Labs
Optimization for Machine Learning
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github.com/epfml/OptML_course

Problem Set 1 – Solutions (Convexity)

Convexity

Exercise 2. *Prove Jensen's inequality (Lemma 1.13)!*

Solution: For $m = 1$, there is nothing to prove, and for $m = 2$, the statement holds by convexity of f. For $m > 2$, we proceed by induction. If $\lambda_m = 1$ (and hence all other λ_i are zero), the statement is trivial. Otherwise, let $\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i$ and define

$$\mathbf{y} = \sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} \mathbf{x}_i.$$

Thus we have $\mathbf{x} = (1-\lambda_m)\mathbf{y} + \lambda_m \mathbf{x}_m$. Also observe that $\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} = 1$. By convexity and Jensens's inequality that we inductively assume to hold for $m-1$ terms, we get

$$\begin{aligned} f(\mathbf{x}) &= f((1-\lambda_m)\mathbf{y} + \lambda_m \mathbf{x}_m) \\ &\leq (1-\lambda_m) f\left(\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} \mathbf{x}_i\right) + \lambda_m f(\mathbf{x}_m) \\ &\leq (1-\lambda_m)\left(\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} f(\mathbf{x}_i)\right) + \lambda_m f(\mathbf{x}_m) = \sum_{i=1}^m \lambda_i f(\mathbf{x}_i). \end{aligned}$$

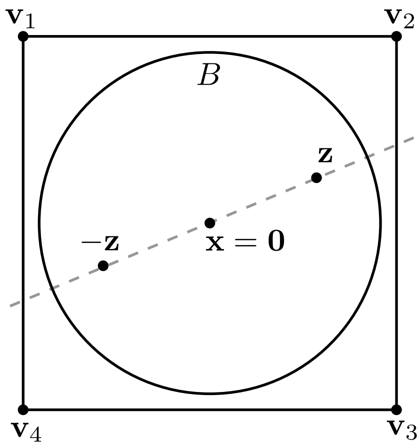
Exercise 3. *Prove that a convex function (with $\mathbf{dom}(f)$ open) is continuous (Lemma 1.14)!*

Hint: *First prove that a convex function f is bounded on any cube $C = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_d, u_d] \subseteq \mathbf{dom}(f)$, with the maximum value occurring on some corner of the cube (a point $\mathbf{z}$ such that $z_i \in \{l_i, u_i\}$ for all i). Then use this fact to show that—given $\mathbf{x} \in \mathbf{dom}(f)$ and $\varepsilon > 0$—all $\mathbf{y}$ in a sufficiently small ball around $\mathbf{x}$ satisfy $|f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$.*

Solution: We will prove that, for any $\mathbf{x} \in \mathbf{dom}(f)$ the function f is continuous at point $\mathbf{x}$. For that we will prove:

1. There exists a ball $B \subset \mathbf{dom}(f)$ with center $\mathbf{x}$ with some radius $R > 0$ for which function difference is bounded, i.e. $|f(\mathbf{y}) - f(\mathbf{x})| \leq \gamma \; \forall \mathbf{y} \in B$ for some finite $\gamma \geq 0$.
2. If $\gamma > \varepsilon$, any point $\mathbf{y}$ in the smaller ball B' with center $\mathbf{x}$ with radius $\frac{R\varepsilon}{\gamma}$ satisfy $|f(\mathbf{y}) - f(\mathbf{x})| \leq \varepsilon$, so f is continuous at $\mathbf{x}$.

1. Existence of B



Assume without loss of generality that $\mathbf{x} = 0$ and $f(\mathbf{x}) = 0$. Now $f(\mathbf{y}) = f(\mathbf{y}) - f(\mathbf{x})$ and $\|y\| = \|y - x\|$.

Since the domain of f is open, there exists a cube with center $\mathbf{x} = \mathbf{0}$ that lies inside the domain. Because a cube is a convex set, any point $\mathbf{p}$ inside it can be written as a convex sum of the cube's 2^d vertices $\mathbf{v}_i$: $\mathbf{p} = \sum_{i=1}^{2^d} \lambda_i \mathbf{v}_i$, where $\lambda_i \geq 0 \; \forall i$ and $\sum_{i=1}^{2^d} \lambda_i = 1$. Due to convexity of f,

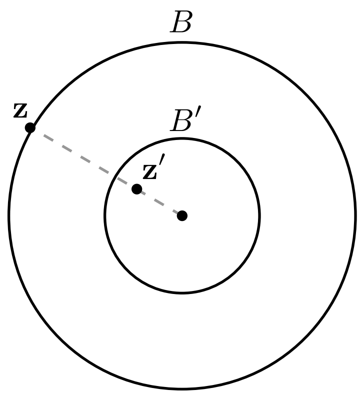
$$f(\mathbf{p}) \leq \sum_{i=1}^{2^d} \lambda_i f(\mathbf{v}_i) \leq \sum_{i=1}^{2^d} \lambda_i \max_i f(\mathbf{v}_i) = \max_i f(\mathbf{v}_i).$$

Because a cube has a finite number of vertices, this maximum exists, and the value of f inside the cube is bounded.

There exists a ball B with center $\mathbf{x}$ inside the cube with some radius R. Because the ball is a subset of the cube, f is bounded from above in the ball as well: $f(\mathbf{y}) \leq (\gamma := \max_i f(\mathbf{v}_i))$ for all $\mathbf{y} \in B$.

We will now show that f inside the ball is also bounded from below to finish this part of the proof. Consider any point $\mathbf{z} \in B$. By symmetry, $-\mathbf{z} \in B$ as well. Because the midpoint $\frac{1}{2}(\mathbf{z} + -\mathbf{z}) = \mathbf{0}$ is a convex combination of these two points, $0 = f(\mathbf{0}) \leq \frac{1}{2} f(\mathbf{z}) + \frac{1}{2} f(-\mathbf{z})$, or $f(\mathbf{z}) \geq -f(-\mathbf{z})$. This turns the upper bound $f(-\mathbf{z}) \leq \gamma$ into a lower bound $f(\mathbf{z}) \geq -\gamma$ for all $\mathbf{z} \in B$.

2. Shrinking of the ball



Again, assume without loss of generality that $\mathbf{x} = 0$ and $f(\mathbf{x}) = 0$. We use the first part of the proof to construct a ball B around the origin with radius R and $|f(\mathbf{y})| \leq \gamma$ for all $\mathbf{y} \in B$ and some $\gamma > 0$.

Consider the smaller ball B' around the origin with radius $r = \frac{R\varepsilon}{\gamma}$. We will use convexity to show that $|f(\mathbf{z}')| \leq \varepsilon$ for all $\mathbf{z}' \in B'$. Any point $\mathbf{z}' \in B'$ can be written as $\lambda \mathbf{z}$, where $\mathbf{z}$ is a point on the perimeter of the big ball B. The scale factor $\lambda \leq \frac{r}{R} = \frac{\varepsilon}{\gamma}$. Note that $0 \leq \lambda < 1$, so

$$f(\mathbf{z}') = f(\lambda \mathbf{z} + (1 - \lambda)\mathbf{0}) \leq \lambda f(\mathbf{z}) \leq \frac{\varepsilon}{\gamma} f(\mathbf{z}) \leq \varepsilon.$$

This is an upper bound $f(\mathbf{z}') \leq \varepsilon$ for $\mathbf{z}' \in B'$. To finish the proof, we just need to get a lower bound $f(\mathbf{z}') \geq -\epsilon$ as well. In part 1 of the proof, we turned an upper bound γ on the large ball B into a lower bound $-\gamma$. We can use the same argumentation here on the smaller ball B' with the previously derived upper bound ε to finish the proof.

Exercise 4. *Prove that the function $d_{\mathbf{y}} : \mathbb{R}^d \to \mathbb{R}$, $\mathbf{x} \mapsto \|\mathbf{x} - \mathbf{y}\|^2$ is strictly convex for any $\mathbf{y} \in \mathbb{R}^d$. (Use Lemma 1.25.)*

Solution: By Lemma 1.25, it suffices to show that $\nabla^2 d_{\mathbf{y}}(\mathbf{x})$ is positive definite for every $\mathbf{x} \in \mathbb{R}^d$ with $\mathbf{x} \neq \mathbf{0}$. We compute

$$d_{\mathbf{y}}(\mathbf{x}) = (\mathbf{x} - \mathbf{y})^\top (\mathbf{x} - \mathbf{y}), \quad \nabla d_{\mathbf{y}}(\mathbf{x}) = 2(\mathbf{x} - \mathbf{y}), \quad \nabla^2 d_{\mathbf{y}}(\mathbf{x}) = 2I \succ 0,$$

where I denotes the identity matrix. The claim follows.

Exercise 5. *Prove Lemma 1.19. Can (ii) be generalized to show that for two convex functions f, g, the function $f \circ g$ is convex as well?*

Solution:

(i) For $f = \max_{i=1}^m f_i$, we compute

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \max_{i=1}^{m} f_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\ &\leq \max_{i=1}^{m} (\lambda f_i(\mathbf{x}) + (1-\lambda) f_i(\mathbf{y})) \\ &= \lambda f_j(\mathbf{x}) + (1-\lambda) f_j(\mathbf{y}) \quad \text{(for some } j\text{)} \\ &\leq \lambda \max_{i=1}^{m} f_i(\mathbf{x}) + (1-\lambda) \max_{i=1}^{m} f_i(\mathbf{y}) \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \end{aligned}$$

For $f = \sum_{i=1}^m \lambda_i f_i$, we compute

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \sum_{i=1}^{m} \lambda_i f_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\ &\leq \sum_{i=1}^{m} \lambda_i \left(\lambda f_i(\mathbf{x}) + (1-\lambda) f_i(\mathbf{y})\right) \\ &= \lambda \cdot \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})}_{f(\mathbf{x})} + (1-\lambda) \cdot \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\mathbf{y})}_{f(\mathbf{y})}, \end{aligned}$$

where the inequality makes use of convexity of the individual f_i and of the fact that the λ_i are non-negative.

(ii) Let $\mathbf{x}, \mathbf{y} \in \mathbf{dom}(f \circ g)$ and $\lambda \in [0, 1]$ be arbitrary. We simply compute

$$\begin{aligned} (f \circ g)(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= f(A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + \mathbf{b}) \\ &= f\big(\lambda \cdot (A\mathbf{x} + \mathbf{b}) + (1-\lambda) \cdot (A\mathbf{y} + \mathbf{b})\big) \\ &\leq \lambda \cdot \underbrace{f(A\mathbf{x} + \mathbf{b})}_{(f \circ g)(\mathbf{x})} + (1-\lambda) \cdot \underbrace{f(A\mathbf{y} + \mathbf{b})}_{(f \circ g)(\mathbf{y})}, \end{aligned}$$

where the inequality makes use of convexity of f and of the fact that both $g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and $g(\mathbf{y}) = A\mathbf{y} + \mathbf{b}$ are in the domain of f.

If two functions f and g are both convex, then their composition $f \circ g$ is not necessarily also convex. Consider for example convex functions $f(x) = x^2$ and $g(x) = x^2 - 1$. Then, the composition

$$(f \circ g)(x) = x^4 - 2x^2 + 1$$

satisfies $(f \circ g)(-1) = (f \circ g)(1) = 0$ and $(f \circ g)(0) = 1$, which is a clear violation of convexity.

Exercise 8. *Prove that the function* $f(\mathbf{x}) = \|\mathbf{x}\|_1 = \sum_{i=1}^{d} |x_i|$ *(ℓ_1-norm) is convex!*

Solution: It suffices to prove that $f_i(\mathbf{x}) = |x_i|$ is convex and then use Lemma 1.19. Equivalently, that $f(x) = |x|$ is convex. For $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, we compute

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= |\lambda x + (1-\lambda)y| \\ &\leq |\lambda x| + |(1-\lambda)y| \quad \text{(triangle inequality)} \\ &= |\lambda||x| + |(1-\lambda)||y| \\ &= \lambda|x| + (1-\lambda)|y| \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Exercise 10. *A seminorm is a function* $f : \mathbb{R}^d \to \mathbb{R}$ *satisfying the following two properties for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *and all* $\lambda \in \mathbb{R}$.

(i) $f(\lambda \mathbf{x}) = |\lambda| f(\mathbf{x})$*,*

(ii) $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ *(triangle inequality).*

Prove that every seminorm is convex!

Solution: This just generalizes the previous exercise and shows what is actually going on. For $\lambda \in [0,1]$ we get

$$\begin{aligned} f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) &\leq f(\lambda \mathbf{x}) + f((1-\lambda)\mathbf{y}) \quad \text{(triangle inequality)} \\ &= |\lambda| f(\mathbf{x}) + |(1-\lambda)| f(\mathbf{y}) \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \end{aligned}$$