Házifeladat

Ockhamist axioms

O :Ockhamist bundled trees

PC: Classical logic

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

(MP) $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

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$\mathbf{K4.3_{F,P}}$: Temporal logic of linear frames

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(C) $\mathbf{PG}\varphi \to \varphi$, $\quad \mathbf{FH}\varphi \to \varphi$

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(.3) $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$,
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Irreflexivity rule:

(IRR) $\dfrac{(p \wedge \mathbf{H}\neg p) \to \varphi}{\varphi}$ where p does not occur in φ

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Full-blooded Ockhamist axioms

(UPP) $\varphi \to \Box\varphi$ where $\mathbf{F}$ does not occur in φ.
unpreventability of past

(WDC) $\varphi \to \mathbf{G}\Box\mathbf{P}\Diamond\varphi$
weak diagram completion

(WDC+) $(p \wedge \mathbf{H}\neg p \wedge \Box\varphi) \to$
$\to \mathbf{G}\Box\mathbf{H}((p \wedge \mathbf{H}\neg p) \to \varphi)$

(MB) $\mathbf{G}\bot \to \Box\mathbf{G}\bot$ maximality of branches

Plan

1. We construct an irreflexive submodel $\mathfrak{M}_K^-$ of the canonical Kamp model $\mathfrak{M}_K$. We will prove that we can use that model to prove a completeness proof, i.e., we prove a
   1. Lindenbaum-type lemma
   2. Existence Lemma
   3. Truth Lemma
   4. The Canonical model is almost a Kamp model – canonicity proofs for all property except the maximality of histories.
2. We transform $\mathfrak{M}_K^-$ into an $\mathrm{M}(\mathfrak{M}_K^-)$ in which the histories are maximal.
3. We prove that $\mathfrak{M}_K^-$ is a zig-zag image of $\mathrm{M}(\mathfrak{M}_K^-)$.

   We can conclude a completeness theorem for Kamp semantics
4. We construct a bundled tree model from $\mathfrak{M}_K$ which satisfy the same formulas as $\mathrm{M}(\mathfrak{M}_K^-)$.

   We can conclude a completeness theorem for bundled tree semantics

Irreflexivity

Irreflexivity again

A canonical model usually has loops, and since we can not define irreflexivity, we can not remove those loops by adding new axioms to the logic. Previous solutions were:

Unraveling In case of K4, We *unravelled* the canonical model into an irreflexive, intransitive, antisymmetric tree, and then we took the transitive closure of the alternative relation, so we got an SPO. And since the canonical model of K4 is a zig-zag-image of this SPO, they satisfy the same formulas, so we can use it to prove completeness.

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Bulldozing In case of K4.3, We *bulldozed* the canonical model's clusters (so the loops as well) into backward and forward infinite lines. Since .3 formulas were canonical for non-branching, we got an STO. And since the canonical model of K4.3 is a zig-zag-image of this STO, they satisfy the same formulas, so we can use it to prove completeness.

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IRR-rule The philosophy of the previous ideas was that the canonical model is too compressed: They unravelled it, or replaced the clusters with infinite routes. This idea is that we can sort out the worlds that have loops without losing the useful properties of canonical model. So this solution is about to make the canonical model smaller, not larger.

IRR-THEORIES

Let us consider a loop as a sin.

A canonical world (m.c.s.) Γ can prove its innocence easily iff

$$p \wedge \mathbf{H}\neg p \in \Gamma \text{ for some } p \in \mathrm{At}.$$

A canonical world (m.c.s.) Γ is in the company of easily provable innocents iff

$$\frac{\mathrm{M}_1(\varphi_1 \wedge \mathrm{M}_2(\varphi_2 \wedge \cdots \wedge \mathrm{M}_{n-1}(\varphi_{n-1} \wedge \mathrm{M}_n\varphi_n) \dots)) \in \Gamma}{\begin{array}{l}\text{There is a } p \in \mathrm{At} \text{ not occurring in } \varphi_1, \dots, \varphi_n, \text{ s.t.} \\ \mathrm{M}_1(\varphi_1 \wedge \mathrm{M}_2(\varphi_2 \wedge \cdots \wedge \mathrm{M}_{n-1}(\varphi_{n-1} \wedge \mathrm{M}_n(\varphi_n \wedge p \wedge \mathbf{H}\neg p)) \dots)) \in \Gamma\end{array}}$$

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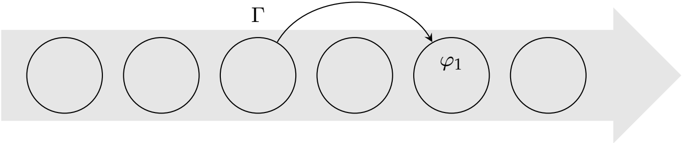

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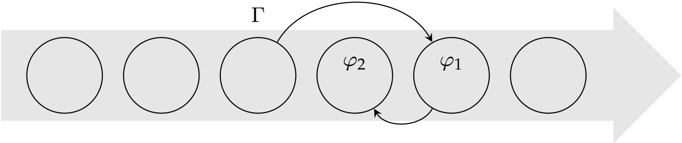

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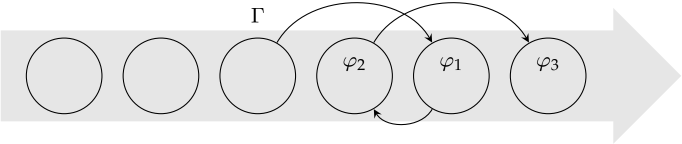

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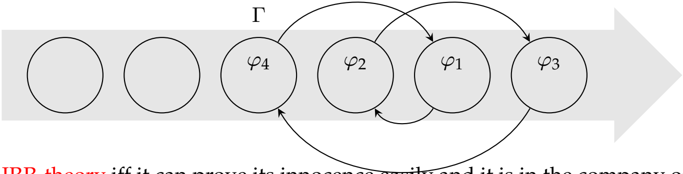

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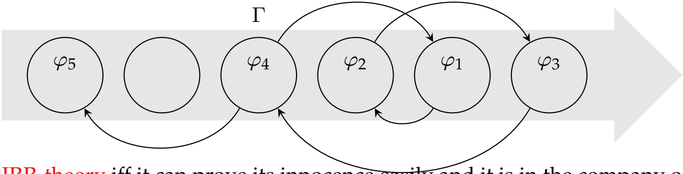

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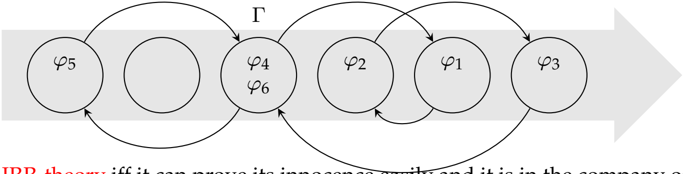

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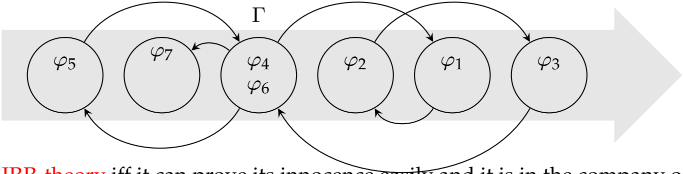

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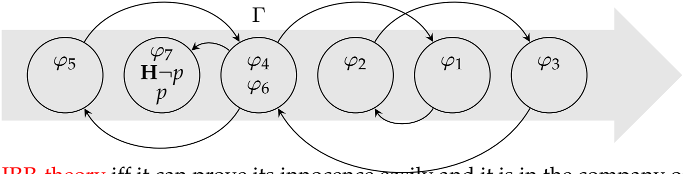

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IRR-THEORIES

ABBREVIATIONS

Let us quickly define the templates the IRR theories are talking about. We will call them scanner templates. To define the act where we put something deeply nested in the parentheses, we will use a placeholder $\bigcirc$, which is formally a syntactical object not occurring in the Kamp language.

- $\bigcirc$ is a scanner template.
- If T is a scanner template, then $\mathbf{F}(\varphi \wedge \mathrm{T})$, $\mathbf{P}(\varphi \wedge \mathrm{T})$ and $\Diamond(\varphi \wedge \mathrm{T})$ are scanner templates for any φ.

If T is a template, then let $\mathrm{T}(\varphi)$ denote $\mathrm{T}[\bigcirc/\varphi]$, i.e., the formula in which we replaced the occurrence of $\bigcirc$ by φ.

Using this definition, the definition of IRR theories can be put in a simple way:

DEFINITION: A maximally consistent set Γ is an IRR theory iff for all scanner templates T-s

$$\frac{\mathrm{T}(\top) \in \Gamma}{\mathrm{T}(p \wedge \mathbf{H}\neg p) \in \Gamma}$$

IRR-THEORIES

LINDENBAUM-TYPE LEMMA: A consistent theory in which an infinite number of atomic propositions do not occur, can be extended to a maximally consistent IRR-theory.

PROOF: Idea: We put an evidence for irreflexivity in Γ (we can do so, since there are atomic sentences to which Γ is indifferent) by hand, and we continue with the standard Lindenbaum's lemma in a way to ensure that the resulting set will be an IRR theory. So let Γ be a consistent theory described above, and let p an atom not occurring in Γ.

Let $\Sigma_0 \stackrel{\text{def}}{=} \Gamma \cup \{p \wedge \mathbf{H}\neg p\}$. This is consistent, for if

$$
\begin{array}{rcll}
\Gamma \cup \{p \wedge \mathbf{H}\neg p\} & \vdash & \bot & \text{ind.ass.} \\
\Gamma & \vdash & \neg(p \wedge \mathbf{H}\neg p) & \text{Ded.thm.} \\
\exists \Gamma_{\text{finite}} \supseteq \Gamma & \vdash & \neg(p \wedge \mathbf{H}\neg p) & \text{def.of} \vdash \\
 & \vdash & \bigwedge \Gamma_{\text{finite}} \to \neg(p \wedge \mathbf{H}\neg p) & \text{def.of} \vdash \\
 & \vdash & (p \wedge \mathbf{H}\neg p) \to \neg \bigwedge \Gamma_{\text{finite}} & \text{contraposition} \\
 & \vdash & \neg \bigwedge \Gamma_{\text{finite}} & \text{That is a valid rule, if } p \text{ does not occur in } \Gamma! \\
\Gamma_{\text{finite}} & \vdash & \bot & \text{ded.thm} \\
\Gamma & \vdash & \bot &
\end{array}
$$

and that contradicts to the assumption that Γ was consistent. So let us prove that this rule is valid indeed!

IRR-THEORIES

LEMMA: The following rule is valid:

$$(IRR) \qquad \frac{(p \wedge \mathbf{H}\neg p) \to \varphi}{\varphi} \qquad \text{where } p \text{ does not occur in } \varphi$$

PROOF: Suppose that $(p \wedge \mathbf{H}\neg p) \to \varphi$ is valid on a Kamp-frame $\mathfrak{K}$, i.e., true in all worlds w.r.t. any Kamp-valuation. Now take an arbitrary but fixed world w and Kamp-valuation V. We will prove that $\mathfrak{K}, V, w \models^{K} \varphi$

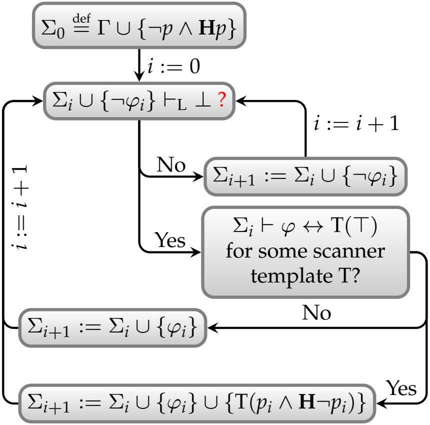
$$\begin{array}{rcll}
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & (p \wedge \mathbf{H}\neg p) \to \varphi & \text{assumption} \\
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & p \wedge \mathbf{H}\neg p & \text{By } V[p \mapsto \{v : w \equiv v\}] \\
\mathfrak{K}, V[p \mapsto \{v : w \equiv v\}], w & \models^{K} & \varphi & \text{modus ponens} \\
\mathfrak{K}, V, w & \models^{K} & \varphi & p \text{ did not occur in } \varphi
\end{array}$$

■

IRR-THEORIES

A consistent theory in which an infinite number of atomic propositions do not occur, can be extended to a maximally consistent IRR-theory.

- Let φ be an enumeration of the well-formed formulas.
- Let p be an enumeration of the atomic formulas *not occurring in* Γ.
- If the first answer is No, then Σ_{i+1} is consistent.
- If the answers are Yes - No, then Σ_{i+1} is consistent.
- If the answers are Yes - Yes, then Σ_{i+1} is consistent by basically because of the validity of the IRR-rule, but the proof is a little bit more complicated than that - solve it at home!
- Let $\Gamma^+ \stackrel{\text{def}}{=} \bigcup\{\Sigma_i : i \in \omega\}$. This is clearly a maximally consistent IRR-theory.



Irreflexive canonical submodel

Canonical Kamp model

$$\mathfrak{M}_{\mathbf{O}} \overset{\text{def}}{=} (W_{\mathbf{O}}, <_{\mathbf{O}}, \equiv_{\mathbf{O}}, V_{\mathbf{O}})$$

where

- $W_{\mathbf{O}} \overset{\text{def}}{=} \{\Gamma : \Gamma$ is a maximally $\mathbf{O}$-consistent IRR-theory$\}$,
- $\Gamma <_{\mathbf{O}} \Gamma'$ iff $\mathbf{G}^-(\Gamma) \subseteq \Gamma'$ Remember that these are equivalent:
$$\begin{array}{rcl} \mathbf{G}^-(\Gamma) & \subseteq & \Gamma' \\ \Gamma & \supseteq & \mathbf{F}^+(\Gamma') \\ \Gamma & \supseteq & \mathbf{H}^-(\Gamma') \\ \mathbf{P}^+(\Gamma) & \subseteq & \Gamma' \end{array}$$
- $\Gamma \equiv_{\mathbf{O}} \Gamma'$ iff $\Box^-(\Gamma) \subseteq \Gamma'$, Similarly: $\begin{array}{rcl} \Box^-(\Gamma) & \subseteq & \Gamma' \\ \Gamma & \supseteq & \Diamond^+(\Gamma') \end{array}$
- $\Gamma \in V_{\mathbf{O}}(p) \overset{\text{def}}{\Leftrightarrow} p \in \Gamma$.

THE OLD EXISTENCE LEMMAS

Let L denote the dual pair of M. The new Existence Lemmas:

$$\mathrm{M}\psi \in \Gamma \Longrightarrow (\exists \Gamma' \in W_{\mathbf{O}})\left[\Gamma' \supseteq \mathrm{L}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\right]$$

The old proofs ensured only the existence of a max. con. set,

but now we need the existence of a max.con. IRR-theory!

But we can reuse the proof to state the following lemma very useful lemma:

LEMMA: If Γ is a max.con. set, then

$$\mathrm{M}\varphi \in \Gamma \quad \Longrightarrow \quad \mathrm{L}^{-}(\Gamma) \cup \{\varphi\} \nvdash_{\mathbf{O}} \bot$$

PROOF:

$\mathrm{L}^{-}(\Gamma) \cup \{\varphi\}$	$\vdash_{\mathbf{O}}$	$\bot$	indirect assumption
$\mathrm{L}^{-}(\Gamma)$	$\vdash_{\mathbf{O}}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\vdash_{\mathbf{O}}$	$\neg\varphi$	def.of $\mathrm{L}^{-}(\Gamma) \vdash_{\mathbf{O}}$
	$\vdash_{\mathbf{O}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \vdash_{\mathbf{O}}$
	$\vdash_{\mathbf{O}}$	$\mathrm{L}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathrm{L}\neg\varphi$	Lemmon
	$\vdash_{\mathbf{O}}$	$(\mathrm{L}\chi_1 \wedge \cdots \wedge \mathrm{L}\chi_n) \to \mathrm{L}\neg\varphi$	A-axiom
$\mathrm{L}\chi_1, \cdots, \mathrm{L}\chi_n$	$\vdash_{\mathbf{O}}$	$\mathrm{L}\neg\varphi$	def.of $(\mathrm{L}\chi_1 \wedge \cdots \wedge \mathrm{L}\chi_n) \vdash_{\mathbf{O}}$
Γ	$\vdash_{\mathbf{O}}$	$\mathrm{L}\neg\varphi$	$\chi \in \mathrm{L}^{-}(\Gamma) \Leftrightarrow \mathrm{L}\chi \in \Gamma$
Γ	$\vdash_{\mathbf{O}}$	$\neg\mathbf{F}\varphi$	Duality
$\Gamma \cup \{\mathbf{F}\varphi\}$	$\vdash_{\mathbf{O}}$	$\bot$	Deduction theorem
Γ	$\vdash_{\mathbf{O}}$	$\bot$	we assumed that $\mathbf{F}\varphi \in \Gamma$

THE NEW EXISTENCE LEMMAS

Let L denote the dual pair of M.

$$\mathrm{M}\psi \in \Gamma \Longrightarrow (\exists \Gamma' \in W_{\mathbf{O}})\left[\Gamma' \supseteq \mathrm{L}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\right]$$

The old proof ensured the existence of a max. con. set, but now we need the existence of a max.con. IRR-theory!

$$\begin{array}{rcll} \Gamma & \ni & \mathrm{M}\varphi & \text{assumption} \\ \Gamma & \ni & \mathrm{M}(\varphi \wedge \top) & \\ (\exists p \in \mathrm{At})\ \Gamma & \ni & \mathrm{M}(\underbrace{\varphi \wedge p \wedge \mathbf{H}\neg p}_{\psi}) & \Gamma \text{ is IRR } (\mathrm{T} = \mathbf{F}\bigcirc) \end{array}$$

Remember that this p does not occur in φ! Now let $\psi \stackrel{\text{def}}{=} \varphi \wedge p \wedge \mathbf{H}\neg p$. Then we have from our old proof that $\Sigma_0 \stackrel{\text{def}}{=} \mathrm{L}^{-}(\Gamma) \cup \{\psi\}$ is **O**-consistent.

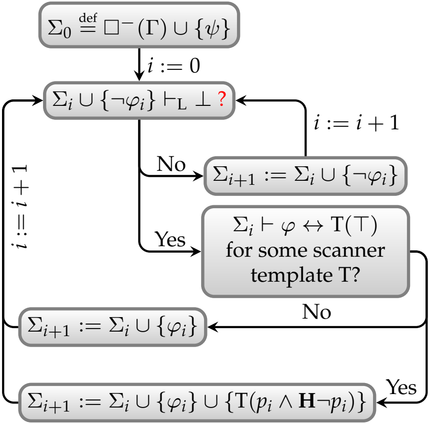
We cannot use the IRR version of Lindenbaum lemma since it is possible that *only finitely many* atomic sentences do not occur in $\mathrm{L}^{-}(\Gamma)$.

The point is, that we have to choose the desired p-s wisely, because they can occur $\mathrm{L}^{-}(\Gamma)$. In fact, If we start with L^{-}, then the IRR property of Γ will provide us a nice infinite set of suitable p-s.

IRR-THEORIES

It is clear by the construction that $\bigcup\{\Sigma_i : i \in \omega\}$ is consistent. Now we have to prove that it is an IRR theory.
It is enough to show that in every step, at the Yes-Yes-case we can find a suitable p_i (even if it occurs in Σ_i).

- If we are at the Yes-Yes case, $\Sigma_i \vdash_O \varphi_i$, i.e., $\Sigma_i \vdash_O \mathrm{T}(\top)$.



IRR-THEORIES

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- If we are at the Yes-Yes case, $\Sigma_i \vdash_O \varphi_i$, i.e., $\Sigma_i \vdash_O T(\top)$.
- Using that we can show that $M(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \varphi_i) \in \Gamma$:

$$
\begin{array}{rcll}
\Gamma & \not\ni & M(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \varphi_i) & \text{indirect assumption} \\
\Gamma & \ni & \neg M(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \varphi_i) & \Gamma \text{ is max.con.} \\
\Gamma & \ni & L((\psi \wedge \bigwedge(\Sigma_i - \Sigma_0)) \to \neg\varphi_i) & \text{push in the } \neg \\
\Box^-(\Gamma) & \ni & (\psi \wedge \bigwedge(\Sigma_i - \Sigma_0)) \to \neg\varphi_i & \text{def.of } \Box^- \\
\Sigma_0 & \ni & (\psi \wedge \bigwedge(\Sigma_i - \Sigma_0)) \to \neg\varphi_i & \Sigma_0 \overset{\text{def}}{=} L^-(\Gamma) \cup \{\psi\} \\
\Sigma_0 & \ni & \psi & \Sigma_0 \overset{\text{def}}{=} L^-(\Gamma) \cup \{\psi\} \\
\Sigma_0 & \vdash_O & \bigwedge(\Sigma_i - \Sigma_0) \to \neg\varphi_i & \text{modus ponens} \\
\Sigma_i & \vdash_O & \bigwedge(\Sigma_i - \Sigma_0) \to \neg\varphi_i & \Sigma_i \supseteq \Sigma_0 \\
\Sigma_i & \vdash_O & \bigwedge(\Sigma_i - \Sigma_0) & \text{all conjuncts of } \bigwedge(\Sigma_i - \Sigma_0) \text{ are in } \Sigma_i \\
\Sigma_i & \vdash_O & \neg\varphi_i & \text{modus ponens, contradiction}
\end{array}
$$

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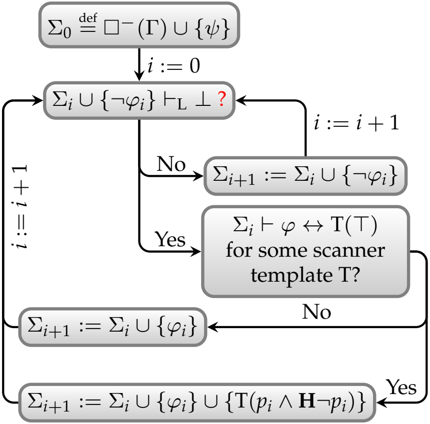
- If we are at the Yes-Yes case, $\Sigma_i \vdash_{\mathrm{O}} \varphi_i$, i.e., $\Sigma_i \vdash_{\mathrm{O}} \mathrm{T}(\top)$.
- Using that we can show that $\mathrm{M}(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \varphi_i) \in \Gamma$:
- But $\mathrm{M}(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \varphi_i)$, i.e., $\mathrm{M}(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \mathrm{T}(\top))$ is a scanner template.



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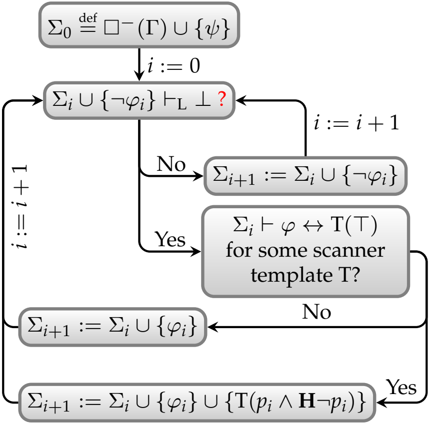
- Then since Γ itself was IRR, there is a p_i not in $\psi \wedge \bigwedge(\Sigma_i - \Sigma_0)$, s.t. $\mathrm{M}(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge \mathrm{T}(p_i \wedge \mathbf{H}\neg p_i)) \in \Gamma$.



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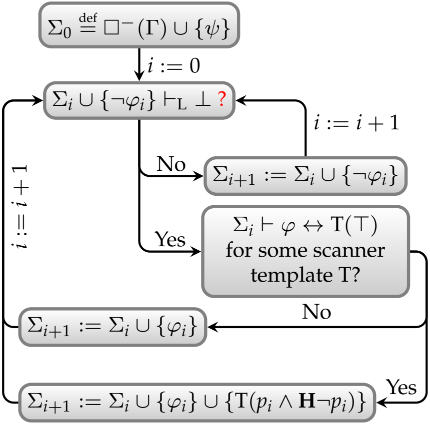

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- Then since Γ itself was IRR, there is a p_i not in $\psi \wedge \bigwedge(\Sigma_i - \Sigma_0)$, s.t. $M(\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge T(p_i \wedge \mathbf{H}\neg p_i)) \in \Gamma$.
- Then we have that $\Box^-(\Gamma) \cup \{\psi \wedge \bigwedge(\Sigma_i - \Sigma_0) \wedge T(p_i \wedge \mathbf{H}\neg p_i)\}$ is consistent.

IRR-THEORIES

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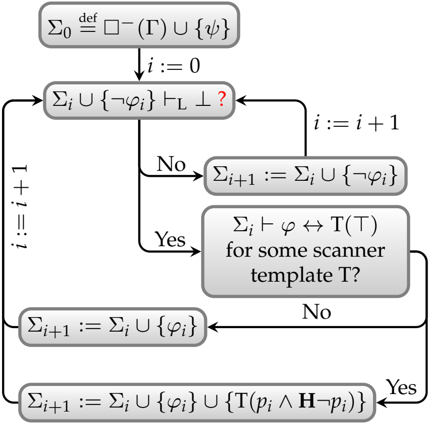
It is enough to show that in every step, at the Yes-Yes-case we can find a suitable p_i (even if it occurs in Σ_i).



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- Then we have that $\underbrace{\underbrace{\Box^-(\Gamma) \cup \{\psi\}}_{\Sigma_0} \cup (\Sigma_i - \Sigma_0)}_{\Sigma_i} \cup \{T(p_i \wedge \mathbf{H}\neg p_i)\}$ is consistent.

Existence Lemma

$$\mathbf{P}\psi \in \Gamma \iff \psi \in \mathbf{H}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{H}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

$\mathbf{H}^{-}(\Gamma) \cup \{\varphi\}$ is $\mathbf{K}$-consistent. For if

$\mathbf{H}^{-}(\Gamma) \cup \{\varphi\}$	$\vdash_{\mathbf{K}}$	$\bot$	indirect assumption
$\mathbf{H}^{-}(\Gamma)$	$\vdash_{\mathbf{K}}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\vdash_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{H}^{-}(\Gamma) \vdash_{\mathbf{K}}$
	$\vdash_{\mathbf{K}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \vdash_{\mathbf{K}}$
	$\vdash_{\mathbf{K}}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{H}\neg\varphi$	See below!
	$\vdash_{\mathbf{K}}$	$(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \to \mathbf{H}\neg\varphi$	See below!
$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\vdash_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \vdash_{\mathbf{K}}$
Γ	$\vdash_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^{-}(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
Γ	$\vdash_{\mathbf{K}}$	$\neg\mathbf{P}\varphi$	Duality
$\Gamma \cup \{\mathbf{P}\varphi\}$	$\vdash_{\mathbf{K}}$	$\bot$	Deduction theorem
Γ	$\vdash_{\mathbf{K}}$	$\bot$	we assumed that $\mathbf{P}\varphi \in \Gamma$

Remember that we relied on basically the following two logical rule:

$$\vdash_{\mathbf{K}} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi) \qquad \frac{\vdash_{\mathbf{K}} \varphi \to \psi}{\vdash_{\mathbf{K}} \mathbf{H}\varphi \to \mathbf{H}\psi}$$

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$$\mathbf{P}\psi \in \Gamma \iff \psi \in \mathbf{H}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{H}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

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$\exists \chi_1, \cdots, \chi_n$	$\vdash_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{H}^{-}(\Gamma) \vdash_{\mathbf{K}}$
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	$\vdash_{\mathbf{K}}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{H}\neg\varphi$	See below!
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$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\vdash_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \vdash_{\mathbf{K}}$
Γ	$\vdash_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^{-}(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
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$\Gamma \cup \{\mathbf{P}\varphi\}$	$\vdash_{\mathbf{K}}$	$\bot$	Deduction theorem
Γ	$\vdash_{\mathbf{K}}$	$\bot$	we assumed that $\mathbf{P}\varphi \in \Gamma$

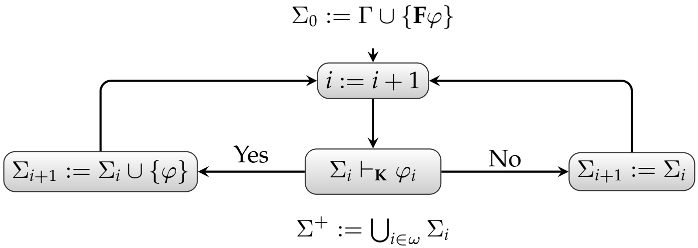
Remember that we relied on basically the following two logical rule:

$$\vdash_{\mathbf{K}} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi) \qquad \frac{\vdash_{\mathbf{K}} \varphi \to \psi}{\vdash_{\mathbf{K}} \mathbf{H}\varphi \to \mathbf{H}\psi}$$

Prove that these are valid indeed!

LINDENBAUM'S LEMMA

Since $\mathbf{G}^-(\Gamma) \cup \{\varphi\}$ is **K**-consistent, it can be extended into a maximally consistent set Γ'. Just list all the formulas and start the following procedure: take the first formula: Is it consistent with $\Sigma_0 \overset{\text{def}}{=} \mathbf{G}^-(\Gamma) \cup \{\varphi\}$? If it is, then extend Σ_0 with that formula, if not, then don't. Repeat this into the infinity. Your m.c.s. will be $\mathbf{G}^-(\Gamma) \cup \{\varphi\}$ the one will contain every formula with which you would extend.

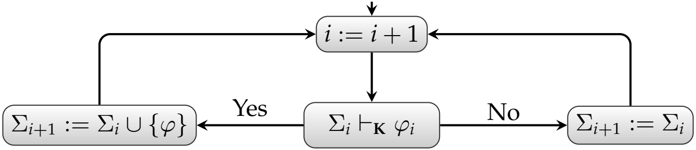


Similarly for **H**.

Lindenbaum's Lemma

Since $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ is **K**-consistent, it can be extended into a maximally consistent set Γ'. Just list all the formulas and start the following procedure: take the first formula: Is it consistent with $\Sigma_0 \stackrel{\text{def}}{=} \mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$? If it is, then extend Σ_0 with that formula, if not, then don't. Repeat this into the infinity. Your m.c.s. will be $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ the one will contain every formula with which you would extend.

$$\Sigma_0 := \Gamma \cup \{\mathbf{F}\varphi\}$$



$$\Sigma^{+} := \bigcup_{i \in \omega} \Sigma_i$$

Similarly for **H**.

Prove that Σ^{+} must be consistent!

Absolute freedom / Characterization / canonical model theorem

Only (Exactly) the valid formulas are true in $\mathfrak{M}_{\mathbf{K}}$.

$$\mathfrak{M}_{\mathbf{K}} \models \varphi \iff \vdash_{\mathbf{K}} \varphi$$

We show that

$$\mathfrak{M}_{\mathbf{K}} \not\models \varphi \iff \nvdash_{\mathbf{K}} \varphi.$$

Since the construction called "canonical model" is a real *model* indeed, we have the $\Rightarrow$ direction.
If $\nvdash_{\mathbf{K}} \varphi$, then $\{\neg\varphi\}$ is **K**-consistent. Therefore we can extend it to a maximally **K**-consistent $\Gamma^{\neg\varphi}$ set by Lindenbaum's lemma. But this set is a world in the canonical model $\mathfrak{M}_{\mathbf{K}}$. And since this world contains $\neg\varphi$, it is *true* in it by the Truth lemma:

$$\neg\varphi \in \Gamma^{\neg\varphi} \Longrightarrow \mathfrak{M}_{\mathbf{K}}, \Gamma^{\neg\varphi} \models \neg\varphi$$

And we are ready, since we found a world of $\mathfrak{M}_{\mathbf{K}}$ where $\neg\varphi$ is true, i.e., φ is not true neither in that world nor in the whole model.

Logics

LOGICS

<u>DEFINITION</u>: A *normal temporal propositional logic* is a set of formulas that contains every **K**-valid formula and is closed under the rules of **K**.

<u>THEOREM</u>: **K** is the smallest normal temporal propositional logic.

<u>DEFINITION</u>: We denote the smallest normal temporal propositional logic that contains (the syntactically defined) **K** and φ with $\mathbf{K} + (\varphi)$

<u>DEFINITION</u>: A formula φ is *canonical* for a property P, iff besides that φ is valid on P-frames, 'by taking it as an axiom the canonical model of that new logic becomes P':

$$(\forall L \supseteq \mathbf{K} + (\varphi))\ \mathfrak{M}_L \text{ has the property } P$$

Axiomatizing Transitivity

THEOREM: $\mathbf{G}\varphi \to \mathbf{GG}\varphi$ is canonical for $wRw'Rw'' \Rightarrow wRw''$

PROOF: defining Let L be a n.t.p. logic that contains the scheme (4) and let Γ, Γ', Γ'' be arbitrary canonical worlds s.t. $\Gamma R_{\mathrm{L}} \Gamma' R_{\mathrm{L}} \Gamma''$. We have to prove that $\mathbf{G}^{-}(\Gamma) \subseteq \Gamma''$. Take a $\mathbf{G}\varphi \in \Gamma$. Then by (4), $\mathbf{GG}\varphi \in \Gamma$, therefore $\mathbf{G}\varphi \in \mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$ and $\varphi \in \mathbf{G}^{-}(\Gamma') \subseteq \Gamma''$.

COROLLARY: $\mathbf{K} + (4)$ axiomatizes the logic of transitive frames.

(Because $\mathfrak{M}_{\mathbf{K}+(4)}$ will count as a counter-model.)

AXIOMATIZING NON-BRANCHING

THEOREM: $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$ is canonical for $(wRw_1 \text{ and } wRw_2) \Rightarrow (w_1Rw_2 \text{ or } w_1 = w_2 \text{ or } w_1 Я w_2)$,

PROOF: Let L be a n.t.p. logic containing the formula (H.3). Let Γ be arbitrary but fixed, and let Γ_1 and Γ_2 be arbitrary R_L-neighbours of Γ.

If $\Gamma_1 = \Gamma_2$, then we are ready. If $\Gamma_1 \neq \Gamma_2$, then suppose indirectly that they are not related by R_L at all. That would mean that there is a formula $\mathbf{H}\varphi \in \Gamma_1$ for which $\varphi \notin \Gamma_2$, and similarly, that there is a formula $\mathbf{H}\psi \in \Gamma_2$ for which $\psi \notin \Gamma_1$. So $\mathbf{H}\varphi, \neg\psi \in \Gamma_1$ and $\mathbf{H}\psi, \neg\varphi \in \Gamma_2$.

In this case we would have that $\neg(\underline{\mathbf{H}}\psi \to \varphi) \in \Gamma_1$ and $\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma_2$, therefore, since both of Γ_1 and Γ_2 are R_L-related to Γ we have that $\mathbf{P}\neg(\underline{\mathbf{H}}\psi \to \varphi) \in \Gamma$ and $\mathbf{P}\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$, i.e., even $\mathbf{P}\neg(\underline{\mathbf{H}}\psi \to \varphi) \wedge \mathbf{P}\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$, hence $\neg\mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi) \wedge \neg\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$ which makes Γ inconsistent.

COROLLARY: $\mathbf{K} + (H.3)$ axiomatizes the logic of those frames where there are no branching in the past.