

Problem Set 6 — Solutions (Non-Convex Optimization)

Theoretical Exercises

Exercise 38. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ twice differentiable, with $X \subseteq \text{dom}(f)$ an open convex set, and suppose that f is smooth with parameter L over X . Prove that under these conditions, the largest eigenvalue of the Hessian $\lambda_{\max}(\nabla^2 f(\mathbf{x})) \leq L$ for all $\mathbf{x} \in X$.

Solution: To prove that the largest eigenvalue of $\nabla^2 f(\mathbf{x})$ is $\leq L$ for all $\mathbf{x} \in X$, we need to show that $\mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq L \|\mathbf{y}\|_2^2$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in \mathbb{R}^d$.

Since X is an open set, for any $\mathbf{x} \in X$, there exists a $c > 0$ such that for all $\lambda \in [0, c]$ and $\mathbf{y} \in \mathbb{R}^d$, $\mathbf{x} + \lambda \mathbf{y} \in X$. For any such \mathbf{x}, \mathbf{y} , and λ , we can use Taylor's theorem and write

$$f(\mathbf{x} + \lambda \mathbf{y}) = f(\mathbf{x}) + \lambda \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle + \frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} + o(\lambda^2).$$

Using the smoothness of f over X , we also have that

$$f(\mathbf{x} + \lambda \mathbf{y}) \leq f(\mathbf{x}) + \lambda \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle + \frac{L\lambda^2}{2} \|\mathbf{y}\|_2^2.$$

These two expressions can be combined into the observation that

$$\frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq \frac{L\lambda^2}{2} \|\mathbf{y}\|_2^2 - o(\lambda^2).$$

For any $\lambda > 0$, we can divide both sides by $\frac{\lambda^2}{2}$ and tend λ to zero to obtain:

$$\mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq \lim_{\lambda \rightarrow 0} L \|\mathbf{y}\|_2^2 - 2 \frac{o(\lambda^2)}{\lambda^2} = L \|\mathbf{y}\|_2^2.$$

This proves the exercise.

Exercise 39. Prove that the statement of Theorem 6.2 implies that

$$\lim_{t \rightarrow \infty} \|\nabla f(\mathbf{x}_t)\|^2 = 0.$$

Solution: Let $\nabla_t = \|\nabla f(\mathbf{x}_t)\|^2$. The statement of Theorem 6.2 is that

$$S_T := \sum_{t=0}^{T-1} \nabla_t \leq K := 2L(f(\mathbf{x}_0) - f(\mathbf{x}^*)), \quad T > 0.$$

Hence, $(S_T)_{T>0}$ is a monotone increasing and bounded sequence and as such has a limit $K' \leq K$. This means that for all $\varepsilon > 0$, there exists T_0 such that $S_T > K' - \varepsilon$ for all $T \geq T_0$. As $S_{T+1} \leq K'$ for all T , we have

$$\nabla_T = S_{T+1} - S_T < \varepsilon$$

for all $T \geq T_0$. And this means that

$$\lim_{T \rightarrow \infty} \nabla_T = 0.$$

Practical Exercises

The theory of non-convex optimization is unfortunately not very illuminative. However, their practical performance is usually unmatched by convex methods. In this exercise, we will use the PyTorch framework to train a small neural network on some simple datasets.

Problem 1 (PyTorch Refresher):

Installation instructions:

pytorch.org

We recommend using the following online tutorial:

pytorch.org/tutorials/beginner/pytorch_with_examples.html

You can also refer to the following exercise from the Machine Learning course:

https://github.com/epfml/ML_course/blob/master/labs/ex12/

Problem 2 (Simple Neural Network):

Follow the notebook provided here:

https://github.com/epfml/OptML_course/tree/master/labs/ex06