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# Portfolio Optimization: Acknowledging The Practical Value of Markowitz's Theory

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2020-08-09

**Abstract**

(Abstract) [DeMiguel et al., 2009] offer one of the most comprehensive studies of Modern Portfolio Theory, in which they analyze 14 relevant extensions of Markowitz's rule and report than none of the models can consistently outperform a naive 1/N strategy. For this reason, they conclude that "there are still many miles to go before the gains promised by optimal portfolio choice can actually be realized out of sample". Despite, the subsequent academic literature acts in accordance with this conclusion, the financial industry has already recognized the usefulness of Markowitz's theory in practice. Namely, Markowitz's minimum variance portfolio is a basis for an ETF (USMV) offered by iShares in 2011 that, as of 31st of March 2020, has more than USD30 billion of net assets under management. Similarly, among the 14 optimized models covered in the above-mentioned research, the minimum variance technique tends to generate the best out of sample results. However, the estimated minimum variance portfolio is only based on the sample covariance matrix. Consequently, our study focusses on more sophisticated minimum variance portfolios that could further improve the result and create superior risk-return properties than the naive 1/N strategy, regardless of the number or characteristic of assets included in the sample universe. Particularly, we use six different ways to estimate the covariance matrix, among which some methods are recommended by the academic literature, while some are introduced in this paper. Hence, our research may be perceived as supplementing the contemporary academic literature and finally acknowledging the practical value of Markowitz's theory.

# A Introduction

Although Modern Portfolio Theory has been introduced almost seven decades ago, the framework is still relevant and recently it has drawn renewed interest by academia, whether it be in combining optimized models with other strategies (as in [Kan and Zhou, 2007];[DeMiguel et al., 2009]; [Tu and Zhou, 2011]) or in the context of reducing parameter uncertainty ([Jagannathan and Ma, 2003];[Ledoit a [DeMiguel et al., 2013]; [Kempf et al., 2015]). Our paper tries to utilize the results of the recent research in the field and to counter the statement saying "there are still many miles to go before the gains promised by optimal portfolio choice can actually be realized out of sample" ([DeMiguel et al., 2009]). Despite opposing their conclusion, our study can be interpreted as an extension to the [DeMiguel et al., 2009] paper which serves as a platform for our research. Although they do not find a model that could consistently outperform 1/N rule in terms of the Sharpe ratio, their comprehensive comparison of various optimization techniques implies that the best out of sample result is generated by the techniques implementing moments' restriction. Particularly, minimum variance models tend to outperform other portfolios, implying that it is useful to completely ignore return estimation and focus on risk reduction. The solution of the minimum variance problem is presented in chapter 3, however, the intuition why the minimum variance portfolio generates better results than mean variance portfolio is that optimization process is very sensitive to the change in the return forecast. Namely, the forecast is susceptible to error and a minuscule change in the expected return can substantially alter portfolio weights ([Fabozzi et al., 2010]) which is why [Michaud and Michaud, 2008] describes the optimization process as "a molehill of garbage in, a mountain of garbage out". Nevertheless, for estimating minimum variance portfolio [DeMiguel et al., 2009] use only a sample covariance matrix as a base for deriving portfolio weights. Moreover, while it is documented that higher frequency data leads to a better estimate ([Jagannathan and Ma, 2003]), only monthly historical returns are used to derive the sample covariance matrices. Also, they use 60 or 120 months of data and advocate for as long as possible estimation intervals, which neglects the fact that covariances among securities and individual volatilities can change over time. On the other hand, in estimating the sample covariance matrix, our study uses daily rather than monthly data and shortens the estimation interval from 120 months to 12 months to capture the time-varying component of individual volatilities and covariances among stocks. Moreover, we incorporate methods that are recommended in the literature such as [Ledoit and Wolf, 2004] estimate and zero correlation matrix ([DeMiguel et al., 2013]. Also, in chapter 4 we introduce a partially-implied model, portfolio of covariance matrices, and momentum adjusted covariance matrix, to generate minimum variance portfolio that out-of-sample produce better results that the naive model. We test the performance of respective strategies over the period covering the first two decades of 21st century, while performing monthly portfolio rebalancing. For the robustness part, we include nine samples, among which seven are the samples of individual stocks and two are the ETF samples, as discussed in more detail in chapter 2. The strategies are ranked based on the Sharpe ratio, where the details about backtesting and Sharpe ratio calculation can be found in chapter 2. Furthermore, the study includes a thorough analysis of the effect of market environment on the minimum variance and 1/N strategy. Intuitively, minimum variance portfolios are expected to have low portfolio beta which should make them particularly attractive in times of uncertainty. We confirm this by analyzing the performance of the specific models in crisis versus non-crisis intervals in section 5.2. Considering the identified effect of the market environment on the respective strategies, we proceed to build a net-long weighting scheme investing in both naive and optimized portfolios. Combining 1/N rule and optimized portfolios is not a novel idea. For instance,[DeMiguel et al., 2009] without much success consider a mixture of 1/N rule and minimum variance portfolios, trying to find the appropriate balance so as to maximize investor's expected utility, however out of sample their model does not perform well. On the other hand, [Tu and Zhou, 2011] find that combinations of the naive and optimized rule can create attractive risk-adjusted yields, outperforming both constituents independently. Nevertheless, our procedure is the first one to take into account the behavior of strategies under different circumstances as a basis of building a simple algorithm that out of sample produces much larger return than uncombined counterparts, as shown in chapter 6. In section 5.3 we compare the performance of our best minimum variance approach with the iShares Minimum Volatility ETF (USMV), which is also based on the Markowitz's minimum variance portfolio. The ETF represents a portfolio that is derived based on the covariance matrix estimated using Barra multi-factor model. We show that our hypothetical portfolio generates lower volatility and higher Sharpe ratio than the ETF. However, in chapter 7 we discuss the potential limitations that we have not considered in the study, that could lower the results of our portfolio in the real word trading. Nevertheless, we offer additional example that shows that after accounting for the constraints, the reduction in the performance is not significant enough to affect the previously obtained conclusions. The following part briefly explains the relevant literature without which we would not be able to fully deliver our paper.

[Ledoit and Wolf, 2003] find that the sample covariance matrix should not be used in the portfolio optimization framework, instead, they recommend using shrinkage procedure to derive an improved covariance matrix estimate, not vulnerable to the extreme values. The use of the sample covariance matrix leads to a large error if the number of historical data points is small relative to the number of assets included in the sample. The intuition behind the shrinkage procedure is that the extreme positive/negative estimates in the sample covariance matrix need to be pulled towards the center, where the center is a specific highly structured estimator. More formally:

$$\delta F + (1 - \delta) S \tag{1}$$

where F stands for a highly structured covariance matrix, while S represents the sample covariance matrix. When it comes to the intensity of shrinkage, [Ledoit and Wolf, 2003] offer a sophisticated mathematical solution for finding $\delta$ parameter, however, it is out of the scope of our paper to thoroughly derive mathematical proof. The intuition is that the larger the number of stocks relative to the length of the estimation period, the larger the $\delta$. When it comes to selecting the shrinkage target - F, they suggest using Sharpe's single-factor model. Forecasting of the covariance matrix using the single-factor model was introduced by [Elton and Urich, 1978], and has been extensively used ever since. The evidence implies that it generates a superior estimate of a covariance matrix, well-balanced portfolio allocations, and leads to more diversified portfolios ([Fabozzi et al., 2010]). According to the Single-Factor Model, the return of individual stocks can be explained by the following equation:

$$R_i = \alpha_i + \beta_i * R_m + \varepsilon_i \tag{2}$$

where $R_i$ stands for the return of a security, intercept represents the part of return independent of market return, the slope illustrates the sensitivity to market return, $R_m$ represents a market return, while the error term is considered the uncertain (random) segment of the return. The underlying assumption of the model is that error terms of individual securities are independent of each other, meaning that stocks move together, solely because of collective correlation with the market. This leads to a fairly simple formula for estimating covariances among securities:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \tag{3}$$

where $\sigma_m^2$ represents the variance of market return. However, similarly to the security's return, its variance consists of a market-related part and an idiosyncratic part:

$$\sigma^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \tag{4}$$

where mean squared error serves as an estimate of $\sigma_{ei}^2$ (Elton, Gruber, Brown and Goetzmann, 2014). Specifically, the formula is:

$$\sigma_{ei}^2 = 1/n \sum_{t=1}^{n} [R_{it} - (\alpha_i + \beta_i * R_{mt})]^2 \tag{5}$$

However, a year later [Ledoit and Wolf, 2004] introduced a new shrinkage target - constant correlation model. Namely, they find that both single-factor and constant correlation models offer similar performance, however, the latter has a more straightforward implementation. Particularly, all correlations in the model are identical, equal to the average sample correlation. Due to the widespread application of their model in portfolio management, computation of a Ledoit Wolf covariance estimate is a built-in function in many statistical packages, which makes its application relatively simple. Rather than relying on a single method for estimating covariance matrix, [Jagannathan and Ma, 2000] propose building equally weighted portfolios of covariance matrix estimators in order to diversify the error to which all estimators are prone. They suggest using the following estimators as portfolio constituents: sample covariance matrix, single-factor generated covariance matrix and a matrix having solely diagonal of the sample covariance estimate ([Fabozzi et al., 2010]). In another study, [Jagannathan and Ma, 2003] analyze the effect of portfolio constraints on the performance of minimum variance portfolios and find that short-sale constraints produce the identical effect as if artificially decreasing the estimated covariances among securities while imposing upper bounds have an opposite effect. Intuitively, the optimization process would allocate large negative weights to the assets having a high level of correlation, however, if the correlation among all assets is relatively low, all of them will be assigned a positive weight.

In more detail, the optimization procedure generally assigns large negative weights to the securities having high covariance with other assets and in a case when the high covariance estimate is a consequence of the estimation error, imposing short-selling constraints leads to better portfolio allocation. Furthermore, [Jagannathan and Ma, 2003] find that when short-sale constraints are implemented, the sample covariance generates similar results in terms of minimum variance portfolios as covariance matrices based on factor models and shrinkage estimators. On the other hand, the effect of the upper bound constraint is compared to artificially increasing covariance among stocks that would reduce the weight they would have received otherwise. [DeMiguel et al., 2013] also focus on minimum variance portfolios, however, they try to improve out of sample performance by using option-implied information. Although they find option implied volatilities and correlations as better estimates than the historical ones, the out of sample results of the portfolios obtained from these estimates are unsatisfactory. The issue comes from the large variability of the implied estimates that leads to unsteady covariance matrices. Nevertheless, they find an alternative way of exploiting option-implied information in the portfolio optimization framework. Specifically, considering that securities having larger volatility risk premium, generally, outperform the low premium ones ([?]; [Goyal and Saretto, 2009]), they scale up (down) the volatilities of stocks having lowest (highest) volatility risk premium. In detail, assets with the highest volatility risk premium (top decile) are scaled to $\sigma(1-\delta)$, while securities from the bottom decile are scaled to $\sigma(1+\delta)$, where $\sigma$ stands for the volatility of the specific security, while $\delta$ represents the scaling factor. The $\delta$ is selected arbitrarily and they test the results for values ranging from 0.1 to 0.9 and conclude that minimum variance portfolios derived from volatility risk premium adjusted covariance matrix estimates perform better than in the non-adjusted case, no matter the size of the scaling factor. Inspired by these results, we use a similar framework for incorporating additional information into the optimization framework, where instead of the volatility risk premium we adjust the volatility of the stocks based on the momentum factor, as shown in section 4.6. Namely, the momentum implies that the assets that produced the highest returns in the past will continue to do so in the future. It is considered a relatively aggressive, return-driven strategy, which is why we find it interesting to inspect the potential benefits of combining it with a relatively defensive, minimum variance strategy. Nevertheless, in order for the combination to be successful, the momentum itself has to be a reliable strategy over the testing interval.

# B Data and Performance Metrics

As our first sample, we choose the stocks that have been constituents of the Dow Jones Index consistently from the beginning of 1999 until the end of 2019, which includes 19 stocks in our first sample. Afterward, based on the constituents of the SandP 500 at the end of 2019, we collect daily stock prices for the companies that have been listed prior to 1999, which totals 370 stocks. From the aggregate, we randomly create several subsamples:

- three samples of 50 stocks (non-overlapping),
- two samples of 150 stocks (non-overlapping),
- the sample including all 370 stocks.

By doing so, it is possible to see whether strategies manifest specific behavior for different sizes of the samples, or the results are robust no matter the size of the sample universe. According to [DeMiguel et al., 2009] the smaller the number of assets the less attractive naive diversification is relative to the optimization techniques due to the lower estimation error. This implies that increasing the number of sample constituents should generate a larger gap between naïve and optimized strategies, in favor of the first one. We test this hypothesis. The start and end date of our testing period are February, 1. 2000. and December, 12. 2019., respectively. We perform monthly rebalancing, assuming that one month has 21 trading days, which gives us in total 238 rebalancing intervals. Prior to the start of each interval, we calculate weights for every strategy based on the new estimates of covariance matrices. Finally, we also consider two ETF samples in order to challenge [DeMiguel et al., 2009] result which states that optimal diversification can outperform 1/N strategy only for very high levels of idiosyncratic volatility. Since ETFs diversify most of the underlying firm-specific risks, it is interesting to observe the behavior of our strategies over this asset class. The first ETF sample includes 12 sectors ETFs, provided by iShares and SPDR, while the second one incorporates all iShares US related ETFs that satisfy data requirements (30 funds). Due to the limited historical data, we had to move the starting date for a later period for both ETF samples, so that sector ETF testing starts on January, 2. 2003., while iShares US ETFs samples are analyzed starting from January, 2. 2002. Daily stock prices have been extracted from Yahoo Finance and immediately converted to continuously compounded returns. For every out of sample interval (21 days), we calculate portfolio daily return following the procedure recommended by Risk Metrics, where they assume that a portfolio return is equal to a weighted average of continuously compounded (log) returns:

$$r_{pt} = \sum_{i=1}^{N} w_i r_{it} \tag{6}$$

Therefore, in the case when we have 238 out of sample intervals, we have data about 4998 (238x21) portfolio daily returns. To evaluate every strategy's performance we use Sharpe ratio, assuming risk-free rate equal to zero:

$$SR = \frac{\mu_P}{\sigma_P} \tag{7}$$

where we calculate portfolio volatility as a standard deviation of portfolio daily returns. Having calculated the parameter, it is possible to rank portfolio strategies according to their respective risk-adjusted returns over almost 20 years of data. Besides, considering the performance for the whole period, we also distinguish between crisis and non-crisis intervals. Namely, according to [Yardeni, 2020] during the first two decades of the 21st century, there were two bear markets and six correction periods . They define corrections and bear markets as a decline in the SandP 500 of 10% or more, and 20% or more, respectively. The hypothesis is that optimized portfolios have a dominant performance during the crisis periods relative to 1/N strategy, since we aim to find minimum variance portfolios, that are also supposed to be the low beta portfolios.

Table 1: Crisis Intervals and Their Respective Length in Days

| Downward Markets | Length |
|---|---|
| March 2000 - October 2002 Crisis | 929 |
| November 2002 - March 2003 Market Correction | 104 |
| October 2007 - March 2009 Crisis | 517 |
| April 2010 - July 2010 Market Correction | 70 |
| April 2011 - October 2011 Market Correction | 157 |
| May 2015 - August 2015 Market Correction | 96 |
| November 2015 - February 2016 Market Correction | 100 |

# C Portfolio optimization problem; Minimizing variance

Harry Markowitz shows that a combination of various assets with appealing risk-return properties does not necessarily mean an optimal portfolio. The important aspect is the relationships among portfolio constituents. While we can define portfolio return as the weighted average return of the portfolio components, the same is not true for the portfolio risk (standard deviation). Namely, the crucial segment in risk determination is covariance among stocks, especially for the large portfolios where the risk can be defined as the total of the weighted covariances ([Reilly and Brown, 2011]). The formula for calculating portfolio standard deviation reduces to:

$$\sigma_P = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j cov_{ij}} \tag{8}$$

Alternatively, using matrix operations the same calculation can be performed in the following way:

$$\sigma_P = \sqrt{w^T \Sigma w} \tag{9}$$

where $\sum$ represents covariance matrix. Therefore, the effect of including new security in the portfolio depends on its average covariance with the assets already incorporated in the portfolio. Finding the minimum variance portfolio can be interpreted as an optimization problem, where we search for the weights that can minimize the above function subject to the constraint $w^T \hat{1} = 1$. Alternatively, it can be shown that the global minimum variance portfolio can be computed using the formula:

$$w_{GMVP} = \frac{\sum^{-1} \hat{1}}{\hat{1}^T \sum^{-1} \hat{1}} \tag{10}$$

where $\sum^{-1}$ stands for the inverse of covariance matrix, while $\hat{1}$ is a columns vector of ones (where the length of rows is equal to the number of securities). Consequently, in order to find the global minimum variance portfolio, we need to forecast the covariance matrix. The gap between estimated and true global minimum variance portfolio depends on how accurate is the covariance estimate. In the next section we introduce the covariance matrix estimators that will be used as a basis for calculating minimum variance portfolios.

# D Estimating covariance matrix

## D.1 Historical model

We use 252 most recent daily returns prior to the start of a new interval to calculate the historical covariance matrix that we assume will be the representative of the next month's covariance matrix. The estimate is used to build minimum variance portfolio $hist$ and minimum variance short-sale constrained portfolios $hist_C$ since the literature suggests that imposing portfolio constraints leads to the performance improvement ([Frost and Savarino, 1988]; [Gupta and Eichhorn, 1998]; [Grauer and Shen, 2000]; [Jagannathan and Ma, 2003]; [DeMiguel et al., 2009]; [Fabozzi et al., 2010]). Nevertheless, the problem occurs for our largest sample of 370 stocks, considering that the number of assets is larger than the length of data used for estimation. Consequently, this situation causes the optimization technique to assign extreme weights to certain securities which is why the literature recommends imposing more structure on the covariance matrix, using shrinkage technique or imposing non-negativity constraints. However, we choose an alternative approach where we simply run the constrained optimization having the lower and upper bounds equal to -1 and 1, respectively.

## D.2 Partially implied model

Our next model aims to use option-implied information and at the same time to resolve the covariance matrix instability issue identified by [DeMiguel et al., 2013]. In particular, we introduce a partially implied model $part_imp$, using Sharpe's single-factor model and option implied-volatility to estimate the covariance matrix. Namely, using the factor model we impose more structure and stability on the covariance matrix, while implied volatility offers option traders' estimate of the next month market volatility. Therefore, in order to create a covariance matrix for our sample universe, we have to make ex-ante estimates of securities' betas and unique risk, which is why we run regression analysis prior to each of our rebalancing dates. [Ledoit and Wolf, 2003], use ten years of monthly historical returns to estimate the parameters, which makes betas less responsive to the current situation. For instance, individual securities betas could increase once a company raises its leverage, however using several years of returns for making beta estimates prevents us from capturing this dynamic ([Elton et al., 2009]). Consequently, we increase data frequency and use shorter time intervals while estimating parameters. In detail, each regression analysis uses 60 days of historical stock returns, which is approximately three months of data. We try to balance between having enough data points to be able to derive solid estimates, and not relying on a too long historical interval to overcome the issue of parameters non-stationarity over time. In regards to market variance, our basis for estimation is VIX index. As described in CBOE (Chicago Board Options Exchange) White Paper, the VIX index represents model-free implied volatility and is based on a wide range of strike prices on the SandP 500 index's puts and calls. The literature suggests that model-free implied volatility (as in case of the VIX index) serves as a better estimate of the future volatility than the implied volatility derived from the Black-Scholes formula ([Jiang and Tian, 2005]; [Vanden, 2008]; [DeMiguel et al., 2013]). CBOE White Paper offers detailed computation of the VIX index, however, what is important for our study is how successful is the index in predicting the realized volatility. At each day, the index level shows market expectations of the volatility for the next 30 days. For instance, if today's value of the index is 15, the meaning is that expected volatility (standard deviation) of the SandP 500 index is 15%. It is documented that the VIX index tends to overestimate realized volatility because of the volatility risk premium. Namely, [Chernov, 2007] shows that this premium exists because implied volatilities are not forecasting objective volatilities, but risk-neutral volatilities. Therefore, we adjust the implied volatility by implementing a similar procedure as [DeMiguel et al., 2013]. To resolve this bias, the procedure is as follows:

- prior to each estimation date, we calculate the historical variance risk premium $HVRP^2$ for each of the past 12 months, dividing VIX implied variance by the realized variance:

$$HVRP_t^2 = VIX_t^2 / RV_t^2 \tag{11}$$

- then, we use the average HVRP over the past 12 months as an estimate of the volatility risk premium for the next one month. Therefore, we calculate the expected volatility for the next period as:

$$VIX_{adj} = \frac{VIX}{1/12 \sum_{i=1}^{12} \sqrt{HVRP^2}} \tag{12}$$

We test the predictive power of the VIX and adjusted VIX, additionally we include EWMA and GARCH (1, 1) models and historical one-month volatility as benchmarks. The computation procedure related to EWMA and GARCH (1, 1) model is available at Appendix A. For comparison purposes, we regress the realized monthly volatility on the respective estimates, over the whole period of interest, and calculate RMSE and R2 as a way of evaluating the models. We assume a regression equation where intercept is equal zero, and slope equals 1.

Table 2: Predicting Next Month Volatility; Models Comparison

| | RMSE | R2 |
|---|---|---|
| VIX | 0.074 | 0.444 |
| VIXadj | 0.065 | 0.565 |
| EWMA | 0.066 | 0.553 |
| GARCH | 0.068 | 0.524 |
| Historical | 0.071 | 0.477 |

As illustrated above we see that adjusting implied volatility for volatility risk premium leads to a large improvement, and overall, it is the best among the analyzed models in terms of both RMSE and R2. Nevertheless, it should be noted that the EWMA model comes immediately after with only slightly worse performance. Therefore, before we decide to use adjusted implied volatility $VIX_adj$ in the portfolio optimization context, we also check whether there is a benefit in combining it with EWMA. Namely, we aim to find w:

$$EV = w * VIX_{adj} + (1 - w) * EWMA \tag{13}$$

Finding w is an optimization problem, that we solve in the following fashion:

- we find the w that minimizes MSE for the five years before out of sample initial date,
- we apply the optimized w for the five years out of sample, and we repeat the procedure for the rest of the interval,
- as a final step we compare the performance of the combined model in estimating monthly volatility over the whole period:

| | | | |
|---|---|---|---|
| **Dep. Variable:** | S&P | **R-squared:** | 0.644 |
| **Model:** | OLS | **Adj. R-squared:** | 0.643 |
| **Method:** | Least Squares | **F-statistic:** | 427.5 |
| **Date:** | Mon, 05 Oct 2020 | **Prob (F-statistic):** | 6.90e-55 |
| **Time:** | 19:54:41 | **Log-Likelihood:** | 336.34 |
| **No. Observations:** | 238 | **AIC:** | -668.7 |
| **Df Residuals:** | 236 | **BIC:** | -661.7 |
| **Df Model:** | 1 | | |

| | **coef** | **std err** | **t** | **P> \|t\|** | **[0.025** | **0.975]** |
|---|---|---|---|---|---|---|
| **const** | 0.0093 | 0.008 | 1.153 | 0.250 | -0.007 | 0.025 |
| **EWMA_implied** | 0.9998 | 0.048 | 20.675 | 0.000 | 0.905 | 1.095 |

| | | | |
|---|---|---|---|
| **Omnibus:** | 109.360 | **Durbin-Watson:** | 1.653 |
| **Prob(Omnibus):** | 0.000 | **Jarque-Bera (JB):** | 867.201 |
| **Skew:** | 1.609 | **Prob(JB):** | 4.89e-189 |
| **Kurtosis:** | 11.780 | **Cond. No.** | 12.9 |

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

As shown above, combining two estimators leads to a non-negligible improvement in R2 (has a value of about 0.64). Similarly, there is a decrease in RMSE to approximately 0.059. Considering the dominant performance of the new model (in further text $EWMA_Implied$), we decide to utilize it in the portfolio optimization framework to estimate market variance. An interesting observation about the model is that despite we might expect implied volatility estimator to work particularly well during crisis periods (due to the expected faster information flow in the options markets), ex-post analysis showing the optimal weight during the crisis periods illustrates that EWMA model, despite being built only based on the recent historic data, often offers better predictions in times of uncertainty, implying information efficiency of spot market.

Table 3: Ex-post Optimal Weight During Crisis Periods

| Crisis | Weight |
|---|---|
| 2000.03. - 2002.09. | 0.558 |
| 2002.10. - 2003.03. | 0.339 |
| 2007.09. - 2009.03. | 0.364 |
| 2010.04. - 2010.07. | 0.355 |
| 2011.04. - 2011.09. | 0.003 |
| 2015.05. - 2015.08. | 0.005 |
| 2015.10. - 2016.02. | 0.004 |

## D.3 Zero correlation matrix

The next portfolio is zero correlation portfolio (zero), where we set all matrix elements, except the diagonal, equal zero. Empirical research demonstrates that this technique leads to more diversified portfolios which usually produce sound out of sample results ([DeMiguel et al., 2013]).

We set diagonal equal to the variances estimated by the partially implied model. The intuition for setting correlations among stocks equal to zero is that by doing so we are forcing the optimization process to diversify more than usual. Namely, as Harry Markowitz shows, the crucial segment of creating an optimal portfolio is the correlation among stocks. The more a certain stock is correlated with the existing stocks in the portfolio, the less likely is it that the optimization process will allocate the positive weight to the specific stock. Consequently, setting correlations artificially equal to zero will likely produce portfolios that are very similar to the 1/N strategy. The potential improvement over the 1/N model comes from accounting for the volatility when assigning the weight, which is why the zero-correlation portfolio will likely avoid securities having large idiosyncratic risk.

## D.4 Ledoit and Wolf covariance matrix

As described earlier, [Ledoit and Wolf, 2004] estimate has found a widespread application in practical portfolio management which is why we incorporate it in our study. Again, compared to the original study we increase data frequency and reduce the length of estimation interval to capture the time-varying dynamics of covariances and variances. Similar to the historical sample model, we use one year of historical daily data (252 trading days) no matter the sample size. The portfolio derived based on [Ledoit and Wolf, 2004] covariance matrix, will be referred to as LW.

## D.5 Portfolio of covariance matrices

Inspired by [Jagannathan and Ma, 2000] we introduce a portfolio of covariance matrices, where we take the equal weights in the partially implied model and in the [Ledoit and Wolf, 2004] covariance estimator, in order to test the potential benefit of diversifying among two highly structured estimates. We label the corresponding minimum variance portfolio as JM.

## D.6 Momentum Adjusted Covariance Matrix

As discussed earlier, [DeMiguel et al., 2013] show a simple way of incorporating additional information into a minimum variance framework and we adopt a similar procedure for incorporating momentum. For identifying momentum, we follow [Moskowitz and Daniel, 2016] approach, meaning that firstly we sort the stocks according to their cumulative returns from 252 days before to 21 days before our testing start. The reasoning for not including the data of the most recent 21 days is to circumvent the short-term reversal ([Jegadeesh, 1990]; [Lehmann, 1990]; [Moskowitz and Daniel, 2016]). Afterward, we adjust the volatilities of the stocks included in the top decile (the 'winners) and bottom decile (the 'losers') to $\sigma(1-\delta)$ and $\sigma(1+\delta)$, respectively. Again, similar to [DeMiguel et al., 2013] we test the results for $\delta$ taking values between 0.1 and 0.9. The intuition is that the optimization process will favor the securities having the lower volatility, which is the reason why we lower the volatility for the "winning" stocks. Having adjusted the covariance matrix estimate for the next period we rerun optimization to generate new portfolio weights. Nevertheless, we do not find the benefit of including momentum in our main strategies. Despite some periods it results in larger returns, it always leads to a large increase in portfolio volatility. In order to examine the reason behind it, we also run momentum only strategy, where we take a long position in the 'winning' stocks and short the 'losers'. Backtesting shows that momentum does not create a positive return in all samples considered, also even when it does, it is accompanied by large volatility, especially for the larger samples. Therefore, the effect momentum has on our portfolios does not come as a surprise. This is consistent with [Barroso and Santa-Clara, 2015], who show that the 2000s have been one of the most chaotic decades for the momentum strategy. Also, we have observed that the stocks in the top and bottom decile, generally, have larger volatilities than the other deciles. Moreover, the stocks within the same decile are likely to have high correlation which is why there are no large diversification benefits of holding them as a part of a portfolio. Since there are no evident benefits of combining momentum with portfolio optimization, we exclude the results from our report. However, as a final consideration, it could be interesting to test this strategy for some emerging markets where momentum is relevant even today, or for the US during the 20th century.

# E Empirical results

## E.1 Whole period

In regards to the whole period out of sample performance, relative to up to date academic research, our analysis generates surprising results. Namely, 1/N strategy is the worst option among analyzed strategies (excluding momentum adjusted portfolios) across all nine samples, which is contradicting the analysis offered by [DeMiguel et al., 2009] who do not find an optimized strategy that is able to consistently outperform equally weighted portfolio. Moreover, our results are not sensitive to the number of securities nor to the level of idiosyncratic risk. While [DeMiguel et al., 2009] suggest that the larger the number of assets the more attractive is 1/N strategy relative to the optimized models, our findings illustrate that increasing the sample universe can even lead to the more dominant performance of optimized strategies. Also, we prove that optimized portfolios or at least minimum variance portfolios can beat an equally weighted strategy for the samples of low idiosyncratic volatility. Among optimized strategies, minimum variance portfolio based on the portfolio of covariance matrices (JM) appears as the best choice, having the largest Sharpe ratio in six out of nine samples. The good performance was primarily caused by the low risk, considering that in each sample JM portfolio was the lowest volatile strategy. This result is particularly important from the perspective of portfolio and risk management, especially taking into account the robustness to different sample universes. It suggests that further research in portfolios of covariance matrices could yield significant results for both academia and practitioners. On the other hand, the naive model returns generated the largest standard deviation across nine samples included in the study, however, as illustrated by the Sharpe ratio the risk was not compensated properly. Further, we see that despite the criticism, portfolios based on historical samples had a solid performance in each sample. Surprisingly, in most cases it even had the largest return, showing that minimum variance portfolios can dominate the naive rule both in terms of return and risk. Interestingly, despite [Jagannathan and Ma, 2003] and [DeMiguel et al., 2009] propose using short-sale constraints, our research illustrates that imposing constraints, generally, results in a lower out of sample performance. The peculiar result can be attributed to the low sampling error of our estimates. Namely, imposing constraints leads to a jump in specification error while reducing the sampling error ([Jagannathan and Ma, 2003]), however, in our case, the benefits of the sampling error reduction are more than offset by the increase in the specification error, which is why constrained portfolios tend to underperform non-constrained ones. In case of a larger estimation error, as in the case of mean-variance portfolios, it is advisable to restrict the optimization techniques. Table 5 illustrates Sharpe ratio, returns and volatilities for the respective strategies, while Figure 1 shows each strategy's hypothetical performance, assuming that we entered the market in the early 2000s (2003, 2002, for the sector ETFs and iShares ETFs, respectively) and followed particular strategies (with monthly rebalancing) until the end of 2019. The hypothetical investor in early 2000, would do the worst choosing to follow the equally weighted strategy for the next two decades. The absolute portfolio value of this approach is depicted by the blue line, which in most cases is the bottom one.

Table 4: Whole Period Performance

| **Panel A: Sharpe Ratio** | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 0.41 | 0.50 | 0.52 | 0.48 | 0.50 | 0.57 | 0.53 | 0.54 | 0.40 |
| hist | 0.44 | 1.24 | 0.90 | 0.86 | 0.79 | 1.16 | 1.15 | 0.90 | 0.86 |
| histC | 0.47 | 0.85 | 0.74 | 0.78 | 0.94 | 0.95 | 0.93 | 0.78 | 0.61 |
| partImp | 0.50 | 1.11 | 0.75 | 0.80 | 1.10 | 0.97 | 1.04 | 0.84 | 0.70 |
| zero | 0.45 | 0.67 | 0.61 | 0.64 | 0.65 | 0.70 | 0.67 | 0.63 | 0.45 |
| LW | 0.46 | 1.20 | 0.86 | 0.83 | 1.15 | 1.26 | 1.30 | 0.91 | 0.86 |
| JM | 0.50 | 1.16 | 0.84 | 0.88 | 1.25 | 1.23 | 1.40 | 0.92 | 0.88 |
| **Panel B: Annualized Returns** | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
| 1/N | 7.59 | 10.30 | 9.92 | 9.31 | 10.03 | 11.15 | 10.31 | 9.62 | 7.62 |
| hist | 6.06 | 16.44 | 11.82 | 11.39 | 11.25 | 17.18 | 15.68 | 10.69 | 10.82 |
| histC | 6.56 | 11.49 | 9.56 | 10.07 | 11.32 | 11.37 | 10.72 | 9.88 | 8.50 |
| partImp | 6.73 | 14.20 | 9.54 | 11.08 | 12.54 | 11.48 | 12.15 | 10.27 | 9.91 |
| zero | 7.11 | 11.06 | 9.66 | 10.07 | 10.37 | 11.22 | 10.64 | 9.79 | 8.07 |
| LW | 6.34 | 15.47 | 11.07 | 10.65 | 13.23 | 14.78 | 14.04 | 10.69 | 10.67 |
| JM | 6.61 | 14.24 | 10.27 | 10.87 | 12.85 | 12.92 | 13.10 | 10.65 | 10.81 |
| **Panel C: Annualized Volatilities** | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
| 1/N | 18.36 | 20.56 | 18.98 | 19.31 | 20.04 | 19.46 | 19.49 | 17.78 | 19.13 |
| hist | 13.91 | 13.28 | 13.15 | 13.21 | 14.29 | 14.84 | 13.66 | 11.82 | 12.64 |
| histC | 14.04 | 13.50 | 12.85 | 12.92 | 12.06 | 11.98 | 11.48 | 12.60 | 13.95 |
| partImp | 13.50 | 12.76 | 12.65 | 13.92 | 11.43 | 11.85 | 11.67 | 12.24 | 14.13 |
| zero | 15.94 | 16.41 | 15.74 | 15.80 | 16.06 | 15.92 | 15.77 | 15.52 | 17.98 |
| LW | 13.77 | 12.89 | 12.84 | 12.85 | 11.53 | 11.77 | 10.80 | 11.76 | 12.42 |
| JM | 13.27 | 12.25 | 12.17 | 12.33 | 10.28 | 10.49 | 9.36 | 11.64 | 12.26 |

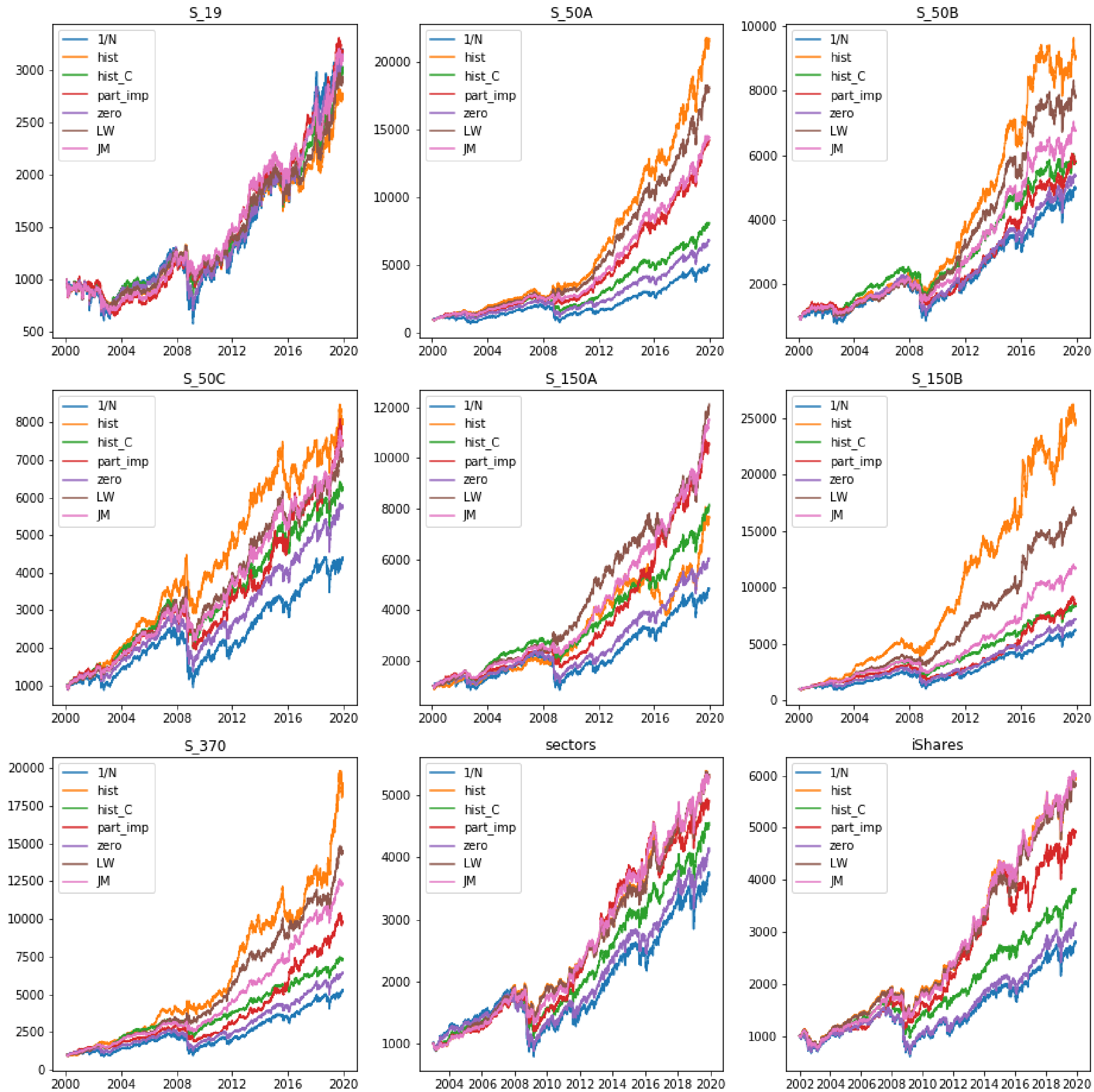


Figure 1: Hypothetical Portfolio Performance Assuming Investment of USD1000 at The Initial Date; Whole Period Model Comparison

## E.2 Segmented analysis

The further analysis aims to examine whether results are dependent on a specific market environment, particular interest is to determine the effect of downward/upward market on individual strategies performance. As explained earlier, within our sample span we identify seven downward markets (as per [Yardeni, 2020]). Portfolios' daily returns within these crisis intervals are extracted and combined for the separate analysis. Specifically, the procedure is the same as for the whole period, the only difference being the time interval. As expected, the bear markets are the difference maker in favor of the minimum variance – low beta portfolios.

On the one hand, the equally weighted model produces a large negative return during crisis intervals, which creates a nonpositive Sharpe ratio across all samples. On the other hand, our minimum variance portfolios (except for the zero-correlation portfolio) were even able to produce positive returns for several samples. When it comes to zero correlation model, it is apparent that setting correlation equal to zero leads to weight allocation very similar to the 1/N rule, since both portfolios consistently demonstrate similar patterns. In respect of the Sharpe ratio, the historical model was able to produce the highest Sharpe ratio in four out of seven individual stocks samples, while JM portfolio generated the largest Sharpe ratio for both ETF samples. Intuitively, short sale constraint during bear markets has an adverse effect which is observed in the difference between historical constrained and non-constrained portfolio. When it comes to portfolio volatility, jump in volatility during the periods of uncertainty is evident for every model, however, JM model was again the least volatile strategy in each of the samples, while the opposite is true for 1/N rule. Figure 2 shows the scenario where we combine all seven crisis period as if they had occurred one after another. Similarly to Figure 1, we assume investment of USD1000 in each of the respective strategies and observe the hypothetical movement in portfolio value on a daily basis. The first seven samples include approximately 1200 days (x-axis), while sector ETF and iShares ETF samples have about 600 and 800 days, respectively. The lines represent the absolute value of the theoretical portfolios, while the relative performance of the models can be observed through the reduction/increase in the gap between particular lines. Generally, the gap between optimized portfolios and naive strategy is almost constantly increasing, except in the case of zero correlation portfolio which manifests a similar pattern as the equally weighted portfolio. For instance, the chart associated with the largest sample (370 stocks), demonstrates a giant gap between the orange line representing historical minimum variance strategy and the blue line portraying the movement of the naive portfolio. In about 1200 trading days, the first one moves from USD1000 to more than USD2000, while the 1/N strategy shrinks from USD1000 to about USD250. Appendix B illustrates the Sharpe ratio, returns, and volatilities for the seven downward markets separately, while Appendix C provides the illustration of the hypothetical portfolios' performances during the uncombined crisis intervals. As for non-crisis periods, the results are quite different. We combine all periods that are not classified as a part of the crisis intervals, which is about 75% of data and again we ran the backtesting algorithm in order to compare particular models. There is no model that dominates others according to the risk-adjusted metrics. JM portfolio managed to generate the highest Sharpe ratio in four periods, nevertheless, minimum variance portfolios weren't able to outperform the 1/N rule on a consistent basis, since the naive strategy produced the highest Sharpe for the sample of 19 stocks. Again, it appears that the larger the number of stock the better are minimum variance strategies relative to the 1/N rule. However, in terms of generating return, equally weighted strategy is unequivocally the best choice since it yielded the largest return in all samples considered. As during the bear markets, the zero correlation model follows a similar pattern as the 1/N strategy, having the second-best return in all observed sample universes. On the issue of risk, JM model was once more the lowest volatility portfolio, regardless of the sample. Figure 3 illustrates the hypothetical portfolios performances over combined non-crisis periods, and regardless of the sample, we are able to notice the similar pattern produced by the respective models. There are about 4000 and 3500 non-crisis trading days for the SandP 500 stocks samples and ETF samples, respectively. The segmented analysis of crisis and non-crisis markets supports the assumption that minimum variance portfolios are low-beta portfolios, while the formal analysis in Table 8 confirms it. The table shows that the partially implied model is able to produce the lowest portfolio beta in eight of nine analyzed samples.

Table 5: Crisis Periods Performance

**Panel A: Sharpe Ratio**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | -0.97 | -0.94 | -0.82 | -0.79 | -0.88 | -0.71 | -0.80 | -1.27 | -1.41 |
| hist | -0.37 | 0.93 | 0.03 | 0.35 | 0.49 | 1.10 | 1.02 | -0.27 | -0.58 |
| histC | -0.60 | -0.24 | -0.26 | -0.10 | -0.19 | 0.15 | 0.01 | -0.78 | -0.99 |
| partImp | -0.15 | 0.46 | -0.03 | 0.41 | 0.44 | 0.41 | 0.56 | -0.21 | -0.61 |
| zero | -0.86 | -0.66 | -0.74 | -0.51 | -0.71 | -0.54 | -0.60 | -1.20 | -1.39 |
| LW | -0.39 | 0.77 | -0.08 | 0.13 | 0.94 | 0.85 | 0.93 | -0.31 | -0.61 |
| JM | -0.29 | 0.63 | -0.04 | 0.29 | 0.69 | 0.60 | 0.78 | -0.20 | -0.56 |

**Panel B: Annualized Returns**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | -25.66 | -27.88 | -22.65 | -21.83 | -25.42 | -19.84 | -22.48 | -39.68 | -43.78 |
| hist | -7.16 | 16.08 | 0.49 | 6.09 | 8.96 | 20.05 | 17.43 | -4.97 | -11.49 |
| histC | -12.01 | -4.26 | -4.51 | -1.76 | -3.12 | 2.42 | 0.20 | -16.33 | -22.04 |
| partImp | -2.70 | 7.61 | -0.51 | 7.45 | 6.51 | 6.24 | 8.39 | -4.11 | -12.93 |
| zero | -19.59 | -15.27 | -16.46 | -11.44 | -15.94 | -12.17 | -13.39 | -32.30 | -40.66 |
| LW | -7.49 | 13.11 | -1.31 | 2.25 | 14.14 | 12.89 | 13.02 | -5.84 | -11.86 |
| JM | -5.38 | 10.03 | -0.68 | 4.73 | 9.29 | 8.18 | 9.61 | -3.63 | -10.66 |

**Panel C: Annualized Volatilities**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 26.44 | 29.59 | 27.46 | 27.56 | 28.75 | 28.05 | 27.99 | 31.18 | 30.99 |
| hist | 19.30 | 17.36 | 17.19 | 17.47 | 18.11 | 18.20 | 17.07 | 18.59 | 19.66 |
| histC | 20.04 | 18.13 | 17.23 | 17.42 | 16.28 | 16.17 | 15.49 | 20.93 | 22.21 |
| partImp | 18.44 | 16.44 | 16.58 | 18.01 | 14.91 | 15.14 | 15.04 | 19.28 | 21.04 |
| zero | 22.80 | 22.98 | 22.21 | 22.38 | 22.54 | 22.54 | 22.29 | 27.00 | 29.21 |
| LW | 19.26 | 17.05 | 17.27 | 17.22 | 15.11 | 15.18 | 14.03 | 18.62 | 19.34 |
| JM | 18.33 | 15.83 | 16.12 | 16.32 | 13.52 | 13.67 | 12.33 | 18.42 | 18.89 |

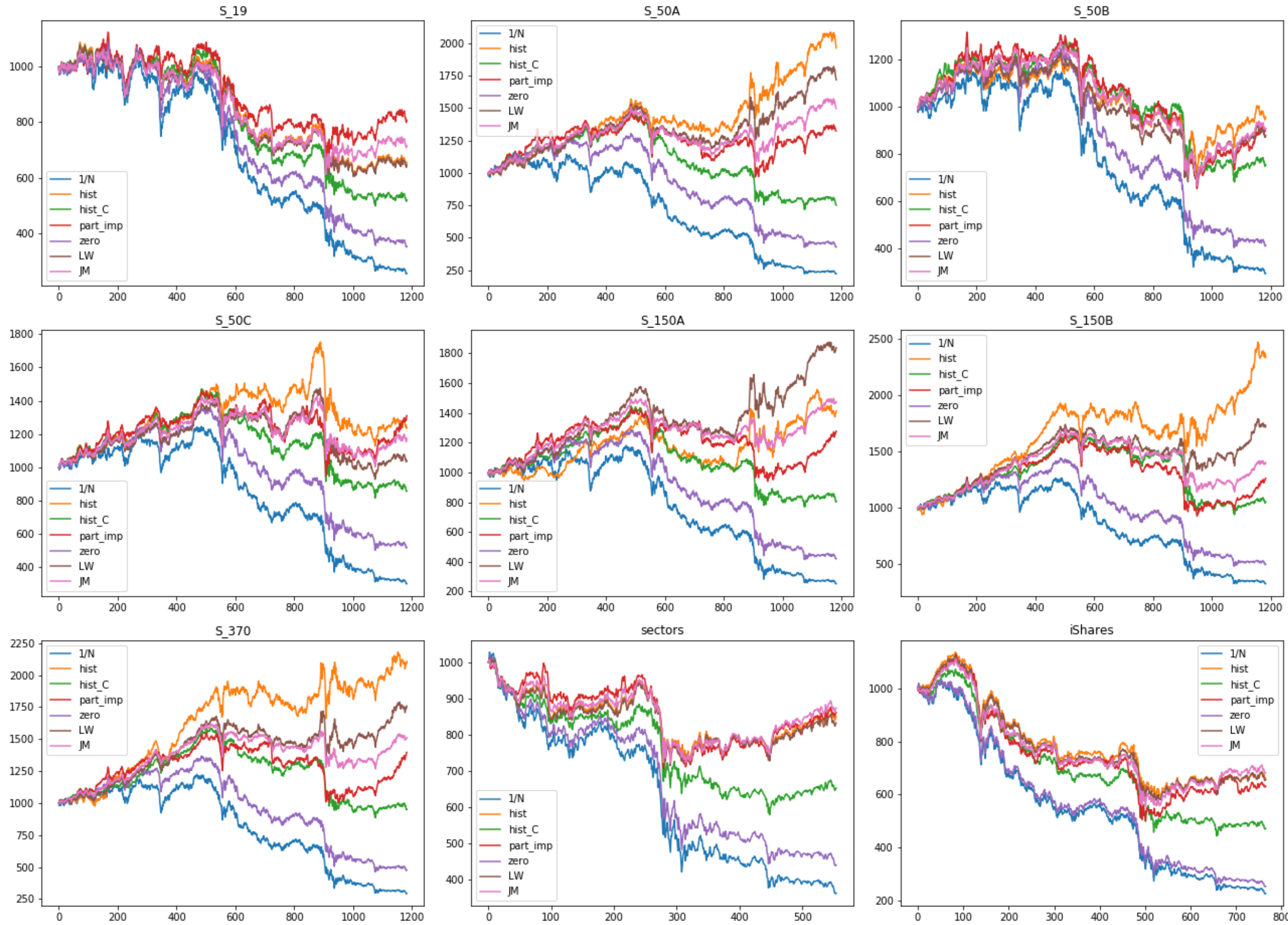


Figure 2: Hypothetical Portfolio Performance Assuming Investment of USD1000 at The Initial Date; Aggregate Crisis Intervals Model Comparison

Table 6: Non-Crisis Periods Performance

**Panel A: Sharpe Ratio**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 1.20 | 1.32 | 1.30 | 1.20 | 1.28 | 1.31 | 1.29 | 1.15 | 1.17 |
| hist | 0.87 | 1.41 | 1.32 | 1.13 | 0.93 | 1.20 | 1.22 | 1.25 | 1.45 |
| histC | 1.07 | 1.40 | 1.25 | 1.23 | 1.52 | 1.37 | 1.41 | 1.28 | 1.28 |
| partImp | 0.84 | 1.43 | 1.14 | 0.99 | 1.43 | 1.23 | 1.28 | 1.15 | 1.19 |
| zero | 1.18 | 1.40 | 1.36 | 1.28 | 1.38 | 1.40 | 1.38 | 1.25 | 1.24 |
| LW | 0.92 | 1.44 | 1.34 | 1.19 | 1.27 | 1.46 | 1.50 | 1.27 | 1.46 |
| JM | 0.92 | 1.43 | 1.28 | 1.18 | 1.55 | 1.55 | 1.72 | 1.25 | 1.47 |

**Panel B: Annualized Returns**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 17.90 | 22.14 | 20.01 | 18.97 | 21.02 | 20.75 | 20.48 | 16.99 | 18.13 |
| hist | 10.16 | 16.55 | 15.33 | 13.04 | 11.96 | 16.29 | 15.14 | 13.03 | 15.38 |
| histC | 12.31 | 16.38 | 13.92 | 13.74 | 15.80 | 14.15 | 13.98 | 13.80 | 14.74 |
| partImp | 9.65 | 16.24 | 12.66 | 12.20 | 14.41 | 13.10 | 13.32 | 12.42 | 14.57 |
| zero | 15.39 | 19.22 | 17.76 | 16.74 | 18.53 | 18.47 | 18.09 | 16.09 | 18.02 |
| LW | 10.62 | 16.21 | 14.91 | 13.26 | 12.94 | 15.37 | 14.35 | 13.17 | 15.27 |
| JM | 10.33 | 15.54 | 13.66 | 12.77 | 13.96 | 14.39 | 14.18 | 12.79 | 15.20 |

**Panel C: Annualized Volatilities**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 14.94 | 16.74 | 15.39 | 15.86 | 16.37 | 15.83 | 15.91 | 14.72 | 15.56 |
| hist | 11.73 | 11.73 | 11.61 | 11.58 | 12.87 | 13.63 | 12.41 | 10.43 | 10.64 |
| histC | 11.54 | 11.67 | 11.14 | 11.16 | 10.40 | 10.34 | 9.91 | 10.80 | 11.52 |
| partImp | 11.55 | 11.37 | 11.14 | 12.38 | 10.11 | 10.62 | 10.41 | 10.79 | 12.24 |
| zero | 13.06 | 13.72 | 13.07 | 13.08 | 13.39 | 13.18 | 13.08 | 12.91 | 14.59 |
| LW | 11.55 | 11.29 | 11.11 | 11.15 | 10.18 | 10.50 | 9.58 | 10.35 | 10.43 |
| JM | 11.24 | 10.90 | 10.64 | 10.80 | 9.04 | 9.28 | 8.22 | 10.25 | 10.37 |

Table 7: Portfolio Betas; Whole Period

| | 1/N | hist | histC | partImp | zero | LW | JM |
|---|---|---|---|---|---|---|---|
| S19 | 0.93 | 0.47 | 0.59 | 0.38 | 0.78 | 0.49 | 0.43 |
| S50A | 1.05 | 0.33 | 0.55 | 0.30 | 0.82 | 0.38 | 0.34 |
| S50B | 0.97 | 0.35 | 0.52 | 0.31 | 0.78 | 0.39 | 0.35 |
| S50C | 0.97 | 0.37 | 0.51 | 0.30 | 0.77 | 0.41 | 0.37 |
| S150A | 1.03 | 0.23 | 0.50 | 0.24 | 0.81 | 0.27 | 0.26 |
| S150B | 1.00 | 0.24 | 0.49 | 0.23 | 0.80 | 0.28 | 0.27 |
| S370 | 1.00 | 0.21 | 0.47 | 0.21 | 0.80 | 0.25 | 0.24 |
| sectors | 0.97 | 0.41 | 0.59 | 0.37 | 0.84 | 0.43 | 0.40 |
| iShares | 1.01 | 0.41 | 0.67 | 0.34 | 0.96 | 0.44 | 0.39 |

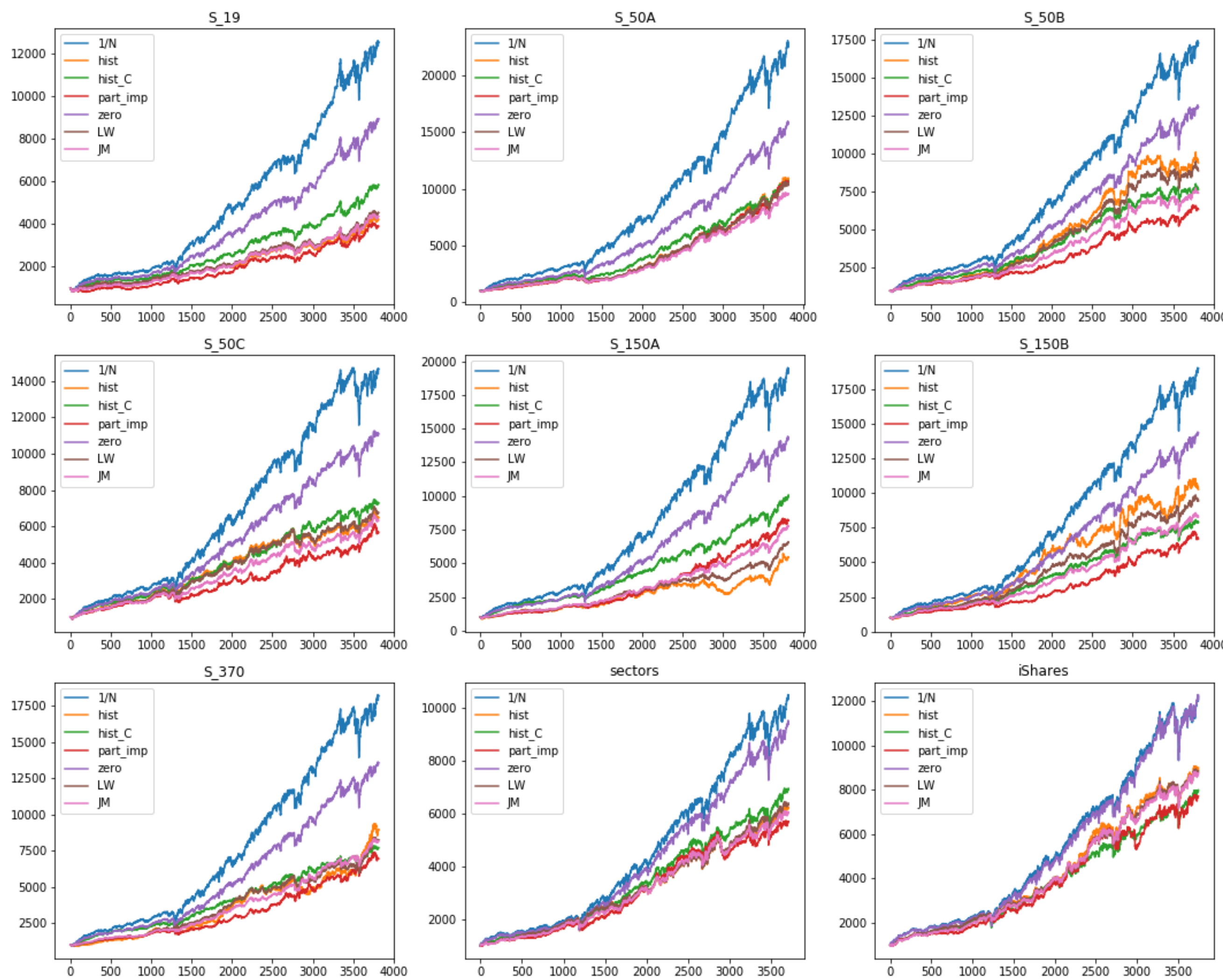


Figure 3: Hypothetical Portfolio Performance Assuming Investment of USD1000 at The Initial Date; Aggregate Non-Crisis Intervals Model Comparison

## E.3 Comparison with iShares Edge MSCI Min Vol USA ETF - USMV

USMV represents an ETF tracking MSCI Minimum Volatility Index, which is also based on Markowitz's minimum variance portfolio. As of 31st of March, 2020 the value of net assets under management exceeds USD30 billion. According to the official document, the minimum volatility portfolio is computed by running the optimization problem on the grounds of the estimated covariance matrix, where the estimate relies on the Barra multi-factor model. Considering that the ETF was introduced in 2011, we compare its standard deviation to the standard deviation of our JM model (sample 370 stocks), starting from 2012 until the end of 2019. However, it should be noted that the difference in portfolios' volatilities cannot be apriori attributed to the quality of the covariance matrix estimate, since the index imposes certain constraints that our model does not take into consideration. Specifically, they impose upper and lower bounds, and constrain the optimization technique to ensure geographical and sector diversification under the condition of a limited turnover . Additionaly, while our JM portfolio relies on the sample with 370 stocks, ETF investment universe relies on the 636 constituents of the MSCI USA Index. Since we have observed that the larger the sample, the lower the volatility generated by our portfolios, it is necessary to account for this dissimilarities when analyzing the results. Despite the apparent differences between the models, Figure 4 illustrating daily return flucutations can be used as a reference. As shown, the USMV ETF tends to dominate our JM portfolio in terms of return volatility. More formally, over the period the annualized standard deviation was 10.25% and 8.45%, respectively. When it comes to annualized return over the period of interest, USMV ETF and JM portfolio, have generated 13.14% and 12.57%, respectively. Consequently, the models have Sharpe Ratio of 1.28 and 1.49, respectively. As for the illustration, Figure 5 depicts the pattern exibited by USMV ETF and our hypothetical JM portfolio.


Figure 4: Return Fluctuation: USMV vs JM

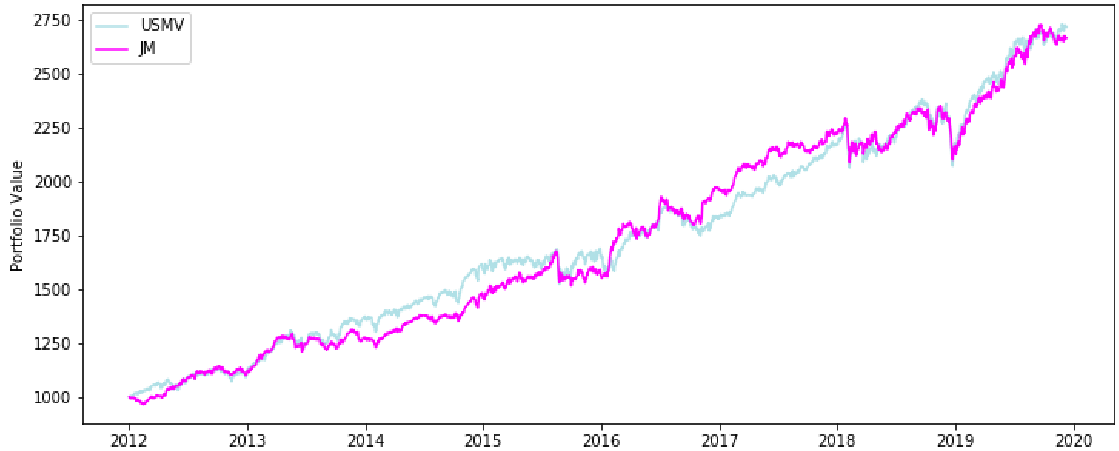


Figure 5: USMV vs JM Portfolio Performance Assuming Investment of USD1000 at The Initial Date

# F Introducing new strategies

After examining the influence of the market environment on particular models, the next step is to try to utilize those characteristics in an investment strategy. However, in order for the strategy to be practically implementable, we have to determine the market environment (bull or bear market) ex-ante. For this purpose, we go back to the VIX index. Namely, despite VIX index itself doesn't offer the most accurate estimate of the future volatility, it is widely used as a measurement of the market sentiment, which is why it has earned a reputation of a 'fear index'. As shown in Figure 6 during crisis intervals VIX level was almost constantly above 20 (the yellow line), implying that high VIX levels and crisis periods often occur simultaneously, or in other words that the 'fear' is justified. Consequently, our aim is to use certain values of the index as a triggering point for aggressive/defensive tactics. Considering that until now, our models were focused primarily on minimizing the volatility, now we try to improve its returns properties, and test whether the additional return could be legitimized on a risk-adjusted basis. Firstly, we observe VIX index deciles prior to the start of our tests (1992-2000) with the intention to use the observed values as an indicator of the market environment in the future. Specifically, 25.73 has been a threshold to the top decile during the historical sample, meaning than in only 10% of cases VIX index has been beyond this value. Therefore, we consider the VIX value above this level as an indicator of large market risk. Likewise, VIX values between the 9th and 4th decile of historical data will be considered as a signal of medium risk, while the values below the 4th decile characterize the low-risk environment. Considering that minimum variance models have a solid performance during the crisis, while equally weighted strategy generates large losses during these periods, we favorize the first ones during the high-risk environments. In parallel, in terms of returns 1/N rule dominates all other strategies during the non-crisis periods, which is why we increase the exposure to this strategy for the lower risk levels. Our weight adjustment scheme between the portfolios (sum of weights always equals 1 – net long exposure) is following the form below:

- high risk – short 1/N (-1), long minimum variance (2)
- medium risk – long 1/N (0.5), long minimum variance (0.5)

- low risk – long 1/N (2), short minimum variance (-1).

We report combinations of 1/N with historical minimum variance (gray line) and JM model (yellow line). As demonstrated in Table 9 the new strategies always lead to a remarkable increase in return, especially considering the potential for compounding effect over a long-term period. The results are similar if we choose a more gradual approach or decide to move the thresholds in a reasonable range (e.g. high-risk 8th decile instead of 9th decile), however, we report values only for the simple weighting form described above. Return increment is particularly apparent in Figure 7, for the charts representing a larger number of stocks. For example, the hypothetical investment of USD1000 at the beginning of 2000 would increase about 80 times until the end of 2019, for the second sample including 150 stocks ($sample_150B$). As a less extreme example, in the case of 370 stocks, we see an increase of initial investment of about 35 times for the period of approximately 20 years. Nevertheless, consistently with our earlier discussion our main measurement of risk-adjusted return is Sharpe ratio. The new strategies are not able to beat the old ones in terms of this parameter (except for the first sample of 19 stocks). In order to control the volatility, one could use additional technical indicators for determining market character, a smoother weighting procedure and more sophisticated extensions. However, a reader should have in mind the potential issues with the metrics itself, especially having a long-term investment horizon in mind. For instance, hardly any investor would be immune to a portfolio having properties such as the one depicted by the gray line for the second sample of 150 stocks, however, following the Sharpe ratio investor would never choose that portfolio over the simple historical minimum variance strategy. The Sharpe ratio does not distinguish between desirable and undesirable volatility. Namely, the large upward movement is treated the same as the large downward return. The solution to the issue could be to use the Sortino ratio which adjusts the returns only for the downside risk. However, at this point, we continue using the Sharpe ratio since it is more broadly used in academic research.

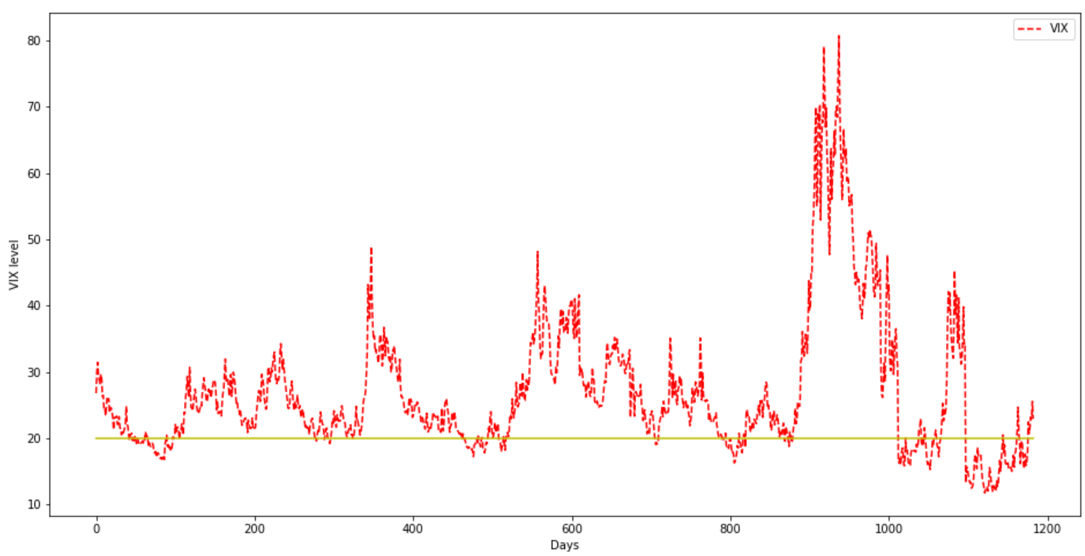


Figure 6: VIX During Crisis Periods

Table 8: Whole Period Including New Strategies

**Panel A: Sharpe Ratio**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 0.41 | 0.50 | 0.52 | 0.48 | 0.50 | 0.57 | 0.53 | 0.54 | 0.40 |
| hist | 0.44 | 1.24 | 0.90 | 0.86 | 0.79 | 1.16 | 1.15 | 0.90 | 0.86 |
| histC | 0.47 | 0.85 | 0.74 | 0.78 | 0.94 | 0.95 | 0.93 | 0.78 | 0.61 |
| partImp | 0.50 | 1.11 | 0.75 | 0.80 | 1.10 | 0.97 | 1.04 | 0.84 | 0.70 |
| zero | 0.45 | 0.67 | 0.61 | 0.64 | 0.65 | 0.70 | 0.67 | 0.63 | 0.45 |
| LW | 0.46 | 1.20 | 0.86 | 0.83 | 1.15 | 1.26 | 1.30 | 0.91 | 0.86 |
| JM | 0.50 | 1.16 | 0.84 | 0.88 | 1.25 | 1.23 | 1.40 | 0.92 | 0.88 |
| histMix | 0.56 | 0.97 | 0.86 | 0.78 | 0.75 | 1.04 | 0.88 | 0.80 | 0.58 |
| JMMix | 0.60 | 0.86 | 0.80 | 0.81 | 0.88 | 0.97 | 0.96 | 0.83 | 0.65 |

**Panel B: Annualized Returns**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 7.59 | 10.30 | 9.92 | 9.31 | 10.03 | 11.15 | 10.31 | 9.62 | 7.62 |
| hist | 6.06 | 16.44 | 11.82 | 11.39 | 11.25 | 17.18 | 15.68 | 10.69 | 10.82 |
| histC | 6.56 | 11.49 | 9.56 | 10.07 | 11.32 | 11.37 | 10.72 | 9.88 | 8.50 |
| partImp | 6.73 | 14.20 | 9.54 | 11.08 | 12.54 | 11.48 | 12.15 | 10.27 | 9.91 |
| zero | 7.11 | 11.06 | 9.66 | 10.07 | 10.37 | 11.22 | 10.64 | 9.79 | 8.07 |
| LW | 6.34 | 15.47 | 11.07 | 10.65 | 13.23 | 14.78 | 14.04 | 10.69 | 10.67 |
| JM | 6.61 | 14.24 | 10.27 | 10.87 | 12.85 | 12.92 | 13.10 | 10.65 | 10.81 |
| histMix | 10.08 | 21.46 | 17.24 | 16.11 | 18.20 | 25.17 | 20.42 | 14.10 | 11.77 |
| JMMix | 10.96 | 18.14 | 15.18 | 15.74 | 17.70 | 18.81 | 18.13 | 14.61 | 13.23 |

**Panel C: Annualized Volatilities**

| | S19 | S50A | S50B | S50C | S150A | S150B | S370 | sectors | iShares |
|---|---|---|---|---|---|---|---|---|---|
| 1/N | 18.36 | 20.56 | 18.98 | 19.31 | 20.04 | 19.46 | 19.49 | 17.78 | 19.13 |
| hist | 13.91 | 13.28 | 13.15 | 13.21 | 14.29 | 14.84 | 13.66 | 11.82 | 12.64 |
| histC | 14.04 | 13.50 | 12.85 | 12.92 | 12.06 | 11.98 | 11.48 | 12.60 | 13.95 |
| partImp | 13.50 | 12.76 | 12.65 | 13.92 | 11.43 | 11.85 | 11.67 | 12.24 | 14.13 |
| zero | 15.94 | 16.41 | 15.74 | 15.80 | 16.06 | 15.92 | 15.77 | 15.52 | 17.98 |
| LW | 13.77 | 12.89 | 12.84 | 12.85 | 11.53 | 11.77 | 10.80 | 11.76 | 12.42 |
| JM | 13.27 | 12.25 | 12.17 | 12.33 | 10.28 | 10.49 | 9.36 | 11.64 | 12.26 |
| histMix | 18.11 | 22.18 | 19.96 | 20.60 | 24.41 | 24.19 | 23.11 | 17.59 | 20.18 |
| JMMix | 18.17 | 21.01 | 18.87 | 19.40 | 20.13 | 19.32 | 18.89 | 17.51 | 20.28 |

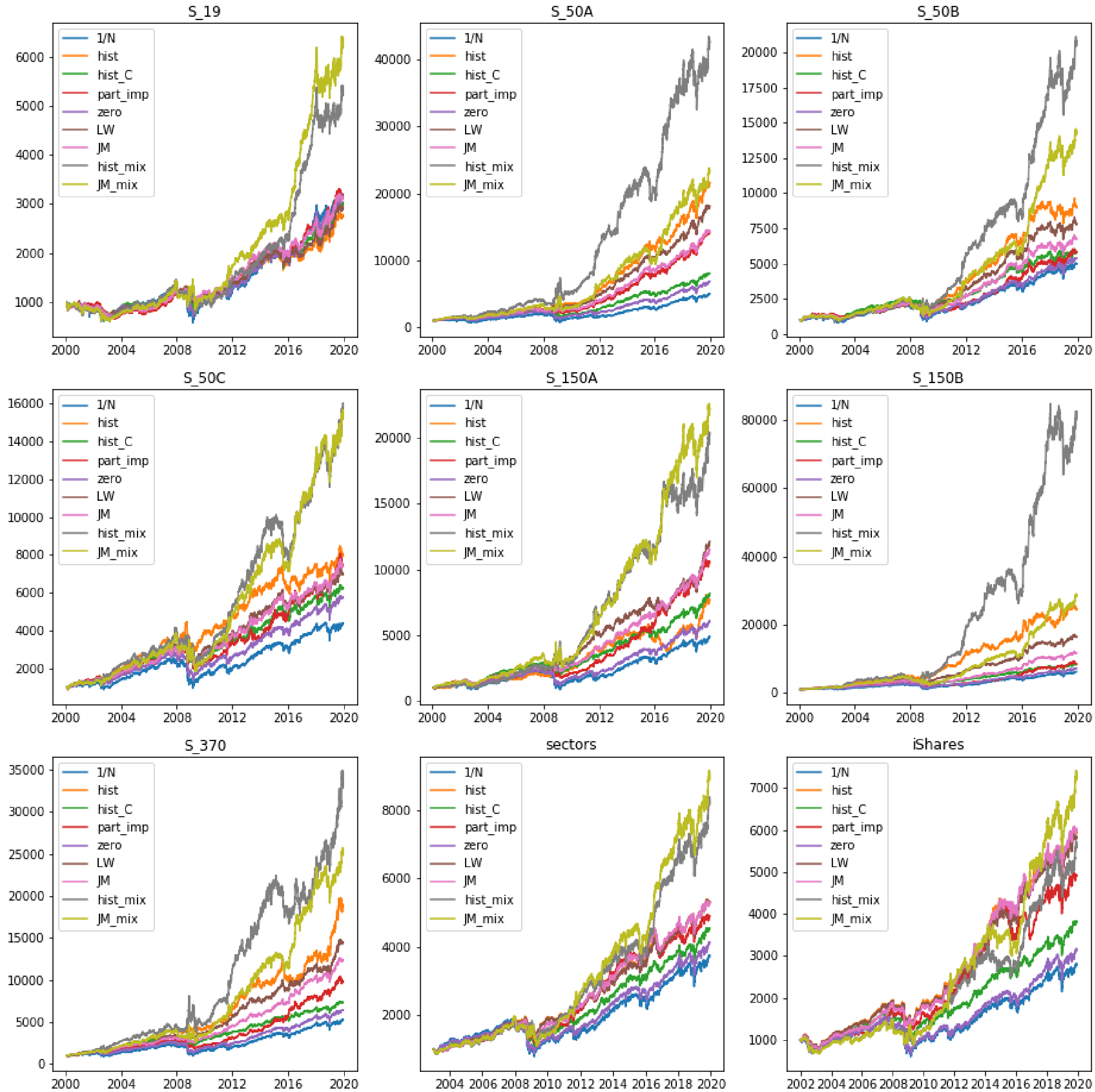


Figure 7: Hypothetical Portfolio Performance Assuming Investment of USD1000 at The Initial Date; Whole Period Model Comparison Including New Strategies

# G Limitations

Several trading constraints would prevent our theoretical portfolios to achieve the same performance as illustrated in the previous sections. Firstly, we assume that the counterparties for the respective trades are always available to trade at the price, quantity and time we desire. However, due to the price movement, the deal can be settled at a different price than originally intended, whether it be due to the actions of other market participants or the price impact of our investment. Additionally, our order might be fulfilled only partially.

For instance, supposing that optimization procedure requests a purchase of 1000 shares of company A, however, we might be able to buy only 500 shares immediately, while gradually by the end of the day we could execute 90% of the aggregate order. Furthermore, our analysis does not account for the trading cost (commissions), which would lower the returns of each strategy under consideration and affect the relative performance among the models by punishing those having larger weights fluctuations. Also, most of our portfolios include short positions, however, we do not discuss leverage problems, that might trigger margin calls (in case there is a large increase in the price of the shorted security) and unable us to proceed with a certain strategy. Finally, there is a slight survivorship bias, in particular if we were to perform optimization in real-time, it is unlikely that we would choose the same universe of 370 stocks. Even if our focus had been only on the SandP 500 constituents that are, generally, considered as safe investments, they are not immune to failure. For instance, our sample universe would probably incorporate Lehman Brothers in 2008, which would pull down (up) the results of portfolios having a long (short) position in the firm. To get the sense of how large is the potential effects of trading constraints on our strategy, we run an example using Quantopian – platform specialized in algorithmic trading. For simplicity, we test only the LW strategy starting from the beginning of 2010 until the end of 2019. Quantopian enables us to include slippage and commission costs into the analysis. Slippage refers to the price impact and the fill rate of trades and we assume the fixed slippage of 0.05% and the volume share limit equal to 10%. In respect of commission, we assume that the fee per share is USD0.001. Again, we perform monthly rebalancing and as a sample choose 10 stocks having the largest market value at each point of time. First, we consider the no-cost scenario that results in a 240.18% return over the 10 years of interest, which is slightly above 13% per annum. Once we include the transaction cost the aggregate return drops to 231.3%, on an annual basis that is about 12.7%. Considering that in both cases volatility is approximately equal to 12%, the trading cost has caused a slight loss in the Sharpe ratio, from 1.09 to 1.06. For the second case, on rare occasions we would receive a message similar to the following: "Your order for 7392 shares of $HSBC_PRA$ has been partially filled. 4060 shares were successfully purchased. 3332 shares were not filled by the end of day and were canceled." Although we observe that the effect of the trading obstacles in the example is relatively small, it should be noted that our portfolio starting value is set to USD100.000. Hence, the larger portfolios should count for a more substantial effect.

# H Conclusion

During the 20 years included in our sample, the downward market was present in about 25% of the time (approximately five years). Furthermore, at the time of writing the paper, the world is facing another crisis caused by the global pandemic. Having in mind that bear markets are an inevitable part of the investing process and considering the devastating effects it can have on investors' portfolios, it amplifies the necessity to develop all-weather strategies that deliver sound results independent of the market environment. We show that minimum variance strategies are worth considering (especially from the perspective of passive and conservative investors) as a reliable investment strategy which in the long-run can even produce larger returns with less risk than the equally weighted strategy. From the perspective of active portfolio management, minimum variance strategy could prove useful in times of uncertainty, while otherwise alpha seeking strategies could be pursued. We show that adjusting the weights between minimum variance and equally weighted portfolios based on the market outlook could lead to a large increase in return and due to the compounding effect enormous benefits for investors in the long horizon. More importantly, considering that up to date academic research favors 1/N investing over the sophisticated extensions of Markowitz's rule, (after all, according to [Childs, 2018] even Harry Markowitz himself follows the equal-weight approach), the unique contribution of our study is reflected in proving that investment strategies based on the Markowitz's groundbreaking theory are practically implementable with more appealing risk-return properties than 1/N rule. We offer the evidence opposing the statement "there are still many miles to go before the gains promised by optimal portfolio choice can actually be realized out of sample" (DeMiguel, Garlappi and Uppal, 2009), and explain the main effect causing minimum variance technique to outperform 1/N rule. Moreover, while doing so we generate a portfolio of very robust properties – JM portfolio, that is able to generate the lowest volatility no matter the sample nor market environment. This shows that further research in combining the covariance matrices could generate even better results, and considering the importance of covariance forecast in portfolio management, the potential benefit for the industry would be considerable.

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