MATHEMATICS

SECTION A

February 2, 2024

1 Vectors

1. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle.
2. If $\mathbf{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\mathbf{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, then express $\mathbf{b}$ in the form of $\mathbf{b} = \mathbf{b_1} + \mathbf{b_2}$, where $\mathbf{b_1}$ is parallel to $\mathbf{a}$ and $\mathbf{b_2}$ is perpendicular to $\mathbf{a}$.

2 Geometry

3. The x-coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find is z-coordinate.

3 Differentiation

4. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $\left[-\sqrt{3}, 0\right]$.
5. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.
6. If $e^y(x + 1 = 1)$, then show that $\frac{d^2y}{dx^2} = \frac{dy}{dx}^2$.
7. find the general solution of the differential equation

$$\frac{dy}{dx} - y = \sin x.$$

8. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$.

4 Integration

9. Find

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

10. Find:

$$\int \frac{dx}{5 - 8x - x^2}$$

11. Evaluate :

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

12. Evaluate :

$$\int_1^4 \left(\left|x-1\right| + \left|x-2\right| + \left|x-4\right|\right) dx$$

13. Find:

$$\int \frac{e^x dx}{(e^x-1)^2(e^x+2)}$$

14. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are $A\,(4,1)\,, B\,(6,6)\,, C\,(8,4)$

5 Functions

15. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f\,(x) = \begin{cases} \frac{(x+3)^2-36}{x-3} & , x \neq 3 \\ k & , x = 3 \end{cases}$$

16. Show the function $f\,(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$.

17. Consider $f : \mathbf{R} - \left\{-\frac{4}{3}\right\} \rightarrow R - \left\{\frac{4}{3}\right\}$ given by $f\,(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}\,(0)$ and x such that $f^{-1}\,(x) = 2$.

18. Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b)\,, (c, d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to $*$ on A.
    (i) Find the identity element in A.
    (ii) Find the invertible elements of A.

6 Matrices

19. If for any 2×2 square matrix A, $A\,(adjA) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, then write the value of $\left|A\right|$.

20. If A is a skew-symmetric matrix of order 3, then prove that $\det A = 0$.

21. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

22. Find matrix A such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

23. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, then find A^{-1} and hence solve the system of linear equations $2x-3y+5z = 11$, $3x+2y-4z = -5$ and $x + y - 2z = -3$.

7 Intersection of conics

24. Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

8 Discrete

25. The volume of a sphere is increasing at the rate of $8cm^3/s$. Find the rate at which its surface area is increasing when the radius of the sphere is $12cm$.

9 Probability

26. A die, whose faces are marked $1, 2, 3$ in red and $4, 5, 6$ in green, is tossed. Let A be the event ”number obtained is even” and b be the event ”number obtained is red”. Find if A and B are independent events.

27. There are 4 cards numbered $1, 3, 5$ and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

28. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

10 Algebra

29. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2 \cdot 5y + 5z = 2$.

30. If $\tan^{-1} \frac{x-3}{x-4} + \tan^{-1} \frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x.

31. find the coordinates of the point where the line through the points $(3, -4, 5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3)$, $(4, 2, -3)$ and $(0, 4, 3)$.

32. A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$.

33. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is $10m$. Find the dimensions of the window to admit maximum light through the whole opening.

11 Optimization

34. Two tailors, A and B, earn ₹300 and ₹400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

35. Solve the following linear programming problem graphically :

Maximise $Z = 7x + 10y$

subject to the constraints

$$
\begin{aligned}
&4x + 6y \leq 240 \\
&6x + 3y \leq 240 \\
&x \geq 10 \\
&x \geq 0, y \geq 0
\end{aligned}
$$