HEINRICH HEINE
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Assignment Sheet Nr. 3

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Contents

1 Exercise 1

1.1 Exercise 1 (a)

3 1 a) For harm. osc. $f(t_n, V_n) = f(Y_n)$

$$Y_{n+1} = \begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix}$$

$$= Y_n + \frac{\delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\begin{aligned} = Y_n + \frac{\delta t}{6}\Big(& f(Y_n) \\ & + 2 f\left(Y_n + \frac{\delta t}{2} f(Y_n)\right) \\ & + 2 f\left(Y_n + \frac{\delta t}{2} f\left(Y_n + \frac{\delta t}{2} f(Y_n)\right)\right) \\ & + f\left(Y_n + \delta t\, f\left(Y_n + \frac{\delta t}{2} f\left(Y_n + \frac{\delta t}{2} f(Y_n)\right)\right)\right)\Big) \end{aligned}$$

$$\left\{ f(Y_n) = \dot{y}(t_n) = \begin{pmatrix} v \\ -x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} Y_n \right\}$$

$$\begin{aligned} = Y_n + \frac{\delta t}{6}\Big(& A Y_n \\ & + 2 A\left(Y_n + \frac{\delta t}{2} A Y_n\right) \\ & + 2 A\left(Y_n + \frac{\delta t}{2} A\left(Y_n + \frac{\delta t}{2} A Y_n\right)\right) \\ & + A\left(Y_n + \delta t\, A\left(Y_n + \frac{\delta t}{2} A\left(Y_n + \frac{\delta t}{2} A(Y_n)\right)\right)\right)\Big) \end{aligned}$$

$$\begin{aligned} = Y_n + \frac{\delta t}{6}\Big(& A Y_n + 2 A Y_n + \delta t A^2 Y_n \\ & + 2A Y_n + \delta t A^2 \left(Y_n + \frac{\delta t}{2} A Y_n\right) \\ & + A Y_n + \delta t A^2 \left(Y_n + \frac{\delta t}{2} A \left(Y_n + \frac{\delta t}{2} A Y_n\right)\right)\Big) \end{aligned}$$

(1)

Figure 1.1: first part of the derivation of the formula

$$= Y_n + \frac{\delta t}{6}\Big(3A\,Y_n + \delta t\,A^2 Y_n + 2A\,Y_n + \delta t\,A^2 Y_n$$
$$+ \frac{\delta t^2}{2} A^3 Y_n + A\,Y_n + \delta t\,A^2 Y_n$$
$$+ \frac{\delta t^2}{2} A^3 \left(Y_n + \frac{\delta t}{2} A\,Y_n\right) \Big)$$

$$= Y_n + \frac{\delta t}{6}\Big(6A\,Y_n + 3\delta t\,A^2 Y_n + \delta t^2 A^3 Y_n$$
$$+ \frac{1}{4}\delta t^3 A^4 Y_n \Big)$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^3 = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A^4 = -A^2 = \mathbb{1}$$

$$= \left(1 + \frac{\delta t^4}{24}\right) Y_n + \delta t\,A\,Y_n + \frac{1}{2}\delta t^2 A^2 Y_n$$
$$+ \frac{1}{6}\delta t^3 A^3 Y_n$$

$$= \left(1 + \frac{\delta t^4}{24}\right)\begin{pmatrix} x_n \\ v_n \end{pmatrix} + \delta t \begin{pmatrix} v_n \\ -x_n \end{pmatrix} - \frac{\delta t^2}{2}\begin{pmatrix} x_n \\ v_n \end{pmatrix}$$
$$+ \frac{1}{6}\delta t^3 \begin{pmatrix} -v_n \\ x_n \end{pmatrix}$$

which is equiv to
the formula in 2 1 a

(2)

Figure 1.2: second part of the derivation of the formula

1.2 Exercise 1 (b)

1.2.1 Code

```
#include <iostream>
#include <cmath>
#include <fstream>
using namespace std;

int main()
{
double tmin     = 0.0;
double tmax     = 50.0;
const int N     = 500;
double dt       = (double)(tmax - tmin)/(N-1);

double y[N][4];
y[0][0] = 1.0;
y[0][1] = 0.0;
y[0][2] = 0.5*(y[0][0]*y[0][0]+y[0][1]*y[0][1]);
y[0][3] = 0.0;

for (int i=0;i<N-1;i++)
{
  y[i+1][0] = y[i][0] + y[i][1]*dt -0.5*y[i][0]*dt*dt-
     (1.0/6.0)*y[i][1]*pow(dt,3.0)+
     (1.0/24.0)*y[i][0]*pow(dt,4.0);
  y[i+1][1] = y[i][1] - y[i][0]*dt -0.5*y[i][1]*dt*dt+
     (1.0/6.0)*y[i][0]*pow(dt,3.0)+
     (1.0/24.0)*y[i][1]*pow(dt,4.0);
  y[i+1][2] = 0.5*(y[i+1][0]*y[i+1][0]+y[i+1][1]*y[i+1][1]);
  y[i+1][3] = dt*i;

}
ofstream outputfile;
outputfile.open("3_1_b_results.txt");

for (int i=0; i<N; i++)
{
 outputfile << y[i][0] << "  " << y[i][1]
 << "  " << y[i][2] << "  " << y[i][3] << endl;
}
outputfile.close();
return 0;
}
```

1.2.2 Results

The following results are achived via the code in the previous subsection.

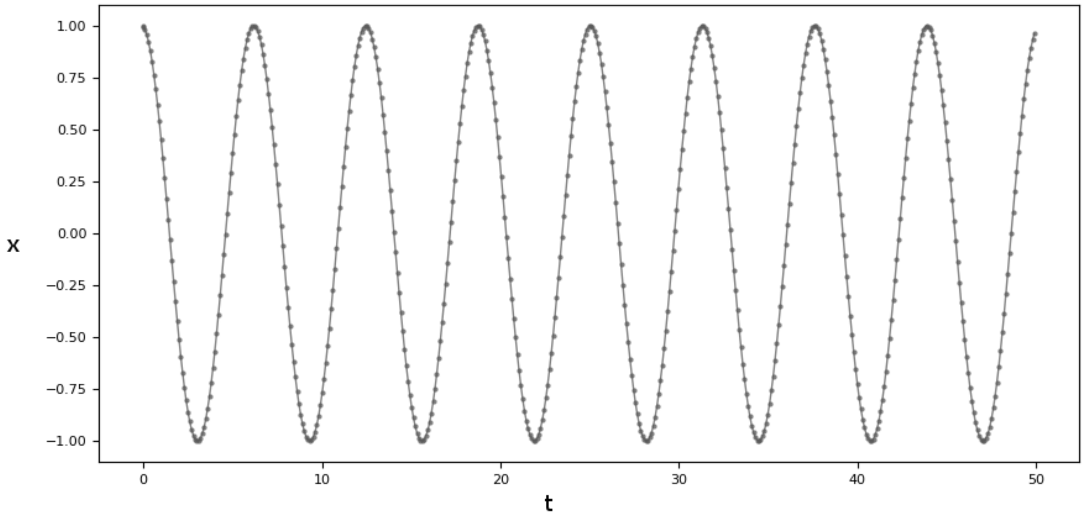

Figure 1.3: time evolution of the x coordinate of the harmonic oscillator

Figure 3.3 shows the time evolution of the x coordinate of a one dimensional harmonic oscillator solved with the Runge-Kutta 4 scheme. There is no apparent divergence from the analytical solution visible in the timeframe of the simulation.

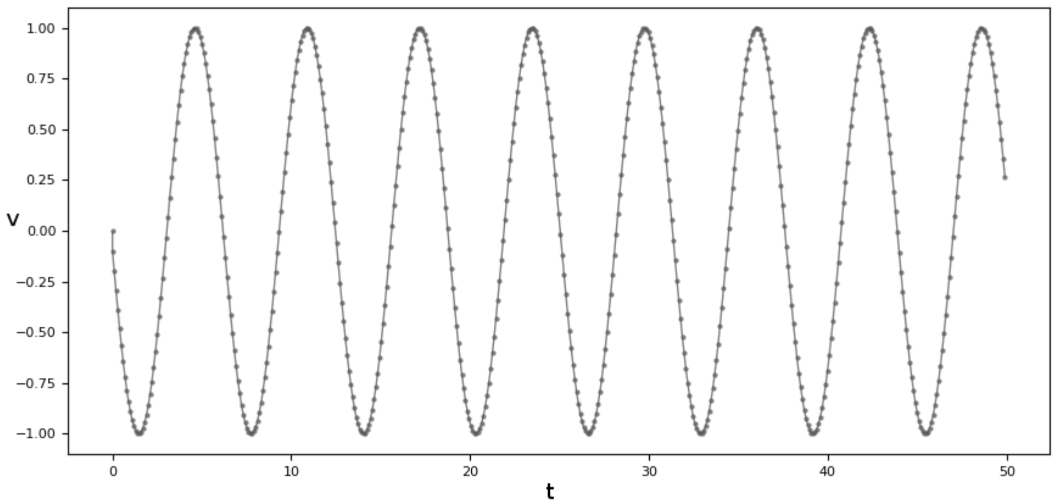

Figure 1.4: time evolution of the velocity of the harmonic oscillator

Figure 3.4 shows the time evolution of the velocity of a one dimensional harmonic oscillator solved with the Runge-Kutta 4 scheme. Similarly to the x coordinate, there is no visible instability in the timeframe of the simulation period.

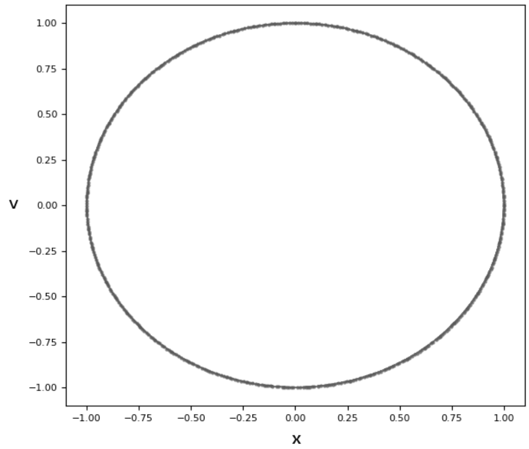

Figure 1.5: phase space trajectory of the harmonic oscillator

Figure 3.5 shows the phase space profile for the simulation results above. The trajectory seems reasonably spherical.

2 Exercise 2