

Notes on conservative, bounded, advection schemes

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As described in [1], the conservation of a tracer q can be most easily derived from the conservative form of the advection equations

$$(\rho q)_t + \nabla \cdot (\mathbf{u} \rho q) = 0, \quad (1)$$

where the density satisfies the continuity equation

$$\rho_t + \nabla \cdot (\mathbf{u} \rho) = 0. \quad (2)$$

On the other hand, the L_∞ bounds for q are most easily derived from the characteristic form

$$q_t + \mathbf{u} \cdot \nabla q = 0. \quad (3)$$

1 Continuity equation

The first step is that we need a positive scheme for ρ , *i.e.*, $\rho(x, t) > 0$ if $\rho(x, 0) > 0$.

The time-continuous discontinuous Galerkin (DG) scheme for $\rho \in DG(n)$ is defined on an element e with boundary ∂e by

$$\frac{d}{dt} \int_e \phi \rho \, dx - \int_e \nabla \phi \cdot \mathbf{u} \rho \, dx + \int_{\partial e} \phi \mathbf{u} \cdot \mathbf{n} \tilde{\rho} \, ds = 0, \quad \forall \phi \in DG(n), \quad (4)$$

where \mathbf{n} is the normal to the boundary, and $\tilde{\rho}$ is the value of ρ on the upwind side of ∂e .

Here we consider the forward Euler time discretisation, given by

$$\int_e \phi \rho^{n+1} \, dx = \int_e \phi \rho^n \, dx + \Delta t \int_e \nabla \phi \cdot \mathbf{u} \rho^n \, dx - \Delta t \int_{\partial e} \phi \mathbf{u} \cdot \mathbf{n} \tilde{\rho}^n \, ds = 0, \quad \forall \phi \in DG(n). \quad (5)$$

Forward Euler steps can be concatenated in SSPRK schemes to obtain higher-order time discretisations.

The above linear scheme is not positivity-preserving. To obtain a positivity-preserving scheme, we take the following steps:

1. A pre-processing limiter applied to ρ , such as the Barth-Jespersen limiter or the Kuzmin vertex-based limiter, that sets bounds $(\rho_{\min}, \rho_{\max})$ on each element e , based on the mean value $\bar{\rho}_e^n$, defined by

$$\bar{\rho}^n = \frac{1}{|e|} \int_e \rho^n \, dx, \quad (6)$$

and the value in selected surrounding elements. The limiter is then defined by

$$\Pi_1 \rho^n|_e = \bar{\rho}^n + \alpha(\rho^n - \bar{\rho}^n), \quad (7)$$

with $0 \leq \alpha \leq 1$ chosen as the maximum value such that $\Pi \rho^n$ lies within the bounds in element e . Note that such a limiter does not alter the element integral of ρ , and so does not violate conservation.

If the element mean values of ρ^n are all positive, then so are the bounds, and therefore $\Pi \rho^n$ is positive everywhere.

2. A second pre-processing limiter Π_2 to guarantee that the element means $\bar{\rho}^{n+1}$ will all be positive following the Euler step. The goal of this section is to introduce such a limiter.

3. The application of the forward Euler step,

$$\int_e \phi \rho^{n+1} dx = \int_e \phi \rho^* dx + \Delta t \int_e \nabla \phi \cdot \mathbf{u} \rho^* dx - \Delta t \int_{\partial e} \phi \mathbf{u} \cdot \mathbf{n} \rho^* ds = 0, \quad \forall \phi \in DG(n), \quad (8)$$

where $\rho^* = \Pi_2 \hat{\rho} = \Pi_2 \Pi_1 \rho^n$.

We now develop the limiter Π_2 .

If we choose ϕ as the indicator function for element e , *i.e.*,

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in e, \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

then Equation (8) becomes

$$\int_e \rho^{n+1} dx = \int_e \rho^* dx - \Delta t \int_{\partial e} \mathbf{u} \cdot \mathbf{n} \rho^* ds = 0. \quad (10)$$

This may be written in the form

$$\bar{\rho}^{n+1} = \bar{\rho}^n (1 + c^- - c^+), \quad (11)$$

where

$$c^\pm = \pm \frac{\Delta t}{|e|} \int_{\partial e^\pm} \mathbf{u} \cdot \mathbf{n} \frac{\bar{\rho}^*}{\bar{\rho}^n} ds, \quad (12)$$

where ∂e^+ , ∂e^- are the parts of ∂e where $\mathbf{u} \cdot \mathbf{n}$ is positive and negative respectively. Both c^+ and c^- are positive if ρ is positive everywhere.

For $\bar{\rho}^{n+1}$ to be positive, we need

$$c^+ < 1 + c^-. \quad (13)$$

This will always be satisfied for sufficiently small Δt . However, it may be violated if there are dramatic jumps in ρ , even for moderate values of the upwind/downwind Courant numbers

$$\tilde{c}^\pm = \pm \frac{\Delta t}{|e|} \int_{\partial e^\pm} \mathbf{u} \cdot \mathbf{n} ds. \quad (14)$$

Hence, we use a failsafe limiter to adjust the slope in each element where Equation (13) is not satisfied, to ensure positivity.

Since c^- depends on values of ρ from outside element e , it will not be altered by limiting the slope in element e . Hence, if the positivity condition (13) is not satisfied, applying a limiter

$$\rho^* = \bar{\rho} + \beta (\tilde{\rho} - \bar{\rho}), \quad (15)$$

where $0 \leq \beta \leq 1$, results in replacing c^+ by

$$\tilde{c}^+ + \beta(c^+ - \tilde{c}^+). \quad (16)$$

Hence, the positivity condition becomes

$$\tilde{c}^+ + \beta(c^+ - \tilde{c}^+) < 1 + c^-. \quad (17)$$

It is always possible to achieve this condition by setting $\beta = 0$, in which case the condition becomes

$$\tilde{c}^+ < 1 + c^-, \quad (18)$$

for which the CFL condition

$$\tilde{c}^+ < 1, \quad (19)$$

is sufficient. Hence, we choose $\beta \leq 1$ to be as large as possible provided that Equation (17) is satisfied, *i.e.*,

$$\beta = \left\{ \max \left(0, \min \left(1, \frac{1+c^- - \tilde{c}^+}{c^+ - \tilde{c}^+} \right) \right) \right\}. \quad (20)$$

2 Reconstruction of flux

If the advection equation for the tracer q wish to be solved provided the continuity equation is solved for density ρ , the flux in the advection equation needs to be reconstructed. following from the paper NEED CITATION HERE but

3 Bounded advection

The forward Euler conservative DG discretisation for the transport equation for q is,

$$\int_e \phi \rho^{n+1} q^{n+1} dx = \int_e \phi \rho^n q^n dx + \Delta t \int_e \nabla \phi \cdot \mathbf{F}^n q^n dx - \Delta t \int_{\partial e} \phi \mathbf{F}^n \cdot \mathbf{n} \tilde{q}^n ds = 0, \quad \forall \phi \in DG(n), \quad (21)$$

where \tilde{q}^n is the value of q^n on the upwind side of ∂e .

Since this linear scheme does not guarantee bounds on q^{n+1} , we also apply limiters to ensure this, following the same two step strategy, first ensuring that all values of q lie between bounds defined from element mean values, then adjusting the slope so that the outflow values guarantee positive element mean values in the next timestep. We use ρ -weighted element mean values defined by

$$\bar{q} = \frac{1}{\bar{\rho}^n} \int_e \rho^n q^n dx, \quad (22)$$

so that

$$\bar{\rho}^n \bar{q} = \int_e \rho^n q^n dx. \quad (23)$$

In this case, replacing q by the limited function

$$\Pi_1 q = \bar{q} + \alpha(q - \bar{q}), \quad (24)$$

does not violate conservation, since

$$\int_e \rho^n \Pi_1 q dx = \bar{q} \int_e \rho^n dx + \alpha \left(\int_e \rho^n q dx - \bar{q} \int_e \rho^n dx \right), \quad (25)$$

$$= \bar{\rho}^n \bar{q} + \alpha(\bar{\rho}^n \bar{q} - \bar{\rho}^n \bar{q}) = \bar{\rho}^n \bar{q}, \quad (26)$$

as required.

The forward Euler timestep for q is then as follows:

1. A pre-processing limiter applied to q , such as the Barth-Jespersen limiter or the Kuzmin vertex-based limiter, that sets bounds (q_{\min}, q_{\max}) on each element e , based on the mean value \bar{q}_e^n , and the value in selected surrounding elements. The limiter is then defined by

$$\Pi_1 q^n|_e = \bar{q}^n + \alpha(q^n - \bar{q}^n), \quad (27)$$

with $0 \leq \alpha \leq 1$ chosen as the maximum value such that Πq^n lies within the bounds in element e .

If the element mean values of q^n are all positive, then so are the bounds, and therefore Πq^n is positive everywhere.

2. A second pre-processing limiter Π_2 to guarantee that the element means \bar{q}^{n+1} will lie within their bounds at the following the Euler step. The goal of this section is to introduce such a limiter.
3. The application of the forward Euler step,

$$\int_e \phi \rho^{n+1} q^{n+1} dx = \int_e \phi \rho^n q^* dx + \Delta t \int_e \nabla \phi \cdot \mathbf{u} \rho^n q^* dx - \Delta t \int_{\partial e} \phi \mathbf{u} \cdot \mathbf{n} \rho^n q^* ds = 0, \quad \forall \phi \in DG(n), \quad (28)$$

where $q^* = \Pi_2 \hat{q} = \Pi_2 \Pi_1 q^n$.

We now develop the limiter Π_2 for q , assuming that q already satisfies the bounds in each element.

If we choose ϕ as the indicator function for element e , then Equation (28) becomes

$$\int_e \rho^{n+1} q^{n+1} dx = \int_e \rho^n q^* dx - \Delta t \int_{\partial e} \mathbf{u} \cdot \mathbf{n} \rho^n q^* ds = 0. \quad (29)$$

This may be written in the form

$$\bar{\rho}^{n+1} \bar{q}^{n+1} = \bar{\rho}^n (\bar{q}^n + c^- q^- - c^+ q^+), \quad (30)$$

where

$$\overline{q^{n+1}} = \frac{1}{\overline{\rho^{n+1}}} \int_e \rho^{n+1} q^{n+1} \mathrm{d} x, \quad (31)$$

$$q^\pm = \pm \frac{1}{c^\pm} \int_{\partial e^\pm} \mathbf{F} \cdot \mathbf{n} \tilde{q} \mathrm{d} s, \quad (32)$$

and $c^\pm > 0$ are defined as before. Finally, making use of Equation (11), we obtain

$$(1 + c^- - c^+) \overline{q^{n+1}} = \bar{q}^n + c^- q^- - c^+ q^+, \quad (33)$$

where positivity ensures that $1 + c^- - c^+ > 0$. We see that \bar{q}^{n+1} might exceed the bounds if q^+ is too far below \bar{q}^n .

If our limiter scheme has an imposed maximum bound $\bar{q}^{n+1} \leq q^{\max}$, we now need to adjust the slope in q^n ,

$$q^n \mapsto \bar{q}^n + \beta(q^n - \bar{q}^n), \quad (34)$$

where $0 \leq \beta \leq 1$, in order to replace

$$q^+ \mapsto \bar{q}^n + \beta(q^+ - \bar{q}^n). \quad (35)$$

Since $q^- \leq q^{\max}$, since we have already limited the slope so that q satisfies the element bounds, which are the same on each side of each element boundary. Therefore, if we choose $\beta = 0$, the update becomes

$$(1 + c^- - c^+) \bar{q}^{n+1} = \bar{q}^n(1 - c^-) + c^- q^- < q^{\max}(1 + c^- - c^+), \quad (36)$$

and the bound will be satisfied as required. We choose the maximum value of $\beta \leq 1$ such that

$$\bar{q}^n(1 - c^+) + c^- q^- - c^+ \beta(q^+ - \bar{q}^n) \leq (1 + c^- - c^+) q^{\max}, \quad (37)$$

i.e.,

$$\beta = \min \left(1, \frac{(1 + c^- - c^+) q^{\max} - (\bar{q}^n(1 - c^+) + c^- q^-)}{c^+ (\bar{q}^n - q^+)} \right). \quad (38)$$

References

- [1] John Thuburn. Multidimensional flux-limited advection schemes. *J. Comput. Phys.*, 123(1):74–83, jan 1996.