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Kinematics and Aerodynamics of Velocity-Vector Roll

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The velocity-vector roll is defined as an angular rotation of an airplane about its instantaneous velocity vector, constrained to be performed at constant angle of attack (AOA), no sideslip, and constant velocity. Consideration of the aerodynamic force equations leads to requirements for body-axis yawing and pitching rotations that must be present to satisfy these constraints. Here, the body-axis rotations and the constraints are used in the moment equations to determine the aerodynamic moments required to perform the velocity-vector roll. The total aerodynamic moments, represented in the reference body-axis coordinate system, are then analyzed to determine the conditions under which their maxima occur. It is shown, for representative tactical airplanes, that the conditions for maximum pitching moment are strongly a function of the orientation of the airplane, occurring at about 90 deg of bank in a level trajectory. Maximum required pitching moment occurs at peak roll rate and is achieved at an AOA in excess of 45 deg. The conditions for maximum rolling moment depend on the value of the roll mode time constant. For a small time constant (fast response) the maximum rolling moment occurs at maximum roll acceleration and zero AOA, largely independent of airplane orientation; for a large time constant, maximum required rolling moment occurs at maximum roll rate, at maximum AOA, and at 180 deg of bank in level flight. The maximum yawing moment occurs at maximum roll acceleration and maximum AOA and is largely independent of airplane orientation. Results are compared with those obtained using conventional assumptions of zero pitch and yaw rates and show significant improvement, especially in the prediction of maximum-pitching-moment requirements.

Nomenclature

Notation in this paper is as defined in Ref. 1, with the following exceptions:

$p_{w,\text{ss}}$	= magnitude of maximum steady-state roll rate
R	= subscript meaning reference body axes (except in δ_R)
ss	= subscript meaning steady state
w	= subscript meaning wind axes
δ_R	= generic roll controller, ± 1
μ, γ, χ	= Euler angles associated with rotation from F_V to F_W
τ	= roll mode time constant

Introduction

HIGH angle-of-attack (AOA) capabilities of modern tactical aircraft have focused attention on the problem of rolling the airplane while at a high AOA. A roll about the x axis of any body axis system generally produces unacceptable sideslip unless other body-axis rates (pitch and yaw) are generated to eliminate the sideslip. A satisfactory rolling maneuver is usually defined in terms of the largest acceptable amount of sideslip generated during the roll, on the order of 3–5 deg. The AOA and velocity are required to be held nearly constant. The usual assumptions in analyzing the roll are that the sideslip angle is zero and AOA and velocity are constant at their reference conditions. The roll that results is commonly termed a velocity-vector roll, since the orientation of the airplane with respect to the velocity vector is constant.

By the definitions of the wind and stability axes,[1] the two coordinate systems remain coincident throughout a velocity-vector roll, and the term stability-axis roll is used interchangeably with velocity-vector roll. If the stability-axis system is chosen for the body axis in the analysis, then the kinematic, force, and moment equations for the airplane motion are identical in both the wind and the stability axes.

Rolling performance of airplanes is usually discussed in terms of the parameters of a first-order differential equation, the time constant, and the steady-state roll rate. Total moment estimates are typically made with the additional assumption that the pitch and yaw rates are zero.[2,3] In the present analysis, we allow the pitch and yaw rates (and their derivatives) to be nonzero and base the estimates on the assumption that terms involving the quantity g^2/V^2 are negligible in comparison with others in the moment equations. The resulting equations are analyzed to determine the conditions under which maxima occur. The estimates of maximum aerodynamic moment requirements are compared with the actual maxima and with those obtained with the assumption that pitch and yaw rates are zero. The results will show that the estimates obtained by neglecting g^2/V^2 are good to within 10% at worst-case conditions and are far superior (especially in pitching moment prediction) to estimates obtained by neglecting pitch and yaw rates.

Dynamics

We will refer to both the wind-axis (subscript w) and reference body-axis (subscript R) coordinate systems in this analysis. The reference body-axis system, often referred to as "the" body-axis system, is the one in which aerodynamic stability and control derivatives are normally represented. In most cases the reference body axes are nearly coincident with the principal axes and will be assumed to be exactly coincident in the following. We begin with the kinematic, force, and moment equations in the wind axes. The relevant equations[1] are

$$\begin{aligned}\dot{\mu} &= p_w + (q_w \sin\mu + r_w \cos\mu)\tan\gamma \\ \dot{\gamma} &= q_w \cos\mu - r_w \sin\mu \\ \dot{\chi} &= (q_w \sin\mu + r_w \cos\mu)\sec\gamma\end{aligned} \tag{1}$$

$$\begin{aligned}T_{xw} - D - mg\sin\gamma &= m\dot{V} \\ T_{yw} - C + mg\cos\gamma\sin\mu &= mVr_w \\ T_{zw} - L + mg\cos\gamma\cos\mu &= -mVq_w\end{aligned} \tag{2}$$

$$\begin{aligned}L_w &= I_x\dot{p}_w - I_{xz}(\dot{r}_w + p_w q_w) - (I_y - I_z)q_w r_w \\ M_w &= I_y\dot{q}_w - I_{xz}\left(r_w^2 - p_w^2\right) - (I_z - I_x)r_w p_w \\ N_w &= I_z\dot{r}_w - I_{xz}(\dot{p}_w - q_w r_w) - (I_x - I_y)p_w q_w\end{aligned} \tag{3}$$

There are no derivatives of the moments of inertia because the wind-axis alignment with respect to the body is constant. The

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moments and cross product of inertia in the moment equations must be evaluated for the particular orientation of the body axis chosen according to

$$\begin{aligned} I_x &= I_{xp}\cos^2\alpha + I_{zp}\sin^2\alpha \\ I_y &= I_{yp} \\ I_z &= I_{zp}\cos^2\alpha + I_{xp}\sin^2\alpha \\ I_{xz} &= \tfrac{1}{2}(I_{xp} - I_{zp})\sin 2\alpha \end{aligned} \tag{4}$$

Here α is measured from the wind x axis to the reference body x axis. There are no cross products of inertia other than I_{xz} since $\beta = 0$.

Analysis

We assume the initial, reference conditions are specified. The desired time history of the roll rate $p_w(t)$ (and hence $\dot{p}_w$) is also assumed to be specified. With the conditions of constant α and velocity, the lift and drag will vary only as the pitch rate q, and to first order lift and drag are constant and assumed known. We will assume the thrust in the z_w direction is also constant, so that $Z_w = -T_{zw} - L = -n_{zw} \times$ weight = const. Finally, since $\beta = 0$ and most tactical airplanes have little or no direct control of side force, we will take the aerodynamic side force C to be zero as well and neglect T_{yw}.

For the roll rate specification, we take

$$\dot{p}_w = -p_w(t)/\tau + p_{w,\mathrm{ss}}\delta_R/\tau \tag{5}$$

We assume a unit step input $\delta_R(t) = 1$, $t \geq 0$. Then,

$$p_w(t) = p_w(0)e^{-t/\tau} + p_{w,\mathrm{ss}}(1 - e^{-t/\tau}) \tag{6}$$

The Y_w and Z_w force equations are solved for q_w and r_w to yield the angular rates that must be present for the Y_w force to remain zero and the Z_w force to remain constant:

$$r_w(t) = \frac{mg\cos\gamma(t)\sin\mu(t)}{mV} \tag{7}$$

$$q_w(t) = -\frac{(Z_w + mg\cos\gamma(t)\cos\mu(t))}{mV} \tag{8}$$

Thus $p_w(t)$ is specified, and $q_w(t)$ and $r_w(t)$ vary with time only through $\mu(t)$ and $\gamma(t)$. Therefore $\dot{\mu}$ and $\dot{\gamma}$ (as well as $\dot{\chi}$) are functions only of $\mu(t)$, $\gamma(t)$, and the specified $p_w(t)$. Only the Euler angle equations need to be integrated to determine the evolution of the states. With the indicated operations, we have

$$\dot{\mu} = p_w(t) - \frac{Z_w\sin\mu(t)\tan\gamma(t)}{mV} \tag{9}$$

$$\dot{\gamma} = -\frac{Z_w\cos\mu(t) + mg\cos\gamma(t)}{mV} \tag{10}$$

$$\dot{\chi} = -\frac{Z_w\sin\mu(t)}{mV}\sec\gamma(t) \tag{11}$$

To determine the required aerodynamic moments, we need to know the total angular acceleration of the airplane. The roll acceleration, $\dot{p}_w$ is specified. The expressions for q_w and r_w are differentiated with respect to time to yield

$$\dot{r}_w = \frac{g[\cos\gamma(t)\cos\mu(t)\dot{\mu} - \sin\gamma(t)\sin\mu(t)\dot{\gamma}]}{V} \tag{12}$$

$$\dot{q}_w = \frac{g[\cos\gamma(t)\sin\mu(t)\dot{\mu} + \sin\gamma(t)\cos\mu(t)\dot{\gamma}]}{V} \tag{13}$$

The expressions for $\dot{\gamma}$ and $\dot{\mu}$ are now substituted into the relationships for $\dot{q}_w$ and $\dot{r}_w$ in Eqs. (12) and (13). The resulting equations are then substituted into the moment equations (3). The results are quite complicated and offer no insight. We may generalize that, for a given velocity and AOA and some prescribed initial attitude, there is only one trajectory the airplane can fly and satisfy the requirements for a velocity-vector roll. In performing this roll, the moments vary with time only through the dependence on $\mu(t)$, $\gamma(t)$, and the specified $p_w(t)$ and $\dot{p}_w$. The aerodynamic moment requirements may therefore be considered as functions of the following factors:

1) All other considerations being equal, the AOA, measured from the reference body-axis system to the wind-axis system, has the effect of changing the moments and cross product of inertia in the wind-axis system [Eq. (3)].

2) Kinematically, the flight velocity will strongly influence the angular rates q_w and r_w, with lower velocities requiring higher angular rates [Eqs. (7) and (8)]. The angular rates, and their derivatives, directly determine the aerodynamic moment requirements [Eq. (3)].

3) The instantaneous values of p_w and $\dot{p}_w$ in a first order response [Eqs. (5) and (6)] enter into the aerodynamic moment equations [Eq. (3)].

4) Along with the flight velocity, the instantaneous orientation terms of the airplane, μ and γ, determine the angular rates q_w and r_w [Eqs. (7) and (8)].

5) The effect of increasing load factor n_{zw} is to increase the Z_w force in Eq. (8), thus changing q_w and $\dot{q}_w$.

Because of the complexity of the interrelationships between these factors, direct analysis is difficult. To gain some insight, we first examine steady-state solutions to the problem and then examine the conditions under which maximum aerodynamic moment requirements occur.

Steady-State Solutions

By steady-state conditions, we mean that only the dynamically influential variables γ and μ are constant, and we allow χ to vary. From Eq. (10), steady-state can occur only if $Z_w\cos\mu(t) + mg\cos\gamma(t) = 0$, or

$$\cos\gamma_{\mathrm{ss}} = -\frac{Z_w}{mg}\cos\mu_{\mathrm{ss}} \tag{14}$$

The actual magnitude of the angles is found from Eq. (9),

$$Z_w\sin\mu_{\mathrm{ss}}\tan\gamma_{\mathrm{ss}} = mVp_{w,\mathrm{ss}} \tag{15}$$

Equation (14) gives the relationship between γ and μ that must be satisfied if the airplane is to be rolling with constant AOA and no side force. For example, if $Z_w = -mg$ (loosely, a load factor of 1), then the airplane must be in a descending (or ascending) spiral with the bank angle equal in magnitude to the flight path angle. Simulation results have shown that at high AOA and low speed, the trajectory of the airplane tends toward this spiraling motion within the first 180 deg of "roll."

Equation (15) must be solved simultaneously with Eq. (14). If $Z_w = -mg$ and we take $\mu = \gamma$, then we have

$$\sin\mu_{\mathrm{ss}}\tan\mu_{\mathrm{ss}} = -Vp_{w,\mathrm{ss}}/g$$

For example, with $V = 100$ ft/s, $p_{w,\mathrm{ss}} = -1.0$ rad/s, we find $\mu_{\mathrm{ss}} \cong 73$ deg. In general, higher velocities and higher roll rates will increase the required bank and flight path angles.

Novice instrument pilots sometimes find that flying by the seat-of-the-pants can have disastrous consequences. By simply maintaining 1-g flight and no sideslip and ignoring (or not believing) their instruments, they find themselves in the aptly named "graveyard spiral." Attempts to "pull up" just change the Z_w force in Eq. (15) and tighten up the spiral. They would probably receive little comfort in knowing they were actually performing a perfect velocity-vector roll.

Maximum Aerodynamic Moment Requirements

The functional dependencies of the aerodynamic moments are complex and offer no insights into the conditions under which maximum moment requirements occur, and simplifying assumptions are needed. We begin by representing the required moments in the reference body-axis system (subscript R), hereafter assumed to be the principal axis system. This is done according to

$$
\begin{aligned}
L_R &= L_w \cos\alpha - N_w \sin\alpha \\
M_R &= M_w \qquad (16) \\
N_R &= N_w \cos\alpha + L_w \sin\alpha
\end{aligned}
$$

Note that the roll performance is specified in the wind-axis system, so that all references to roll mode time constant and steady-state roll rate are understood to be those associated with a roll around the velocity vector. The AOA used subsequently is the angle from the reference body-axis system to the wind-axis system.

When all of the functional dependencies are included and Eq. (16) are expanded, the multipliers of the principal moments of inertia include several terms in which the factor g^2/V^2 appears. This factor is part of a complicated product involving the sines and cosines of angles, sometimes with the load factor included as well. For example, the expression multiplying I_{yp} in the expansion of L_R is

$$
(g/V)p_w \sin\alpha(\cos\mu\cos\gamma - n_{zw}) + (g/V)^2 \times \cos\alpha\sin\mu\cos\gamma(\cos\mu\cos\gamma - n_{zw})
$$

The velocities of interest in high-AOA maneuvering will be low, but typically greater than 100 ft/s. For a number of reasons, $V = 100$ ft/s is probably unreasonably low and represents an extreme case. The ratio g/V will then be smaller than $\frac{1}{3}$. At these low velocities the load factor will almost certainly be no greater than $1g$, higher load factors will require higher velocities, and the value of $(g^2/V^2)\, n_{zw}$ will be limited by $n_{z,\max}$. We tentatively assume that terms involving g^2/V^2 may be neglected, subject to later verification. With this assumption, the moment equations in reference body-axis coordinates become

$$
L_R \cong I_{xp}\cos\alpha\dot p_w + (g/V)(I_{yp} - I_{zp} - I_{xp})\cos\mu \times \cos\gamma\sin\alpha p_w(t) - (g/V)(I_{yp} - I_{zp})n_{zw}\sin\alpha p_w(t) \qquad (17)
$$

$$
M_R \cong (I_{xp} - I_{zp})\sin 2\alpha p_w(t)^2/2 + (g/V)(I_{xp} - I_{zp}) \times \sin\mu\cos\gamma\cos 2\alpha p_w(t) + (g/V)I_{yp}\sin\mu\cos\gamma p_w(t) \qquad (18)
$$

$$
N_R \cong I_{zp}\sin\alpha\dot p_w + (g/V)(I_{xp} - I_{yp} + I_{zp})\cos\mu \times \cos\gamma\cos\alpha p_w(t) - (g/V)(I_{xp} - I_{yp})n_{zw}\cos\alpha p_w(t) \qquad (19)
$$

Clearly, the mass distribution of the airplane will drive the relative importance of each of the terms in the three moment equations. In modern tactical airplanes, I_{yp} and I_{zp} are usually on the order of 4–7 times greater than I_{xp}. In the case of some aircraft, such as the A-4, I_{yp} and I_{zp} are nearly equal, which will further simplify the rolling and yawing moment equations. The F-16 and F-18 airplanes have principal moments of inertia in the ratios $I_{xp} : I_{yp} : I_{zp} \cong 1:6:7$ and $1:5:6$, respectively. The latter will be used in the subsequent analysis to illustrate the results.

We now analyze each moment equation to determine the conditions under which maximum aerodynamic moment is required.

Rolling Moment

The rolling moment equation is given by

$$
L_R = I_{xp}\cos\alpha\dot p_w + (g/V)(I_{yp} - I_{zp} - I_{xp}) \times \cos\mu\cos\gamma\sin\alpha p_w(t) - (g/V)(I_{yp} - I_{zp})n_{zw}\sin\alpha p_w(t)
$$

This equation is of the form

$$
f(t) = C_1\dot p_w + C_2 p_w(t)
$$

We will use this expression to show that the maximum of L_R occurs at $p_w(t) = 0$, $\dot p_w = \dot p_{w,\max}$, or at $\dot p_w = 0$, $p_w(t) = p_{w,\mathrm{ss}}$, except for one special case in which the moment requirement is constant throughout the roll.

Here, C_1 is a positive number (for positive AOA), and C_2 may be positive, negative, or zero. We wish to determine $\max|f(t)|$. We have assumed a first-order response; with initial conditions, we have

$$
p_w(t) = p_w(0)e^{-t/\tau} + p_{w,\mathrm{ss}}\delta_r(1 - e^{-t/\tau})
$$

$$
\dot p_w = \frac{p_{w,\mathrm{ss}}\delta_r - p_w(0)}{\tau}e^{-t/\tau}
$$

Combining

$$
f(t) = C_3 + C_4 e^{-t/\tau}
$$

where

$$
C_3 = C_2 p_{w,\mathrm{ss}}\delta_r
$$

$$
C_4 = [p_{w,\mathrm{ss}}\delta_r - p_w(0)]\left(\frac{C_1}{\tau} - C_2\right)
$$

$$
|f(0)| = |C_3 + C_4|
$$

$$
|f(\infty)| = |C_3|
$$

If a maximum occurs in $0 < t < \infty$, then $\dot f = 0$, which requires $C_4 = 0$ and $f(t)$ is constant. If the first factor of C_4 is zero, then $p_{w,\mathrm{ss}}\delta_r = p_w(0)$, and $\dot p_w = 0$, $p_w(t) = p_{w,\mathrm{ss}}\delta_r$. This is the condition at $t = \infty$. Otherwise $\dot f = 0$ if $C_1 = C_2\tau$. If $C_1 \neq C_2\tau$, then

$$
\max|f(t)| = \max(|f(\infty)|, |f(0)|),
$$

or

$$
\max|f(t)| = \max(|C_3|, |C_3 + C_4|)
$$

Consider $p_w(0) = 0$, $\delta_r = +1$ (a roll to the left is symmetric). With these conditions,

$$
C_3 = C_2 p_{w,\mathrm{ss}}
$$

$$
C_4 = p_{w,\mathrm{ss}}\left(\frac{C_1}{\tau} - C_2\right) = \frac{p_{w,\mathrm{ss}}C_1}{\tau} - C_3
$$

$$
C_3 + C_4 = \frac{p_{w,\mathrm{ss}}C_1}{\tau}
$$

Then $|C_3 + C_4| > |C_3|$ if and only if $C_1 > |C_2|\tau$; otherwise the reverse is true. To find the maximum rolling moment, we just compare its magnitude with $p_w(t) = 0$, $\dot p_w = \dot p_{w,\max}$, against that with $\dot p_w = 0$, $p_w(t) = p_{w,\mathrm{ss}}$. If $p_w(t) = 0$, then $L_R = I_{xp}\cos\alpha\dot p_w$, which is a maximum at $\alpha = 0$:

$$
L_R = I_{xp}\dot p_{w,\max} \qquad \text{if} \qquad \alpha = 0 \qquad (20)
$$

at any attitude. If $\dot p_w = 0$, then

$$
L_r = (g/V)[(I_{yp} - I_{zp} - I_{xp})\cos\mu\cos\gamma - (I_{yp} - I_{zp})n_{zw}]\sin\alpha p_w(t)
$$

This expression is maximum at $\alpha = \alpha_{\max}$, $p_w(t) = p_{w,\mathrm{ss}}$. We also need to find the maximum value of the expression in the square brackets. The rolling moment is stationary with respect to μ and γ only if 1) $\gamma = 0$, $\mu = 0$; 2) $\gamma = 0$, $\mu = \pi$; or 3) $\gamma = \pm\pi/2$, $\mu = \pm\pi/2$. With these values,

$$
L_R = \begin{cases}
-(g/V_{\min})I_{xp}\sin\alpha_{\max}p_{w,\mathrm{ss}} \\
\quad \text{if} \quad n_{zw} = 1, \gamma = 0, \mu = 0 & (21) \\
-(g/V_{\min})(2I_{yp} - 2I_{zp} - I_{xp})\sin\alpha_{\max}p_{w,\mathrm{ss}} \\
\quad \text{if} \quad n_{zw} = 1, \gamma = 0, \mu = \pi & (22) \\
-(g/V_{\min})(I_{yp} - I_{zp})\sin\alpha_{\max}p_{w,\mathrm{ss}} \\
\quad \text{if} \quad n_{zw} = 1, \gamma = \pm\pi/2, \mu = \pm\pi/2 & (23)
\end{cases}
$$

We therefore must pick the greatest of the four candidate maxima found. For a 1:5:6 ratio of moments of inertia, the magnitude of the inertia terms in Eqs. (20–23) are related as 1:1:3:1, so neither Eq. (21) nor Eq. (23) will ever be a maximum. The ratio $g/V_{\min}$ is assumed to be smaller than $\frac{1}{3}$. For Eq. (22) to be greater in magnitude than Eq. (20), we would require $|p_{w,ss}|\sin\alpha_{\max} > |\dot{p}_{w,\max}|$. The parameters $p_{w,ss}$ and $\dot{p}_w$ are related according to

$$\dot{p}_w = \frac{p_{w,ss}\delta_R - p_w(0)}{\tau}e^{-t/\tau}$$

For a roll from rest $[p_w(0) = 0, \delta_R = 1]$, maximum $\dot{p}_w$ occurs at $t = 0$. Therefore $|\dot{p}_{w,\max}| = |p_{w,ss}|/\tau$, and

$$|p_{w,ss}|\sin\alpha_{\max} > |\dot{p}_{w,\max}| \Leftrightarrow \tau > 1/\sin\alpha_{\max}$$

This expression for τ is really a worst case, since we have hypothesized a very low value for V. At higher velocities, the roll mode time constant will have to be appreciably greater than 1 s before the steady-state roll becomes the maximizing condition. The more general criterion for the cut-off value of the roll mode time constant is $\tau > \tau^*$, where

$$\tau^* = \frac{-I_{xp}}{(g/V)(2I_{yp} - 2I_{zp} - I_{xp})\sin\alpha_{\max}} \tag{24}$$

We conclude that, for aircraft of interest at realistic speeds, with roll mode time constants on the order of 1s, the maximum aerodynamic rolling moment is found at $\alpha = 0$: $L_{R,\max} = I_{xp}\dot{p}_{w,\max}$.

For values of τ greater than τ^*, the maximum aerodynamic rolling moment is found at $\alpha = \alpha_{\max}$:

$$L_{R,\max} = -(g/V_{\min})(2I_{yp} - 2I_{zp} - I_{xp})\sin\alpha_{\max}p_{w,ss}$$

The condition $\alpha = 0$ causes the derivatives with respect to γ and μ to vanish for all γ and μ, so it appears that this maximum moment will be independent of the airplane orientation. There actually is a small effect due to γ and μ, owing to the neglected terms involving g^2/V^2.

Yawing Moment

The yawing moment equation is given by

$$N_R = I_{zp}\sin\alpha\dot{p}_w + (g/V)(I_{xp} - I_{yp} + I_{zp})$$
$$\times \cos\mu\cos\gamma\cos\alpha p_w(t) - (g/V)(I_{xp} - I_{yp})$$
$$\times n_{zw}\cos\alpha p_w(t)$$

The yawing moment equation is similar to the rolling moment equation, and only the results will be presented here:

$$N_R = \begin{cases} I_{zp}\sin\alpha_{\max}\dot{p}_{w,\max} \\ \quad \text{if} \quad p_w(t) = 0 & (25) \\ (g/V_{\min})I_{zp}p_{w,ss} \\ \quad \text{if} \quad \alpha = 0, n_{zw} = 1, \gamma = 0, \mu = 0 & (26) \\ -(g/V_{\min})(2I_{xp} - 2I_{yp} + I_{zp})p_{w,ss} \\ \quad \text{if} \quad \alpha = 0, n_{zw} = 1, \gamma = 0, \mu = \pm\pi & (27) \\ -(g/V_{\min})(I_{xp} - I_{yp})p_{w,ss} \\ \quad \text{if} \quad \alpha = 0, n_{zw} = 1, \gamma = \pm\pi/2, \mu = \pm\pi/2 & (28) \end{cases}$$

Again we must pick the greatest of the four candidate maxima found. Assuming a 1:5:6 ratio of principal moments of inertia, the magnitude of the inertia terms in Eqs. (25–28) are related as 6:6:2:4, so Eq. (26) is the closest challenger to Eq. (25) for a maximum. For Eq. (26) to be a maximum, with $g/V_{\min}$ assumed to be smaller than $\frac{1}{3}$, we get a much more stringent requirement for the roll mode time constant: $\tau > 3/\sin\alpha_{\max}$.

We conclude that, for aircraft of interest, the maximum yawing moment is found at $\alpha = \alpha_{\max}$, where the moment is given by: $N_{R,\max} = I_{zp}\sin\alpha_{\max}\dot{p}_{w,\max}$.

In dealing with the rolling and yawing moment equations, we took $\dot{p}_{w,\max}$ to be the roll acceleration that occurs from rest, $p_w(0) = 0$. The greatest value of $\dot{p}_{w,\max}$, however, occurs during roll reversal: application of full opposite roll control from a steady roll condition. The roll reversal yields $|\dot{p}_{w,\max}| = 2|p_{w,ss}|/\tau$, where $\dot{p}_w$ and $p_{w,ss}$ have opposite signs. This case was not considered in the rolling moment analysis because the additional roll acceleration comes from roll-rate damping, and we are concerned with control power requirements here. If structural loads are to be determined, then the roll reversal will give the greatest total moment in roll. The rolling reversal is not a critical maneuver with respect to yawing moment because of the signs of the terms in the expression for N_R and the fact that $\dot{p}_w$ and $p_{w,ss}$ have opposite signs.

Pitching Moment

The pitching moment is given by

$$M_R = (I_{xp} - I_{zp})\sin 2\alpha p_w(t)^2/2 + (g/V)(I_{xp} - I_{zp})$$
$$\times \cos\gamma\sin\mu\cos 2\alpha p_w(t) + (g/V)I_{yp}\cos\gamma\sin\mu p_w(t)$$

We note that this equation is not stationary with respect to V or p_w, unless $p_w = 0$ (and the moment is zero). We therefore take $V = V_{\min}$, $p_w = p_{w,ss}$. Maximizing with respect to μ, γ, and α, we have

$$\frac{\partial M_R}{\partial\gamma} = -\frac{g}{V}[(I_{xp} - I_{zp})\cos 2\alpha + I_{yp}]\sin\gamma\sin\mu p_w(t) = 0$$

Either $\sin\gamma$ or $\sin\mu = 0$, or $\cos 2\alpha = -I_{yp}/(I_{xp} - I_{zp})$. Even if the angle is defined (i.e., if $I_{yp} < |I_{xp} - I_{zp}|$), we cannot impose a value on α unless it also makes the function stationary with respect to α, which this value does not do. So we consider only the conditions on γ and μ. The derivative with respect to μ is

$$\frac{\partial M_R}{\partial\mu} = \frac{g}{V}[(I_{xp} - I_{zp})\cos 2\alpha + I_{yp}]\cos\gamma\cos\mu p_w(t) = 0$$

The analysis is the same as above, but either $\cos\gamma = 0$ or $\cos\mu = 0$. We therefore require at least one of the following conditions to hold: 1) $\mu = 0, \pi$ and $\gamma = \pm\pi/2$ or 2) $\mu = \pm\pi/2$ and $\gamma = 0$.

$$\frac{\partial M_R}{\partial\alpha} = (I_{xp} - I_{zp})\cos 2\alpha p_w(t)^2$$
$$-2\frac{g}{V}(I_{xp} - I_{zp})\cos\gamma\sin\mu\sin 2\alpha p_w(t) = 0$$

From the conditions on γ and μ, $\cos\gamma\sin\mu = -1, 0, +1$. For the derivative with respect to α to vanish,

$$\tan 2\alpha = \frac{p_w(t)}{2\cos\gamma\sin\mu g/V}$$

We can maximize the magnitude of M_R by making all three terms in the expression for M_R nonzero and of the same sign. Since $\sin 2\alpha \geq 0$ and $I_{zp} > I_{xp}$, the first term in M_R is always negative. The other two depend on the sign of $\cos 2\alpha$. This in turn is determined by the choice we make of γ and μ. For positive $p_w(t)$, the sign of $\tan 2\alpha$ is determined by the sign of $\cos\gamma\sin\mu$:

1. If $\cos\gamma\sin\mu = 0$, then $2\alpha = \pi/2$, $\alpha = \pi/4$, and $\cos 2\alpha = 0$.
2. If $\cos\gamma\sin\mu = -1$, then $\pi/2 < 2\alpha < \pi$, or $\pi/4 < \alpha < \pi/2$, and $\cos 2\alpha < 0$.
3. If $\cos\gamma\sin\mu = +1$, then $0 < 2\alpha < \pi/2$, or $0 < \alpha < \pi/4$, and $\cos 2\alpha > 0$.

We can make all three terms nonzero and negative by taking $\cos\gamma\sin\mu = -1$. As a result, the maximum pitching moment requirement for a roll to right occurs at $\gamma = 0$, $\mu = -\pi/2$, and

$$\tan 2\alpha = \frac{-V_{\min}p_{w,\max}}{2g}$$

Table 1 Summary of formulas

Aerodynamic moment	α	p_w	q_w	r_w	γ	μ	Maximum required
L_R	0 if $\tau \leq \tau^*$	0	—	$\pm g/V$	—	—	$I_{xp}\dot{p}_{w,\max}$ if $\tau \leq \tau^*$, otherwise
	$\alpha_{\max}$	$p_{w,\mathrm{ss}}$	—	0	0	π	$-(g/V_{\min})(2I_{yp} - 2I_{zp} - I_{xp}) \times \sin\alpha_{\max} p_{w,\mathrm{ss}}$
M_R	$\tan 2\alpha = -\dfrac{V_{\min} p_{w,\max}}{2g}$	$p_{w,\mathrm{ss}}$	g/V	—	0	$-\pi/2$	$(I_{xp} - I_{zp})\sin 2\alpha p_{w,\mathrm{ss}}^2/2 - (g/V_{\min})(I_{xp} - I_{zp})\cos 2\alpha p_{w,\mathrm{ss}} - (g/V_{\min}) I_{yp} p_{w,\mathrm{ss}}$
N_R	$\alpha_{\max}$	0	—	$\pm g/V$	—	—	$I_{zp}\sin\alpha_{\max}\dot{p}_{w,\max}$

Table 2 Pitching moment: actual vs estimated

		Actual				Estimated (from Table 1)			
V, ft/s	τ, s	α, deg	μ, deg	γ, deg	Maximum, ft-lb	α, deg	μ, deg	γ, deg	Maximum, ft-lb
100	1.0	62	−94	−15	−106,700	61	−90	0	−111,300
200	1.0	54	−91	−8	−80,800	54	−90	0	−83,000

Table 3 Rolling moment: actual vs estimated

		Actual				Estimated (from Table 1)			
V, ft/s	τ, s	α, deg	μ, deg	γ, deg	Maximum, ft-lb	α, deg	μ, deg	γ, deg	Maximum, ft-lb
100	1.0	0	120	0	25,800	0	—	—	23,200
100	1.5	70	174	−1	18,700	70	180	0	18,700
200	1.0	0	120	0	23,800	0	—	—	23,200
200	3.0	70	177	0	9,300	70	180	0	9,340

Table 4 Yawing moment: actual vs estimated

		Actual				Estimated (from Table 1)			
V, ft/s	τ, s	α, deg	μ, deg	γ, deg	Maximum, ft-lb	α, deg	μ, deg	γ, deg	Maximum, ft-lb
100	1.0	70	−118	−14	147,900	70	—	—	134,600
200	1.0	70	−118	−14	137,900	70	—	—	134,600

Using this value of α, we may calculate

$$M_{R,\max} = (I_{xp} - I_{zp})\sin 2\alpha p_{w,\mathrm{ss}}^2/2 - \frac{g}{V_{\min}}(I_{xp} - I_{zp}) \times \cos 2\alpha p_{w,\mathrm{ss}} - \frac{g}{V_{\min}} I_{yp} p_{w,\mathrm{ss}} \tag{29}$$

Unlike the conditions for maximum rolling and yawing moments, the maximum pitching moment is highly dependent on the orientation of the airplane. The required pitching moment as the airplane passes through $\gamma = 0$, $\mu = -\pi/2$ at $p_{w,\mathrm{ss}}$ can be several times greater than that required at $\gamma = 0$, $\mu = +\pi/2$.

Summary of Results

Table 1 summarizes the results obtained above. The formulas estimate the total aerodynamic moments required for a velocity-vector roll. Principal moments of inertia were taken in the ratios 1: 5: 6. The load factor was assumed to be 1.0. The pitching moment data assume a roll to the right.

Verification

The maximum-moment requirements were determined using data representative of an F-18 airplane: I_{xp} = 23, 168 slug-ft^2, I_{yp} = 123,936 slug-ft^2, I_{zp} = 143,239 slug-ft^2. The determination of actual maximum-moment requirements was made using a brute-force search over all values of the relevant parameters at V = 100 ft/s and again at V = 200 ft/s. The maximum AOA, $\alpha_{\max}$, was taken to be 70 deg. The steady-state roll rate was $p_{w,\mathrm{ss}} = 1.0$. These results were compared with the estimates obtained using the equations in Table 1. The results are shown in Tables 2–4.

From Tables 2–4, it is seen that the greatest error in estimating the actual maximum moments occurs in estimating the rolling moment at 100 ft/s, $\tau = 1.0$, with an error of about 10%. The yawing moment error is just under 10% at the same speed. Pitching moment estimates are good to under 5% error.

Comparisons

In the determination of maximum moments for the velocity-vector roll, it has been common practice to assume that the stability-axis pitch and yaw rates (equivalent to $q_w = r_w = 0$) and their derivatives are negligible.[2,3] This assumption has the velocity vector fixed with respect to the local horizontal while the airplane rolls around it. The frequent use of the $q_w = r_w = 0$ assumption has contributed to some confusion when test pilots speak of the maneuver in terms of heading change, whereas the engineers are visualizing an aileron roll of sorts. The notable exception to this misperception is found in Kalviste's analysis.[4] With the $q_w = r_w = 0$ assumption, the moment equations (3) become simply

$$\begin{aligned} L_w &= I_x\dot{p}_w = (I_{xp}\cos^2\alpha + I_{zp}\sin^2\alpha)\dot{p}_w \\ M_w &= I_{xz}p_w^2 = \tfrac{1}{2}(I_{xp} - I_{zp})\sin 2\alpha p_w^2 \\ N_w &= I_{xz}\dot{p}_w = \tfrac{1}{2}(I_{xp} - I_{zp})\sin 2\alpha\dot{p}_w \end{aligned} \tag{30}$$

Table 5 Comparison of estimates

		Maximum, actual			Maximum, Table 1			Maximum, $q_w = r_w = 0$, assumption		
V, ft/s	τ, s	Roll	Pitch	Yaw	Roll	Pitch	Yaw	Roll	Pitch	Yaw
100	1.0	25,800	−106,700	147,900	23,200	−111,300	134,600	23,200	−60,000	134,600
200	1.0	23,800	−80,800	137,900	23,200	−83,000	134,600	23,200	−60,000	134,600
100	1.5	18,700	—	—	18,700	—	—	15,400		
200	3.0	9,300	—	—	9,340	—	—	7,720		

The rolling and yawing moments in Eq. (3) are represented in the wind-axis system; in the reference body-axis system they are

$$L_R = I_{xp} \cos\alpha\, \dot{p}_w \qquad N_R = I_{zp} \sin\alpha\, \dot{p}_w \tag{31}$$

Thus the $q_w = r_w = 0$ assumption yields the first of the terms for each moment as given by Eqs. (17–19). The $q_w = r_w = 0$ assumption predicts maximum rolling moment at $\alpha = 0$, which is only true if the roll mode time constant is short compared to the value of τ^* given in Eq. (24). It predicts maximum pitching moment at $\alpha = 45$ deg, which may be far less than the actual AOA for maximum, depending on the speed, orientation and mass properties of the airplane. The $q_w = r_w = 0$ assumption does accurately predict the maximum yawing moment for the aircraft of interest.

To illustrate this, the moments given in Tables 2–4 are compared with those estimated by the $q_w = r_w = 0$ assumption. The same conditions used in Tables 2–4 apply, and results are in Table 5.

Table 5 confirms the observations made above. The yawing moment predictions are the same, as expected. The rolling moment predictions are the same for fast time constants, but for slower responses, the $q_w = r_w = 0$ assumption is in error by 17% at both 100 and 200 ft/s.

The differences in the maximum-pitching-moment predictions are striking. The $q_w = r_w = 0$ assumption predicts only 56% of the actual maximum required moment at 100 ft/s. Even at 200 ft/s, a fairly representative speed for high-AOA maneuvering, the $q_w = r_w = 0$ assumption accounts for only 75% of the actual maximum required moment.

Conclusions

The equations given for estimating the aerodynamic rolling, yawing, and pitching moments [Eqs. (17–19)] demonstrate a significant improvement over the assumption $q_w = r_w = 0$ often used. The primary utility of such estimates is in predicting the maximum moments required to perform the velocity-vector roll for use in defining the control power requirements. The area of most notable improvement is in estimating the pitching moment, especially at lower speeds, at which previous methods can be in error by almost 50%. The equations presented also show improvements in estimating the rolling moment requirements, especially if the speed is low and the roll mode time constant is large.

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