Constructive Geometric Sequence Notation for the Universal Matrix

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Introduction

This document formalises the *Universal Matrix* sequence originally introduced as a visual heuristic in *Universal Matrix Definition & Visual Proof, Visual Heuristic, Logical Chain* [1]. By translating every picture-only step into precise Euclidean notation, we prove that the sequence is **canonically determined** by four global principles:

P1: *Least Action* – add the minimum number of operations.

P2: *No Ambiguity* – at each stage exactly one configuration satisfies all rules; every rival construction is rejected.

P3: *Self-Reflection* – Steps 7–12 mirror Steps 1–6 inside the same hexagon.

P4: *Inheritance* – each state $\mathcal{X}_n$ strictly contains $\mathcal{X}_{n-1}$.

The result is a twelve-shape cycle that leaves *no room for alternatives*: any deviation violates at least one principle.

Notation

- S_n – set of points at Step n.
- $\mathcal{L}_n$ – set of straight segments or whole lines at Step n.
- $\mathcal{X}_n = S_n \cup \mathcal{L}_n$ – full configuration.
- $\overline{AB}$ denotes the *segment* with endpoints A, B; ℓ_{AB} (or $\overleftrightarrow{AB}$) denotes the *infinite line* through A, B.

Constructive Geometric Sequence Notation

We define a sequence of sets $\{S_n\}_{n=1}^{\infty}$, where each S_n represents a geometric configuration at step n.

Step 1 – A Point

Let

$$S_1 = \{A\}, \quad \text{where } A \in \mathbb{R}^2$$

A single point. The base of the construction.

Step 2 – A Line Segment

Add a second distinct point:

$$S_2 = S_1 \cup \{B\} = \{A, B\}, \quad A \neq B$$

Define line segment:

$$\mathcal{L}_2 = \{\overline{AB}\}, \qquad \ell_{AB} = \overleftrightarrow{AB}.$$

Step 3a – Rejected Case: Three Collinear Points

Add point C such that

$$C \in \ell_{AB} \quad \Rightarrow \quad \text{Points } A, B, C \text{ are collinear}$$

Define:

$$S_3' = S_2 \cup \{C\}, \quad \text{with } C \in \ell_{AB}$$

This case is **rejected** since it yields no new directions, only extending ℓ_{AB}:

$$\boxed{S_3' \text{ is a dead end. No new shape is possible. Terminate.}}$$

Step 3b – Accepted Case: Triangle

Instead, add point C such that

$$C \notin \ell_{AB} \quad \Rightarrow \quad \text{Points } A, B, C \text{ are non-collinear}$$

Define:

$$S_3 = \{A, B, C\}$$

And triangle:

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$$

This forms the first **enclosed shape** in the system.

Summary of Rejection Logic (Visual Proof Component)

To embed visual contradiction logic into formal notation:

- Let $\text{Collinear}(X, Y, Z)$ denote the predicate: "points X, Y, Z are collinear".
- If $\text{Collinear}(A, B, C) = \text{True}$, then no area is defined; the configuration remains linear:
$$\exists t \in \mathbb{R} : C = A + t(B - A)$$
- Such a configuration does not enable further intersection-based construction and is therefore excluded from the Universal Matrix:
$$\boxed{S_3' \notin \mathcal{U}} \quad \text{where } \mathcal{U} = \text{valid states of the Universal Matrix}$$

Step 4 – Constructing the Quadrilateral

From the previous step:

$$S_3 = \{A, B, C\} \quad \text{with } \triangle ABC \text{ non-degenerate}$$

Add Fourth Point Outside the Triangle

Let
$$D \notin \triangle ABC \quad \text{and} \quad D \notin \text{any line or edge of } \triangle ABC$$
Construct:
$$S_4 = S_3 \cup \{D\} = \{A, B, C, D\}$$

This creates a **convex quadrilateral**:

$$Q = \square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

Or more generally:

$$Q = \bigcup_{i=1}^{4} \overline{P_i P_{i+1}} \quad \text{with } P_1 = A,\ P_2 = B,\ P_3 = C,\ P_4 = D,\ P_5 = A$$

Principle of Optimality (Least Ambiguity)

To formalize the justification for rejecting alternatives:

Rejected Case: Adding Point Inside the Triangle

Let

$$D' \in \text{Interior}(\triangle ABC)$$

This leads to a requirement to define intersection points:

- Must define diagonals (not present yet).
- Must calculate intersection:

$$O = \overline{AC} \cap \overline{BD}$$

- Requires multiple operations and introduces **ambiguity**:
  - Where exactly inside?
  - Which diagonals to prioritize?

We state:

$$\boxed{D' \in \triangle ABC \Rightarrow \text{Rejected due to higher complexity and lower clarity.}}$$

Conclusion for Step 4

$$\boxed{S_4 = \{A, B, C, D\} \quad \text{with } D \notin \triangle ABC}$$

$$\boxed{\square ABCD \in \mathcal{U} \quad \text{(valid Universal Matrix state)}}$$

All other configurations at step 4 are either ambiguous or require premature rule-breaking (e.g., internal intersections), and are therefore **excluded** by the visual logic principle.

Step 5 – Central Point from Diagonal Intersection

We start with:

$$S_4 = \{A, B, C, D\} \quad \text{defining a convex quadrilateral } \square ABCD$$

Construct Diagonals

Let:

$$d_1 = \overline{AC}, \quad d_2 = \overline{BD}$$

Then the intersection point:

$$E = d_1 \cap d_2$$

We define:

$$S_5 = S_4 \cup \{E\} = \{A, B, C, D, E\}$$

This yields a structure where:

- All original vertices A, B, C, D remain.
- E lies in the **center**, unambiguously defined by the intersection.
- No ambiguity: diagonals intersect in exactly **one** point.

The resulting structure is:

$$\mathcal{X}_5 = \square ABCD \cup \{E\} \quad \text{where } E = \overline{AC} \cap \overline{BD}$$

This satisfies:

- **1 action** (draw two diagonals)
- **1 point** added
- **No alternatives**: only one point of intersection

Rejected Case: Adding a Fifth Dot to Form a Pentagon

Attempt to Add Point P Outside Quadrilateral

Let:

$$P \notin \square ABCD$$

Then attempt to form:

$$S_5' = S_4 \cup \{P\} \quad \text{with edges such as } \overline{AB}, \overline{BC}, \overline{CD}, \overline{DP}, \overline{PA} \Rightarrow \text{Pentagon}$$

Why Pentagon Is Rejected

1. **Ambiguity**:
   - Where to place P? Above, below, or near which edge?
   - Multiple interpretations exist
   - No defined method yields a **unique** point like diagonal intersection
2. **Extra Construction Steps**:
   - Constructing a regular or visually consistent pentagon requires:
     - Circular arc construction, or
     - Compass and angle-division methods
   - This contradicts the principle of **least action**
3. **Logical Chain Break**:
   - A pentagon introduces 5-way symmetry
   - The Universal Matrix logic depends on **central convergence**, such as diagonal intersections
   - Arbitrary point addition leads to inconsistency in subsequent steps

Final Statement

$$\boxed{\text{The pentagon is rejected due to ambiguity, higher complexity, and incompatibility with further development.}}$$

Summary of Step 5 Notation

Valid Construction:

$$S_5 = \{A, B, C, D, E\} \quad \text{where } E = \overline{AC} \cap \overline{BD}$$

$$\mathcal{X}_5 = \square ABCD \cup \{E\}$$

Rejected:

$$S_5' = S_4 \cup \{P\}, \quad P \notin \square ABCD \Rightarrow \text{Ambiguous pentagon} \notin \mathcal{U}$$

Step 6 – Construction of the Hexagon

From the previous step:

- We have quadrilateral $\square ABCD$
- Point $E = \overline{AC} \cap \overline{BD}$, the center

We now embed this into a regular hexagonal structure that:

- Uses 6 points
- Has strong symmetry
- Grows unambiguously

Let the center point be:

$$O = E$$

Construction Logic

1. From center O, construct 6 equidistant points on a circle around O
   - Let radius r be arbitrary but fixed
   - Angle between adjacent points: $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$
2. Define the 6 vertices:

$$P_k = O + r \cdot \left(\cos\left(\frac{2\pi(k-1)}{6}\right), \sin\left(\frac{2\pi(k-1)}{6}\right) \right), \quad k = 1, \ldots, 6$$

These 6 points define the regular hexagon:

$$H = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

With sides:

$$\mathcal{L}_6 = \{\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_6P_1}\}$$

Define:

$$S_6 = H \cup \{O\}$$

This construction is:

- **Unambiguous** (defined by fixed radius and angle)
- **Natural** continuation from diagonal intersection in Step 5
- **Symmetric** and foundational for future intersection-based constructions

Rejected Alternatives

Case 1: Add One More Point Outside the Quadrilateral

Let F be a new point such that:

$$F \notin \square ABCD, \quad F \notin \text{existing diagonals}, \quad F \notin H$$

This creates:

- 5 external points with internal center E
- Ambiguity: where to place F?
- Multiple possible outcomes: star, warped hexagon, etc.

$$\boxed{\text{Ambiguous shape with fake symmetry — rejected.}}$$

Case 2: Pentagon With Center (Trying to Salvage Step 5)

Attempting:

$$S_6' = \text{Pentagon } P_1 \dots P_5 \cup \{O\}$$

Issues:

- Appears constructive
- But breaks rotational symmetry: 5-fold + center = contradiction
- Intersections cannot align cleanly in further steps

$$\boxed{\text{Pentagon + intersection is not constructible from prior logic — rejected.}}$$

Case 3: Add a Triangle On Top of Quadrilateral (”House” Shape)

- Involves extra point and three lines
- Must choose an edge $\to$ introduces ambiguity
- Already rejected in Step 3 (triangle paths do not evolve)

$$\boxed{\text{Triangle-topped quadrilateral is asymmetric and rejected by prior logic.}}$$

Final Construction for Step 6

$$\boxed{S_6 = \{O, P_1, P_2, P_3, P_4, P_5, P_6\} \quad \text{(Hexagon with center)}}$$

Where:

- O is the intersection of the diagonals of $\square ABCD$
- Each P_k is placed at angular spacing $\frac{\pi}{3}$ around O

This is the first fully symmetric, enclosed structure, enabling the mirror-like geometric growth of steps 7–12.

Step 7 – First Internal Intersection of Hexagon Diagonals

From the previous step:

$$S_6 = \{O, P_1, P_2, P_3, P_4, P_5, P_6\}$$

where the points P_i form a regular hexagon around the center O, which originated in Step 5.

Construction of Diagonals

Define the **major diagonals** of the regular hexagon:

$$d_1 = \overline{P_1P_4}, \quad d_2 = \overline{P_2P_5}, \quad d_3 = \overline{P_3P_6}$$

Each diagonal connects a pair of opposite vertices in the hexagon. These are the longest internal chords.

All three intersect at the center:

$$P = d_1 \cap d_2 \cap d_3 = O$$

Define:

$$S_7 = S_6 \cup \{P\} = S_6 \quad \text{(since } P = O \text{ already existed)}$$

Although the point set remains unchanged, this step introduces a shift in **structural interpretation**: the point O now serves as the *intersection of hexagonal diagonals*, not merely a quadrilateral center. It is elevated to a **central node of symmetric intersections**.

Why This Step Is Unique and Unambiguous

- Only 3 major diagonals exist between opposite hexagon vertices.
- These diagonals intersect at a single, clearly defined point O.
- No other point satisfies the rotational and reflectional symmetry constraints.
- No new vertex or edge can be added without violating symmetry or logic.

Therefore, the structure at Step 7 is:

$$\mathcal{X}_7 = \text{Regular hexagon } P_1 \dots P_6 \text{ with intersecting diagonals at } O$$

Rejected Alternatives

There are none at this stage.

Attempting to Add Another Point

- Would require arbitrary placement.
- Breaks symmetry.
- Lacks geometric justification by intersection or rotation.

$\boxed{S_7 = S_6 \cup \{\text{internal diagonals and point } O\}}$ with no alternative construction path

Step 8 – Internal Lines Reflecting Step 2

At this stage we have

- the regular hexagon from Step 6 with vertices $P_1, P_2, P_3, P_4, P_5, P_6$ listed counter-clockwise, P_1 at the top;
- the central point O from Step 7 (common intersection of the three major diagonals).

Objective

Re-enact the "single straight segment" of **Step 2** inside the hexagon while *(a)* preserving the global orientation fixed in Step 6 and *(b)* introducing no new construction rules.

Directional Ambiguity

The hexagon possesses three opposite-vertex diagonals

$$d_1 = \overline{P_1P_4}, \qquad d_2 = \overline{P_2P_5}, \qquad d_3 = \overline{P_3P_6}.$$

Any of them could serve as the "internal line", and rigid rotations would map one choice to another. A tie-breaker is required that costs *zero* additional operations.

Resolution: Adopt the Orientation Fixed in Step 6

Step 6 placed vertex P_1 exactly at the top, giving a canonical vertical axis. Hence we *must* choose

$$d_1 = \overline{P_1P_4}$$

as the primary internal line.

To complete the reflection of a straight two-point segment while maintaining bilateral symmetry, draw the two chords

$$\ell_1 = \overline{P_2P_6}, \qquad \ell_2 = \overline{P_3P_5},$$

which are perpendicular to d_1.

New intersection points. These chords cut d_1 at two distinct points:

$$I_1 = d_1 \cap \ell_1, \qquad I_2 = d_1 \cap \ell_2.$$

Both lie strictly between P_1 and P_4 and satisfy $I_1, O, I_2 \subset d_1$.

Internal line set

$$\mathcal{L}_8 \ = \ \big\{\overline{P_1P_4},\ \overline{P_2P_6},\ \overline{P_3P_5}\big\}, \qquad \text{new point set } \mathcal{P}_8^{\text{new}} = \{I_1, I_2\}.$$

Rejected Alternatives

- *Choosing d_2 or d_3 as the main axis.* Merely yields an isomorphic configuration rotated by $\pm 60°$; the canonical orientation would be lost.
- *Adding extra chords.* Any further chord either duplicates an existing diagonal or bypasses the centre O, breaking symmetry and violating the least-action principle.

Resulting Structure

$$\boxed{\mathcal{X}_8 \ = \ \mathcal{X}_7 \ \cup \ \mathcal{L}_8 \quad \text{and} \quad S_8 \ = \ S_7 \cup \{I_1, I_2\}.}$$

This finalises the internal analogue of Step 2: a single collinear triple (P_1, O, P_4) intersected by two perpendicular chords, producing the uniquely defined points I_1 and I_2 that drive Steps 10–12.

Step 9 – Completing the Central Three-DiagonalStar

State after Step8. We currently have the internal line-set

$$\mathcal{L}_8 = \{ \underbrace{\overline{P_1P_4}}_{\text{vertical diagonal}}, \underbrace{\overline{P_2P_6}}_{\text{horizontal chord}}, \underbrace{\overline{P_3P_5}}_{\text{horizontal chord}} \},$$

together with the centre O (common intersection of the three chords from Step7) and the two points

$$I_1 = \overline{P_1P_4} \cap \overline{P_2P_6}, \quad I_2 = \overline{P_1P_4} \cap \overline{P_3P_5}.$$

Objective. Insert the two *remaining* opposite-vertex diagonals so that all three major diagonals concur at O. This mirrors the triadic structure introduced in Step3.

Construction. Add the line set

$$\mathcal{L}_9 = \{\overline{P_2P_5}, \overline{P_3P_6}\}.$$

Both diagonals pass through O; hence

$$O = \overline{P_1P_4} \cap \overline{P_2P_5} \cap \overline{P_3P_6}.$$

Resulting structure.

$$\boxed{\mathcal{X}_9 = \mathcal{X}_8 \cup \mathcal{L}_9}$$

The interior now contains a three-line star (the major diagonals) centred at O plus the two perpendicular chords from Step8. Points I_1, I_2 remain on $\overline{P_1P_4}$.

Why *both* diagonals are required. Adding only one of $\overline{P_2P_5}$ or $\overline{P_3P_6}$ would leave a mirror-image choice unresolved and invite further alterations, violating the least-ambiguity principle. Including both closes the symmetry and fixes the configuration uniquely.

Step 10 – Internal Quadrangle Mirroring Step 4

State carried from Step8. Two points lie on the vertical diagonal:

$$I_1 = \overline{P_1P_4} \cap \overline{P_2P_6}, \quad I_2 = \overline{P_1P_4} \cap \overline{P_3P_5}.$$

The global centre O remains from Step7 but takes no part in the current construction.

Objective. Construct the minimal, unambiguous four-edge figure inside the hexagon that plays the role of the outer quadrangle in Step4—using *only* existing points.

Construction. Draw

$$\mathcal{L}_{10} = \{\overline{I_1P_3},\ \overline{I_1P_5},\ \overline{I_2P_2},\ \overline{I_2P_6}\}.$$

These four segments intersect pairwise to create exactly two new points:

$$V_1 = \overline{I_1P_3} \cap \overline{I_2P_2}, \quad V_2 = \overline{I_1P_5} \cap \overline{I_2P_6}.$$

New point set.

$$\mathcal{P}_{10}^{\text{new}} \ = \ \{\, V_1,\, V_2\,\}, \quad S_{10} = S_8 \cup \mathcal{P}_{10}^{\text{new}}$$

(no additional "central" point is added; O remains from earlier steps).

Resulting quadrangle.

$$Q_{10} = \{I_1,\ V_1,\ I_2,\ V_2\}, \quad \mathcal{E}_{10} = \{\overline{I_1V_1},\ \overline{V_1I_2},\ \overline{I_2V_2},\ \overline{V_2I_1}\}.$$

The pair $(Q_{10}, \mathcal{E}_{10})$ forms a convex quadrilateral centred on the vertical axis, uniquely determined by the four straight connections specified above.

Minimality. Removing any of the four segments leaves an open three-sided shape; adding a fifth produces redundancy or an extra vertex, violating the least-action rule. Consequently this quadrangle is the sole unambiguous analogue of Step4 inside the hexagon.

Step11 – Diagonal Intersection Mirroring Step 5

State after Step10. We possess the quadrilateral

$$Q_{10} = \{I_1,\, V_1,\, I_2,\, V_2\}$$

with existing edge set $\mathcal{E}_{10} = \{\overline{I_1V_1}, \overline{V_1I_2}, \overline{I_2V_2}, \overline{V_2I_1}\}$ and its vertical diagonal

$$d_1 = \overline{I_1I_2} \subset \overline{P_1P_4},$$

which already contains the global centre O.

Objective. Replicate the logic of **Step 5**: introduce the second diagonal of the quadrangle, obtain its intersection with the first, and thereby anchor the structure centrally.

Construction. Add the single segment

$$d_2 = \overline{V_1V_2}.$$

Because Q_{10} is mirror-symmetric about the vertical axis,

$$d_1 \cap d_2 \ = \ \{\, O\,\}.$$

No new point is created; the intersection coincides with the pre-existing centre O.

Updated line and point sets.

$$\mathcal{L}_{11} = \{\, d_2 \,\}, \qquad \mathcal{X}_{11} = \mathcal{X}_{10} \cup \mathcal{L}_{11}, \quad S_{11} = S_{10}$$

(points unchanged).

Minimality. Adding d_2 closes the quadrangle's pair of diagonals with a single action. Any additional segment would either duplicate an edge/diagonal or introduce an unnecessary vertex, breaching the least-action and no-ambiguity principles.

$$\boxed{\text{Step11 complete: } Q_{10} \text{ now carries both diagonals } d_1, d_2 \text{ intersecting at } O.}$$

Step 12 – Inner Hexagon via Skip-One Vertex Connections

Starting point. Retain the outer regular hexagon from Step6 with vertices $P_1, P_2, P_3, P_4, P_5, P_6$ in counter-clockwise order (P_1 at the top).

Objective. Complete the twelve-dot sequence by the least-action scheme: re-connect *existing* vertices so that an internal hexagon, rotated by 90°, appears—mirroring Step6 without adding any new construction rules.

Construction: two interlaced triangles

Draw the six "skip-one" edges

$$\mathcal{L}_{12} = \Big\{ \underbrace{\overline{P_1P_3},\ \overline{P_3P_5},\ \overline{P_5P_1}}_{\triangle_\uparrow},\ \underbrace{\overline{P_2P_4},\ \overline{P_4P_6},\ \overline{P_6P_2}}_{\triangle_\downarrow} \Big\}.$$

These six segments form two equilateral triangles $\triangle_\uparrow$ and $\triangle_\downarrow$, often visualised as a "Star of David" but here used purely for their intersection pattern.

New intersection points. Each edge of $\triangle_\uparrow$ meets two edges of $\triangle_\downarrow$ inside the original hexagon; the six pairwise intersections, ordered counter-clockwise, are

$$\begin{aligned} R_1 &= \overline{P_1P_3} \cap \overline{P_6P_2}, \quad & R_2 &= \overline{P_1P_3} \cap \overline{P_2P_4}, \\ R_3 &= \overline{P_3P_5} \cap \overline{P_2P_4}, \quad & R_4 &= \overline{P_3P_5} \cap \overline{P_4P_6}, \\ R_5 &= \overline{P_5P_1} \cap \overline{P_4P_6}, \quad & R_6 &= \overline{P_5P_1} \cap \overline{P_6P_2}. \end{aligned}$$

Set $\mathcal{P}_{12}^{\text{new}} = \{R_1, R_2, R_3, R_4, R_5, R_6\}, \quad S_{12} = S_{11} \cup \mathcal{P}_{12}^{\text{new}}$.

Resulting inner hexagon

$$H_{\text{inner}} = \{R_1, R_2, R_3, R_4, R_5, R_6\}, \qquad \mathcal{E}_{\text{inner}} = \big\{ \overline{R_1R_2},\ \overline{R_2R_3},\ \overline{R_3R_4},\ \overline{R_4R_5},\ \overline{R_5R_6},\ \overline{R_6R_1} \big\}.$$

Basic angle-chasing (or symmetry) shows all $\|R_i - R_j\|$ for adjacent indices are equal, so H_{inner} is a *regular* hexagon whose orientation is rotated by 30° (one sixth-turn) relative to the outer one from Step6.

Minimality and uniqueness. No extra segments are needed: the six skip-one edges both create the inner hexagon and saturate the symmetry. Adding fewer edges leaves an open six-point star; adding more duplicates existing work or yields non-essential chords, violating least-action.

$$\boxed{\mathcal{X}_{12} = \mathcal{X}_{11} \cup \mathcal{L}_{12}, \quad S_{12} = S_{11} \cup \{R_1, \ldots, R_6\}}$$

Step12 thus finalises the Universal-Matrix cycle: the interior now contains a second regular hexagon, mirrored and rotated, completing the twelve-dot sequence with perfect internal–external symmetry.

References

[1] Kraskov, Artur. *Universal Matrix Definition & Visual Proof, Visual Heuristic, Logical Chain.* 2024. DOI: https://doi.org/10.13140/RG.2.2.10045.47843