TER
Clustering of inertial particles in turbulence

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Acknowledgements

Acknowledgements

Abstract

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1 Introduction

Particles are considered to be randomly arranged in a cube with a fluid in a Poisson distribution. Then a turbulence is applied to the fluid. After a given time and depending on the parameters, we can observe void areas and areas where the particles are grouped together. The different parameters used are the Reynolds number that characterizes the turbulence, and the Stokes number that characterizes the ability of a particle to follow the movement of the fluid. The purpose of this TER is to quantify the void areas and to characterize the clustering using a Voronoi Tesselation for different Reynolds number and Stokes number values. We will be based on data and the scientific articles of our research tutors.

2 Study of initial data

This section is intended to study how the data produced by our research tutors was produced.

2.1 Fluid mechanics

The droplet arrangement in this cube depends on different physical factors, which we will define, mainly the Stokes and Reynolds number.

Definition 1. *Reynolds number*
The Reynolds number characterizes a flow, in particular the nature of its regime (laminar, transitional, turbulent) and therefore its degree of turbulence. Thus, for a strong turbulence, the Reynolds number will be large. The Reynolds number is defined as such:

$$Re = \frac{V \times L}{\nu}$$

where :
V is the velocity of the fluid with respect to the object (m/s)
L is a characteristic linear dimension (m)
ν is the kinematic viscosity of the fluid (m^2/s)

Definition 2. *Time constant*
In physics, a time constant, noted τ, is a quantity that characterizes the speed of the evolution of a physical quantity over time, especially when this evolution is exponential.

Definition 3. *Viscous time*
Time $\tau_{viscous}$ is the characteristic duration of the exponential decrease in the velocity of a particle subjected to viscous friction.

$$\tau_{viscous} = \frac{\rho \times d^2}{18 \times \mu}$$

where :
ρ is the density of the particle
d is the characteristic length of the particle
μ is the dynamic viscosity of the fluid

Definition 4. *Inertia time*
Time $\tau_{inertia}$ is the characteristic duration of inertia.

$$\tau_{inertia} = \frac{L}{v}$$

where :
v is the fluid velocity
L is the characteristic lengt

Definition 5. *Stokes number*
The Stokes number is used to study the behaviour of a particle in a fluid. It represents the ratio between the kinetic energy of the particle and the energy dissipated by friction with the fluid. Thus, the Stokes number characterizes the inertia of a particle in a fluid. The more inertia a particle has, the greater the Stokes number will be. If $St = 0$ the particle have no mass. The Stokes number is defined as such:

$$St = \frac{\tau_{viscous}}{\tau_{inertia}}$$

There are two different regimes:
- The viscous regime ($St < 1$): the particles follow the movement of the fluid.
- The inertial regime ($St > 1$):the particles are entrained by their inertia and their trajectory is not very influenced by the movement of the fluid.

Definition 6. *Navier-Stokes equations*
The governing equations of turbulent airflow are the continuity and Navier-Stokes equations for three dimensional incompressible flows:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho_a}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2} + F_i$$

where :
u_i is the fluid velocity in the ith direction
ρ_a is the air density
p is the pressure
ν is the kinematic viscosity
F_i is the external forcing term.

2.2 Conditions of the experiment

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!! A REVOIR !!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
difference between Re_λ and Re ???

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho_a}\nabla p + \nu\nabla^2 u$$

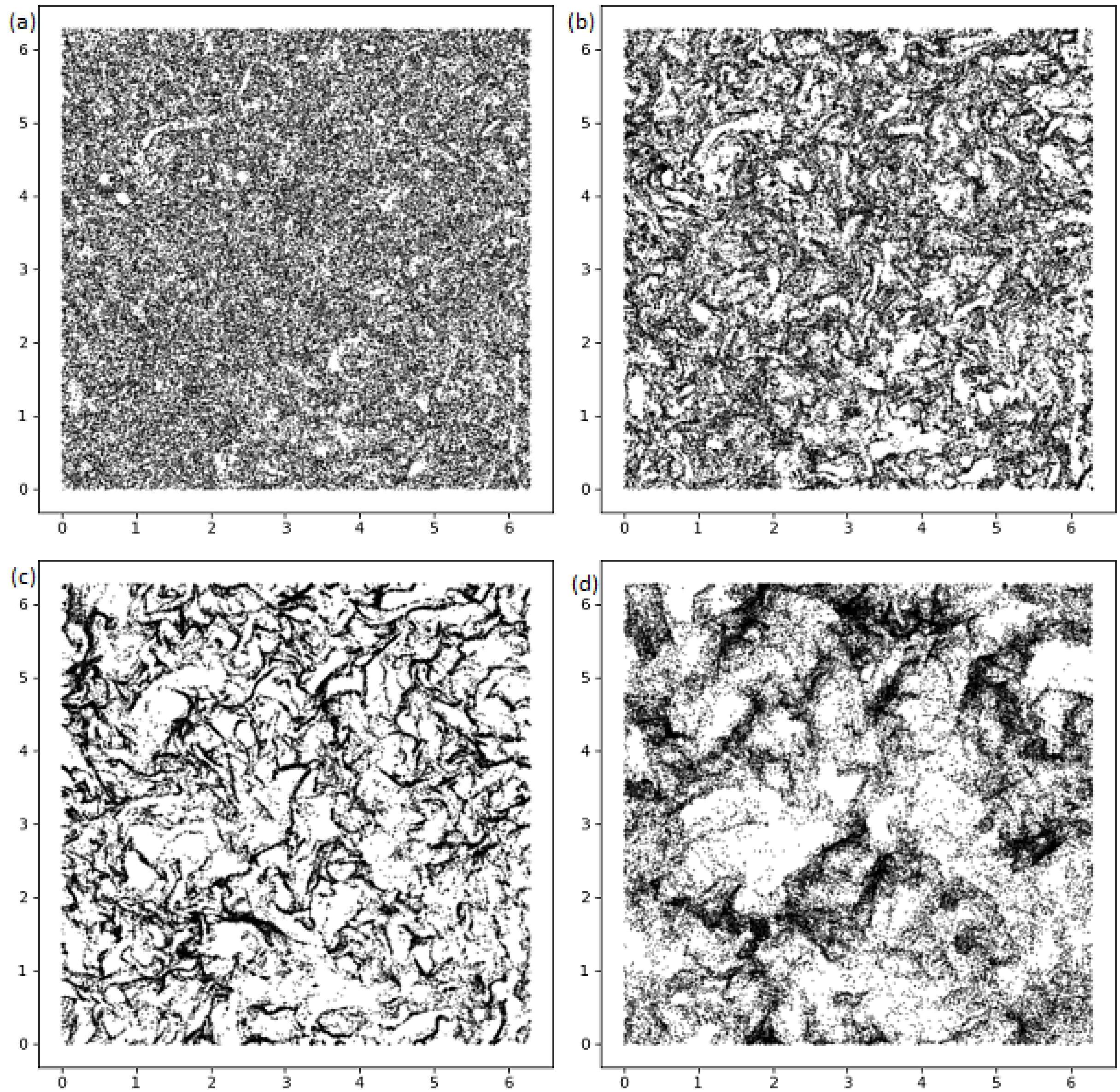
$$\frac{\partial x_i}{\partial t} = v_i \qquad\qquad \frac{\partial v_i}{\partial t} = -\frac{\partial v_i - u_i}{\tau_p}$$

In this simulation we have :

$$Re_\lambda = \frac{l_\lambda u_{rms}}{\nu} \qquad\qquad St = \frac{\tau_p}{\tau_\eta}$$

where :
l_λ is Taylor microscale
u_{rms} is RMS value of velocity fluctuation
ν is kinematic viscosity
τ_p is relaxation time of droplet motion
τ_η is Kolmogorov time

The simulation is done in a cube with edges of length $2\pi L_0$ where L_0 is the representative length scale and with the 3 pairs of opposite faces glued, so this is a 3-torus. The medium is considered isotropic, i. e. it has the same properties in all directions. The data we will study were simulated in an environment without gravity in order not to alter the isotropic nature of the environment. We take a large number of inertial particles, i.e. with a higher density than the fluid, with a random spatial distribution that follows a Poisson probability distribution function (PDF) and consider them as Stokes particles. Collisions between particles will be neglected. Simulations for values from St=0.05, 0.2, 0.5, 1.0, 2.0 and 5.0 at Re=204 were done.



Spatial distribution of dropets for $St = (a)0.005, (b)0.02, (c)1, (d)5$ *at* $Re_\lambda = 204$

For St=1.0 we can clearly obsered void areas. For St<1.0, the void areas are less clear. For St>1.0 they are larger but less clear than St=1.0. We can ask ourselves how to quantify the clustering in fonction of the Stokes and Reynolds number.

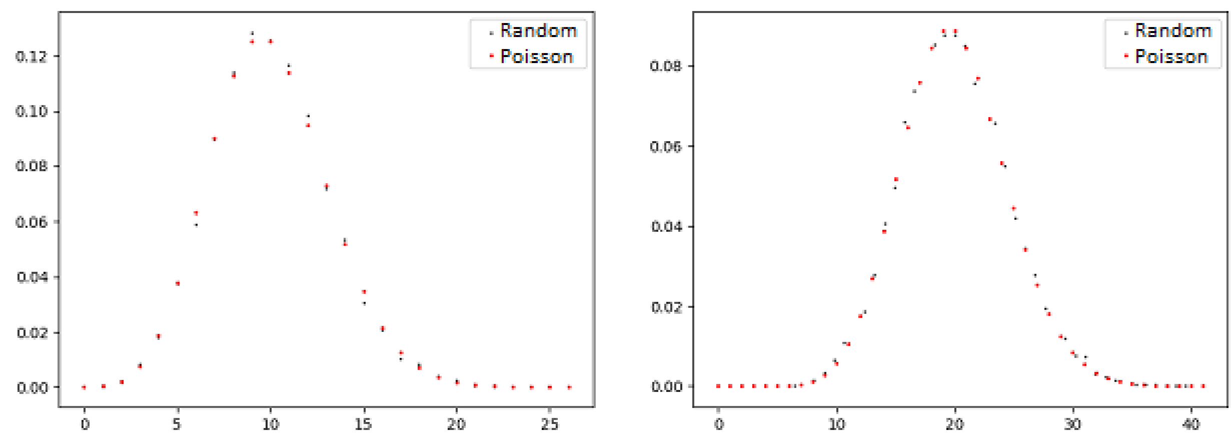
3 Clustering study

Afterwards we will take a $\frac{2*\pi}{200}$ size range of the studied data. We will see two methods to quantify the clustering to highlight the influence of the Stokes and Reynolds number. Firstly, we will compute the density by cutting the area in boxes and counting the number of particles per cell. Secondly, we will use the Voronoi diagram approach, indeed the more the particles are grouped, the smaller the area of their cell will be.

3.1 Density

3.1.1 Before turbulence

We will calculate the particle density following an uniform distribution knowing that this is the initial condition before the turbulence.



To the letf (respectively right), in red a Poisson density of parameter $\lambda = 10$ ($\lambda = 20$), in black the density of 100000 points (200000 points) with 100×100 subdivisions.

Propertie 1. *The density of an uniform distribution follow a Poisson distribution.*

$$B(n,p) = P(\lambda) \text{ with } \lambda = np$$

Proof 1. *The density of an uniform distribution follow a binomial distribution because we independently repeat several identical random experiments. Let X following binomial distribution of parameter n and p.*
Let $\lambda = np$

$$\begin{aligned}
\mathbb{P}(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \frac{n!}{(n-k)!} \frac{1}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} \quad = \quad \frac{\lambda^k}{k!} \frac{n!}{(n-k)!} \frac{1}{n^n} (n-k)^{n-k}
\end{aligned}$$

Stirling's approximation give us :

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$\frac{n!}{(n-k)!} \approx \sqrt{\frac{n}{n-k}} \frac{n^n}{(n-k)^{n-k}} e^{-k}$$

$$\lim_{n\to+\infty} \frac{\lambda^k}{k!}\frac{n!}{(n-k)!}\frac{1}{n^n}(n-k)^{n-k} = \lim_{n\to+\infty} \frac{\lambda^k}{k!}\sqrt{\frac{n}{n-k}}\frac{n^n}{(n-k)^{n-k}}e^{-k}\frac{1}{n^n}(n-k)^{n-k}$$
$$= \lim_{n\to+\infty} \frac{\lambda^k}{k!}\left(\frac{n-\lambda}{n-k}\right)^{n-k} e^{-k}$$

$$\left(\frac{n-\lambda}{n-k}\right)^{n-k} = e^{(n-k)ln\left(\frac{n-\lambda}{n-k}\right)} and\ we\ know\ that\ ln(1+x) \approx x\ when\ x \to 0$$

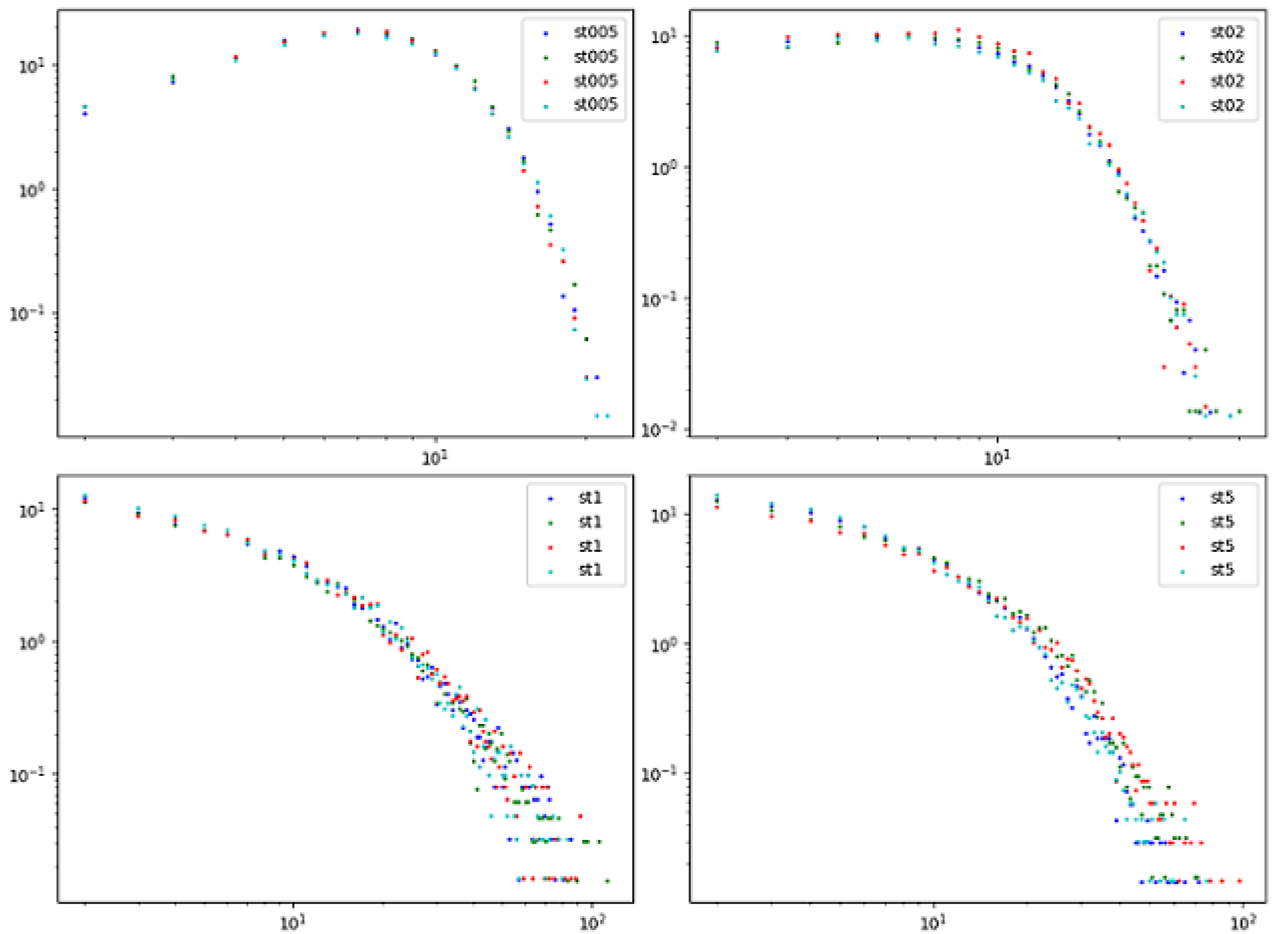
$$So\ \lim_{n\to+\infty} ln\left(\frac{n-\lambda}{n-k}\right) = \frac{n-\lambda}{n-k} - 1 = \frac{k-\lambda}{n-k}\ and\ \lim_{n\to+\infty} e^{(n-k)ln\left(\frac{n-\lambda}{n-k}\right)} = e^{\lambda+k}$$

And $\lim_{n\to+\infty} \frac{\lambda^k}{k!}\left(\frac{n-\lambda}{n-k}\right)^{n-k} e^{-k} = \frac{\lambda^k}{k!}e^{-\lambda}$
We can conclude that :

$$B(n,p) = P(np) = P(\lambda)$$

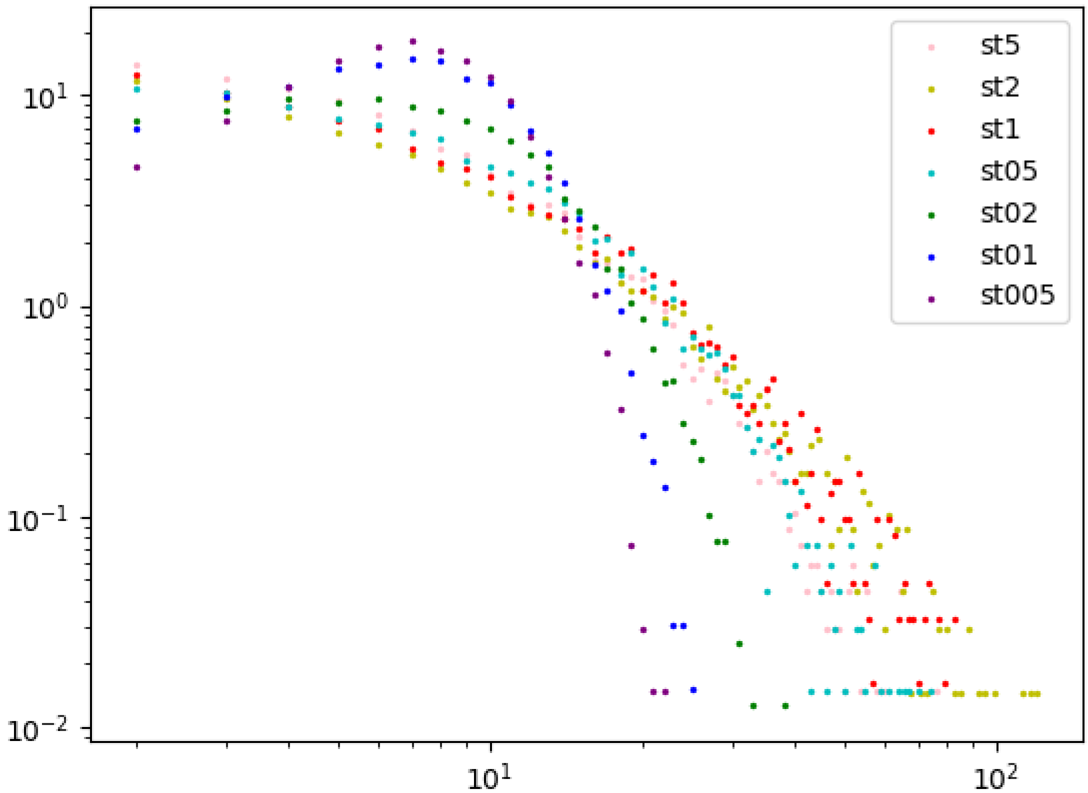
3.1.2 After turbulence

In this subsection we will calculate the density after the turbulence for differente Stokes number and then for different Reynolds number.



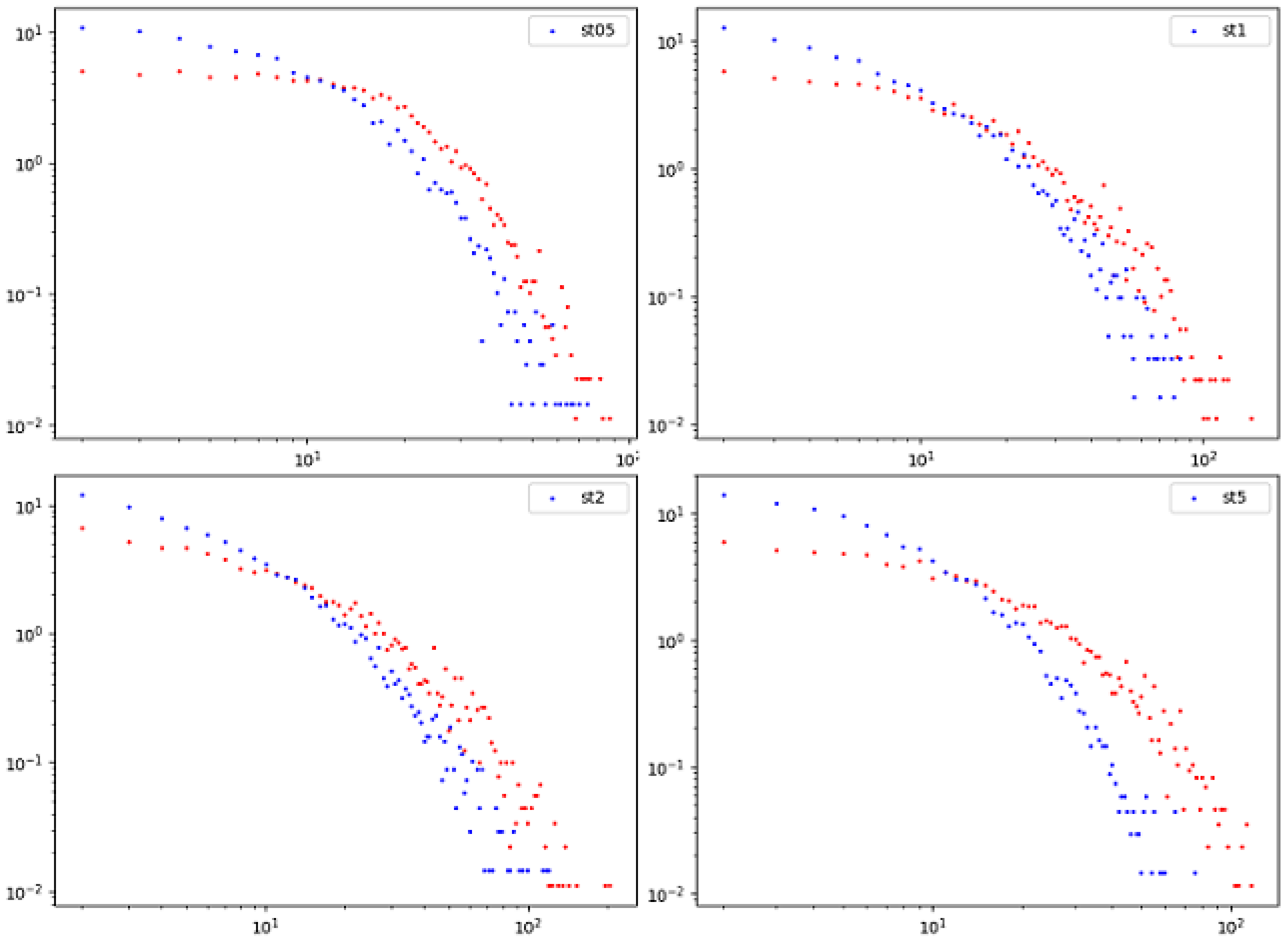
Different density at different times.

The density for a same Stokes number at different times are stable.



Histogram of the density of particles using a logarithmic scale.

As expected nearest St are to 1, more there are dens and void areas.



Different density for Re=204 in blue and Re=328 in red for ST=0.05,1,2 and 5.

When the turbulence become stronger, there are less void area but there are more hight density area.

3.2 Voronoi diagram approach

The Voronoi diagram approach are intersting because the area of cells depend to the position of particules contrary to the density approche whitch depend to the dimmention of boxes.

3.2.1 Voronoi diagram construction

A Voronoi diagram is a paving of the plane built from a finite number of points called sites or germs. For each germ p_i, a Voronoi cell is the group of points of the plane that are closer to the germ p_i than to all the other germs in the plane. The paving of the plane by Voronoi cells is called a Voronoi diagram. This diagram is named after the Russian mathematician Georgi Fedoseevich Voronoi (1868 - 1908). Voronoi diagrams are used in many disciplines and have many applications (robotics, biology, plant growth, medical imaging...). A Voronoi cell delimits the area of influence of a point. Note that Voronoi diagrams can be generalized to other spaces and norm.

General definitions

We suppose known the coordinates of a set of points $P = \{p_i, 1 \leq i \leq n\}$ of $\mathbb{R}^d$.

Definition 7. *Voronoi cells*
We call Voronoi cell of the point $p_i \in P$, which is noted C_i, all the points of the space closer to p_i than all the other points of P : $C_i = \{q \in \mathbb{E}, \forall j, ||qp_i|| \leq ||qp_j||\}$. The point p_i associated with the cell C_i is called the germ of this cell.

Definition 8. *Voronoi diagrams*
We call Voronoi diagram of the set P, the cutting of the space in cell C_i associated to the points.

Definition 9. *Convex set*
A set C in $\mathbb{R}^d$ is said to be convex if, $\forall (a, b) \in C \times C, \forall t \in [0, 1], ta + (1 - t)b \in C$

Afterwards we will use the Euclidean norm.

Voronoi diagram in $\mathbb{R}^2$

Definition 10. *Voronoi vertex and edge:*
The intersection of two Voronoi cells is empty or equal to a segment, a half right, or a right. In the latter three cases, this intersection will be called Voronoi edge. The intersection of two Voronoi edge, if it is not empty, is called the Voronoi vertex.

A Voronoi diagram shows the following properties :

Propertie 2. *A Voronoi edge, separating two Voronoi cells C_i and C_j, is the perpendicular bisector p_ip_j.*

Proof 2. *All points on this Voronoi edge are at equal distance of p_i and p_j. So they are on the perpendicular bisector p_ip_j.*

Propertie 3. *The Voronoi vertex common to three cells C_i, C_j and C_k is the centre of the circumscribed circle to the triangle of vertices p_i, p_j and p_k.*

Proof 3. *The intersection points of two Voronoi edges are on the perpendicular bisector p_ip_j and p_jp_k, so it is the center of the circumscribed circle.*

Note 1. *If the points p_i, p_j and p_k are aligned, the perpendicular bisector p_ip_j and p_ip_k are parallel. The Voronoi vertex does not exist.*

Propertie 4. *A Voronoi diagram is a convex subdivision of plane. A bounded Voronoi cell is a polygon.*

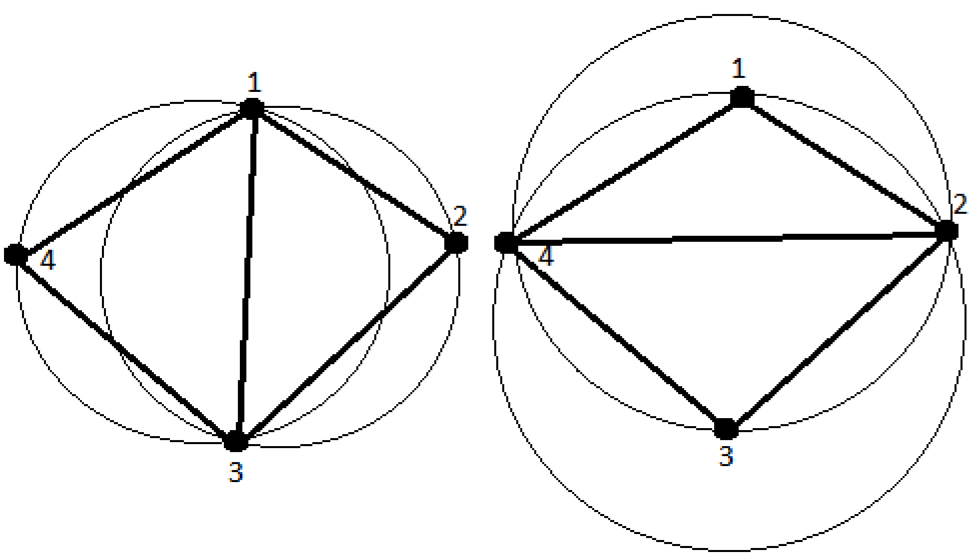
Proof 4. *C_i is the intersection of a finite number of half-plane, so it is a convex region. The border is made up of a series of Voronoi edges and Voronoi vertex. If C_i is bounded, its boundary is closed; C_i is therefore a convex polygon.*

Delaunay triangulation in $\mathbb{R}^2$

Like the Voronoi diagrams, Delaunay triangulation is a division of the plane into cells associated with points $p_i \in P$. The Delaunay triangulation of a set of points $p_i \in P$ of the plane is a triangulation T such that no point of P is inside the circumscribed circle of one of the triangles of T. This triangulation was invented by the Russian mathematician Boris Delaunay (1890 - 1980) in an article published in 1934.

Definition 11. *Delaunay triangle*
We call Delaunay triangle, a triangle noted D_i, which has as vertex three of the germs $p_a, p_b, p_c \in P$ and such that its circumscribed circle has no germs inside it.

Definition 12. *Delaunay triangulation*
We call Delaunay triangulation of the set P, the cutting of the plane in cell D_i, i.e. all the Delaunay triangles associated with the points P.



Here are two examples of triangulation, the one on the left is a Delaunay triangulation, the one on the right is not because point 3 is inside the circle circumscribed to the triangle formed by points 1,2,4.

From Delaunay to Voronoi in $\mathbb{R}^d$

Definition 13. $d-Simplex$
A simplex is the convex hull of a set of $(d+1)$ points used to form an affine coordinate system in an affine space of dimension d.

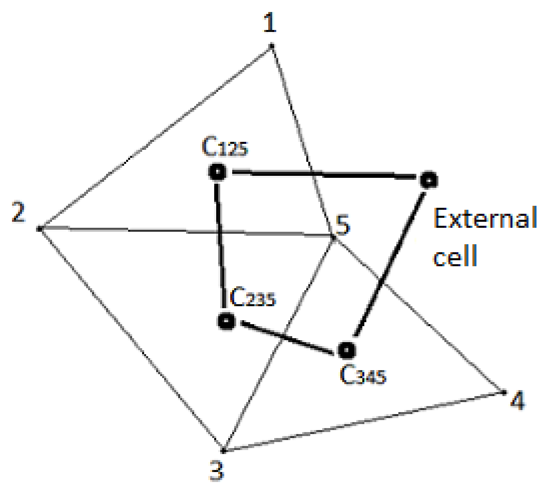
Definition 14. *Open $d-Ball$*
The open $d-Ball$ of centre c_0 and radius r noted $B^d(c_0, r)$ is define as :

$$B^d(c_0, r) := \{p \in \mathbb{R}^d \text{ with } ||c_0 - p|| \leq r\}$$

Definition 15. *Delaunay cells*
We call Delaunay cell, a simplex noted D_i, which has for vertices $(d+1)$ germs such as there are no other germ inside the open $d-Ball$ generated the this vertices.

Definition 16. *Undirected graph*
An undirected graph G is a pair $(V; E)$, where V is a set and E is a symmetrical binary relation define on V, i.e. $(u, v) \in E \Longrightarrow (v, u) \in E$. The elements of V are called vertices and the elements of E are called the edges of G.

Definition 17. *Dual graph*
Tthe dual graph G' of G, is a graph whose vertices are the faces of the previous graph (including the outer region), and whose edges are the edges of the previous graph, each edge connecting the two faces it borders.



Exemple in 2D of a graphe and this dual.

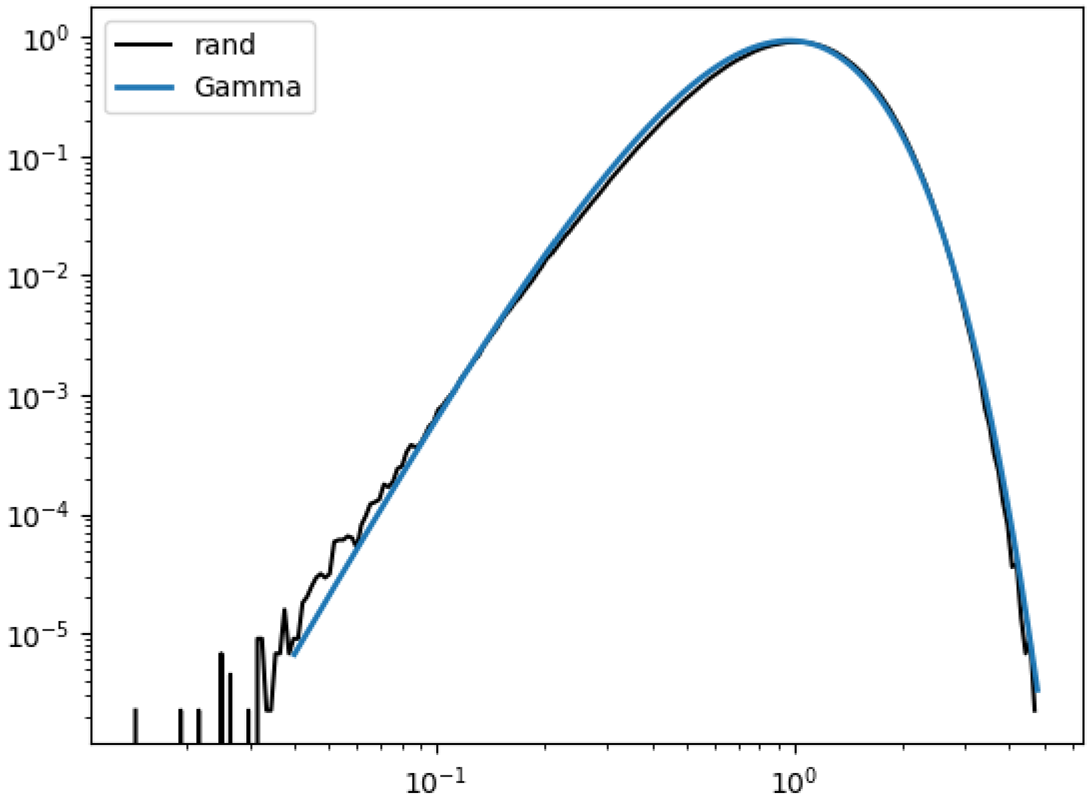
Definition 18. *Voronoi diagrams*
For a set of points P, the dual of the Delaunay triangulation is the Voronoi diagram.

Definition 19. *Voronoi cells*
We can deduct that the Voronoi cell C_i is the convex hull of all the centers of the circumscribed circles generated by the simplexes of the germ p_i.

The algorithm of Guibas and Stolfi where be used. This algorithm is based on the principle of divide-and-conquer, which allows us to have an algorithmic complexity of $O(nln(n))$, which will allow us to work on a large number of data.

3.2.2 Before turbulence

We will calculate the area of Voronoi cells following an uniform distribution knowing that this is the initial condition before the turbulence.



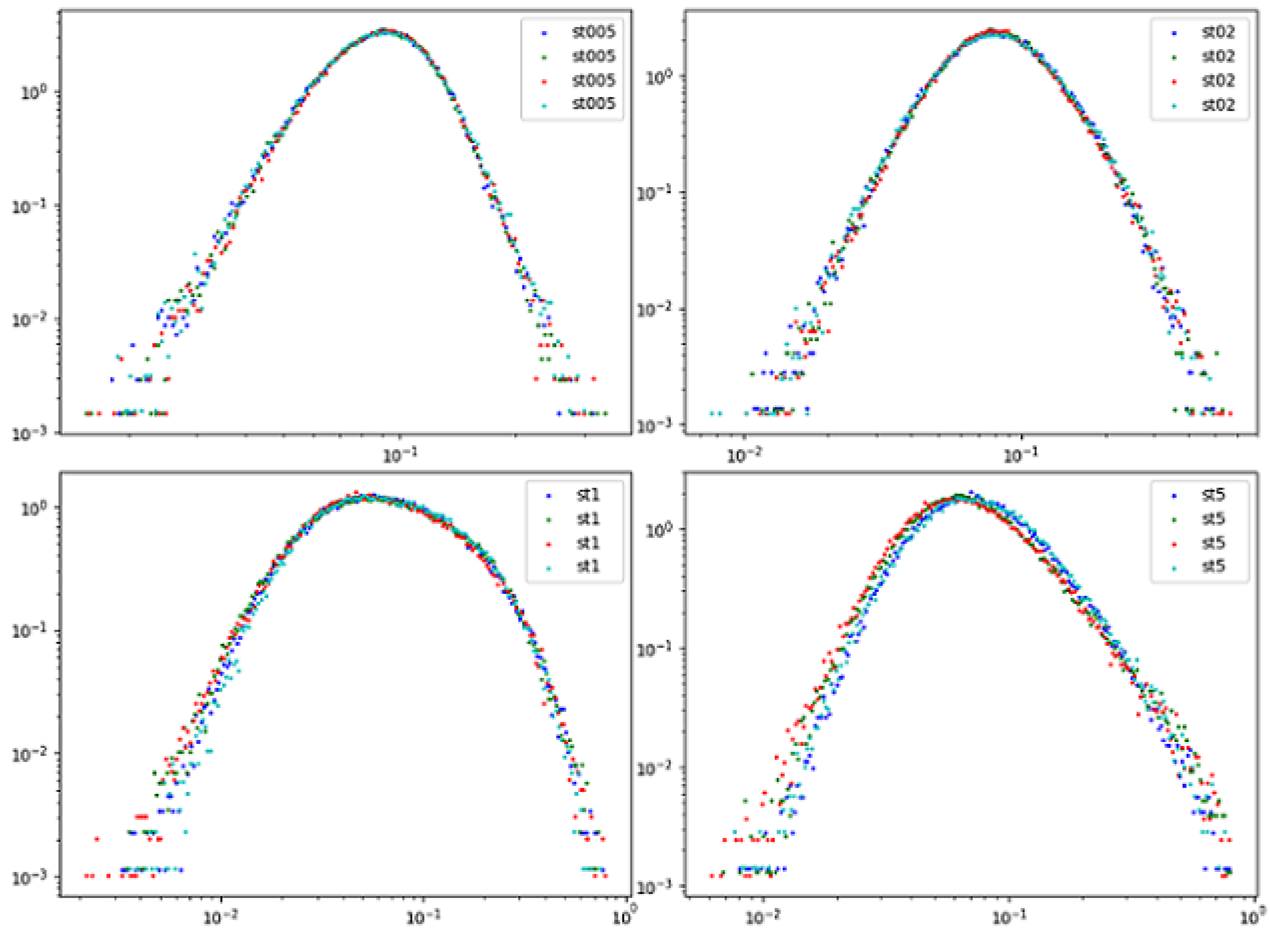
Area of Voronoi cell normaled by the mean for an uniforme distribution and an approche by a Gamma distribution.

Propertie 5. *The area of Voronoi cells following an uniform distribution follow a gamma distribution.*

Proof 5. *In 1D*
demonstration

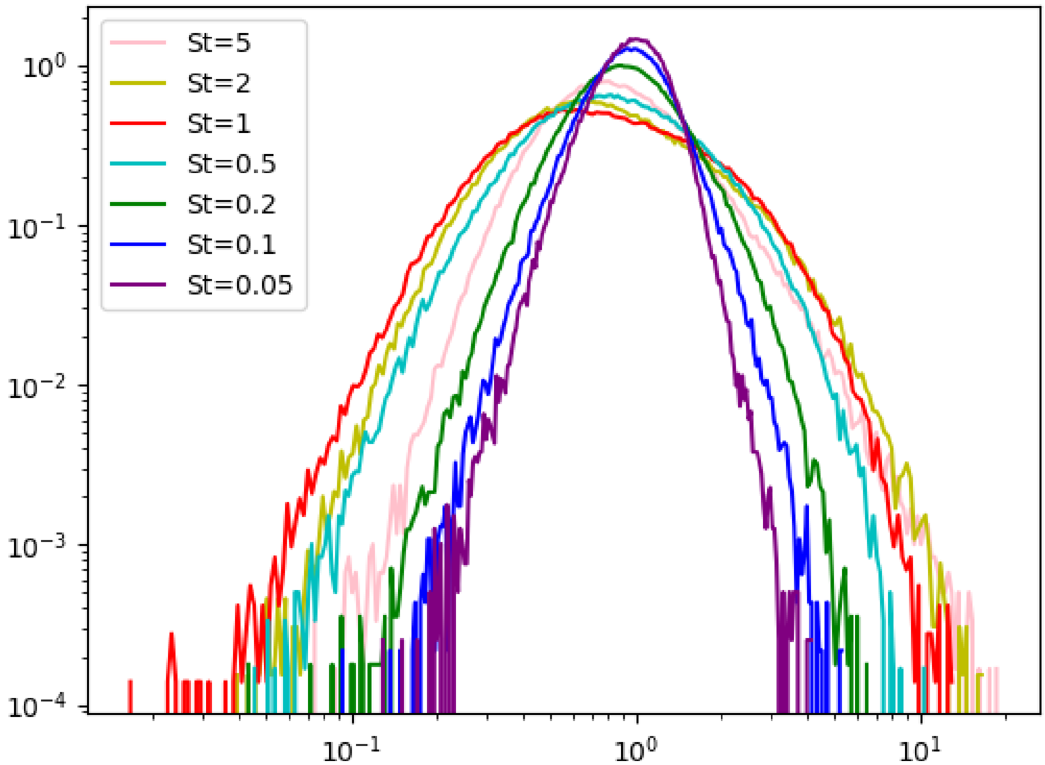
3.2.3 After turbulence

In this subsection we will calculate the area of Voronoi cells after the turbulence for differente Stokes number and then for different Reynolds number. For this section when we speak about standard deviation or mean we speak about the standard deviation an the mean of the logarithme of the area.



Different area of Voronoi cell at different times.

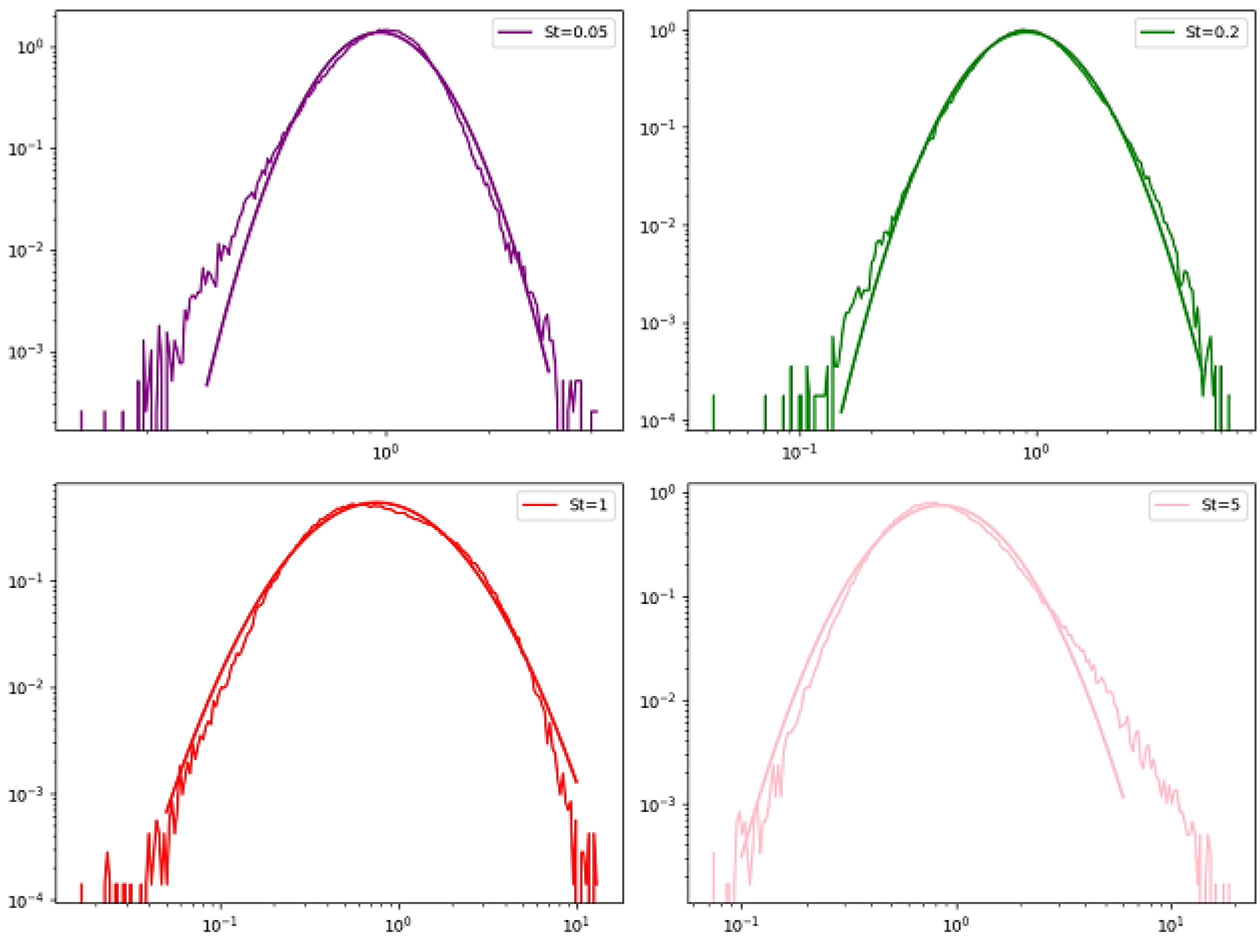
As the density the area of Voronoi cells are stable for a same Stokes number.



Histogram of the area of Voronoi cell using a logarithmic scale normaled by the mean.

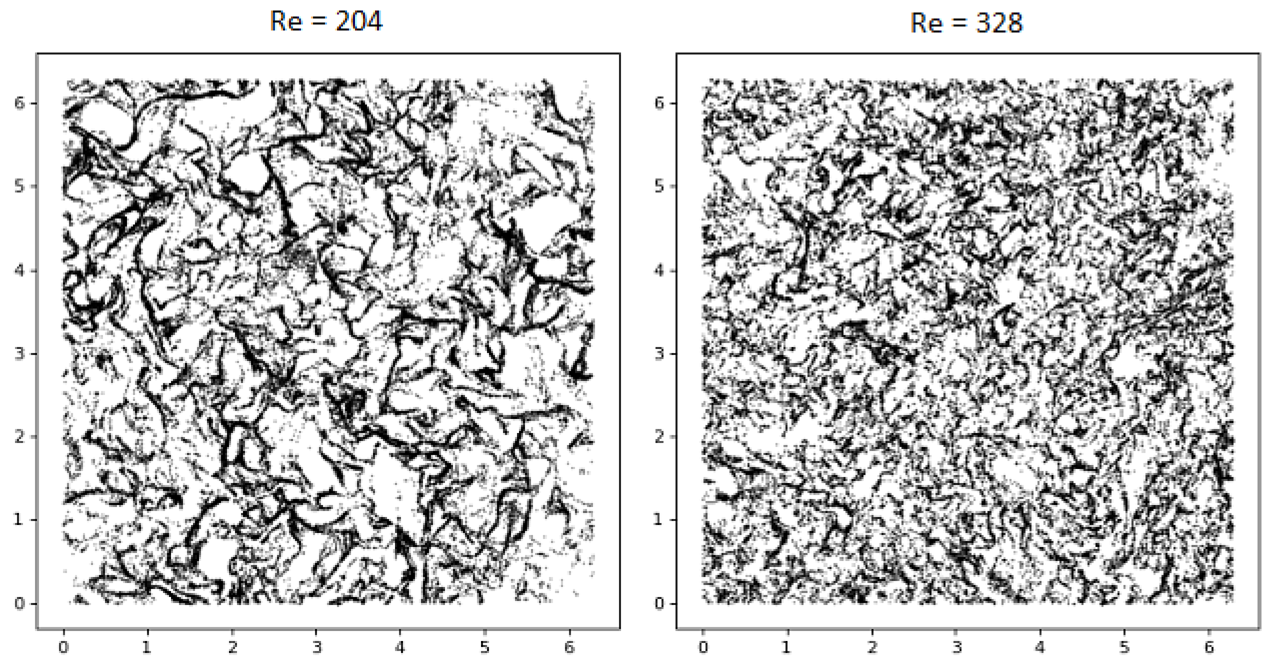
As expected when the Stokes number are weak the standard deviation are small, and nearest Stokes number are to 1 more the standard deviation are. It should be noted that for $St = 5$ the standard deviation become again small. We can think that if we do a simulation for an higher Stokes number we will found a small the standard deviation as when the Stokes number a weak.

We can try to approche the PDF with Log-normal distribution.

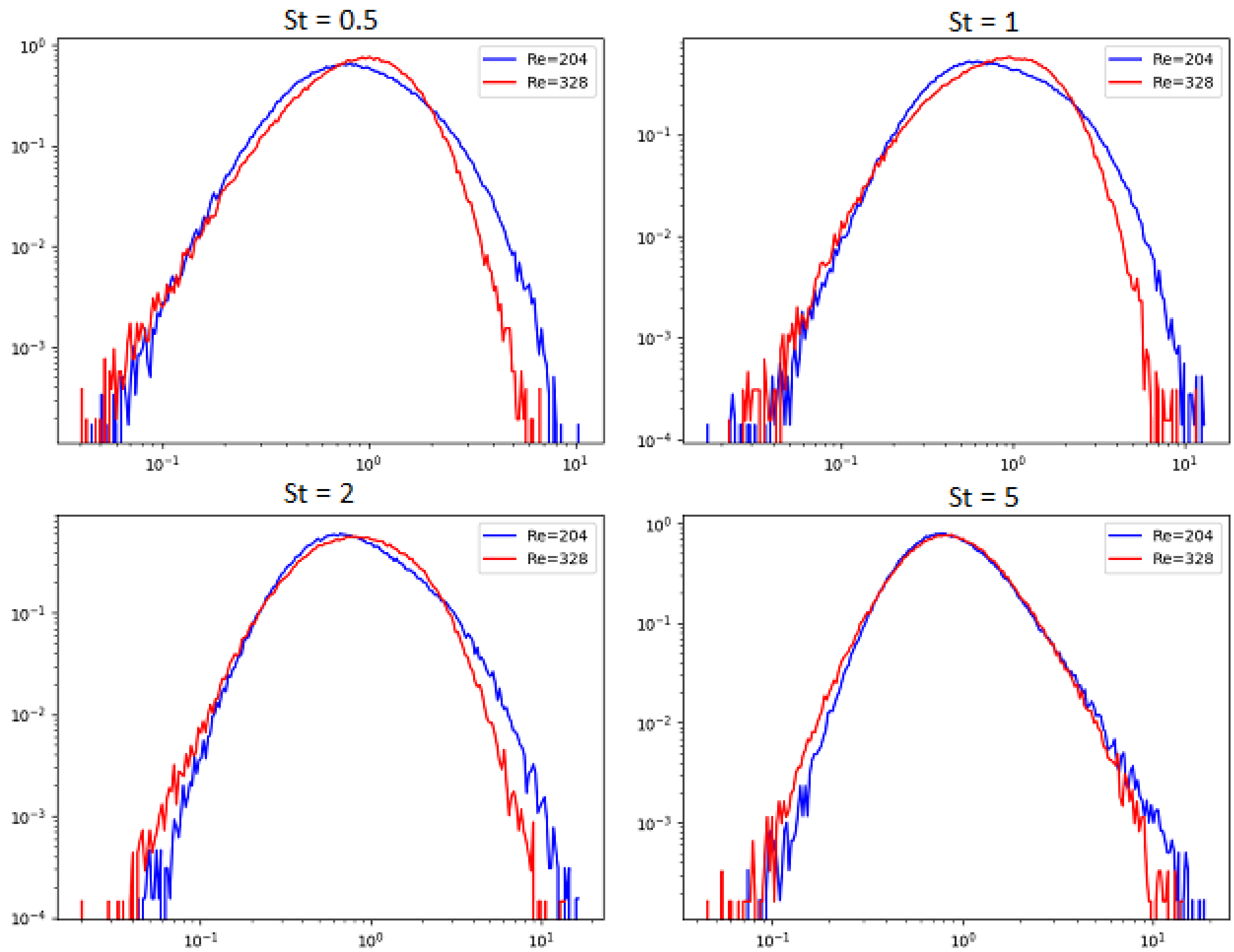


Different area of Voronoi cell for Re=204 and the Log-normal distribution associate.

We can conlude the the area of Voronoi cells clearly don't follow a Log-normal distribution even if the Log-normal distribution is preaty near to the PDF when the Stokes number are near to 1. This aspect is much more visible on 3D data (q.v. annexe).



Spatial distribution of dropets for $St = 1$ *at* $Re_\lambda = 204$ *and* $Re_\lambda = 328$



Different area of Voronoi cell for Re=204 in blue and Re=328 in red for ST=0.05,1,2 and 5.

We can see that Reynolds number have an influence on number of large area of Voronoi cells.