Games Problem Sheet

Preface

This problem set is longer and more difficult than is probably reasonable for a problem set.

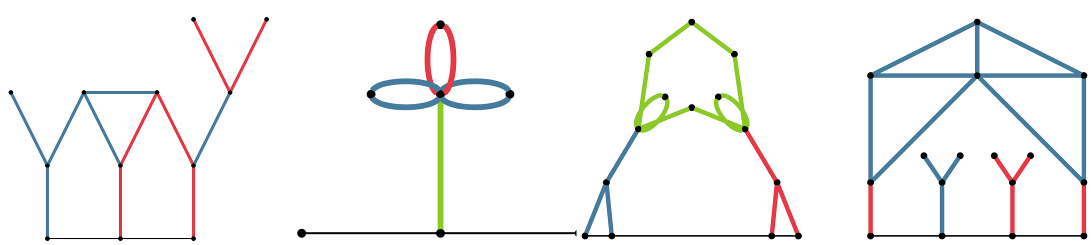
The idea is that there are some easy problems, but if you're intrigued by certain elements of this workshop, then there are questions there to pique your interest.

Problems preceded by more 🌶s are more difficult.

1 Hackenbush Hijinks

1.1 Hackenbush Strategies

1. 🌶Figure out for each of the following games, whether the winning strategy ensures a win for the 1st, 2nd, Blue or Red player:



2. 🌶Prove that $a + (-a) = 0$. In other words, for any hackenbush game which has two copies, one with the red/blue edges flipped, that there is a strategy for the 2nd player to win.

3. 🌶🌶Prove the following statements about games with values X, Y and Z. From this, you can deduce that all 2nd player winning games are equivalent, in that they have no effect on play. So equating them to zero is reasonable.

- If X has a strategy for the 2nd player to win, and Y has a strategy for the 2nd player to win, then $Z = X + Y$ has a strategy for the 2nd player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the 1st player to win, then $Z = X + Y$ has a strategy for the 1st player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the Blue player to win, then $Z = X + Y$ has a strategy for the Blue player to win.
- If X has a strategy for the 2nd player to win, and Y has a strategy for the Red player to win, then $Z = X + Y$ has a strategy for the Red player to win.

4. Evaluate the following position, and use this to formulate an equation involving $*$, $*_2$ and $*_3$

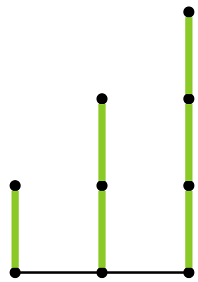


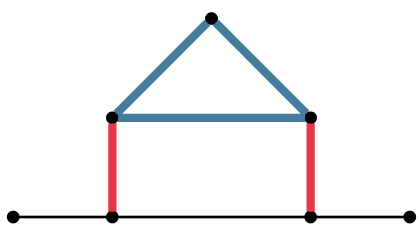
Figure 1

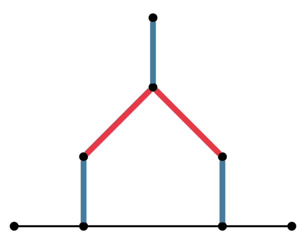
1.2 Evaluating Positions

5. Create Hackenbush positions with the following values:

- $3 + \frac{1}{4}$
- $\frac{1}{8}$
- $-\frac{15}{16}$
- $2 + * + *_3$

6. Evaluate the following hackenbush positions by comparing them to known games (are they less than game X, greater than game Y?)





7. Prove that the flower petal position plus $*$, is positive, but less than every positive number generated in blue-red hackenbush

8. Prove or Disprove the following statements, by coming up with a winning strategy, or giving a game example:

a. If $X = Y$ and $Y = Z$ then $X = Z$
b. If $X > Y$ and $Y > Z$ then $X > Z$
c. If $X||Y$ and $Y||Z$ then $X||Z$

2 Treasure Trove of Numbers

2.1 Treasure

1. Create Treasure games with the following values:

- $\frac{5}{4}$
- $\Uparrow$
- ω
- $\omega + \omega$

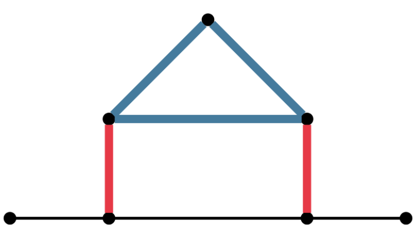
2. Prove that any formulation of bracket notation (purely built up from $\{|\}$) can be expressed as a game of Treasure.

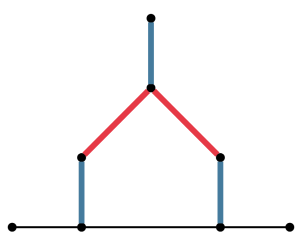
3. Provide an algorithm to combine two Treasure games A and B into a single game with value $A + B$.

2.2 A bracing garden

4. Evaluate the following hackenbush positions by recursively evaluating their left/right options in the brace notation.

Remember to use the rule of $X = \{a, b, c|d, e, f\}$ satisfying $a, b, c < X < d, e, f$, when a through f are real numbers, and X being the simplest possible solution.





5. Let's revisit the flower petal example. If X is the single petal flower, prove that $X = \uparrow *$. You may need to compare X with $\uparrow, 0, *$.

6. If Y is the double petal flower, prove that $\uparrow * < Y < \Uparrow *$, $Y || \uparrow$ and $Y || \Uparrow$

2.3 Nimbers

Now that we've got some nice notation to hand we can do some more complicated problems on simple numbers. For the next for problems, we'll use the formulation of nimbers:

$$0 = \{|\}, * = \{0|0\}, *_2 = \{0, *|0, *\}, *_3 = \{0, *, *_2|0, *, *_2\}$$

7. Prove the following facts about Nimbers:

a. Prove that $*_a$ is it's own negative for every positive integer a, in other words, show that $*_a + *_a = 0$

b. Prove that $* + *_4 + *_5 = 0$

8. 🌶🌶A super neat tool with identifying nimber games is the following rule:

If $X = \{a, b, c, ... | a, b, c, ...\}$, and $a, b, c, ...$ are all nimbers, then the value of X is simply the smallest nimber not found in $a, b, c, ...$ (Here 0 is counted as the first nimber).

For example, $\{0, *, *_2, *_4, *_7, *_8 | 0, *, *_2, *_4, *_7, *_8\}$ has value $*_3$, because that is the smallest excluded nimber. We can prove this by induction, fill in the following steps in logic:

a. Prove the statement holds when $X = 0$ (When $a, b, c, ...$ does not contain 0, then $X = 0$)
b. Suppose that the statement holds when $X = *_a$ for all $0 \leq a < N$. Prove the statement holds when $X = *_n$.

9. 🌶🌶🌶The factoid in the previous problem lets us generate a table of nimber additions in a rather neat manner: We simply look at all nimber values in the row/column before what we are computing, then find the minimal excluded value (why does this work?).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7

Prove that the Exclusive OR (XOR) function on natural numbers follows this same rule.

2.4 Omega Flower, Sans Undergrowth

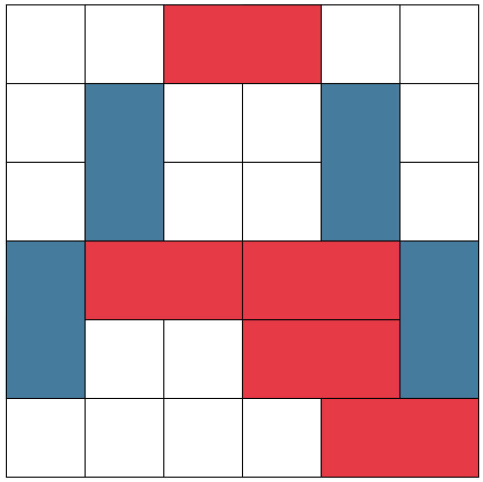
10. 🌶Have a think about what the value $\frac{1}{\omega}$ would look like in bracket notation and draw it in hackenbush, and in treasure. Prove that $\omega + 1 > \omega + \frac{1}{\omega} > \omega$

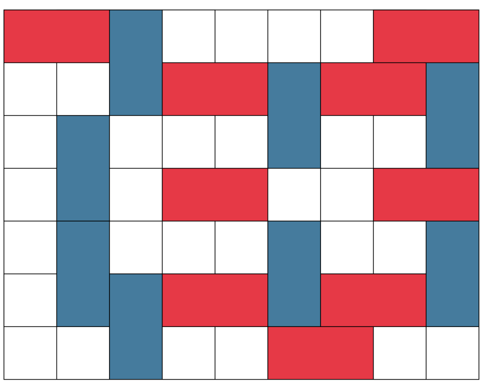
11. 🌶Let's think about the hackenbush positions with an infinite amount of blue stems touching the ground (Kind of like if you put ω through a shredder). Call this game `on'. What is the bracket notation of `on'?

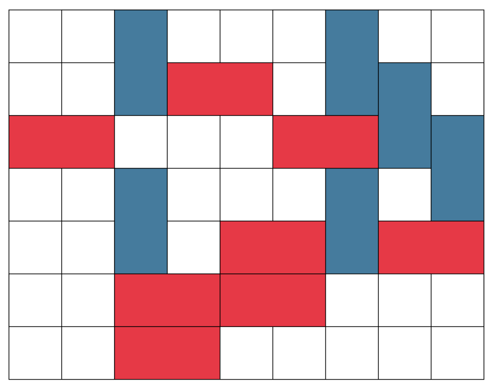
3 The Effect of Dominoes

3.1 Dominoes

1. Figure out the total value of the following games and where blue/red might place their piece first.







2. Find the values of the following generic shapes:

a. The $1 \times c$ row
b. The $r \times 1$ column
c. The \`L' shape with a height of h and width of w.
d. The hollow rectangle of height h and width w.

3.2 Switches

3. Create games with the following properties:

a. $X || -3, X || -2, X < -4, , X > -1$
b. $X || 2, X || -2 + \frac{1}{\omega}, X > -2, X < 2 + \frac{1}{\Uparrow}$
c. $\uparrow < X < \Uparrow$

4. 🌶️🌶️By adding only switches ($\pm a$), make this game have a purely positive / negative value. What is this value?

$$\{-3|1\} + \left\{0 \middle| \frac{1}{2}\right\} + \{-3* | -2*\} + \{*2| * 2\} + \{0, *|0, *\}$$

3.3 Up Miny Down Tiny

5. 🌶️🌶️Prove that $+_{\uparrow} || * 2$
6. 🌶️🌶️🌶️Prove that for any positive fraction a and integer b, and integer $c \geq 2$, that $+_{a+b\cdot\uparrow} || * c$ if and only if $+_{a+b\cdot\uparrow} || * 2$

3.4 Reversible Moves

7. 🌶️🌶️The formal statement of reversible moves is as follows:

For a game $G = \{A, B, C, ... | D, E, F, ...\}$, suppose there is some left option of D, call it $D_L = \{U, V, W, ... | X, Y, Z ...\}$, such that $D_L \geq G$.

Then $H = \{A, B, C, ... | X, Y, Z, ..., E, F, ...\} = G$.

Essentially, if right picks D, then left can immediately pick D_L, so essentially we can just shortcut this and say that right has the option of X, Y, Z in the first turn.

Prove that this statement is true by evaluating the game $G - H$, keeping in mind $D_L \geq G$.

8. 🌶️🌶️🌶️Prove that in a game with finite left and right options, if $\{A, B, C, ... | D, E, F ...\} = G = H = \{U, V, W, ... | X, Y, Z, ...\}$ and no moves in G or H are reversible ($G_{LR} \leq G$) or dominated ($G_{L_1} < G_{L_2}$), then the formulations G and H have the exact same options. That is, $A, B, C ... = U, V, W, ...$ and $D, E, F ... = X, Y, Z ...$

Postface

Hope you enjoyed the problems! If you did, or just generally found the theory interesting, Winning Ways for your Mathematical Plays is an excellent book that covers much, much, much more of this, with a lot more whimsy, wit, and weird/wonderful diagrams 😁