Proof

Riccati

$$H_k = L_k + \lambda_{k+1}^T f(x_k, u_k)$$

$$H_k = \frac{1}{2} x_k^T Q x_k + q^T x_k + \frac{1}{2} u_k^T R u_k + r^T u_k + \lambda_{k+1}^T (A x_k + B u_k)$$

$$\frac{\partial H_k}{\partial \lambda_{k+1}} = x_{k+1} = A x_k + B u_k \qquad (1)$$

$$\frac{\partial H_k}{\partial x_k} = \lambda_k = Q x_k + q_k + A^T \lambda_{k+1} \qquad (2)$$

$$\frac{\partial H_k}{\partial u_k} = 0 = R u_k + r_k + B^T \lambda_{k+1} \qquad (3)$$

Assuming $\lambda_k = P_k x_k + s_k$ $^{(4)}$; $\lambda_{k+1} = P_{k+1} x_{k+1} + s_{k+1}$ (4)

$$\lambda_{k+1} = P_{k+1}(A x_k + B u_k) + s_{k+1} \qquad (5)$$

(5) → (3)

$$R u_k + r_k + B^T(P_{k+1} A x_k + P_{k+1} B u_k + s_{k+1}) = 0$$

$$R u_k + B^T P_{k+1} B u_k = -(r_k + B^T P_{k+1} A x_k + B^T s_{k+1})$$

$$\underbrace{(R + B^T P_{k+1} B)}_{M} u_k = -(r_k + B^T P_{k+1} A x_k + B^T s_{k+1})$$

$$M u_k = -(B^T P_{k+1} A x_k) - (r_k + B^T s_{k+1})$$

$$u_k = -\underbrace{M^{-1}(B^T P_{k+1} A)}_{K_k} x_k - \underbrace{M^{-1}(r_k + B^T s_{k+1})}_{d_k} \qquad (6)$$

So
$$\boxed{\begin{aligned} K_k &= (R + B^T P_{k+1} B)^{-1}(B^T P_{k+1} A) \\ d_k &= (R + B^T P_{k+1} B)^{-1}(B^T s_{k+1} + r_k) \end{aligned}}$$

(5) → (2)

$$\lambda_k = Q x_k + q_k + A^T(P_{k+1} A x_k + P_{k+1} B u_k + s_{k+1}) \quad (7)$$

(4) → (7)

$$P_k x_k + s_k = Q x_k + q_k + A^T P_{k+1} A x_k + A^T P_{k+1} B u_k + A^T s_{k+1} \quad (8)$$

(6) → (8)

$$P_k x_k + s_k = Q x_k + q_k + A^T P_{k+1} A x_k + A^T P_{k+1} B(-K_k x_k - d_k) + A^T s_{k+1}$$

$$P_k x_k + s_k = Q x_k + A^T P_{k+1} A x_k - A^T P_{k+1} B K_k x_k + q_k - A^T P_{k+1} B d_k + A^T s_{k+1}$$

$$[P_k] x_k + \{s_k\} = [Q + A^T P_{k+1} A - A^T P_{k+1} B K_k] x_k + \{q_k - A^T P_{k+1} B d_k + A^T s_{k+1}\}$$

$$\boxed{\begin{aligned} P_k &= Q + A^T P_{k+1} A - A^T P_{k+1} B K_k \\ s_k &= q_k - A^T P_{k+1} B d_k + A^T s_{k+1} \end{aligned}}$$