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Problem 3.1: **Runge-Kutta**

Consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{f}(\mathbf{y}(t), t) \tag{1}$$

with initial condition $\mathbf{y}(0) = \mathbf{y}_0$. The 4th order Runge-Kutta (RK4) algorithm for Eq. (1) is:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(t_n, \mathbf{y}_n\right) \\
\mathbf{k}_2 &= \mathbf{f}\left(t_n + \delta t/2, \mathbf{y}_n + \delta t \mathbf{k}_1/2\right) \\
\mathbf{k}_3 &= \mathbf{f}\left(t_n + \delta t/2, \mathbf{y}_n + \delta t \mathbf{k}_2/2\right) \\
\mathbf{k}_4 &= \mathbf{f}\left(t_n + \delta t, \mathbf{y}_n + \delta t \mathbf{k}_3\right) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{\delta t}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right) \ .
\end{aligned} \tag{2}$$

a) Consider the harmonic oscillator described by the equations

$$\begin{aligned}
\frac{dv}{dt} &= -x \\
\frac{dx}{dt} &= v,
\end{aligned}$$

with x the displacement from the rest position and v the velocity of the oscillator. Show that in this case, Eqs. (2) can be written as follows

$$\begin{aligned}
x_{n+1} &= x_n + v_n\delta t - \frac{1}{2}x_n\delta t^2 - \frac{1}{6}v_n\delta t^3 + \frac{1}{24}x_n\delta t^4 \\
v_{n+1} &= v_n - x_n\delta t - \frac{1}{2}v_n\delta t^2 + \frac{1}{6}x_n\delta t^3 + \frac{1}{24}v_n\delta t^4 \ .
\end{aligned} \tag{3}$$

b) Solve Eqs. (3) numerically for the initial conditions $v_0 = 0$ and $x_0 = 1$ using the RK4 algorithm and compare the time evolution of the total energy as well as the trajectory $x(t)$ with those obtained in problem 2.2 (Euler and Euler-Cromer algorithms).

Problem 3.2: **Time Reversibility**

Consider a mathematical pendulum Fig. 1 as described by the equations

$$\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{\omega} &= -\frac{g}{L}\sin(\theta).
\end{aligned} \tag{4}$$

with θ, ω and $\alpha = \dot{\omega}$ the angular displacement, angular velocity and torque, respectively.

a) With initial conditions $\omega(0) = \omega_0$ and $\theta(0) = \theta_0$ show analytically that

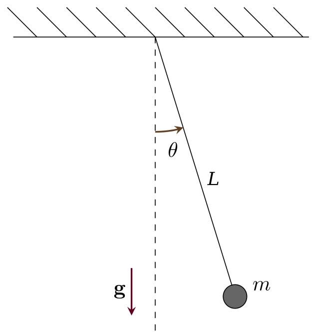

Figure 1:

i) the 2^{nd} order Runge-Kutta algorithm

$$\begin{aligned}\theta_{n+1} &= \theta_n + \omega_n \delta t - \frac{1}{2}\sin(\theta_n)\delta t^2 \\ \omega_{n+1} &= \omega_n - \sin\left(\theta_n + \frac{1}{2}\omega_n \delta t\right)\delta t\end{aligned} \tag{5}$$

is not time reversible.

ii) the velocity-Verlet algorithm

$$\begin{aligned}\theta_{n+1} &= \theta_n + \omega_n \delta t - \frac{1}{2}\sin(\theta_n)\delta t^2 \\ \omega_{n+1} &= \omega_n - \frac{1}{2}\left(\sin(\theta_n) + \sin(\theta_{n+1})\right)\delta t\end{aligned} \tag{6}$$

is time reversible.

b) Using Euler and Euler-Cromer algorithms solve Eqs. (4) for $g = 10$ and $L = 10$. Use the time step $\delta t = 0.5$ and the initial conditions $\omega_0 = 0$ and $\theta_0 = 7°$.

c) Plot the results for θ, ω and also energy E and compare them to the analytical solution. (For the analytical solution consider the small angle approximation $\sin(\theta) \approx \theta$.)

d) Redo the previous steps for the different time steps $\delta t = 0.1, 0.05, 0.01, 0.005$ and explain your observation.

e) The behavior of a mathematical pendulum generally can be described as linear, oscillatory and rotational corresponding to small (where the harmonic approximation holds), intermediate and large values of θ_0. Plot the phase portrait for these three different regimes, considering proper θ_0.

f) The choice of $\theta_0 = \pi$ is a very interesting case (the so called Separatrix angle). How long will it take for the pendulum to reach $\theta = \pi$?