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github.com/epfml/OptML_course

# Problem Set 10 — Solutions (Convex conjugate)

For a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ (which is not necessarily convex !), we consider its **convex conjugate** which for $y \in \mathbb{R}^d$ is defined as

$$f^*(y) = \sup_{x \in \mathbb{R}^d} (\langle x, y \rangle - f(x)) \in \mathbb{R} \cup \{+\infty\}$$

Prove the following properties.

1. Show that $f^*$ is convex.

   **Proof:** Note that $f^*$ is the pointwise supremum of **affine functions** $y \mapsto \langle x, y \rangle - f(x)$. As seen in the first class, the pointwise supremum of convex functions is convex. Therefore $f^*$ is convex.

2. Show that for $x, y \in \mathbb{R}^d$, $f(x) + f^*(y) \geq \langle x, y \rangle$. This is known as the Fenchel inequality.

   **Proof:** For $y \in \mathbb{R}^d$, $f^*(y) = \sup_{x \in \mathbb{R}^d}(\langle x, y \rangle - f(x)) \geq \langle x, y \rangle - f(x)$ for all $x \in \mathbb{R}^d$.

3. Show that the biconjugate $f^{**}$ (the conjugate of the conjugate) is such that $f^{**} \leq f$.

   **Proof:** From the previous inequality we have that for all $x, y \in \mathbb{R}^d$, $f(x) \geq \langle x, y \rangle - f^*(y)$, we can therefore take the supremum over $y$ of the left hand side: $f(x) \geq \sup_{y \in \mathbb{R}^d}(\langle y, x \rangle - f^*(y)) = f^{**}(x)$

The Fenchel-Moreau theorem (which we will not prove here) states that $f = f^{**}$ if and only if $f$ is convex and closed. It will turn out to be useful to show the following property.

4. Assume that $f$ is closed and convex. Then show that for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned} y \in \partial f(x) &\Leftrightarrow x \in \partial f^*(y) \\ &\Leftrightarrow f(x) + f^*(y) = \langle x, y \rangle \end{aligned}$$

   **Proof that** $y \in \partial f(x) \Rightarrow f(x) + f^*(y) = \langle x, y \rangle$ **:** Assume that $y \in \partial f(x)$, then we have that for all $z \in \mathbb{R}^d$, $f(z) \geq f(x) + \langle y, z - x \rangle$. Therefore for all $z \in \mathbb{R}^d$, $\langle y, x \rangle - f(x) \geq \langle z, y \rangle - f(z)$. We can therefore take the supremum of the left hand size which gives that $\langle y, x \rangle - f(x) \geq \sup_z(\langle z, y \rangle - f(z))$ which also means that $\langle y, x \rangle - f(x) = \sup_z \langle z, y \rangle - f(z) = f^*(y)$ which proves the first part of the result.

   **Proof that** $f(x) + f^*(y) = \langle x, y \rangle \Rightarrow y \in \partial f(x)$ **:** We basically do the previous reasoning the other way round. Let $x, y \in \mathbb{R}^d$ such that $f(x) + f^*(y) = \langle x, y \rangle$. Therefore $\langle x, y \rangle - f(x) = f^*(y) = \sup_z(\langle z, y \rangle - f(z)) \geq \langle z, y \rangle - f(z)$ for all $z \in \mathbb{R}^d$. Rearranging we get that for all $z \in \mathbb{R}^d$, $f(z) \geq f(x) + \langle y, z - x \rangle$ which means that $y \in \partial f(x)$.

   Hence we have shown that $y \in \partial f(x) \Leftrightarrow f(x) + f^*(y) = \langle x, y \rangle$. Now we can apply this same result to $f^*$: $x \in \partial f^*(y) \Leftrightarrow f^*(y) + f^{**}(x) = \langle y, x \rangle$. Since $f$ is closed and convex, by the Fenchel-Moreau theorem we have that $f = f^{**}$, hence $x \in \partial f^*(y) \Leftrightarrow f^*(y) + f(x) = \langle y, x \rangle$. Therefore all the implications are proven.