Temporal Logic

Introduction

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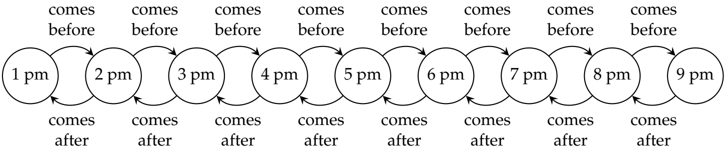
Language

McTaggart 1908

There are two ways of speaking about time:

A-series: with singular predicates: "…is past", "…is present", "…is future" (maybe builted in tenses "was", "is", "will"). Note that the truth of these sentences depends on the time of the utterance. **Local** perspective.

B-series: with ordering relations: "…comes before …", "…comes after …". The truth of these sentences does not depend on the time of the utterance. **Global** perspective.



Logics of tenses / Tense logics / Temporal logics: A-theories of time
Semantics of tense logics, first-order theories of orderings: B-theories of time

Temporal language

(the A-perspective)

Basic Temporal Language

Readings:

φ :	"It is the case that φ."
$\bot$:	the contradiction
$\varphi \to \psi$:	"if φ then ψ"
$\mathbf{F}\varphi$:	"It will be the case that φ."
$\mathbf{P}\varphi$:	"It was the case that φ."

- Symbols:
    - Atomic sentences $p, q, r, \ldots$ $\quad At \stackrel{\text{def}}{=} \{p_i : i \in \omega\}$
    - Logical symbols: $\neg, \wedge, \mathbf{F}, \mathbf{P}$
    - Other symbols: $(,)$
- Formulas:

$$\varphi ::= p \mid \bot \mid (\varphi \to \psi) \mid \mathbf{F}\varphi \mid \mathbf{P}\varphi$$

Defined connectives

Abbreviations:

$\neg\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\varphi \to \bot$	"it is not true that φ"
$\varphi \wedge \psi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\neg(\varphi \to \neg\psi)$	"φ and ψ"
$\varphi \vee \psi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\neg\varphi \to \psi$	"φ or ψ (or both of them) are true."
$\top$	$\stackrel{\text{def}}{\Leftrightarrow}$	$p \vee \neg p$	the tautology, the true, or verum
$\varphi \leftrightarrow \psi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$(\varphi \to \psi) \wedge (\psi \to \varphi)$	"It is the case that φ if and only if ψ is the case."
$\mathbf{G}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\neg\mathbf{F}\neg\varphi$	"It will always **G**oing to be the case that φ."
$\mathbf{H}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\neg\mathbf{P}\neg\varphi$	"It **H**as always been the case that φ."
$\underline{\mathbf{F}}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\varphi \vee \mathbf{F}\varphi$	"It is or will be the case that φ."
$\underline{\mathbf{P}}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\varphi \vee \mathbf{P}\varphi$	"It is or was the case that φ."
$\underline{\mathbf{G}}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\varphi \wedge \mathbf{G}\varphi$	"It is and always going to be the case that φ."
$\underline{\mathbf{H}}\varphi$	$\stackrel{\text{def}}{\Leftrightarrow}$	$\varphi \wedge \mathbf{H}\varphi$	"It is and always has been the case that φ."

Check (using classical logic) that $\neg\underline{\mathbf{F}}\neg\varphi \iff \underline{\mathbf{G}}\varphi$!

Examples

Interplay of tense and logic

Which one of the followings sounds true?

	$\mathbf{G}(\varphi \wedge \psi) \to (\mathbf{G}\varphi \wedge \mathbf{G}\psi)$	fine
	$\mathbf{G}(\varphi \wedge \psi) \leftarrow (\mathbf{G}\varphi \wedge \mathbf{G}\psi)$	fine
	$\mathbf{G}(\varphi \vee \psi) \to (\mathbf{G}\varphi \vee \mathbf{G}\psi)$	strange
	$\mathbf{G}(\varphi \vee \psi) \leftarrow (\mathbf{G}\varphi \vee \mathbf{G}\psi)$	fine
	$\mathbf{F}(\varphi \vee \psi) \to (\mathbf{F}\varphi \vee \mathbf{F}\psi)$	fine
	$\mathbf{F}(\varphi \vee \psi) \leftarrow (\mathbf{F}\varphi \vee \mathbf{F}\psi)$	fine
	$\mathbf{F}(\varphi \wedge \psi) \to (\mathbf{F}\varphi \wedge \mathbf{F}\psi)$	fine
	$\mathbf{F}(\varphi \wedge \psi) \leftarrow (\mathbf{F}\varphi \wedge \mathbf{F}\psi)$	strange
(K)	$\mathbf{G}(\varphi \to \psi) \to (\mathbf{G}\varphi \to \mathbf{G}\psi)$	fine
	$\mathbf{G}(\varphi \to \psi) \leftarrow (\mathbf{G}\varphi \to \mathbf{G}\psi)$	strange

Memorization Trick: If **F** and $\vee$ are **weak**, **G** and $\wedge$ are **strong**, then

"weak likes the weak, and strong likes the strong"

$$\mathbf{G}(\varphi \wedge \psi) \leftrightarrow (\mathbf{G}\varphi \wedge \mathbf{G}\psi)$$

$$\text{(A2)}\quad \mathbf{F}(\varphi \vee \psi) \leftrightarrow (\mathbf{F}\varphi \vee \mathbf{F}\psi)$$

And "WeakStrong $\to$ StrongWeak": $\mathbf{F}\wedge \to \wedge\mathbf{F}$, and $\vee\mathbf{G} \to \mathbf{G}\vee$

That is quite usual in logic: $\exists x \forall y\, xRy \to \forall y \exists x\, xRy$ but not vice versa.

Interplay of tense and tense

Which one of the followings sounds true?

(M)	$\mathbf{GF}\varphi \to \mathbf{FG}\varphi$	strange
(G)	$\mathbf{FG}\varphi \to \mathbf{GF}\varphi$	fine
(B)	$\varphi \to \mathbf{GF}\varphi$	strange
(T)	$\mathbf{G}\varphi \to \varphi$	strange
	$\underline{\mathbf{G}}\varphi \to \varphi$	trivial
(4)	$\mathbf{FF}\varphi \to \mathbf{F}\varphi$	fine
(Den)	$\mathbf{F}\varphi \to \mathbf{FF}\varphi$	fine
(E)	$\mathbf{F}\varphi \to \mathbf{GF}\varphi$	strange
$(\mathrm{C})_{\mathrm{F}}$	$\varphi \to \mathbf{HF}\varphi$	fine
$(\mathrm{C})_{\mathrm{P}}$	$\varphi \to \mathbf{GP}\varphi$	fine
$(\mathrm{D})_{\mathrm{F}}$	$\mathbf{G}\varphi \to \mathbf{F}\varphi$	fine
$(\mathrm{H})_{\mathrm{F}}$	$(\mathbf{F}\varphi \wedge \mathbf{F}\psi) \to (\mathbf{F}(\mathbf{F}\varphi \wedge \psi) \vee \mathbf{F}(\varphi \wedge \mathbf{F}\psi) \vee \mathbf{F}(\varphi \wedge \psi))$	fine
$(.3)_{\mathrm{F}}$	$\mathbf{G}(\underline{\mathbf{G}}\varphi \to \psi) \ \vee \ \mathbf{G}(\underline{\mathbf{G}}\psi \to \varphi)$	?

It's time to use precise semantics instead of "sense the Truth behind".

Structures

(The B-perspective)

Frames and models

A **frame** is a pair $\langle W, R\rangle$, where

- W is not empty, its elements are called **worlds** or **moments** and
- R is a binary relation on W, sometimes called **alternative** or **accessibility** relation.

If wRv, then we say that "w sees v" or "v is seen by w".

A **strict partial ordering** (**SPO**) is a **frame** $\langle T, <\rangle$, where $<$ is

- irreflexive: $\forall w\, \neg w < w$
- transitive: $\forall w, v, u\big((w < v \wedge v < u) \to w < u\big)$

A SPO $\langle T, <\rangle$ is **treelike** or **is a forest** if

- there is no branching to the past:
  $\forall w, v, u\big((w < u \wedge v < u) \to (w < v \vee w = v \vee w > v)\big)$

$w \leq v \overset{\text{def}}{\Leftrightarrow} w < v \vee w = v$

A **tree** is a treelike SPO $\langle T, <\rangle$ where

- every two different element has a 'root':
  $\forall w, v\,\big(w \neq v \to \exists u(u \leq w \wedge u \leq v)\big)$

A **flow of time** or **strict total order** (**STO**) is a SPO $\langle T, <\rangle$, where

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a) treelike
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A SPO $\langle T, < \rangle$ is treelike or is a forest iff

- there is no branching to the past:
  $\forall w, v, u ((w < u \wedge v < u) \rightarrow (w < v \vee w = v \vee w > v))$

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The easiest way to solve the homeworks, is to draw a lot first!!

Closures

The **reflexive closure** R^r of a relation R is the smallest reflexive relation that contains it, i.e.,

- wR^rv whenever wRv,
- R^r is reflexive: $\forall w\ wRw$
- Whenever a relation Q has these two property above, it can not have less arrows than R^r, i.e. wR^rv implies wQv.

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- $wR^{rts}v$ whenever wRv,
- R^{rts} is reflexive,
- R^{rts} is transitive,
- R^{rts} is symmetric: $\forall w, v(wR^{rts}v \to vR^{rts}w)$
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- Whenever a relation Q has these four property above, $wR^{rts}v$ implies wQv.

Show that
a) all trees are connected,
b) not all treelike SPO's are connected.

A **frame** $\langle W, R\rangle$ is **connected** iff $\forall w \forall v\, wR^{rts}v$

Models

Models

We'll use frames to determine the meaning of the formulas. To establish the connection, what we need is an **interpretation** or **evaluation** V.

The job of V is to tell for every formula φ, whether it is true or not in a given moment of a frame or not. So this will be a function which assigns a truth value 0 or 1 to every formula p and moment $w \in W$, i.e.,

$$V : \mathrm{At} \times W \to \{0, 1\}.$$

Another perspective is the following: Let the job of V be to tell for every formula φ, what is the set of worlds in which it is true, i.e.,

$$V : \mathrm{At} \to \mathcal{P}(W).$$

Hereby we have the (first step for a) mathematical representation of that connection between the syntax (At), and the semantics ($\langle W, R \rangle$).

According to the latter then, $w \in V(p)$ will represent the fact that p is true at w with respect to $\langle W, R \rangle$ and V. We will abbreviate this by

$$W, R, V, w \models p.$$

To simplify the notation, we will call the frame+interpretation pairs **models**.

MODELS

A **model** $\mathfrak{M}$ is a pair $\langle \mathfrak{F}, V \rangle$ where

- $\mathfrak{F}$ is a frame $\mathfrak{F} = \langle W, R \rangle$,
- V is an evaluation $V : \mathrm{At} \to \mathcal{P}(W)$.

We define the **satisfaction** or **local truth** relation in the following way:

$$
\begin{array}{lcl}
\mathfrak{M}, w \models p & \stackrel{\text{def}}{\Leftrightarrow} & w \in V(p) \\
\mathfrak{M}, w \not\models \bot & & \\
\mathfrak{M}, w \models \varphi \to \psi & \stackrel{\text{def}}{\Leftrightarrow} & \mathfrak{M}, w \models \varphi \text{ implies } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \mathbf{F}\varphi & \stackrel{\text{def}}{\Leftrightarrow} & \exists v \big(wRv \wedge \mathfrak{M}, w \models \varphi \big) \\
\mathfrak{M}, w \models \mathbf{P}\varphi & \stackrel{\text{def}}{\Leftrightarrow} & \exists v \big(vRw \wedge \mathfrak{M}, w \models \varphi \big)
\end{array}
$$

We define the **global truth** or just simply the **truth** relation based on the local truth:

$$\mathfrak{M} \models \varphi \iff \forall w \, \mathfrak{M}, w \models \varphi$$

And the most important: we say that φ is valid of $\mathfrak{F}$ iff it is true *no matter what are the meanings of its atomic particles*:

$$\mathfrak{F} \models \varphi \iff \forall V \, \mathfrak{F}, V \models \varphi$$

Why is the latter so important? Because only the structure matters here. So by investigating validities, we will able to investigate the structure of time, while we keep the local perspective of the modal language.

MODELS

Give a countermodel
a) for every formula what we labelled 'strange', such that
b) for some formula what we labelled 'fine'.
(i.e., give a model in which the formula in question is not true (i.e., false in some world of it))

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Standard Translation

B LANGUAGE

Every temporal model $\mathfrak{M}$ can be viewed as a classical first-order model:

$$
\begin{array}{lll}
\mathfrak{M} & = & \langle W, R, V \rangle \\
& \simeq & \langle W, R, V(p), V(q), \dots \rangle_{p,q,\dots \in \mathrm{At}} & \text{"unpack" } V \\
& \leftrightsquigarrow & \langle W, I(\mathrm{R}), V(p), V(q), \dots \rangle_{\substack{p,q,\dots \in \mathrm{At} \\ R \in \mathrm{Pred}^2}} & \text{consider } R \text{ as a meaning of an R} \\
& \leftrightsquigarrow & \langle W, I(\mathrm{R}), I(P), I(Q), \dots \rangle_{\substack{P,Q,\dots \in \mathrm{Pred}^1 \\ R \in \mathrm{Pred}^2}} & \text{consider At as monadic predicates} \\
& \simeq & \langle W, I \rangle & \text{"pack" } I
\end{array}
$$

So the corresponding (object linguistic!) FOL language is

- Symbols:
  - Monadic predicates: $P, Q, R, \dots$
  - Binary predicates: R
  - Variables: $w, v, u, \dots$
  - Logical symbols: $\perp, \rightarrow, =, \forall,$
  - Other symbols: (,)
- Formulas:

$$\varphi ::= \ w = v \ \mid \ P(w) \ \mid \ wRv \ \mid \ \perp \ \mid \ (\varphi \rightarrow \psi) \ \mid \ \forall w \varphi$$

Standard Translation

$$
\begin{array}{lcll}
\mathrm{ST}_x(p) & \stackrel{\text{def}}{=} & P(x) & \\
\mathrm{ST}_x(\bot) & \stackrel{\text{def}}{=} & \bot & \\
\mathrm{ST}_x(\varphi \to \psi) & \stackrel{\text{def}}{=} & \mathrm{ST}_x(\varphi) \to \mathrm{ST}_x(\psi) & \\
\mathrm{ST}_x(\mathbf{G}\varphi) & \stackrel{\text{def}}{=} & \forall v(xRv \to \mathrm{ST}_v(\varphi)) & \text{where } v \text{ is a fresh variable} \\
\mathrm{ST}_x(\mathbf{H}\varphi) & \stackrel{\text{def}}{=} & \forall v(vЯx \to \mathrm{ST}_v(\varphi)) & \text{where } v \text{ is a fresh variable}
\end{array}
$$

Homeworks:

$$
\begin{array}{lrcl}
\underline{\text{THEOREM}}: & \mathfrak{M}, w \models \varphi & \iff & \mathfrak{M} \models \mathrm{ST}_x(\varphi) \quad [\sigma[x \mapsto w]] \\
\underline{\text{COROLLARY}}: & \mathfrak{M} \models \varphi & \iff & \mathfrak{M} \models \forall x\, \mathrm{ST}_x(\varphi) \\
\underline{\text{COROLLARY}}: & \mathfrak{F} \models \varphi & \iff & \mathfrak{M} \models \forall P \forall Q \ldots \forall x\, \mathrm{ST}_x(\varphi)
\end{array}
$$

The last is in Second Order Logic!!!! I.e., in frame semantics we quantify over subsets of W! SOL is a powerful language, but it has a tons of disadvantages, just some of them: Truths of formulas depends on that which ZFC model are we in, it can articulate non-logical statements, what is more, ZFC-independent statements like continuum hypothesis, no completeness theorem, no compactness, etc.

But, the fragment corresponding to TL is free from all of these, while it can maintain some of SOL's power. And sometime second order statements defined by TL are just equivalent to FOL statements...

FOL ABBREVIATIONS

Of course, we always omit the outermost brackets.

$$
\begin{array}{lcl}
\exists x\varphi & \overset{\text{def}}{\Leftrightarrow} & \neg\forall x\neg\varphi \\
\exists xy\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists x\exists y\varphi \\
\exists xyz\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists x\exists y\exists z\varphi \\
 & \vdots & \\
(\exists x \in \varphi)\psi & \overset{\text{def}}{\Leftrightarrow} & \exists x(\varphi(x) \wedge \psi)
\end{array}
\qquad
\begin{array}{lcl}
\exists x, y\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists x\exists y\varphi \\
\exists x, y, z\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists x\exists y\exists z\varphi \\
 & \vdots & \\
(\forall x \in \varphi)\psi & \overset{\text{def}}{\Leftrightarrow} & \forall x(\varphi(x) \rightarrow \psi)
\end{array}
$$

And a full stop after a logical symbol means an opening bracket whose scope is the longest as possible (i.e., ends before the first closing bracket), e.g.

$$\exists x.\varphi \rightarrow \psi \iff \exists x(\varphi \rightarrow \psi)$$

Or the 3rd Frege-Hilbert axiom

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

can be written up as

$$(\varphi \rightarrow .\psi \rightarrow \chi) \rightarrow .(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$$

$$(\varphi \rightarrow .\psi \rightarrow \chi) \rightarrow .(\varphi \rightarrow \psi) \rightarrow .\varphi \rightarrow \chi$$

A-B Correspondences (modal definability)

Difficulty	Name	TL formula	FOL formula	Name
Easy	**T**	$\Box\varphi \to \varphi$	$\forall w\ wRw$	reflexive
Easy	**4**	$\Box\varphi \to \Box\Box\varphi$	$\forall wvu.\ wRvRu \to wRu$	transitive
Normal	**Den**	$\Box\Box\varphi \to \Box\varphi$	$\forall wv.\ wRu \to (\exists v)wRvRu$	dense
Easy	**B**	$\varphi \to \Box\Diamond\varphi$	$\forall wv.\ wRv \to vRw$	symmetric
Normal	**E**	$\Diamond\varphi \to \Box\Diamond\varphi$	$\forall wv.\ u\text{Я}wRv \to vRu$	euclidean
Normal	**G**	$\Diamond\Box\varphi \to \Box\Diamond\varphi$	$\forall wvu.\ v\text{Я}wRu \to (\exists u')(vRu'\text{Я}u)$	convergent
Normal	**.3**	$\Diamond\varphi \wedge \Diamond\psi \to \big(\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\Diamond\varphi \wedge \psi)\big)$	$\forall wvu.\ v\text{Я}wRu \to (vRu \vee v\text{Я}u \vee u = v)$	no branching to the right
Hard	**.3**	$\Box(\underline{\Box}\varphi \to \psi) \vee \Box(\underline{\Box}\psi \to \varphi)$	$\forall wvu.\ v\text{Я}wRu \to (vRu \vee v\text{Я}u \vee u = v)$	no branching to the right
Easy	**D**	$\Box\varphi \to \Diamond\varphi$	$\forall w \exists v\ wRv$	serial
Easy	**D$^+$**	$\Box(\Box\varphi \to \varphi)$	$\forall wv.\ wRv \to vRv$	secondary reflexive
Beautiful	**GL**	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	$\forall wvu(wRvRu \to wRu) \wedge \neg\exists P(\forall w \in P)(\exists v\text{Я}w)\, P(v)$	Noetherian SPO
Beautiful	**Grz**	$\Box(\Box(\varphi \to \Box\varphi) \to \to \varphi) \to \varphi$	$\forall w\ wRw \wedge$ $\forall wvu\ (wRvRu \to wRu) \wedge$ $\neg\exists P(\forall w \in P)(\exists v\text{Я}w)(w \neq v \wedge P(v))$	reflexive Noetherian partial ordering
Easy	**V**	$\Box\varphi$	$\forall wv\ \neg wRv$	empty
Easy	**Tr**	$\varphi \to \Box\varphi$	$\forall wv.\ wRv \to w = v$	diagonal
Normal	**1.1**	$\Diamond\varphi \to \Box\varphi$	$\forall wvu.\ v\text{Я}wRu \to v = u$	partial function
Normal	**ijkl**	$\Diamond^i\Box^j\varphi \to \Box^k\Diamond^l\varphi$	$\forall wvu.\ v\text{Я}^i wR^k u \to (\exists u')(vR^j u'\text{Я}^l u)$	*ijkl*-convergent