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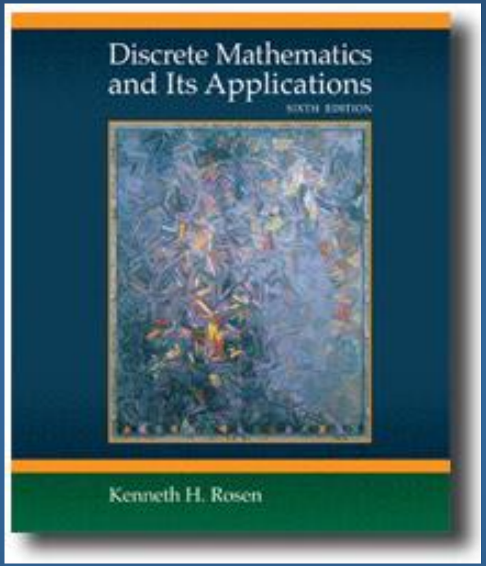
Discrete Mathematics and Its Applications



Chapter 9
Relations

Relations (9.1) (cont.)

- Relations & their properties

  - Definition 1

    Let A and B be sets. A binary relation from A to B is a subset of A * B.

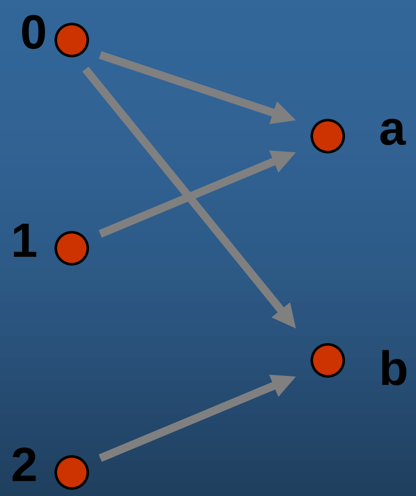
    In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Relations (9.1) (cont.)

– Notation:

$$aRb \Leftrightarrow (a, b) \in R$$

$$a \not R b \Leftrightarrow (a, b) \notin R$$



R	a	b
0	X	X
1	X	
2		X

Relations (9.1) (cont.)

– Example:

A = set of all cities

B = set of the 50 states in the USA

Define the relation R by specifying that (a, b) belongs to R if city a is in state b.

$$\left.\begin{array}{l} (\textit{Boulder}, \textit{Colorado}) \\ (\textit{Bangor}, \textit{Maine}) \\ (\textit{Ann Arbor}, \textit{Michigan}) \\ (\textit{Cupertino}, \textit{California}) \\ \textit{Red Bank}, \textit{New Jersey}) \end{array}\right\} \textit{are in R.}$$

Relations (9.1) (cont.)

- Functions as relations
  - The graph of a function f is the set of ordered pairs (a, b) such that b = f(a)
  - The graph of f is a subset of A * B $\Rightarrow$ it is a relation from A to B
  - Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph

Relations (9.1) (cont.)

- Relations on a set
  - Definition 2

    A relation on the set A is a relation from A to A.

  - Example: A = set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}

    Solution: Since (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b

    R = {(1,1), (1,2), (1.3), (1.4), (2,2), (2,4), (3,3), (4,4)}

Relations (9.1) (cont.)

- Properties of Relations
  - Definition 3

A relation R on a set A is called reflexive if (a, a) $\in$ R for every element a $\in$ A.

Relations (9.1) (cont.)

– Example (a): Consider the following relations on {1, 2, 3, 4}

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Which of these relations are reflexive?

Relations (9.1) (cont.)

Solution:

R_3 and R_5: reflexive $\Leftarrow$ both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4).

R_1, R_2, R_4 and R_6: not reflexive $\Leftarrow$ not contain all of these ordered pairs. (3,3) is not in any of these relations.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (9.1) (cont.)

– Definition 4:

A relation R on a set A is called symmetric if (b, a) $\in$ R whenever (a, b) $\in$ R, for all a, b $\in$ A.
A relation R on a set A such that (a, b) $\in$ R and (b, a) $\in$ R only if a = b, for all a, b $\in$ A, is called antisymmetric.

Relations (9.1) (cont.)

– Example: Which of the relations from example (a) are symmetric and which are antisymmetric?

Solution:

- R_2 & R_3: symmetric $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.
For R_2: only thing to check that both (1,2) & (2,1) belong to the relation
For R_3: it is necessary to check that both (1,2) & (2,1) belong to the relation.
None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (9.1) (cont.)

Solution (cont.):

- R_4, R_5 and R_6: antisymmetric $\Leftarrow$ for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.
  None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (9.1) (cont.)

– Definition 5:

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b,c) \in R$, then $(a, c) \in R$, for all $a, b, c \in R$.

Relations (9.1) (cont.)

– Example: Which of the relations in example (a) are transitive?

- R_4 , R_5 & R_6 : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation
  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4.
  Same reasoning for R_5 and R_6.
- R_1 : not transitive $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
- R_2 : not transitive $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
- R_3 : not transitive $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (9.1) (cont.)

- Combining relations
  - Example:

    Let A = {1, 2, 3} and B = {1, 2, 3, 4, }. The relations R_1 = {(1,1), (2,2), (3,3)} and
    R_2 = {(1,1), (1,2), (1,3), (1,4)} can be combined to obtain:

    $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

    $R_1 \cap R_2 = \{(1,1)\}$

    $R_1 - R_2 = \{(2,2), (3,3)\}$

    $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Relations (9.1) (cont.)

– Definition 6:

Let R be a relation from a set A to a set B and S a relation from B to a set C.
The composite of **R and S** is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by **S ° R**.

Relations (9.1) (cont.)

- **Example:** What is the composite of the **relations R and S** where R is the relation from {1,2,3} to {1,2,3,4} with R = {(1,1), (1,4), (2,3), (3,1), (3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S = {(1,0), (2,0), (3,1), (3,2), (4,1)}?

  Solution: **S ° R** is constructed using all ordered pairs in R and ordered pairs in S, where the **second element of the ordered in R agrees with the first element of the ordered pair in S**.
  For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in S ° R. Computing all the ordered pairs in the composite, we find

  S ° R = ((1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}

9.3 Representing Relations

- To represent binary relations
  - Ordered pairs
  - Tables
  - Zero-one matrices
  - Directed graphs

Representing Relations Using Matrices

- Suppose that R is a relation from $A=\{a_1,a_2,\ldots,a_m\}$ to $B=\{b_1,b_2,\ldots,b_n\}$. R can be represented by the matrix $\mathbf{M}_R=[m_{ij}]$, where $m_{ij}=1$ if $(a_i,b_j)\in R$, or 0 if $(a_i,b_j)\notin R$.
  - Ex.1-6

FIGURE 1 (9.3)

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

FIGURE 1 The Zero-One Matrix for a Reflexive Relation.

FIGURE 2 (9.3)

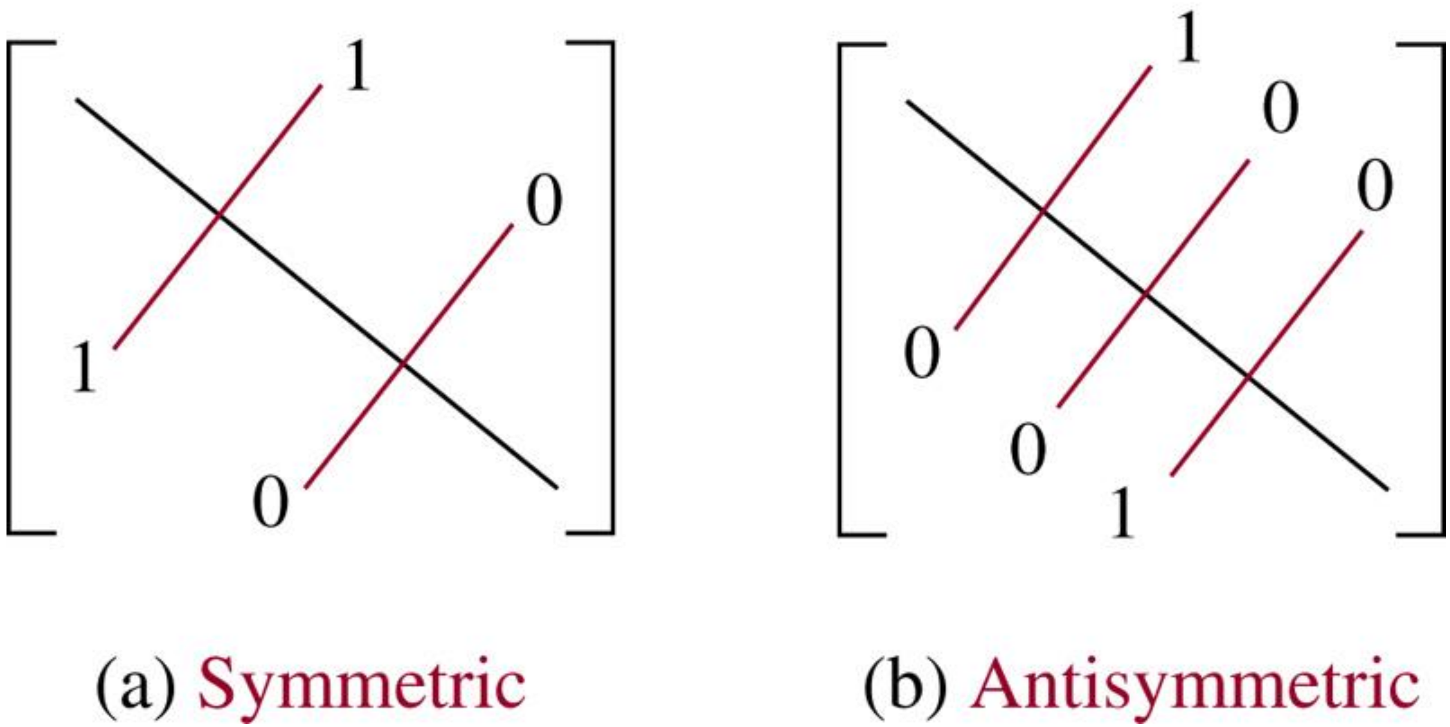


FIGURE 2 The Zero-One Matrices for Symmetric and Antisymmetric Relations.

Representing Relations Using Digraphs

- Definition 1: A *directed graph* (*digraph*) consists of a set V of *vertices* (or *nodes*) and a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.
  - Ex.7-10

FIGURE 3 (9.3)

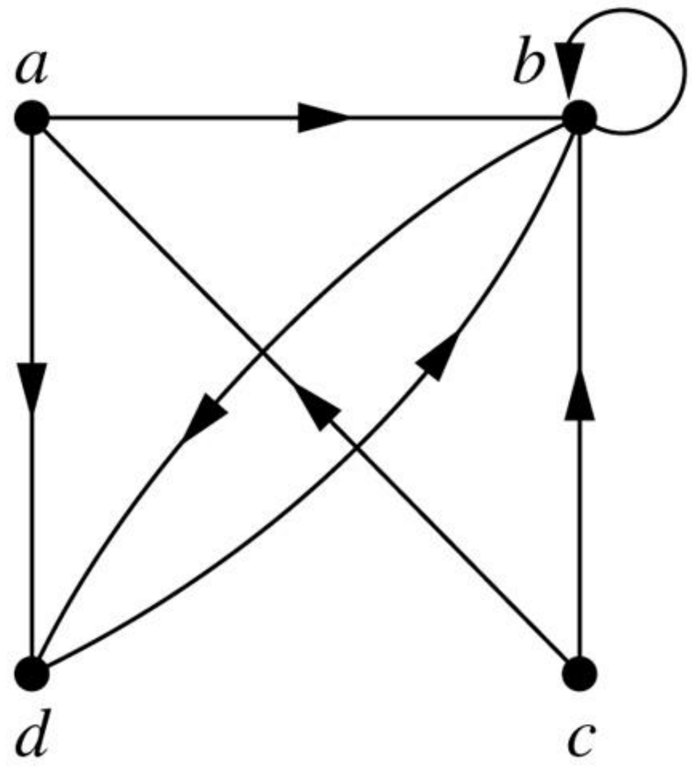


FIGURE 3 The Directed Graph.

FIGURE 4 (9.3)

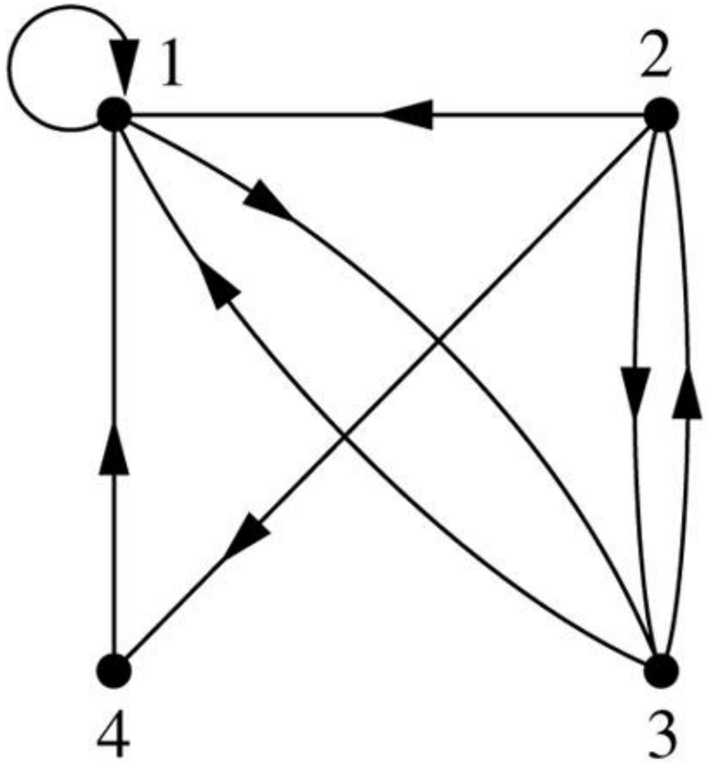


FIGURE 4 The Directed Graph of the Relations *R*.

FIGURE 5 (9.3)

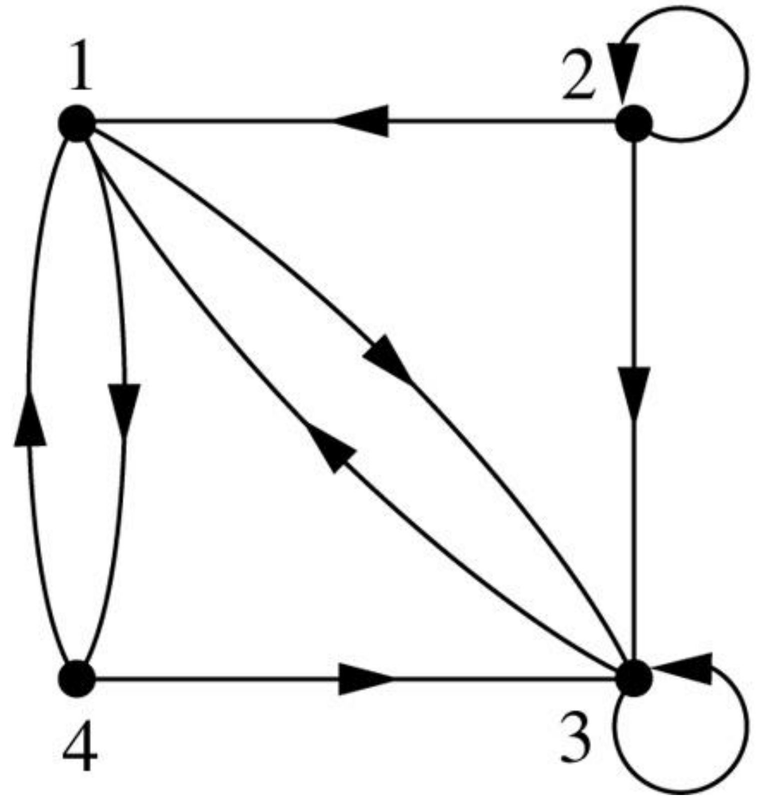
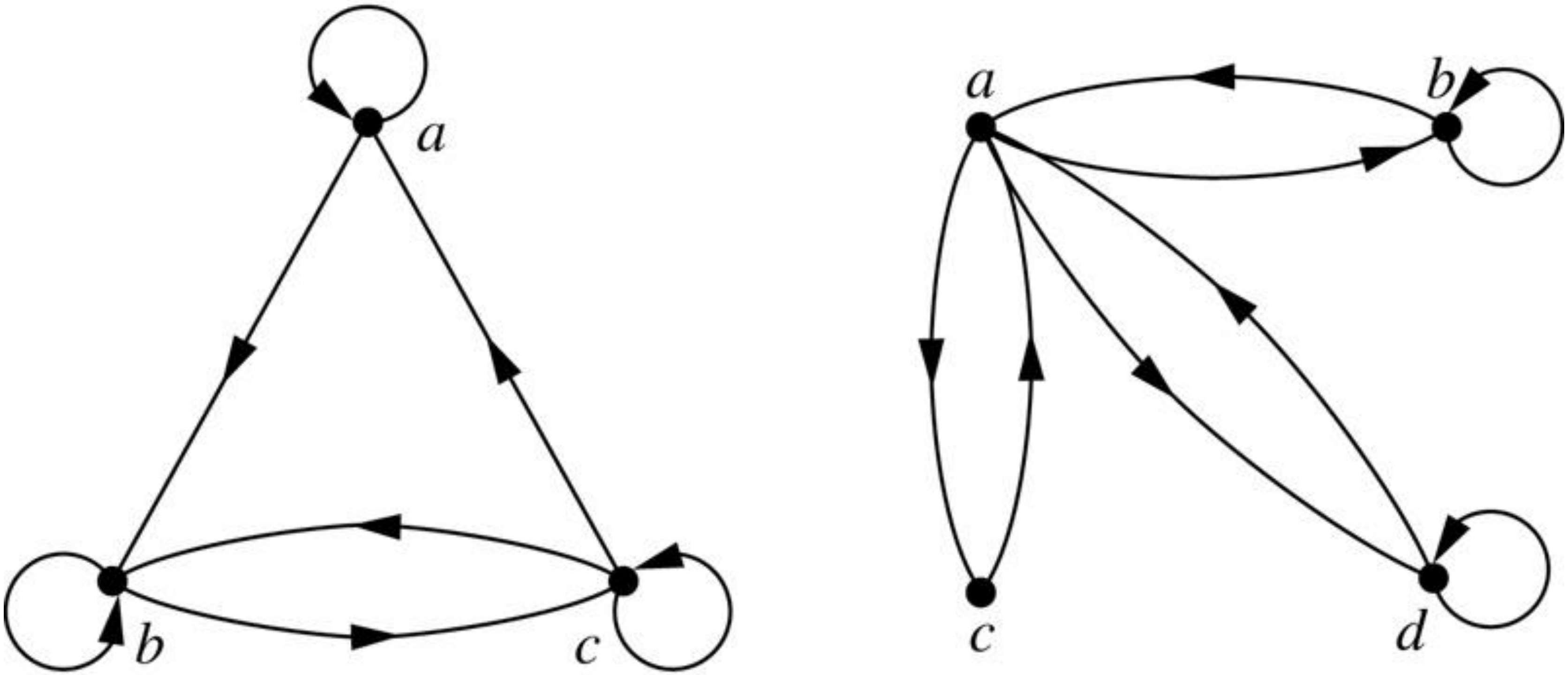


FIGURE 5 The Directed Graph of the Relations *R*.

FIGURE 6 (9.3)



(a) Directed graph of *R* (b) Directed graph of *S*

FIGURE 6 The Directed Graph of the Relations *R* and *S*.

9.5 Equivalence Relations

- Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.
- Definition 2: Two elements a and b are related by an equivalence relation are called *equivalent*.
  - The notation $a \sim b$: to denote that a and b are equivalent elements with respect to a particular equivalence relation
  - Ex. 1-5: equivalence relations
  - Ex. 6-7: not equivalence relations

Equivalence Classes

- Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a.
  - Denoted by $[a]_R$ or $[a]$
  - $[a]_R = \{s \mid (a,s) \in R\}$
  - If $b \in [a]_R$, then b is called a representative of the equivalence class
  - Ex. 8-11

9.6 Partial Orderings

- Definition 1: A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive.
  - A set S together with its partial ordering R is called a *partially ordered set*, or *poset*, (S, R).
  - Proof.
  - Ex.1-3: $\geq$, $|$, $\subseteq$
  - Ex.4: older than
  - Notation: $\preceq$
    - $\prec$: a $\leq$ b, but a$\neq$b

Partial Orderings (cont.)

- Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
  - If neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are *incomparable*. ("partial")
  - Ex. 5
  - When every two elements in the set are comparable, the relation is called a *total ordering*

Partial Orderings (cont.)

- Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set* (or a *chain*), and $\preccurlyeq$ is called a *total order* or a *linear order*.
  - Ex.6: $(\mathbf{Z}, \leq)$
  - Ex.7: $(\mathbf{Z}+, \mid)$
- Definition 4: $(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.
  - Ex.8

Partial Orderings (cont.)

- Theorem 1: (The Principle of Well-Ordered Induction) Suppose that S is a well-ordered set. Then P(x) is true for all $x \in S$, if
  (inductive step): For every $y \in S$, if P(x) is true for all $x \in S$ with $x \prec y$, then P(y) is true.
  - Proof.

Lexicographic Order

- $(A_1, \preceq_1), (A_2, \preceq_2)$
  - $(a_1, a_2) \prec (b_1, b_2)$
    - Either if $a_1 \prec_1 b_1$
    - Or if $a_1 = b_1$ and $a_2 \prec_2 b_2$
  - Ex.9-10
  - n-tuples (Fig. 1)
  - Ex.11

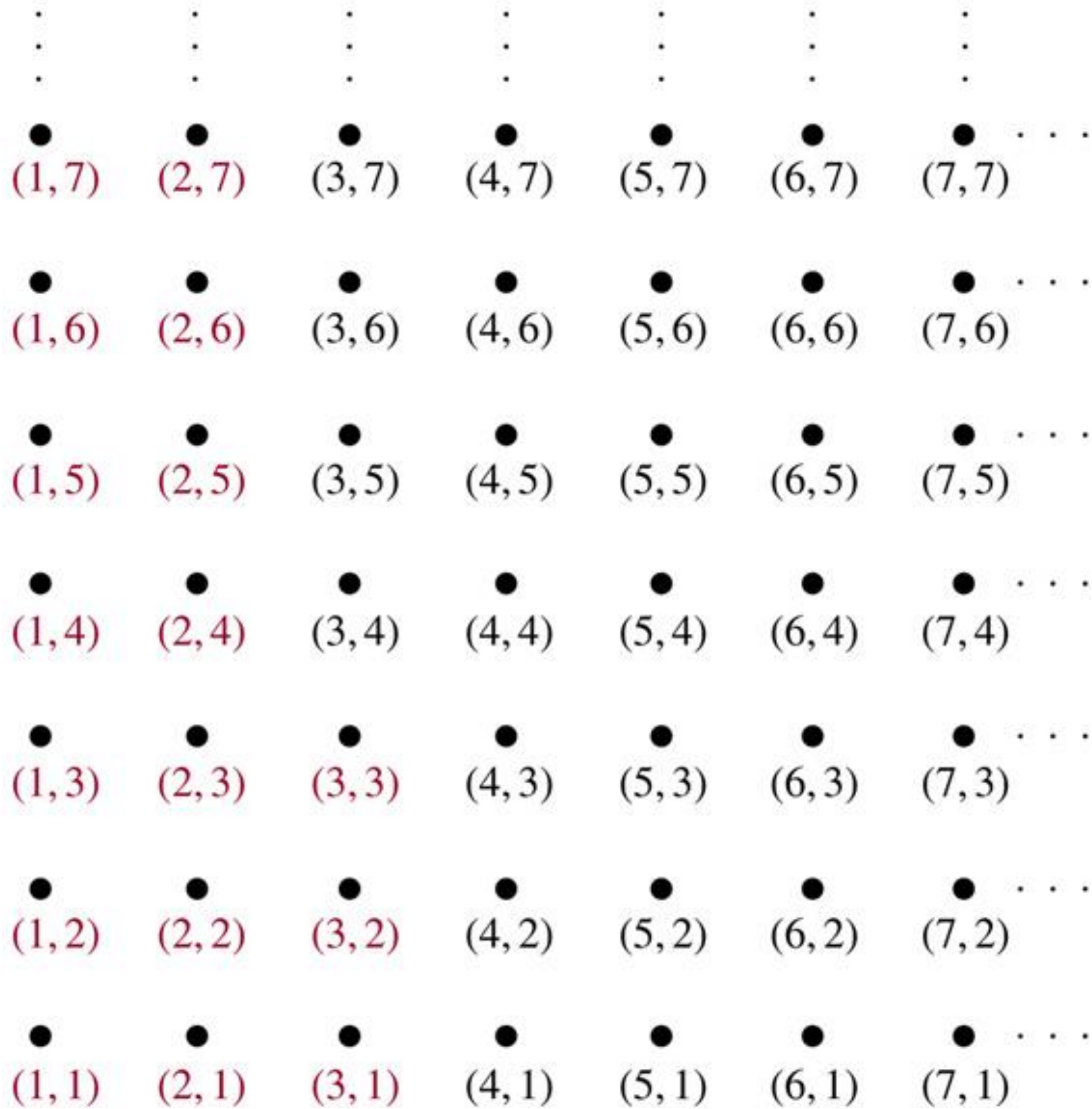


FIGURE 1 The Ordered Pairs Less Than (3,4) in Lexicographic Order.

Hasse Diagrams

- Hasse Diagrams: some edges do not have to be shown
  - Reflexive: loops can be removed
  - Transitive: if (a,b) and (b,c), then (a,c) can be removed
  - Remove all the arrows
  - Ex. 12: (Fig. 3)
  - Ex. 13: (Fig. 4)

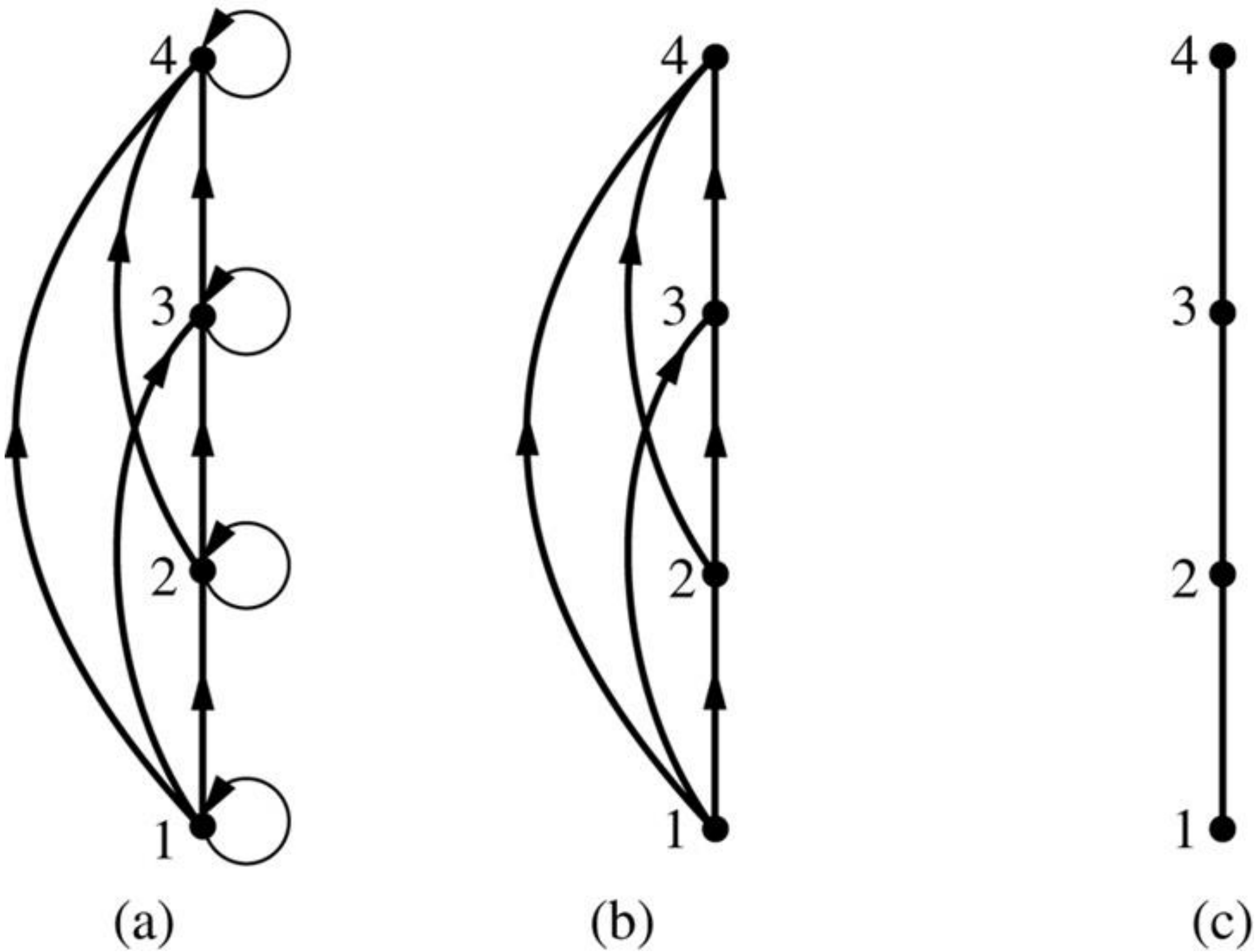


FIGURE 2 Constructing the Hasse Diagram for ({1,2,3,4},≦).

FIGURE 3 (9.6)

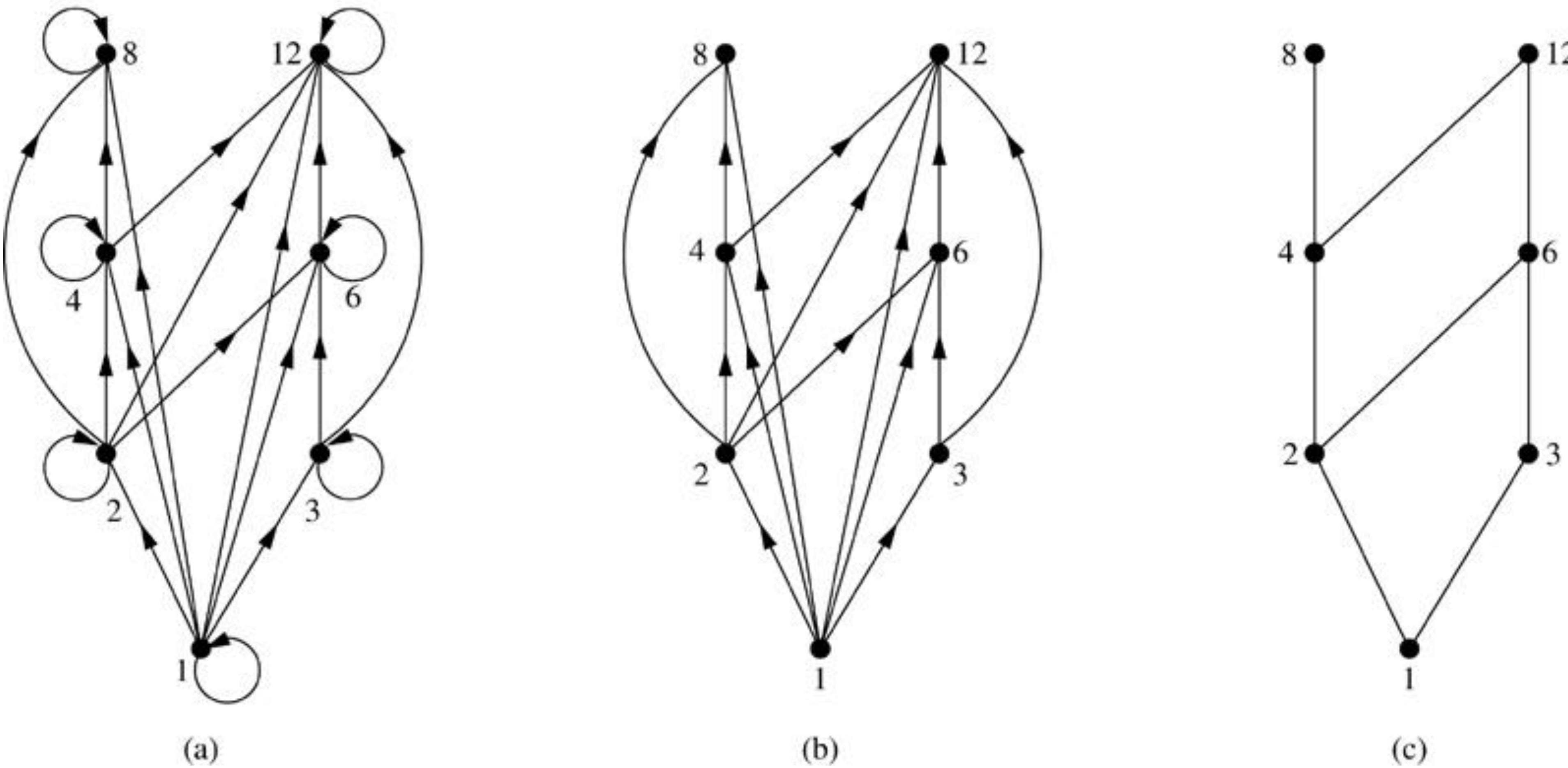


FIGURE 3 Constructing the Hasse Diagram of ({1,2,3,4,6,8,12}, |).

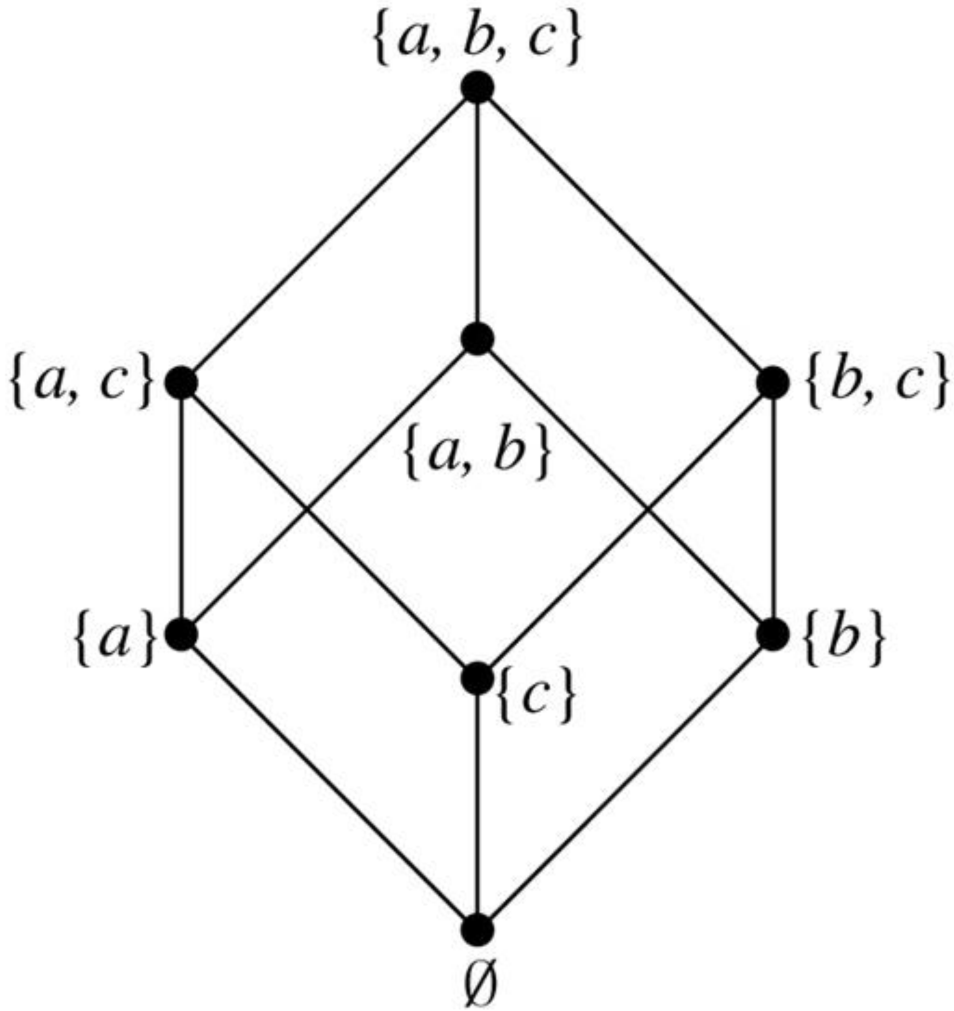


FIGURE 4 The Hasse Diagram of $(P(\{a,b,c\}), \subseteq)$.

Maximal and Minimal Elements

- a is maximal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.
- a is minimal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$.
  - Ex. 14: (Fig. 5)
  - Ex. 15: (Fig. 6)
  - Ex. 16-17

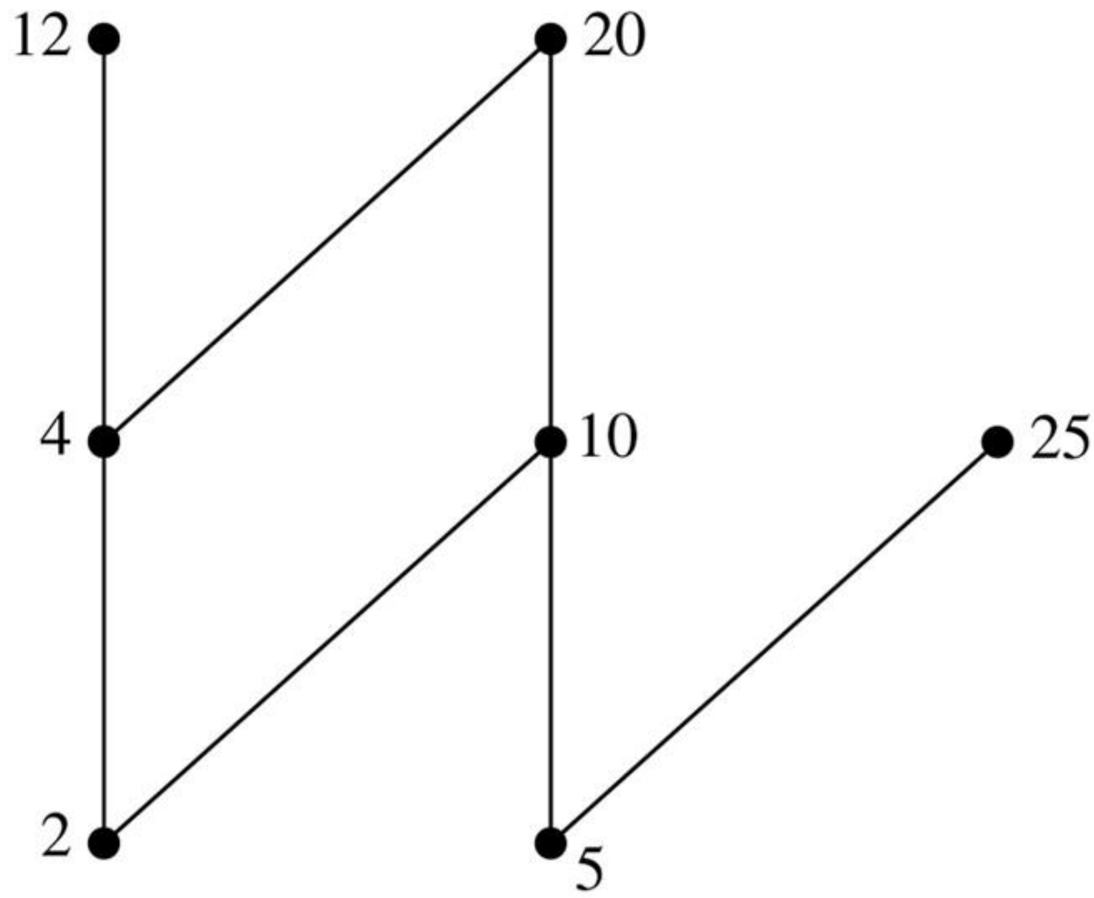


FIGURE 5 The Hasse Diagram of a Poset.

FIGURE 6 (9.6)

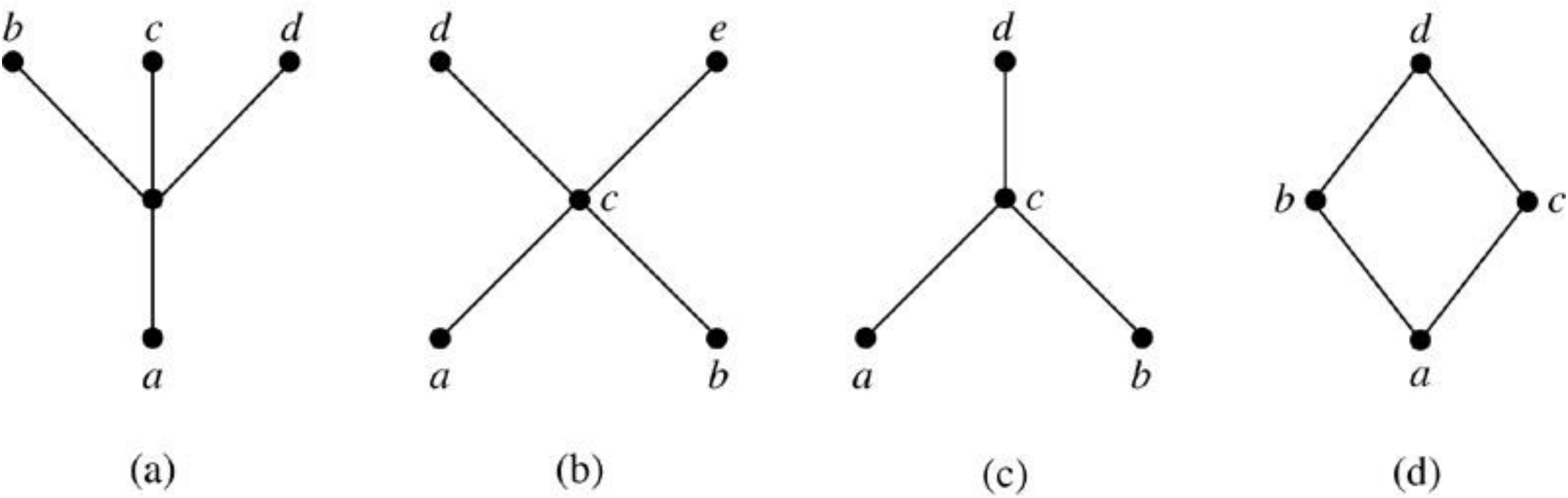


FIGURE 6 Hasse Diagrams of Four Posets.

Maximal and Minimal Elements (cont.)

- In a subset A of a poset (S, $\preccurlyeq$), if u is an element of S such that a $\preccurlyeq$ u for all a$\in$A, then u is an *upper bound* of A.
- In a subset A of a poset (S, $\preccurlyeq$), if l is an element of S such that l $\preccurlyeq$ a for all a$\in$A, then l is an *lower bound* of A.
  - Ex. 18: (Fig. 7)
  - Ex. 19: (Fig. 7)
  - Ex. 20

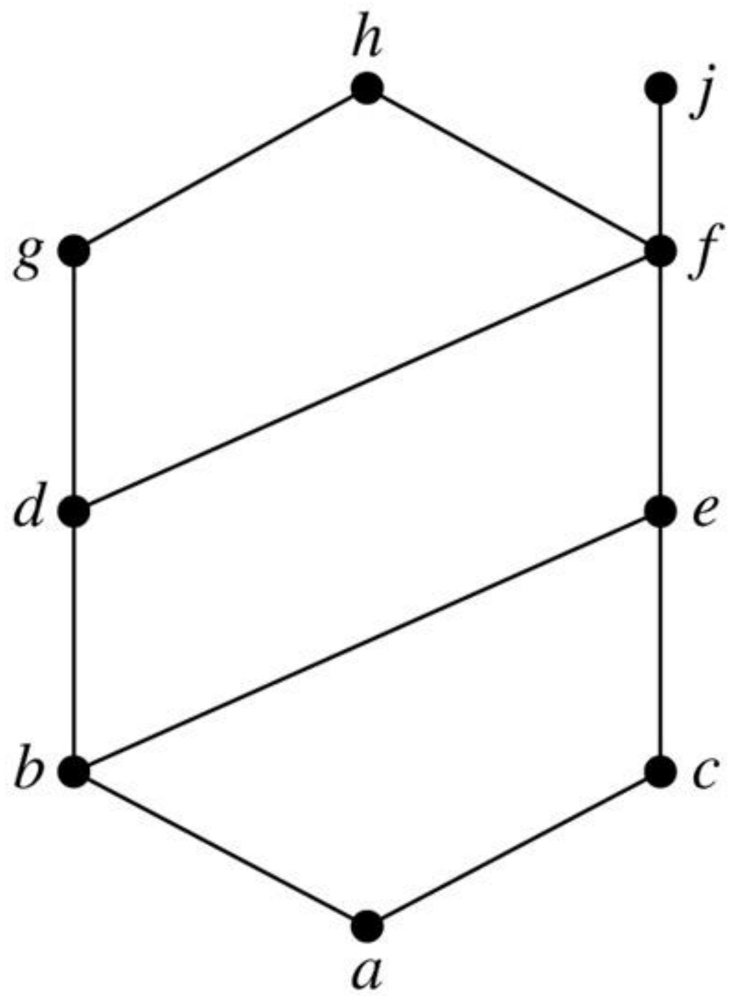


FIGURE 7 The Hasse Diagram of a Poset.

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*
  - Ex. 21: (Fig. 8)
  - Ex. 22-25

FIGURE 8 (9.6)

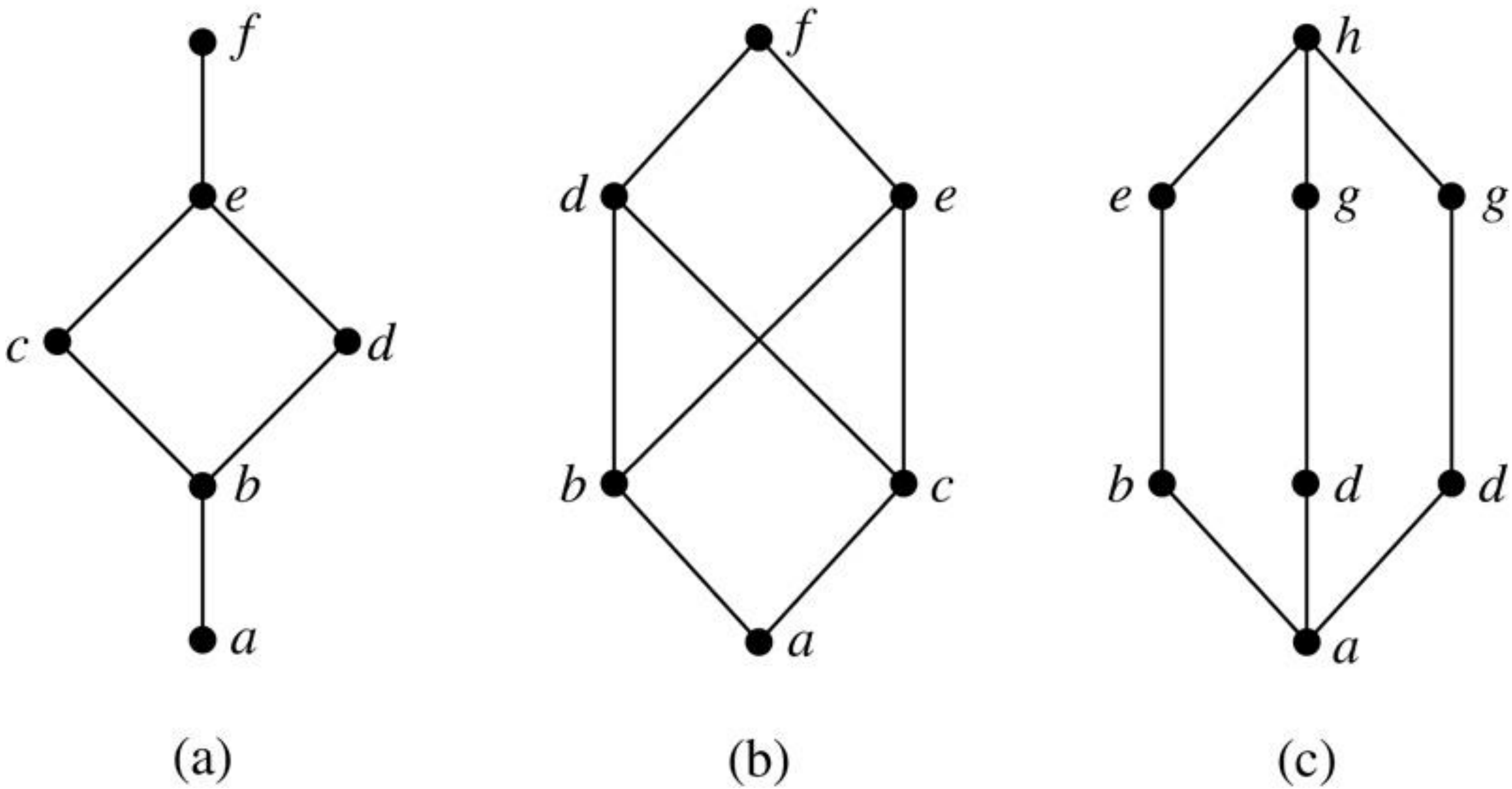


FIGURE 8 Hasse Diagrams of Three Posets.

Thank You