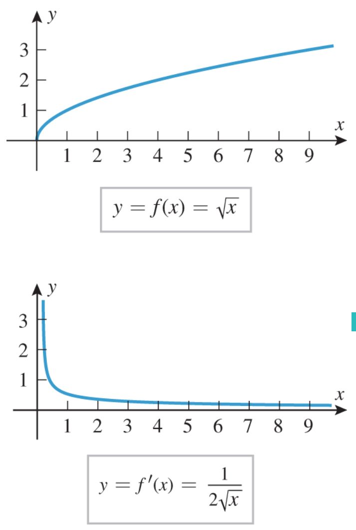
Solution (b). The slope of the tangent line at $x = 9$ is $f'(9)$. From part (a), this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.

Solution (c). The graphs of $f(x) = \sqrt{x}$ and $f'(x) = 1/(2\sqrt{x})$ are shown in Figure 2.2.5. Observe that $f'(x) > 0$ if $x > 0$, which means that all tangent lines to the graph of $y = \sqrt{x}$ have positive slope at all points in this interval. Since

$$\lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{2\sqrt{x}} = 0$$

the graph of f becomes more and more vertical as $x \to 0^+$ and more and more horizontal as $x \to +\infty$. ◀



▲ **Figure 2.2.5**

COMPUTING INSTANTANEOUS VELOCITY

It follows from Formula (5) of Section 2.1 (with t replacing t_0) that if $s = f(t)$ is the position function of a particle in rectilinear motion, then the instantaneous velocity at an arbitrary time t is given by

$$v_{\text{inst}} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Since the right side of this equation is the derivative of the function f (with t rather than x as the independent variable), it follows that if $f(t)$ is the position function of a particle in rectilinear motion, then the function

$$v(t) = f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{4}$$

represents the instantaneous velocity of the particle at time t. Accordingly, we call (4) the ***instantaneous velocity function*** or, more simply, the ***velocity function*** of the particle.

▶ **Example 5** Recall the particle from Example 4 of Section 2.1 with position function $s = f(t) = 1 + 5t - 2t^2$. Here $f(t)$ is measured in meters and t is measured in seconds. Find the velocity function of the particle.

Solution. It follows from (4) that the velocity function is

$$\begin{aligned} v(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{[1 + 5(t+h) - 2(t+h)^2] - [1 + 5t - 2t^2]}{h} \\ &= \lim_{h \to 0} \frac{-2[t^2 + 2th + h^2 - t^2] + 5h}{h} = \lim_{h \to 0} \frac{-4th - 2h^2 + 5h}{h} \\ &= \lim_{h \to 0} (-4t - 2h + 5) = 5 - 4t \end{aligned}$$

where the units of velocity are meters per second. ◀

DIFFERENTIABILITY

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined. To account for this possibility we make the following definition.

2.2.2 DEFINITION A function f is said to be ***differentiable at x_0*** if the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on (a, b)***, and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

Geometrically, a function f is differentiable at x_0 if the graph of f has a tangent line at x_0. Thus, f is not differentiable at any point x_0 where the secant lines from $P(x_0, f(x_0))$ to points $Q(x, f(x))$ distinct from P do not approach a unique *nonvertical* limiting position as $x \to x_0$. Figure 2.2.6 illustrates two common ways in which a function that is continuous at x_0 can fail to be differentiable at x_0. These can be described informally as

- corner points
- points of vertical tangency

At a corner point, the slopes of the secant lines have different limits from the left and from the right, and hence the *two-sided* limit that defines the derivative does not exist (Figure 2.2.7). At a point of vertical tangency the slopes of the secant lines approach $+\infty$ or $-\infty$ from the left and from the right (Figure 2.2.8), so again the limit that defines the derivative does not exist.

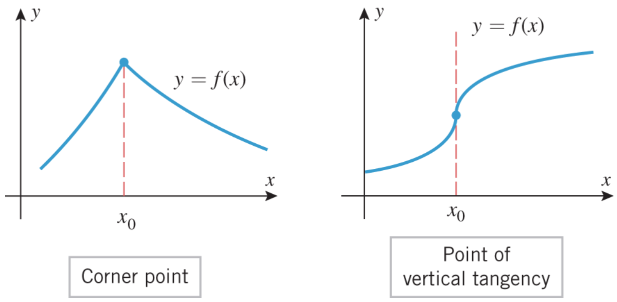


Figure 2.2.6

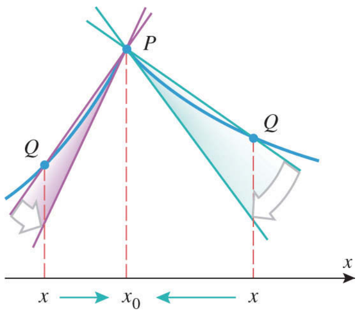


Figure 2.2.7

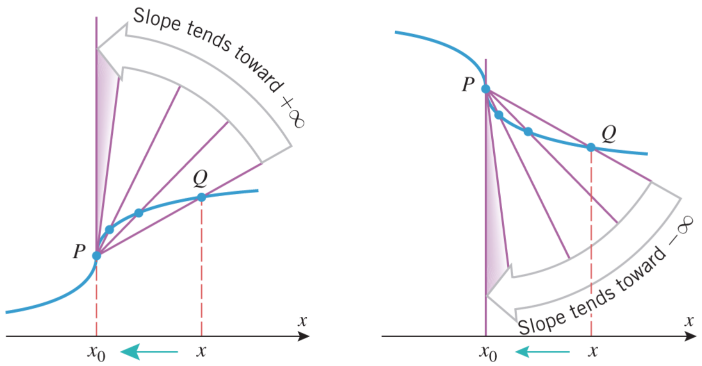


Figure 2.2.8

There are other less obvious circumstances under which a function may fail to be differentiable. (See Exercise 49, for example.)

Differentiability at x_0 can also be described informally in terms of the behavior of the graph of f under increasingly stronger magnification at the point $P(x_0, f(x_0))$ (Figure 2.2.9). If f is differentiable at x_0, then under sufficiently strong magnification at P the graph looks like a nonvertical line (the tangent line); if a corner point occurs at x_0, then no matter how great the magnification at P the corner persists and the graph never looks like a nonvertical line; and if vertical tangency occurs at x_0, then the graph of f looks like a vertical line under sufficiently strong magnification at P.

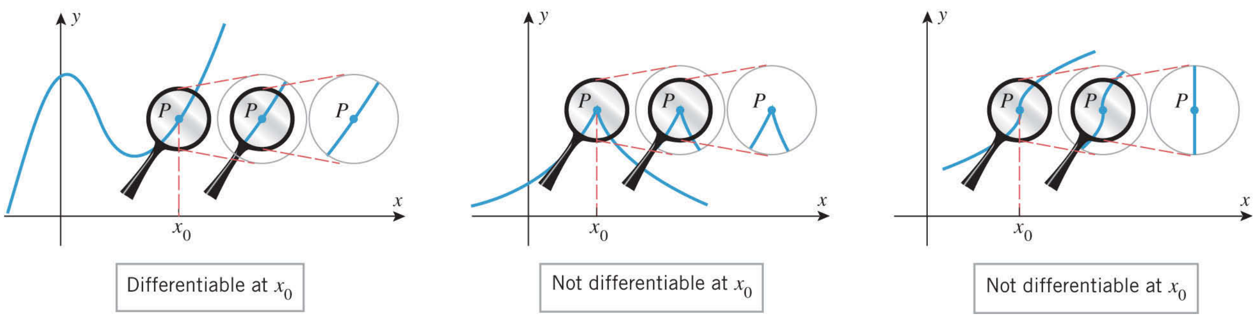
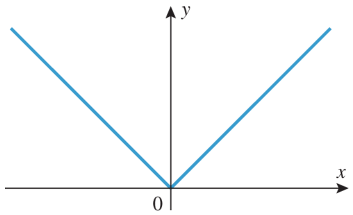


Figure 2.2.9



▲ Figure 2.2.10

Example 6 The graph of $y = |x|$ in Figure 2.2.10 has a corner at $x = 0$, which implies that $f(x) = |x|$ is not differentiable at $x = 0$.

(a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by showing that the limit in Definition 2.2.2 does not exist at $x = 0$.

(b) Find a formula for $f'(x)$.

Solution (a). From Formula (5) with $x_0 = 0$, the value of $f'(0)$, if it were to exist, would be given by

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h} \tag{6}$$

But

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

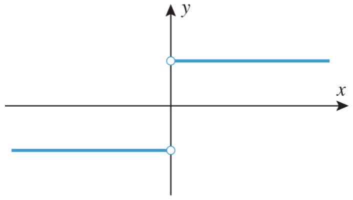
so that

$$\lim_{h\to 0^-} \frac{|h|}{h} = -1 \quad \text{and} \quad \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

Since these one-sided limits are not equal, the two-sided limit in (5) does not exist, and hence f is not differentiable at $x = 0$.

Solution (b). A formula for the derivative of $f(x) = |x|$ can be obtained by writing $|x|$ in piecewise form and treating the cases $x > 0$ and $x < 0$ separately. If $x > 0$, then $f(x) = x$ and $f'(x) = 1$; if $x < 0$, then $f(x) = -x$ and $f'(x) = -1$. Thus,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



▲ Figure 2.2.11

The graph of f' is shown in Figure 2.2.11. Observe that f' is not continuous at $x = 0$, so this example shows that a function that is continuous everywhere may have a derivative that fails to be continuous everywhere. ◀

THE RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

We already know that functions are not differentiable at corner points and points of vertical tangency. The next theorem shows that functions are not differentiable at points of discontinuity. We will do this by proving that if f is differentiable at a point, then it must be continuous at that point.

A theorem that says "If statement A is true, then statement B is true" is equivalent to the theorem that says "If statement B is not true, then statement A is not true." The two theorems are called ***contrapositive forms*** of one another. Thus, Theorem 2.2.3 can be rewritten in contrapositive form as "If a function f is not continuous at x_0, then f is not differentiable at x_0."

2.2.3 THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

PROOF We are given that f is differentiable at x_0, so it follows from (5) that $f'(x_0)$ exists and is given by

$$f'(x_0) = \lim_{h\to 0} \left[\frac{f(x_0+h) - f(x_0)}{h}\right] \tag{7}$$

To show that f is continuous at x_0, we must show that $\lim_{x\to x_0} f(x) = f(x_0)$ or, equivalently,

$$\lim_{x\to x_0} [f(x) - f(x_0)] = 0$$

Expressing this in terms of the variable $h = x - x_0$, we must prove that

$$\lim_{h\to 0} [f(x_0+h) - f(x_0)] = 0$$

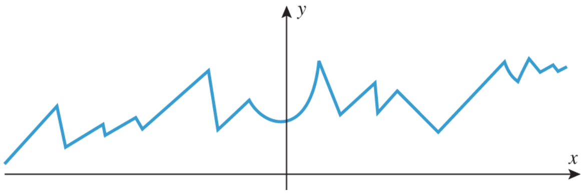
However, this can be proved using (7) as follows:

$$\begin{aligned} \lim_{h\to 0} [f(x_0+h) - f(x_0)] &= \lim_{h\to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \cdot h\right] \\ &= \lim_{h\to 0} \left[\frac{f(x_0+h) - f(x_0)}{h}\right] \cdot \lim_{h\to 0} h \\ &= f'(x_0) \cdot 0 = 0 \quad \blacksquare \end{aligned}$$

WARNING

The converse of Theorem 2.2.3 is false; that is, *a function may be continuous at a point but not differentiable at that point.* This occurs, for example, at corner points of continuous functions. For instance, $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there (Example 6).

The relationship between continuity and differentiability was of great historical significance in the development of calculus. In the early nineteenth century mathematicians believed that if a continuous function had many points of nondifferentiability, these points, like the tips of a sawblade, would have to be separated from one another and joined by smooth curve segments (Figure 2.2.12). This misconception was corrected by a series of discoveries beginning in 1834. In that year a Bohemian priest, philosopher, and mathematician named Bernhard Bolzano discovered a procedure for constructing a continuous function that is not differentiable at any point. Later, in 1860, the great German mathematician Karl Weierstrass (biography on p. 32) produced the first formula for such a function. The graphs of such functions are impossible to draw; it is as if the corners are so numerous that any segment of the curve, when suitably enlarged, reveals more corners. The discovery of these functions was important in that it made mathematicians distrustful of their geometric intuition and more reliant on precise mathematical proof. Recently, such functions have started to play a fundamental role in the study of geometric objects called ***fractals***. Fractals have revealed an order to natural phenomena that were previously dismissed as random and chaotic.

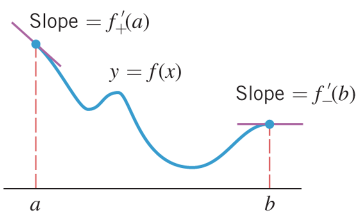


▶ **Figure 2.2.12**

DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ but not outside that interval, then f' is not defined at the endpoints of the interval because derivatives are two-sided limits. To deal with this we define ***left-hand derivatives*** and ***right-hand derivatives*** by

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$



▲ **Figure 2.2.13**

respectively. These are called ***one-sided derivatives***. Geometrically, $f'_-(x)$ is the limit of the slopes of the secant lines as x is approached from the left and $f'_+(x)$ is the limit of the slopes of the secant lines as x is approached from the right. For a closed interval $[a, b]$, we will understand the derivative at the left endpoint to be $f'_+(a)$ and at the right endpoint to be $f'_-(b)$ (Figure 2.2.13).

In general, we will say that f is ***differentiable*** on an interval of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if it is differentiable at all points inside the interval and the appropriate one-sided derivative exists at each included endpoint.

It can be proved that a function f is continuous from the left at those points where the left-hand derivative exists and is continuous from the right at those points where the right-hand derivative exists.

Bernhard Bolzano (1781–1848) Bolzano, the son of an art dealer, was born in Prague, Bohemia (Czech Republic). He was educated at the University of Prague, and eventually won enough mathematical fame to be recommended for a mathematics chair there. However, Bolzano became an ordained Roman Catholic priest, and in 1805 he was appointed to a chair of Philosophy at the University of Prague. Bolzano was a man of great human compassion; he spoke out for educational reform, he voiced the right of individual conscience over government demands, and he lectured on the absurdity of war and militarism. His views so disenchanted Emperor Franz I of Austria that the emperor pressed the Archbishop of Prague to have Bolzano recant his statements. Bolzano refused and was then forced to retire in 1824 on a small pension. Bolzano's main contribution to mathematics was philosophical. His work helped convince mathematicians that sound mathematics must ultimately rest on rigorous proof rather than intuition. In addition to his work in mathematics, Bolzano investigated problems concerning space, force, and wave propagation.

[Image: Juulijs/Fotolia]

OTHER DERIVATIVE NOTATIONS

The process of finding a derivative is called ***differentiation***. You can think of differentiation as an *operation* on functions that associates a function f' with a function f. When the independent variable is x, the differentiation operation is also commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)]$$

In the case where there is a dependent variable $y = f(x)$, the derivative is also commonly denoted by

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}$$

Later, the symbols dy and dx will be given specific meanings. However, for the time being do not regard dy/dx as a ratio, but rather as a single symbol denoting the derivative.

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0}, \quad f'(x_0) = D_x[f(x)]\big|_{x=x_0}, \quad f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}$$

If a variable w changes from some initial value w_0 to some final value w_1, then the final value minus the initial value is called an ***increment*** in w and is denoted by

$$\Delta w = w_1 - w_0 \tag{8}$$

Increments can be positive or negative, depending on whether the final value is larger or smaller than the initial value. The increment symbol in (8) should not be interpreted as a product; rather, Δw should be regarded as a single symbol representing the change in the value of w.

It is common to regard the variable h in the derivative formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

as an increment Δx in x and write (9) as

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{10}$$

Moreover, if $y = f(x)$, then the numerator in (10) can be regarded as the increment

$$\Delta y = f(x+\Delta x) - f(x) \tag{11}$$

in which case

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{12}$$

The geometric interpretations of Δx and Δy are shown in Figure 2.2.14.

Sometimes it is desirable to express derivatives in a form that does not use increments at all. For example, if we let $w = x + h$ in Formula (9), then $w \to x$ as $h \to 0$, so we can rewrite that formula as

$$f'(x) = \lim_{w\to x} \frac{f(w) - f(x)}{w - x} \tag{13}$$

(Compare Figures 2.2.14 and 2.2.15.)

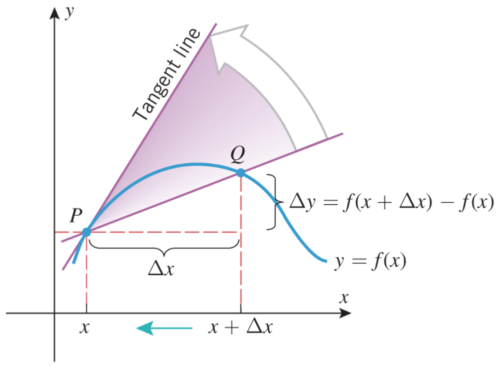
When letters other than x and y are used for the independent and dependent variables, the derivative notations must be adjusted accordingly. Thus, for example, if $s = f(t)$ is the position function for a particle in rectilinear motion, then the velocity function $v(t)$ in (4) can be expressed as

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{14}$$

Newton and Leibniz each used a different notation when they published their discoveries of calculus, thereby creating a notational divide between Britain and the European continent that lasted for more than 50 years. The ***Leibniz notation*** dy/dx eventually prevailed because it reflects the underlying concepts in a natural way, the equation

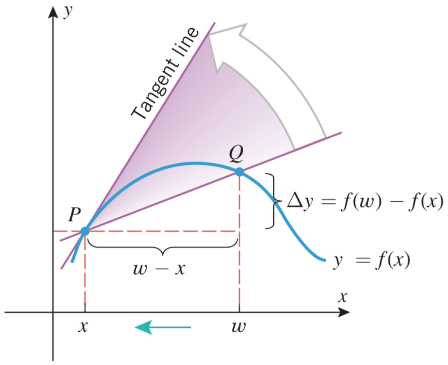
$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$$

being one example.



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

▲ **Figure 2.2.14**



$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

▲ **Figure 2.2.15**

✓ QUICK CHECK EXERCISES 2.2 (See page 100 for answers.)

1. The function $f'(x)$ is defined by the formula
$$f'(x) = \lim_{h \to 0} \underline{\qquad\qquad}$$

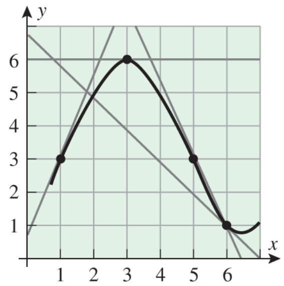
2. (a) The derivative of $f(x) = x^2$ is $f'(x) =$ ______.
   (b) The derivative of $f(x) = \sqrt{x}$ is $f'(x) =$ ______.

3. Suppose that the line $2x + 3y = 5$ is tangent to the graph of $y = f(x)$ at $x = 1$. The value of $f(1)$ is ______ and the value of $f'(1)$ is ______.

4. Which theorem guarantees us that if
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$
exists, then $\lim_{x \to x_0} f(x) = f(x_0)$?

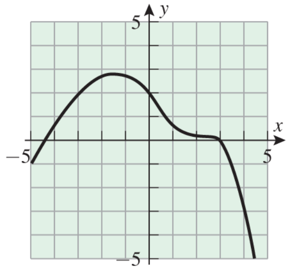
EXERCISE SET 2.2 Graphing Utility

1. Use the graph of $y = f(x)$ in the accompanying figure to estimate the value of $f'(1)$, $f'(3)$, $f'(5)$, and $f'(6)$.



◀ **Figure Ex-1**

2. For the function graphed in the accompanying figure, arrange the numbers 0, $f'(-3)$, $f'(0)$, $f'(2)$, and $f'(4)$ in increasing order.



◀ **Figure Ex-2**

FOCUS ON CONCEPTS

3. (a) If you are given an equation for the tangent line at the point $(a, f(a))$ on a curve $y = f(x)$, how would you go about finding $f'(a)$?
   (b) Given that the tangent line to the graph of $y = f(x)$ at the point $(2, 5)$ has the equation $y = 3x - 1$, find $f'(2)$.
   (c) For the function $y = f(x)$ in part (b), what is the instantaneous rate of change of y with respect to x at $x = 2$?

4. Given that the tangent line to $y = f(x)$ at the point $(1, 2)$ passes through the point $(-1, -1)$, find $f'(1)$.

5. Sketch the graph of a function f for which $f(0) = -1$, $f'(0) = 0$, $f'(x) < 0$ if $x < 0$, and $f'(x) > 0$ if $x > 0$.

6. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 0$, and $f'(x) > 0$ if $x < 0$ or $x > 0$.

7. Given that $f(3) = -1$ and $f'(3) = 5$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = 3$.

8. Given that $f(-2) = 3$ and $f'(-2) = -4$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = -2$.

9–14 Use Definition 2.2.1 to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = a$. ■

9. $f(x) = 2x^2;\ a = 1$ **10.** $f(x) = 1/x^2;\ a = -1$

11. $f(x) = x^3;\ a = 0$ **12.** $f(x) = 2x^3 + 1;\ a = -1$

13. $f(x) = \sqrt{x+1};\ a = 8$ **14.** $f(x) = \sqrt{2x+1};\ a = 4$

15–20 Use Formula (12) to find dy/dx. ■

15. $y = \dfrac{1}{x}$ **16.** $y = \dfrac{1}{x+1}$ **17.** $y = x^2 - x$

18. $y = x^4$ **19.** $y = \dfrac{1}{\sqrt{x}}$ **20.** $y = \dfrac{1}{\sqrt{x-1}}$

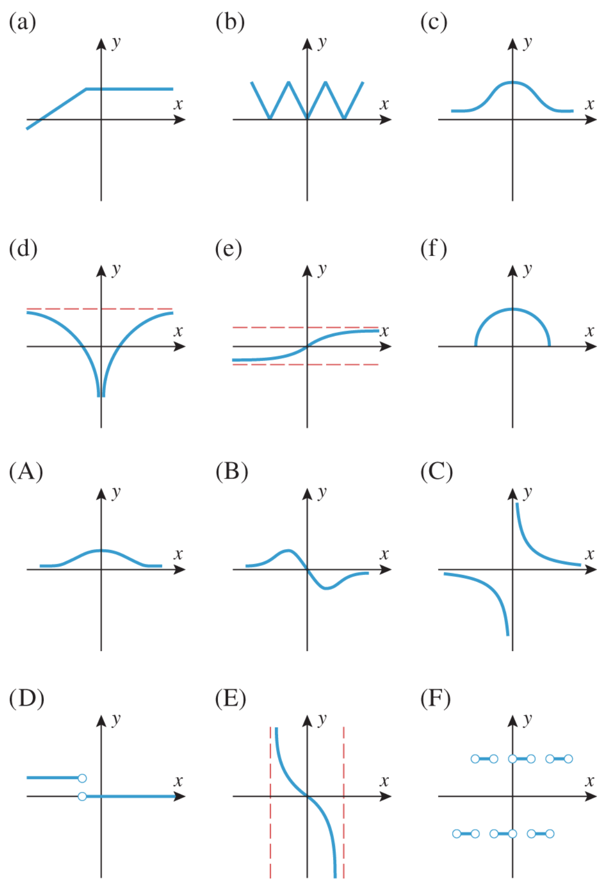
21–22 Use Definition 2.2.1 (with appropriate change in notation) to obtain the derivative requested. ■

21. Find $f'(t)$ if $f(t) = 4t^2 + t$.

22. Find dV/dr if $V = \frac{4}{3}\pi r^3$.

FOCUS ON CONCEPTS

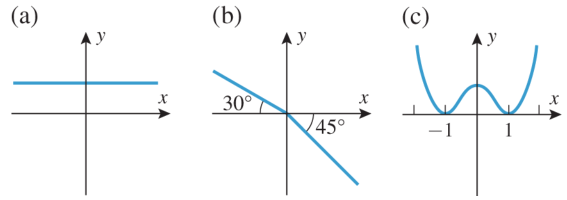
23. Match the graphs of the functions shown in (a)–(f) with the graphs of their derivatives in (A)–(F).



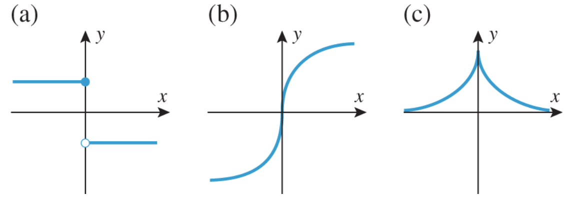
24. Let $f(x) = \sqrt{1 - x^2}$. Use a geometric argument to find $f'(\sqrt{2}/2)$.

25–26 Sketch the graph of the derivative of the function whose graph is shown. ■

25.



26.



27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

27. If a curve $y = f(x)$ has a horizontal tangent line at $x = a$, then $f'(a)$ is not defined.

28. If the tangent line to the graph of $y = f(x)$ at $x = -2$ has negative slope, then $f'(-2) < 0$.

29. If a function f is continuous at $x = 0$, then f is differentiable at $x = 0$.

30. If a function f is differentiable at $x = 0$, then f is continuous at $x = 0$.

31–32 The given limit represents $f'(a)$ for some function f and some number a. Find $f(x)$ and a in each case. ■

31. (a) $\displaystyle\lim_{\Delta x \to 0} \frac{\sqrt{1+\Delta x} - 1}{\Delta x}$ (b) $\displaystyle\lim_{x_1 \to 3} \frac{x_1^2 - 9}{x_1 - 3}$

32. (a) $\displaystyle\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$ (b) $\displaystyle\lim_{x \to 1} \frac{x^7 - 1}{x - 1}$

33. Find $dy/dx|_{x=1}$, given that $y = 1 - x^2$.

34. Find $dy/dx|_{x=-2}$, given that $y = (x+2)/x$.

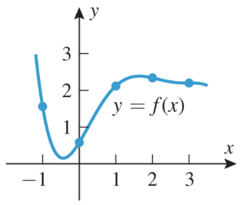
35. Find an equation for the line that is tangent to the curve $y = x^3 - 2x + 1$ at the point $(0, 1)$, and use a graphing utility to graph the curve and its tangent line on the same screen.

36. Use a graphing utility to graph the following on the same screen: the curve $y = x^2/4$, the tangent line to this curve at $x = 1$, and the secant line joining the points $(0, 0)$ and $(2, 1)$ on this curve.

37. Let $f(x) = 2^x$. Estimate $f'(1)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Formula (13) by making a table of values for a succession of values of w approaching 1.

38. Let $f(x) = \sin x$. Estimate $f'(\pi/4)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Formula (13) by making a table of values for a succession of values of w approaching $\pi/4$.

39–40 The function f whose graph is shown below has values as given in the accompanying table.



x	-1	0	1	2	3
$f(x)$	1.56	0.58	2.12	2.34	2.2

39. (a) Use data from the table to calculate the difference quotients

$$\frac{f(3)-f(1)}{3-1}, \quad \frac{f(2)-f(1)}{2-1}, \quad \frac{f(2)-f(0)}{2-0}$$

(b) Using the graph of $y = f(x)$, indicate which difference quotient in part (a) best approximates $f'(1)$ and which difference quotient gives the worst approximation to $f'(1)$.

40. Use data from the table to approximate the derivative values.
(a) $f'(0.5)$ (b) $f'(2.5)$

FOCUS ON CONCEPTS

41. Suppose that the cost of drilling x feet for an oil well is $C = f(x)$ dollars.
(a) What are the units of $f'(x)$?
(b) In practical terms, what does $f'(x)$ mean in this case?
(c) What can you say about the sign of $f'(x)$?
(d) Estimate the cost of drilling an additional foot, starting at a depth of 300 ft, given that $f'(300) = 1000$.

42. A paint manufacturing company estimates that it can sell $g = f(p)$ gallons of paint at a price of p dollars per gallon.
(a) What are the units of dg/dp?
(b) In practical terms, what does dg/dp mean in this case?
(c) What can you say about the sign of dg/dp?
(d) Given that $dg/dp|_{p=10} = -100$, what can you say about the effect of increasing the price from \$10 per gallon to \$11 per gallon?

43. It is a fact that when a flexible rope is wrapped around a rough cylinder, a small force of magnitude F_0 at one end can resist a large force of magnitude F at the other end. The size of F depends on the angle θ through which the rope is wrapped around the cylinder (see the accompanying figure). The figure shows the graph of F (in pounds) versus θ (in radians), where F is the magnitude of the force that can be resisted by a force with magnitude $F_0 = 10$ lb for a certain rope and cylinder.
(a) Estimate the values of F and $dF/d\theta$ when the angle $\theta = 10$ radians.
(b) It can be shown that the force F satisfies the equation $dF/d\theta = \mu F$, where the constant μ is called the ***coefficient of friction***. Use the results in part (a) to estimate the value of μ.

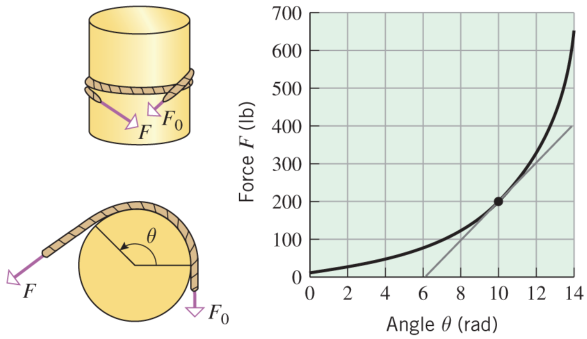


Figure Ex-43

44. The accompanying figure shows the velocity versus time curve for a rocket in outer space where the only significant force on the rocket is from its engines. It can be shown that the mass $M(t)$ (in slugs) of the rocket at time t seconds satisfies the equation

$$M(t) = \frac{T}{dv/dt}$$

where T is the thrust (in lb) of the rocket's engines and v is the velocity (in ft/s) of the rocket. The thrust of the first stage of a *Saturn V* rocket is $T = 7{,}680{,}982$ lb. Use this value of T and the line segment in the figure to estimate the mass of the rocket at time $t = 100$.

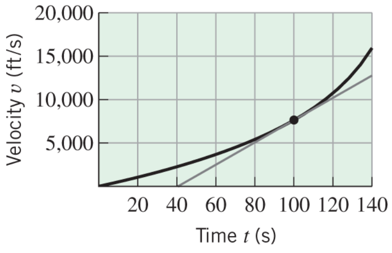


Figure Ex-44

45. According to ***Newton's Law of Cooling***, the rate of change of an object's temperature is proportional to the difference between the temperature of the object and that of the surrounding medium. The accompanying figure on the next page shows the graph of the temperature T (in degrees Fahrenheit) versus time t (in minutes) for a cup of coffee, initially with a temperature of 200°F, that is allowed to cool in a room with a constant temperature of 75°F.
(a) Estimate T and dT/dt when $t = 10$ min.
(b) Newton's Law of Cooling can be expressed as

$$\frac{dT}{dt} = k(T - T_0)$$

where k is the constant of proportionality and T_0 is the temperature (assumed constant) of the surrounding medium. Use the results in part (a) to estimate the value of k.

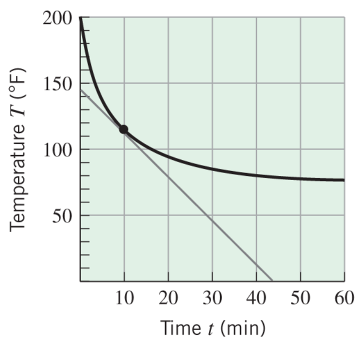


Figure Ex-45

46. Show that $f(x)$ is continuous but not differentiable at the indicated point. Sketch the graph of f.
(a) $f(x) = \sqrt[3]{x},\ x = 0$
(b) $f(x) = \sqrt[3]{(x-2)^2},\ x = 2$

47. Show that

$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Sketch the graph of f.

48. Show that

$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Sketch the graph of f.

49. Show that

$$f(x) = \begin{cases} x \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. Sketch the graph of f near $x = 0$. (See Figure 1.6.6 and the remark following Example 3 in Section 1.6.)

50. Show that

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous and differentiable at $x = 0$. Sketch the graph of f near $x = 0$.

FOCUS ON CONCEPTS

51. Suppose that a function f is differentiable at x_0 and that $f'(x_0) > 0$. Prove that there exists an open interval containing x_0 such that if x_1 and x_2 are any two points in this interval with $x_1 < x_0 < x_2$, then $f(x_1) < f(x_0) < f(x_2)$.

52. Suppose that a function f is differentiable at x_0 and define $g(x) = f(mx + b)$, where m and b are constants. Prove that if x_1 is a point at which $mx_1 + b = x_0$, then $g(x)$ is differentiable at x_1 and $g'(x_1) = mf'(x_0)$.

53. Suppose that a function f is differentiable at $x = 0$ with $f(0) = f'(0) = 0$, and let $y = mx$, $m \ne 0$, denote any line of nonzero slope through the origin.
(a) Prove that there exists an open interval containing 0 such that for all nonzero x in this interval $|f(x)| < \left|\frac{1}{2}mx\right|$. [*Hint:* Let $\epsilon = \frac{1}{2}|m|$ and apply Definition 1.4.1 to (5) with $x_0 = 0$.]
(b) Conclude from part (a) and the triangle inequality that there exists an open interval containing 0 such that $|f(x)| < |f(x) - mx|$ for all x in this interval.
(c) Explain why the result obtained in part (b) may be interpreted to mean that the tangent line to the graph of f at the origin is the best *linear* approximation to f at that point.

54. Suppose that f is differentiable at x_0. Modify the argument of Exercise 53 to prove that the tangent line to the graph of f at the point $P(x_0, f(x_0))$ provides the best linear approximation to f at P. [*Hint:* Suppose that $y = f(x_0) + m(x - x_0)$ is any line through $P(x_0, f(x_0))$ with slope $m \ne f'(x_0)$. Apply Definition 1.4.1 to (5) with $x = x_0 + h$ and $\epsilon = \frac{1}{2}|f'(x_0) - m|$.]

55. **Writing** Write a paragraph that explains what it means for a function to be differentiable. Include examples of functions that are not differentiable as well as examples of functions that are differentiable.

56. **Writing** Explain the relationship between continuity and differentiability.

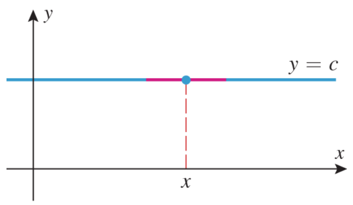
✓ QUICK CHECK ANSWERS 2.2 **1.** $\dfrac{f(x+h) - f(x)}{h}$ **2.** (a) $2x$ (b) $\dfrac{1}{2\sqrt{x}}$ **3.** $1;\ -\frac{2}{3}$
4. Theorem 2.2.3: If f is differentiable at x_0, then f is continuous at x_0.

2.3 INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION

In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

DERIVATIVE OF A CONSTANT

The simplest kind of function is a constant function $f(x) = c$. Since the graph of f is a horizontal line of slope 0, the tangent line to the graph of f has slope 0 for every x;



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Figure 2.3.1

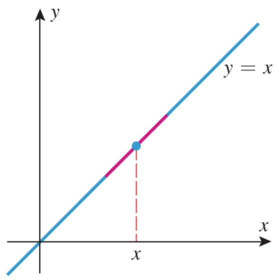
and hence we can see geometrically that $f'(x) = 0$ (Figure 2.3.1). We can also see this algebraically since

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Thus, we have established the following result.

2.3.1 THEOREM *The derivative of a constant function is* 0; *that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0 \tag{1}$$



The tangent line to the graph of $f(x) = x$ has slope 1 for all x.

Figure 2.3.2

▶ Example 1

$$\frac{d}{dx}[1] = 0, \quad \frac{d}{dx}[-3] = 0, \quad \frac{d}{dx}[\pi] = 0, \quad \frac{d}{dx}[-\sqrt{2}] = 0 \blacktriangleleft$$

DERIVATIVES OF POWER FUNCTIONS

The simplest power function is $f(x) = x$. Since the graph of f is a line of slope 1, it follows from Example 3 of Section 2.2 that $f'(x) = 1$ for all x (Figure 2.3.2). In other words,

$$\frac{d}{dx}[x] = 1 \tag{2}$$

Example 1 of Section 2.2 shows that the power function $f(x) = x^2$ has derivative $f'(x) = 2x$. From Example 2 in that section one can infer that the power function $f(x) = x^3$ has derivative $f'(x) = 3x^2$. That is,

$$\frac{d}{dx}[x^2] = 2x \quad \text{and} \quad \frac{d}{dx}[x^3] = 3x^2 \tag{3–4}$$

These results are special cases of the following more general result.

2.3.2 THEOREM (*The Power Rule*) *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{5}$$

Verify that Formulas (2), (3), and (4) are the special cases of (5) in which $n = 1$, 2, and 3.

PROOF Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x+h)^n$, we obtain

$$\begin{aligned}
\frac{d}{dx}[x^n] = f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h \to 0} \frac{nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1} + 0 + \cdots + 0 + 0 \\
&= nx^{n-1} \quad \blacksquare
\end{aligned}$$

The binomial formula can be found on the front endpaper of the text. Replacing y by h in this formula yields the identity used in the proof of Theorem 2.3.2.

▶ Example 2

$$\frac{d}{dx}[x^4] = 4x^3, \quad \frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dt}[t^{12}] = 12t^{11} \blacktriangleleft$$

Although our proof of the power rule in Formula (5) applies only to *positive* integer powers of x, it is not difficult to show that the same formula holds for all integer powers of x (Exercise 84). Also, we saw in Example 4 of Section 2.2 that

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{6}$$

which can be expressed as

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{(1/2)-1}$$

Thus, Formula (5) is valid for $n = \frac{1}{2}$, as well. In fact, it can be shown that this formula holds for any real exponent. We state this more general result for our use now, although we won't be prepared to prove it until Chapter 3.

2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

In words, *to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.*

▶ Example 3

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \blacktriangleleft$$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

Formula (8) can also be expressed in function notation as

$$(cf)' = cf'$$

2.3.4 THEOREM (*Constant Multiple Rule*) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \tag{8}$$

PROOF

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\frac{d}{dx}[f(x)] \quad ■\end{aligned}$$

A constant factor can be moved through a limit sign.

In words, *a constant factor can be moved through a derivative sign.*

Example 4

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi\frac{d}{dx}[x^{-1}] = \pi(-x^{-2}) = -\frac{\pi}{x^2} \quad ◀$$

DERIVATIVES OF SUMS AND DIFFERENCES

2.3.5 THEOREM (*Sum and Difference Rules*) *If f and g are differentiable at x, then so are f + g and f − g and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{9}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

Formulas (9) and (10) can also be expressed as

$$(f+g)' = f' + g'$$
$$(f-g)' = f' - g'$$

PROOF Formula (9) can be proved as follows:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\end{aligned}$$

The limit of a sum is the sum of the limits.

Formula (10) can be proved in a similar manner or, alternatively, by writing $f(x) - g(x)$ as $f(x) + (-1)g(x)$ and then applying Formulas (8) and (9). ■

In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

▶ Example 5

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\frac{d}{dx}\left[\frac{\sqrt{x} - 2x}{\sqrt{x}}\right] = \frac{d}{dx}[1 - 2\sqrt{x}]$$

$$= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}$$

See Formula (6). ◀

Although Formulas (9) and (10) are stated for sums and differences of two functions, they can be extended to any finite number of functions. For example, by grouping and applying Formula (9) twice we obtain

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

As illustrated in the following example, the constant multiple rule together with the extended versions of the sum and difference rules can be used to differentiate any polynomial.

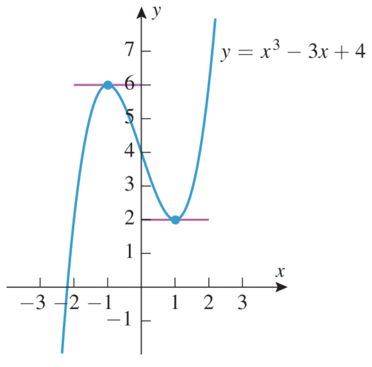
▶ Example 6 Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution.

$$\frac{dy}{dx} = \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1]$$

$$= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]$$

$$= 24x^7 - 10x^4 + 6$$ ◀



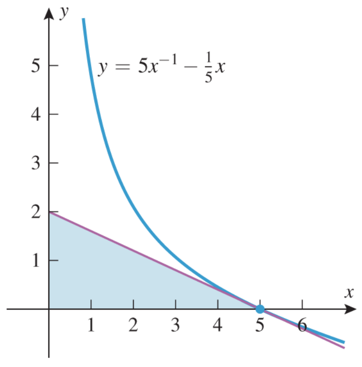
▲ **Figure 2.3.3**

▶ Example 7 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

Solution. Horizontal tangent lines have slope zero, so we must find those values of x for which $y'(x) = 0$. Differentiating yields

$$y'(x) = \frac{d}{dx}[x^3 - 3x + 4] = 3x^2 - 3$$

Thus, horizontal tangent lines occur at those values of x for which $3x^2 - 3 = 0$, that is, if $x = -1$ or $x = 1$. The corresponding points on the curve $y = x^3 - 3x + 4$ are $(-1, 6)$ and $(1, 2)$ (see Figure 2.3.3). ◀



▲ **Figure 2.3.4**

▶ Example 8 Find the area of the triangle formed from the coordinate axes and the tangent line to the curve $y = 5x^{-1} - \frac{1}{5}x$ at the point $(5, 0)$.

Solution. Since the derivative of y with respect to x is

$$y'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right] = \frac{d}{dx}[5x^{-1}] - \frac{d}{dx}\left[\frac{1}{5}x\right] = -5x^{-2} - \frac{1}{5}$$

the slope of the tangent line at the point $(5, 0)$ is $y'(5) = -\frac{2}{5}$. Thus, the equation of the tangent line at this point is

$$y - 0 = -\frac{2}{5}(x - 5) \quad \text{or equivalently} \quad y = -\frac{2}{5}x + 2$$

Since the y-intercept of this line is 2, the right triangle formed from the coordinate axes and the tangent line has legs of length 5 and 2, so its area is $\frac{1}{2}(5)(2) = 5$ (Figure 2.3.4). ◀

HIGHER DERIVATIVES

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f. These successive derivatives are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

If $y = f(x)$, then successive derivatives can also be denoted by

$$y', \quad y'', \quad y''', \quad y^{(4)}, \quad y^{(5)}, \ldots$$

Other common notations are

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{d}{dx}[f(x)] \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)] \\
y''' &= \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)] \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

These are called, in succession, the *first derivative*, the *second derivative*, the *third derivative*, and so forth. The number of times that f is differentiated is called the ***order*** of the derivative. A general nth order derivative can be denoted by

$$\frac{d^ny}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)] \tag{11}$$

and the value of a general nth order derivative at a specific point $x = x_0$ can be denoted by

$$\left.\frac{d^ny}{dx^n}\right|_{x=x_0} = f^{(n)}(x_0) = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=x_0} \tag{12}$$

Example 9 If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\
f''(x) &= 36x^2 - 12x + 2 \\
f'''(x) &= 72x - 12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0 \\
&\vdots \\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$

We will discuss the significance of second derivatives and those of higher order in later sections.

QUICK CHECK EXERCISES 2.3

(See page 108 for answers.)

1. In each part, determine $f'(x)$.
   (a) $f(x) = \sqrt{6}$
   (b) $f(x) = \sqrt{6x}$
   (c) $f(x) = 6\sqrt{x}$
   (d) $f(x) = \sqrt{6x}$
2. In parts (a)–(d), determine $f'(x)$.
   (a) $f(x) = x^3 + 5$
   (b) $f(x) = x^2(x^3 + 5)$
   (c) $f(x) = \dfrac{x^3 + 5}{2}$
   (d) $f(x) = \dfrac{x^3 + 5}{x^2}$
3. The slope of the tangent line to the curve $y = x^2 + 4x + 7$ at $x = 1$ is ________.
4. If $f(x) = 3x^3 - 3x^2 + x + 1$, then $f''(x) =$ ________.

EXERCISE SET 2.3 Graphing Utility

1–8 Find dy/dx.

1. $y = 4x^7$
2. $y = -3x^{12}$
3. $y = 3x^8 + 2x + 1$
4. $y = \frac{1}{2}(x^4 + 7)$
5. $y = \pi^3$
6. $y = \sqrt{2}x + (1/\sqrt{2})$
7. $y = -\frac{1}{3}(x^7 + 2x - 9)$
8. $y = \dfrac{x^2+1}{5}$

9–16 Find $f'(x)$.

9. $f(x) = x^{-3} + \dfrac{1}{x^7}$
10. $f(x) = \sqrt{x} + \dfrac{1}{x}$
11. $f(x) = -3x^{-8} + 2\sqrt{x}$
12. $f(x) = 7x^{-6} - 5\sqrt{x}$
13. $f(x) = x^e + \dfrac{1}{x^{\sqrt{10}}}$
14. $f(x) = \sqrt[3]{\dfrac{8}{x}}$
15. $f(x) = (3x^2 + 1)^2$
16. $f(x) = ax^3 + bx^2 + cx + d \quad (a, b, c, d \text{ constant})$

17–18 Find $y'(1)$.

17. $y = 5x^2 - 3x + 1$
18. $y = \dfrac{x^{3/2}+2}{x}$

19–20 Find dx/dt.

19. $x = t^2 - t$
20. $x = \dfrac{t^2+1}{3t}$

21–24 Find $dy/dx|_{x=1}$.

21. $y = 1 + x + x^2 + x^3 + x^4 + x^5$
22. $y = \dfrac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3}$
23. $y = (1-x)(1+x)(1+x^2)(1+x^4)$
24. $y = x^{24} + 2x^{12} + 3x^8 + 4x^6$

25–26 Approximate $f'(1)$ by considering the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

for values of h near 0, and then find the exact value of $f'(1)$ by differentiating.

25. $f(x) = x^3 - 3x + 1$
26. $f(x) = \dfrac{1}{x^2}$

27–28 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

27. $f(x) = \dfrac{x^2+1}{x}$
28. $f(x) = \dfrac{x + 2x^{3/2}}{\sqrt{x}}$

29–32 Find the indicated derivative.

29. $\dfrac{d}{dt}[16t^2]$
30. $\dfrac{dC}{dr}$, where $C = 2\pi r$
31. $V'(r)$, where $V = \pi r^3$
32. $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If f and g are differentiable at $x = 2$, then
$$\frac{d}{dx}[f(x) - 8g(x)]\bigg|_{x=2} = f'(2) - 8g'(2)$$

34. If $f(x)$ is a cubic polynomial, then $f'(x)$ is a quadratic polynomial.

35. If $f'(2) = 5$, then
$$\frac{d}{dx}[4f(x) + x^3]\bigg|_{x=2} = \frac{d}{dx}[4f(x) + 8]\bigg|_{x=2} = 4f'(2) = 20$$

36. If $f(x) = x^2(x^4 - x)$, then
$$f''(x) = \frac{d}{dx}[x^2] \cdot \frac{d}{dx}[x^4 - x] = 2x(4x^3 - 1)$$

37. A spherical balloon is being inflated.
    (a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.
    (b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.

38. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$ (λ_0 is constant).

39. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

40. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 2$ if $f(2) = -2$ and $f'(2) = -1$.

41–42 Find d^2y/dx^2.

41. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$
    (c) $y = \dfrac{x+1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$
    (c) $y = \dfrac{3x-2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

43–44 Find y'''.

43. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$
    (c) $y = ax^3 + bx + c \quad (a, b, c \text{ constant})$

44. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$
    (c) $y = ax^4 + bx^2 + c \quad (a, b, c \text{ constant})$

45. Find
    (a) $f'''(2)$, where $f(x) = 3x^2 - 2$
    (b) $\dfrac{d^2y}{dx^2}\bigg|_{x=1}$, where $y = 6x^5 - 4x^2$
    (c) $\dfrac{d^4}{dx^4}[x^{-3}]\bigg|_{x=1}$.

46. Find
    (a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$
    (b) $\dfrac{d^4y}{dx^4}\bigg|_{x=1}$, where $y = \dfrac{6}{x^4}$.

47. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

48. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

49–50 Use a graphing utility to make rough estimates of the locations of all horizontal tangent lines, and then find their exact locations by differentiating.

49. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ **50.** $y = \dfrac{x^2+9}{x}$

FOCUS ON CONCEPTS

51. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

52. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

53. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

54. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

55. Find the coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

56. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

57. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

58. Show that the segment cut off by the coordinate axes from any tangent line to the graph of $y = 1/x$ is bisected by the point of tangency.

59. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

60. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

61. Newton's Law of Universal Gravitation states that the magnitude F of the force exerted by a point with mass M on a point with mass m is

$$F = \frac{GmM}{r^2}$$

where G is a constant and r is the distance between the points. Assuming that the points are moving, find a formula for the instantaneous rate of change of F with respect to r.

62. In the temperature range between 0°C and 700°C the resistance R [in ohms (Ω)] of a certain platinum resistance thermometer is given by

$$R = 10 + 0.04124T - 1.779 \times 10^{-5}T^2$$

where T is the temperature in degrees Celsius. Where in the interval from 0°C to 700°C is the resistance of the thermometer most sensitive and least sensitive to temperature changes? [*Hint:* Consider the size of dR/dT in the interval $0 \le T \le 700$.]

63. A stuntman estimates the time T in seconds for him to fall x meters by $T = 0.453\sqrt{x}$. Use this formula to find the instantaneous rate of change of T with respect to x when $x = 9$ meters.

64. The mean orbital radius r (in units of 10^5 km) of a moon of Saturn can be modeled by the equation $r = 1.93t^{2/3}$, where t is the time in (Earth) days for the moon to complete one orbit about the planet. Use this model to estimate the instantaneous rate of change of r with respect to t when $t = 0.602$ day (the orbital period of Saturn's moon Atlas).

65–66 Use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

65. $f(x) = x - \dfrac{1}{x}$ **66.** $f(x) = x^3 - 3x$

67–70 You are asked in these exercises to determine whether a piecewise-defined function f is differentiable at a value $x = x_0$, where f is defined by different formulas on different sides of x_0. You may use without proof the following result, which is a consequence of the Mean-Value Theorem (discussed in Section 4.8). ***Theorem.*** *Let f be continuous at x_0 and suppose that $\lim_{x\to x_0} f'(x)$ exists. Then f is differentiable at x_0, and $f'(x_0) = \lim_{x\to x_0} f'(x)$.*

67. Show that

$$f(x) = \begin{cases} x^2 + x + 1, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f.

68. Let
$$f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \ge 9 \end{cases}$$

Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

69. Let
$$f(x) = \begin{cases} x^2, & x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

70. Let
$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \ge \frac{1}{2} \end{cases}$$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

71. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

72. In each part, compute f', f'', f''', and then state the formula for $f^{(n)}$.
(a) $f(x) = 1/x$ (b) $f(x) = 1/x^2$
[*Hint:* The expression $(-1)^n$ has a value of 1 if n is even and -1 if n is odd. Use this expression in your answer.]

73. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

74. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{w\to 2} \frac{f'(w) - f'(2)}{w-2}$$

75. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$, $n = 1, 2, 3, \ldots$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

76. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

77. Let $f(x) = (mx + b)^n$, where m and b are constants and n is an integer. Use the result of Exercise 52 in Section 2.2 to prove that $f'(x) = nm(mx + b)^{n-1}$.

78–79 Verify the result of Exercise 77 for $f(x)$.

78. $f(x) = (2x + 3)^2$ **79.** $f(x) = (3x - 1)^3$

80–83 Use the result of Exercise 77 to compute the derivative of the given function $f(x)$.

80. $f(x) = \dfrac{1}{x-1}$

81. $f(x) = \dfrac{3}{(2x+1)^2}$

82. $f(x) = \dfrac{x}{x+1}$

83. $f(x) = \dfrac{2x^2+4x+3}{x^2+2x+1}$

84. The purpose of this exercise is to extend the power rule (Theorem 2.3.2) to any integer exponent. Let $f(x) = x^n$, where n is any integer. If $n > 0$, then $f'(x) = nx^{n-1}$ by Theorem 2.3.2.
(a) Show that the conclusion of Theorem 2.3.2 holds in the case $n = 0$.
(b) Suppose that $n < 0$ and set $m = -n$ so that

$$f(x) = x^n = x^{-m} = \frac{1}{x^m}$$

Use Definition 2.2.1 and Theorem 2.3.2 to show that

$$\frac{d}{dx}\left[\frac{1}{x^m}\right] = -mx^{m-1}\cdot\frac{1}{x^{2m}}$$

and conclude that $f'(x) = nx^{n-1}$.

✓ QUICK CHECK ANSWERS 2.3 **1.** (a) 0 (b) $\sqrt{6}$ (c) $3/\sqrt{x}$ (d) $\sqrt{6}/(2\sqrt{x})$ **2.** (a) $3x^2$ (b) $5x^4 + 10x$ (c) $\frac{3}{2}x^2$ (d) $1 - 10x^{-3}$ **3.** 6 **4.** $18x - 6$

2.4 THE PRODUCT AND QUOTIENT RULES

In this section we will develop techniques for differentiating products and quotients of functions whose derivatives are known.

DERIVATIVE OF A PRODUCT

You might be tempted to conjecture that the derivative of a product of two functions is the product of their derivatives. However, a simple example will show this to be false. Consider the functions

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

The product of their derivatives is

$$f'(x)g'(x) = (1)(2x) = 2x$$

but their product is $h(x) = f(x)g(x) = x^3$, so the derivative of the product is

$$h'(x) = 3x^2$$

Thus, the derivative of the product is not equal to the product of the derivatives. The correct relationship, which is credited to Leibniz, is given by the following theorem.

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{1}$$

Formula (1) can also be expressed as

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

PROOF Whereas the proofs of the derivative rules in the last section were straightforward applications of the derivative definition, a key step in this proof involves adding and subtracting the quantity $f(x+h)g(x)$ to the numerator in the derivative definition. This yields

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[f(x+h)\cdot \frac{g(x+h)-g(x)}{h} + g(x)\cdot \frac{f(x+h)-f(x)}{h}\right] \\
&= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \left[\lim_{h\to 0} f(x+h)\right] \frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x)\right] \frac{d}{dx}[f(x)] \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\end{aligned}$$

[*Note:* In the last step $f(x+h)\to f(x)$ as $h\to 0$ because f is continuous at x by Theorem 2.2.3. Also, $g(x)\to g(x)$ as $h\to 0$ because $g(x)$ does not involve h and hence is treated as constant for the limit.] ■

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

▶ Example 1 Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method 1. *(Using the Product Rule)*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[(4x^2-1)(7x^3+x)] \\
&= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] \\
&= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1
\end{aligned}$$

Method 2. *(Multiplying First)*

$$y = (4x^2-1)(7x^3+x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

▶ **Example 2** Find ds/dt if $s = (1+t)\sqrt{t}$.

Solution. Applying the product rule yields

$$\begin{aligned}\frac{ds}{dt} &= \frac{d}{dt}[(1+t)\sqrt{t}] \\ &= (1+t)\frac{d}{dt}[\sqrt{t}] + \sqrt{t}\frac{d}{dt}[1+t] \\ &= \frac{1+t}{2\sqrt{t}} + \sqrt{t} = \frac{1+3t}{2\sqrt{t}}\end{aligned}$$ ◀

DERIVATIVE OF A QUOTIENT

Just as the derivative of a product is not generally the product of the derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship is given by the following theorem.

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2} \tag{2}$$

Formula (2) can also be expressed as

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

PROOF

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Adding and subtracting $f(x)\cdot g(x)$ in the numerator yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)} \\ &= \lim_{h\to 0}\frac{\left[g(x)\cdot\dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot\dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)} \\ &= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)} \\ &= \frac{\left[\lim\limits_{h\to 0} g(x)\right]\cdot\dfrac{d}{dx}[f(x)] - \left[\lim\limits_{h\to 0} f(x)\right]\cdot\dfrac{d}{dx}[g(x)]}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)} \\ &= \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}\end{aligned}$$

[See the note at the end of the proof of Theorem 2.4.1 for an explanation of the last step.] ■

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

Sometimes it is better to simplify a function first than to apply the quotient rule immediately. For example, it is easier to differentiate

$$f(x) = \frac{x^{3/2} + x}{\sqrt{x}}$$

by rewriting it as

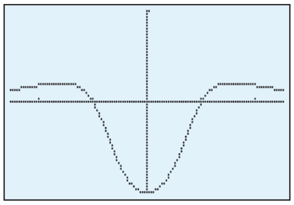
$$f(x) = x + \sqrt{x}$$

as opposed to using the quotient rule.

▶ Example 3 Find $y'(x)$ for $y = \dfrac{x^3 + 2x^2 - 1}{x + 5}$.

Solution. Applying the quotient rule yields

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3 + 2x^2 - 1}{x + 5}\right] &= \frac{(x+5)\dfrac{d}{dx}[x^3 + 2x^2 - 1] - (x^3 + 2x^2 - 1)\dfrac{d}{dx}[x+5]}{(x+5)^2} \\
&= \frac{(x+5)(3x^2 + 4x) - (x^3 + 2x^2 - 1)(1)}{(x+5)^2} \\
&= \frac{(3x^3 + 19x^2 + 20x) - (x^3 + 2x^2 - 1)}{(x+5)^2} \\
&= \frac{2x^3 + 17x^2 + 20x + 1}{(x+5)^2} \quad ◀
\end{aligned}$$



$[-2.5, 2.5] \times [-1, 1]$
xScl $= 1$, yScl $= 1$

$$y = \frac{x^2 - 1}{x^4 + 1}$$

▲ **Figure 2.4.1**

▶ Example 4 Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

(a) Graph $y = f(x)$, and use your graph to make rough estimates of the locations of all horizontal tangent lines.

(b) By differentiating, find the exact locations of the horizontal tangent lines.

Solution (a). In Figure 2.4.1 we have shown the graph of the equation $y = f(x)$ in the window $[-2.5, 2.5] \times [-1, 1]$. This graph suggests that horizontal tangent lines occur at $x = 0, x \approx 1.5$, and $x \approx -1.5$.

Solution (b). To find the exact locations of the horizontal tangent lines, we must find the points where $dy/dx = 0$. We start by finding dy/dx:

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] &= \frac{(x^4+1)\dfrac{d}{dx}[x^2 - 1] - (x^2 - 1)\dfrac{d}{dx}[x^4 + 1]}{(x^4+1)^2} \\
&= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2} \\
&= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2} = -\frac{2x(x^4 - 2x^2 - 1)}{(x^4+1)^2}
\end{aligned}$$

The differentiation is complete. The rest is simplification.

Now we will set $dy/dx = 0$ and solve for x. We obtain

$$-\frac{2x(x^4 - 2x^2 - 1)}{(x^4+1)^2} = 0$$

The solutions of this equation are the values of x for which the numerator is 0, that is,

$$2x(x^4 - 2x^2 - 1) = 0$$

The first factor yields the solution $x = 0$. Other solutions can be found by solving the equation

$$x^4 - 2x^2 - 1 = 0$$

This can be treated as a quadratic equation in x^2 and solved by the quadratic formula. This yields

$$x^2 = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

The minus sign yields imaginary values of x, which we ignore since they are not relevant to the problem. The plus sign yields the solutions

$$x = \pm\sqrt{1+\sqrt{2}}$$

In summary, horizontal tangent lines occur at

$$x = 0, \quad x = \sqrt{1+\sqrt{2}} \approx 1.55, \quad \text{and} \quad x = -\sqrt{1+\sqrt{2}} \approx -1.55$$

which is consistent with the rough estimates that we obtained graphically in part (a). ◀

Derive the following rule for differentiating a reciprocal:

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

Use it to find the derivative of

$$f(x) = \frac{1}{x^2+1}$$

SUMMARY OF DIFFERENTIATION RULES

The following table summarizes the differentiation rules that we have encountered thus far.

Table 2.4.1
RULES FOR DIFFERENTIATION

$\frac{d}{dx}[c] = 0$	$(f+g)' = f' + g'$	$(f \cdot g)' = f \cdot g' + g \cdot f'$	$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
$(cf)' = cf'$	$(f-g)' = f' - g'$	$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$	$\frac{d}{dx}[x^r] = rx^{r-1}$

✓ QUICK CHECK EXERCISES 2.4

(See page 113 for answers.)

1. (a) $\frac{d}{dx}[x^2 f(x)] =$ ______ (b) $\frac{d}{dx}\left[\frac{f(x)}{x^2+1}\right] =$ ______
   (c) $\frac{d}{dx}\left[\frac{x^2+1}{f(x)}\right] =$ ______

2. Find $F'(1)$ given that $f(1) = -1$, $f'(1) = 2$, $g(1) = 3$, and $g'(1) = -1$.
   (a) $F(x) = 2f(x) - 3g(x)$ (b) $F(x) = [f(x)]^2$
   (c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

EXERCISE SET 2.4

Graphing Utility

1–4 Compute the derivative of the given function $f(x)$ by (a) multiplying and then differentiating and (b) using the product rule. Verify that (a) and (b) yield the same result.

1. $f(x) = (x+1)(2x-1)$
2. $f(x) = (3x^2-1)(x^2+2)$
3. $f(x) = (x^2+1)(x^2-1)$
4. $f(x) = (x+1)(x^2-x+1)$

5–20 Find $f'(x)$.

5. $f(x) = (3x^2+6)\left(2x - \frac{1}{4}\right)$
6. $f(x) = (2 - x - 3x^3)(7 + x^5)$
7. $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$
8. $f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$
9. $f(x) = (x-2)(x^2+2x+4)$
10. $f(x) = (x^2+x)(x^2-x)$
11. $f(x) = \frac{3x+4}{x^2+1}$
12. $f(x) = \frac{x-2}{x^4+x+1}$
13. $f(x) = \frac{x^2}{3x-4}$
14. $f(x) = \frac{2x^2+5}{3x-4}$
15. $f(x) = \frac{(2\sqrt{x}+1)(x-1)}{x+3}$
16. $f(x) = (2\sqrt{x}+1)\left(\frac{2-x}{x^2+3x}\right)$
17. $f(x) = (2x+1)\left(1+\frac{1}{x}\right)(x^{-3}+7)$
18. $f(x) = x^{-5}(x^2+2x)(4-3x)(2x^9+1)$
19. $f(x) = (x^7+2x-3)^3$
20. $f(x) = (x^2+1)^4$

21–24 Find $dy/dx|_{x=1}$.

21. $y = \frac{2x-1}{x+3}$
22. $y = \frac{4x+1}{x^2-5}$
23. $y = \left(\frac{3x+2}{x}\right)(x^{-5}+1)$
24. $y = (2x^7 - x^2)\left(\frac{x-1}{x+1}\right)$

25–26 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

25. $f(x) = \frac{x}{x^2+1}$
26. $f(x) = \frac{x^2-1}{x^2+1}$

27. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$.
(a) $g(x) = \sqrt{x} f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$

28. Find $g'(3)$ given that $f(3) = -2$ and $f'(3) = 4$.
(a) $g(x) = 3x^2 - 5f(x)$ (b) $g(x) = \dfrac{2x+1}{f(x)}$

29. In parts (a)–(d), $F(x)$ is expressed in terms of $f(x)$ and $g(x)$. Find $F'(2)$ given that $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, and $g'(2) = -5$.
(a) $F(x) = 5f(x) + 2g(x)$ (b) $F(x) = f(x) - 3g(x)$
(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

30. Find $F'(\pi)$ given that $f(\pi) = 10$, $f'(\pi) = -1$, $g(\pi) = -3$, and $g'(\pi) = 2$.
(a) $F(x) = 6f(x) - 5g(x)$ (b) $F(x) = x(f(x) + g(x))$
(c) $F(x) = 2f(x)g(x)$ (d) $F(x) = \dfrac{f(x)}{4 + g(x)}$

31–36 Find all values of x at which the tangent line to the given curve satisfies the stated property.

31. $y = \dfrac{x^2 - 1}{x + 2}$; horizontal
32. $y = \dfrac{x^2 + 1}{x - 1}$; horizontal
33. $y = \dfrac{x^2 + 1}{x + 1}$; parallel to the line $y = x$
34. $y = \dfrac{x + 3}{x + 2}$; perpendicular to the line $y = x$
35. $y = \dfrac{1}{x + 4}$; passes through the origin
36. $y = \dfrac{2x + 5}{x + 2}$; y-intercept 2

FOCUS ON CONCEPTS

37. (a) What should it mean to say that two curves intersect at right angles?
(b) Show that the curves $y = 1/x$ and $y = 1/(2 - x)$ intersect at right angles.

38. Find all values of a such that the curves $y = a/(x - 1)$ and $y = x^2 - 2x + 1$ intersect at right angles.

39. Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and f' are differentiable at x.

40. Suppose that the function f is differentiable everywhere and $F(x) = xf(x)$.
(a) Express $F'''(x)$ in terms of x and derivatives of f.
(b) For $n \geq 2$, conjecture a formula for $F^{(n)}(x)$.

41. A manufacturer of athletic footwear finds that the sales of their ZipStride brand running shoes is a function $f(p)$ of the selling price p (in dollars) for a pair of shoes. Suppose that $f(120) = 9000$ pairs of shoes and $f'(120) = -60$ pairs of shoes per dollar. The revenue that the manufacturer will receive for selling $f(p)$ pairs of shoes at p dollars per pair is $R(p) = p \cdot f(p)$. Find $R'(120)$. What impact would a small increase in price have on the manufacturer's revenue?

42. Solve the problem in Exercise 41 under the assumption that $f(120) = 9000$ and $f'(120) = -80$.

43. Use the quotient rule (Theorem 2.4.2) to derive the formula for the derivative of $f(x) = x^{-n}$, where n is a positive integer.

✓ QUICK CHECK ANSWERS 2.4 1. (a) $x^2 f'(x) + 2xf(x)$ (b) $\dfrac{(x^2+1)f'(x) - 2xf(x)}{(x^2+1)^2}$ (c) $\dfrac{2xf(x) - (x^2+1)f'(x)}{[f(x)^2]}$
2. (a) 7 (b) −4 (c) 7 (d) $\frac{5}{9}$

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions. If needed, you will find a review of trigonometric functions in Appendix A.

We will assume in this section that the variable x in the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ is measured in radians. Also, we will need the limits in Theorem 1.6.3, but restated as follows using h rather than x as the variable:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \qquad (1\text{–}2)$$

Let us start with the problem of differentiating $f(x) = \sin x$. Using the definition of the derivative we obtain

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{By the addition formula for sine} \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h\to 0} \left[\cos x \left(\frac{\sin h}{h}\right) - \sin x \left(\frac{1 - \cos h}{h}\right)\right] && \text{Algebraic reorganization} \\
&= \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} - \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= \left(\lim_{h\to 0} \cos x\right)(1) - \left(\lim_{h\to 0} \sin x\right)(0) && \text{Formulas (1) and (2)} \\
&= \lim_{h\to 0} \cos x = \cos x && \text{$\cos x$ does not involve the variable h and hence is treated as a constant in the limit computation.}
\end{aligned}$$

Formulas (1) and (2) and the derivation of Formulas (3) and (4) are only valid if h and x are in radians. See Exercise 49 for how Formulas (3) and (4) change when x is measured in degrees.

Thus, we have shown that

$$\frac{d}{dx}[\sin x] = \cos x \tag{3}$$

In the exercises we will ask you to use the same method to derive the following formula for the derivative of $\cos x$:

$$\frac{d}{dx}[\cos x] = -\sin x \tag{4}$$

▶ Example 1 Find dy/dx if $y = x \sin x$.

Solution. Using Formula (3) and the product rule we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[x \sin x] \\
&= x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] \\
&= x\cos x + \sin x \blacktriangleleft
\end{aligned}$$

▶ Example 2 Find dy/dx if $y = \dfrac{\sin x}{1 + \cos x}$.

Solution. Using the quotient rule together with Formulas (3) and (4) we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1+\cos x)\cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1 + \cos x]}{(1+\cos x)^2} \\
&= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} \\
&= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x} \blacktriangleleft
\end{aligned}$$

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x \tag{5–6}$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x \tag{7–8}$$

Since Formulas (3) and (4) are valid only if x is in radians, the same is true for Formulas (5)–(8).

These can all be obtained using the definition of the derivative, but it is easier to use Formulas (3) and (4) and apply the quotient rule to the relationships

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

For example,

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Example 3 Find $f''(\pi/4)$ if $f(x) = \sec x$.

$$f'(x) = \sec x \tan x$$

$$f''(x) = \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

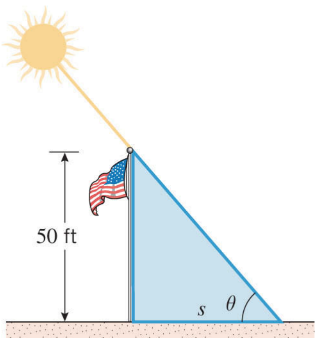
$$= \sec^3 x + \sec x \tan^2 x$$

Thus,

$$f''(\pi/4) = \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)$$

$$= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2} \blacktriangleleft$$

When finding the value of a derivative at a specific point $x = x_0$, it is important to substitute x_0 *after* the derivative is obtained. Thus, in Example 3 we made the substitution $x = \pi/4$ after f'' was calculated. What would have happened had we *incorrectly* substituted $x = \pi/4$ into $f'(x)$ before calculating f''?



▲ **Figure 2.5.1**

Example 4 On a sunny day, a 50 ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun (Figure 2.5.1). Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45°$. Express your answer in units of feet/degree.

Solution. The variables s and θ are related by $\tan\theta = 50/s$ or, equivalently,

$$s = 50\cot\theta \tag{9}$$

If θ is measured in radians, then Formula (7) is applicable, which yields

$$\frac{ds}{d\theta} = -50\csc^2\theta$$

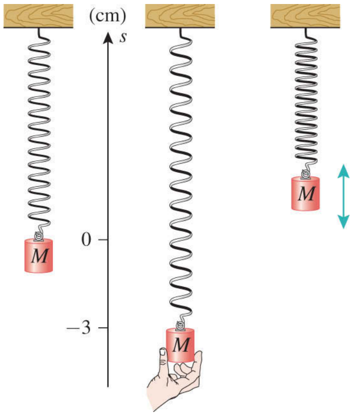
which is the rate of change of shadow length with respect to the elevation angle θ in units of feet/radian. When $\theta = 45°$ (or equivalently $\theta = \pi/4$ radians), we obtain

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/4} = -50\csc^2(\pi/4) = -100 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

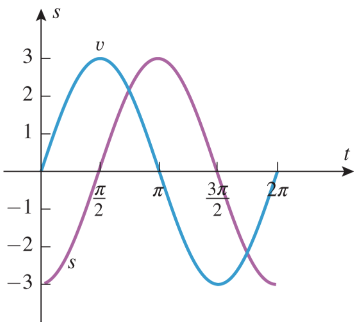
$$-100\,\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\,\frac{\text{rad}}{\text{deg}} = -\frac{5}{9}\pi\,\frac{\text{ft}}{\text{deg}} \approx -1.75 \text{ ft/deg}$$

Thus, when $\theta = 45°$, the shadow length is decreasing (because of the minus sign) at an approximate rate of 1.75 ft/deg increase in the angle of elevation. ◀



▲ **Figure 2.5.2**

Example 5 As illustrated in Figure 2.5.2, suppose that a spring with an attached mass is stretched 3 cm beyond its rest position and released at time $t = 0$. Assuming that



▲ Figure 2.5.3

the position function of the top of the attached mass is

$$s = -3\cos t \tag{10}$$

where s is in centimeters and t is in seconds, find the velocity function and discuss the motion of the attached mass.

Solution. The velocity function is

$$v = \frac{ds}{dt} = \frac{d}{dt}[-3\cos t] = 3\sin t$$

Figure 2.5.3 shows the graphs of the position and velocity functions. The position function tells us that the top of the mass oscillates between a low point of $s = -3$ and a high point of $s = 3$ with one complete oscillation occuring every 2π seconds [the period of (10)]. The top of the mass is moving up (the positive s-direction) when v is positive, is moving down when v is negative, and is at a high or low point when $v = 0$. Thus, for example, the top of the mass moves up from time $t = 0$ to time $t = \pi$, at which time it reaches the high point $s = 3$ and then moves down until time $t = 2\pi$, at which time it reaches the low point of $s = -3$. The motion then repeats periodically. ◀

In Example 5, the top of the mass has its maximum speed when it passes through its rest position. Why? What is that maximum speed?

✓ QUICK CHECK EXERCISES 2.5 (See page 118 for answers.)

1. Find dy/dx.
   (a) $y = \sin x$ (b) $y = \cos x$
   (c) $y = \tan x$ (d) $y = \sec x$
2. Find $f'(x)$ and $f'(\pi/3)$ if $f(x) = \sin x \cos x$.
3. Use a derivative to evaluate each limit.
   (a) $\displaystyle\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$ (b) $\displaystyle\lim_{h\to 0}\frac{\csc(x+h)-\csc x}{h}$

EXERCISE SET 2.5 Graphing Utility

1–18 Find $f'(x)$.

1. $f(x) = 4\cos x + 2\sin x$
2. $f(x) = \dfrac{5}{x^2} + \sin x$
3. $f(x) = -4x^2\cos x$
4. $f(x) = 2\sin^2 x$
5. $f(x) = \dfrac{5-\cos x}{5+\sin x}$
6. $f(x) = \dfrac{\sin x}{x^2+\sin x}$
7. $f(x) = \sec x - \sqrt{2}\tan x$
8. $f(x) = (x^2+1)\sec x$
9. $f(x) = 4\csc x - \cot x$
10. $f(x) = \cos x - x\csc x$
11. $f(x) = \sec x \tan x$
12. $f(x) = \csc x \cot x$
13. $f(x) = \dfrac{\cot x}{1+\csc x}$
14. $f(x) = \dfrac{\sec x}{1+\tan x}$
15. $f(x) = \sin^2 x + \cos^2 x$
16. $f(x) = \sec^2 x - \tan^2 x$
17. $f(x) = \dfrac{\sin x \sec x}{1 + x\tan x}$
18. $f(x) = \dfrac{(x^2+1)\cot x}{3-\cos x\csc x}$

19–24 Find d^2y/dx^2.

19. $y = x\cos x$
20. $y = \csc x$
21. $y = x\sin x - 3\cos x$
22. $y = x^2\cos x + 4\sin x$
23. $y = \sin x \cos x$
24. $y = \tan x$

25. Find the equation of the line tangent to the graph of $\tan x$ at
   (a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.

26. Find the equation of the line tangent to the graph of $\sin x$ at
   (a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.

27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.
   (b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.

28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
   (b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.

29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
   (a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
   (c) $f(x) = \tan x$ (d) $f(x) = \sec x$

30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.
   (b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10 ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure on the next page. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

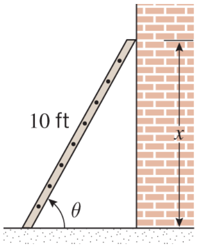


Figure Ex-31

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30^\circ$? Express the answer in units of feet/degree.

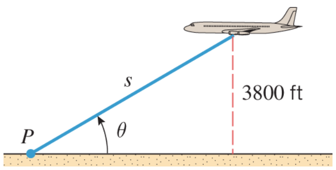


Figure Ex-32

33. A searchlight is trained on the side of a tall building. As the light rotates, the spot it illuminates moves up and down the side of the building. That is, the distance D between ground level and the illuminated spot on the side of the building is a function of the angle θ formed by the light beam and the horizontal (see the accompanying figure). If the searchlight is located 50 m from the building, find the rate at which D is changing with respect to θ when $\theta = 45^\circ$. Express your answer in units of meters/degree.

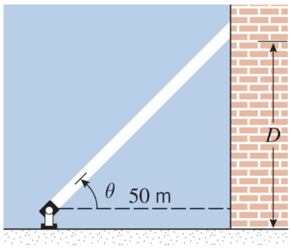


Figure Ex-33

34. An Earth-observing satellite can see only a portion of the Earth's surface. The satellite has horizon sensors that can detect the angle θ shown in the accompanying figure. Let r be the radius of the Earth (assumed spherical) and h the distance of the satellite from the Earth's surface.
(a) Show that $h = r(\csc\theta - 1)$.
(b) Using $r = 6378$ km, find the rate at which h is changing with respect to θ when $\theta = 30^\circ$. Express the answer in units of kilometers/degree.

Source: Adapted from *Space Mathematics*, NASA, 1985.

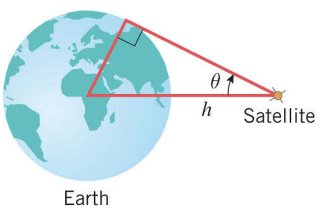


Figure Ex-34

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. If $g(x) = f(x)\sin x$, then $g'(x) = f'(x)\cos x$.

36. If $g(x) = f(x)\sin x$, then $g'(0) = f(0)$.

37. If $f(x)\cos x = \sin x$, then $f'(x) = \sec^2 x$.

38. Suppose that $g(x) = f(x)\sec x$, where $f(0) = 8$ and $f'(0) = -2$. Then

$$g'(0) = \lim_{h\to 0}\frac{f(h)\sec h - f(0)}{h} = \lim_{h\to 0}\frac{8(\sec h - 1)}{h}$$
$$= 8\cdot\frac{d}{dx}[\sec x]\bigg|_{x=0} = 8\sec 0\tan 0 = 0$$

39–40 Make a conjecture about the derivative by calculating the first few derivatives and observing the resulting pattern.

39. $\dfrac{d^{87}}{dx^{87}}[\sin x]$ **40.** $\dfrac{d^{100}}{dx^{100}}[\cos x]$

41. Let $f(x) = \cos x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

42. Let $f(x) = \sin x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

FOCUS ON CONCEPTS

43. In each part, determine where f is differentiable.
(a) $f(x) = \sin x$ (b) $f(x) = \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \cot x$
(e) $f(x) = \sec x$ (f) $f(x) = \csc x$
(g) $f(x) = \dfrac{1}{1+\cos x}$ (h) $f(x) = \dfrac{1}{\sin x\cos x}$
(i) $f(x) = \dfrac{\cos x}{2-\sin x}$

44. (a) Derive Formula (4) using the definition of a derivative.
(b) Use Formulas (3) and (4) to obtain (7).
(c) Use Formula (4) to obtain (6).
(d) Use Formula (3) to obtain (8).

45. Use Formula (1), the alternative form for the definition of derivative given in Formula (13) of Section 2.2, that is,
$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x}$$
and the difference identity
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
to show that $\dfrac{d}{dx}[\sin x] = \cos x$.

46. Follow the directions of Exercise 45 using the difference identity
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$
to show that $\dfrac{d}{dx}[\cos x] = -\sin x$.

47. (a) Show that $\lim_{h\to 0} \dfrac{\tan h}{h} = 1$.
(b) Use the result in part (a) to help derive the formula for the derivative of $\tan x$ directly from the definition of a derivative.

48. Without using any trigonometric identities, find

$$\lim_{x\to 0} \frac{\tan(x+y) - \tan y}{x}$$

[*Hint:* Relate the given limit to the definition of the derivative of an appropriate function of y.]

49. The derivative formulas for $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ were obtained under the assumption that x is measured in radians. If x is measured in degrees, then

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\pi}{180}$$

(See Exercise 39 of Section 1.6). Use this result to prove that if x is measured in degrees, then
(a) $\dfrac{d}{dx}[\sin x] = \dfrac{\pi}{180}\cos x$
(b) $\dfrac{d}{dx}[\cos x] = -\dfrac{\pi}{180}\sin x$.

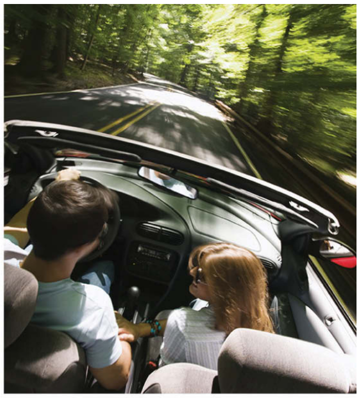
50. Writing Suppose that f is a function that is differentiable everywhere. Explain the relationship, if any, between the periodicity of f and that of f'. That is, if f is periodic, must f' also be periodic? If f' is periodic, must f also be periodic?

✓ **QUICK CHECK ANSWERS 2.5** **1.** (a) $\cos x$ (b) $-\sin x$ (c) $\sec^2 x$ (d) $\sec x \tan x$ **2.** $f'(x) = \cos^2 x - \sin^2 x$, $f'(\pi/3) = -\frac{1}{2}$
3. (a) $\left.\dfrac{d}{dx}[\sin x]\right|_{x=\pi/2} = 0$ (b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

2.6 THE CHAIN RULE

In this section we will derive a formula that expresses the derivative of a composition $f \circ g$ in terms of the derivatives of f and g. This formula will enable us to differentiate complicated functions using known derivatives of simpler functions.

DERIVATIVES OF COMPOSITIONS



Comstock Images/Getty Images
The cost of a car trip is a combination of fuel efficiency and the cost of gasoline.

Suppose you are traveling to school in your car, which gets 20 miles per gallon of gasoline. The number of miles you can travel in your car without refueling is a function of the number of gallons of gas you have in the gas tank. In symbols, if y is the number of miles you can travel and u is the number of gallons of gas you have initially, then y is a function of u, or $y = f(u)$. As you continue your travels, you note that your local service station is selling gasoline for \$4 per gallon. The number of gallons of gas you have initially is a function of the amount of money you spend for that gas. If x is the number of dollars you spend on gas, then $u = g(x)$. Now 20 miles per gallon is the rate at which your mileage changes with respect to the amount of gasoline you use, so

$$f'(u) = \frac{dy}{du} = 20 \text{ miles per gallon}$$

Similarly, since gasoline costs \$4 per gallon, each dollar you spend will give you $1/4$ of a gallon of gas, and

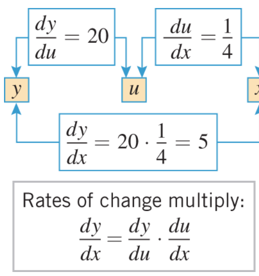
$$g'(x) = \frac{du}{dx} = \frac{1}{4} \text{ gallons per dollar}$$

Notice that the number of miles you can travel is also a function of the number of dollars you spend on gasoline. This fact is expressible as the composition of functions

$$y = f(u) = f(g(x))$$

You might be interested in how many miles you can travel per dollar, which is dy/dx. Intuition suggests that rates of change multiply in this case (see Figure 2.6.1), so

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{20 \text{ miles}}{1 \text{ gallon}}\cdot\frac{1 \text{ gallon}}{4 \text{ dollars}} = \frac{20 \text{ miles}}{4 \text{ dollars}} = 5 \text{ miles per dollar}$$



▲ **Figure 2.6.1**

The following theorem, the proof of which is given in Web Appendix L, formalizes the preceding ideas.

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \textit{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

The name "chain rule" is appropriate because the desired derivative is obtained by a two-link "chain" of simpler derivatives.

Formula (1) is easy to remember because the left side is exactly what results if we "cancel" the du's on the right side. This "canceling" device provides a good way of deducing the correct form of the chain rule when different variables are used. For example, if w is a function of x and x is a function of t, then the chain rule takes the form

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

Example 1 Find dy/dx if $y = \cos(x^3)$.

Solution. Let $u = x^3$ and express y as $y = \cos u$. Applying Formula (1) yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[x^3] \\ &= (-\sin u) \cdot (3x^2) \\ &= (-\sin(x^3)) \cdot (3x^2) = -3x^2 \sin(x^3) \end{aligned}$$

Example 2 Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule computations take the form

$$\begin{aligned} \frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} \\ &= \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\ &= (\sec^2 x) \cdot (12t^2 + 1) \\ &= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t) \end{aligned}$$

AN ALTERNATIVE VERSION OF THE CHAIN RULE

Formula (1) for the chain rule can be unwieldy in some problems because it involves so many variables. As you become more comfortable with the chain rule, you may want to dispense with writing out the dependent variables by expressing (1) in the form

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x) \tag{2}$$

Confirm that (2) is an alternative version of (1) by letting $y = f(g(x))$ and $u = g(x)$.

A convenient way to remember this formula is to call f the "outside function" and g the "inside function" in the composition $f(g(x))$ and then express (2) in words as:

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside function}}$$

Example 3 (*Example 1 revisited*) Find $h'(x)$ if $h(x) = \cos(x^3)$.

Solution. We can think of h as a composition $f(g(x))$ in which $g(x) = x^3$ is the inside function and $f(x) = \cos x$ is the outside function. Thus, Formula (2) yields

$$\begin{aligned} h'(x) &= \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside function}} \\ &= f'(x^3) \cdot 3x^2 \\ &= -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3) \end{aligned}$$

which agrees with the result obtained in Example 1. ◀

Example 4

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}[(\tan x)^2] = \underbrace{(2\tan x)}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{(\sec^2 x)}_{\text{Derivative of the inside function}} = 2\tan x \sec^2 x$$

$$\frac{d}{dx}\left[\sqrt{x^2+1}\right] = \underbrace{\frac{1}{2\sqrt{x^2+1}}}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{2x}_{\text{Derivative of the inside function}} = \frac{x}{\sqrt{x^2+1}}$$

See Formula (6) of Section 2.3. ◀

GENERALIZED DERIVATIVE FORMULAS

There is a useful third variation of the chain rule that strikes a middle ground between Formulas (1) and (2). If we let $u = g(x)$ in (2), then we can rewrite that formula as

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \tag{3}$$

This result, called the ***generalized derivative formula*** for f, provides a way of using the derivative of $f(x)$ to produce the derivative of $f(u)$, where u is a function of x. Table 2.6.1 gives some examples of this formula.

Table 2.6.1

GENERALIZED DERIVATIVE FORMULAS

$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$	
$\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u \frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u \frac{du}{dx}$

▶ Example 5 Find

(a) $\dfrac{d}{dx}[\sin(2x)]$ (b) $\dfrac{d}{dx}[\tan(x^2+1)]$ (c) $\dfrac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$

(d) $\dfrac{d}{dx}[x^2-x+2]^{3/4}$ (e) $\dfrac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Solution (b). Taking $u = x^2+1$ in the generalized derivative formula for $\tan u$ yields

$$\begin{aligned}\frac{d}{dx}[\tan(x^2+1)] &= \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}\\ &= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x\\ &= 2x\sec^2(x^2+1)\end{aligned}$$

Solution (c). Taking $u = x^3+\csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right] &= \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]\\ &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}\end{aligned}$$

Solution (d). Taking $u = x^2-x+2$ in the generalized derivative formula for $u^{3/4}$ yields

$$\begin{aligned}\frac{d}{dx}[x^2-x+2]^{3/4} &= \frac{d}{dx}[u^{3/4}] = \frac{3}{4}u^{-1/4}\frac{du}{dx}\\ &= \frac{3}{4}(x^2-x+2)^{-1/4}\cdot\frac{d}{dx}[x^2-x+2]\\ &= \frac{3}{4}(x^2-x+2)^{-1/4}(2x-1)\end{aligned}$$

Solution (e). Taking $u = 1+x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\begin{aligned}\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right] &= \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}\\ &= -8(1+x^5\cot x)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]\\ &= -8(1+x^5\cot x)^{-9}\cdot[x^5(-\csc^2 x)+5x^4\cot x]\\ &= (8x^5\csc^2 x-40x^4\cot x)(1+x^5\cot x)^{-9}\end{aligned}$$ ◀

Sometimes you will have to make adjustments in notation or apply the chain rule more than once to calculate a derivative.

▶ Example 6 Find

(a) $\dfrac{d}{dx}\left[\sin\left(\sqrt{1+\cos x}\right)\right]$ (b) $\dfrac{d\mu}{dt}$ if $\mu = \sec\sqrt{\omega t}$ (ω constant)

Solution (a). Taking $u = \sqrt{1+\cos x}$ in the generalized derivative formula for $\sin u$ yields

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(\sqrt{1+\cos x}\right)\right] &= \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}\\ &= \cos\left(\sqrt{1+\cos x}\right)\cdot\frac{d}{dx}\left[\sqrt{1+\cos x}\right]\\ &= \cos\left(\sqrt{1+\cos x}\right)\cdot\frac{-\sin x}{2\sqrt{1+\cos x}}\\ &= -\frac{\sin x\cos\left(\sqrt{1+\cos x}\right)}{2\sqrt{1+\cos x}}\end{aligned}$$

We used the generalized derivative formula for $\sqrt{u}$ with $u = 1+\cos x$.

Solution (b).

$$\begin{aligned}\frac{d\mu}{dt} &= \frac{d}{dt}[\sec\sqrt{\omega t}] = \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{d}{dt}[\sqrt{\omega t}]\\ &= \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{\omega}{2\sqrt{\omega t}}\end{aligned}$$

We used the generalized derivative formula for $\sec u$ with $u = \sqrt{\omega t}$.

We used the generalized derivative formula for $\sqrt{u}$ with $u = \omega t$. ◀

DIFFERENTIATING USING COMPUTER ALGEBRA SYSTEMS

Even with the chain rule and other differentiation rules, some derivative computations can be tedious to perform. For complicated derivatives, engineers and scientists often use computer algebra systems such as *Mathematica*, *Maple*, or *Sage*. For example, although we have all the mathematical tools to compute

$$\frac{d}{dx}\left[\frac{(x^2+1)^{10}\sin^3(\sqrt{x})}{\sqrt{1+\csc x}}\right] \tag{4}$$

by hand, the computation is sufficiently involved that it may be more efficient (and less error-prone) to use a computer algebra system.

TECHNOLOGY MASTERY

If you have a CAS, use it to perform the differentiation in (4).

✓ QUICK CHECK EXERCISES 2.6

(See page 125 for answers.)

1. The chain rule states that the derivative of the composition of two functions is the derivative of the ______ function evaluated at the ______ function times the derivative of the ______ function.

2. If y is a differentiable function of u, and u is a differentiable function of x, then

$$\frac{dy}{dx} = \_\_\_\_\_\_ \cdot \_\_\_\_\_\_$$

3. Find dy/dx.
(a) $y = (x^2+5)^{10}$ (b) $y = \sqrt{1+6x}$

4. Find dy/dx.
(a) $y = \sin(3x+2)$ (b) $y = (x^2\tan x)^4$

5. Suppose that $f(2) = 3$, $f'(2) = 4$, $g(3) = 6$, and $g'(3) = -5$. Evaluate
(a) $h'(2)$, where $h(x) = g(f(x))$
(b) $k'(3)$, where $k(x) = f\left(\frac{1}{3}g(x)\right)$.

EXERCISE SET 2.6

Graphing Utility C CAS

1. Given that

$$f'(0) = 2, \quad g(0) = 0 \quad \text{and} \quad g'(0) = 3$$

find $(f\circ g)'(0)$.

2. Given that

$$f'(9) = 5, \quad g(2) = 9 \quad \text{and} \quad g'(2) = -3$$

find $(f\circ g)'(2)$.

3. Let $f(x) = x^5$ and $g(x) = 2x-3$.
(a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
(b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.

4. Let $f(x) = 5\sqrt{x}$ and $g(x) = 4+\cos x$.
(a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
(b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.

FOCUS ON CONCEPTS

5. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	−2	5	7
5	3	−1	12	4

(a) $F'(3)$, where $F(x) = f(g(x))$
(b) $G'(3)$, where $G(x) = g(f(x))$

6. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−1	2	3	2	−3
2	0	4	1	−5

(a) $F'(-1)$, where $F(x) = f(g(x))$
(b) $G'(-1)$, where $G(x) = g(f(x))$

7–26 Find $f'(x)$.

7. $f(x) = (x^3 + 2x)^{37}$

8. $f(x) = (3x^2 + 2x - 1)^6$

9. $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$

10. $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$

11. $f(x) = \dfrac{4}{(3x^2 - 2x + 1)^3}$

12. $f(x) = \sqrt{x^3 - 2x + 5}$

13. $f(x) = \sqrt{4 + \sqrt{3x}}$

14. $f(x) = \sqrt[3]{12 + \sqrt{x}}$

15. $f(x) = \sin\left(\dfrac{1}{x^2}\right)$

16. $f(x) = \tan\sqrt{x}$

17. $f(x) = 4\cos^5 x$

18. $f(x) = 4x + 5\sin^4 x$

19. $f(x) = \cos^2(3\sqrt{x})$

20. $f(x) = \tan^4(x^3)$

21. $f(x) = 2\sec^2(x^7)$

22. $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$

23. $f(x) = \sqrt{\cos(5x)}$

24. $f(x) = \sqrt{3x - \sin^2(4x)}$

25. $f(x) = [x + \csc(x^3 + 3)]^{-3}$

26. $f(x) = [x^4 - \sec(4x^2 - 2)]^{-4}$

27–40 Find dy/dx.

27. $y = x^3 \sin^2(5x)$

28. $y = \sqrt{x}\tan^3(\sqrt{x})$

29. $y = x^5 \sec(1/x)$

30. $y = \dfrac{\sin x}{\sec(3x+1)}$

31. $y = \cos(\cos x)$

32. $y = \sin(\tan 3x)$

33. $y = \cos^3(\sin 2x)$

34. $y = \dfrac{1 + \csc(x^2)}{1 - \cot(x^2)}$

35. $y = (5x+8)^7 (1 - \sqrt{x})^6$

36. $y = (x^2 + x)^5 \sin^8 x$

37. $y = \left(\dfrac{x-5}{2x+1}\right)^3$

38. $y = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$

39. $y = \dfrac{(2x+3)^3}{(4x^2-1)^8}$

40. $y = [1 + \sin^3(x^5)]^{12}$

C **41–42** Use a CAS to find dy/dx.

41. $y = [x\sin 2x + \tan^4(x^7)]^5$

42. $y = \tan^4\left(2 + \dfrac{(7-x)\sqrt{3x^2+5}}{x^3 + \sin x}\right)$

43–50 Find an equation for the tangent line to the graph at the specified value of x.

43. $y = x\cos 3x,\ x = \pi$

44. $y = \sin(1 + x^3),\ x = -3$

45. $y = \sec^3\left(\dfrac{\pi}{2} - x\right),\ x = -\dfrac{\pi}{2}$

46. $y = \left(x - \dfrac{1}{x}\right)^3,\ x = 2$

47. $y = \tan(4x^2),\ x = \sqrt{\pi}$

48. $y = 3\cot^4 x,\ x = \dfrac{\pi}{4}$

49. $y = x^2\sqrt{5 - x^2},\ x = 1$

50. $y = \dfrac{x}{\sqrt{1 - x^2}},\ x = 0$

51–54 Find d^2y/dx^2.

51. $y = x\cos(5x) - \sin^2 x$

52. $y = \sin(3x^2)$

53. $y = \dfrac{1+x}{1-x}$

54. $y = x\tan\left(\dfrac{1}{x}\right)$

55–58 Find the indicated derivative.

55. $y = \cot^3(\pi - \theta)$; find $\dfrac{dy}{d\theta}$.

56. $\lambda = \left(\dfrac{au+b}{cu+d}\right)^6$; find $\dfrac{d\lambda}{du}$ (a, b, c, d constants).

57. $\dfrac{d}{d\omega}[a\cos^2 \pi\omega + b\sin^2 \pi\omega]$ (a, b constants)

58. $x = \csc^2\left(\dfrac{\pi}{3} - y\right)$; find $\dfrac{dx}{dy}$.

59. (a) Use a graphing utility to obtain the graph of the function $f(x) = x\sqrt{4 - x^2}$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f'.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f'.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together.

60. (a) Use a graphing utility to obtain the graph of the function $f(x) = \sin x^2 \cos x$ over the interval $[-\pi/2, \pi/2]$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f' over the interval.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f' over the interval.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together over the interval.

61–64 True–False Determine whether the statement is true or false. Explain your answer.

61. If $y = f(x)$, then $\dfrac{d}{dx}[\sqrt{y}] = \sqrt{f'(x)}$.

62. If $y = f(u)$ and $u = g(x)$, then $dy/dx = f'(x) \cdot g'(x)$.

63. If $y = \cos[g(x)]$, then $dy/dx = -\sin[g'(x)]$.

64. If $y = \sin^3(3x^3)$, then $dy/dx = 27x^2 \sin^2(3x^3)\cos(3x^3)$.

65. If an object suspended from a spring is displaced vertically from its equilibrium position by a small amount and released, and if the air resistance and the mass of the spring are ignored, then the resulting oscillation of the object is called ***simple harmonic motion***. Under appropriate conditions the displacement y from equilibrium in terms of time t is given by

$$y = A\cos\omega t$$

where A is the initial displacement at time $t = 0$, and ω is a constant that depends on the mass of the object and the stiffness of the spring (see the accompanying figure). The constant $|A|$ is called the ***amplitude*** of the motion and ω the ***angular frequency***.

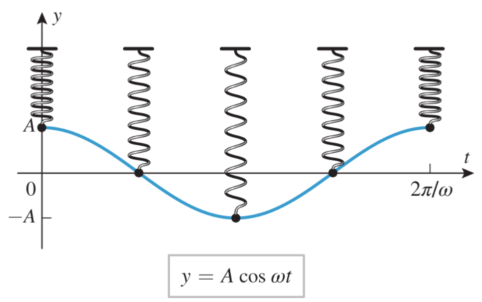
(a) Show that

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

(b) The ***period*** T is the time required to make one complete oscillation. Show that $T = 2\pi/\omega$.

(c) The ***frequency*** f of the vibration is the number of oscillations per unit time. Find f in terms of the period T.

(d) Find the amplitude, period, and frequency of an object that is executing simple harmonic motion given by $y = 0.6\cos 15t$, where t is in seconds and y is in centimeters.



▲ **Figure Ex-65**

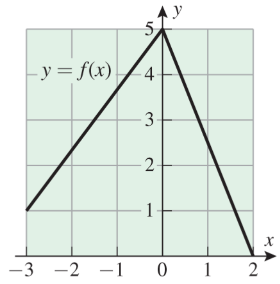
66. Find the value of the constant A so that $y = A\sin 3t$ satisfies the equation

$$\frac{d^2y}{dt^2} + 2y = 4\sin 3t$$

FOCUS ON CONCEPTS

67. Use the graph of the function f in the accompanying figure to evaluate

$$\frac{d}{dx}\left[\sqrt{x + f(x)}\right]\bigg|_{x=-1}$$



◀ **Figure Ex-67**

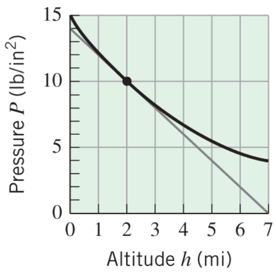
68. Using the function f in Exercise 67, evaluate

$$\frac{d}{dx}[f(2\sin x)]\bigg|_{x=\pi/6}$$

69. The accompanying figure shows the graph of atmospheric pressure p (lb/in^2) versus the altitude h (mi) above sea level.

(a) From the graph and the tangent line at $h = 2$ shown on the graph, estimate the values of p and dp/dh at an altitude of 2 mi.

(b) If the altitude of a space vehicle is increasing at the rate of 0.3 mi/s at the instant when it is 2 mi above sea level, how fast is the pressure changing with time at this instant?



◀ **Figure Ex-69**

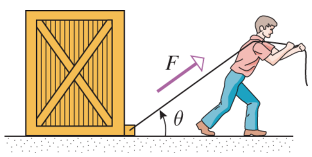
70. The force F (in pounds) acting at an angle θ with the horizontal that is needed to drag a crate weighing W pounds along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

where μ is a constant called the ***coefficient of sliding friction*** between the crate and the surface (see the accompanying figure). Suppose that the crate weighs 150 lb and that $\mu = 0.3$.

(a) Find $dF/d\theta$ when $\theta = 30°$. Express the answer in units of pounds/degree.

(b) Find dF/dt when $\theta = 30°$ if θ is decreasing at the rate of $0.5°$/s at this instant.



◀ **Figure Ex-70**

71. Recall that
$$\frac{d}{dx}(|x|) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$
Use this result and the chain rule to find
$$\frac{d}{dx}(|\sin x|)$$
for nonzero x in the interval $(-\pi, \pi)$.

72. Use the derivative formula for $\sin x$ and the identity
$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$
to obtain the derivative formula for $\cos x$.

73. Let
$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
(a) Show that f is continuous at $x = 0$.
(b) Use Definition 2.2.1 to show that $f'(0)$ does not exist.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Determine whether $\lim_{x \to 0} f'(x)$ exists.

74. Let
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
(a) Show that f is continuous at $x = 0$.
(b) Use Definition 2.2.1 to find $f'(0)$.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Show that f' is not continuous at $x = 0$.

75. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$
2	1	7
8	5	–3

(a) $g'(2)$, where $g(x) = [f(x)]^3$
(b) $h'(2)$, where $h(x) = f(x^3)$

76. Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

77. Given that $f'(x) = \dfrac{x}{x^2 + 1}$ and $g(x) = \sqrt{3x - 1}$, find $F'(x)$ if $F(x) = f(g(x))$.

78. Find $f'(x^2)$ if $\dfrac{d}{dx}[f(x^2)] = x^2$.

79. Find $\dfrac{d}{dx}[f(x)]$ if $\dfrac{d}{dx}[f(3x)] = 6x$.

80. Recall that a function f is ***even*** if $f(-x) = f(x)$ and ***odd*** if $f(-x) = -f(x)$, for all x in the domain of f. Assuming that f is differentiable, prove:
(a) f' is odd if f is even
(b) f' is even if f is odd.

81. Draw some pictures to illustrate the results in Exercise 80, and write a paragraph that gives an informal explanation of why the results are true.

82. Let $y = f_1(u)$, $u = f_2(v)$, $v = f_3(w)$, and $w = f_4(x)$. Express dy/dx in terms of dy/du, dw/dx, du/dv, and dv/dw.

83. Find a formula for
$$\frac{d}{dx}[f(g(h(x)))]$$

84. Writing The "co" in "cosine" comes from "complementary," since the cosine of an angle is the sine of the complementary angle, and vice versa:
$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$
Suppose that we define a function g to be a *cofunction* of a function f if
$$g(x) = f\left(\frac{\pi}{2} - x\right) \quad \text{for all } x$$
Thus, cosine and sine are cofunctions of each other, as are cotangent and tangent, and also cosecant and secant. If g is the cofunction of f, state a formula that relates g' and the cofunction of f'. Discuss how this relationship is exhibited by the derivatives of the cosine, cotangent, and cosecant functions.

✓ QUICK CHECK ANSWERS 2.6 **1.** outside; inside; inside **2.** $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$ **3.** (a) $10(x^2+5)^9 \cdot 2x = 20x(x^2+5)^9$ (b) $\dfrac{1}{2\sqrt{1+6x}} \cdot 6 = \dfrac{3}{\sqrt{1+6x}}$ **4.** (a) $3\cos(3x+2)$ (b) $4(x^2 \tan x)^3(2x \tan x + x^2 \sec^2 x)$ **5.** (a) $g'(f(2))f'(2) = -20$ (b) $f'\left(\dfrac{1}{3}g(3)\right) \cdot \dfrac{1}{3}g'(3) = -\dfrac{20}{3}$

CHAPTER 2 REVIEW EXERCISES

Graphing Utility   C CAS

1. Explain the difference between average and instantaneous rates of change, and discuss how they are calculated.

2. In parts (a)–(d), use the function $y = \frac{1}{2}x^2$.
(a) Find the average rate of change of y with respect to x over the interval $[3, 4]$.
(b) Find the instantaneous rate of change of y with respect to x at $x = 3$.

(cont.)

(c) Find the instantaneous rate of change of y with respect to x at a general x-value.
(d) Sketch the graph of $y = \frac{1}{2}x^2$ together with the secant line whose slope is given by the result in part (a), and indicate graphically the slope of the tangent line that corresponds to the result in part (b).

3. Complete each part for the function $f(x) = x^2 + 1$.
(a) Find the slope of the tangent line to the graph of f at a general x-value.
(b) Find the slope of the tangent line to the graph of f at $x = 2$.

4. A car is traveling on a straight road that is 120 mi long. For the first 100 mi the car travels at an average velocity of 50 mi/h. Show that no matter how fast the car travels for the final 20 mi it cannot bring the average velocity up to 60 mi/h for the entire trip.

5. At time $t = 0$ a car moves into the passing lane to pass a slow-moving truck. The average velocity of the car from $t = 1$ to $t = 1 + h$ is

$$v_{\text{ave}} = \frac{3(h+1)^{2.5} + 580h - 3}{10h}$$

Estimate the instantaneous velocity of the car at $t = 1$, where time is in seconds and distance is in feet.

6. A skydiver jumps from an airplane. Suppose that the distance she falls during the first t seconds before her parachute opens is $s(t) = 976((0.835)^t - 1) + 176t$, where s is in feet. Graph s versus t for $0 \le t \le 20$, and use your graph to estimate the instantaneous velocity at $t = 15$.

7. A particle moves on a line away from its initial position so that after t hours it is $s = 3t^2 + t$ miles from its initial position.
(a) Find the average velocity of the particle over the interval $[1, 3]$.
(b) Find the instantaneous velocity at $t = 1$.

8. State the definition of a derivative, and give two interpretations of it.

9. Use the definition of a derivative to find dy/dx, and check your answer by calculating the derivative using appropriate derivative formulas.
(a) $y = \sqrt{9 - 4x}$ (b) $y = \dfrac{x}{x+1}$

10. Suppose that $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x-1), & x > 1. \end{cases}$
For what values of k is f
(a) continuous? (b) differentiable?

11. The accompanying figure shows the graph of $y = f'(x)$ for an unspecified function f.
(a) For what values of x does the curve $y = f(x)$ have a horizontal tangent line?
(b) Over what intervals does the curve $y = f(x)$ have tangent lines with positive slope?
(c) Over what intervals does the curve $y = f(x)$ have tangent lines with negative slope?
(d) Given that $g(x) = f(x)\sin x$, find $g''(0)$.

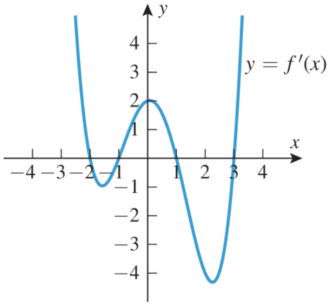


Figure Ex-11

12. Sketch the graph of a function f for which $f(0) = 1$, $f'(0) = 0$, $f'(x) > 0$ if $x < 0$, and $f'(x) < 0$ if $x > 0$.

13. According to the U.S. Bureau of the Census, the estimated and projected midyear world population, N, in billions for the years 1950, 1975, 2000, 2025, and 2050 was 2.555, 4.088, 6.080, 7.841, and 9.104, respectively. Although the increase in population is not a continuous function of the time t, we can apply the ideas in this section if we are willing to approximate the graph of N versus t by a continuous curve, as shown in the accompanying figure.
(a) Use the tangent line at $t = 2000$ shown in the figure to approximate the value of dN/dt there. Interpret your result as a rate of change.
(b) The instantaneous ***growth rate*** is defined as

$$\frac{dN/dt}{N}$$

Use your answer to part (a) to approximate the instantaneous growth rate at the start of the year 2000. Express the result as a percentage and include the proper units.

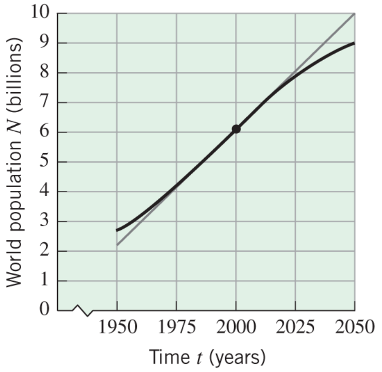


Figure Ex-13

14. Use a graphing utility to graph the function

$$f(x) = |x^4 - x - 1| - x$$

and estimate the values of x where the derivative of this function does not exist.

C **15–18** (a) Use a CAS to find $f'(x)$ via Definition 2.2.1; (b) check the result by finding the derivative by hand; (c) use the CAS to find $f''(x)$.

15. $f(x) = x^2 \sin x$

16. $f(x) = \sqrt{x} + \cos^2 x$

17. $f(x) = \dfrac{2x^2 - x + 5}{3x + 2}$

18. $f(x) = \dfrac{\tan x}{1 + x^2}$

19. The amount of water in a tank t minutes after it has started to drain is given by $W = 100(t - 15)^2$ gal.
(a) At what rate is the water running out at the end of 5 min?
(b) What is the average rate at which the water flows out during the first 5 min?

20. Use the formula $V = l^3$ for the volume of a cube of side l to find
(a) the average rate at which the volume of a cube changes with l as l increases from $l = 2$ to $l = 4$
(b) the instantaneous rate at which the volume of a cube changes with l when $l = 5$.

21–22 Zoom in on the graph of f on an interval containing $x = x_0$ until the graph looks like a straight line. Estimate the slope of this line and then check your answer by finding the exact value of $f'(x_0)$.

21. (a) $f(x) = x^2 - 1,\ x_0 = 1.8$
(b) $f(x) = \dfrac{x^2}{x - 2},\ x_0 = 3.5$

22. (a) $f(x) = x^3 - x^2 + 1,\ x_0 = 2.3$
(b) $f(x) = \dfrac{x}{x^2 + 1},\ x_0 = -0.5$

23. Suppose that a function f is differentiable at $x = 1$ and

$$\lim_{h \to 0} \frac{f(1 + h)}{h} = 5$$

Find $f(1)$ and $f'(1)$.

24. Suppose that a function f is differentiable at $x = 2$ and

$$\lim_{x \to 2} \frac{x^3 f(x) - 24}{x - 2} = 28$$

Find $f(2)$ and $f'(2)$.

25. Find the equations of all lines through the origin that are tangent to the curve $y = x^3 - 9x^2 - 16x$.

26. Find all values of x for which the tangent line to the curve $y = 2x^3 - x^2$ is perpendicular to the line $x + 4y = 10$.

27. Let $f(x) = x^2$. Show that for any distinct values of a and b, the slope of the tangent line to $y = f(x)$ at $x = \frac{1}{2}(a + b)$ is equal to the slope of the secant line through the points (a, a^2) and (b, b^2). Draw a picture to illustrate this result.

28. In each part, evaluate the expression given that $f(1) = 1$, $g(1) = -2$, $f'(1) = 3$, and $g'(1) = -1$.
(a) $\dfrac{d}{dx}[f(x)g(x)]\Big|_{x=1}$
(b) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\Big|_{x=1}$
(c) $\dfrac{d}{dx}\left[\sqrt{f(x)}\right]\Big|_{x=1}$
(d) $\dfrac{d}{dx}[f(1)g'(1)]$

29–32 Find $f'(x)$.

29. (a) $f(x) = x^8 - 3\sqrt{x} + 5x^{-3}$
(b) $f(x) = (2x + 1)^{101}(5x^2 - 7)$

30. (a) $f(x) = \sin x + 2\cos^3 x$
(b) $f(x) = (1 + \sec x)(x^2 - \tan x)$

31. (a) $f(x) = \sqrt{3x + 1}(x - 1)^2$
(b) $f(x) = \left(\dfrac{3x + 1}{x^2}\right)^3$

32. (a) $f(x) = \cot\left(\dfrac{\csc 2x}{x^3 + 5}\right)$
(b) $f(x) = \dfrac{1}{2x + \sin^3 x}$

33–34 Find the values of x at which the curve $y = f(x)$ has a horizontal tangent line.

33. $f(x) = (2x + 7)^6(x - 2)^5$

34. $f(x) = \dfrac{(x - 3)^4}{x^2 + 2x}$

35. Find all lines that are simultaneously tangent to the graph of $y = x^2 + 1$ and to the graph of $y = -x^2 - 1$.

36. (a) Let n denote an even positive integer. Generalize the result of Exercise 35 by finding all lines that are simultaneously tangent to the graph of $y = x^n + n - 1$ and to the graph of $y = -x^n - n + 1$.
(b) Let n denote an odd positive integer. Are there any lines that are simultaneously tangent to the graph of $y = x^n + n - 1$ and to the graph of $y = -x^n - n + 1$? Explain.

37. Find all values of x for which the line that is tangent to $y = 3x - \tan x$ is parallel to the line $y - x = 2$.

38. Approximate the values of x at which the tangent line to the graph of $y = x^3 - \sin x$ is horizontal.

39. Suppose that $f(x) = M\sin x + N\cos x$ for some constants M and N. If $f(\pi/4) = 3$ and $f'(\pi/4) = 1$, find an equation for the tangent line to $y = f(x)$ at $x = 3\pi/4$.

40. Suppose that $f(x) = M\tan x + N\sec x$ for some constants M and N. If $f(\pi/4) = 2$ and $f'(\pi/4) = 0$, find an equation for the tangent line to $y = f(x)$ at $x = 0$.

41. Suppose that $f'(x) = 2x \cdot f(x)$ and $f(2) = 5$.
(a) Find $g'(\pi/3)$ if $g(x) = f(\sec x)$.
(b) Find $h'(2)$ if $h(x) = [f(x)/(x - 1)]^4$.

CHAPTER 2 MAKING CONNECTIONS

1. Suppose that f is a function with the properties (i) f is differentiable everywhere, (ii) $f(x+y) = f(x)f(y)$ for all values of x and y, (iii) $f(0) \neq 0$, and (iv) $f'(0) = 1$.
(a) Show that $f(0) = 1$. [*Hint:* Consider $f(0+0)$.]
(b) Show that $f(x) > 0$ for all values of x. [*Hint:* First show that $f(x) \neq 0$ for any x by considering $f(x-x)$.]
(c) Use the definition of derivative (Definition 2.2.1) to show that $f'(x) = f(x)$ for all values of x.

2. Suppose that f and g are functions each of which has the properties (i)–(iv) in Exercise 1.
(a) Show that $y = f(2x)$ satisfies the equation $y' = 2y$ in two ways: using property (ii), and by directly applying the chain rule (Theorem 2.6.1).
(b) If k is any constant, show that $y = f(kx)$ satisfies the equation $y' = ky$.
(c) Find a value of k such that $y = f(x)g(x)$ satisfies the equation $y' = ky$.
(d) If $h = f/g$, find $h'(x)$. Make a conjecture about the relationship between f and g.

3. (a) Apply the product rule (Theorem 2.4.1) twice to show that if f, g, and h are differentiable functions, then $f \cdot g \cdot h$ is differentiable and

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

(b) Suppose that f, g, h, and k are differentiable functions. Derive a formula for $(f \cdot g \cdot h \cdot k)'$.
(c) Based on the result in part (a), make a conjecture about a formula differentiating a product of n functions. Prove your formula using induction.

4. (a) Apply the quotient rule (Theorem 2.4.2) twice to show that if f, g, and h are differentiable functions, then $(f/g)/h$ is differentiable where it is defined and

$$[(f/g)/h]' = \frac{f' \cdot g \cdot h - f \cdot g' \cdot h - f \cdot g \cdot h'}{g^2 h^2}$$

(b) Derive the derivative formula of part (a) by first simplifying $(f/g)/h$ and then applying the quotient and product rules.
(c) Apply the quotient rule (Theorem 2.4.2) twice to derive a formula for $[f/(g/h)]'$.
(d) Derive the derivative formula of part (c) by first simplifying $f/(g/h)$ and then applying the quotient and product rules.

5. Assume that $h(x) = f(x)/g(x)$ is differentiable. Derive the quotient rule formula for $h'(x)$ (Theorem 2.4.2) in two ways:
(a) Write $h(x) = f(x) \cdot [g(x)]^{-1}$ and use the product and chain rules (Theorems 2.4.1 and 2.6.1) to differentiate h.
(b) Write $f(x) = h(x) \cdot g(x)$ and use the product rule to derive a formula for $h'(x)$.

EXPANDING THE CALCULUS HORIZON

To learn how derivatives can be used in the field of robotics, see the module entitled **Robotics** at:

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