Archglacor back to back arms problem

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Suppose the Archglacor does N specs, what is the probability of getting the arms spec back to back at least once during those N specs?

Let's label the specs with letters, so let A denote the arms spec and B to E denote the other specs in some fixed order (the order won't matter). We know the Archglacor will always start by doing a set of its 5 specs in some random order without repeats, and after that do another set of random 5 specs without repeats and so on. So essentially this is equivalent to generating a sequence of random permutations of $\{A, B, C, D, E\}$, and we assume all permutations are equally likely, then we are looking for sequences of the form:

$$[AEDBC][DBEAC][BECDA][ACEBD]\dots$$

where at some point in the chain of specs or letters, we get $\dots, A][A, \dots$ at least once. It's clear back to back arms or back to back A can only occur at 'intersections'][, where you get A at the end of a set of 5 spec chain, AND at the start of the next set of 5 spec chain. In N specs there are $n = \lfloor \frac{N-1}{5} \rfloor$ 'intersections' where back to back arms can occur.

Let $P(n)$ denote the probability of at least one back to back arms spec in n 'intersections', (equivalently $\dots, A][A, \dots$ observed at least once in n 'intersections'). To get a feel for the problem, start with small cases. For $P(1)$, this is simply $0.2 * 0.2 = 0.04$ (0.2 chance of getting A at the end of the first set of 5 specs, and 0.2 chance of getting A at the start of the second set of 5 specs, and these events are independent). For $P(2)$, we have $P(2) = p(1) + p(2)$, where $p(i)$ is the probability that the first back to back arms spec occurs at the ith intersection. Clearly $p(1) = P(1) = 0.04$, and it is not hard to see $p(2) = 0.2 * 0.2 = 0.04$ as well, since once the specs at the second intersection are given to be $A][A$, which has probability of 0.04, the specs in the first set of 5 specs can be any permutation of $\{A, B, C, D, E\}$ (so probability 1), and so $P(2) = 0.08$. Similarly, $P(n) = \sum_{i=1}^{n} p(i)$

To compute $p(n)$ for $n > 2$, we have $p(n) = 0.04(1 - P(n-2))$, since there is again probability 0.04 of back to back arms occurring at the nth intersection, but we must also have no back to back arms occurring before that, which has probability $(1-P(n-2))$. More precisely, $p(n) = P(\text{b2b arms at nth intersection} \cap \text{no b2b arms before } nth \text{ intersection}) = P(\text{b2b arms at nth intersection})P(\text{no b2b arms before } nth \text{ intersection}|\text{b2b arms at nth intersection})$. This gives the recurrence relation:

$$p(n) = 0.04(1 - P(n-2)) = 0.04(1 - \sum_{i=1}^{n-2} p(i)) \tag{1}$$

with 'initial conditions' $p(1) = p(2) = 0.04$. At this point one can write a program recursively to solve for $p(n)$ and therefore $P(n)$. However, an analytical solution for $P(n)$ is in fact available. Notice we also have:

$$p(n-1) = 0.04(1 - P(n-3)) = 0.04(1 - \sum_{i=1}^{n-3} p(i)) \tag{2}$$

subtracting (2) from (1) yields:

$$p(n) - p(n-1) = -0.04p(n-2) \tag{3}$$

This is a 2nd order homogeneous difference equation, which can be solved via standard methods. The difference equation (3) has auxiliary equation:

$$x^2 - x + 0.04 = 0 \tag{4}$$

which has roots $r_1 = \frac{5+\sqrt{21}}{10}$ and $r_2 = \frac{5-\sqrt{21}}{10}$. So the solution to (3) is of the form:

$$p(n) = ar_1^n + br_2^n \tag{5}$$

where a and b can be obtained from the initial conditions, to be $a = \frac{0.04(1-r_2)}{r_1(r_1-r_2)}$ and $b = \frac{0.04(1-r_1)}{r_2(r_2-r_1)}$ respectively. Now that we have $p(n)$, the formula for $P(n) = \sum_{i=1}^{n} p(i)$ can be obtained by the sum of geometric series formula:

$$P(n) = ar_1 \frac{(1-r_1^n)}{1-r_1} + br_2 \frac{(1-r_2^n)}{1-r_2} \tag{6}$$

As an example, for Evil Lucario's world first 4000% Archglacor kill, the Archglacor did between $56-60$ specs (unless I counted incorrectly), which gives $P(11) \approx 0.3732$, so there was about a 37.32% chance of getting back to back arms that kill.

Markov Chains approach

There is an alternative method using Markov Chains (perhaps easier, but less accessible, since you need to know what a Markov Chain is, or at least have some basic knowledge of linear algebra in addition to elementary discrete probability). Suppose state 1 is not getting $A]$, state 2 is getting $A]$ and state 3 is getting $A][A$. Let $p_{ij}^{(n)}$ be the probability of moving from state i to state j in n steps, where a 'step' is a new set of $\{A, B, C, D, E\}$ being generated. The transition matrix of this Markov Chain is (sorry but I can't be bothered to draw the Markov Chain in LaTeX):

$$\begin{bmatrix} 4/5 & 1/5 & 0 \\ 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 \end{bmatrix}$$

Note the probability we are interested in is $p(n) = \frac{1}{25}(p_{11}^{(n-1)} + p_{12}^{(n-1)})$. To solve for $p_{11}^{(n-1)}$ and $p_{12}^{(n-1)}$, we have:

$$p_{11}^{(n)} = p_{11}^{(n-1)} p_{11} + p_{12}^{(n-1)} p_{21} = p_{11}^{(n-1)} \frac{4}{5} + p_{12}^{(n-1)} \frac{3}{5}$$

$$p_{12}^{(n)} = p_{11}^{(n-1)} p_{12} + p_{12}^{(n-1)} p_{22} = p_{11}^{(n-1)} \frac{1}{5} + p_{12}^{(n-1)} \frac{1}{5}$$

We can write these equations in matrix form:

$$\begin{bmatrix} p_{11}^{(n)} \\ p_{12}^{(n)} \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} p_{11}^{(n-1)} \\ p_{12}^{(n-1)} \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 4/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

then clearly we have

$$\begin{bmatrix} p_{11}^{(n)} \\ p_{12}^{(n)} \end{bmatrix} = A^n \begin{bmatrix} p_{11}^{(0)} \\ p_{12}^{(0)} \end{bmatrix}$$

A^n is tedious to compute. To get around this we can diagonalise the matrix A, in other words, find a diagonal matrix:

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

such that $A = P^{-1}DP$ and λ_1 and λ_2 are the eigenvalues of A. So $A^n = P^{-1}D^nP$, but D^n is easy to compute since

$$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

Note the eigenvalues λ of this matrix satisfy $\det(A - \lambda I) = 0$, in other words:

$$\begin{aligned} (\frac{4}{5} - \lambda)(\frac{1}{5} - \lambda) - \frac{3}{5}\frac{1}{5} &= 0 \\ \lambda^2 - \lambda + \frac{1}{25} &= 0 \end{aligned}$$

which unsurprisingly is exactly the same equation as (4) and hopefully it's obvious how to get to the same answer from here, by recalling that $p(n) = \frac{1}{25}(p_{11}^{(n-1)} + p_{12}^{(n-1)}), p_{11}^{(0)} = 1, p_{12}^{(0)} = 0, p_{11}^{(1)} = 4/5, p_{12}^{(1)} = 1/5$.

Using Markov Chains it's also easy to compute the expected number of steps until the first back to back arms spec. Let a be the expected number of steps to hit double arms (state 3) if you're at state 1, and let b be the expected number of steps to hit double arms (state 3) if you're at state 2, then we have:

$$a = 1 + \frac{4}{5}a + \frac{1}{5}b$$
$$b = 1 + \frac{1}{5}b + \frac{3}{5}a + \frac{1}{5}0$$

which gives $a = 25$ and $b = 20$. So you expect to get the first back to back arms spec at the $25th$ set of 5 specs.