Notes on conservative, bounded, advection schemes

Colin Cotter

January 7, 2023

As described in [1], the conservation of a tracer q can be most easily derived from the conservative form of the advection equations

$$(\rho q)_t + \nabla \cdot (\boldsymbol{u}\rho q) = 0, \tag{1}$$

where the density satisfies the continuity equation

$$\rho_t + \nabla \cdot (\boldsymbol{u}\rho) = 0. \tag{2}$$

On the other hand, the L_∞ bounds for q are most easily derived from the characteristic form

$$q_t + \boldsymbol{u} \cdot \nabla q = 0. \tag{3}$$

1 Continuity equation

The first step is that we need a positive scheme for ρ, *i.e.*, $\rho(x,t) > 0$ if $\rho(x,0) > 0$.

The time-continuous discontinuous Galerkin (DG) scheme for $\rho \in DG(n)$ is defined on an element e with boundary ∂e by

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_e \phi\rho \,\mathrm{d}\,x - \int_e \nabla\phi \cdot \boldsymbol{u}\rho \,\mathrm{d}\,x + \int_{\partial e} \phi \boldsymbol{u}\cdot\boldsymbol{n}\tilde{\rho}\,\mathrm{d}\,s = 0, \quad \forall \phi \in DG(n), \tag{4}$$

where $\boldsymbol{n}$ is the normal to the boundary, and $\tilde{\rho}$ is the value of ρ on the upwind side of ∂e.

Here we consider the forward Euler time discretisation, given by

$$\int_e \phi\rho^{n+1} \,\mathrm{d}\,x = \int_e \phi\rho^n \,\mathrm{d}\,x + \Delta t \int_e \nabla\phi \cdot \boldsymbol{u}\rho^n \,\mathrm{d}\,x - \Delta t \int_{\partial e} \phi \boldsymbol{u}\cdot\boldsymbol{n}\tilde{\rho}^n\,\mathrm{d}\,s = 0, \quad \forall \phi \in DG(n). \tag{5}$$

Forward Euler steps can be concatenated in SSPRK schemes to obtain higher-order time discretisations.

The above linear scheme is not positivity-preserving. To obtain a positivity-preserving scheme, we take the following steps:

1. A pre-processing limiter applied to ρ, such as the Barth-Jespersen limiter or the Kuzmin vertex-based limiter, that sets bounds $(\rho_{\min}, \rho_{\max})$ on each element e, based on the mean value $\overline{\rho}^n_e$, defined by

$$\overline{\rho}^n = \frac{1}{|e|}\int_e \rho^n \,\mathrm{d}\,x, \tag{6}$$

   and the value in selected surrounding elements. The limiter is then defined by

$$\Pi_1 \rho^n|_e = \overline{\rho}^n + \alpha(\rho^n - \overline{\rho}^n), \tag{7}$$

   with $0 \le \alpha \le 1$ chosen as the maximum value such that $\Pi\rho^n$ lies within the bounds in element e. Note that such a limiter does not alter the element integral of ρ, and so does not violate conservation.

   If the element mean values of ρ^n are all positive, then so are the bounds, and therefore $\Pi\rho^n$ is positive everywhere.

2. A second pre-processing limiter Π_2 to guarantee that the element means $\overline{\rho}^{n+1}$ will all be positive following the Euler step. The goal of this section is to introduce such a limiter.

3. The application of the forward Euler step,
$$\int_e \phi\rho^{n+1}\,\mathrm{d}\,x = \int_e \phi\rho^*\,\mathrm{d}\,x + \Delta t\int_e \nabla\phi\cdot\boldsymbol{u}\rho^*\,\mathrm{d}\,x - \Delta t\int_{\partial e}\phi\boldsymbol{u}\cdot\boldsymbol{n}\rho^*\,\mathrm{d}\,s = 0, \quad \forall\phi\in DG(n), \tag{8}$$
where $\rho^* = \Pi_2\hat{\rho} = \Pi_2\Pi_1\rho^n$.

We now develop the limiter Π_2.

If we choose ϕ as the indicator function for element e, *i.e.*,
$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x}\in e,\\ 0 & \text{otherwise,}\end{cases} \tag{9}$$
then Equation (8) becomes
$$\int_e \rho^{n+1}\,\mathrm{d}\,x = \int_e \rho^*\,\mathrm{d}\,x - \Delta t\int_{\partial e}\boldsymbol{u}\cdot\boldsymbol{n}\rho^*\,\mathrm{d}\,s = 0. \tag{10}$$
This may be written in the form
$$\overline{\rho}^{n+1} = \overline{\rho}^n\left(1 + c^- - c^+\right), \tag{11}$$
where
$$c^\pm = \pm\frac{\Delta t}{|e|}\int_{\partial e^\pm}\boldsymbol{u}\cdot\boldsymbol{n}\frac{\tilde{\rho}^*}{\overline{\rho}^n}\,\mathrm{d}\,s, \tag{12}$$
where ∂e^+, ∂e^- are the parts of ∂e where $\boldsymbol{u}\cdot\boldsymbol{n}$ is positive and negative respectively. Both c^+ and c^- are positive if ρ is positive everywhere.

For $\overline{\rho}^{n+1}$ to be positive, we need
$$c^+ < 1 + c^-. \tag{13}$$
This will always be satisfied for sufficiently small Δt. However, it may be violated if there are dramatic jumps in ρ, even for moderate values of the upwind/downwind Courant numbers
$$\tilde{c}^\pm = \pm\frac{\Delta t}{|e|}\int_{\partial e^\pm}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\,s. \tag{14}$$
Hence, we use a failsafe limiter to adjust the slope in each element where Equation (13) is not satisfied, to ensure positivity.

Since c^- depends on values of ρ from outside element e, it will not be altered by limiting the slope in element e. Hence, if the positivity condition (13) is not satisfied, applying a limiter
$$\rho^* = \overline{\tilde{\rho}} + \beta\left(\tilde{\rho} - \overline{\tilde{\rho}}\right), \tag{15}$$
where $0\le\beta\le 1$, results in replacing c^+ by
$$\tilde{c}^+ + \beta(c^+ - \tilde{c}^+). \tag{16}$$
Hence, the positivity condition becomes
$$\tilde{c}^+ + \beta(c^+ - \tilde{c}^+) < 1 + c^-. \tag{17}$$
It is always possible to achieve this condition by setting $\beta = 0$, in which case the condition becomes
$$\tilde{c}^+ < 1 + c^-, \tag{18}$$
for which the CFL condition
$$\tilde{c}^+ < 1, \tag{19}$$
is sufficient. Hence, we choose $\beta\le 1$ to be as large as possible provided that Equation (17) is satisfied, *i.e.*,
$$\beta = \left\{\ \max\left(0, \min\left(1, \tfrac{1+c^- - \tilde{c}^+}{c^+ - \tilde{c}^+}\right)\right)\right. \quad . \tag{20}$$

2 Reconstruction of flux

If the advection equation for the tracer q wish to be solved provided the conti nuity equation is solved for density ρ, the flux in the advection equation needs to be reconstructed. following from the paper NEED CITATION HERE

3 Bounded advection

The forward Euler conservative DG discretisation for the transport equation for q is,

$$\int_e \phi\rho^{n+1}q^{n+1}\,\mathrm{d}\,x = \int_e \phi\rho^n q^n\,\mathrm{d}\,x + \Delta t\int_e \nabla\phi\cdot\boldsymbol{F}^n q^n\,\mathrm{d}\,x - \Delta t\int_{\partial e}\phi\boldsymbol{F}^n\cdot\boldsymbol{n}\tilde{q}^n\,\mathrm{d}\,s = 0, \quad \forall\phi\in DG(n), \tag{21}$$

where $\tilde{q}^n$ is the value of q^n on the upwind side of ∂e.

Since this linear scheme does not guarantee bounds on q^{n+1}, we also apply limiters to ensure this, following the same two step strategy, first ensuring that all values of q lie between bounds defined from element mean values, then adjusting the slope so that the outflow values guarantee positive element mean values in the next timestep. We use ρ-weighted element mean values defined by

$$\overline{q} = \frac{1}{\overline{\rho^n}}\int_e \rho^n q\,\mathrm{d}\,x, \tag{22}$$

so that

$$\overline{\rho^n}\,\overline{q} = \int_e \rho^n q\,\mathrm{d}\,x. \tag{23}$$

In this case, replacing q by the limited function

$$\Pi_1 q = \overline{q} + \alpha(q-\overline{q}), \tag{24}$$

does not violate conservation, since

$$\int_e \rho^n\Pi_1 q\,\mathrm{d}\,x = \overline{q}\int_e \rho^n\,\mathrm{d}\,x + \alpha\left(\int_e \rho^n q\,\mathrm{d}\,x - \overline{q}\int_e \rho^n\,\mathrm{d}\,x\right), \tag{25}$$

$$= \overline{\rho^n}\,\overline{q} + \alpha\left(\overline{\rho^n}\,\overline{q} - \overline{\rho^n}\,\overline{q}\right) = \overline{\rho^n}\,\overline{q}, \tag{26}$$

as required.

The forward Euler timestep for q is then as follows:

1. A pre-processing limiter applied to q, such as the Barth-Jespersen limiter or the Kuzmin vertex-based limiter, that sets bounds $(q_{\min}, q_{\max})$ on each element e, based on the mean value $\overline{q}^n_e$, and the value in selected surrounding elements. The limiter is then defined by

   $$\Pi_1 q^n|_e = \overline{q}^n + \alpha(q^n - \overline{q}^n), \tag{27}$$

   with $0\le\alpha\le 1$ chosen as the maximum value such that Πq^n lies within the bounds in element e.

   If the element mean values of q^n are all positive, then so are the bounds, and therefore Πq^n is positive everywhere.

2. A second pre-processing limiter Π_2 to guarantee that the element means $\overline{q}^{n+1}$ will lie within their bounds at the following the Euler step. The goal of this section is to introduce such a limiter.

3. The application of the forward Euler step,

   $$\int_e \phi\rho^{n+1}q^{n+1}\,\mathrm{d}\,x = \int_e \phi\rho^n q^*\,\mathrm{d}\,x + \Delta t\int_e \nabla\phi\cdot\boldsymbol{u}\rho^n q^*\,\mathrm{d}\,x - \Delta t\int_{\partial e}\phi\boldsymbol{u}\cdot\boldsymbol{n}\rho^n q^*\,\mathrm{d}\,s = 0, \quad \forall\phi\in DG(n), \tag{28}$$

   where $q^* = \Pi_2\hat{q} = \Pi_2\Pi_1 q^n$.

We now develop the limiter Π_2 for q, assuming that q already satisfies the bounds in each element.

If we choose ϕ as the indicator function for element e, then Equation (28) becomes

$$\int_e \rho^{n+1}q^{n+1}\,\mathrm{d}\,x = \int_e \rho^n q^*\,\mathrm{d}\,x - \Delta t\int_{\partial e}\boldsymbol{u}\cdot\boldsymbol{n}\rho^n q^*\,\mathrm{d}\,s = 0. \tag{29}$$

This may be written in the form

$$\overline{\rho}^{n+1}\overline{q^{n+1}} = \overline{\rho}^n\left(q^n + c^- q^- - c^+ q^+\right), \tag{30}$$

where

$$\overline{q^{n+1}} = \frac{1}{\overline{\rho^{n+1}}} \int_e \rho^{n+1} q^{n+1} \,\mathrm{d}\, x, \tag{31}$$

$$q^{\pm} = \pm \frac{1}{c^{\pm}} \int_{\partial e^{\pm}} \boldsymbol{F} \cdot \boldsymbol{n} \tilde{q} \,\mathrm{d}\, s, \tag{32}$$

and $c^{\pm} > 0$ are defined as before. Finally, making use of Equation (11), we obtain

$$(1 + c^{-} - c^{+})\overline{q^{n+1}} = \bar{q}^{n} + c^{-} q^{-} - c^{+} q^{+}, \tag{33}$$

where positivity ensures that $1 + c^{-} - c^{+} > 0$. We see that $\bar{q}^{n+1}$ might exceed the bounds if q^{+} is too far below $\bar{q}^{n}$.

If our limiter scheme has an imposed maximum bound $\bar{q}^{n+1} \leq q^{\max}$, we now need to adjust the slope in q^{n},

$$q^{n} \mapsto \bar{q}^{n} + \beta(q^{n} - \bar{q}^{n}), \tag{34}$$

where $0 \leq \beta \leq 1$, in order to replace

$$q^{+} \mapsto \bar{q}^{n} + \beta(q^{+} - \bar{q}^{n}). \tag{35}$$

Since $q^{-} \leq q^{\max}$, since we have already limited the slope so that q satisfies the element bounds, which are the same on each side of each element boundary. Therefore, if we choose $\beta = 0$, the update becomes

$$(1 + c^{-} - c^{+})\bar{q}^{n+1} = \bar{q}^{n}(1 - c^{-}) + c^{-} q^{-} < q^{\max}(1 + c^{-} - c^{+}), \tag{36}$$

and the bound will be satisfied as required. We choose the maximum value of $\beta \leq 1$ such that

$$\bar{q}^{n}(1 - c^{+}) + c^{-} q^{-} - c^{+} \beta(q^{+} - \bar{q}^{n}) \leq (1 + c^{-} - c^{+}) q^{\max}, \tag{37}$$

i.e.,

$$\beta = \min\left(1, \frac{(1 + c^{-} - c^{+})q^{\max} - (\bar{q}^{n}(1 - c^{+}) + c^{-} q^{-})}{c^{+}(\bar{q}^{n} - q^{+})}\right). \tag{38}$$

References

[1] John Thuburn. Multidimensional flux-limited advection schemes. *J. Comput. Phys.*, 123(1):74–83, jan 1996.