

# Archglacor back to back arms problem

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Suppose the Archglacor does  $N$  specs, what is the probability of getting the arms spec back to back at least once during those  $N$  specs?

Let's label the specs with letters, so let  $A$  denote the arms spec and  $B$  to  $E$  denote the other specs in some fixed order (the order won't matter). We know the Archglacor will always start by doing a set of its 5 specs in some random order without repeats, and after that do another set of random 5 specs without repeats and so on. So essentially this is equivalent to generating a sequence of random permutations of  $\{A, B, C, D, E\}$ , and we assume all permutations are equally likely, then we are looking for sequences of the form:

$$[AEDBC][DBEAC][BECDA][ACEBD] \dots$$

where at some point in the chain of specs or letters, we get  $\dots, A][A, \dots$  at least once. It's clear back to back arms or back to back  $A$  can only occur at 'intersections' ], where you get  $A$  at the end of a set of 5 spec chain, AND at the start of the next set of 5 spec chain. In  $N$  specs there are  $n = \lfloor \frac{N-1}{5} \rfloor$  'intersections' where back to back arms can occur.

Let  $P(n)$  denote the probability of at least one back to back arms spec in  $n$  'intersections', (equivalently  $\dots, A][A, \dots$  observed at least once in  $n$  'intersections'). To get a feel for the problem, start with small cases. For  $P(1)$ , this is simply  $0.2 * 0.2 = 0.04$  (0.2 chance of getting  $A$  at the end of the first set of 5 specs, and 0.2 chance of getting  $A$  at the start of the second set of 5 specs, and these events are independent). For  $P(2)$ , we have  $P(2) = p(1) + p(2)$ , where  $p(i)$  is the probability that the first back to back arms spec occurs at the  $i$ th intersection. Clearly  $p(1) = P(1) = 0.04$ , and it is not hard to see  $p(2) = 0.2 * 0.2 = 0.04$  as well, since once the specs at the second intersection are given to be  $A][A$ , which has probability of 0.04, the specs in the first set of 5 specs can be any permutation of  $\{A, B, C, D, E\}$  (so probability 1), and so  $P(2) = 0.08$ . Similarly,  $P(n) = \sum_{i=1}^n p(i)$

To compute  $p(n)$  for  $n > 2$ , we have  $p(n) = 0.04(1 - P(n-2))$ , since there is again probability 0.04 of back to back arms occurring at the  $n$ th intersection, but we must also have no back to back arms occurring before that, which has probability  $(1 - P(n-2))$ . More precisely,  $p(n) = P(\text{b2b arms at } n\text{th intersection} \cap \text{no b2b arms before } n\text{th intersection}) = P(\text{b2b arms at } n\text{th intersection})P(\text{no b2b arms before } n\text{th intersection} | \text{b2b arms at } n\text{th intersection})$ . This gives the recurrence relation:

$$p(n) = 0.04(1 - P(n-2)) = 0.04(1 - \sum_{i=1}^{n-2} p(i)) \quad (1)$$

with 'initial conditions'  $p(1) = p(2) = 0.04$ . At this point one can write a program recursively to solve for  $p(n)$  and therefore  $P(n)$ . However, an analytical solution for  $P(n)$  is in fact available. Notice we also have:

$$p(n-1) = 0.04(1 - P(n-3)) = 0.04(1 - \sum_{i=1}^{n-3} p(i)) \quad (2)$$

subtracting (2) from (1) yields:

$$p(n) - p(n-1) = -0.04p(n-2) \quad (3)$$

This is a 2nd order homogeneous difference equation, which can be solved via standard methods. The difference equation (3) has auxiliary equation:

$$x^2 - x + 0.04 = 0 \quad (4)$$

which has roots  $r_1 = \frac{5+\sqrt{21}}{10}$  and  $r_2 = \frac{5-\sqrt{21}}{10}$ . So the solution to (3) is of the form:

$$p(n) = ar_1^n + br_2^n \quad (5)$$

where  $a$  and  $b$  can be obtained from the initial conditions, to be  $a = \frac{0.04(1-r_2)}{r_1(r_1-r_2)}$  and  $b = \frac{0.04(1-r_1)}{r_2(r_2-r_1)}$  respectively. Now that we have  $p(n)$ , the formula for  $P(n) = \sum_{i=1}^n p(i)$  can be obtained by the sum of geometric series formula:

$$P(n) = ar_1 \frac{(1-r_1^n)}{1-r_1} + br_2 \frac{(1-r_2^n)}{1-r_2} \quad (6)$$

As an example, for Evil Lucario's world first 4000% Archglacor kill, the Archglacor did between 56–60 specs (unless I counted incorrectly), which gives  $P(11) \approx 0.3732$ , so there was about a 37.32% chance of getting back to back arms that kill.

## Markov Chains approach

There is an alternative method using Markov Chains (perhaps easier, but less accessible, since you need to know what a Markov Chain is, or at least have some basic knowledge of linear algebra in addition to elementary discrete probability). Suppose state 1 is not getting  $A$ , state 2 is getting  $A$  and state 3 is getting  $A$ [ $A$ . Let  $p_{ij}^{(n)}$  be the probability of moving from state  $i$  to state  $j$  in  $n$  steps, where a 'step' is a new permutation of  $\{A, B, C, D, E\}$  being generated, representing the 5 unique specs from the Archglacor in some order. The transition matrix of

this Markov Chain is (sorry but I can't be bothered to draw the Markov Chain in LaTeX):

$$\begin{bmatrix} 4/5 & 1/5 & 0 \\ 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 \end{bmatrix}$$

Note the probability we are interested in is  $p(n) = \frac{1}{25}(p_{11}^{(n-1)} + p_{12}^{(n-1)})$ . To solve for  $p_{11}^{(n-1)}$  and  $p_{12}^{(n-1)}$ , we have:

$$\begin{aligned} p_{11}^{(n)} &= p_{11}^{(n-1)}p_{11} + p_{12}^{(n-1)}p_{21} = p_{11}^{(n-1)}\frac{4}{5} + p_{12}^{(n-1)}\frac{3}{5} \\ p_{12}^{(n)} &= p_{11}^{(n-1)}p_{12} + p_{12}^{(n-1)}p_{22} = p_{11}^{(n-1)}\frac{1}{5} + p_{12}^{(n-1)}\frac{1}{5} \end{aligned}$$

We can write these equations in matrix form:

$$\begin{bmatrix} p_{11}^{(n)} \\ p_{12}^{(n)} \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} p_{11}^{(n-1)} \\ p_{12}^{(n-1)} \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 4/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

then clearly we have

$$\begin{bmatrix} p_{11}^{(n)} \\ p_{12}^{(n)} \end{bmatrix} = A^n \begin{bmatrix} p_{11}^{(0)} \\ p_{12}^{(0)} \end{bmatrix}$$

$A^n$  is tedious to compute. To get around this we can diagonalise the matrix  $A$ , in other words, find a diagonal matrix:

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

such that  $A = P^{-1}DP$  and  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $A$ . So  $A^n = P^{-1}D^nP$ , but  $D^n$  is easy to compute since

$$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

Note the eigenvalues  $\lambda$  of this matrix satisfy  $\det(A - \lambda I) = 0$ , in other words:

$$\begin{aligned} \left(\frac{4}{5} - \lambda\right)\left(\frac{1}{5} - \lambda\right) - \frac{3}{5}\frac{1}{5} &= 0 \\ \lambda^2 - \lambda + \frac{1}{25} &= 0 \end{aligned}$$

which unsurprisingly is exactly the same equation as (4) and hopefully it's obvious how to get to the same answer from here, by recalling that  $p(n) = \frac{1}{25}(p_{11}^{(n-1)} + p_{12}^{(n-1)})$ ,  $p_{11}^{(0)} = 1, p_{12}^{(0)} = 0, p_{11}^{(1)} = 4/5, p_{12}^{(1)} = 1/5$ .

Using Markov Chains it's also easy to compute the expected number of steps until the first back to back arms spec. Let  $a$  be the expected number of steps to hit double arms (state 3) if you're at state 1, and let  $b$  be the expected number of steps to hit double arms (state 3) if you're at state 2, then we have:

$$a = 1 + \frac{4}{5}a + \frac{1}{5}b$$

$$b = 1 + \frac{1}{5}b + \frac{3}{5}a + \frac{1}{5}0$$

which gives  $a = 25$  and  $b = 20$ . So you expect to get the first back to back arms spec at the 25th set of 5 specs.