Logic of branching time

Ways of Indeterminism

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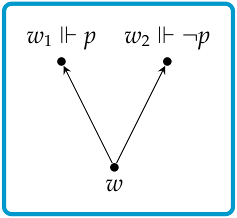
Tree of Time

Indeterminist frames



Consider the tree on the right. Let p represent the sentence "There is a sea battle". Suppose that w is today, and w_1 and w_2 are the two possible tomorrows. We have that $w \models \mathbf{F}p \wedge \mathbf{F}\neg p$, therefore,

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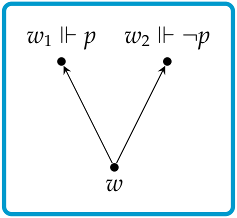

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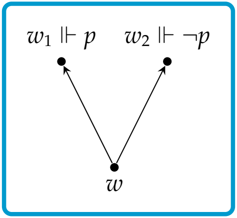

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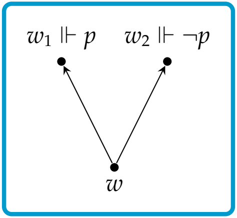

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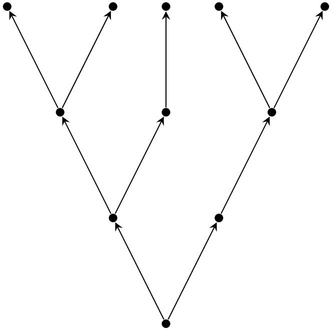
So trees are appropriate drawings but somehow not "will" is the appropriate word for **F**. But then what is the meaning of "will" here?.

HISTORIES

Consider the trees as a **bundle** of linear frames, that are called **histories** in that context.



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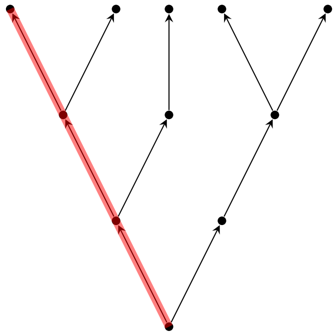

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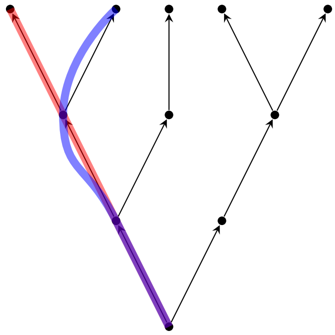

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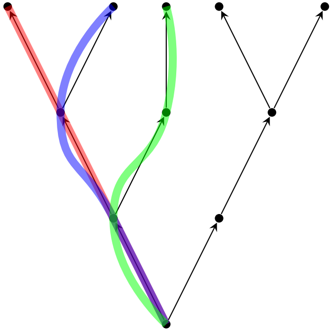

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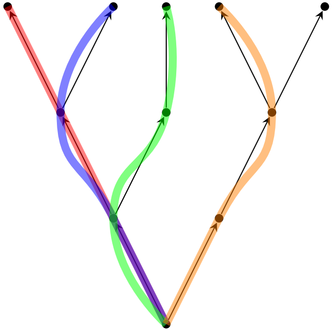

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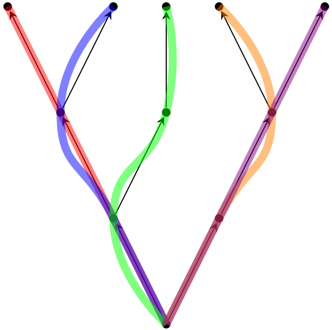

HISTORIES

Let $\mathfrak{F} = (W, <)$ be a tree.

<u>DEFINITION</u>: A **history** h is a maximally linear subset of W, i.e.,

- linear: $(\forall w, v \in h)\ w < v \vee w = v \vee w > v$.
- there is no proper linear extension of it:

$$(\forall h' \supseteq h)\big[h' \text{ is linear } \rightarrow h' \subseteq h.\big]$$

$h \overset{w}{\sim} h'$ will mean that histories h and h' share the same past until w. Since we are working with trees, this can be formalized simply by

$$h \overset{w}{\sim} h' \overset{\text{def}}{\Leftrightarrow} w \in h \cap h'$$

The set of all histories of a frame will be denoted by $\mathrm{H}(\mathfrak{F})$:

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show that $\overset{w}{\sim}$ is an equivalence relation.

Indeterminist interpretations of the tense “will”.

Read $\mathbf{F}\varphi$ as “it will be the case that φ”.
Is it plausible that

$$(\mathbf{F}\varphi \wedge \mathbf{F}\psi) \rightarrow \left[\mathbf{F}(\varphi \wedge \mathbf{F}\psi) \vee \mathbf{F}(\varphi \wedge \psi) \vee \mathbf{F}(\mathbf{F}\varphi \wedge \psi)\right] \quad ?$$

If φ will be true and ψ will be true, then at least one of the followings is true:

- φ will be true, and after that ψ will true.
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Yes: Ockhamist future

No: Peircean future

All the other options are variations of these two.

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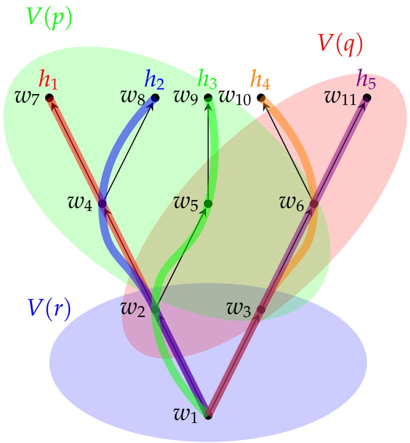
Indeterminism comes into the picture when we change the "specified possible future" (history) with an operator $\Diamond$.
Let $\mathfrak{M} = (W, <, V)$ be a tree model.

$$\mathfrak{M}, h, w \models^{\mathrm{O}} p \quad \overset{\text{def}}{\Leftrightarrow} \quad w \in V(p)$$
$$\mathfrak{M}, h, w \models^{\mathrm{O}} \neg\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad \text{it is not true that } \mathfrak{M}, h, w \models^{\mathrm{O}} \varphi$$
$$\mathfrak{M}, h, w \models^{\mathrm{O}} \varphi \wedge \psi \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathfrak{M}, h, w \models^{\mathrm{O}} \varphi \text{ and } \mathfrak{M}, h, w \models^{\mathrm{O}} \psi$$
$$\mathfrak{M}, h, w \models^{\mathrm{O}} \mathbf{P}\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists v \big(v < w \wedge \mathfrak{M}, h, v \models^{\mathrm{O}} \varphi\big)$$
$$\mathfrak{M}, h, w \models^{\mathrm{O}} \mathbf{F}\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad (\exists v \in h)\big(w < v \wedge \mathfrak{M}, h, v \models^{\mathrm{O}} \varphi\big)$$
$$\mathfrak{M}, h, w \models^{\mathrm{O}} \Diamond\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad (\exists h' \overset{w}{\sim} h)\ \mathfrak{M}, h', w \models^{\mathrm{O}} \varphi$$

Training



OCKHAMIST "WILL"

$$(\mathbf{F}\varphi \wedge \mathbf{F}\psi) \ \rightarrow \ \left[\mathbf{F}(\varphi \wedge \mathbf{F}\psi) \vee \mathbf{F}(\varphi \wedge \psi) \vee \mathbf{F}(\mathbf{F}\varphi \wedge \psi)\right] \quad ?$$

is valid because outside of φ and ψ there are no $\Diamond$-s, so once the meaning of φ and ψ is given, the meaning of the formula above is evaluated on a given history, which is a linear order of moments.

Peircean future

PEIRCEAN FUTURE

> From the Peircean point of view, [...] "φ will happen" is short for "φ will happen, no matter what possible future of w is considered", which is true just in case every possible future of w contains a moment at which φ is true.
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> ZANARDO 1996 (about PRIOR 1967)

So even if we use the histories to give the semantics of **F**, we do not relativize the truth of formulas to certain histories. Truth of a temporal statement is history-independent.

A set of worlds X **bars** w iff every history containing w goes through X:

$$X \text{ bars } w \overset{\text{def}}{\Leftrightarrow} (\forall h \in \mathrm{H}(\mathfrak{F}))(w \in h \to h \cap X \neq \varnothing)$$

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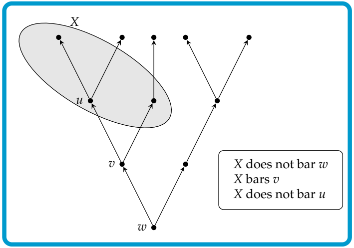
\end{array}$$

where $[\![\varphi]\!]^{\mathfrak{M}}_{\mathrm{P}} \overset{\text{def}}{=} \{w : \mathfrak{M}, w \models^{\mathrm{P}} \varphi\}$, i.e, the set of those worlds in which φ is true.



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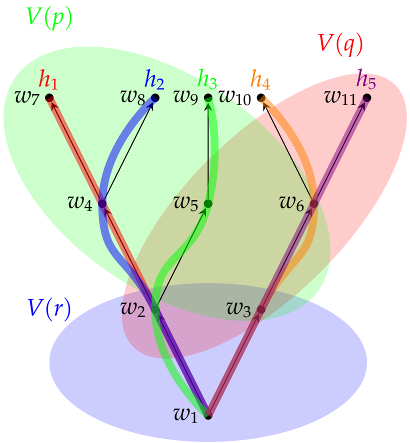

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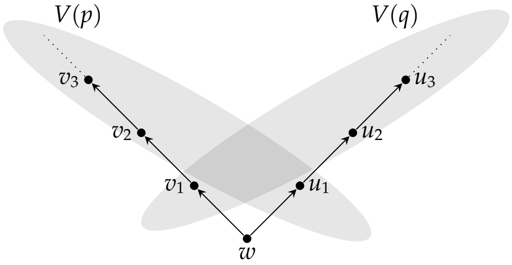
is invalid; let $\mathfrak{M}$ be the following "twin lines"-model:



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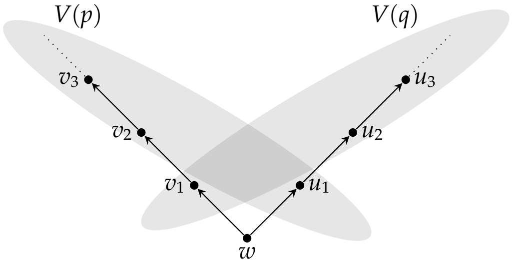

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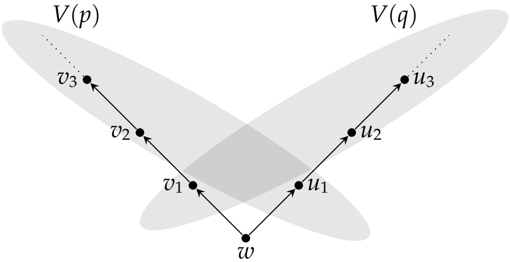

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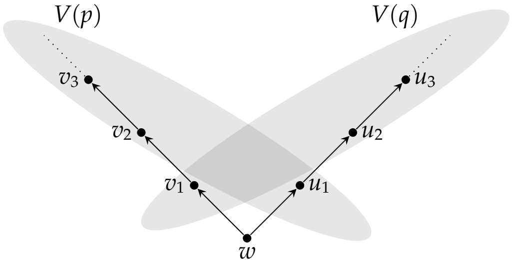

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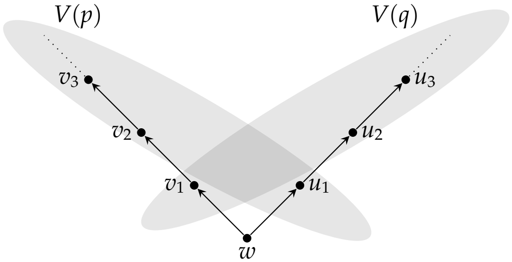

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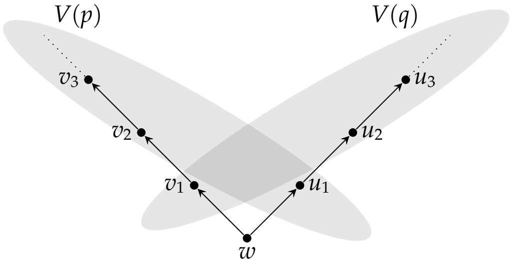

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But neither of these bars w, so $\mathfrak{M}, w \not\models^{\mathrm{P}} \mathbf{F}(\mathbf{F}p \wedge q)$ and $\mathfrak{M}, w \not\models^{\mathrm{P}} \mathbf{F}(p \wedge \mathbf{F}q)$

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How should we formalize the statement "There won't be a 3rd World War."?

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None of the above is correct, because the first is speaking about only some 'escape'-history, and the second talks about only some moments in the future in which there is no 3rd World war. The reason of course is that we interpret **F** with a $\forall\exists$-way, and no matter how we negate it, the mixed nature of it will survive.

If we want to formalize the 'won't-s, and "always going to be"-s, we need the old history-independent strong future operator for that purpose:

$$\mathfrak{M}, w \models^{\mathrm{P}} \mathbf{G}\varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall v \big(w < v \wedge \mathfrak{M}, v \models^{\mathrm{P}} \varphi \big)$$

Leibnizian

Parallel histories

Kamp frames instead of trees

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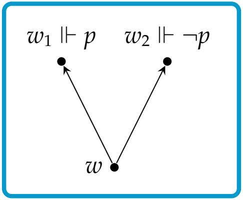

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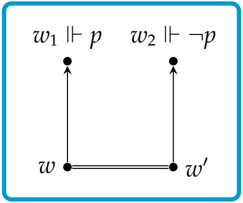
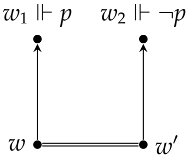


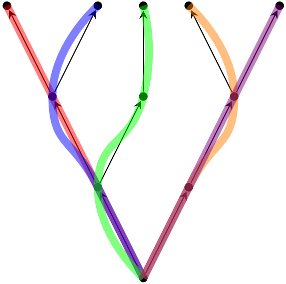
Parallel histories

Kamp frames instead of trees

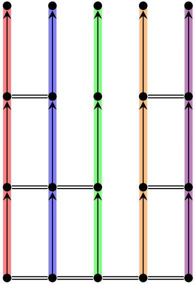
Consider the tree on the right. Let p represent the sentence "There is a sea battle". Suppose that w and w' are today, and w_1 and w_2 are the two possible tomorrows.



Standard (tree) frame
$(W, <)$



Kamp-frame
$(W, <, \equiv)$



KAMP-FRAMES

A Kamp-frame is a triplet $(W, <, \equiv)$ where

- $<$ is irreflexive, transitive and non-branching:
  - $w \not< w$
  - $(w < v \wedge v < u) \rightarrow w < u$
  - $(w < v \wedge w < u) \rightarrow (v < u \vee v = u \vee v > u)$
  - $(w > v \wedge w > u) \rightarrow (v < u \vee v = u \vee v > u)$
- $\equiv$ is reflexive, transitive and symmetric:
  - $w \equiv w$
  - $(w \equiv v \wedge v \equiv u) \rightarrow w \equiv u$
  - $w \equiv v \rightarrow v \equiv w$
- $x \equiv y \rightarrow x \not< y$ — class irreflexivity
- $(w \equiv v \wedge w' < w) \rightarrow (\exists v' < v)\ w' \equiv v'$ — "sharing the same past"
- $(\forall w, v)(\exists w' < w)(\exists v' < v)\ w \equiv v$ — class common root
- $(\forall w, v)(w \equiv v \wedge w \neq v)(\exists w' > w)(\forall v' > v)\ w' \not\equiv v'$ — maximality of histories

$$\begin{tikzcd} w \arrow[r, equal] & v \\ w' \arrow[u] \arrow[r, equal] & \exists v' \arrow[u] \end{tikzcd}$$

"sharing the same past"

$$\begin{tikzcd} w & v \\ \exists w' \arrow[u] \arrow[r, equal] & \exists v' \arrow[u] \end{tikzcd}$$

class common root

$$\begin{tikzcd} \exists w' \arrow[r, phantom, "\not="] & \forall v \\ w \arrow[u] \arrow[r, equal, "\neq"'] & v \arrow[u] \end{tikzcd}$$

maximality of histories

Kamp-frames

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- $\equiv$ is reflexive, transitive and symmetric:
  - $w \equiv w$
  - $(w \equiv v \wedge v \equiv u) \rightarrow w \equiv u$
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maximality of histories

Show that
- class irreflexivity implies irreflexivity
- maximality of histories implies class irreflexivity

Kamp-models

Let $\mathfrak{K} = (W, <, \equiv)$ be a Kamp-frame. A Kamp-valuation is a $V : \mathrm{At} \to \wp W$ for which the following additional property holds:

$$w \in V(p) \Rightarrow (\forall v \equiv w)\ v \in V(p) \qquad \text{for all } p \in \mathrm{At}$$

a Kamp-frame $\mathfrak{K} = (W, <, \equiv)$ together with such a valuation V is a Kamp-model $\mathfrak{M}_K = (\mathfrak{K}, V)$.

$$
\begin{array}{lcl}
\mathfrak{M}_K, w \models^{\mathrm{K}} p & \overset{\text{def}}{\Leftrightarrow} & w \in V(p) \\
\mathfrak{M}_K, w \models^{\mathrm{K}} \neg\varphi & \overset{\text{def}}{\Leftrightarrow} & \text{it is not true that } \mathfrak{M}_K, w \models^{\mathrm{K}} \varphi \\
\mathfrak{M}_K, w \models^{\mathrm{K}} \varphi \wedge \psi & \overset{\text{def}}{\Leftrightarrow} & \mathfrak{M}_K, w \models^{\mathrm{K}} \varphi \text{ and } \mathfrak{M}_K, w \models^{\mathrm{K}} \psi \\
\mathfrak{M}_K, w \models^{\mathrm{K}} \mathbf{P}\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v < w)\ \mathfrak{M}_K, v \models^{\mathrm{K}} \varphi \\
\mathfrak{M}_K, w \models^{\mathrm{K}} \mathbf{F}\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v > w)\ \mathfrak{M}_K, v \models^{\mathrm{K}} \varphi \\
\mathfrak{M}_K, w \models^{\mathrm{K}} \Diamond\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v \equiv w)\ \mathfrak{M}_K, v \models^{\mathrm{K}} \varphi
\end{array}
$$

Kamp-frames

Theorem: Every $\overset{K}{\models}$-valid formula is $\overset{O}{\models}$-valid.

Kamp-frames

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This was expected: $\Diamond$ in $\models^{K}$ quantifies over worlds,
but $\models^{O}$ quantify over sets of worlds.

$\Diamond$ is interpreted with a 1st order quantification in $\models^{K}$
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Show that every history in a finite tree is uniquely identifiable with a world.

Counterexample: Infinite binary trees

$W \stackrel{\text{def}}{=} \{w : w \text{ is a route to a point}\}$
$= \{\langle w_1, \ldots, w_n \rangle : n \in \omega, (\forall i \leq n) w_i \in \{U, R\}\}$

$w \sqsubseteq v \stackrel{\text{def}}{\Leftrightarrow} v$ is a continuation of w, i.e.,
iff $(\forall i \leq n) w_i = v_i$ where n is the length of w.

Note that histories correspond to infinite routes!

Also note we can not name the histories by worlds (as was the case in the finite cases)! There are (infinitely) many infinite continutations of finite routes.

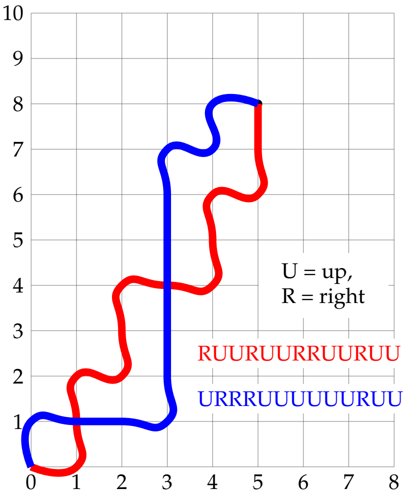
A set of histories $B \subseteq \mathrm{H}(\mathfrak{F})$ is called a **bundle** iff

$$\bigcup B = W,$$

that is, for every $w \in W$ there is a history $h \in B$ s.t. $w \in h$.

We can find a proper bundle, which is in fact can be named by worlds:

$$\{h \in \mathrm{H}(\mathfrak{F}) : \exists w (\forall v > w) v = \langle w, U, \ldots, U \rangle\}$$



Bundled trees

DEFINITION: A bundled tree is a triplet $\mathfrak{F}_B = (W, <, B)$ where $(W, <)$ is a tree and $B \subseteq \mathrm{H}(W, <)$ is a bundle. A bundled model is a quadruple $(\mathfrak{F}_B, V)$ where $\mathfrak{F}_B$ is a bundled frame and $V : \mathrm{At} \to \wp W$ is a valuation.

$$
\begin{array}{lcl}
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\mathfrak{M}, h, w \models^{B} \mathbf{P}\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists v \big(v < w \wedge \mathfrak{M}, h, v \models^{B} \varphi\big) \\
\mathfrak{M}, h, w \models^{B} \mathbf{F}\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v \in h)\big(w < v \wedge \mathfrak{M}, h, v \models^{B} \varphi\big) \\
\mathfrak{M}, h, w \models^{B} \Diamond\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists h' \overset{w}{\sim} h)\ \big(h \in B \wedge \mathfrak{M}, h', w \models^{B} \varphi\big)
\end{array}
$$

PROPOSITION: $\models^{K}$-validity corresponds to $\models^{B}$-validity.

QUOTE: “Belnap et al. have argued that it is implausible to assume that there could be some property which could »justify treating some maximal chains as real possibilities and others as not« (Belnap et al. 2001, p. 205)”

Hirokazu Nishimura (1979) – Stanford Enc.

Defense of Bundled trees

Suppose we have discrete models (we are thinking in days instead of moments). Are these two sentences contradictory?

- "Inevitably, if today there is life on earth, then either this is the last day (of life on earth), or the last day will come."
- "At any possible day on which there is life on earth, it is possible that there will be life on earth the following day."

Hirokazu Nishimura (1979)– Stanford Enc.

Let p be "there is life on earth". Let h be some history (it does not matter actually)

- $\mathfrak{M}, h, w \models \Box(p \rightarrow \underline{\mathbf{F}}\mathbf{G}\neg p)$
- For all $v \in W$: $\mathfrak{M}, h, v \models p \rightarrow \Diamond \mathbf{F} p$

Defense of Bundled trees

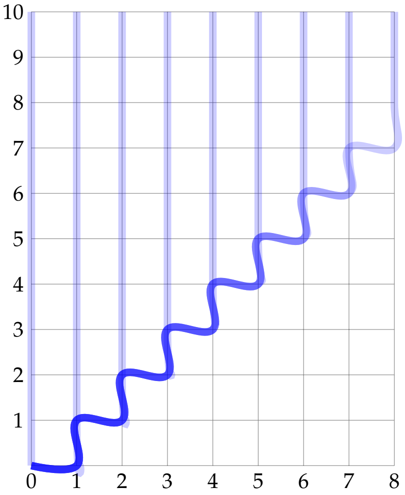
finite zigzags: $\langle R, U, R, U, \ldots, R, U\rangle$
infinite columns: $\langle U, U, U, U, \ldots, \rangle$
W: finite routes
R: continuation

$$B = \left\{ w \oplus v : \begin{array}{l} w \text{ fin. route,} \\ v \text{ inf. column} \end{array} \right\}$$

$V(p)$: the finite zigzags
life on Earth $\leftrightsquigarrow$ zigzagging
$\Box(p \to \underline{\mathbf{F}}\mathbf{G}\neg p)$ $\leftrightsquigarrow$ in every history, sooner or later life (zigzags) will permanently end
$p \to \Diamond\mathbf{F}p$ $\leftrightsquigarrow$ one more day of life (zigzag) is always possible
The Ockhamist will find the infinite zigzag, which refutes

$$(p \to \underline{\mathbf{F}}\mathbf{G}\neg p)$$

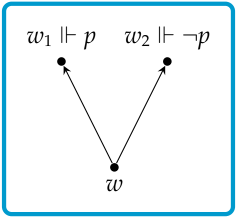
But the bundled treehugger won't.



Thomason

Truth value gaps for the future contingents

Consider the tree on the right. Let p represent the sentence "There is a sea battle". Suppose that w is today, and w_1 and w_2 are the two possible tomorrows.



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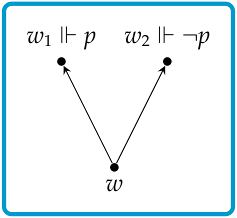

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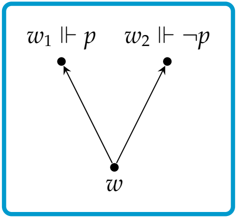

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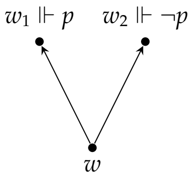

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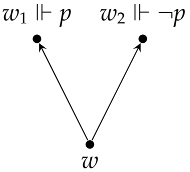

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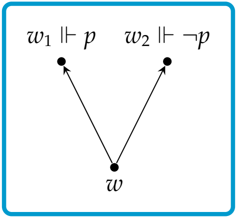

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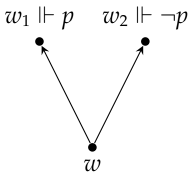

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Partial semantics: What if neither of them bar the moment of utterance?

Then $\mathbf{F}p$ is undefined in the moment of utterance. Because these statements' truth values are not settled yet. Tomorrow they will be settled, but now they are not.

Imre Ruzsa, the founder of this department, is a (the?) champion of partial modal semantics.

Supervaluation

We will consider the word undefined as a 3rd truth value.

Ruzsa was a hardcore classical logician as well: He did not acknowledged any formal system as a logic if it did not fulfilled the Law of (Non-)Contradiction and the Law of Excluded Middle. But how did he give an account about these 3-valued thing as a logic? These classical laws are fulfilled – on those formulas where the interpretation is defined.

Supervaluation

We will consider the word undefined as a 3rd truth value.

Notational problem: we can not represent 3 value with our $\models$ sign, so we switch to the intension-notation:

$$\mathfrak{M}, w \models \varphi \leftrightsquigarrow [\![\varphi]\!]^{\mathfrak{M}}_{w} = \text{true}$$

Cheat: Instead of starting everything from the beginning, we can define truth via Ockhamist truth!

...Thomasonian truth supervenes on Ockhamist truth...

$$[\![\varphi]\!]^{\mathfrak{M}}_{w} \stackrel{\text{def}}{=} \begin{cases} \mathrm{T} & \text{if } (\forall h \ni w)\mathfrak{M}, h, w \models^{\mathrm{O}} \varphi \\ \mathrm{F} & \text{if } (\forall h \ni w)\mathfrak{M}, h, w \models^{\mathrm{O}} \neg\varphi \\ \mathrm{U} & \text{otherwise} \end{cases}$$

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Compositionality

Our cheat had a cost: the compositionality.

<u>Definition</u>: A meaning function $[\![]\!]_w$ is compositional iff the meaning $[\![f(\varphi, \psi, \ldots)]\!]_w$ of a complex expression $f(\varphi, \psi, \ldots)$ is determined by the meanings $[\![\varphi]\!]_w, [\![\psi]\!]_w, \ldots$ of the constituents φ, ψ, …

That is a very general definition which is common in the Andréka–Németi–Madarász–Sain school

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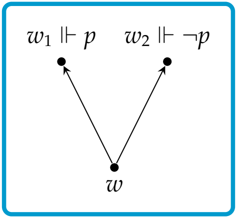
Remember to the sea battle model:

$$[\![\mathbf{F}p]\!]_w = \mathrm{U}, \qquad [\![\mathbf{F}\neg p]\!]_w = \mathrm{U}, \qquad [\![\mathbf{F}p \vee \mathbf{F}\neg p]\!]_w = \mathrm{T}$$

But

$$[\![\mathbf{F}p]\!]_w = \mathrm{U}, \qquad [\![\mathbf{F}p]\!]_w = \mathrm{U}, \qquad [\![\mathbf{F}p \vee \mathbf{F}p]\!]_w = \mathrm{U}$$

In non-compositional logics you have to "look into" the formula



Łukasiewitz?

Modelling the undefined – Kleene's logic(s)

Atoms and Negation

Every atom is either true or false, given by the model's valuation V.

		$\neg\varphi$
φ	F	T
	U	U
	T	F

Modelling the undefined – Kleene's logic(s)

Conjunction and Disjunction

$\varphi \wedge \psi$		ψ F	ψ U	ψ T
φ	F	F	?	F
φ	U	?	U	U
φ	T	F	U	T

$\varphi \vee \psi$		ψ F	ψ U	ψ T
φ	F	F	U	T
φ	U	U	U	?
φ	T	T	?	T

Modelling the undefined – Kleene's logic(s)

Conjunction and Disjunction

$\varphi \wedge \psi$		ψ F	ψ U	ψ T
φ	F	F	?	F
	U	?	U	U
	T	F	U	T

$\varphi \vee \psi$		ψ F	ψ U	ψ T
φ	F	F	U	T
	U	U	U	?
	T	T	?	T

Weak conjectives

$\varphi \wedge \psi$		ψ F	ψ U	ψ T
φ	F	F	U	F
	U	U	U	U
	T	F	U	T

$\varphi \vee \psi$		ψ F	ψ U	ψ T
φ	F	F	U	T
	U	U	U	U
	T	T	U	T

Modelling the undefined – Kleene's logic(s)

Conjunction and Disjunction

$\varphi \wedge \psi$	ψ: F	U	T
φ: F	F	?	F
U	?	U	U
T	F	U	T

$\varphi \vee \psi$	ψ: F	U	T
φ: F	F	U	T
U	U	U	?
T	T	?	T

Weak conjectives

$\varphi \wedge \psi$	ψ: F	U	T
φ: F	F	U	F
U	U	U	U
T	F	U	T

$\varphi \vee \psi$	ψ: F	U	T
φ: F	F	U	T
U	U	U	U
T	T	U	T

Strong conjectives

$\varphi \wedge \psi$	ψ: F	U	T
φ: F	F	F	F
U	F	U	U
T	F	U	T

$\varphi \vee \psi$	ψ: F	U	T
φ: F	F	U	T
U	U	U	T
T	T	T	T

Modelling the undefined – Kleene's logic(s)

Variations for implication

weak $\neg\varphi \vee \psi$		ψ F	ψ U	ψ T
φ	F	T	U	T
φ	U	U	U	U
φ	T	F	U	T

strong $\neg\varphi \vee \psi$		ψ F	ψ U	ψ T
φ	F	T	T	T
φ	U	U	U	T
φ	T	F	U	T

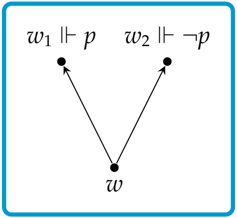
Łukasiewicz $\varphi \to \psi$		ψ F	ψ U	ψ T
φ	F	T	T	T
φ	U	U	T	T
φ	T	F	U	T

About future

$$
[\![\mathbf{F}\varphi]\!]_w^{\mathfrak{M}} \stackrel{\text{def}}{=} \begin{cases} \mathrm{T} & \text{if } (\forall h \ni w)(\exists v > w)\ [\![\varphi]\!]_w^{\mathfrak{M}} = \mathrm{T} \\ \mathrm{F} & \text{if } (\forall h \ni w)(\exists v > w)\ [\![\varphi]\!]_w^{\mathfrak{M}} = \mathrm{F} \\ \mathrm{U} & \text{otherwise} \end{cases}
$$

About future

$$[\![\mathbf{F}\varphi]\!]^{\mathfrak{M}}_w \stackrel{\text{def}}{=} \begin{cases} \text{T} & \text{if } (\forall h \ni w)(\exists v > w)\ [\![\varphi]\!]^{\mathfrak{M}}_w = \text{T} \\ \text{F} & \text{if } (\forall h \ni w)(\exists v > w)\ [\![\varphi]\!]^{\mathfrak{M}}_w = \text{F} \\ \text{U} & \text{otherwise} \end{cases}$$



Now go back to the sea battle

$$[\![\mathbf{F}p]\!]_w = \text{U}, \qquad [\![\mathbf{F}\neg p]\!]_w = \text{U}, \qquad [\![\mathbf{F}p \vee \mathbf{F}\neg p]\!]_w = \text{U}$$