Ex:

$X \sim \mathcal{E}(\lambda) \quad \lambda > 0$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_0^{+\infty} t \times \lambda e^{-\lambda t} \, dt$$

$$u(t) = t \Rightarrow u'(t) = 1$$

$$v'(t) = \lambda e^{-\lambda t} \Leftarrow v(t) = -e^{-\lambda t}$$

$$\mathbb{E}[X] = \underbrace{\left[ t\left(-e^{-\lambda t}\right) \right]_0^{+\infty}}_{=0} - \int_0^{+\infty} 1 \times \left(-e^{-\lambda t}\right) dt$$

$$E[X] = \int_0^{+\infty} e^{-\lambda t}\,dt = \left[-\frac{e^{-\lambda t}}{\lambda}\right]_0^{+\infty}$$

$$\boxed{E[X] = \frac{1}{\lambda}}$$

$$V[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

$$= \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

proof:

$$\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \left(\mathbb{E}[X]\right)^2\right]$$

$$V[X] = E[X^2] - 2E[X \times \underbrace{E[X]}_{\text{is a constant}}] + E[\underbrace{(E[X])^2}_{\text{is a constant}}]$$

$$= E[X^2] - 2E[X] \times E[X] + (E[X])^2$$

$\blacksquare$

$$\mathbb{E}[X^2] = \int_0^{+\infty} t^2 \lambda e^{-\lambda t} \, dt$$

$$u(t) = t^2 \Rightarrow u'(t) = 2t$$

$$v'(t) = \lambda e^{-\lambda t} \Leftarrow v(t) = -e^{-\lambda t}$$

$$\mathbb{E}[X^2] = \underbrace{\left[ t^2 \left( -e^{-\lambda t} \right) \right]_0^{+\infty}}_{=0} - \int_0^{+\infty} 2t \left( -e^{-\lambda t} \right) dt$$

$$\mathbb{E}[X^2] = \frac{2}{\lambda} \underbrace{\int_0^{+\infty} t\left(\lambda e^{-\lambda t}\right) dt}_{= \mathbb{E}[X]}$$

$$= \frac{2}{\lambda^2}$$

$$\boxed{V[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}}$$

## Proposition

let $X$ be a r.v whose distribution $f^{ct}$ is $F$.

Then if $U \sim \mathcal{U}([0;1])$ and if $F^{-1}$ exists $\left( F^{-1} \circ F(x) = x \text{ and } F \circ F^{-1}(x) = x \right)$

$\Rightarrow F^{-1}(U)$ has the same law of $X$

<u>proof:</u>

$$Y = F^{-1}(U)$$

$$\forall t \in \mathbb{R}, \quad P(Y \leq t) = P(F^{-1}(U) \leq t)$$

$$= P(U \leq F(t))$$

$$= F(t)$$

it is the distributeon fct of $\mathcal{U}([0;1])$ evaluated at point $F(t)$

Thus:

$$\forall t \in \mathbb{R},\ F_Y(t) = F_X(t)$$

$\Rightarrow$ $Y$ and $X$ have the same distribution

$\boxed{\diagup}$

To simulate observations of an exponential r.v with parameter $\lambda$ thanks to a $\mathcal{U}([0;1])$, we need:

1) to compute the analytic expression of the associated distribution fct $F$.
2) to compute the analytic expression of $F^{-1}$.
3) to compute $F^{-1}(U)$

i)

Let $t \in \mathbb{R}$, $F(t) = P(X \leq t)$ with $X \sim \mathcal{E}(\lambda)$

- if $t < 0$, $F(t) = 0$
- if $t \geq 0$, $F(t) = \int_0^t \lambda e^{-x\lambda}\, dx$

$$= \left[-e^{-\lambda x}\right]_0^t = 1 - e^{-\lambda t}$$

$$\boxed{F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

2) $F^{-1} : [0,1] \longrightarrow \mathbb{R}^+$

let $y \in [0,1]$ such that $F(x) = y$ with $x \in \mathbb{R}^+$

$\Rightarrow x = F^{-1}(y)$

$$1 - e^{-\lambda x} = y$$

$$e^{-\lambda x} = 1 - y$$

$$-\lambda x = \ln(1-y)$$

$$x = -\frac{1}{\lambda} \ln(1-y) \Rightarrow F^{-1} : [0,1] \to \mathbb{R}^{+}$$

$$y \to -\frac{1}{\lambda} \ln(1-y)$$

3) To simulate observations of X $\left(X \sim \mathcal{E}(\lambda)\right)$ we can compute

$$-\frac{1}{\lambda}\ln\left(1-U\right) \text{ with } U \sim \mathcal{U}\left([0;1]\right)$$

<u>Rk</u>: often, we do $\boxed{-\frac{1}{\lambda}\ln(U)}$

this is because if $U \sim \mathcal{U}([0;1])$

$\Rightarrow 1-U \sim \mathcal{U}([0;1])$

**Rk:** To prove that $X$ and $Y$ have the same distribution, we compute $F_X$ and $F_Y$

$F_X = F_Y \iff X$ and $Y$ have the same distribution!

let $U \sim \mathcal{U}([0;1])$

Define $Y$ by $Y = 1 - U$

$\forall t \in \mathbb{R}$, $P(Y \leq t) = P(1 - U \leq t)$

$= P(U \geq 1-t) = 1 - P(U \leq 1-t)$

We know that $F_U(z) = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{if } z \in [0;1] \\ 1 & \text{if } z \geq 1 \end{cases}$

- $1-t<0 \iff t>1$

$P(Y \leqslant t) = 1 - 0 = 1$

- $1-t>1 \iff t<0$

$P(Y \leqslant t) = 1-1 = 0$

- $1-t \in [0;1] \iff t \in [0;1]$

$P(Y \leqslant t) = 1-(1-t) = t$

$$P(Y \leqslant t) = \begin{cases} 0 \text{ if } t<0 \\ t \text{ if } t \in [0;1] \\ 1 \text{ if } t>1 \end{cases}$$

$\Rightarrow Y \sim \mathcal{U}([0;1])$

# Reject method

**aim:** to simulate observations associated to the density $f$ of $f$.

**proposition:**

we assume that there exists a density function $g$ and a constant $c$ such that:

- we know how to simulate observation from $g$
- $f \leq c \times g$

The method is :

1) let $z$ be a simulation from $g$

2) let $u$ be a simulation from $\mathcal{U}([0;1])$

if $u \times c \times g(z) < f(z)$ then $z$ is a simulation from $f$

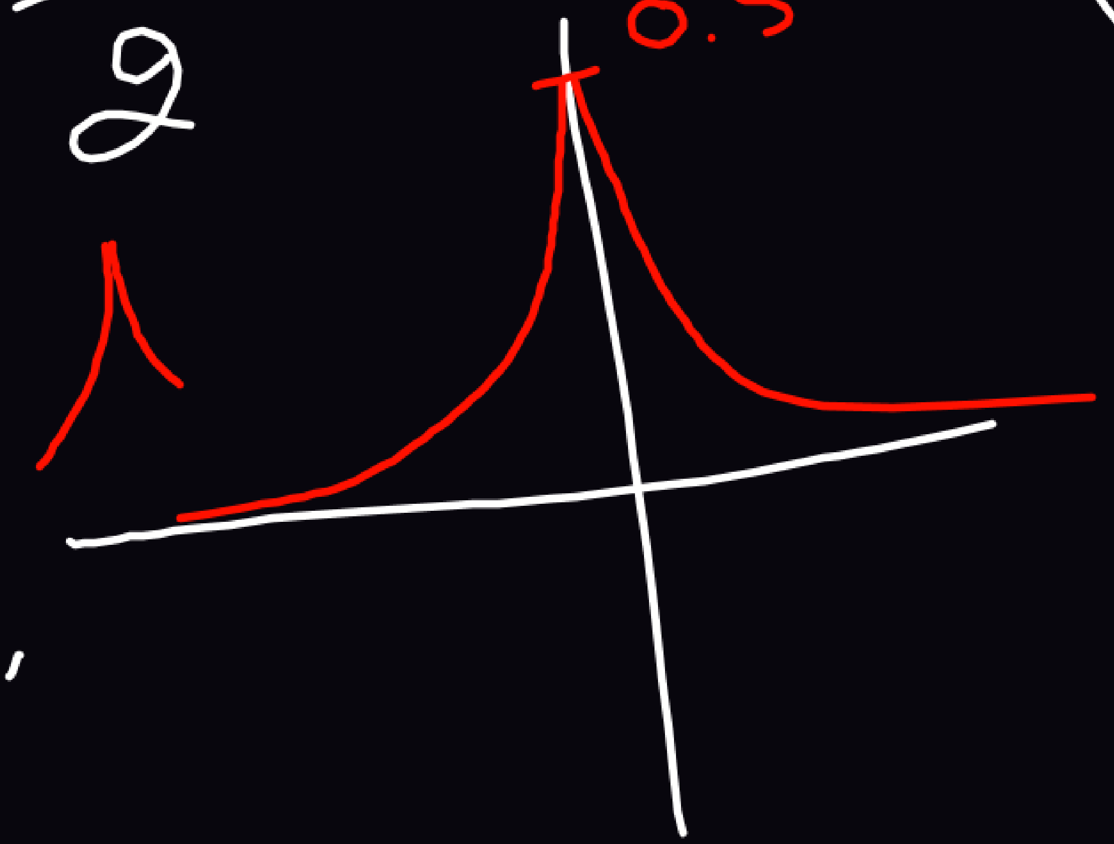
otherwise we begin again !

## Application:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$g(x) = \frac{1}{2} e^{-|x|}$$

(double exponential or Laplace)




$$c = \sqrt{\frac{2e}{\pi}}$$

$$h(x) = \exp\left(-\frac{x^2}{2}\right) \times \frac{1}{\sqrt{2\pi}}$$

$$- \sqrt{\frac{2e}{\pi}} \frac{1}{2} \exp\left(-|x|\right)$$

→ compute $h'$ and see the variations of $h$.

To make simulations from the Laplace density, we can use the following property:

- let $\varepsilon$ be a r.v such that
$$P(\varepsilon = -1) = P(\varepsilon = 1) = \frac{1}{2}$$

+ independent

- let $Z \sim \mathcal{E}(1)$

$(\varepsilon \perp\!\!\!\perp Z)$

- Define $Y$ by $Y = \varepsilon Z$ $\rightarrow$ $Y$ is a r.v with density $g$.

$$g(x) = \frac{1}{2} e^{-|x|}$$

$$= \underbrace{\frac{1}{2} e^{-x} \cdot \mathbb{1}_{x>0}}_{\text{density of an exponential (1)}} + \underbrace{\frac{1}{2} e^{-(-x)} \cdot \mathbb{1}_{x \leq 0}}_{\text{density associated to - an exponential (1)}}$$

mixture

$$g(x) = \sum_{i=1}^{n} p_i f_i(x)$$

density for $f$

other way to see that $Y$ comes from $g$.

$\Rightarrow$ we compute the distribution $f^{ct}$ of $Y$ to obtain then the density $f^{ct}$ of $Y$.

$$\forall t \in \mathbb{R},\ P(Y \leq t) = P(\varepsilon Z \leq t)$$

$$= P\big((Z \leq t \text{ and } \varepsilon = 1) \text{ or } (-Z \leq t \text{ and } \varepsilon = -1)\big)$$

$$= P(Z \leq t \text{ and } \varepsilon = 1) + P(-Z \leq t \text{ and } \varepsilon = -1)$$

$$= P(Z \leq t)\, P(\varepsilon = 1)$$

$$+ P(-Z \leq t)\, P(\varepsilon = -1)$$

$$= \frac{1}{2}\left( P(Z \leq t) + P(Z \geq -t) \right)$$

$$= \frac{1}{2}\left( P(Z \leq t) + 1 - P(Z \leq -t) \right)$$

$$= \frac{1}{2}\left( 1 + F_Z(t) - F_Z(-t) \right)$$

$$P(Y < t) = \frac{1}{2}\left(1 + F_Z(t) - F_Z(-t)\right)$$

$$f_Y(t) = \frac{1}{2}\left(f_Z(t) - \left(-f_Z(-t)\right)\right)$$

$$= \frac{1}{2}\left(f_Z(t) + f_Z(-t)\right)$$

$$= g(t)$$

## other method

1) We determine the analytic expression of the distribution f$^{ct}$ associated to g.

2) We compute it inverse denoted $F^{-1}$

3) We compute $F^{-1}(U)$

1) Computation of the distribution $f^{ct}$.

let $t \in \mathbb{R}$, $X \sim g$

$$F(t) = P(X \leq t) = \int_{-\infty}^{t} \frac{1}{2} e^{-|x|} dx$$

- if $t < 0$

$$F(t) = \frac{1}{2}\int_{-\infty}^{t} e^{x}\,dx = \frac{1}{2}\left[e^{x}\right]_{-\infty}^{t} = \frac{1}{2}e^{t}$$

- if $t \geq 0$

$$F(t) = \underbrace{\frac{1}{2}\int_{-\infty}^{0} e^{x}\,dx}_{1/2} + \underbrace{\frac{1}{2}\int_{0}^{t} e^{-x}\,dx}_{= \frac{1}{2}\left[-e^{-x}\right]_{0}^{t} = \frac{1}{2}\left(1 - e^{-t}\right)}$$

$$F(t) = \begin{cases} \frac{1}{2} e^{t} & \text{if } t < 0 \\ 1 - \frac{1}{2} e^{-t} & \text{if } t \geq 0 \end{cases}$$

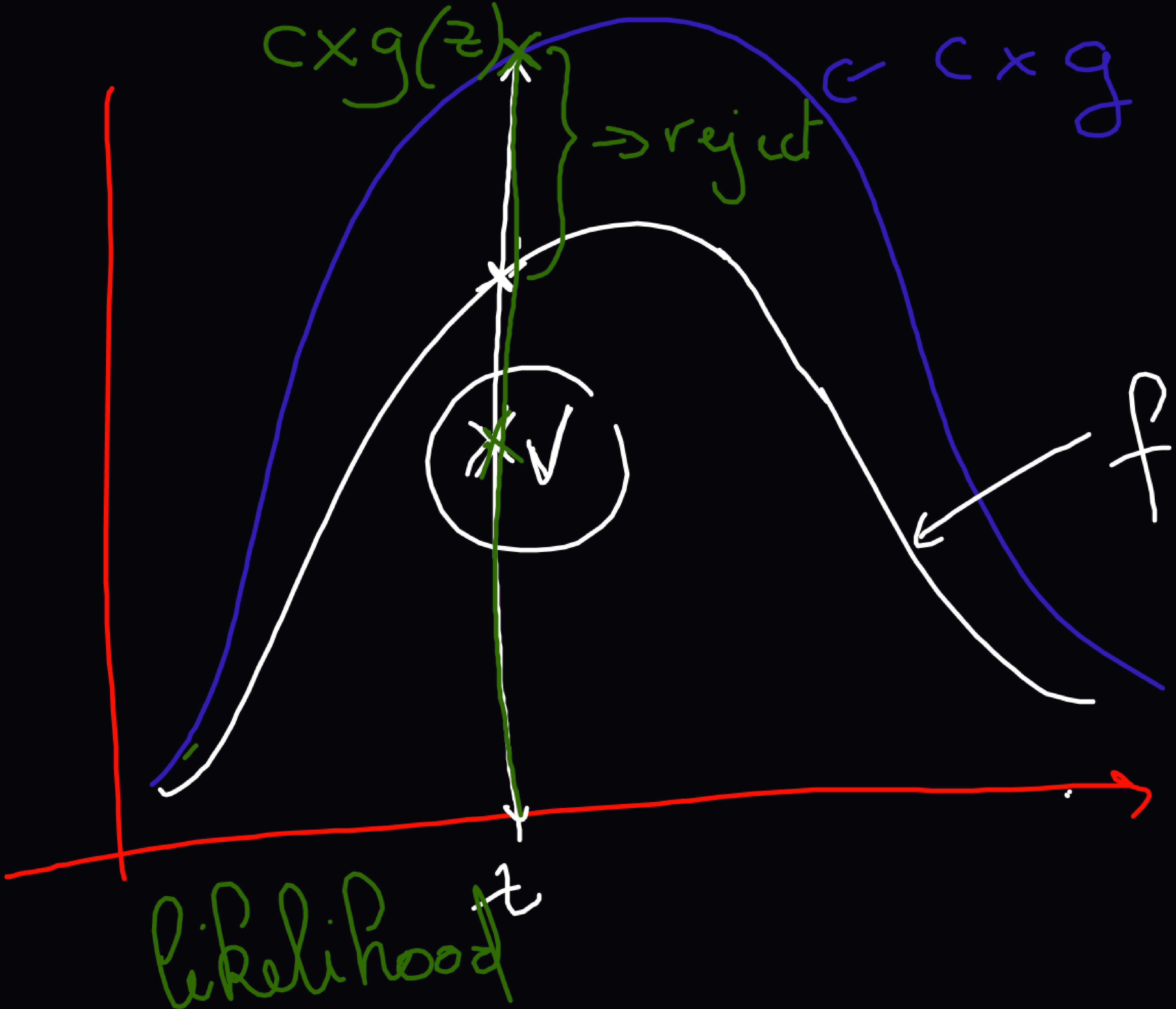
$$F(t) = F_1(t) + F_2(t)$$

$$]-\infty, 0[ \to ]0, \tfrac{1}{2}] \qquad [0, +\infty[ \to [\tfrac{1}{2}, 1[$$

$$\frac{1}{2}e^{t} = y \iff t = \ln(2y)$$

$$1-\frac{1}{2}e^{-t} = y \iff t = -2\ln(1-y)$$

$$F^{-1}(y) = \begin{cases} \ln(2y) & \text{if } y \in \left]0; \tfrac{1}{2}\right] \\ -2\ln(1-y) & \text{if } y \in \left[\tfrac{1}{2}; 1\right[ \end{cases}$$




$$V = c \times g(z) \times U$$

## proposition : Box-Muller

let $U_1 \sim \mathcal{U}([0,1])$

let $U_2 \sim \mathcal{U}([0,1])$

$\Big\}\ U_1 \perp\!\!\!\perp U_2$

let define $Y_1$ and $Y_2$ by :

$$Y_1 = \sqrt{-2\ln U_1} \times \cos(2\pi U_2)$$

$$Y_2 = \sqrt{-2\ln U_1} \times \sin(2\pi U_2)$$

$\Rightarrow$ $Y_1$ and $Y_2$ are 2 standard gaussian r.v

and $Y_1 \perp\!\!\!\perp Y_2$

# Detection of outliers

→ quantiles (quartiles)

<u>definition:</u> let $\alpha \in [0;1]$

the quantile of order $\alpha$ is the observable value such that $100 \times \alpha$ % of the population takes values under it!

3 quartiles: $Q_1$, $Q_2$, $Q_3$

$\alpha = 0.25$     Median     $\alpha = 0.75$

In practice, to determine the quantile of order $\alpha$

. for a discrete variable:

→ this is the value associated to the cumulative percent immidiately upper than $\alpha$.

· for a continuous variable :

→ we define the class associated to the quantile. It is the class corresponding to the cumulative percent immediately upper to $\alpha \Rightarrow [a, b[$

→ to determine the value of the quantile by linear interpolation on $[a, b[$, find $p$ and $q$

such that $F(x) = px + q$

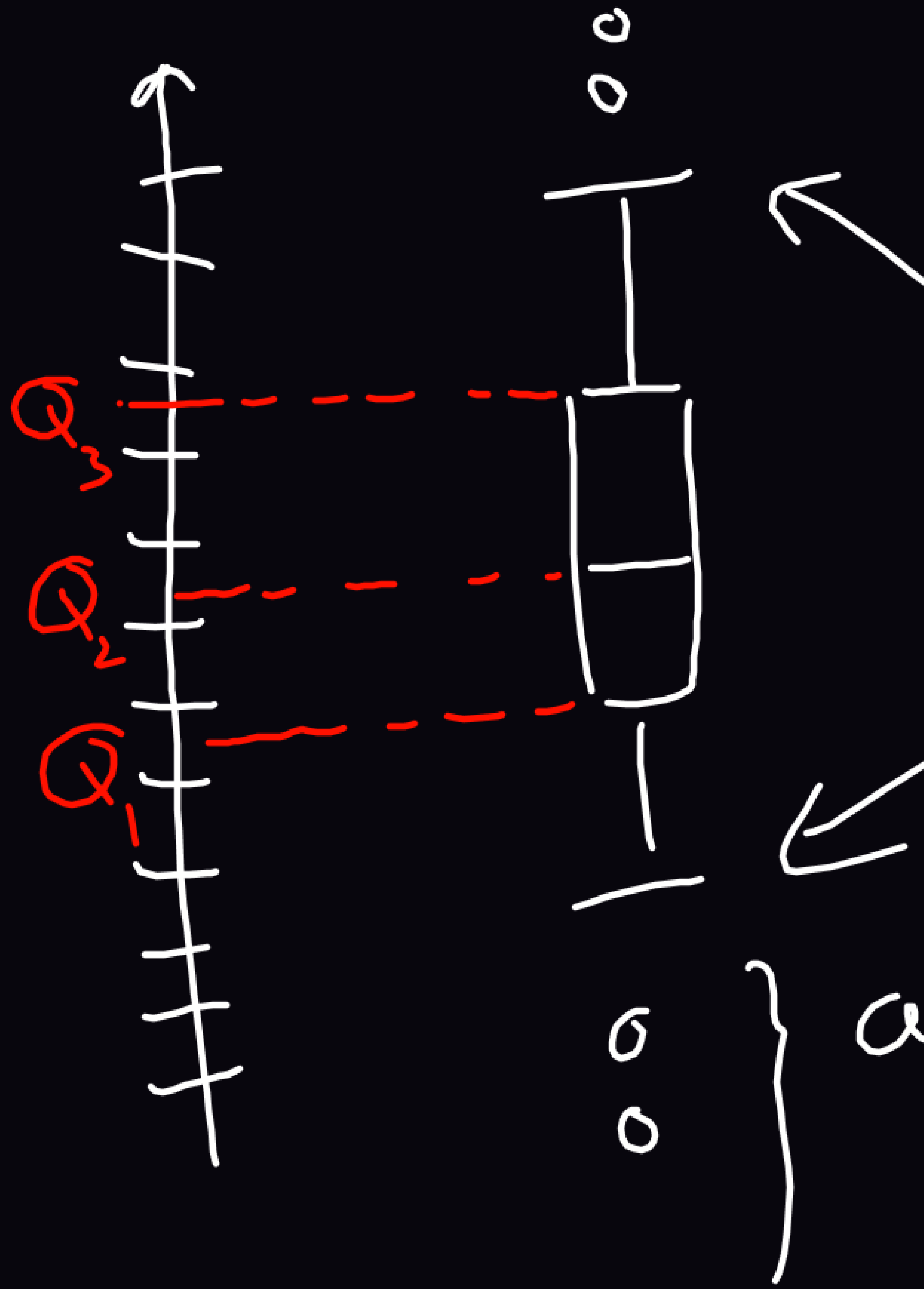
with $F$ the cumulative percent.

Then find $x$ such that $F(x) = \alpha$.

$\Rightarrow Q_1, Q_3, Q_2$.

↳ boxplot

(In R Software : boxplot)




$D_9 \leftarrow$ quantile of order 0.9

$D_1 \leftarrow$ quantile of order 0.1

} are potentialy outliers

an observation may be an outlier if:

- the observation $< Q_1 - 1.5 \times (Q_3 - Q_1)$

or

- the observation $> Q_3 + 1.5 \times (Q_3 - Q_1)$

- variance / standard deviation

individual data : $\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{K} n_i (a_i - \bar{x})^2$

$n_i$ : the frequency associated to the value $a_i$

$n = \sum n_i$

grouped data : $\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{K} n_i (c_i - \bar{x})^2$

$c_i$ : midle of the class number $i$

$n_i$ : frequency

standard deviation $= \sqrt{\text{variance}}$ : $\sigma$

coefficient of variation $= 100 \times \dfrac{\sigma}{\overline{X}}$

if $< 20\% \rightarrow$ concentrated around $\overline{X}$.

$> 33\% \rightarrow$ dispersion of the data

# Kurtosis and skewness

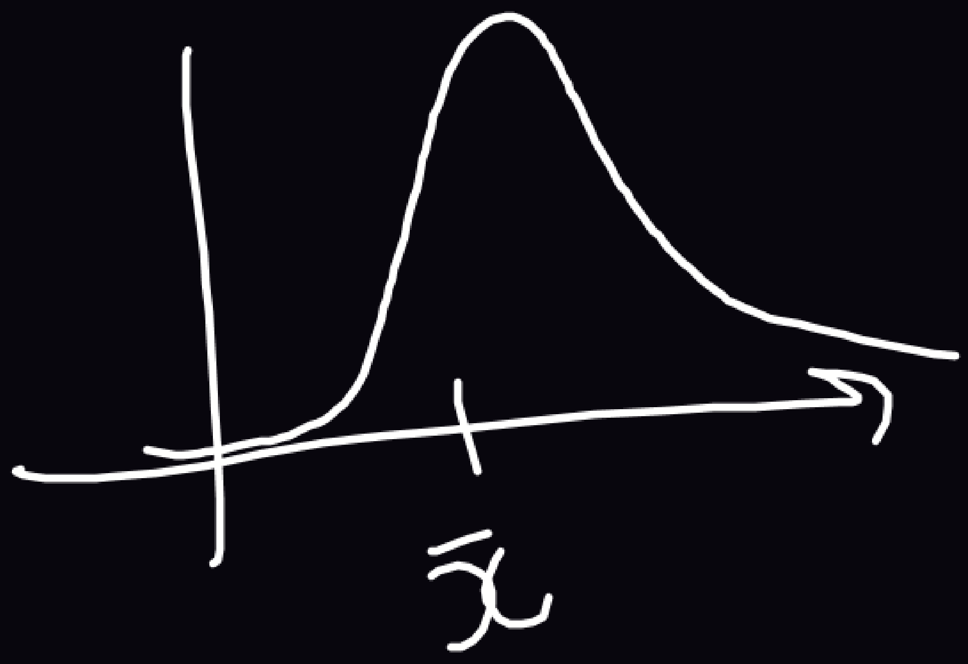
↳ parameters for the shape of the distribution.

skewness : → symetry

kurtosis → gaussian distribution

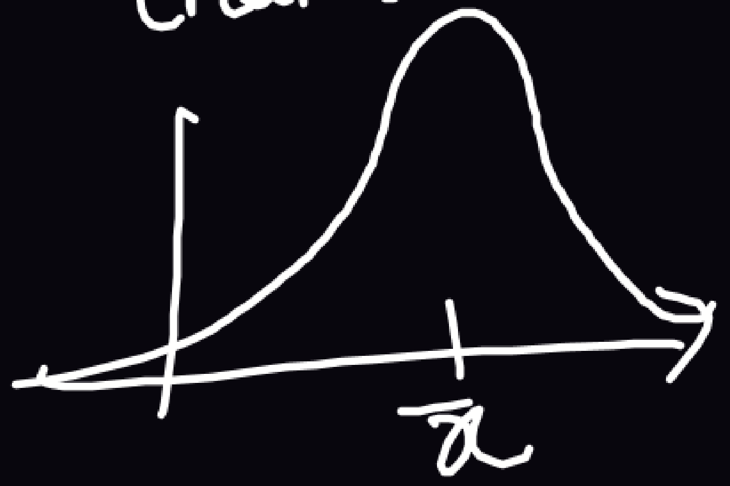
both cases: comparison with 0

skweeness : $\rightarrow$ close to 0
you can consider a symetry in the distribution of the data




$> 0 \rightarrow$ your data have a tendency to take values bigger than $\bar{x}$.

$< 0$



## Kurtosis :

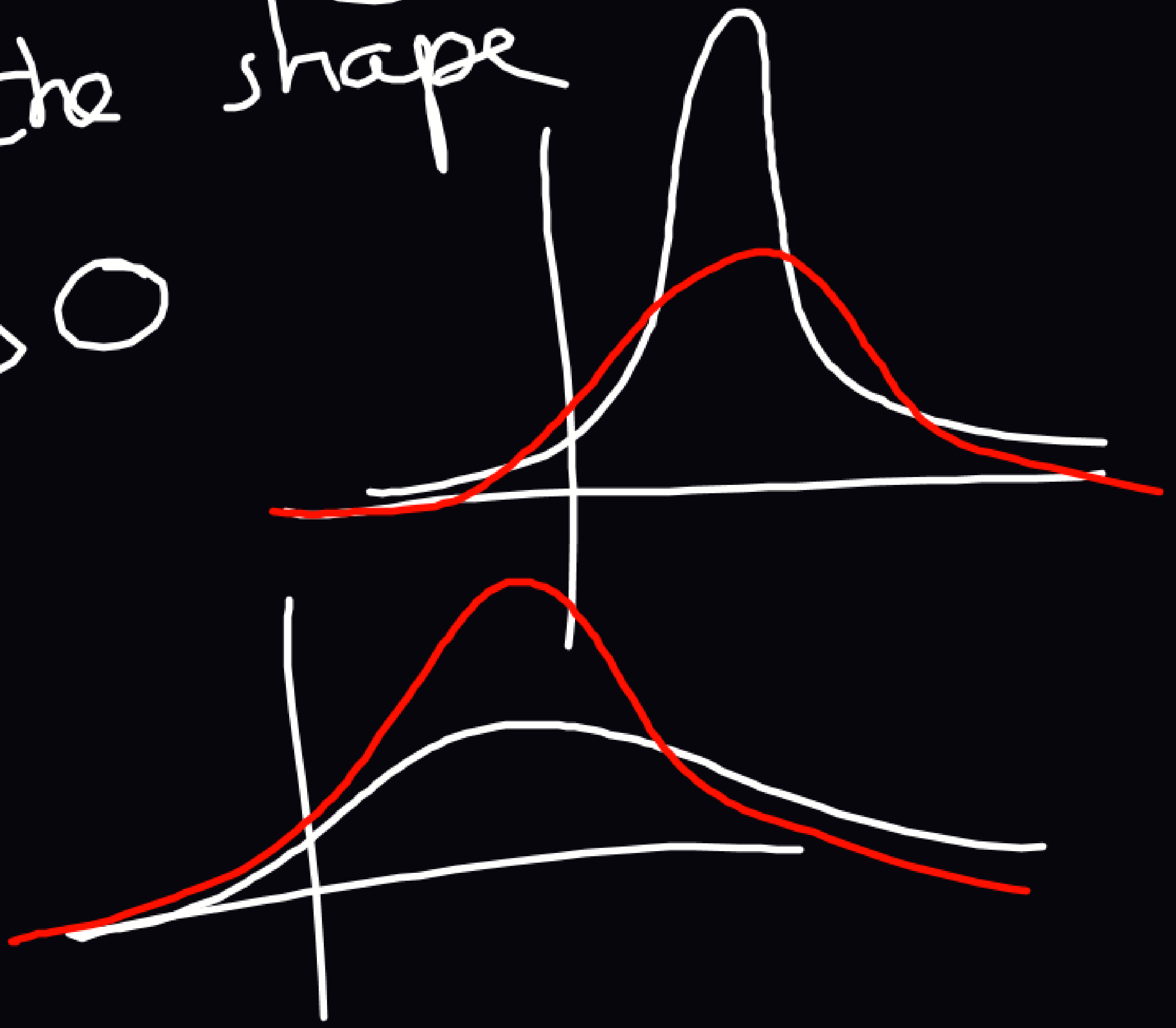
$\rightarrow$ close to 0

$$\sum \left(\frac{x_i - \bar{x}}{\sigma}\right)^4 \ldots$$

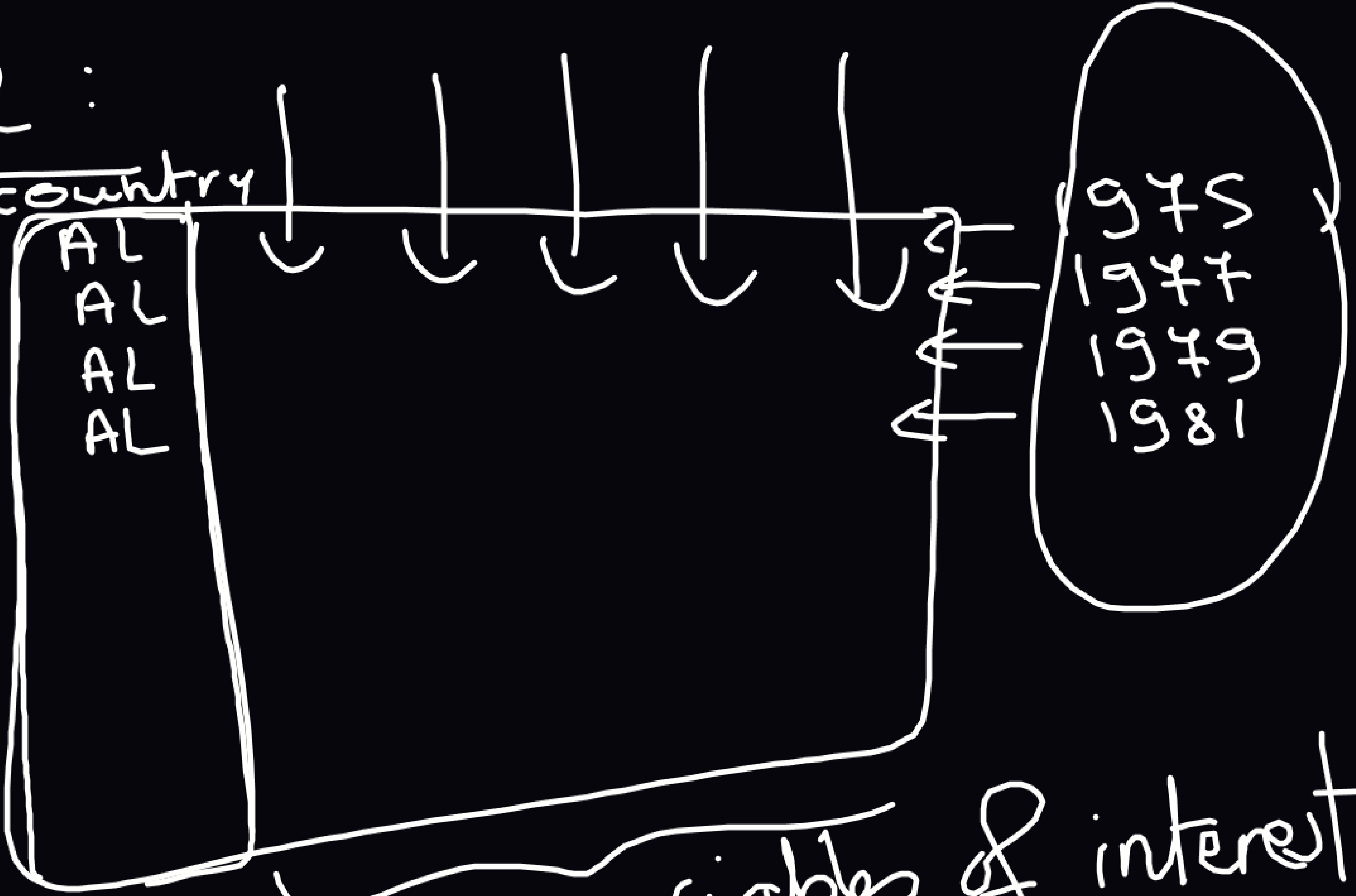
the distribution is similar to a gaussian with respect to the shape

$> 0$

$< 0$



ocde :




variables of interest

kernel

density

# Chapter: Estimation

- point estimation
  - method of moments
  - likelihood method
- confidence interval

## def of an estimator

Let $x_1, \dots, x_n$ n observations associated to the r.v $X_1, \dots, X_n$ which are supposed to be iid. We assume that the density of $X_1, \dots, X_n$ depends on an unknown parameter $\theta$.
An estimator of $\theta$ is just a function of $X_1, \dots, X_n$.

<u>Rk</u>: · an estimator is a r.v.

an estimation is an observation of the estimator.

· an estimator of $\theta$ based on $X_1, \ldots, X_n$ is denoted $\hat{\theta}_n$

## Properties :

. biais : we say that $\hat{\theta}_n$ is an unbaised estimator if

$$\mathbb{E}[\hat{\theta}_n] = \theta$$

**Ex:** let $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow \bar{X}_n$ is an unbaised estimator of $\mathbb{E}[X_1]$

proof:

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$\hat{\theta}_n$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i]$$

$\theta$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_1] = \mathbb{E}[X_1]$$

$\blacksquare$

**Rk:**

Sometimes we just have $\hat{\Theta}_n$ which is asymptoticaly unbaised.

$$E[\hat{\Theta}_n] \xrightarrow[n\to+\infty]{} \Theta$$

Ex: $$E\Big[\underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}_{\hat{\Theta}_n}\Big] = \overbrace{\frac{n-1}{n}}^{1\ \uparrow_{n\to+\infty}}\ \overbrace{V[X_1]}^{\Theta}$$

$$\Rightarrow \frac{n}{n-1}\mathbb{E}\left[\hat{\Theta}_n\right] = \Theta$$

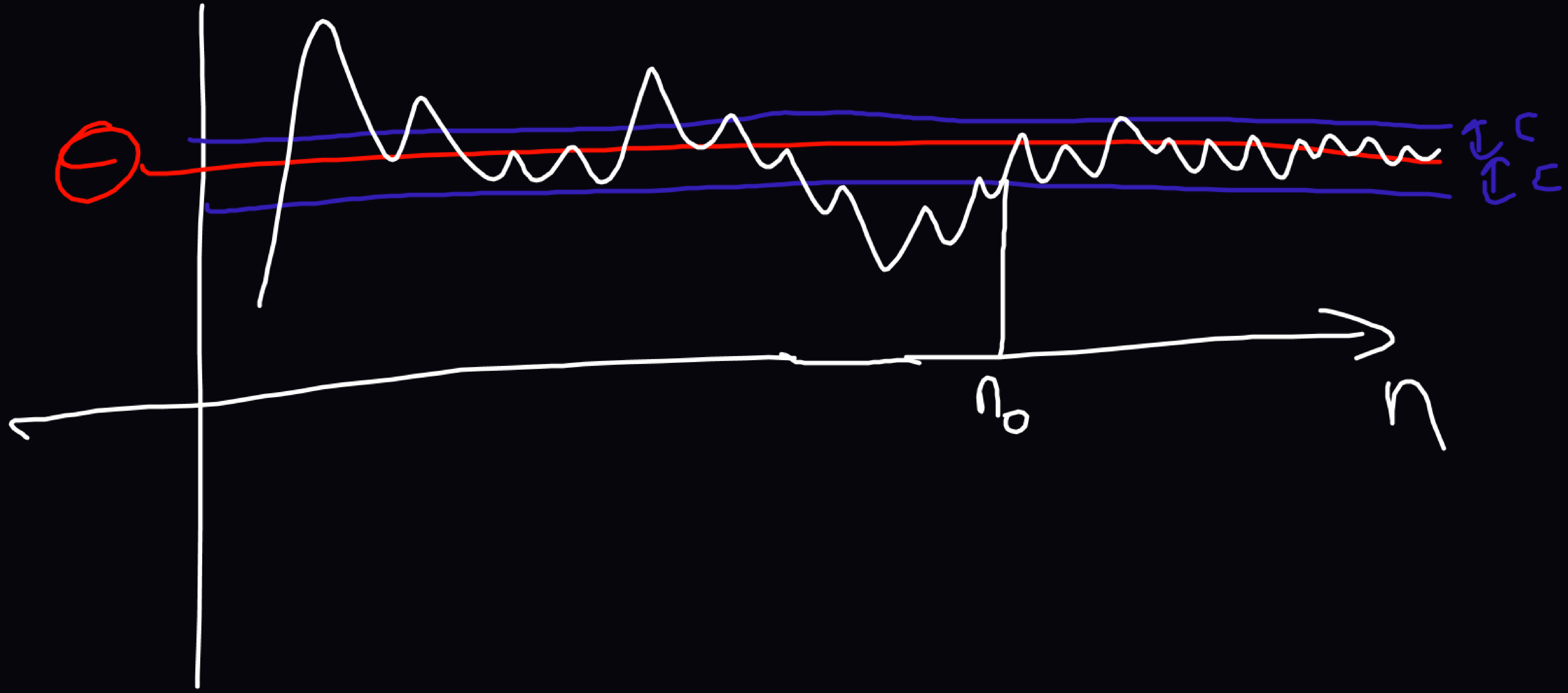
$$\Rightarrow \tilde{\Theta}_n = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$$

We say that

$\hat{\theta}_n \xrightarrow[P]{} \theta$ if $\forall \varepsilon > 0$,

$$P\left(|\hat{\theta}_n - \theta| > \varepsilon\right) \xrightarrow[n \to +\infty]{} 0$$

In this case, we say that $\hat{\theta}_n$ is consistent!



θ
ε
ε
$n_0$
n

**Th:**

Let $\hat{\Theta}_n$ be an estimator on $\Theta$.

If:

- $\mathbb{E}[\hat{\Theta}_n] = \Theta$
- $V[\hat{\Theta}_n] \xrightarrow[n \to +\infty]{} 0$

Then $\hat{\Theta}_n$ is consistent.

Proof:

$$\forall \varepsilon > 0, \ P\left(\left|\hat{\theta}_n - \mathbb{E}[\hat{\theta}_n]\right| > \varepsilon\right) \leq \frac{Var(\hat{\theta}_n)}{\varepsilon^2}$$

Rk: family of density is

$$\left\{ \mathcal{N}(\mu, \sigma^2),\ \mu = 2,\ \sigma^2 > 0 \right\}$$

$$\to \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$$

. family of density is:

$$\left\{ \mathcal{P}(\mu, \sigma^2) ; \mu \in \mathbb{R}, \sigma^2 > 0 \right\}$$

$$\hat{\mu} = \overline{X}_n$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

# Two methods to construct estimators

- method of moments

let $X$ be a r.v with density $f^{ct}$ $f$ that depends on an unknown parameter $\theta$.
let $X_1, \dots, X_n$ iid r.v with the same density $f^{ct}$ than $X$.

the moment of order $k$ of $X$ is:

$$\mu_k = \mathbb{E}[X^k]$$

the centered moment of order $k$ of $X$ is:

$$c_k = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^k\right]$$

let $j$ be the first integer such that $\mu_j = h(\theta)$ or $c_j = t(\theta)$

In this case, we define $\hat{\theta}_n$ an estimator of $\theta$ by:

$$h(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} X_i^j \text{ or } t(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^j$$

Ex: let $X \sim \mathcal{U}([0,\theta])$

$$\mathbb{E}[X] = \frac{\theta}{2} \quad \text{and} \quad V[X] = \frac{\theta^2}{12}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$h(\theta) = \frac{\theta}{2} \qquad\qquad t(\theta) = \frac{\theta^2}{12}$$

$$\frac{\hat{\theta}_n}{2} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad \frac{\hat{\theta}_n^2}{12} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

$$\hookrightarrow \boxed{\hat{\theta}_n = 2 \times \overline{X}_n} \qquad\qquad \Rightarrow \boxed{\hat{\theta}_n = \sqrt{\frac{12}{n}\sum (X_i - \overline{X}_n)^2}}$$

$$\mathbb{E}[X^3] = \int_0^\theta \frac{t^3}{\theta}\,dt = \left[\frac{t^4}{4\theta}\right]_0^\theta$$

$$= \frac{\theta^3}{4}$$

$$\Rightarrow \hat{\theta}_n = \sqrt[3]{\frac{4}{n}\sum_{i=1}^{n} X_i^3}$$

$X \sim \mathcal{U}([-\theta, \theta])$

$\hookrightarrow \mathbb{E}[X] = 0 \Rightarrow$ not a $f^{ct}$ of $\theta$!

<u>Rk</u>: There exists also some estimator defined just with the definition of the density $f^{ct}$.

For examplе, if $X \sim \mathcal{U}([0,\theta])$

$$\Rightarrow \hat{\theta}_n = \max_{i=1,\dots,n}(X_i)$$

<u>Rk</u>: For tomorrow determine the density of $\hat{\theta}_n = \max(X_i)$

# TCL

Let $X_1, .., X_n$ be $n$ iid r.v with expectation $\mu$ and variance $\sigma^2$

$$\sqrt{n}\,\frac{\overline{X_n}-\mu}{\sigma} \xrightarrow[\mathcal{L}]{} \mathcal{N}(0,1) \iff \overline{X_n} \xrightarrow[\mathcal{L}]{} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$Y_n \xrightarrow[\mathcal{L}]{} Y \text{ if } \forall t \in \mathbb{R},\ F_n(t) \xrightarrow[n\to+\infty]{} F(t)$$

$F_n(t)$: distribution $f^{ct}$ of $Y_n$

$F(t)$: distribution $f^{ct}$ of $Y$

# Maximum likelihood method

Let $X_1, \dots, X_n$ be $n$ iid r.v. whose density $f^{ct}$ is $f_\theta$.

The likelihood of $X_1, \dots, X_n$ is defined by:

$$\mathcal{L}(X_1 = x_1, \dots, X_n = x_n, \theta) = \begin{cases} \prod_{i=1}^{n} f_\theta(x_i) \\ \prod_{i=1}^{n} P(X_i = x_i) \end{cases}$$

Ex: $f_\theta$ is the density of $\mathcal{E}(\theta)$

$$\mathcal{L}(X_1 = x_1, \dots, X_n = x_n; \theta) = \prod_{i=1}^{n} \left[\left(\theta e^{-\theta x_i}\right) \mathbb{1}_{x_i \geq 0}\right] = \left[\prod_{i=1}^{n} \left[\theta e^{-\theta x_i}\right]\right] \mathbb{1}_{\min(x_i) \geq 0} = \theta^n e^{-\theta(x_1 + \dots + x_n)} \times \mathbb{1}_{\min(x_i) \geq 0}$$

$$\mathcal{L}(X_1 = x_1, \dots, X_n = x_n ; \theta)$$

$$\approx \lim_{\Delta t \to 0} P\left(X_1 \in [x_1 - \Delta t, x_1 + \Delta t], \dots, X_n \in [x_n - \Delta t, x_n + \Delta t]\right)$$

$$\hookrightarrow \hat{\theta}_n = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}(X_1 = x_1, \dots, X_n = x_n ; \theta)$$

Rk : $X \sim \mathcal{B}(p)$

$$f_X(t) = (1-p)^{1-t} \times p^{t} \quad \text{with } t \in \{0,1\}$$

Ex : $X_1, \dots, X_n \sim \mathscr{E}(\lambda)$

$$\mathscr{L}(X_1 = x_1, \dots, X_n = x_n ; \lambda) = \lambda^n e^{-\lambda \sum x_i} \mathbb{1}_{\min(x_i)}$$

$$g(\lambda) = \ln \mathscr{L}(X_1 = x_1, \dots, X_n = x_n ; \lambda)$$

$$= n \ln \lambda - \lambda \sum x_i$$

$$g'(\lambda) = \frac{n}{\lambda} - \sum x_i$$

$$g'(\lambda) = 0 \iff \lambda = \frac{n}{\sum x_i}$$

$$g''(\lambda) = -\frac{n}{\lambda^2} < 0$$

$$\Rightarrow \hat{\lambda}_n = \frac{1}{\overline{X}_n}$$

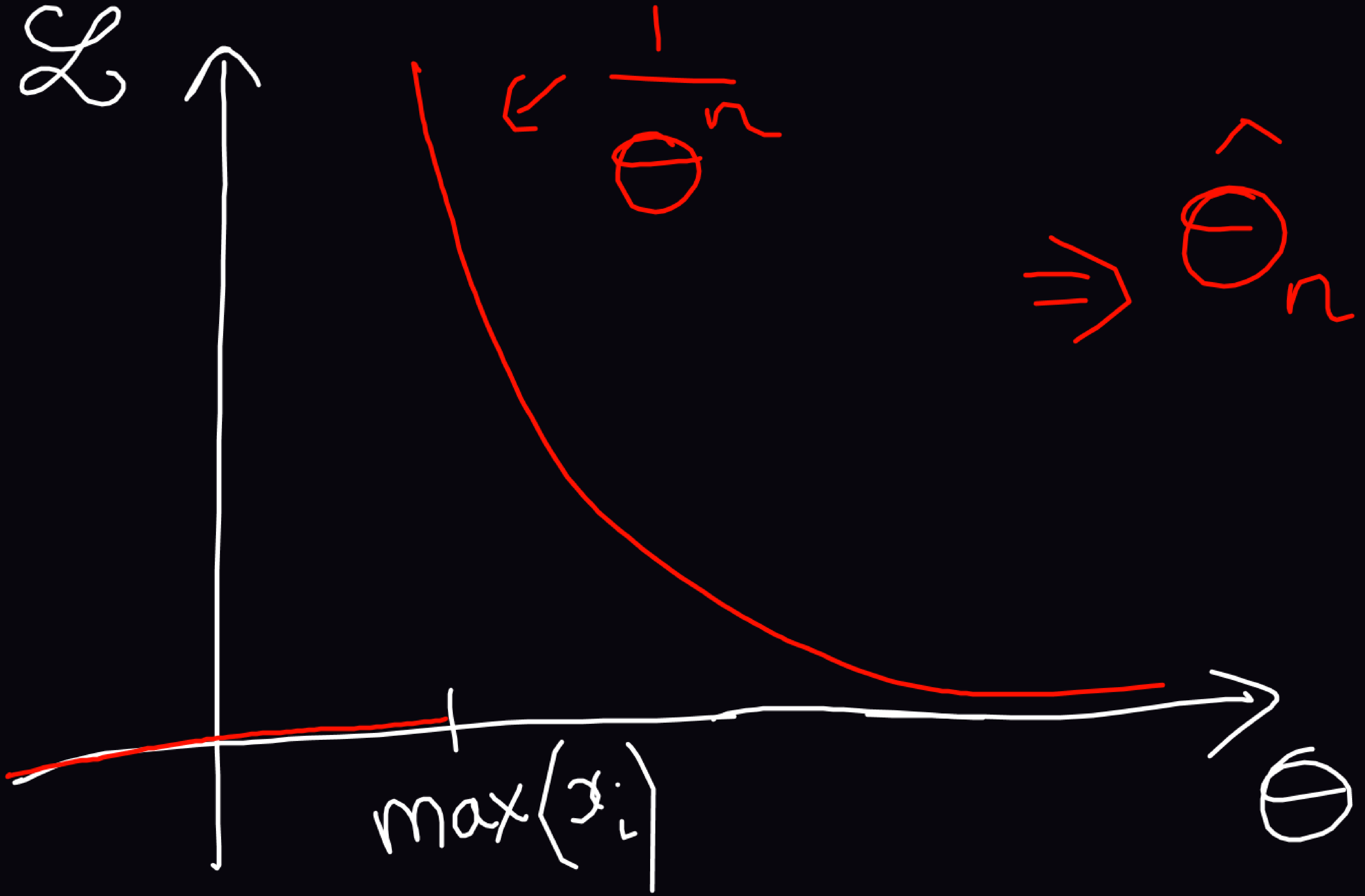
- $X_1, \dots, X_n \sim \mathcal{U}([0,\theta])$

$$\mathcal{L}(X_1 = x_1, \dots, X_n = x_n; \theta) = \prod_{i=1}^{n} \left( \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x_i) \right)$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\min(x_i) \geq 0} \mathbb{1}_{\max(x_i) \leq \theta}$$

Rk:

$$\mathbb{1}_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$




$$\Rightarrow \hat{\theta}_n = \max(X_i)$$

$P^b$ with point estimation (value of an estimator)

fluctuation

solution : confidence interval

<u>def</u>: $I$ is