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1-11-2010

Islamic University (IU)

Course title: **Elements of Statistics and Probability**

Chapter: **Probability**

Experiment / Trial: An experiment or trial is an act that can be repeated under given identical conditions.

Example: Throwing a die, tossing a coin are the examples of experiment or trial.

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Outcome: An outcome is the results of an experiment.

Examples

- If we throw a die we get 1 or 2 or 3 or 4 or 5 or 6. So that individually 1 is an outcome, 2 is an outcome.
- If we toss a coin we get head or tail. Individually head and trial are two outcomes.

Sample space: A sample space is the set of all outcomes.

Example

- If we throw a die the outcomes are 1, 2, 3, 4, 5 and 6. Then $S = \{1,2,3,4,5,6\}$ is a sample space.
- If we toss a coin then the outcomes are head (H) and tail (T). Then $S = \{H,T\}$ is a sample space.

Event: An event is the collection of one or more outcomes of an experiment.

Example

If we throw a die the outcomes are 1, 2, 3, 4, 5, and 6. Then the outcomes of even numbers are 2, 4, 6. Then $A = \{2,4,6\}$ is called an event of even numbers.

Mutually Exclusive event: When an event occurs and none of the other events will occure at the same time, then the event is called mutually exclusive event.

Example

If we toss a coin two outcomes head (H) and tail (T) are mutually exclusive event. Because if it appears head (H) or tail (T) not both head and tail at the same time.

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Formally, two events A and B are mutually exclusive if and only if $A \cap B = \phi$.

Exhaustive events

The total number of possible outcomes in any trial is known as exhaustive events or exhaustive cases.

Example

- In tossing of a coin there are two exhaustive cases viz, head and tail.
- In throwing of a die, there are six exhaustive cases since any one of the 6 faccs 1, 2, 3, 4, 5, 6 may come upper most.

Definition of Probability

There are mainly two definitions of probability, namely

1) Mathematical or classical or a priori definition of probability
2) Statistical or empirical or a posteriori definition of probability.

Classical or a priori probability

If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favorable to the happening of an event E, then the probability p of happening of E is given by

$$p = p(E) = \frac{\text{Favorable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}.$$

Limitations

⇨ The classical probability fails to define probability when the total numbers of possible outcomes are infinite.

⇨It is not always to enumerate. → গণনা,

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Statistical or empirical probability

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event happens to the number of trials as the number of trials become indefinitely large is called the probability of happening of the event. It is assumed that the limit is finite and unique.

Symbolically, if in n trials an event E happens m times, then the probability p of the happening of E is given by

$$p = p(E) = \lim_{n \to \infty} \frac{m}{n}.$$

Subjective probability

The probability that a person assigns to an event which is the possible outcomes of some processes on the basis of his own judgment, beliefs and information about the processes is known as subjective probability.

For example, one fine morning Mr. X may well be prepared for rain, but his friend Mr. Y may not.

Properties of probability

Let E be an experiment. Also let S be a sample space associated with E, with each event A we associate a real number, designed by $P(A)$ and called the probability of A satisfying the following properties:

- $0 \le p(A) \le 1$.
- $p(S) = 1$.
- If A and B are mutually exclusive events, $p(A \cup B) = p(A) + p(B)$.

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Example

A bag contains 4 white and 6 black balls. If one ball is drawn at random from the bag, what is the probability that the ball is

(a) black (b) white (c) white or black and (d) red.

Solution

Hence the total numbers of balls are 10. Since one ball is drawn from the bag, there are 10 mutually exclusive, equally likely and exhaustive outcomes of this experiment.

(a) Let A be the event that the ball is black, then the number of favorable outcomes to A is 6. So that

$$p(A) = \frac{\text{number of black balls}}{\text{total number of balls}} = \frac{6}{10}.$$

(b) Let B be the event that the ball is white, then the favorable outcomes corresponding to B are 4. Therefore

$$p(B) = \frac{4}{10}.$$

(c) Let C be the event that the ball is white or black, then the favorable outcomes corresponding to C are 10. Therefore

$$p(C) = \frac{10}{10} = 1.$$

(d) Let D be the event that the drawn ball is red, then the favorable outcomes corresponding to D is zero. Therefore

$$p(D) = \frac{0}{10} = 0.$$

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Example

Three contractor A, B and C are biding for the construction of a new cinema hall. Some expert in this industry believes that A has exactly half the chance that B, in turn is $\frac{4}{5}$th as like as C to win the contract. What is the probability for each to win the contract if the expert's estimates are accurate?

Solution: Let the probability of C's wining the contract is x. Then $p(C)=x$, $p(B)=\frac{4}{5}x$ and $p(A)=\frac{4}{5}\cdot\frac{1}{2}x$. We know that total probability is one. So that

$$x+\frac{4}{5}x+\frac{2}{5}x=1$$

$$\Rightarrow \frac{5x+4x+2x}{5}=1$$

$$\Rightarrow \frac{11x}{5}=1$$

$$\therefore x=\frac{5}{11}$$

Hence, $p(C)=\frac{5}{11}$, $p(B)=\frac{4}{11}$ and $p(A)=\frac{2}{11}$.

Example

Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has

1) An odd number
2) A number 4 or multiple of 4
3) A number which is greater than 70 and
4) A number which is square?

Since there are 100 tickets, the total number of cases ...

1) Let A denote the event that the ticket drawn an odd number. Since there are 50 odd number tickets so the number of cases favorable to the event A is 50.

$$\therefore p(A)=\frac{50}{100}=0.5.$$

2) Let B denote the event that the ticket drawn has a number 4 or multiple of 4. The numbers favorable to event B are $4, 8, 12, 16, 20, \dots, 92, 96, 100$. The total number of cases will be $\frac{100}{4}=25$.

$$\therefore p(B)=\frac{25}{100}=0.25.$$

3) Let C denote the event that the drawn ticket has a number greater than 70. Since the number greater than 70 are $71, 72, 73, \dots, 100$. Therefore, 30 cases are favorable to the event C.

$$\therefore p(C)=\frac{30}{100}=0.3.$$

4) Let D denote the event that the drawn ticket has a number which is a square. Since the squares between 1 and 100 are 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100. So the cases favorable to event D are 10 in number. Hence,

$$\therefore p(D)=\frac{10}{100}=0.1.$$

Conditional probability

Let A and B be two events. The conditional probability of event A given that B has occurred, is defined by the symbol $p(A|B)$ and is found to be:

$$p(A|B)=\frac{p(A\cap B)}{p(B)}; \text{ provided } p(B)>0.$$

Similarly, $p(B|A)=\dfrac{p(A\cap B)}{p(A)}$; provided $p(A)>0$.

Example

A hamburger chain found that 75% of all customers use mustard, 80% use ketchup and 65% use both. What are the probabilities that a ketchup user uses mustard and that a mustard user uses ketchup?

Solution

Let A be the event "customer uses mustard" and B be the event "customer uses ketchup". Thus, we have, $p(A)=0.75$, $p(B)=0.80$ and $p(A\cap B)=0.65$.

The probability that a ketchup user uses mustard is the conditional probability of event A, given event B is

$$p(A|B)=\frac{p(A\cap B)}{p(B)}=\frac{0.65}{0.80}=0.8125.$$

Similarly, the probability that a mustard user use ketchup is

$$p(B|A)=\frac{p(A\cap B)}{p(A)}=\frac{0.65}{0.75}=0.8667$$

Rules of Addition

Special rule of Addition

If two events A and B are mutually exclusive, the special rule of addition states that the probability of one or the other events occurring equals the sum of their probabilities i., e.,

$$p(A \text{ or } B)=p(A)+p(B)$$

$$\text{or } p(A\cup B)=p(A)+p(B).$$

For three mutually exclusive events designated A, B and C the rule is written as

$$p(A \text{ or } B \text{ or } C) = p(A) + p(B) + P(C)$$

$$\text{or } p(A \cup B \cup C) = p(A) + p(B) + P(C).$$

Example

If we toss a coin then what is the probability of head or tail?

Solution

Here there are two events, namely event $A = H$ and event $B = T$. So that

$$p(A \text{ or } B) = p(A) + p(B)$$
$$= \frac{1}{2} + \frac{1}{2} = 1.$$

The general rules for addition

When two or more events are not mutually exclusive then we use the general rule for addition. The rule is

$$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$$

$$\text{or } p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

Example

Mr. X feels that the probability that he will pass Mathematics is $\frac{2}{3}$ and Statistics is $\frac{5}{6}$. If the probability that he will pass both the course is $\frac{3}{5}$. What is the probability that he will pass at least one of the course?

Solution

Let M and S be the events that he will pass the courses Mathematics and Statistics respectively. The event $M \cup S$ means that at least one of M or S occurs. Therefore

$$p(M \cup S) = p(M \text{ or } S)$$
$$= p(\text{he pass at least one of the course})$$

$$= p(M)+p(S)-p(M \text{ or } S)$$
$$= \frac{2}{3}+\frac{5}{6}-\frac{3}{5}=\frac{9}{10}.$$

Example

Mr. Y feels that the probability that he will get A in Calculus is $\frac{3}{4}$, A in Statistics is $\frac{4}{5}$ and A in both the courses is $\frac{3}{5}$. What is the probability that Mr. Y will get

a) At least one A
b) No A's?

Solution

Let C be the event that Mr. Y will get A in Calculus and S be the event that he will get A in Statistics. We have, $p(C)=\frac{3}{4}$, $p(S)=\frac{4}{5}$ and $p(C \text{ and } S)=p(C\cap S)=\frac{3}{5}$.

a) $p(\text{at least one } A)=p(C\cup S)$

$$= p(C)+p(S)-p(C\overset{\cap}{\cup}S)$$
$$= \frac{3}{4}+\frac{4}{5}-\frac{3}{5}=\frac{19}{20}$$

b)

$$p(\text{no } A\text{'s})=p(\overline{CS})$$
$$=1-p(C\cup S)=1-\frac{19}{20}=\frac{1}{20}$$

Complement rule

The complement rule is used to determine the probability of an event occurring by subtracting the probability of the event not occurring from 1 i., e. $p(A)=1-p(\overline{A})$.

Example

Weight	Event	Probability
Underweight	A	0.025
Satisfactory	B	??
Overweight	C	0.075

Find $p(B)$.

Solution

We know that $p(B) = 1 - p(\bar{B}) = 1 - (0.025 + 0.075) = 0.90$.

Rules of Multiplication

Special rule of multiplication requires that two events A and B are independent (two events are independent if the occurrence of one event does not alter the probability of the occurrence of the other event).

For two independent events A and B, the probability that A and B will both occur is found by multiplying the two probabilities i., e., $p(A \text{ and } B) = p(A) * p(B)$.

For three events A, B and C the special rule of multiplication used to determine the probability that all the events will occur is:

$$p(A \text{ and } B \text{ and } C) = p(A) * p(B) * p(C)$$

Example

A company has two large computers. The probability that the newer one will breakdown on any particular month is 0.05, the probability that the older one will breakdown on any particular month is 0.1. What is the probability that they will both breakdown in a particular month?

Solution

Let, Event A is the newer one will breakdown and Event B is the older one will breakdown. So that $p(A) = 0.05$ and $p(B) = 0.1$.

$$\therefore p(A \text{ and } B) = p(A) * p(B) = 0.05 * 0.1 = 0.005.$$

General rule of Multiplication

The general rule of multiplication states that for two events A and B, the joint probability that both events will happen is found by multiplying the probability of event A will happen by the conditional probability of event B occurring given that event A has occurred. Symbolically, the joint probability is

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$$p(A \text{ and } B) = p(A) * p(B|A).$$

Example

There are 10 rolls of film in a box, 3 of which are defective. Two rolls are to be selected one after another. What is the probability of selecting a defective roll followed by another defective roll?

Solution

The first roll of film selected from the box being found defective is event D_1.

$$\therefore p(D_1) = \frac{3}{10}.$$

The second roll selected being found defective is event D_2. Therefore, $p(D_2|D_1) = \frac{2}{9}$. Since, after the first selection was found to be defective, only 2 defective rolls of film remained in the box containing 9 rolls.

So the probability of two defectives is

$$\begin{aligned} &= p(D_1 \text{ and } D_2) \\ &= p(D_1) * p(D_2|D_1) \\ &= \frac{3}{10} * \frac{2}{9} = \frac{6}{90} = 0.07. \end{aligned}$$

Bayes Theorem

If $E_1, E_2, \ldots, E_n$ are mutually disjoint events with $p(E_i) \neq 0$; $i = 1, 2, \ldots, n$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $p(A) > 0$, we have

$$p(E_i|A) = \frac{p(E_i)\, p(A|E_i)}{\sum_{i=1}^{n} p(E_i)\, p(A|E_i)}, \quad i = 1, 2, \ldots, n.$$

Example

In a bolt factory machines A, B and C manufacturer respectively 25%, 35% and 40% of the total. Of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at randomly from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

Solution

Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event of its being defective. Then we have, $p(E_1) = 0.25$, $p(E_2) = 0.35$ and $p(E_3) = 0.40$.

The probability of drawing a defective bolt manufactured by machine A is $p(E|E_1) = 0.05$. Similarly, we have $p(E|E_2) = 0.04$ and $p(E|E_3) = 0.02$.

Hence, the probability that a defective bolt selected at random is manufactured by machine A is given by

$$p(E_1|E) = \frac{p(E_1)p(E|E_1)}{\sum_{i=1}^{3} p(E_i)p(E|E_i)}$$

$$= \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.362$$

Similarly, $p(E_2|E) = \dfrac{p(E_2)p(E|E_2)}{\sum_{i=1}^{3} p(E_i)p(E|E_i)}$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.406$$

and $p(E_3|E) = \dfrac{p(E_3)p(E|E_3)}{\sum_{i=1}^{3} p(E_i)p(E|E_i)}$

$$= \frac{0.40 \times 0.02}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.232$$

Exercise

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