Q.1 2009 — Class-lecture

Variable:- A variable is a quantity whose value varies from Individual to individual:

Example: the hight of the third year student because it varies student to student

C.V

Stochastic variable:- A stochastic variable is variable whose values is a number determined by out comes, time and space of the experiment

Example: let us consider an experiment of throwing a true die. let X_n be the ~~number of~~ sing first out comes of the nth throw, $n \geq 1$. For a distinct value of $n = 1, 2, \ldots$ we get distinct binomial variable. $X_n : \{X_n, n \geq 1\}$. Here X_n is a stochastic variable.

Question:- what is stochastic process? Describe the classification of stochastic process and give an example in each case

A stochastic process is a process which has some random or stochastic element involved in it's structure.

mathematically: a stochastic process is a process is a set of Random variables $\{x_t\}$ or $\{x(t)\}$ depending on some real parameters like the time and state

In other word, Families of random variable which are the functions of say time and state are known as stochastic process.

$P_{ex}(s) =$
$P_n(t)$
$P_n(t) =$

Example: consider a simple experiment like throwing a true die. Let X_n be the number of outcome of the nth throw, $n \geq 1$. For a distinct value of $n = 1, 2, \cdots$, we get we get a distinct binomial variable X_n: $\{X_n, n \geq 1\}$ which gives a family of Random.

2010/09 ~~Question~~ Question:- Discuss the classification of stochastic processes:-

Answer:- One dimentional stochastic process can be classified into four types of process which are as follows:=

(I) Discrete time and discrete state space:- Let X_n be the total no. of sixes in the first n throwing a die. Then the set of possible values of X_n is the finite of non negative Integer $0, 1, 2, \cdots n$. Here the state space of X_n is discrete and time n is discreate.

(II) Discreate time and continious state space:- Let X_n be random variable which represents the weight of nth student. Here n discreate time and the state space is continious

(III) Continious time and discreate state space

Suppose that $X(t)$ is the number of ~~all~~ autos in $(0, t)$. Here the $X(t)$ is discreate through the $X(t)$ is defined for a continious range of time.

(IV) continious time and continious state space

suppose that $X(t)$ represents the maximum temperature of a particular place in $(0,t)$ the set of possible values of $X(t)$ is continious. Here the time and state continious

~~Q.~~ Define Markov chain and Markov process.?

10 Answer:-

2010,12,08

Markov chain :- The stochastic process $\{X_n ; n=0,1,2\cdots\}$ is called a Markov chain if for $j, k, j_1, j_2, \cdots j_{n-1} \in N$ which are the outcomes or results of the experiment.

$$\Pr\{X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, \cdots ; X_0 = j_{n-1}\}$$
$$= \Pr\{X_n = k \mid X_{n-1} = j\}$$
$$= P_{jk} \text{ (say)}$$

wherever the first member is defined.

Markove process :- (MC) :

If $\{X(t), t \in T\}$ is a stochastic process such that given the value $X(s)$, the values of $X(t), t>s$ do not depend on the values of $X(u), u<s$ then the process is said to be MC.

The uses of MC are given bellow:

i) MC is widely used in physics, Engineering, Marketing and many other related areas.

ii) In business the MC is used powerfully. It is used to analyse the current states of product in an effort to predict its future.

iii) It is used to forecast the behaviour of customers from stand point.

$X_{n-1} = j$ $X_n = K$ or

Transition :-

Let $\{X_n; n = 0, 1, 2, \cdots\}$ be a markov chain. If $X_{n-1} = j$ and $X_n = K$, then the system is called transition when the state j at (n-1)-the trail go to the state K at nth trail.

Transition probability :-

$P(H) = P$

The probabilities of transition from the state j at (n-1) th trial to state K at n-th trail is called transition probability. The transition probability is basic to the study of the structure of the markov chain.

Transition probability matrix (t.p.m) :-

The transition probabilities P_{jK} satisfy $P_{jK} > 0$ and $\sum_K P_{jK} = 1$ $\forall$ j. There probabilities may be written as,

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

which is called the t.p.m of the M.C.

stochastic matrix :-

A square matrix with non negative elements and unit row sums is called stochastic matrix. Such that transition prob. matrix is stochastic matrix.

$$\text{i.e. } P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ \cdots & & & \cdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}.$$

2-set

Marcov chai

M.C-(2)

Math problem: (1)

(খুবই জরুরি)

305

pgf →

poisson →

Defn: (i) order of markov chain:

(ii) (chapman-Kolmogorov)

Q. What do you mean by markov chain?

Markov chain: The stochastic process $\{x_n : n = 0, 1, 2, 3 \cdots\}$ is called a markov chain if for $j, k, i_{n-1} \in N$

$$P_r\{x_n = k \mid x_{n-1} = j, x_{n-2} = i_1, \cdots x_0 = i_{n-1}\}$$

whenever the first member is defined.

The outcomes are called the state of the markov chain; if x_n has the outcome j i.e. $(x_n = j)$ the process is said to be at state j at nth trail. To a pair of states (j, k) at the two sucessive trails (say nth and $(n+1)$th trail). There is an associated conditional probability P_{jk}. It is the probability of transition from the state j at nth trail to the state k at $(n+1)$th trail.

Q. What do you mean by Homogeneous and non-homogeneous markov chain.

Homogeneous markov chain: The transition probabilities may or may not be independent of n. If the transition probability P_{jk} is the independent of n, then the markov chain is called / said to be homogeneous markov chain or stationary transition probabilities.

(5)

Non-homogeneous markov chain: The transition prob may or may not be independent of n. If the transition probability P_{jk} is dependen of n. then the Markov chain is called non-homogeneous markov chain.

Q. Define unit-step transition probability and m-step transition probability.

Unit-step transition probability: To a pair of states (j,k) at the two sucessive trails (nth and $(n+1)$th trals) there is an ~~associated~~ probability P_{jk}. It is the probability of transition from the state j at nth trail and to the state k at $(n+1)$th trail. Then the corresponding transition probability $P_{jk}^{(1)}$ Then the corresponding transition probability. It is is called unit-step transition probability. It is denoted by $P_{jk}^{(1)}$ and is defined as

$$P_{jk}^{(1)} = P_r\{x_{n+1} = k \mid x_n = j\}.$$

m-step transition probability: To a pair of states (j,k) at the two sucessive trails (nth and $(n+m)$th trail) there is an associated probability. It is the prob of transition from the state j at nth trail to the state k at $(n+m)$th trail. Then the corresponding transition probability is called m-unit step transition probability and it is denoted by $P_{jk}^{(m)}$ and is defined as:-

$$P_{jk}^{(m)} = P_r\{x_{n+m} = k \mid x_n = j\}.$$

Q. Define t.p.m with an example.

Ans: The transition probabilities P_{jk} satisfies the two condition

(i) $P_{jk} \geq 0$

(ii) $\sum_{k=0}^{n} P_{jk} = 1 \quad \forall j$

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots \end{pmatrix}$$

This is called transition probability matrix or matrix of transition probability. Let P is a stochastic matrix i.e a square matrix with non negative elements and unit row sums.

Example:

$$P = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \end{array} = 1$$

(3) P P

Q. Write the properties of transition probability matrix.

Ans: The properties of transition probability matrix are given below:-

(i) It is square matrix

(ii) It has non-negative elements.

(iii) It's row sum unit

Q. Define unit step transition matrix and m step transition matrix.

Soln:

Unit step transition matrix: Let x_n be a unit step markov chain assuming values $(n=0,1,2 \ldots m-1)$. Then its transition probability matrix p is given by

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0,m-1} \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1,m-1} \\ \vdots & & \vdots & & \vdots \\ P_{m-1,0} & P_{m-2,1} & & \cdots & P_{m-1,m-1} \end{pmatrix}$$

Where,

$$P_{jk}^{(1)} = \{ x_{n+1} = k \mid x_n = j \} \; ; \; i, k = 0, 1, 2 \ldots (m-1)$$

which is called unit step transition matrix.

m step transition matrix: Let x_n be a m step markov chain assuming values $(n=0,1,2 \cdots m-1)$ and let its transition proby matrix p is given by:-

$$P^{(m)} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0,(m-1)} \\ P_{10} & P_{11} & \cdots & P_{1,(m-1)} \\ \vdots & \vdots & & \vdots \\ P_{m,0} & P_{m,1} & \cdots & P_{m,m-1} \end{pmatrix}$$

where,

$$P_{jk}^{(m)} = \boxed{\{ x_{n+m} = k \mid x_n = j \}}$$

which is called m-step transition matrix.

Q. Define double stochastic matrix.

Ans: A non-negative square matrix is said to be double stochastic matrix if all the rows and columns sum are unity.

Example:

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

The above square is said to be double stochastic matrix since its row and column sums are unity.

Tutorial

Q. How do you calculate proby distn with the help of transition probability.

Ans. The proby distn of $x_r, x_{r+1} \cdots x_{r+n}$ can be computed interms of the transition probability matrix P_{jk} and the initial distn of x_0. suppose for simplicity $r=0$ then we get,

$$P_r\{x_0=a, x_1=b \cdots x_{n-2}=i, x_{n-1}=j, x_n=k\}$$

$$= P_r\{x_n=k \mid x_{n-1}=j, x_{n-2}=i \cdots x_0=a\}$$
$$P_r\{x_{n-1}=j, \cdots x_0=a\}.$$

$$= P_r\{x_n=k \mid x_{n-1}=j\} \cdot P_r\{x_{n-1}=j \mid x_{n-2}=i, \cdots x_0=a\}$$
$$P_r\{x_{n-2}=i, x_{n-3}=l \cdots x_0=a\}$$

$$= P_r\{x_n=k \mid x_{n-1}=j\} P_r\{x_{n-1}=j \mid x_{n-2}=i\}$$
$$P_r\{x_{n-2}=i \cdots x_0=a\}$$

$$= P_r\{x_n=k \mid x_{n-1}=j\} P_r\{x_{n-1}=j \mid x_{n-2}=i\} \cdots$$
$$P_r\{x_1=b \mid x_0=a\} P_r\{x_0=a\}$$

$$= P_r\{x_0=a\} p_{ab} \cdots p_{ij} \, p_{jk}$$

Thus

$$P_r\{x_r=a, x_{r+1}=b \cdots x_{r+n-2}=i, x_{r+n-1}=j, x_{r+n}=k\}$$
$$= P_r\{x_0=a\} p_{ab} \cdots p_{ij} \cdot p_{jk}.$$

(showed)

~~problem~~: Let $\{x_n, n \geq 0\}$ be a Markov chain with three states 0, 1 and 2 with transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

ass

and initial distn $P_r\{x_0=i\} = 1/3 ; i=0,1,2$. Find

(i) $p_{12}^{(2)}$ (ii) $p_{13}^{(2)}$ (iii) $p_{3}^{(2)}$ (iv) $p_{21}^{(1)}$

(v) $p_{12}^{(1)}$ (vi) $P_r\{x_2=2, x_1=1, x_0=2\}$

(vii) $P_r\{x_3=1, x_2=2, x_1=1, x_0=2\}$.

Solution: we have

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} \frac{3}{4} & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

state

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{10}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{pmatrix} \end{array}$$

(i) $P_{11}^{(2)} = P_r\{x_2 = 1 \mid x_0 = 1\} = \frac{8}{16}$

(ii) $P_{12}^{(2)} = P_r\{x_3 = 2 \mid x_1 = 1\} = 3/16$

$\cancel{x_2} = 2 \quad x_0 = 1$

(iii) $P_{10}^{(2)} = P_r\{x_4 = 0 \mid x_2 = 1\} = 5/16$

(iv) $P_{21}^{(1)} = P_r\{x_1 = 1 \mid x_0 = 2\} = 3/4.$

(v) $P_{12}^{(1)} = P_r\{x_2 = 2 \mid x_1 = 1\} = 1/4$

(vi) $P_r\{x_2 = 2, x_1 = 1, x_0 = 2\}$

$= P_r\{x_2 = 2 \mid x_1 = 1\} \cdot P_r\{x_1 = 1 \mid x_0 = 2\} \; P_r\{x_0 = 2\}$

$= P_{12}^{(1)} \times P_{21}^{(1)} \times P_r(x_0 = 2)$

$= \frac{3}{4} \times \frac{1}{4} \times 1/3$

$= 1/16$

(vii) $P_r\{x_3 = 1, x_2 = 2, x_1 = 1, x_0 = 2\}$

$= P_r\{x_3 = 1 \mid x_2 = 2\} \; P_r\{x_2 = 2 \mid x_1 = 1\} \; P_r\{x_1 = 1 \mid x_0 = 2\}$ $P_r\{x_0 = 2\}$

$= P_{21}^{(1)} \times P_{12}^{(1)} \times P_{21}^{(1)} \times P_r(x_0 = 2)$

$= \frac{3}{4} \times \frac{1}{4} \times 3/4 \times 1/3$

$= \frac{3}{64}$ Ans:-

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Problem: Given the transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ with states 1, 2, 3 and intial dist^n $Pr\{X_0 = i\}$; $i = 1, 2, 3$

Find (I) $P_{13}^{(1)}$ (II) $P_{13}^{(2)}$ (III) $P_{13}^{(3)}$ (IV) $P_{23}^{(1)}$ (V) $P_{23}^{(2)}$ (VI) $P_{23}^{(3)}$ (VII) $P_{33}^{(1)}$ (VIII) $P_{33}^{(2)}$ (IX) $P_{33}^{(3)}$.

Solution: We have,

states

$$P = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

$$\therefore P^3 = P^2 . P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 3/8 & 4/8 & 1/8 \\ 1/4 & 5/8 & 1/8 \\ 1/4 & 2/4 & 1/4 \end{pmatrix} \end{array}$$

(I) $P_{13}^{(1)} = Pr\{x_2 = 3 | x_1 = 1\} = 1/2$

(II) $P_{13}^{(2)} = Pr\{x_3 = 3 | x_1 = 1\} = 0$

(III) $P_{13}^{(3)} = Pr\{x_4 = 3 | x_1 = 1\} = 1/8$

(IV) $P_{23}^{(1)} = Pr\{x_2 = 3 | x_1 = 2\} = 0$

(V) $P_{23}^{(2)} = Pr\{x_3 = 3 | x_1 = 2\} = 1/4$

(VI) $P_{23}^{(3)} = Pr\{x_4 = 3 | x_1 = 2\} = 1/8$

(VII) $P_{33}^{(1)} = Pr\{x_2 = 3 | x_1 = 3\} = 0$

(VIII) $P_{33}^{(2)} = Pr\{x_3 = 3 | x_1 = 3\} = 0$

(IX) $P_{33}^{(3)} = Pr\{x_4 = 3 | x_1 = 3\} = 1/4$

Qustion : How to construct the t.p.m ?

2011 : Qustion (নম্বর ৫০)

Soln :- 2009 problem : Let x_n be the position of a particle performs a random walk with absorbing barriers at states 0 and 4. The unit step transition probabilities are given below :-

$$P_r\{x_n = r+1 \mid x_{n-1} = r\} = p$$

$$P_r\{x_n = r-1 \mid x_{n-1} = r\} = q$$

$$P_r\{x_n = 0 \mid x_{n-1} = 0\} = 1 = P_{00}^{(1)} = 1$$

$$P_r\{x_n = 4 \mid x_{n-1} = 4\} = 1 = P_{44}^{(1)} = 1$$

Solution : The transition matrix is given by

states of x_n

$$\text{states of } x_{n-1} \quad \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ P_{00} & P_{01} & P_{02} & P_{03} & P_{04} \\ P_{20} & P_{21} & P_{22} & P_{23} & P_{24} \\ P_{30} & P_{31} & P_{32} & P_{33} & P_{34} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} \end{pmatrix}$$

$$= \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 \\ 3 & 0 & 0 & q & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(5a) Define order of a Markov chain. ok

Ans : A Markov chain $\{x_n\}$ is said to be of order s $(s = 1, 2, 3)$ if for all n.

$$P_r\{x_n = k \mid x_{n-1} = j,\ x_{n-2} = j_1 \cdots x_{n-s} = j_{s-1} \cdots\}$$

$$= P_r\{x_n = k \mid x_{n-1} = j \cdots x_{n-s} = j_{s-1}\}.$$

whenever the l.h.s is defined.

A Markov chain $\{x_n\}$ is said to be of order one (simply a Markov chain) if

$$P_r\{x_n = k \mid x_{n-1} = j,\ x_{n-2} = j_1 \cdots\}$$

$$= P_r\{x_n = k \mid x_{n-1} = j\}$$

$$= P_{jk}$$

2010 ok

Problem: Draw the transition graph of the following Markov chain

$$\begin{pmatrix} 5/8 & 3/8 & 0 \\ 3/8 & 1/4 & 3/8 \\ 0 & 7/8 & 1/8 \end{pmatrix}$$

Solution: We have,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 5/8 & 3/8 & 0 \\ 3/8 & 1/4 & 3/8 \\ 0 & 7/8 & 1/8 \end{pmatrix} \end{array}$$

The graph of the above chain is

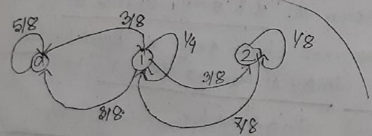


Fig-1: Transition graph of Markov chain.

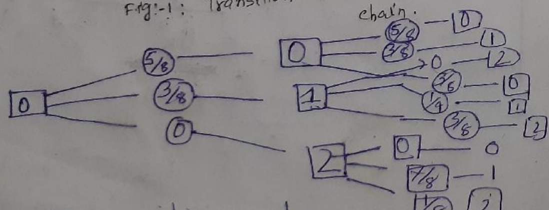


Fig: Transition trees:-

Problem: Draw the transition graph of the following Markov chain

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

Solution: Let us define

$$\text{states}\ \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \text{states} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

The graph of the above chain is

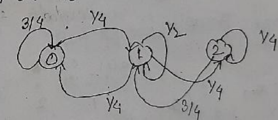
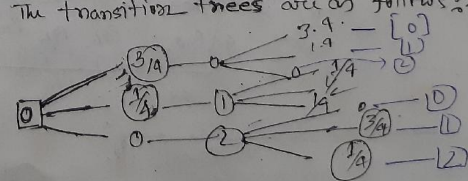


Fig: Transition graph of Markov chain.

The transition trees are as follows:-



Problem: Draw the transition graph of the following Markov chain

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

Solution: Let us define

$$\begin{array}{cc} & \text{states} \\ & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \text{states} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

The transition graph of the above chain is

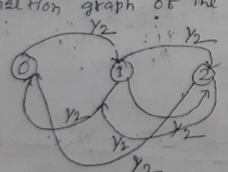
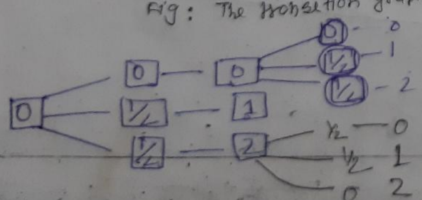


Fig: The transition graph of Markov chain.



2010, 2011, 2012

State and prove that Chapman-Kolmogorov equation. Show how these equations are used to give n-step transition probability.

Statement: For a unit step transition prob''s gives the prob from the state j at a trail state k at the unit following trail.

Now,

The $(m+n)$ step-transition probabilities is

$$P_{jk}^{(m+n)} = \sum_r P_{rk}^{(n)} \cdot P_{rj}^{(m)} = \sum P_{jr}^{(m)} P_{rk}^{(n)}.$$

Proof: If state k can be reached from the state j in two step through some intermediate state r then we can write for fixed r

$$P_r\left[x_{n+2}=k,\ x_{n+1}=r \mid x_n=j\right]$$

$$= P_r\left[x_{n+2}=k \mid x_{n+1}=r\right] P_r\left[x_{n+1}=r \mid x_n=j\right]$$

$$= P_r\left[x_{n+2}=k \mid x_{n+1}=r\right] P_r\left[x_{n+1}=r \mid x_n=j\right]$$

$$P_{jk}^{(2)} = P_{jr}^{(1)} P_{rk}^{(1)}$$

Since the intermediate states r can assume value $r = 1, 2 \cdots$ then we get

$$P_{jk}^{(2)} = P_r\left[x_{n+2}=k \mid x_n=j\right]$$

$$= \sum_r P_r[x_{n+2} = k, x_{n+1} = r \mid x_n = j]$$

$$= \sum_r P_r[x_{n+2} = k \mid x_{n+1} = r]\, P_r[x_{n+1} = r \mid x_n = j]$$

$$= \sum p_{jr}^{(1)}\, p_{rk}^{(1)}$$

which is the chapman kolmogorov equation for two step.

Again, we can write,

$$p_{jk}^{(m+1)} = \sum P_r[x_{m+n+1} = k \mid x_n = j]$$

$$= \sum P_r[x_{m+n+1} = k \mid x_{n+m} = r] \cdot P_r[x_{n+m} = r \mid x_n = j]$$

$$= \sum p_{rk}^{(1)}\, p_{jr}^{(m)}$$

similarly we get

$$p_{jk}^{(m+1)} = \sum p_{jr}^{(m)} \cdot p_{rk}^{(1)}$$

which is the chapman kolmogorov equation for (m+1) step.

In general we have

$$p_{jk}^{(m+n)} = \sum_r p_{rk}^{(n)}\, p_{jr}^{(m)}$$

$$= \sum_r p_{jr}^{(m)}\, p_{rk}^{(n)} \quad \cdots (1).$$

which is the chapman - kolmogorov equation for (m+n) step.

Note:

From (1) we have

$$p_{jk}^{m+n} \geq p_{jr}^{(m)}\, p_{rk}^{(n)}$$

Let $P = p_{jk}$ be the transition matrix of the unit-step transition and $p^{(m)} = p_{jk}^{(m)}$ be the transition matrix for m unit-step transition. put $m = 2$ then we get $p^{(2)}$.

i.e $p^{(2)} = p \cdot p = p^2$

similarly

$$p^{m+1} = p^m \cdot p = p \cdot p^m$$

and

$$p^{m+n} = p^m \cdot p^n = p^n \cdot p^m$$

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Problem: Let $\{X_n, n \geq 0\}$ be a homogeneous two-state Markov chain with unit-step transition matrix

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

Show that

$$P^n = \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^n}{2} & \frac{1}{2} - \frac{(q-p)^n}{2} \\ \frac{1}{2} - \frac{(q-p)^n}{2} & \frac{1}{2} + \frac{(q-p)^n}{2} \end{pmatrix}$$

Solution: We have

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

Now,

$$P^2 = P.P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}\begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

$$= \begin{pmatrix} q^2+p^2 & pq+pq \\ pq+pq & p^2+q^2 \end{pmatrix}$$

$$= \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} \frac{(p+q)^2}{2} + \frac{(q-p)^2}{2} & \frac{(q+p)^2}{2} - \frac{(q-p)^2}{2} \\ \frac{(q+p)^2}{2} - \frac{(q-p)^2}{2} & \frac{(p+q)^2}{2} + \frac{(q-p)^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^2}{2} & \frac{1}{2} - \frac{(q-p)^2}{2} \\ \frac{1}{2} - \frac{(q-p)^2}{2} & \frac{1}{2} + \frac{(q-p)^2}{2} \end{pmatrix}$$

$$P^3 = P^2.P$$

$$= \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix}\begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

$$= \begin{pmatrix} q^3+p^2q+2p^2q & p^3+pq^2+2pq^2 \\ p^3+pq^2+2pq^2 & q^3+p^2q+2p^2q \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} q^3+3p^2q & p^3+3pq^2 \\ p^3+3pq^2 & q^3+3pq^2 \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{(p+q)^3}{2} + \frac{(q-p)^3}{2} & \frac{(p+q)^3}{2} - \frac{(q-p)^3}{2} \\ \frac{(p+q)^3}{2} - \frac{(q-p)^3}{2} & \frac{(p+q)^3}{2} + \frac{(q-p)^3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^3}{2} & \frac{1}{2} - \frac{(q-p)^3}{2} \\ \frac{1}{2} - \frac{(q-p)^3}{2} & \frac{1}{2} + \frac{(q-p)^3}{2} \end{pmatrix}$$

similarly

$$P^4 = \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^4}{2} & \frac{1}{2} - \frac{(q-p)^4}{2} \\ \frac{1}{2} - \frac{(q-p)^4}{2} & \frac{1}{2} + \frac{(q-p)^4}{2} \end{pmatrix}$$

In general we have

$$P^n = \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^n}{2} & \frac{1}{2} - \frac{(q-p)^n}{2} \\ \frac{1}{2} - \frac{(q-p)^n}{2} & \frac{1}{2} - \frac{(q-p)^n}{2} \end{pmatrix}$$

(showed)

2010

Q. Suppose that the initial distribution

$$P_r\{x_0 = 0\} = a$$

and $P_r\{x_0 = 1\} = b = 1-a$.

Applying Bayes rule, show that

$$P_r\{x_0 = 0 \mid x_n = 0\} = \frac{a\{1 + (q-p)^n\}}{1 + (a-b)(q-p)^n}$$

Solution: we have,

$$P^n = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \frac{1}{2} + \frac{(q-p)^n}{2} & \frac{1}{2} - \frac{1}{2}(q-p)^n \\ \frac{1}{2} - \frac{1}{2}(q-p)^n & \frac{1}{2} + \frac{1}{2}(q-p)^n \end{pmatrix} \end{array}$$

Now,

$$P_r\{x_0 = 0\} = a$$

$$P_r\{x_0 = 1\} = b = 1-a$$

$$\therefore P_r\{x_n = 0, x_0 = 0\} = P_r(x_n = 0 \mid x_0 = 0)\, P_r(x_0 = 0)$$

$$= P_{00}^{(n)}\, a$$

$$= a P_{00}^{(n)}.$$

Again,

$$P_r\{x_n = 0, x_0 = 1\} = P_r\{x_n = 0 \mid x_0 = 1\}\, P_r\{x_0 = 1\}$$

$$= P_{10}^{(n)}\, b$$

$$= b P_{10}^{(n)}.$$

Applying Bayes rule we have

$$Pr\{x_0=0 \mid x_n=0\} = \frac{Pr(x_n=0 \mid x_0=0)\, Pr(x_0=0)}{Pr(x_n=0 \mid x_0=0)\, Pr(x_0=0) + Pr(x_n=0 \mid x_1=1)\, Pr(x_1=1)}$$

$$= \frac{a\, p_{00}^{(n)}}{a\, p_{00}^{(n)} + b\, p_{10}^{(n)}}$$

$$= \frac{a\left\{\frac{1}{2}+\frac{1}{2}(2-p)^n\right\}}{a\left\{\frac{1}{2}+\frac{1}{2}(2-p)^n\right\} + b\left\{\frac{1}{2}-\frac{1}{2}(2-p)^n\right\}}$$

$$= \frac{\frac{a}{2}\left\{1+(2-p)^n\right\}}{a\left\{\frac{1}{2}+\frac{1}{2}(2-p)^n\, \frac{1}{2}+\frac{1}{2}(2-p)^n\right\} + \frac{1}{2}-}$$

$$= \frac{\frac{a}{2}\left\{1+(2-p)^n\right\}}{\frac{a}{2}+\frac{a}{2}(2-p)^n+\frac{b}{2}-\frac{b}{2}(2-p)^n}$$

$$= \frac{\frac{a}{2}\left\{1+(2-p)^n\right\}}{\frac{1}{2}(a+1-a)+\frac{(a-b)}{2}(2-p)^n}$$

$$= \frac{a\left\{1+(2-p)^n\right\}}{1+(a-b)(2-p)^n}$$

$$\therefore\ Pr\{x_0=0 \mid x_n=0\} = \frac{a\left\{1+(2-p)^n\right\}}{1+(a-b)(2-p)^n}$$

(showed)

problem: If

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array}$$

$Pr\{x_0=1\} = P_1 = 1 + Pr\{x_0=0\} = 1-q_1$.

Show that $P_n = \frac{a}{a+b} + \left(P_1 - \frac{a}{a+b}\right)(1-a-b)^n$

solution: we have

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{b}{a+b} + (1-a-b)\frac{a}{a+b} & \frac{a}{a+b} - (1-a-b)\frac{a}{a+b} \\ \frac{b}{a+b} - (1-a-b)\frac{b}{a+b} & \frac{a}{a+b} + (1-a-b)\frac{b}{a+b} \end{pmatrix}$$

$$P^2 = P \cdot P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

$$= \begin{pmatrix} (1-a)^2 + ab & a(1-a) + a(1-b) \\ b(1-a) + b(1-b) & ab + (1-b)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2a+a^2+ab & 2a-a^2-ab \\ 2b-b^2-ab & 1-2b+b^2+ab \end{pmatrix}$$

$$= \begin{pmatrix} \frac{b}{a+b} + (1-a-b)^2 \frac{a}{a+b} & \frac{a}{a+b} - (1-a-b)^2 \frac{a}{a+b} \\ \frac{b}{a+b} - (1-a-b)^2 \frac{b}{a+b} & \frac{a}{a+b} + (1-a-b)^2 \frac{b}{a+b} \end{pmatrix}$$

similarly

$$p^n = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \frac{b}{a+b} + (1-a-b)^n \frac{a}{a+b} & \frac{a}{a+b} - (1-a-b)^n \frac{a}{a+b} \\ \frac{b}{a+b} - (1-a-b)^n \frac{b}{a+b} & \frac{a}{a+b} + (1-a-b)^n \frac{b}{a+b} \end{pmatrix} \end{array}$$

we know that

$$P_n = Pr\{x_n = 1\}$$

$$= Pr\{x_n = 1 | x_0 = 0\} \, Pr\{x_0 = 0\} + Pr\{x_n = 1 | x_0 = 1\} \, Pr\{x_0 = 1\}$$

$$= p_{11}^{(n)} Pr\{x_0 = 1\} + p_{01}^{(n)} Pr\{x_0 = 0\}.$$

Now given that

$$Pr\{x_0 = 1\} = P_1 = 1 - q_1$$

Now,

$$P_n = p_{11}^{(n)} P_1 + p_{01}^{(n)} q_1$$

$$= p_{11}^{n} P_1 + p_{01}^{(n)} (1 - P_1)$$

$$= p_{11}^{(n)} P_1 + p_{01}^{(n)} - p_{01}^{(n)} P_1$$

$$= p_{01}^{(n)} + P_1 \{ p_{11}^{(n)} - p_{01}^{(n)} \}$$

$$= \frac{a}{a+b} - (1-a-b)^n \frac{a}{b+a} + P_1 \left\{ \cancel{\frac{a}{a+b}} + (1-a-b)^n \frac{b}{a+b} - \cancel{\frac{a}{a+b}} + (1-a-b)^n \frac{a}{a+b} \right\}$$

$$= \frac{a}{a+b} - (1-a-b)^n \frac{a}{a+b} + P_1 (1-a-b)^n \cdot \left(\frac{a+b}{a+b} \right)$$

$$= \frac{a}{a+b} - (1-a-b)^n \frac{a}{a+b} + P_1 (1-a-b)^n.$$

$$= \frac{a}{a+b} - (1-a-b)^n \frac{a}{a+b} + P_1 (1-a-b)^n.$$

$$\therefore P_n = \frac{a}{a+b} - (1-a-b)^n \frac{a}{a+b} + P_1 (1-a-b)^n$$

shown

Problem: If
$$P = \begin{pmatrix} 1-(1-c)p & (1-c)p \\ (1-c)(1-p) & (1-c)p+c \end{pmatrix}$$
show that
$$p_n = c\,p_{n-1} + (1-c)p.$$

Solution: we have
$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-(1-c)p & (1-c)p \\ 1 & (1-c)(1-p) & (1-c)p+c \end{array}$$

we know that
$$p_n = Pr\{x_n = 1\}$$
$$= Pr\{x_n = 1 \mid x_{n-1} = 1\}\, Pr\{x_{n-1} = 1\} + Pr\{x_n = 1 \mid x_{n-1} = 0\}\, Pr\{x_{n-1} = 0\}.$$
$$= p_{11}^{(1)}\, p_{n-1} + p_{01}^{(1)}\, q_{n-1}$$
$$= \{(1-c)p + c\}\, p_{n-1} + (1-c)p\,(1 - p_{n-1})$$
$$= \cancel{(1-c)p\,p_{n-1}} + c\,p_{n-1} + (1-c)p - \cancel{(1-c)p\,p_{n-1}}$$
$$= c\,p_{n-1} + (1-c)p. \text{ (showed)}$$

Q. Define persistent, transient, null persistent and non-null persistent, ergodic. V.V.g

Ans:

Persistent: A state j is said to be persistent or recurrent if and only if $F_{jj} = 1$ return to state j is certain.

Transient: A state j is said to be transient if and only if $F_{jj} \lneq 1$ return to state j is uncertain.

Null persistent: A persistent state j is said to be null persistent if and only if $\mu_{jj} = \infty$ i.e the mean recurrence time is infinite.

Non-null persistent: A persistent state j is said to be non-null persistent if and only if $\mu_{jj} \leq \infty$ i.e the mean recurrence time is less than infinite.

Ergodic: A non-null persistent and aperiod state of a markov chain is said to be ergodic.

Define recurrent, positive recurrent and null recurrent Markov chain.

Ans:

Recurrent: A state j is said to be recurent if and only if $F_{jj} = 1$ return to the state j is certain.

Positive recurrent: A recurrent state j is said to be positive recurrent if and only if $\mu_{jj} < \infty$ i.e the mean recurrence time is infinite.

Null recurrent: A reurrent state j is said to be null recurrent if and only if $\mu_{jj} \leq \infty$ i.e the mean recurrence time is less than infinite.

Distinguish between a persistent and a transient state.

Ans: The difference between a persistent state and transient state are given below:-

(i) If $F_{jj} = 1$ then the state j is said to be persistent but if $F_{jj} \leq 1$ then the state j is said to be transient state.

(ii) For state j the persistent state return to the state j is certain but transient state is uncertain.

(iii) There are two types of persistent state null persistent and non-null persist but transient has no type.

When is a state null persistent? When it is periodic.

Ans: A persistent state j is said to be null persistent if and only if $\mu_{jj} \leq \infty$ i.e the mean recurrence time is infinite.

Let us consider a transition probability matrix P.

In general $P^{2n} = P^{r}$ and $P = P^{2n+1}$

so that $P^{2n} \geqslant 0$ and $P^{2n+1} = 0$ for each i then the state is said to periodic.

but transient state is

Question : construct the transition matrix with the following statement :-

$\underline{012}\ \underline{00}\underline{1}0\ \underline{22}\ \underline{11}\ 000\ 12\ 1\underline{02}\ \underline{10}$

Answer :-

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 111 = 3/8 & 111 = 3/8 & 11 = 2/8 \\ 1 & 1111 = 4/7 & 1 = 1/7 & 11 = 2/7 \\ 2 & 1 = 1/5 & 111 = 3/5 & 1 = 1/5 \end{array}$$

$$\therefore P = \begin{pmatrix} 3/8 & 3/8 & 2/8 \\ 4/7 & 1/7 & 2/7 \\ 1/5 & 3/5 & 1/5 \end{pmatrix}$$

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \cancel{111} & \cancel{111} & \cancel{11} \\ 1 & \cancel{1111} & 1 & \cancel{11} \\ 2 & 1 & \cancel{111} & \cancel{1} \end{array}$$