Assignment Sheet Nr. 3

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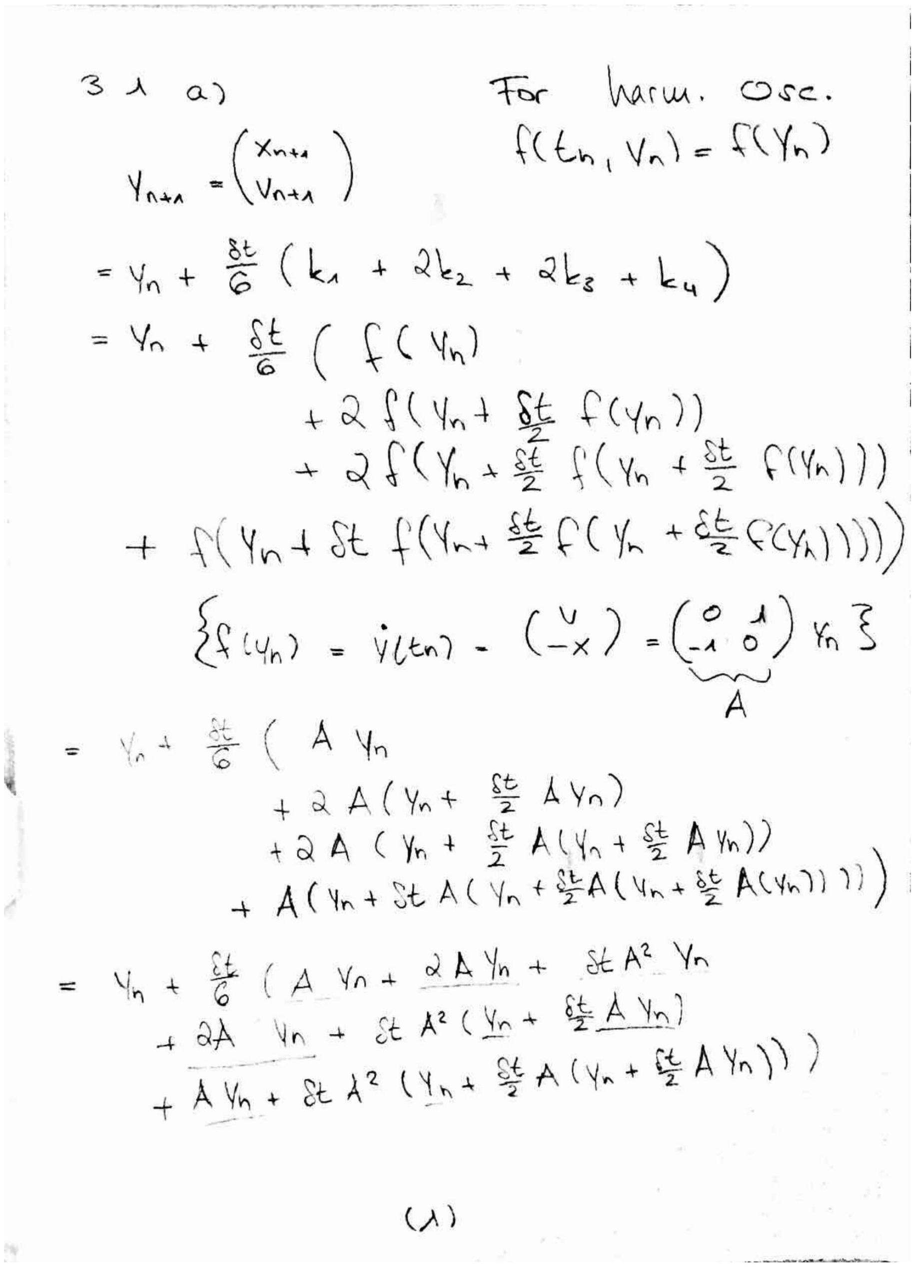
Exercise 3.1 (a)



3 1 a) For harm. osc.

$$f(t_n, y_n) = f(y_n)$$

$$y_{n+1} = \begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix}$$

$$= y_n + \frac{\delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\begin{aligned} = y_n + \frac{\delta t}{6}\Big(& f(y_n) \\ & + 2 f\big(y_n + \tfrac{\delta t}{2} f(y_n)\big) \\ & + 2 f\big(y_n + \tfrac{\delta t}{2} f\big(y_n + \tfrac{\delta t}{2} f(y_n)\big)\big) \\ & + f\big(y_n + \delta t\, f\big(y_n + \tfrac{\delta t}{2} f\big(y_n + \tfrac{\delta t}{2} f(y_n)\big)\big)\big)\Big) \end{aligned}$$

$$\left\{ f(y_n) = \dot{y}(t_n) = \begin{pmatrix} v \\ -x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} y_n \right\}$$

$$\begin{aligned} = y_n + \frac{\delta t}{6}\Big(& A y_n \\ & + 2A\big(y_n + \tfrac{\delta t}{2} A y_n\big) \\ & + 2A\big(y_n + \tfrac{\delta t}{2} A\big(y_n + \tfrac{\delta t}{2} A y_n\big)\big) \\ & + A\big(y_n + \delta t\, A\big(y_n + \tfrac{\delta t}{2} A\big(y_n + \tfrac{\delta t}{2} A(y_n)\big)\big)\big)\Big) \end{aligned}$$

$$\begin{aligned} = y_n + \frac{\delta t}{6}\Big(& A y_n + 2A y_n + \delta t A^2 y_n \\ & + 2A y_n + \delta t A^2\big(y_n + \tfrac{\delta t}{2} A y_n\big) \\ & + A y_n + \delta t A^2\big(y_n + \tfrac{\delta t}{2} A\big(y_n + \tfrac{\delta t}{2} A y_n\big)\big)\Big) \end{aligned}$$

(1)

Figure 3.1: first part of the derivation of the formula

$$= Y_n + \frac{\delta t}{6} \Big(3A Y_n + \delta t A^2 Y_n + 2 A Y_n + \delta t A^2 Y_n$$
$$+ \frac{\delta t^2}{2} A^3 Y_n + A Y_n + \delta t A^2 Y_n$$
$$+ \frac{\delta t^2}{2} A^3 \left(Y_n + \frac{\delta t}{2} A Y_n \right) \Big)$$

$$= Y_n + \frac{\delta t}{6} \Big(6 A Y_n + 3 \delta t A^2 Y_n + \delta t^2 A^3 Y_n$$
$$+ \frac{1}{4} \delta t^3 A^4 Y_n \Big)$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A^3 = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$A^4 = -A^2 = \mathbb{1}$$

$$= \left(1 + \frac{\delta t^4}{24}\right) Y_n + \delta t A Y_n + \frac{1}{2} \delta t^2 A^2 Y_n$$
$$+ \frac{1}{6} \delta t^3 A^3 Y_n$$

$$= \left(1 + \frac{\delta t^4}{24}\right) \begin{pmatrix} x_n \\ v_n \end{pmatrix} + \delta t \begin{pmatrix} v_n \\ -x_n \end{pmatrix} - \frac{\delta t^2}{2} \begin{pmatrix} x_n \\ v_n \end{pmatrix}$$
$$+ \frac{1}{6} \delta t^3 \begin{pmatrix} -v_n \\ x_n \end{pmatrix}$$

which is equiv to
the formula in 3.1 a

(2)

Figure 3.2: second part of the derivation of the formula

Exercise 3.1 (b)

Code

```
#include <iostream>
#include <cmath>
#include <fstream>
using namespace std;

int main()
{
double tmin      = 0.0;
double tmax      = 50.0;
const int N      = 500;
double dt        = (double)(tmax - tmin)/(N-1);

double y[N][4];
y[0][0] = 1.0;
y[0][1] = 0.0;
y[0][2] = 0.5*(y[0][0]*y[0][0]+y[0][1]*y[0][1]);
y[0][3] = 0.0;

for (int i=0;i<N-1;i++)
{
  y[i+1][0] = y[i][0] + y[i][1]*dt -0.5*y[i][0]*dt*dt-
     (1.0/6.0)*y[i][1]*pow(dt,3.0)+
     (1.0/24.0)*y[i][0]*pow(dt,4.0);
  y[i+1][1] = y[i][1] - y[i][0]*dt -0.5*y[i][1]*dt*dt+
     (1.0/6.0)*y[i][0]*pow(dt,3.0)+
     (1.0/24.0)*y[i][1]*pow(dt,4.0);
  y[i+1][2] = 0.5*(y[i+1][0]*y[i+1][0]+y[i+1][1]*y[i+1][1]);
  y[i+1][3] = dt*i;

}
ofstream outputfile;
outputfile.open("3_1_b_results.txt");

for (int i=0; i<N; i++)
{
 outputfile << y[i][0] << "  " << y[i][1]
 << "  " << y[i][2] << "  " << y[i][3] << endl;
}
outputfile.close();
return 0;
}
```

Results

The following results are achived via the code in the previous subsection.

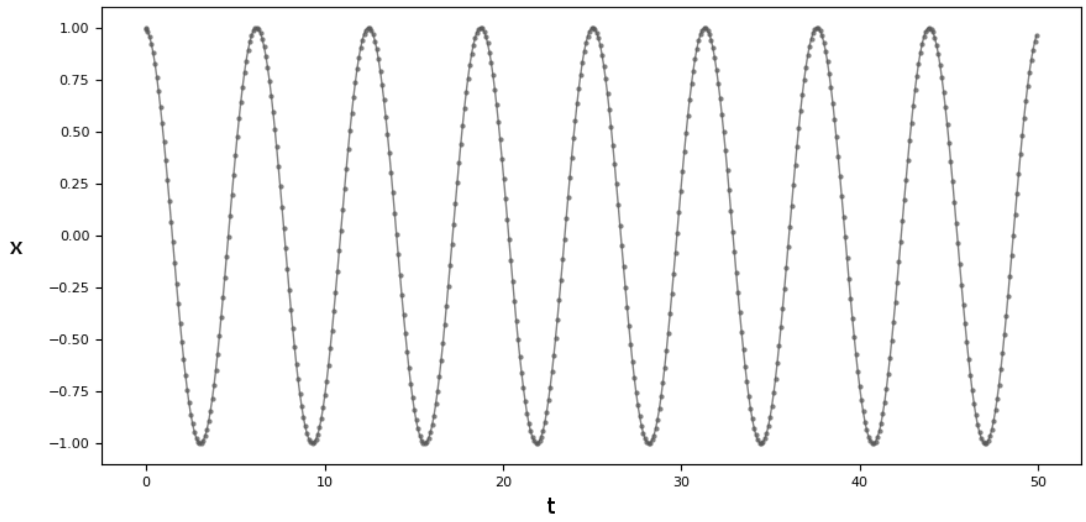


Figure 3.3: time evolution of the x coordinate of the harmonic oscillator

Figure 3.3 shows the time evolution of the x coordinate of a one dimensional harmonic oscillator solved with the Runge-Kutta 4 scheme. There is no apparent divergence from the analytical solution visible in the timeframe of the simulation.

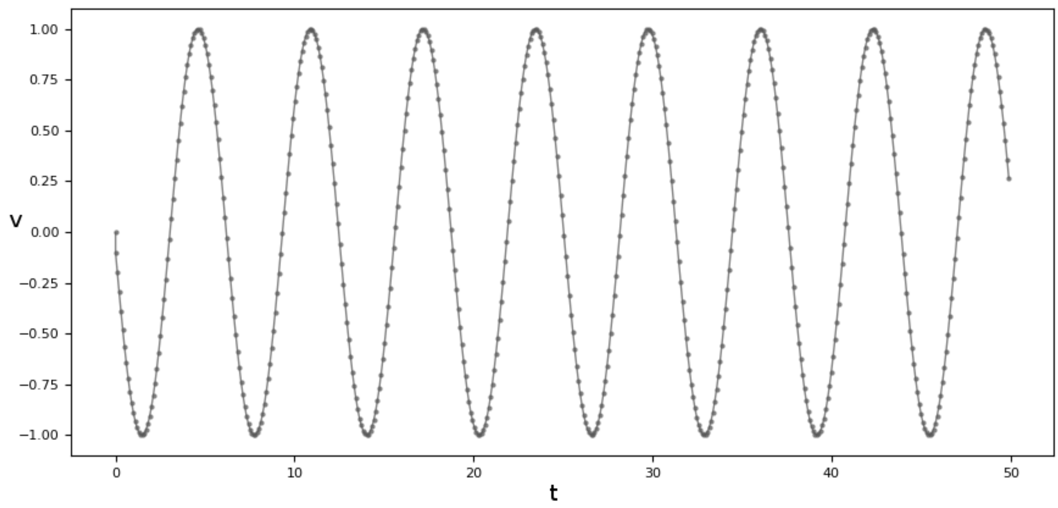


Figure 3.4: time evolution of the velocity of the harmonic oscillator

Figure 3.4 shows the time evolution of the velocity of a one dimensional harmonic oscillator solved with the Runge-Kutta 4 scheme. Similarly to the x coordinate, there is no visible instability in the timeframe of the simulation period.

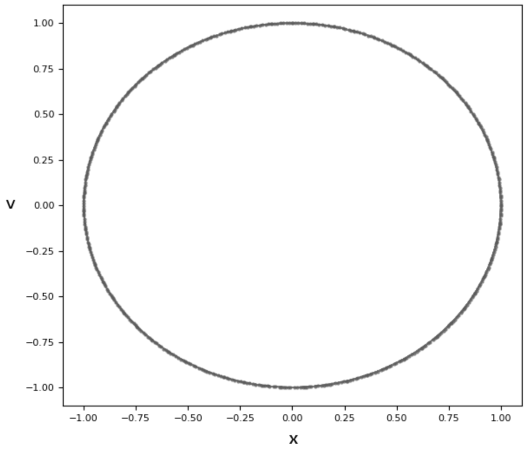


Figure 3.5: phase space trajectory of the harmonic oscillator

Figure 3.5 shows the phase space profile for the simulation results above. The trajectory seems reasonably spherical.