Assignment
Deadline: 1st May 2022

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Question 1: The Binomial Method

Consider a discrete model for a stock price $S_0, S_{t_1}, \ldots, S_{t_M}$ with $t_{i+1} = t_i + \Delta t$ and:

$$\mathbb{P}\left(S_{t_{i+1}} = uS_{t_i}\right) = p$$
$$\mathbb{P}\left(S_{t_{i+1}} = dS_{t_i}\right) = 1 - p$$

for constants $0 < d < u$ such that $u \cdot d = 1$ and a probability $0 < p < 1$. We assume a constant continuously compounded interest rate $r > 0$.

1.1

What is the price of the zero-coupon bond $P(t_i, t_m)$?

1.2

For what values of u, d and p do we obtain the risk neutral dynamics?

1.3

Price a European call at $t_0 = 0$ with maturity $t_M = 1$, strike $K = 70$, interest rate $r = 0.01$, upward move $u = 1.05$ and current spot $S_0 = 50$. Derive numerically the convergence rate as $M \to \infty$.

1.4

Construct a delta hedging strategy. Experiment with Monte-Carlo simulations how well do you replicate the option as $M \to \infty$.

1.5

Use the dynamic programming principle to price an American put with strike $K = 35$, using the same set-up and performing the same analysis as in previous questions.

Question 2: PDEs methods

Assume a constant continuously compounded interest rate $r > 0$ and the GBM model for the stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

2.1

Derive the Black-Scholes PDE for the present value $V(t, S_t)$ of a European call or put option:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

with maturity date T.

2.2

Transform the Black-Scholes PDE into a heat equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}, \quad \tau \geq 0$$

How far τ evolves? What are the initial conditions? What is the asymptotic of y as $x \to \pm\infty$?

2.3

For the heat equation above, derive the explicit, the implicit and the Crank-Nicolson finite-difference schemes with time, space steps $\Delta\tau, \Delta x$ respectively. For each of those schemes:

- How do you choose the boundary conditions for a bounded space domain $x_{min} \leq x \leq x_{max}$
- For what values of $\Delta\tau, \Delta x$ we have numerical stability (i.e. the rounding error is bounded with each iteration step)? Provide numerical examples to show that your condition is sharp.
- What is the order of convergence as $\Delta\tau, \Delta x \to 0$?

2.4

For a non-trivial set of parameters of your choice, price the European call using the Crank-Nicolson method. Compare the solution with the analytic Black-Scholes formula and derive numerically the order of convergence as $\Delta\tau, \Delta x \to 0$ for your implementation.

2.5*

Starting from the Crank-Nicolson method for the European put option, use the dynamic programming principle to derive a scheme to price the American put option. (*Hint: Brennan-Schwartz algorithm*)