Q5

(a):

Solution:

Proof. By induction on n.

Base case: let n be 1, $n^3 + 2n = 1^3 + 2 \cdot 1 = 1 + 2 = 3$.

Inductive step: Suppose for integer n, $n^3 + 2n$ is divisible by 3. For (n+1), (n+

$1)^3 + 2(n + 1) = n^3 + 3n^2 + 3n + 1 + 2n + 2$

$$(n + 1)^3 + 2(n + 1) = n^3 + 3n^2 + 3n + 1 + 2n + 2$$

$$= (n^3 + 2n) + (3n^2 + 3n + 1 + 2)$$
$$= (n^3 + 2n) + 3(n^2 + n + 1)$$

Because $n^3 + 2n$ is divisible by 3 as our previous assumption. $3(n^2 + n + 1)$ is also divisible by 3 because n is integer. Therefore, the entire term $(n + 1)^3 + 2(n + 1)$ is divisible by 3.

□

(b):

Solution:

Proof. By strong induction on integer n that is larger than or equal to 2.

Base case: n=2. 2 is a prime number.

Inductive step: Suppose for integer n, every integer ranging from 2 to (n-1) could be expressed as a product of primes (or itself is a prime). If n is not prime, by definition, it could be written as a product of two integer $m, k \in \{x \in Z | 2 \leq x \leq n\}$. Because all integer ranging from 2 to (n-1) could be expressed as products of primes by assumption. Therefore, $n = m \cdot k$ could also be written as production of primes.

□

Q6

Exercise 7.4.1

(a):
When n=3, the left side of the equation is $1^2+2^2+3^2 = 14$. The right side of the equation is 14 also. The statement is true

(b):

$p(k) = \frac{k(k+1)(2k+1)}{6}$

(c):

$p(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$

(d):

$p(1) = \frac{1(1+1)(2+1)}{6} = 1 = \sum_{j=1}^{1} j^2$

(e):

$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ implies $\sum_{j=1}^{n+1} j^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

(f):

lets suppose $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true.

(g):
solution:

Proof. By induction on positive integer n.

Base case: n=1.

$$p(1) = \frac{1(1+1)(2+1)}{6} = 1 = \sum_{j=1}^{1} j^2.$$

Inductive step:
Suppose for positive integer n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true.

$$
\begin{aligned}
\sum_{j=1}^{n+1} j^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6n^2+12n+6}{6} \\
&= \frac{2n^3+3n^2+n}{6} + \frac{6n^2+12n+6}{6} \\
&= \frac{2n^3+9n^2+13n+6}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}
$$

□

Exercise 7.4.3

(c):

solution:

Proof. By induction on n.

Base case: let n be 1, the inequality is valid

Inductive step:
Suppose for integer n, the inequality is valid. We got:

$$
\begin{aligned}
\sum_{j=1}^{n} \frac{1}{j^2} &\le 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\
&\le 2 - \frac{1}{n} + \frac{1}{n(n+1)}
\end{aligned}
$$

The right hand side of the inequality, $2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{n+1-1}{n(n+1)} = 2 - \frac{1}{(n+1)}$. □

Exercise 7.5.1

(a)
solution:

Proof. By induction on n.

Base case: let n be 1, $3^{2n} - 1 = 3^2 - 1 = 8$. The statement is true.

Inductive step:
Suppose for integer n, $3^{2n} - 1$ is divisible by 4. For (n+1), $3^{2(n+1)} - 1 = 3^2 \cdot 3^{2n} - 1 = 3^2 \cdot (3^{2n} - 1) + 3^2 - 1$
$= 3^{2(n+1)} - 1 = 3^2 \cdot 3^{2n} - 1 = 3^2 \cdot (3^{2n} - 1) + 3^2 - 1 = 3^2 \cdot (3^{2n} - 1) + 8$

By assumption, $3^{2n} - 1$ is divisible by 4, And 8 is divisible by 4. Therefore, $3^2 \cdot (3^{2n} - 1) + 8 = 3^{2(n+1)} - 1$ is divisible by 4.

$\square$