Q7

Exercise 6.1.5

(b):

**Solution:**

If we order the 5-card hand with the three-of-a-kind first, we have 13 ways for the number showing on the first three cards. Since we will have three out of four suits, we have $\binom{4}{3}$ ways to choose the suits. The remaining two cards must show different numbers than the three-of-a- kind and each other. There are $\binom{12}{2}$ ways for these numbers. The last two cards may have any of the four suits.

The probability that the hand is a three of a kind:

$$\frac{13 \cdot \binom{4}{3}\binom{12}{2} \cdot 4 \cdot 4}{\binom{52}{5}} = 0.02112845138$$

(c):

**Solution:**

If we have to select all 5 cards of same suit, then cards can be selected in $\binom{13}{5}$ ways, and the suit can be chosen in 4 ways. $|S| = \binom{52}{5}$

The probability that the hand is a three of a kind:

$$\frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} = 0.00198079231$$

(d):

**Solution:**

If we order the 5-card hand with the two-of-a-kind first, we have 13 ways for the number on the first two cards. Since we will have three out of four suits, we have $\binom{4}{2}$ ways to choose the suits. The remaining three cards must show different numbers than the two-of-a- kind and each other. There are $\binom{12}{3}$ ways for these numbers. The last three cards may have any of the four suits.

The probability that the hand is a two of a kind:

$$\frac{13 \cdot \binom{4}{2}\binom{12}{3} \cdot 4 \cdot 4 \cdot 4}{\binom{52}{5}} = 0.42256902761$$

Exercise 6.2.4

(a):

**Solution:**

The mutually exclusive event is there are no clubs, we have 13 ranks but only 3 suits to choose from, we choose 5 from 39 cards. The probability is:

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

(b):

**Solution:**

The mutually exclusive event is there are no 2 cards that have the same rank. Which means that all 5 cards have different ranks. The probability is:

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{\binom{13}{5}}{\binom{52}{5}}$$

(c):

**Solution:**

There are exactly one club or one spade. The probability of that is exactly 1 club plus the probability of that is exactly 1 spade minus there are both 1 club and 1 spade.

The probability is:

$$p(E) = \frac{\binom{39}{4} + \binom{39}{4} - \binom{26}{3}}{\binom{52}{5}}$$

(d):

**Solution:**

The mutually exclusive event is there are no spades and no clubs. We have 13 ranks and 2 suits to choose from. The probability is:

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{\binom{26}{5}}{\binom{52}{5}}$$

Q8

Exercise 6.3.2

(a):

**Solution:**

P(A) = 1/7

b falls in the middle out of 7 possible ways

P(B) = $\dfrac{6+5+4+3+2+1}{42}$ = 1/2

If b falls in the first place, c can fall in the 2nd, 3rd, 4th, 5th, 6th or 7th place

If b falls in the 2nd place, c can fall in the 3rd, 4th, 5th, 6th or 7th place

...

If b falls in the 6th place, c can fall in the 7th place

P(C) = $\sum_{i=1}^{5}\left(\frac{1}{7}+\frac{1}{6}+\frac{1}{5}\right)$ = 1/42

If d falls in the 1st place, e falls in the 2nd and f in the 3rd place

If d falls in the 2nd place, e falls in the 3rd and f in the 4th place

If d falls in the 3rd place, e falls in the 4th and f in the 5th place

If d falls in the 4th place, e falls in the 5th and f in the 6th place

If d falls in the 5th place, e falls in the 6th and f in the 7th place

(b):

**Solution:**

P(A|C) = P(A∩C)/P(C) = 2*(1/7*1/6*1/5*1/4) / (1/42) = 1/10

If d falls in the 1st place, e in the 2nd, f in the 3rd and b in the 4th place

If b falls in the 4th place, d in the 5th place, e in the 6th, f in the 7th place

(c):

**Solution:**

P(B|C) = P(B∩C)/P(C) = $\sum_{i=1}^{3}\frac{1}{7}\cdot\frac{1}{6}\cdot\frac{1}{5}\left(\sum_{j=i+3}^{6}\frac{1}{4}\cdot\frac{1}{3}\right)$ * P(C) = 1/420/(1/42) = 1/10

(d):

**Solution:**

P(A|B) = P(B∩A)/P(B) = 3*(1/7*1/6) * P(B) = 1/7

If b falls in the 4th place, c in the 5th

If b falls in the 4th, c in the 6th

If b falls in the 4th, c in the 7th

(e):

**Solution:**

P(A∩B) = 1/14 = (1/7)*(1/2) = P(A)P(B), A and B are independent P(A∩C)=1/420≠(1/7)*(1/42)=1/294=P(A)P(C), A and C aren't independent P(B∩C)=1/420≠(1/2)*(1/42)=1/84=P(A)P(C), B and C aren't independent

Exercise 6.3.6

(b):

**Solution:**

$P(E) = (1/3)^5 * (2/3)^5 = 0.00054$
Probability that the first five flips come up heads = $(1/3)^5$
The probability that the last five flips come up tails = $(2/3)^5$

(c):
**Solution:**
$P(E) = 1/3 * (2/3)^9 = 0.0087$
Probability that the first flip comes up heads = 1/3
Probability that the next nine flips come up tails = $(2/3)^9$

Exercise 6.4.2
(a):
**Solution:**
Let A and B be the events
A: "The die is fair"
B: "The die lands on 6 two times out of 6"

$$p(A \mid B) = \frac{0.2009 * 0.5}{0.2009 * 0.5 + 0.2966 * 0.5} = 0.4038$$

We want to determine if the die is fair given that it landed on 6 two times out of 6 tosses, that is P(A | B).

By the Bayes' theorem:

$$p(A \mid B) = \frac{p(B \mid A)p(A)}{p(B \mid A)p(A) + p(B \mid \overline{A})p(\overline{A})}$$

$$p(A) = p(\overline{A}) = 1/2$$

If the die is fair P(B | A) is the probability of getting exactly two six in a binomial experiment with probability of "success" (land on 6) 1/6 and six repeated trials

$$p(B \mid A) = \binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4$$

and $p(B \mid \overline{A})$ is the probability of getting exactly two six in a binomial experiment with probability of "success" (land on 6) 0.25 and six repeated trials

$$p(B \mid \overline{A}) = \binom{6}{2}(0.25)^2(0.75)^4$$

Therefore:

$$p(A \mid B) = \frac{0.2009 * 0.5}{0.2009 * 0.5 + 0.2966 * 0.5} = 0.4038$$

Q9

Exercise 6.5.2

(a):

**Solution:**

The range of A is {0, 1, 2, 3, 4}

There are 4 aces in a deck of cards so we can get at most 4 aces. Therefore the range of number of aces in hand is {0, 1, 2, 3, 4}

(b):

**Solution:**

The distribution over random variable A:

$$p(y) = \frac{C(4, y) \cdot C(48, 5-y)}{C(52, 5)}$$

p(A = 0) = p(0) = $\frac{C(4, 0) \cdot C(48, 5)}{C(52, 5)} \approx 0.6588;$

p(A = 1) = p(1) = $\frac{C(4, 1) \cdot C(48, 4)}{C(52, 5)} \approx 0.2995;$

: $\approx 0.0399$

: $\approx 0.0017$

p(A = 4) = p(4) = $\frac{C(4, 4) \cdot C(48, 1)}{C(52, 5)} \approx 0.00001$

Exercise 6.6.1

(a):

**Solution:**

$E|G| = 0 \cdot p(G = 0) + 1 \cdot p(G = 1) + 2 \cdot p(G = 2) = 1.4$

G is the random variable denoting the number of girls chosen and the range X(G) = [0, 2]. The |S| = $\binom{10}{2}$(since we are choosing two from ten).

1. Choose zero girl and two boys:

$$p(G = 0) = \frac{\binom{7}{0}\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45}$$

2. Choose 1 girl and one boy:

$$p(G = 1) = \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{21}{45}$$

3. Choose 2 girls and zero boy:

$$p(G=2) = \frac{\binom{7}{2}\binom{3}{0}}{\binom{10}{2}} = \frac{21}{45}$$

$$E\left|G\right| = 0 \cdot p\left(G = 0\right) + 1 \cdot p\left(G = 1\right) + 2 \cdot p\left(G = 2\right) = 63/45 = 1.4$$

Exercise 6.6.4

(a):

**Solution:**

Let x be the value of the die, the range of x is {1, 2, 3, 4, 5, 6}. X(x) = x^2 , the range of X(x) is { 1, 4, 9, 16, 25, 36}

And It has uniform probability.

$$E\left|X\right| = \frac{1}{6}\left[1 + 4 + 9 + 16 + 25 + 36\right] \approx 15.167$$

(b):

**Solution:**

$$E\left|Y\right| = y_0 \cdot p\left(y_0\right) + y_1 \cdot p\left(y_1\right) + y_2 \cdot p\left(y_2\right) + y_3 \cdot p\left(y_3\right) = 3$$

The probability of getting a head on a fair coin is 1/2. Denote the value of getting a head on a coin toss by x, thus for three coin tosses we have $0 \le y \le 3$. Note that for random variable Y we have Y (y) = y 2 , thus the range of Y (y) = {0, 1, 4, 9}. Let $y_i$ be the event of getting i heads during three flips, thus $0 \le i \le 3$. The total number of outcomes on three coin flips is 8

$$P\left(y_0\right) = \frac{1}{8}; P\left(y_1\right) = \frac{\binom{3}{1}}{8}; P\left(y_2\right) = \frac{\binom{3}{2}}{8}; P\left(y_3\right) = \frac{1}{8}$$

$$E\left|Y\right| = y_0 \cdot p\left(y_0\right) + y_1 \cdot p\left(y_1\right) + y_2 \cdot p\left(y_2\right) + y_3 \cdot p\left(y_3\right) = 3$$

Exercise 6.7.4

(a):

**Solution:**

$$E\left|X\right| = 10 \cdot E\left[X_1\right] = 1$$

There are ten ways to choose from. Thus for $1 \le i \le 10$, define random variables $X_1, \cdots, X_{10}$ where $X_i = 1$: if coat j is given to a student; $X_i = 0$: if coat j is not given to a student.

P($X_i$= 1) = 1/10 and P($X_i$ = 0) = 1 − 1 /10.

Applying linearity of expectation:

E[$X$] = E[$X_1 + \cdots + X_{10}$ = E[$X_1$] + · · · + E[$X_{10}$] = 10E[$X_1$]

The uniform probability for each coat assigned to a random student is 1/10.

$$E\left|X\right| = \left(\frac{1}{10}\right) \cdot 1 + \left(1 - \frac{1}{10}\right) \cdot 0$$

$$E\left|X\right| = 10 \cdot E\left[X_1\right] = \frac{10}{10} = 1$$

Q10

Exercise 6.8.1

n = 100, p = 1%

X = number of defects, $p(X = k) = \binom{n}{k} p^k \left(1 - p\right)^{n-k}$

(a):

**Solution:**

P(X =2) = 0.1849

(b):

**Solution:**

$p(X \geq 2) = 1 - \left[P\,(X = 0) + P\,(X = 1)\right]$ = 0.2642

At 1 defect is P(X = 0) + P(X = 1) = 0.7358

So $p(X \geq 2) = 1 - \left[P\,(X = 0) + P\,(X = 1)\right]$= 0.2642

(c):

**Solution:**

The expected number of defects(formula for binomial):

E[X] = np = 100 x 0.01 =1

(d):

**Solution:**

n = 50, p = 0.01 and if X ~ B(n, P) then X = 1 means a batch has defect so two boards have defects

Means P(X = 1) = 0.3056 and P(X = 0) = 0.6050

So probability of at least two defects is $p(X \geq 2) = 1 - \left[P\,(X = 0) + P\,(X = 1)\right]$= 0.0894 which is much lower than before

Expected number of defective batches is np = 50 * 0.01 = 0. 5 and each defective batch has 2 defects so Expected number of defective boards is 1

Exercise 6.8.3

(b):

**Solution:**

This probability is equivalent to the probability of getting at least 4 heads in tossing a biased coin 10 times. X~binomial(10, 0.3). The PMF of X is as follows:

$p\,(X = x) = \binom{10}{x} 0.3^x\,(1 - 0.3)^{10-x},\ x = 0,1,....,10$

Therefore, this event is the complement event of getting 0,1,2,3 heads in tossing a biased coin 10 times:

$$\begin{aligned} p\,(X \geq 4) &= 1 - p\,(X < 4) \\ &= 1 - \left[P\,(X = 0) + P\,(X = 1) + P\,(X = 2) + P\,(X = 3)\right] \\ &= 1 - \left[\binom{10}{0}(0.3)^0\,(0.7)^{10} + \binom{10}{1}(0.3)^1\,(0.7)^9 + \binom{10}{2}(0.3)^2\,(0.7)^8 + \binom{10}{3}(0.3)^3\,(0.7)^7\right] \end{aligned}$$

$p\,(X \geq 4) = 0.3504$