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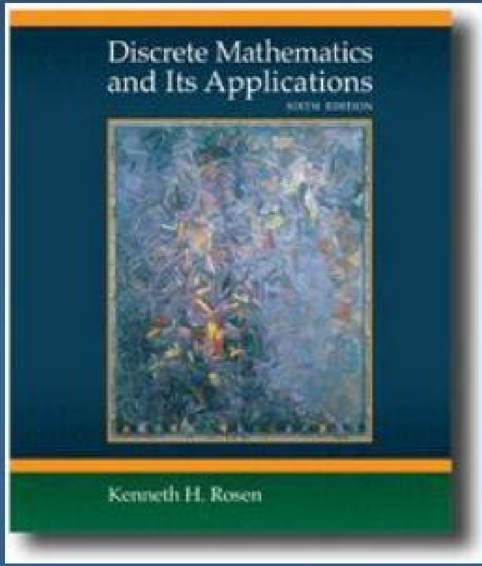
Discrete Mathematics and Its Applications

Sixth Edition

By Kenneth Rosen



Chapter 7
Advanced Counting Techniques

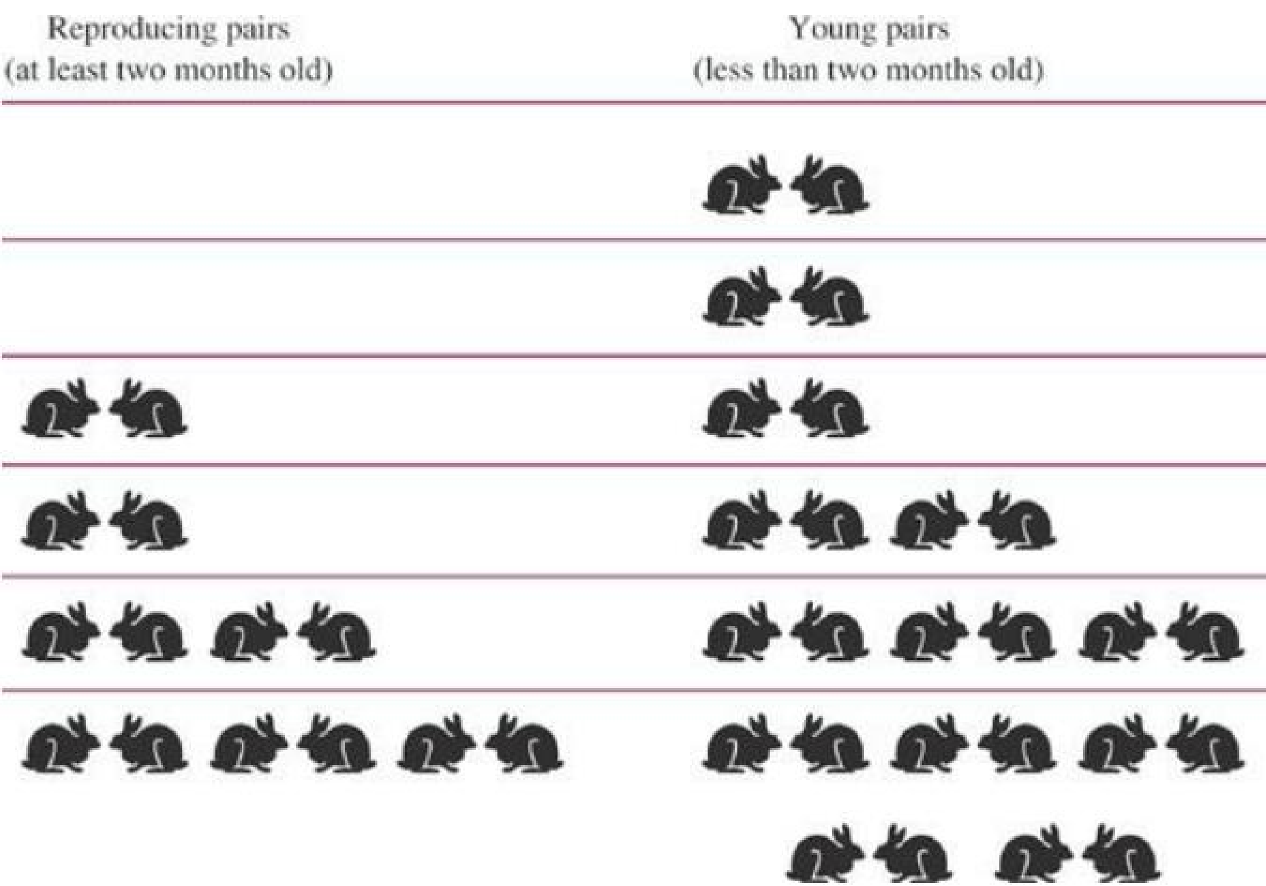
- 7.1 Recurrence Relations
- 7.2 Solving Linear Recurrence Relations
- 7.3 Divide-and-Conquer Algorithms and Recurrence Relations
- 7.4 Generating Functions
- 7.5 Inclusion—Exclusion
- 7.6 Applications of Inclusion—Exclusion

7.1 Recurrence Relations

- Definition 1: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with $n>=n_0$, where n_0 is a nonnegative integer.
  - A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation
  - Ex.1-2

Modeling with Recurrence Relations

- Ex.3: Compound interest
- Ex.4: Rabbits and the Fibonacci numbers
- Ex.5: The Tower of Hanoi
- Ex.6
- Ex.7: Codeword enumeration
- Ex.8



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

FIGURE 1 Rabbits on an Island.

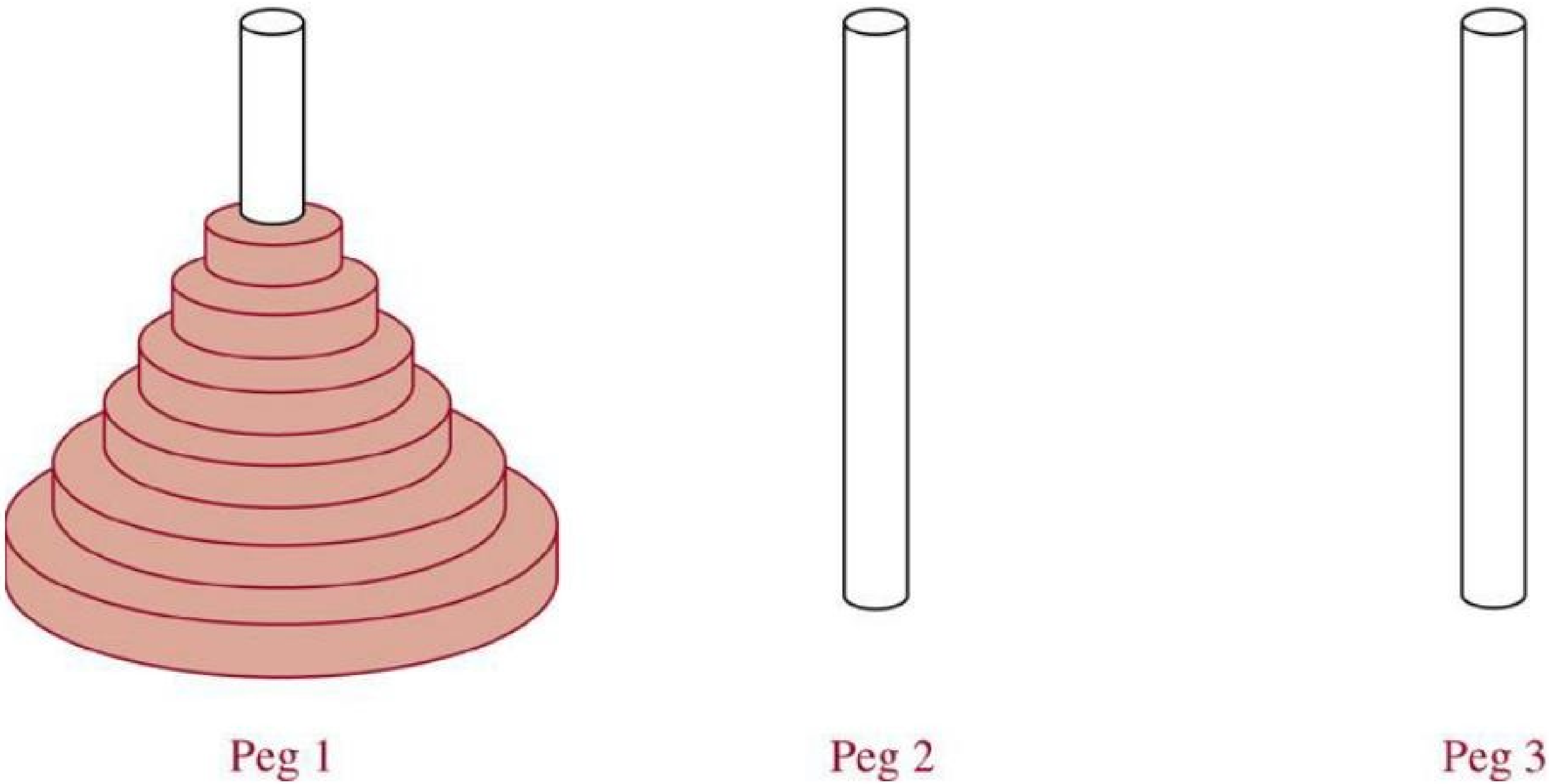


FIGURE 2 The Initial Position in the Tower of Hanoi.

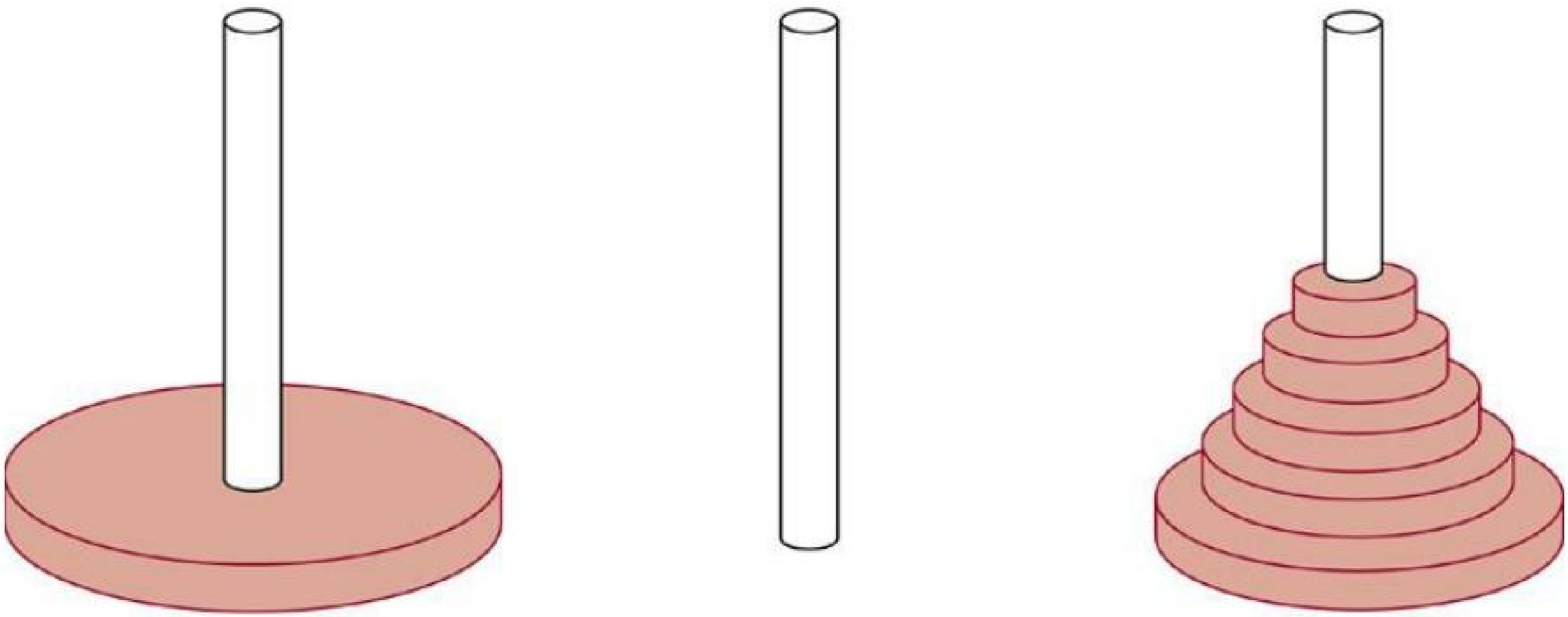


FIGURE 3 An Intermediate Position in the Tower of Hanoi.

			Number of bit strings of length n with no two consecutive 0s:
End with a 1:	Any bit string of length $n-1$ with no two consecutive 0s	1	a_{n-1}
End with a 0:	Any bit string of length $n-2$ with no two consecutive 0s	1 0	a_{n-2}
		Total:	$a_n = a_{n-1} + a_{n-2}$

FIGURE 4 Counting Bit Strings of Length n with No Two Consecutive 0s.

7.2 Solving Linear Recurrence Relations

- Definition 1: A *linear homogeneous recurrence relation of degree k with constant coefficients*:
  $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$,
  where $c_1,c_2,\ldots,c_k$ are real numbers, and $c_k\neq 0$.
  - Ex.1
    - $P_n=(1.11)P_{n-1}$
    - $f_n=f_{n-1}+f_{n-2}$
    - $a_n=a_{n-5}$

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

- $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$
  - $a_n=r^n$ is a solution iff $r^n=c_1r^{n-1}+c_2r^{n-2}+\ldots+c_kr^{n-k}$
  - $r^k-c_1r^{k-1}-c_2r^{k-2}-\ldots-c_{k-1}r-c_k=0$
    - Characteristic equation
    - Characteristic roots
- Theorem 1: Let c_1 and c_2 are real numbers. Suppose $r^2-c_1r-c_2=0$ has two distinct roots r_1 and r_2.
  Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ iff $a_n=\alpha_1r_1^n+\alpha_2r_2^n$ for n=0, 1, 2, …, where α_1 and α_2 are constants.
  - Proof.

- Ex.3: $a_n=a_{n-1}+2a_{n-2}$, $a_0=2$, $a_1=7$.
- Ex.4: Fibonacci numbers.

- Theorem 2: Let c_1 and c_2 are real numbers with $c_2 \neq 0$. Suppose $r^2-c_1r-c_2=0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ iff $a_n=\alpha_1 r_0^n+\alpha_2 n r_0^n$ for n=0, 1, 2, …, where α_1 and α_2 are constants.
  - Ex. 5: $a_n=6a_{n-1}-9a_{n-2}$, $a_0=1$, $a_1=6$.

- Theorem 3: Let $c_1,c_2,\ldots, c_k$ be real numbers. Suppose $r^k-c_1r^{k-1}-\ldots-c_k=0$ has k distinct roots $r_1,r_2,\ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$ iff $a_n=\alpha_1r_1^n+\alpha_2r_2^n+\ldots+\alpha_kr_k^n$ for n=0, 1, 2, …, where $\alpha_1,\alpha_2 \ldots,\alpha_k$ are constants.
  - Ex.6: $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$, $a_0=2$, $a_1=5$, $a_2=15$.

- Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ such that $m_i >= 1$ and $m_1 + m_2 + \ldots + m_t = k$ .Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ iff $a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m2-1}) r_2^n + \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,mt-1}) r_t^n$ for $n = 0, 1, 2, \ldots$, where $\alpha_{i,i}$ are constants.
  - Ex.7
  - Ex.8: $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, $a_0 = 1$, $a_1 = -2$, $a_2 = -1$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- Nonhomogeneous:
  - $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n)$
  - Associated homogeneous recurrence relation: $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$
  - Ex.9: $a_n=a_{n-1}+2^n$, $a_n=a_{n-1}+a_{n-2}+n^2+n+1$, $a_n=3a_{n-1}+n3^n$, $a_n=a_{n-1}+a_{n-2}+a_{n-3}+n!$

- Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation with constant coefficients $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n)$, then every solution is of the form $\{a_n^{(p)}+a_n^{(h)}\}$, where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$.
  - Proof.
  - Ex.10. $a_n=3a_{n-1}+2n$, $a_1=3$.
  - Ex.11: $a_n=5a_{n-1}-6a_{n-2}+7^n$.

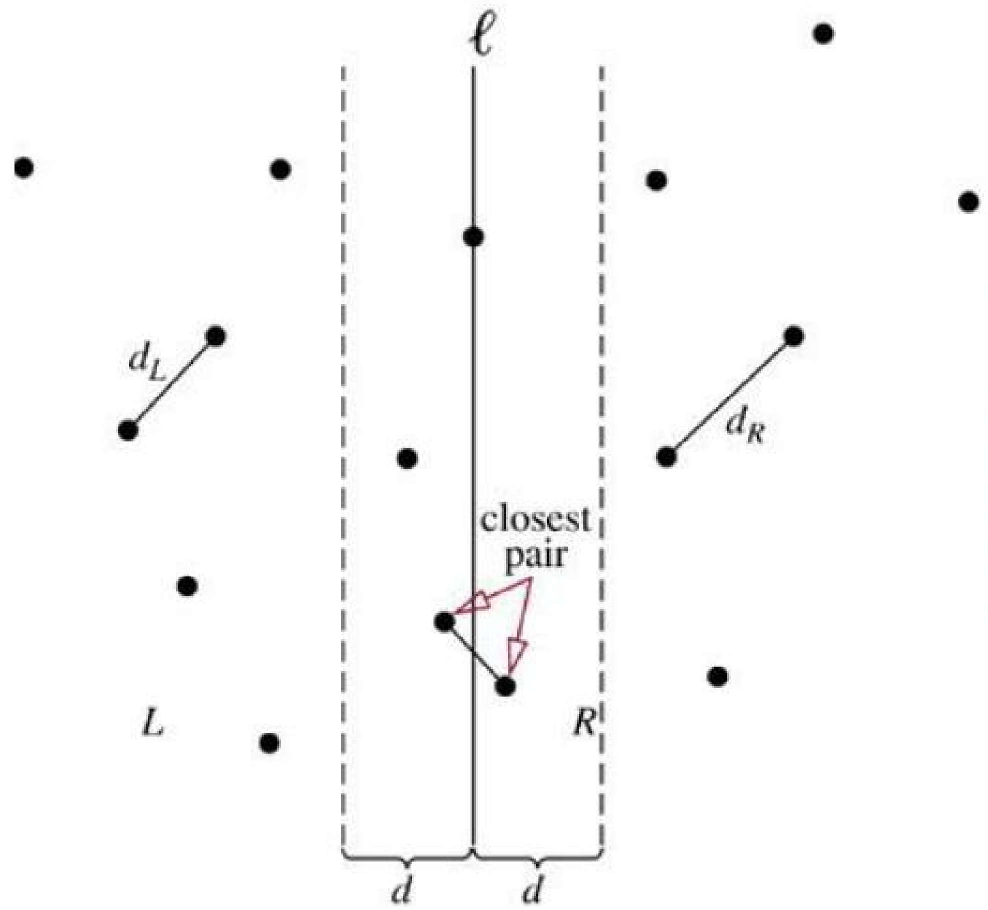
- Theorem 6: Suppose $\{a_n\}$ satisfies the nonhomogeneous recurrence relation with constant coefficients $a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n)$, and $F(n)=(b_tn^t+b_{t-1}n^{t-1}+\ldots+b_1n+b_0)s^n$. When s is not a root of the associate recurrence relation, there is a particular solution of the form $(p_tn^t+p_{t-1}n^{t-1}+\ldots+p_1n+p_0)s^n$.
  When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form $n^m\,(p_tn^t+p_{t-1}n^{t-1}+\ldots+p_1n+p_0)s^n$.
  - Ex.12: $a_n=6a_{n-1}-9a_{n-2}+F(n)$, $F(n)=3^n$, $n3^n$, n^22^n, $(n^2+1)3^n$.
  - Ex.13: $a_n=a_{n-1}+n$.

7.3 Divide-and-Conquer Algorithms and Recurrence Relations

- Divide-and-conquer algorithms
  - Divide: a problem of size n into a subproblems, each of size n/b
  - Conquer: combine the solutions of subproblems
- Divide-and-conquer recurrence relation
  - $f(n)=af(n/b)+g(n)$
  - Ex.1: binary search
  - Ex.2: finding the maximum and minimum of a sequence
  - Ex.3: merge sort
  - Ex.4: fast multiplication of integers
  - Ex.5: fast matrix multiplication

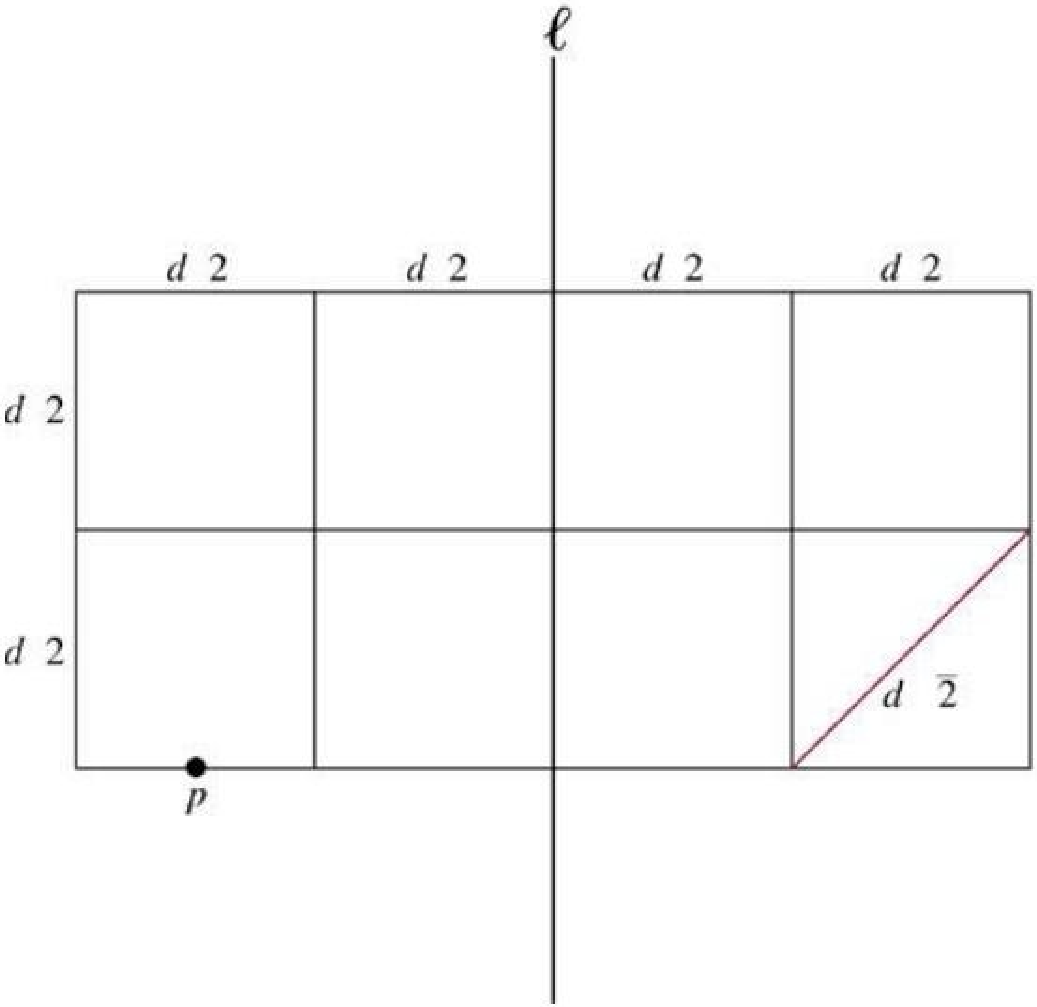
- $n=b^k$
- $f(n)=a^k f(1)+\Sigma_{j=0..k-1} a^j g(n/b^j)$
- Theorem 1: Let f be an increasing function that satisfies the recurrence relation
  $f(n)=af(n/b)+c$. Then
  f(n) is $O(n^{\log_b a})$ if a>1, O(logn) if a=1.
  When $n=b^k$, $f(n)=C_1 n^{\log_b a}+C_2$, where
  $C_1=f(1)+c/(a-1)$, $C_2=-c/(a-1)$
  - Proof.
  - Ex. 6: $f(n)=5f(n/2)+3$, $f(1)=7$. Find $f(2^k)$.
  - Ex.7: binary search
  - Ex.8: locate the maximum and minimum elements in a seqeunce.

- Theorem 2 (Master Theorem): Let f be an increasing function that satisfies the recurrence relation
  $f(n)=af(n/b)+cn^d$. When $n=b^k$, Then
  f(n) is $O(n^d)$ if $a<b^d$, $O(n^d \log n)$ if $a=b^d$, $O(n^{\log_b a})$ if $a>b^d$.
  - Ex.9: merge sort
  - Ex.10: fast multiplication algorithm
  - Ex.11: fast matrix multiplication
  - Ex.12: The Closest-Pair Problem



In this illustration the problem of finding the closest pair in a set of 16 points is reduced to two problems of finding the closest pair in a set of eight points *and* the problem of determining whether there are points closer than $d = \min(d_L, d_R)$ within the strip of width $2d$ centered at .

FIGURE 1 The Recursive Step of the Algorithm for Solving the Closest-Pair Problem.



At most eight points, including p, can lie in or on the $2d \cdot d$ rectangle centered at because at most one point can lie in or on each of the eight $(d\ 2) \cdot (d\ 2)$ squares.

FIGURE 2 Showing That There Are at Most Seven Other Points to Consider for Each Point in the Strip.

7.4 Generating Functions

- To represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series
  - To solve recurrence relations
  - To prove combinatorial identities
  - To study properties of sequences

- Definition 1: The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series $G(x) = a_0+a_1x+\ldots+a_kx^k+\ldots=\Sigma_{k=0..\infty}a_kx^k$.
  - Ordinary generating functions
  - Ex.1
  - Ex.2
  - Ex.3

Useful Facts about Power Series

- Ex.4: $f(x)=1/(1-x)$
- Ex.5: $f(x)=1/(1-ax)$
- Theorem 1: Let $f(x)=\Sigma_{k=0..\infty}a_k x^k$ and $g(x)=\Sigma_{k=0..\infty}b_k x^k$. Then $f(x)+g(x)=\Sigma_{k=0..\infty}(a_k+b_k)x^k$ and $f(x)g(x)=\Sigma_{k=0..\infty}(\Sigma_{j=0..k}a_j b_{k-j})x^k$ and.
  - It's valid only for power series that converge in an interval.
  - Ex.6: $f(x)=1/(1-x)^2$.

- Definition 2: Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient (u,k) is defined by (u,k)=u(u-1)…(u-k+1)/k! if k>0, or 1 if k=0.
  - Ex.7: (-2, 3), (1/2, 3)
  - Ex.8: $(-n, r)=(-1)^rC(n+r-1,r)$
- Theorem 2: (The Extended Binomial Theorem) Let x be a real number with $|x|<1$ and let u be a real number. Then $(1+x)^u= \Sigma_{k=0..\infty}(u,k)\, x^k$.
  - Ex.9: $(1+x)^{-n}$, $(1-x)^{-n}$.

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Counting Problems and Generating Functions

- Counting the r-combinations from n elements when repetition is allowed
  - $e_1+e_2+\ldots+e_n=C$
  - Ex.10: $e_1+e_2+e_3=17$, e_1:2-5, e_2:3-6, e_3:4-7.
  - Ex.11
  - Ex.12
  - Ex.13: k-combinations from n elements.
  - Ex.14: r-combinations from n elements when repetition is allowed.
  - Ex.15: select r objects of n different kinds if we must select at least one object of each kind.

Using Generating Functions to Solve Recurrence Relations

- Ex.16: $a_k=3a_{k-1}$, $a_0=2$.
- Ex.17: $a_n=8a_{n-1}+10^{n-1}$, $a_1=9$.

Proving Identities via Generating Functions

- Ex.18

7.5 Inclusion-Exclusion

- The principle of inclusion-exclusion
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - Ex.1: (Fig.1)
  - Ex.2: (Fig.2)
  - Ex.3

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$

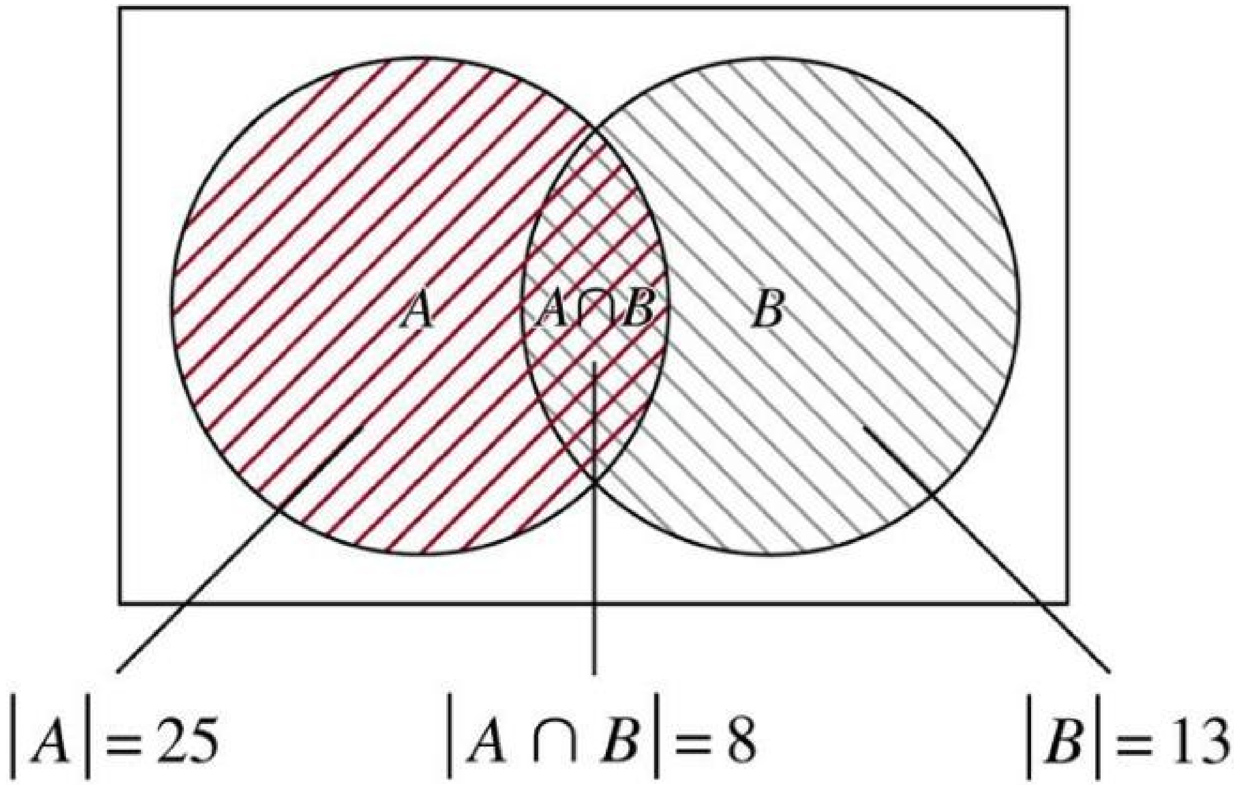


FIGURE 1 The Set of Students in a Discrete Mathematics Class.

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$$

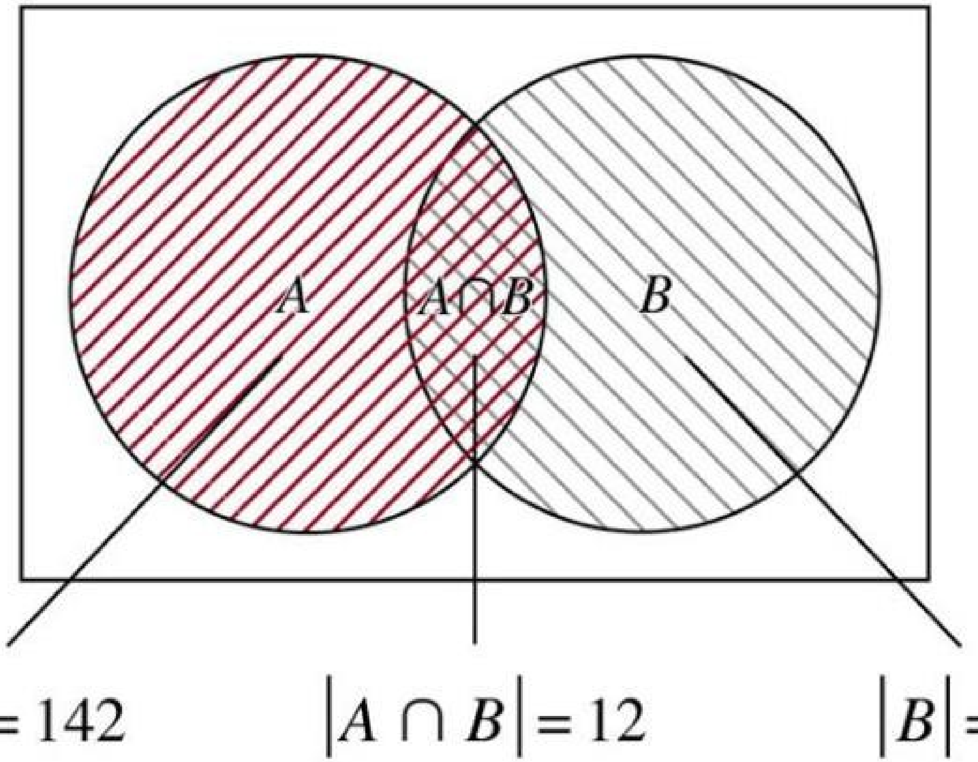
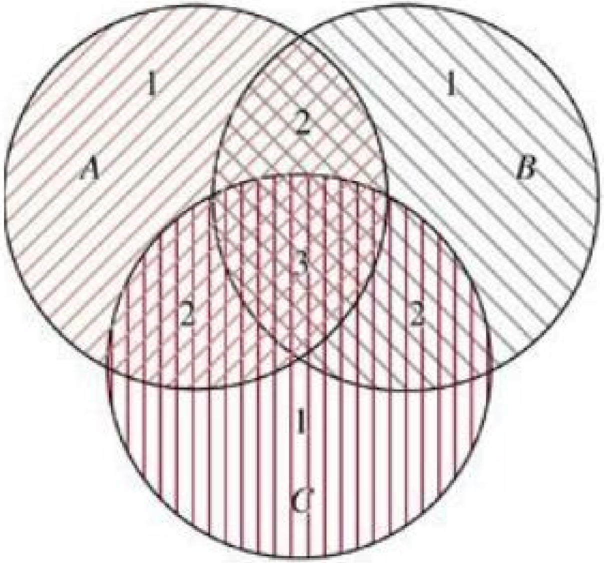
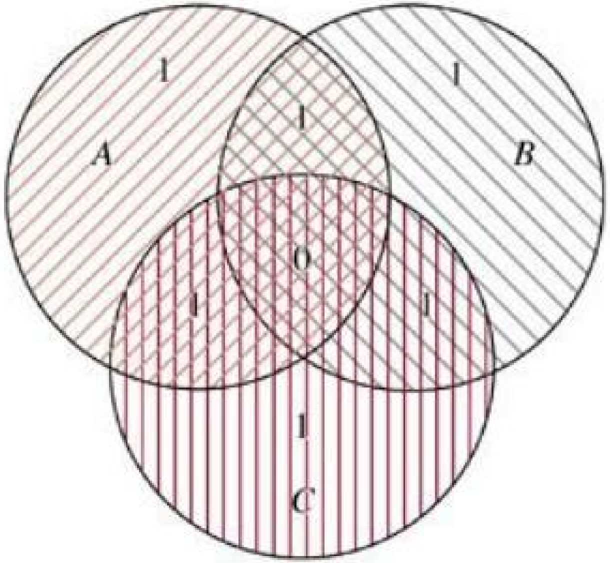


FIGURE 2 The Set of Positive Integers Not Exceeding 1000 Divisible by Either 7 or 11.

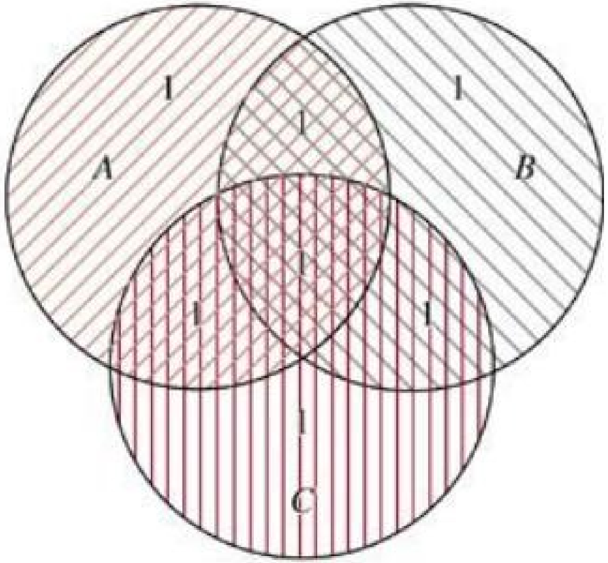
- For three sets: (Fig. 3)
  - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$
  - Ex. 4 (Fig. 4)



(a) Count of elements by $|A|+|B|+|C|$



(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$



(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

FIGURE 3 Finding a Formula for the Number of Elements in the Union of Three Sets.

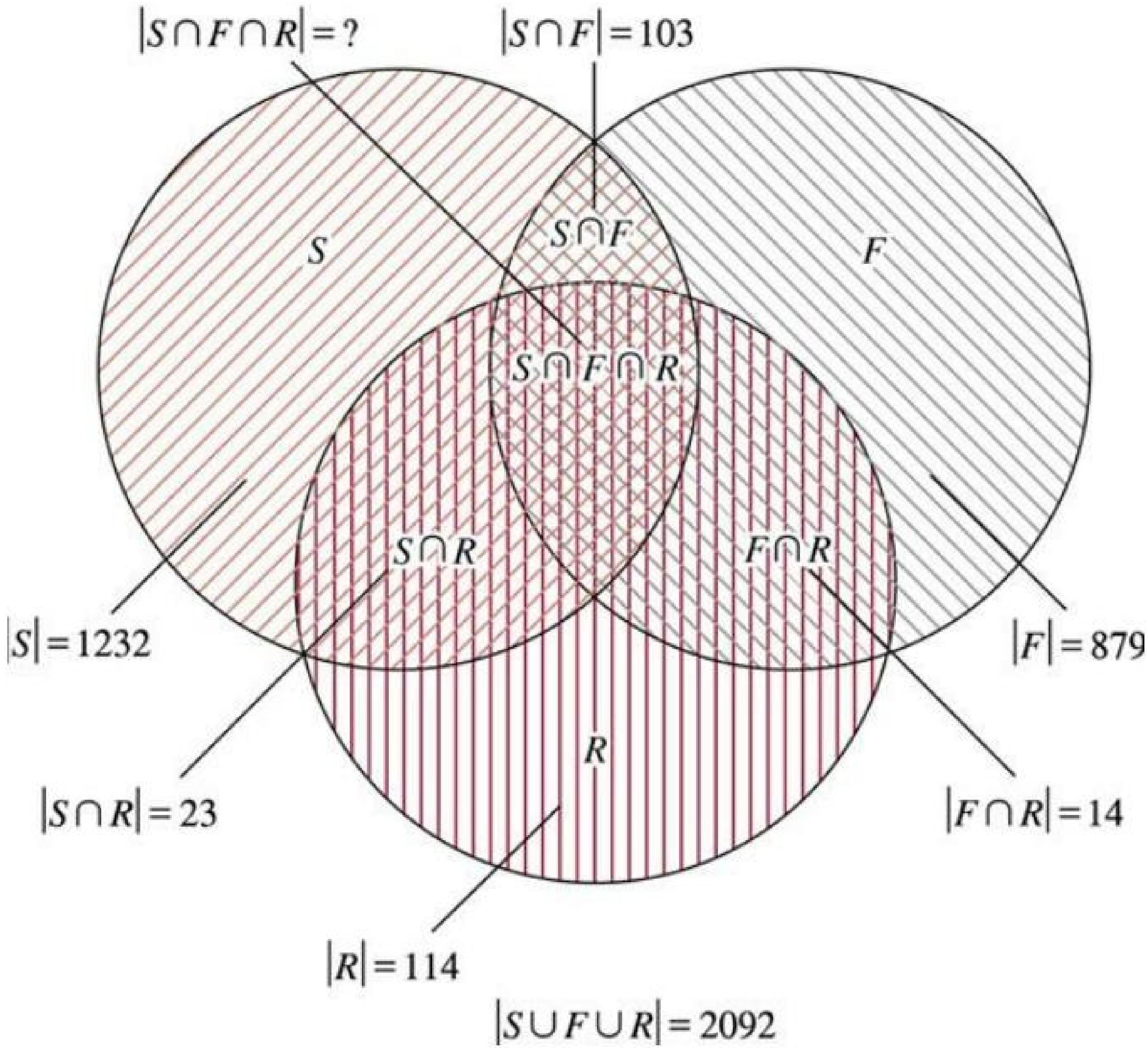


FIGURE 4

The Set of Students Who Have Taken Courses in Spanish, French, and Russian.

- Theorem 1: (The Principle of Inclusion-Exclusion) Let A_1, A_2, …, A_n be finite sets. Then $|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma_{i=1..n}|A_i| - \Sigma_{i,j=1..n}|A_i \cap A_j| + \Sigma_{i,j,k=1..n}|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$.
  - Ex.5

7.6 Applications of Inclusion-Exclusion

- An alternative form of inclusion-exclusion
  - To solve: the number of elements in a set that have none of n properties P1, P2, …, Pn.
    - Let Ai be the subset that have property Pi.
    - The number of elements with all the properties Pi1, Pi2, …, Pik: N(Pi1, Pi2, Pik).
  - $N(P1', P2', \ldots, Pn') = N - |A_1 \cup A_2 \cup \ldots \cup A_n|$ $= N - \Sigma_{i=1..n} N(P_i) + \Sigma_{i,j=1..n} N(P_i P_j) - \Sigma_{i,j,k=1..n} N(P_i P_j P_k) + \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$.
    - Ex.1

The Sieve of Eratosthenes

- To find all primes not exceeding a specified positive integer
  - For example, the primes not exceeding 100
    - P1: divisible by 2
    - P2: divisible by 3
    - P3: divisible by 5
    - P4: divisible by 7

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The Number of Onto Functions

- Ex.2
- Theorem 1: Let m and n be positive integers with m>=n. Then, there are $n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m + \ldots + (-1)^{n-1} C(n,n-1)1^m$ onto functions from a set with m elements to a set with n elements.
  - =n!S(m,n), where S(m,n) is a Stirling number of the second kind
  - Ex.3

Derangement

- *Derangement*: a permutation of n objects that leave no objects in their original positions
  - Ex.4: The Hatcheck problem (later)
  - Ex.5
  - Let D_n denote the number of derangements of n objects, $D_3=2$
- Theorem 2: The number of derangements of a set with n elements is
  $D_n=n![1-1/1!+1/2!-1/3!+\ldots+(-1)^n 1/n!]$
  - Proof.

TABLE 2 **The Probability of a Derangement.**

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

Thanks for Your Attention!