Games Problem Sheet

Preface

This problem set is longer and more difficult than is probably reasonable for a problem set.

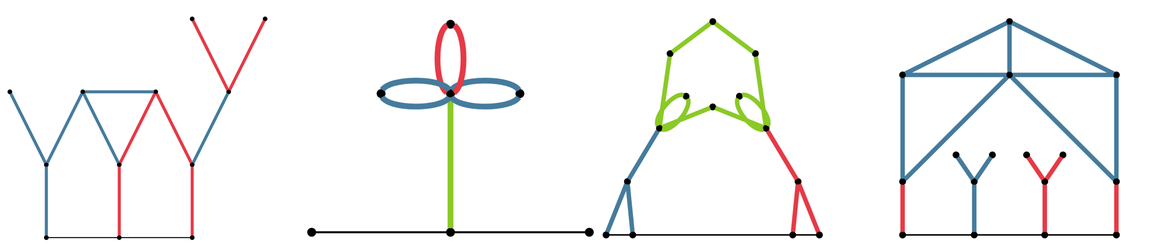
The idea is that there are some easy problems, but if you're intrigued by certain elements of this workshop, then there are questions there to pique your interest.

Problems preceded by more 🌶s are more difficult.

1 Hackenbush Hijinks

1.1 Hackenbush Strategies

1. 🌶Figure out for each of the following games, whether the winning strategy ensures a win for the 1st, 2nd, Blue or Red player:



2. 🌶Prove that $a + (-a) = 0$. In other words, for any hackenbush game which has two copies, one with the red/blue edges flipped, that there is a strategy for the 2nd player to win.

3. 🌶🌶Prove the following statements about games with values X, Y and Z. From this, you can deduce that all 2nd player winning games are equivalent, in that they have no effect on play. So equating them to zero is reasonable.

- If X has a strategy for the 2nd player to win, and Y has a strategy for the 2nd player to win, then $Z = X + Y$ has a strategy for the 2nd player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the 1st player to win, then $Z = X + Y$ has a strategy for the 1st player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the Blue player to win, then $Z = X + Y$ has a strategy for the Blue player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the Red player to win, then $Z = X + Y$ has a strategy for the Red player to win.

4. 🌶Evaluate the following position, and use this to formulate an equation involving $*$, $*2$ and $*3$

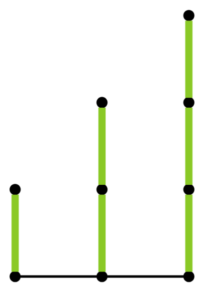


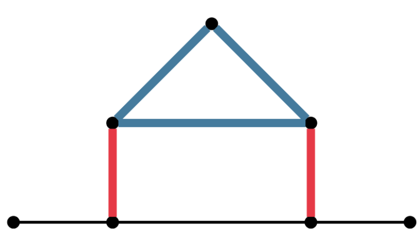
Figure 1

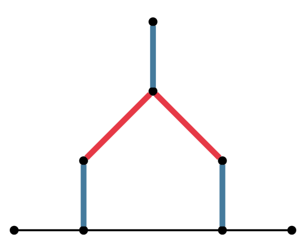
1.2 Evaluating Positions

5. 🌶Create Hackenbush positions with the following values:

- $3 + \frac{1}{4}$
- $\frac{1}{8}$
- $-\frac{15}{16}$
- $2 + * + *3$

6. 🌶🌶Evaluate the following hackenbush positions by comparing them to known games (are they less than game X, greater than game Y?)





7. From the final slide we saw that some fuzzy games, when combined, produce non-fuzzy, positive/negative results. Evaluate the following positions as positive, negative, zero, or fuzzy.

TODO: Games

8. Prove that the flower petal position plus $*$, is positive, but less than every positive number generated in blue-red hackenbush

2 Treasure Trove of Numbers

2.1 Treasure

1. Create Treasure games with the following values:

- $\frac{5}{4}$
- $\Uparrow$
- ω
- $\omega + \omega$

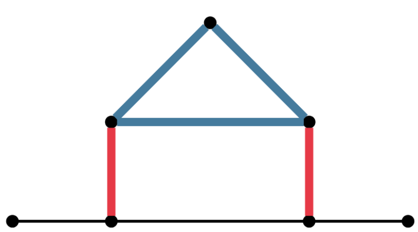
2. Prove that any formulation of bracket notation (purely built up from $\{|\}$) can be expressed as a game of Treasure.

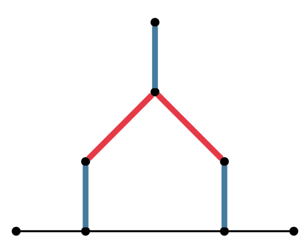
3. Provide an algorithm to combine two Treasure games A and B into a single game with value $A + B$.

2.2 A bracing garden

4. Evaluate the following hackenbush positions by recursively evaluating their left/right options in the brace notation.

Remember to use the rule of $X = \{a, b, c|d, e, f\}$ satisfying $a, b, c < X < d, e, f$, when a through f are real numbers, and X being the simplest possible solution.





5. Let's revisit the flower petal example. If X is the single petal flower, prove that $X = \uparrow *$. You may need to compare X with $\uparrow, 0, *$.

6. If Y is the double petal flower, prove that $\uparrow * < Y < \Uparrow *$, $Y || \uparrow$ and $Y || \Uparrow$

2.3 Nimbers

Now that we've got some nice notation to hand we can do some more complicated problems on simple numbers. For the next for problems, we'll use the formulation of nimbers:

$$0 = \{|\}, * = \{0|0\}, *2 = \{0, *|0, *\}, *3 = \{0, *, *2|0, *, *2\}$$

7. Prove the following facts about Nimbers:

a. Prove that $*a$ is it's own negative for every positive integer a, in other words, show that $*a + *a = 0$.

b. Prove that $* + *4 + *5 = 0$

8. A super neat tool with identifying nimber games is the following rule:

If $X = \{a, b, c, ... |a, b, c, ...\}$, and $a, b, c, ...$ are all nimbers, then the value of X is simply the smallest nimber not found in $a, b, c, ...$. (Here 0 is counted as the first nimber).

For example, $\{0, *, *2, *4, *7, *8|0, *, *2, *4, *7, *8\}$ has value $*3$, because that is the smallest excluded nimber. We can prove this by induction, fill in the following steps in logic:

a. Prove the statement holds when $X = 0$ (When $a, b, c, ...$ does not contain 0, then $X = 0$)
b. Suppose that the statement holds when $X = *a$ for all $0 \leq a < N$. Prove the statement holds when $X = *n.$

9. The factoid in the previous problem lets us generate a table of nimber additions in a rather neat manner: We simply look at all nimber values in the row/column before what we are computing, then find the minimal excluded value (why does this work?).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7

Prove that the Exclusive OR (XOR) function on natural numbers follows this same rule.

2.4 Omega Flower, Sans Undergrowth

10. Have a think about what the value $\frac{1}{\omega}$ would look like in bracket notation and draw it in hackenbush, and in treasure. Prove that $\omega + 1 > \omega + \frac{1}{\omega} > \omega$

11. Let's think about the hackenbush positions with an infinite amount of blue stems touching the ground (Kind of like if you put ω through a shredder). Call this game `on'. What is the bracket notation of `on'?

3 The Effect of Dominoes

** TODO **

3.1 Full Game

Figure out the total value of this game of domineering. Determine the first move that should be made by the winning side.