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Ⓦ ⑤ ⓐ pure birth process → $\boxed{\text{using p.g.f}}$
ⓑ simple birth process.
ⓒ show

Simple Birth, death Process

3-STAT-305
H.M.P

Q. Define Birth, death, Birth process, Death process, Pure Birth process, Pure Death process.

Ans: Birth: A Birth occurs whenever a new number joins the population.

Death: A Death occurs whenever a number leaves the population.

2010 Birth process:
The probabilities are the function of time of the Birth distribution, then the distribution is called Birth process.

Death process:
The probabilities are the function of ~~the~~ time of the Death distribution, then the dist^n are called Death ~~dist~~ process.

Pure Birth Process:
A pure Birth process is one that experience only births no deaths.

Pure Death Process:
A Pure Death Process is one that experiense only deaths no births.

13

Q) Establish the probability distribution formula for pure death process. OR

Derive the probᵗʸ distⁿ of departures (pure death process).

Ans: In this process we assume that there are N customers in the system at time $t=0$. Also assume that no arrivals (births) in the system.

Departures occurs at a rate μ per unit time. We wish to derive the distⁿ of departures from the system on the basis of the following three axioms:

1) $Pr[\text{one departure during } \Delta t] = \mu\Delta t + o(\Delta t)^2$
$= \mu\Delta t$
$[\because o(\Delta t)^2 \text{ is negligible}]$

2) $Pr[\text{more than one departure during } \Delta t]$
$= o(\Delta t)^2 = 0$

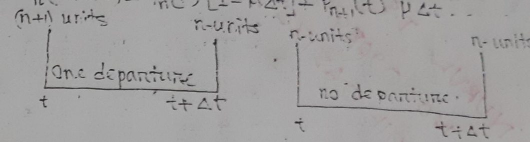
3) The number of departures is non-overlapping intervals are statistically independent, and identically distributed random variable i.e. The process N(t) has independent increment.

Now first obtain the difference equation in three mutually exclusive ways -

Case-I: when $0<n<N$ then

Proceeding exactly as in the pure death process.

$P_n(t+\Delta t) = P_n(t)[1-\mu\Delta t] + P_{n+1}(t)\,\mu\Delta t$ . . . (1)



Case II: when, $n=N$

Since there are exactly N-units in the system,

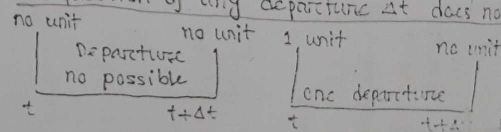
$\therefore P_{n+1}(t) = 0$

$\therefore P_N(t+\Delta t) = P_N(t)[1-\mu\Delta t]$ . . . . (2)

Case III: when, $n=0$ then,

$P_0(t+\Delta t) = P_0(t)1 + P_1(t)\mu\Delta t$ ——— (3)

Since there is no unit in the system at time t, the question of any departure Δt does not arise



Now from (2)

$P_N(t+\Delta t) = P_N(t) - P_N(t)\mu\Delta t$

$\Rightarrow \lim_{\Delta t \to 0} \frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-\mu P_N(t)\Delta t}{\Delta t} = -\mu P_N(t)$

$\Rightarrow P_N'(t) = -\mu P_N(t)$ ——— (4)

and from (1)

$P_n'(t) = -\mu P_n(t) + \mu P_{n+1}(t)$ ——— (5)

Again from (3) we get, $P_0'(t) = \mu P_1(t)$ ——— (6)

Step I: $\frac{P_N'(t)}{P_N(t)} = -\mu$ From (4)

$\Rightarrow \frac{d \log P_N(t)}{dt} = -\mu$

$\Rightarrow \int \frac{d \log P_N(t)}{dt} dt = -\mu \int dt$

$\Rightarrow \log P_N(t) = -\mu t + c$

To get c, the boundary condition is $P_N(0)=1$

then we get, $c=0$

$\therefore \log P_N(t) = -\mu t$

$\Rightarrow P_N(t) = e^{-\mu t}$ ——— (7)

Step-2: Put $n = N-1$ in equation (5) and we get, $P_{N-1}'(t) = -\mu P_{N-1}(t) + \mu P_{N-1+1}(t)$

$\Rightarrow P_{N-1}'(t) = -\mu P_{N-1}(t) + \mu P_N(t)$

$\Rightarrow P_{N-1}'(t) + \mu P_{N-1}(t) = \mu e^{-\mu t}$ ——— (8)

Now multiply bothsides by $e^{\mu t}$,

$e^{\mu t}\left[P_{N-1}'(t) + \mu \cdot P_{N-1}(t)\right] = \mu$

$\Rightarrow \frac{d}{dt}\left[e^{\mu t} P_{N-1}(t)\right] = \mu$

$\Rightarrow e^{\mu t} P_{N-1}(t) = \mu t + c$ [Integrating both sides]

$\Rightarrow$ under boundary condition $t=0$ then $c=0$

$\therefore e^{\mu t} P_{N-1}(t) = \mu t$

$\Rightarrow P_{N-1}(t) = \frac{e^{-\mu t}(\mu t)^1}{1!}$

Step 3: Putting $n = N-2, N-3, \ldots, N-i$ and using induction process,

$P_{N-2}(t) = \frac{e^{-\mu t}(\mu t)^2}{2!}$, $P_{N-3}(t) = \frac{e^{-\mu t}(\mu t)^3}{3!}$ …

$P_{N-i}(t) = \frac{e^{-\mu t}(\mu t)^i}{i!}$; $i = 1, 2, \ldots, N-1.$

Now in general, put $n = N - i$ or $i = N - n$.

$$P_n(t) = \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \quad ; n = 1, 2, \dots, N \qquad \text{——— (9)}$$

Step 4: In order to find $P_0(t)$, using the following procedure, we know

$$\sum_{n=0}^{N} P_n(t) = 1$$

$$\Rightarrow P_0(t) + \sum_{n=1}^{N} P_n(t) = 1$$

$$\Rightarrow P_0(t) = 1 - \sum_{n=1}^{N} P_n(t) =$$

$$= 1 - \sum_{n=1}^{N} \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \qquad \text{——— (10)}$$

Finally combining the result (9) and (10) we get,

$$P_n(t) = \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \quad ; n = 1, 2, 3, \dots, N$$

$$= 1 - \sum_{n=1}^{N} \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \quad ; n = 0$$

Thus the number of departures in time t, follows the Truncated Poisson distribution.

2007, 08

Q: Stating necessary assumptions, write down the difference equations for a pure birth process. Also derive pure birth process.

Ans: The necessary assumptions for a pure birth process are as follows:

(i) Let, there are n-units in the system at time t and $\Pr[\text{Exactly one arrival during } \Delta t]$

$$= \lambda \Delta t + o(\Delta t)$$

$$= \lambda \Delta t \qquad [o(\Delta t) \text{ is negligible}]$$

where, λ is arrival rate.

(ii) $\Pr[\text{more than one arrival during } \Delta t] = o(\Delta t) \simeq 0$

(iii) The number of arrivals in non-overlapping intervals are statistically independent.

Now we obtain the difference equation in two mutually exclusive ways —

Case I: When, $n > 0$ Proceeding exactly as in the pure birth process,

$$P_n(t + \Delta t) = P_n(t)\,[1 - \lambda \Delta t] + P_{n-1}(t)\lambda \Delta t \qquad \text{——— (1)}$$

n-units, n-units: no arrival (t to $t + \Delta t$)

$(n-1)$ units, n-units: one arrival (t to $t + \Delta t$)

Case II: when $n=0$, then

$$P_0(t+\Delta t) = P_0(t)\,[1-\lambda\Delta t] \quad \text{——— (2)}$$

From (1) we get,

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t$$

$$\Rightarrow P_n(t+\Delta t) - P_n(t) = -P_n(t)\lambda\Delta t + P_{n-1}(t)\lambda\Delta t$$

$$\Rightarrow \lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

$$\Rightarrow P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad \text{——— (3)}$$

and from (2) $P_0'(t) = -\lambda P_0(t)$ ——— (4)

Hence the equation (3) and (4) is known as the system of differencial difference equation.

From (4) we get,

$$P_0'(t) = -\lambda P_0(t)$$

$$\Rightarrow \frac{P_0'(t)}{P_0(t)} = -\lambda$$

$$\Rightarrow \frac{d\ln P_0(t)}{dt} = -\lambda$$

$$\Rightarrow \int \frac{d\ln P_0(t)}{dt}\,dt = -\lambda\int dt$$

$$\Rightarrow \ln P_0(t) = -\lambda t + c$$

To get c, the boundary condition $P_0(0)=1$

$\therefore \ln P_0(0) = -\lambda.0 + c \Rightarrow c = \log(1) = 0$

$$\therefore \ln P_0(t) = -\lambda t$$

$$\Rightarrow P_0(t) = e^{-\lambda t}$$

Now put $n=1$ in (3) we get,

$$P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

$$\Rightarrow P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

$$\Rightarrow e^{\lambda t} P_1'(t) + \lambda P_1(t) e^{\lambda t} = \lambda$$

$$\Rightarrow \frac{d}{dt}[P_1(t)e^{\lambda t}] = \lambda$$

$$\Rightarrow \int \frac{d}{dt}[P_1(t)e^{\lambda t}]\,dt = \lambda\int dt$$

$$\Rightarrow P_1(t)e^{\lambda t} = \lambda t + c; \qquad P_1(0)=0$$

Using boundary condition we get $c=0$

$$\therefore P_1(t)e^{\lambda t} = \lambda t$$

$$\Rightarrow P_1(t) = \lambda t e^{-\lambda t} = \frac{e^{-\lambda t}(\lambda t)^1}{1!}$$

Similarly we get, $P_2(t), \ldots P_n(t)$

i.e. $$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Which is Poisson $dist^n$ formulla.

C.L

4·7. DISTRIBUTIONS IN QUEUEING SYSTEMS [Delhi (OR) 19]

The pattern of arrival of customers at a queueing system var[ies] between one system and another, but one pattern of comm[on] occurrence in practice, which turns out to be relatively easy [to] deal with mathematically, is that of 'completely random arrival[s]'. This phrase means something quite specific, and we discuss wh[at] it means before dealing in the subsequent sections with a varie[ty] of queueing systems. In particular, we show that, if arrivals a[re] 'completely random', the number of arrivals in unit time has [a] Poisson distribution, and the intervals between successive arriva[ls] are distributed negative exponentially.

4·7-1. Distribution of Arrivals 'The Poisson Process' (Pure Birth Process).

[Meerut (Stat.) 1983, 7[...]]

In many situations the objective of an analysis consists of merel[y] observing the number of customers that enter the system. Th[e] model in which only arrivals are counted and no departures tak[e] place are called *pure birth models*. The term 'birth' refers to the arrival of a new calling unit in the system, and the 'death' refers to the departure of a served unit. As such pure birth models are not of much importance so far as their applicability to real life situation is concerned, but these are very important in the understanding of completely random arrival problems.

Arrival distribution Theorem. *If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time-interval follows a Poisson distribution.* [Kanpur 1972]

Proof. In order to derive the arrival distribution in queues, we make the following three assumptions (sometimes called the axioms).

1. Assume that there are n units in the system at time t, and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda \Delta t + O(\Delta t)$, where λ is the arrival rate independent of t and $O(\Delta t)$ includes the terms of higher order of Δt.
2. Further assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$, i.e., almost zero.
3. The number of arrivals in non-overlapping intervals are statistically independent, *i.e.*, the process has independent increments.

We now wish to determine the probability on n arrivals in a time interval of length t, denoted by $P_n(t)$. Clearly, n will be an integer greater than or equal to zero. To do so, we shall first develop the differential equations governing the process in two differen situations.

Case I. When $n > 0$.

For $n > 0$, there may be two mutually exclusive ways of having n units at time $t + \Delta t$.

(i) There are n units in the system at time t and no arrival takes place during time interval Δt. Hence, there will be n units at time $t + \Delta t$ also. This situation is better explained in Fig. 4·4 below.

Therefore, the probability of these two combined events will be

= Prob. of n units at time t

× Prob. of no arrival during Δt

$$= P_n(t) \cdot (1 - \lambda \Delta t) \quad \ldots (4\cdot1)$$

$$\left[\begin{array}{ll} \because & \text{Prob. of exactly one arrival in } \Delta t = \lambda \Delta t \\ \therefore & \text{Prob. of no arrival becomes } = 1 - \lambda \Delta t \end{array}\right]$$

n units ↑ no arrival ↑ n units	$n-1$ units ↑ one arrival ↑ n units
t —— $t+\Delta t$	t —— $t+\Delta t$
Fig. 4·4	Fig. 4·5

(ii) Alternately, there are $(n-1)$ units in the system at time t, and one arrival takes place during Δt. Hence there will remain n units in the system at time $t + \Delta t$. This situation is better explained in Fig. 4·5 above.

Therefore, the probability of these two combined events will be

= Prob. of $(n-1)$ units at time t

× Prob. of one arrival in time Δt

$$= P_{n-1}(t) . \lambda \Delta t \quad \ldots (4\cdot2)$$

Note. Since the probability of more than one arrival in Δt is assumed to be negligible, other alternatives do not exist.

Now, adding these two probabilities [given by (4·1) and (4·2)], we get the probability of n arrivals at time $t + \Delta t$,

$$P_n(t+\Delta t) = P_n(t)(1 - \lambda \Delta t) + P_{n-1}(t) \lambda \Delta t \quad \ldots (4\cdot3)$$

Case 2. When $n = 0$.

$P_0(t+\Delta t)$ = Prob. [no unit at time t]

× Prob. [no arrival in time Δt]

$$\therefore \quad P_0(t+\Delta t) = P_0(t)(1 - \lambda \Delta t). \quad \ldots (4\cdot4)$$

Rewriting the equations (4·3) and (4·4) after transposing the terms $P_n(t)$ and $P_0(t)$ to left hand sides, respectively,

$$P_n(t+\Delta t) - P_n(t) = P_n(t)(-\lambda \Delta t) + P_{n-1}(t) \lambda \Delta t, \; n > 0 \quad \ldots (4\cdot4)'$$

$$P_0(t+\Delta t) - P_0(t) = P_0(t)(-\lambda \Delta t) \qquad n = 0 \quad \ldots (4\cdot4)''$$

Dividing both sides by Δt and then taking limit $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad \ldots (4\cdot5)$$

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t)-P(t)}{\Delta t} = -\lambda P_0(t) \qquad \ldots(4{\cdot}6)$$

Since by definition of first derivative,

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \frac{d\,P_n(t)}{dt} = P_n'(t),$$

the equations (4·6) and (4·5) can be respectively written as

$$P_0'(t) = -\lambda P_0(t), \qquad n=0 \qquad \ldots(4{\cdot}7)$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n>0 \qquad \ldots(4{\cdot}8)$$

This is known as the system of differential difference equations.

To solve the equations (4·7) and (4·8) by iterative method :

Equation (4·7) can be written as

$$\frac{P_0'(t)}{P_0(t)} = -\lambda \quad \text{or} \quad \frac{d}{dt} \log P_0(t) = -\lambda \qquad \ldots(4{\cdot}9)$$

Integrating both sides w.r.t. 't',

$$\log P_0(t) = -\lambda t + A \qquad \ldots(4{\cdot}10)$$

The constant of integration can be determined by using the boundary conditions :

$$P_n(0) = \begin{cases} 1, & n=0 \\ 0, & n>0. \end{cases}$$

Substituting $t=0$, $P_0(0)=1$ in (4·10), find $A=0$. Thus, (4·10) gives

$$\log P_0(t) = -\lambda t \quad \text{or} \quad P_0(t) = e^{-\lambda t}. \qquad \ldots(4{\cdot}11)$$

Putting $n=1$ in (4·8),

$$P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

or

$$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}. \qquad \ldots(4{\cdot}12)$$

Since this is the linear differential equation of first order, it can be easily solved by multiplying both sides of this equation by the integrating factor (I.F.) :

$$\text{I.F.} = e^{\int \lambda\, dt} = e^{\lambda t}.$$

Thus, (4·12) becomes

$$e^{\lambda t}[P_1'(t) + \lambda P_1(t)] = \lambda$$

$$\frac{d}{dt}[e^{\lambda t} P_1(t)] = \lambda$$

Now integrating both sides w.r.t. 't'

$$e^{\lambda t} P_1(t) = \lambda t + B, \qquad \ldots(4{\cdot}13)$$

where B is the constant of integration.

In order to determine the constant B, put $t=0$ in (4·13), and get

$$P_1(0) = 0 + B \quad \text{or} \quad B=0 \quad (\because\ P_1(0)=0)$$

Substituting $B=0$ in (4·13),

$$P_1(t) = \frac{\lambda t\, e^{-\lambda t}}{1\,!} \qquad \ldots(4{\cdot}14)$$

Similarly, putting $n=2$ in (4·8) and using the result (4·14), we get the equation

$$P_2'(t) + \lambda P_2(t) = \lambda \cdot \frac{(\lambda t)\, e^{-\lambda t}}{1\,!}$$

or

$$\frac{d}{dt}[e^{\lambda t} P_2(t)] = \frac{\lambda(\lambda t)}{1\,!}$$

Integrating w.r.t. 't'

$$e^{\lambda t} P_2(t) = \frac{(\lambda t)^2}{2\,!} + C,$$

Put $t=0$, $P_2(0)=0$ to obtain $C=0$. Hence

$$P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2\,!}, \text{ for } n=2 \qquad \ldots(4{\cdot}15)$$

Similarly, obtain

$$P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2\,!}$$

$$P_3(t) = \frac{(\lambda t)^3 e^{-\lambda t}}{3\,!}, \text{ for } n=3$$

Likewise, in general

$$P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m\,!}, \text{ for } n=m \qquad \ldots(4{\cdot}16)$$

If, anyhow, it can be proved that the result (4·16) is also true for $n \equiv m+1$, then by induction hypothesis (4·16) will be true for general value of n.

To do so, put $n=m+1$ in (4·8) and get

$$P_{m+1}'(t) + \lambda P_{m+1}(t) = \lambda \frac{(\lambda t)^m}{m\,!} e^{-\lambda t} \qquad \text{[using the results (4·16)]}$$

or

$$\frac{d}{dt}[e^{\lambda t} P_{m+1}(t)] = \frac{(\lambda t)^m}{m\,!}$$

Integrating both sides,

$$e^{\lambda t} P_{m+1}(t) = \frac{(\lambda t)^{m-1}}{(m+1)\, m\,!} + D$$

Again, putting $t=0$, $P_{m+1}(0)=0$, we get $D=0$.

$$\therefore \quad P_{m+1}(t) = \frac{(\lambda t)^{m+1} e^{-\lambda t}}{(m+1)\,!}$$

Hence, in general,

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n\,!}, \qquad \ldots(4{\cdot}17)$$

which is a Poisson distribution formula.

This completes the proof of the theorem.

$$^{*}a(T) = {}^{**}\lambda P_0(T). \qquad \text{.. (4·27)}$$

But, from equation (4·24), $P_0(T) = e^{-\lambda T}$.

Putting this value of $P_0(T)$ in (4·27),

$$a(T) = \lambda e^{-\lambda T}, \qquad \text{.. (4·28)}$$

which is the exponential law of probability for T with mean $1/\lambda$ and variance $1/\lambda^2$, *i.e.*,

$$E(T) = 1/\lambda, \ \text{Var.}\ (T) = 1/\lambda^2.$$

In a similar fashion, the converse of this theorem can be proved.

EXAMINATION QUESTIONS

1. If the number of arrivals in some time interval follows a Poisson distribution, show that the distribution of the time interval between two consecutive arrivals is exponential.
2. Show that if the inter-arrival times are negative exponentially distributed, the number of arrivals in a time period is a Poisson process and conversely. [Calcutta Stat. 1976]
3. If the intervals between successive arrivals are i.i.d. random variables which follow the negative exponential distribution with mean $1/\lambda$ then show that the arrivals form Poisson a Process with mean λt. 2012 [Bhopal (Stat.) 1986]

4·7·4. Markovian Property of Inter-arrival Times

Statement. *The Markovian property of inter-arrival times states that at any instant the time until the next arrival occurs is independent of the time that has elapsed since the occurrence of the last arrival. That is to say.*

$$\text{Prob.}\ [T \geq t_1 \mid T \geq t_0] = \text{Prob.}\ [0 \leq T \leq t_1 - t_0]$$

0 ——— t_0 ——— t_1

← t_0 → | ← $(t_1 - t_0)$ →

Fig. 4·7

Proof. Consider

$$\text{Prob.}\ [T \geq t_1 \mid T \geq t_0] = \frac{\text{Prob.}\ [(T \geq t_1) \text{ and } (T \geq t_0)]}{\text{Prob.}\ [T \geq t_0]} \qquad \text{...(4·29)}$$

(using the formula of conditional probability)

Since the interarrival times are exponentially distributed, the righthand side of equation (4·29) can be written as

$$= \frac{\int_{t_0}^{t_1} \lambda e^{-\lambda t}\, dt}{\int_{t_1}^{\infty} \lambda e^{-\lambda t}\, dt}$$

*According probability distributions $d/dx\,[F(x)] = f(x)$, where $F(x)$ is the 'distribution function' and $f(x)$ is the 'probability density function'. Hence by the similarar gument, we may write $d/dT\,[P_0(T)]$; where $P_0(T)$ is the probability distribution function for no arrival in time T, and $a(T)$ is denoting the corresponding probability density function of T.

**Since 'probability density function' is always non-negative, so neglect the negative sign from right side of equation (4·26).

$$= \frac{e^{-\lambda t_1} - e^{-\lambda t_0}}{e^{-\lambda t_0}}$$

$$\therefore \quad \text{Prob.}\ [T \geq t_1 \mid T \geq t_0] = 1 - e^{-\lambda (t_1 - t_0)} \qquad \text{...(4·30)}$$

But,

$$\text{Prob.}\ [0 \leq T \leq t_1 - t_0] = \int_0^{t_1 - t_0} \lambda e^{-\lambda t}\, dt$$

$$= 1 - e^{-\lambda (t_1 - t_0)} \qquad \text{.. (4·31)}$$

Thus, by virtue of equations (4·30) and (4·31), it can be concluded that

$$\text{Prob.}\ [T \geq t_1 \mid T \geq t_0] = \text{Prob.}\ [0 \leq T \leq t_1 - t_0].$$

This proves the Markovian property of inter-arrival times.

EXAMINATION QUESTION

1. State and prove the Markovian property of inter-arrival times. [Rohilkhand 1982]

4·7·5. Distribution of departures (or Pure Death Process)

In this process assume that there are N customers in the system at time $t = 0$. Also, assume that no arrivals (births) can occur in the system. Departures occur at a rate μ per unit time, *i.e.*, output rate is μ. We wish to derive the distribution of departures from the system on the basis of the following three axioms.

1. Prob. [one departure during Δt] $= \mu \Delta t + O(\Delta t)^2$
   $= \mu \Delta t$
   [$\because$ $O(\Delta t)^2$ is negligible]
2. Prob. [more than one departure during Δt] $= O(\Delta t)^2 \simeq 0$.
3. The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable, *i.e.*, the process $N(t)$ has independent increments.

First obtain the differential difference equation in three mutually exclusive ways :

Case I. When $0 < n < N$.

Proceeding exactly as in the Pure Birth Process,

$$P_n(t + \Delta t) = P_n(t)\,[1 - \mu \Delta t] + P_{n+1}(t)\, \mu \Delta t \qquad \text{...(4·32)}$$

$n+1$ units	n units	n units	n units
↑ one departure	↑	↑ No departure	↑
t	$t + \Delta t$	t	$t + \Delta t$
Fig. 4·8		Fig 4·9	

Case II. When $n = N$.

Since there are exactly N units in the system, $P_{n+1}(t) = 0$,

$$\therefore \quad P_N(t + \Delta t) = P_N(t)\,[1 - \mu \Delta t] \qquad \text{...(4·33)}$$

Case III. When n = 0.

$$P_0(t+\Delta t)=P_0(t)\ 1+P_1\ \mu\Delta t \qquad \ldots(4\cdot34)$$

Since there is no unit in the system at time t, the question of any departure Δt does not arise. Therefore, probability of no departure is unity in this case.

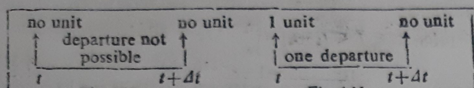


Fig. 4·10 Fig. 4·11

Now, re-arranging the terms and dividing by Δt, and also taking the limit $\Delta t\to0$, the equations (4·33), (4·32) and (4·34), respectively, become

$$P'_N(t)=-\mu P_N(t), \qquad n=N \qquad \ldots(4\cdot35)$$

$$P_n'(t)=-\mu P_n(t)+\mu P_{n+1}(t), \qquad 0<n<N \qquad \ldots(4\cdot36)$$

$$P_0'(t)=\mu P_1(t), \qquad n=0 \qquad \ldots(4\cdot37)$$

To solve the system of equations (4·35), (4·36) and (4·37) :

Interative method can be used to solve this system of three equations.

Step 1. From equation (4·35) obtain

$$\frac{P_N'(t)}{P_N(t)}=-\mu, \quad \text{or} \quad \frac{d}{dt}\log P_N(t)=-\mu$$

Integrating both sides of this equation,

$$\log P_N(t)=-\mu t+A \qquad \ldots(4\cdot38)$$

To determine 'A', use the boundary condition $P_N(0)=1$, and thus get $A=0$ ($\because$ $\log 1=0$),

Therefore, equation (4·38) becomes

$$\log P_N(t)=-\mu t \quad \text{or} \quad P_N(t)=e^{-\mu t} \qquad \ldots(4\cdot39)$$

Step 2. In equation (4·36), put $n=N-1$, and the value of $P_N(t)$ from equation (4·39),

$$P'_{N-1}(t)=-\mu P_{N-1}(t)+\mu P_N(t)$$

$$=-\mu P_{N-1}(t)+\mu e^{-\mu t} \qquad \text{[from equation (4·39)]}$$

or
$$P'_{N-1}(t)+\mu P_{N-1}(t)=\mu e^{-\mu t} \qquad \ldots(4\cdot40)$$

The solution of this equation is given by

$$P_{N-1}\,e^{\mu t}=\int \mu e^{-\mu t}\,e^{\mu t}\,dt+B \qquad (\because\ \text{I.F.}=e^{\mu t})$$

or
$$P_{N-1}(t)=\mu t\,e^{-\mu t}+Be^{-\mu t} \qquad \ldots(4\cdot41)$$

To determine B, put $t=0$, $P_{N-1}=0$ in (4·41) and get $B=0$. Therefore,

$$P_{N-1}(t)=\frac{\mu t\,e^{-\mu t}}{1\,!}$$

Step 3. Putting $n=N-2$ in equation (4·36) and proceeding exactly as in step 2,

$$P_{N-2}(t)=\frac{e^{-\mu t}\,(\mu t)^2}{2\,!}$$

Step 4. Now, putting $n=N-3,\ N-4,\ \ldots,\ N-i$, and using induction process

$$P_{N-3}(t)=\frac{e^{-\mu t}\,(\mu t)^3}{3\,!}$$

$$\ldots \qquad \ldots$$

$$P_{N-i}(t)=\frac{e^{-\mu t}\,(\mu t)^i}{i\,!},\ i=0,1,2,\ldots,N-1$$

$$\ldots \qquad \ldots$$

In general, on letting $n=N-i$

$$P_n(t)=\frac{e^{-\mu t}\,(\mu t)^{N-n}}{(N-n)\,!},\ n=1,2,\ldots,N \qquad \ldots(4\cdot42)$$

Step 5. In order to find $P_0(t)$, use the following procedure,

Since
$$1=\sum_{n=0}^{N}P_n(t)$$

$$=P_0(t)+\sum_{n=1}^{N}P_n(t)$$

or
$$P_0(t)=1-\sum_{n=0}^{N}P_n(t)$$

$$=1-\sum_{n=1}^{N}\frac{(\mu t)^{N-n}\,e^{-\mu t}}{(N-n)\,!} \qquad \ldots(4\cdot43)$$

Finally, combining the results (4·42) and (4·43),

$$P_n(t)=\begin{cases}\dfrac{(\mu t)^{N-n}\,e^{-\mu t}}{(N-n)\,!}, & \text{for } n=1,2,\ldots,N\\[2ex] 1-\displaystyle\sum_{n=1}^{N}\frac{(\mu t)^{N-n}\,e^{-\mu t}}{(N-n)\,!}, & \text{for } n=0\end{cases} \qquad \ldots(4\cdot44)$$

Thus, the number of departures in time t follows the 'Truncated Poisson Distribution.'

EXAMINATION QUESTION

1. Establish the probability distribution formula for Pure-Death Process.

[Rohilkhand 1979]

4·7-6. Derivation of Service Time Distribution

Let T be the random variable denoting the service time and t the possible ok 2013

Q. Find the mean and variance of pure birth process.

Ans: We know the pure birth process is,

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \; n = 0, 1, 2, \ldots$$

$$\text{Mean} = E(n) = \sum_{n=0}^{\infty} n \, P_n(t) = \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t \, e^{-\lambda t} \cdot e^{\lambda t}$$

$$= \lambda t .$$

$$E(n^2) = \sum_{n=0}^{\infty} n^2 P_n(t) = \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t}(\lambda t)^2 \sum_{n=2}^{\infty} \frac{(\lambda t)^{n-2}}{(n-2)!}$$

$$= e^{-\lambda t}(\lambda t)^2 \cdot e^{\lambda t} = \cancel{(\lambda t)}$$

$$= (\lambda t)^2$$

$$E(n^2) = E[n(n-1) + n] = E[n(n-1)] + E(n) .$$

Now,
$$E[n(n-1)] = \sum_{n=0}^{\infty} n(n-1) \, P_n(t)$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$\therefore E[n(n-1)] = e^{-\lambda t}(\lambda t)^2 \sum_{n=2}^{\infty} \frac{(\lambda t)^{n-2}}{(n-2)!}$$

$$= e^{-\lambda t}(\lambda t)^2 e^{\lambda t}$$

$$= (\lambda t)^2$$

$$\therefore E(n^2) = (\lambda t)^2 + \lambda t$$

$$V(n) = E(n^2) - [E(n)]^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t$$

2007, 06

Q. Is any connection between a poisson process and pure birth process?

ok

Ans: Yes, now we explain some causes of connection –

① The form of poisson process $P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ and pure birth process is, $P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$. Hence distn functions are same to the both process.

② The poisson process and pure birth process are both arrival distn.

Q. Define Birth, Death, Birth process, Death process, pure Birth process, pure death process.

Ans:

Birth: A birth occurs whenever a new number joint in the population.

Death: A death occurs whenever a number leaves in the population.

Birth process: The probabilities are the function of time of the birth distn. Then the distn is called birth process.

Death process: The probabilities are the function of time of the death distn. Then the distn is called death process.

pure birth process: A pure birth process is one that experience only births no death.

pure death process: A pure death process is one that experience only death no birth.

Q. Explain the situation when a birth and death process becomes

(i) pure birth process

(ii) simple birth process

(iii) pure death process

(iv) simple death process

(v) Birth and death process.

Ans: Let λ_n be the birth rate and μ_n be the death rate of the population of size n. Then we explain the situation as:—

(i) pure birth process: If $\lambda_n = \lambda$ and $\mu_n = 0$

(ii) simple birth process: If $\lambda_n = n\lambda$ and $\mu_n = 0$

(iii) pure death process: If $\lambda_n = 0$ and $\mu_n = \mu$

(iv) simple death process: If $\lambda_n = 0$ and $\mu_n = n\mu$

(v) simple death and birth process: If $\lambda_n = n\lambda$ and $\mu_n = n\mu$.

Q. Derive pure birth process with the help of probability generating function.

Ans: we have [pure birth prosses এর সমীকরণ [illegible]]

$$P_0'(t) = -P_0(t)\lambda \quad \text{——— (i)}$$

$$P_n'(t) = -P_n(t)\lambda + P_{n-1}(t)\lambda \quad \text{——— (ii)}$$

By definition pgf of $P_n(t)$ is

$$P(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$$

Also

$$P'(z,t) = \sum P_n'(t) z^n$$

multiplying both sides of (ii) by z^n and taking summation for $n = 1,2,3 \cdots \infty$ then

$$\sum_{n=1}^{\infty} P_n'(t) z^n = -\lambda \sum_{n=1}^{\infty} P_n(t) z^n + \lambda \sum_{n=1}^{\infty} P_{n-1}(t) z^n \quad \text{——— (iii)}$$

Adding (i) and (iii) then we get;

$$P_0'(t) + \sum_{n=1}^{\infty} P_n'(t) z^n = -P_0(t)\lambda - \sum P_n \lambda z^n + \lambda \sum P_{n-1}(t) z^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n'(t) z^n = -\lambda \sum_{n=0}^{\infty} P_n(t) z^n + z\lambda \sum_{n=1}^{\infty} P_{n-1}(t) z^{n-1}$$

$$\Rightarrow P'(z,t) = -\lambda P(z,t) + \lambda z P(z,t)$$

$$\Rightarrow \frac{P'(z,t)}{P(z,t)} = -\lambda + \lambda z$$

$$\Rightarrow \frac{d}{dt} \log P(z,t) = \lambda (z-1)$$

Taking integration on both sides w.r.to

$$\log P(z,t) = \lambda (z-1) t + c \quad \text{——— (iv)}$$

Putting $t = 0$ in (iv) then we have

$$\log P(z,0) = c.$$

Now,

$$P(z,0) = \sum_{n=0}^{\infty} z^n P_n(0)$$

$$= P_0(0) + \sum_{n=1}^{\infty} z^n P_n(0)$$

$$= 1 + 0$$

$$= 1$$

Now,

$$\log(1) = c$$

$$\Rightarrow c = 0$$

$$\therefore \log P(z,t) = \lambda (z-1) t$$

$$\Rightarrow P(z,t) = e^{\lambda (z-1) t}$$

$$\therefore P_n(z,t) = e^{\lambda (z-1) t}$$

Now, $P_n(t)$ can be defined as

$$P_n(t) = \frac{1}{n!} \left[\frac{d^n P(z,t)}{dz^n} \right]_{z=0}$$

using this formula

$$P_0(t) = \frac{1}{0!}\left[\frac{d}{dz} e^{\lambda(z-1)t}\right]$$

$$= \frac{1}{0!} e^{\lambda(0-1)t} = e^{-\lambda t}.$$

$$P_1(t) = \frac{1}{1!}\left[\frac{d}{dz} e^{\lambda(z-1)t}\right]$$

$$= \frac{1}{1!}\left[e^{(z-1)\lambda t} \cdot \lambda t\right]_{z=0}$$

$$= \lambda t\, e^{-\lambda t}$$

$$\therefore P_1(t) = \frac{e^{-\lambda t}(\lambda t)^1}{1!}$$

similarly

$$P_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2!} \text{ for } n=2$$

$$P_3(t) = \frac{e^{-\lambda t}(\lambda t)^3}{3!} \text{ for } n=3$$

$$\vdots$$

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \text{ for } n=n.$$

which is the poisson process formula.

(showed)

Q. If $N_1(t)$ and $N_2(t)$ are two independent poisson process with parameter λ_1 and λ_2 respectively. then show that

$$P_r\{N_1(t)=k \mid N_1(t)+N_2(t)=n\} = \binom{n}{k} p^k q^{n-k}$$

where, $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

proof: since $N_1(t)$ and $N_2(t)$ are two poisson process with parameter λ_1 and λ_2 respectively then we have

$$P_{N_1(t)} = \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!}$$

$$P_{N_2(t)} = \frac{e^{-\lambda_2 t}(\lambda_2 t)^k}{k!}$$

Now,

$$P_r\{N_1(t)=k \mid N_1(t)+N_2(t)=n\} = \frac{P_r\{N_1(t)=k \text{ and } N_1(t)+N_2(t)=n\}}{P_r\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{P_r\{N_1(t)=k \text{ and } N_2(t)=n-k\}}{P_r\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{P_r\{N_1(t)=k\} \cdot P_r\{N_2(t)=n-k\}}{P_r\{\frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!}}$$

$$= \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!} \cdot \frac{n!}{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{(\lambda_1 t)^k (\lambda_2 t)^{n-k}}{\{(\lambda_1+\lambda_2)t\}^n}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{(t)^{\cancel{k+n-k}} (\lambda_1)^k (\lambda_2)^{n-k}}{(t)^n (\lambda_1+\lambda_2)^n}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{(\lambda_1)^k (\lambda_2)^{n-k}}{(\lambda_1+\lambda_2)^{n-k+k}}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= \binom{n}{k} p^k q^{n-k}$$

where,

$$p = \frac{\lambda_1}{\lambda_1+\lambda_2} \quad \text{and} \quad q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

(showed)

2008, 2009, 2010, 2006

Q. Explain the situations when a birth and death process becomes: (i) Pure birth process (ii) Simple birth process (iii) Pure death process (iv) simple death process (v) Birth and death process.

Ans: If λ_n be the birth rate and μ_n be the ~~rate~~ death rate of the population size n. Then we can explain the situation as-

(i) Pure birth process:
if $\lambda_n = \lambda$ and $\mu_n = 0$

(ii) Simple birth process:
if $\lambda_n = n\lambda$ and $\mu_n = 0$

(iii) Pure death process:
if $\mu_n = \mu$ and $\lambda_n = 0$

(iv) Simple death process:
if $\mu_n = n\mu$ and $\lambda_n = 0$

(v) Birth and Death process:
if $\lambda_n = n\lambda$ and $\mu_n = n\mu$.

2010

2007, 2006, 2010, 2005

Q. Establish the probability function for a simple birth process. Considering birth-rate proportional to the population size, show that the number of individual present at time t follow a negative binomial distn.

Ans: In general, the birth process follows the following assumption —

(i) Pr [The event occurs one] $= \lambda_n \Delta t + o(\Delta t)$

(ii) Pr [The event does not occur] $= 1 - \lambda_n \Delta t + o(\Delta t)$

(iii) Pr [The event occurs more than one] $= o(\Delta t)$

Let, $P_n(t)$ be the probability that n event occur in time $(0, t]$. Using Chapman-Kolmogorov equation for transitions in the intervals of time $(0, t]$ and $(t, t+\Delta t]$ we can write,

$$P_0(t+\Delta t) = P_0(t)[1-\lambda_0 \Delta t] + o(\Delta t)$$

$$P_n(t+\Delta t) = P_n(t)[1-\lambda_n \Delta t] + P_n(t)\lambda_{n-1}\Delta t + o(\Delta t) \qquad n>0$$

which gives, $P_0'(t) = -\lambda_0 P_0(t)$ ——— (i)

$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$; $n \geq 0$ ——— (ii)

with $P_0(0) = 1$ and $P_n(0) = 0$ for $n > 0$.
Which are the differential equations.

Now for simple birth process i.e. $\lambda_n = n\lambda$
we get from (I) and (II)

$$P_0'(t) = 0 \quad \text{———— (III)} \quad n \geq 0$$

$$P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t) \quad ; n \geq 1$$

Assuming that, $P_i(0) = 1$ and $P_i(0) = 0$; $i \neq 1$
Then the solution can be obtained.

For $n = 1$, we have,

$$P_1'(t) = -\lambda P_1(t)$$

$$\Rightarrow \frac{P_1'(t)}{P_1(t)} = -\lambda$$

$$\therefore P_1(t) = e^{-\lambda t}$$

For $n = 2$ we get

$$P_2'(t) = -2\lambda P_2(t) + \lambda P_1(t)$$

$$\Rightarrow P_2'(t) + 2\lambda P_2(t) = \lambda P_1(t) = \lambda e^{-\lambda t}$$

$$\Rightarrow P_2'(t) e^{2\lambda t} + 2\lambda P_2(t) e^{2\lambda t} = \lambda e^{\lambda t}$$

$$\Rightarrow \int \frac{d}{dt} e^{2\lambda t} P_2(t)\, dt = \lambda \int e^{\lambda t}\, dt$$

$$\Rightarrow e^{2\lambda t} P_2(t) = e^{\lambda t} + c_2$$

for, $P_2(0) = 0$ then $c_2 = -1$

$$\therefore e^{2\lambda t} P_2(t) = e^{\lambda t} - 1$$

$$\Rightarrow P_2(t) = e^{-2\lambda t}(e^{\lambda t} - 1) = e^{-\lambda t}(1 - e^{-\lambda t})^{2-1}$$

Continue this process we get,

$$P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \quad ; n \geq 1$$

$$\text{and } P_0(t) = 0 \quad ; n = 0$$

which is the probability distn of simple birth process.

If $e^{-\lambda t} = p$ and $1 - e^{-\lambda t} = q$ then $P_n(t)$ is geometric distn.

If birth rate is proportional to the population size i (say) then the population size at time t is the sum of i geometric random variables.
This is the negative binomial distn which is

$$P_n(t) = \binom{n-1}{n-i} e^{-i\lambda t}(1 - e^{-\lambda t})^{n-i}$$

Now mean of this process is

$$E[N(t)] = i e^{\lambda t}$$

and variance of the process

$$V[N(t)] = i e^{\lambda t}(e^{\lambda t} - 1)$$

where, $N(t)$ is the number of population.

2008

Q. If $N_1(t), N_2(t)$ are two independent poisson process with parameter λ_1, λ_2 respectively, then show that

$$Pr\{N_1(t)=k \mid N_1(t)+N_2(t)=n\} = \binom{n}{k} p^k q^{n-k}$$

where, $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

Solution: Let $N_1(t)$ and $N_2(t)$ are two independent poisson process with parameters λ_1 and λ_2 respectively, then the additive property, the sum $N_1(t)+N_2(t)$ is also a poisson process with parameter $(\lambda_1+\lambda_2)$. Now

$$Pr\{N_1(t)=k \mid N_1(t)+N_2(t)=n\}$$

$$= \frac{Pr\{N_1(t)=k \text{ and } N_1(t)+N_2(t)=n\}}{Pr\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{Pr\{N_1(t)=k \text{ and } N_2(t)=n-k\}}{Pr\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{Pr\{N_1(t)=k\}\, Pr\{N_2(t)=n-k\}}{Pr\{N_1(t)+N_2(t)=n\}}$$

[$\because$ $N_1(t)$ and $N_2(t)$ are independent]

$$= \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!} \Big/ \frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!}$$

$$= \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= \binom{n}{k} p^k q^{n-k}$$

where, $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

(Proved).

Q. What is pure birth process?

Ans: In many situations of an analysis consist of merely observing the number of customers that enter the system. The model in which only one arrival are counted and no departures take place is called pure birth model and the process is known as pure birth process or poisson process.

$$= e^{-\lambda t} \cdot \frac{1}{e^{-2\lambda t}}$$

$$= e^{\lambda t}.$$

$\therefore$ mean $= e^{\lambda t}$.

Variance :

$$V(n) = E(n^2) - \{E(n)\}^2$$

Now,

$$E(n^2) = \sum n^2 P_n(t)$$

$$= \sum n^2 e^{-\lambda t} (1 - e^{-\lambda t})^{n-1}$$

$$= e^{-\lambda t} \left[1 + 4(1 - e^{-\lambda t}) + 9(1 - e^{-\lambda t})^2 + \cdots\right]$$

$$= e^{-\lambda t} \left[1 + 1 - e^{-\lambda t}\right] (1 - 1 + e^{-\lambda t})^{-3}$$

$$= e^{-\lambda t} (2 - e^{-\lambda t})\, e^{3\lambda t}$$

$$= e^{2\lambda t} (2 - e^{-\lambda t})$$

$$\therefore V(n) = e^{2\lambda t} (2 - e^{-\lambda t}) - (e^{\lambda t})^2$$

$$= 2e^{2\lambda t} - e^{2\lambda t - \lambda t} - e^{2\lambda t}$$

$$= e^{\lambda t} (e^{\lambda t} - 1)$$

Q. Find the pgf of simple birth process.

Ans: We know that the pgf of simple birth process is given by :-

$$P_n(t) = e^{-\lambda t} (1 - e^{-\lambda t})^{n-1}$$

Hence the pdf of the process is given by

$$P(s,t) = \sum_{n=1}^{\infty} P_n(t)\, s^n$$

$$= \sum_{n=1}^{\infty} e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} s^n$$

$$= s e^{-\lambda t} \sum_{n=1}^{\infty} \{s(1 - e^{-\lambda t})\}^{n-1}$$

$$= s e^{-\lambda t} \left[1 + s(1 - e^{-\lambda t}) + \{s(1 - e^{-\lambda t})\}^2 + \cdots\right]$$

$$= s e^{-\lambda t} \{1 - s(1 - e^{-\lambda t})\}^{-1}$$

$$= \frac{s e^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}$$

which is the required pgf of the simple birth process.

Question: Derive the probability distribution of simple death process.

Ans: Let $N(t)$ be the total number of individuals at epoch t starting from $t=0$ and let $P\{N(t)=n\} = P_n(t)$; Here $\lambda_n = 0$ for all n i.e an individual in $(t, \Delta t + t)$ is $\mu\Delta t + o(\Delta t)$. Then if n is individuals are present at time.

(i) The proby of one death in $(t, t+\Delta t)$ is

$$P_n(t) = n\mu\Delta t + o(\Delta t) = n\mu\Delta t \cdot P_{n+1}(t)$$

(ii) The proby that no death will occur time at Δt

$$P_{n}(\Delta t) = (1 - n\mu\Delta t) P_{n}(t)$$

(iii) The proby that more than one death will occur in Δt is $o(\Delta t)$.

Now,
Adding (i) and (ii) and the proby of n units at $(t+\Delta t)$ is

$$P_n(t+\Delta t) = n\mu\Delta t P_{n+1}(t) + (1 - n\mu\Delta t) P_{n}(t)$$

$$= n\mu\Delta t P_{n+1}(t) + P_n(t) - n\mu\Delta t P_n(t)$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = (n+1)\mu P_{n+1}(t) - n\mu P_n(t) + \frac{o(\Delta t)}{\Delta t}$$

Now,
Taking limit as $\Delta t \to 0$ then we get,

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = (n+1)\mu P_{n+1}(t) - n\mu P_n(t) \quad \text{--- (1)}$$

It is clear that the intial population must be greater than zero. suppose it is to be a i.e $N(0) = a$. Then the intial conditions are

$$P_a(0) = 1 \text{ and } P_n(0) = 0 \text{ for } n = a \quad \text{--- (2)}$$

Then from (1) we get,

$$P_a'(t) = -a\mu P_a(t)$$

$$\Rightarrow \frac{P_a'(t)}{P_a(t)} = -a\mu$$

$$\Rightarrow \frac{d}{dt}[\log P_a(t)] = -a\mu$$

Taking integration on both sides w.r to t

$$\log P_a(t) = -a\mu t + c_1 \quad \text{--- (3)}$$

Putting $t=0$ and $P_a(0) = 1$ in (3) then we get

$$\log(1) = 0 + c_1$$

$$\Rightarrow c_1 = 0$$

$$\therefore c_1 = 0$$

Putting the value of c_1 in (3) then we get;

$$\log P_a(t) = -a\mu t$$

$$\Rightarrow P_a(t) = e^{-a\mu t}$$

$$\therefore P_a(t) = e^{-a\mu t}$$

putting $n=a-1$ in (1) then we get;

$$P'_{a-1}(t) = (a-1+1)\mu P_{a-1+1}(t) - (a-1)\mu P_{a-1}(t)$$

$$\Rightarrow P'_{a-1}(t) + (a-1)\mu P_{a-1}(t) = a\mu P_a(t)$$

$$\Rightarrow P'_{a-1}(t) + (a-1)\mu P_{a-1}(t) = a\mu e^{-a\mu t}$$

$$\Rightarrow e^{(a-1)\mu t} P'_{a-1}(t) + (a-1)\mu P_{a-1}(t) e^{(a-1)\mu t} = a\mu e^{-a\mu t} e^{(a-1)\mu t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{(a-1)\mu t} P_{a-1}(t)\right] = a\mu e^{-\cancel{a\mu t}+\cancel{a\mu t}-\mu t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{(a-1)\mu t} P_{a-1}(t)\right] = a\mu e^{-\mu t}$$

Taking integration on both sides

$$e^{(a-1)\mu t} P_{a-1}(t) = \int a\mu e^{-\mu t} dt$$

$$= a\cancel{\mu} \frac{e^{-\mu t}}{-\cancel{\mu}} + c_2$$

$$= -a e^{-\mu t} + c_2 \quad \text{---} (4)$$

putting $t=0$ and $P_{a-1}(0)=0$ in (4) then $\Rightarrow$

$$e^0 \times 0 = -a e^0 + c_2$$

$$\Rightarrow c_2 = a$$

$$\therefore c_2 = a$$

putting the value of c_2 in (4) then we get;

$$e^{(a-1)\mu t} P_{a-1}(t) = -a e^{-\mu t} + a$$

$$\Rightarrow P_{a-1}(t) = -a e^{-\mu t - a\mu t + \mu t} + a e^{-(a-1)\mu t}$$

$$= a e^{-(a-1)\mu t}\left[1 - e^{-\mu t}\right]$$

$$\therefore P_{a-1}(t) = a e^{-(a-1)\mu t}(1 - e^{-\mu t})$$

putting $n=a-2$ in (1) then we get,

$$P'_{a-2}(t) = (a-2+1)\mu P_{a-2+1}(t) - (a-2)\mu P_{a-2}(t)$$

$$\Rightarrow P'_{a-2}(t) = (a-1)\mu P_{a-1}(t) - (a-2)\mu P_{a-2}(t)$$

$$\Rightarrow P'_{a-2}(t) + (a-2)\mu P_{a-2}(t) = (a-1)\mu a e^{-(a-1)\mu t}(1-e^{-\mu t})$$

$$\Rightarrow e^{(a-2)\mu t}\left[P'_{a-2}(t) + (a-2)\mu P_{a-2}(t)\right] = (a-1)\mu a e^{-(a-1)\mu t}(1-e^{-\mu t}) \cdot e^{(a-2)\mu t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{(a-2)\mu t} P_{a-2}(t)\right] = a(a-1)\mu e^{-a\mu t+\mu t+a\mu t-2\mu t}(1-e^{-\mu t})$$

$$= a(a-1)\mu e^{-\mu t}(1-e^{-\mu t})$$

Taking integration on both w.r.to t then we get

$$e^{(a-2)\mu t} P_{a-2}(t) = a(a-1)\left[\mu \int e^{-\mu t} dt - \mu \int e^{-2\mu t} dt\right]$$

$$= a(a-1)\left[\cancel{\mu}\cdot\frac{e^{-\mu t}}{-\cancel{\mu}} - \cancel{\mu}\cdot\frac{e^{-2\mu t}}{-2\cancel{\mu}}\right] + c_3$$

$$= \frac{a(a-1)}{2}\left[e^{-2\mu t} - 2e^{-\mu t}\right] + c_3 \quad \text{——— (5)}$$

putting $t=0$ and $P_{a-2}(0)=0$ in (5) then we get;

$$e^0 \times 0 = \frac{a(a-1)}{2}\left[(1-2)\right] + c_3$$

$$\Rightarrow \frac{a(a-1)}{2} = c_3$$

$$\therefore c_3 = \frac{a(a-1)}{2}$$

putting the value of c_3 in (5) then we get;

$$e^{(a-2)\mu t} P_{a-2}(t) = \frac{a(a-1)}{2}\left[e^{-2\mu t} - 2e^{-\mu t} + 1\right]$$

$$= \frac{a(a-1)}{2!}\cdot\left(e^{-\mu t} - 1\right)^2$$

$$P_{a-2}(t) = \frac{a(a-1)}{2!} e^{-(a-2)\mu t}\left(1 - e^{-\mu t}\right)^2$$

similarly

$$P_{a-3}(t) = \frac{a(a-1)(a-2)}{3!} e^{-(a-3)\mu t}\left(1 - e^{-\mu t}\right)^3$$

In general we get;

$$P_n(t) = \frac{a(a-1)\cdots\{a-(a-n)+1\}}{n!\,(a-n)!} e^{-n\mu t}\left(1 - e^{-\mu t}\right)^{a-n}$$

$$P_n(t) = {}^{a}C_n\, e^{-n\mu t}\left(1 - e^{-\mu t}\right)^{a-n}$$

$$\therefore P_n(t) = {}^{a}C_n\, e^{-n\mu t}\left(1 - e^{-\mu t}\right)^{a-n}$$

which is the prob function of simple death process and which is the form of negative binomial distribution.

Hence the simple death process conforms to negative binomial distn.

operating characteristics (behaviour) are dependent on time. This usually occurs at the early stages of the operation of the system where its behaviour is still dependent on the initial conditions. However, since we are mostly interested in the "long run" behaviour of the system, mainly the attention has been paid toward "steady state" results.

A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $[dP_n(t)/dt]$ or $P_n'(t)$. Then the queueing system is said to become 'stable' eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes $(t \to \infty)$. Mathematically, in steady state

$$\lim_{t\to\infty} P_n(t) = P_n \text{ (independent of } t)$$

$$\Rightarrow \lim_{t\to\infty} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt}$$

$$\Rightarrow \lim_{t\to\infty} P_n'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady sate cannot be reached regardless of the length of the elapsed time. In fact, in this case the queue length will increase with time and theoretically it could build up to infinity. Such case is called the "explosive state".

In this chapter, only the steady state analysis will be considered. We shall not treat the '*transient*' and '*explosive*' states.

4·5. A LIST OF SYMBOLS

Unless otherwise stated, the following symbols and terminology will be used henceforth in connection with the queueing models. The reader is reminded that a queueing system is defined to include both the *queue* and the *service stations*. (See Fig. 4·3).

n = number of units in the system

$P_n(t)$ = transient state probability that exactly n calling units are in the queueing system at time t

E_n = the state in which there are n-calling units in the system

P_n = steady state probability of having n units in the system

λ_n = mean arrival rate (expected number of arrivals per unit time) of customers (when n units are present in the system

μ_n = mean service rate (expected number of customers served per unit time when there are n units in the system

λ = mean arrival rate when λ_n is constant for all n

μ = mean service rate when μ_n is constant for all $n \geq 1$

= number of parallel service stations

= $\lambda/\mu s$ = traffic intensity (or utilization factor) for servers facility, that is, the expected function of time the servers are busy

$p_T(n)$ = probability of n services in time T, given that servicing is going on throughout T

Line length (or queue size)
= number of customers in the queueing system

Queue length
= line length (or queue size)–(number of units being served)

$\psi(w)$ = probability density function (p.d.f.) of waiting time in the system

L_s = expected line length, *i.e.*, expected number of customers in the system

L_q = expected queue length, *i.e.*, expected number of customers in the queue

W_s = expected waiting time per customer in the system

W_q = expected waiting time per customer in the queue

$(W \mid W > 0)$
= expected waiting time of a customer who has to wait

$(L \mid L > 0)$ = expected length of non-empty queues, *i.e.*, expected number of customers in the queue when there is a queue

$P(W>0)$ = probability of a customer having to wait for service

$\binom{n}{r}$ = denotes the binomial coefficient ${}^nc_r$

$$= \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!},$$

for r and n non-negative integers $(r \leq n)$.

4·6. TRAFFIC INTENSITY (or UTILIZATION FACTOR)

An important measure of a simple queue $(M \mid M \mid 1)$ is its traffic intensity, where

$$\text{traffic intensity } (\rho) = \frac{\text{mean arrival rate}}{\text{mean service rate}} = \frac{\lambda}{\mu},$$

i.e.

$$\rho = \frac{1/\mu}{1/\lambda} = \frac{\text{Mean service time}}{\text{Mean inter-arrival time}}.$$

The unit of traffic intensity is *Erlang*. Erlang

It should be noted here carefully that a necessary condition for a system to have settled down to steady state is that $\rho < 1$ or $\lambda/\mu < 1$ or $\lambda < \mu$, *i.e.*, arrival rate < service rate.

If this is not so, *i.e.*, $\rho > 1$, the arrival rate will be greater than the service rate and consequently, the number of units in the queue tends to increase indefinitely as time goes on, provided the rate of service is not affected by the length of queue.

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