CMS Spread Option

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Abstract

The aim of this document is to provide some technical elements required for the reconstruction of Cms Spread Option implied correlation time series according to today's market conventions.

Contents

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1 Introduction

1.1 Notations

$D(t,T)$: Stochastic discount factor at time t for the maturity T.

$P(t,T)$: T-maturity zero coupon bond value at time $t \leq T$.

$L(T_R, T_S, T_S + \tau)$: Simply compounded Libor rate resetting at time T_R for the start date T_S and the tenor τ.

$S(T_R, T_S, T_E)$: Swap rate resetting at time T_R for the start date T_S and the end date T_E.

1.2 Payoff

A CMS spread option is a financial instrument whose payoff is a function of the spread between two swap rates of different tenor.

1.2.1 SingleLook

For a given maturity T and a strike K, a single look CMS spread option is a one year basis caplet with fixing and payment in arrears

Its forward premium value at time t is therefore given by

$$V_{SL}^{SO}(t,T,K) \quad = \quad \frac{1}{P^d(t,T)} \mathbb{E}_t^{Q_d} \left[D^d(t,T) \left(S_T^i - S_T^j - K \right)_+ \right] = \mathbb{E}_t^{Q_d^T} \left[\left(S_T^i - S_T^j - K \right)_+ \right]$$

1.2.2 Cap Floor

A CMS spread cap is a strip of three month basis caplets with fixing in advance and in arrears payment.

Its value at time t is given by

$$V_{CAP}^{SO}(t) \quad = \quad \mathbb{E}_t^{Q_d} \left[\sum_{l=1}^{N} \tau_l D^d(t, T_E^l) \left(S_{T_l}^i - S_{T_l}^j - K \right)_+ \right] = \sum_{l=1}^{N} \tau_l P^d(t, T_E^l) \mathbb{E}_t^{Q_d^{T_E^l}} \left[\left(S_{T_l}^i - S_{T_l}^j - K \right)_+ \right]$$

Market quotes

Both types are quoted by brokers such as Tullet. Both products allow the investor a view on the shape of the yield curve.

and $S_T = S(T, T_S, T_E)$ the swap forward rate

2 CMS Spread Modelling

2.1 Convexity Adjusted CMS

The convexity adjusted constant maturity swap rate is classicaly defined by $CMS_t \triangleq \mathbb{E}_t^{Q_d^T}[S_T]$. Using the fact that $CMS_T = S_T$, a single look CMS spread option value rewrites as

$$V_{SL}^{SO}(t,T,K) \quad = \quad \mathbb{E}_t^{Q_d^T}\left[\left(CMS_T^1 - CMS_T^2 - K\right)_+\right]$$

2.2 Gaussian CMS Spread Model

Assuming that each convexity adjusted CMS rates is gaussian such that $dCMS_t^i = \sigma_N^i \, dW_t^i$ with $\forall i \neq j$, $\left\langle dW_t^i, dW_t^j \right\rangle = \rho_{ij} dt$, the $\mathcal{F}_t$ conditional law of the convexity adjusted CMS spread $X_T = CMS_T^i - CMS_T^j$ is given by

$$X_T \stackrel{d}{=} \mathcal{N}\left(X_t, \sigma_N \sqrt{T-t}\right), \quad \sigma_N \quad = \quad \sqrt{\left(\sigma_N^i\right)^2 + (\sigma_N^j)^2 - 2\rho_{ij}\sigma_N^i \sigma_N^j}$$

The gaussian model provides a convenient common langage for quoting spread options and gives an analytical expression of the correlation ρ_{ij} between CMS^i and CMS^j which is the natural risk factor for this kind of product. In practice, $\sigma_N^{i,j}$ are choosen as atm swaption implied gaussian volatilities and the convexity adjusted CMS spread X_t is obtened by replication. If the CMS spread is assumed to have a Gaussian distribution, the price of a T-maturity caplet spread option with a strike K is given by the Bachelier's formula :

$$C_t^N(X_t, T, K, \sigma_N) = (X_t - K) \; \Phi\left(\frac{X_t - K}{\sigma_N \sqrt{T-t}}\right) + \sigma_N \sqrt{T-t} \; \phi\left(\frac{X_t - K}{\sigma_N \sqrt{T-t}}\right)$$

2.3 Smiled Implied Correlation

For a given maturity T, a natural cubic correlation spline $\rho(T, K_i)$ is used to fit a set of CMS Spread option market prices $\left(V_{SL}^{SO*}\right)_{i=1..n}$, i.e such that

$$\forall i \quad = \quad 1...n, \quad V_{SL}^{SO*}(t,T,K_i) = C_t^N(X_t, T, K_i, \sigma_N(K_i, T))$$

The CMS Spread distribution is then projected on a dynamic range - wrt the money - by building another relative correlation spline for a set of predefined moneyness strike $K_j^m \in [K_{\min}^m, K_{\max}^m]$:

$$\begin{aligned} \rho_{Atm}^T &= \rho(T, K = X_t) \\ \rho_m(T, K_j^m) &= \rho(T, X_t + K_j^m) - \rho_{Atm}^T \end{aligned}$$

2.4 Cap Floor Pricing

2.4.1 Paylag Adjustment

2.4.2 Term Structure Interpolation

3 CMS Spread Correlation Time series

For Butterfly Arbitrage, maturity dependency can be omitted

- fit quickly
- compliant with the cubic spline
- compliant with the cubic splineproduce butterfly arbitrage free prices

3.1 Butterfly Arbitrage condition

Setting $\xi(K) = \frac{X_t - K}{\sigma_N(T,K)\sqrt{T-t}}$, Bachelier's formula rewrittes as

$$C_t^N(X_t, T, K, \sigma_N\ (K,T)) \quad = \quad \sigma_N\ (K,T)\sqrt{T-t}\,(\phi(\xi) + \xi\Phi(\xi))$$

Assuming $\sigma_N\ : \mathbb{R} \to \mathbb{R}_+$ and $\sigma_N\ \in \mathcal{C}^2(\mathbb{R})$ CMS spread Cdf and Pdf can be deduced as follow :

$$\begin{aligned}
\frac{\partial C_t^N}{\partial K} &= \frac{\sigma_N^{'}\ (K)}{\sigma_N\ (K)} C_t^N + \sigma_N\ (K)\Phi(\xi)\frac{\partial \xi}{\partial K} \\
\frac{\partial^2 C_t^N}{\partial K^2} &= 2\sigma_N^{'}\ (K)\Phi(\xi)\frac{\partial \xi}{\partial K} + \sigma_N^{''}\ (K)\,(\phi(\xi) + \xi\Phi(\xi)) + \sigma_N\ (K)\phi(\xi)\left(\frac{\partial \xi}{\partial K}\right)^2 + \sigma_N\ (K)\Phi(\xi)\frac{\partial^2 \xi}{\partial K^2}
\end{aligned}$$

Computing first and second derivatives of ξ

$$\begin{aligned}
\frac{\partial \xi}{\partial K} &= -\frac{1}{\sigma_N\ (K)}\left(1 + \sigma_N^{'}\ (K)\xi\right) \\
\frac{\partial^2 \xi}{\partial K^2} &= -\frac{1}{\sigma_N\ (K)}\left(\sigma_N^{''}\ (K)\xi + 2\sigma_N^{'}\ (K)\frac{\partial \xi}{\partial K}\right)
\end{aligned}$$

leads to the equivalence reliatioship between the price convexity and the gaussian volatility convexity for the butterfly arbitrage condition

$$\frac{\partial^2 C_t^N}{\partial K^2} \quad = \quad \phi(\xi)\left(\sigma_N^{''}\ (K) + \frac{\left(1 + \sigma_N^{'}\ (K)\xi\right)^2}{\sigma_N\ (kK)}\right) \quad \text{and} \quad \frac{\partial^2 C_B}{\partial K^2} > 0 \Leftrightarrow \sigma_N^{''}\ (K) > 0$$

3.2 Implied Correlation Parametrization

As seen before, the relationship between gaussian CMS spread implied volatility and gaussian CMS implied correlation is of the following form

$$\sigma_N(K) = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\rho(K)}, \quad (\alpha, \beta) \in (\mathbb{R}_+)^2$$

then

$$\sigma_N(K) \quad > \quad 0 \Leftrightarrow \rho(K) < 1 + \frac{(\alpha - \beta)^2}{2\alpha\beta}\ \forall K$$

that is always true since $\forall K$, $|\rho(K)| < 1$. The Butterfly arbitrage condition in terms of implied correlation is adressed through the following equations

$$
\begin{aligned}
\sigma_N^{'}(K) &= -\alpha\beta\frac{\rho^{'}(K)}{\sigma_N(K)} \\
\sigma_N^{''}(K) &= -\frac{\alpha\beta}{\sigma_N^2(K)}\left(\rho^{''}(K)\sigma_N(K)+\alpha\beta\frac{\left(\rho^{'}(K)\right)^2}{\sigma_N(K)}\right)
\end{aligned}
$$

Obviously $\rho \in \mathcal{C}^2(\mathbb{R})$ implies that $\sigma_N \in \mathcal{C}^2(\mathbb{R})$ and the volatility convexity is equivalent to

$$
\forall K, \quad \rho^{''}(K)\sigma_N(K)^2 + \alpha\beta\rho^{'}(K)^2 \quad < \quad 0
$$

Given a parabolic parametrization of the correlation $\rho(K) = aK^2 + bK + \rho_{atm}$, this condition rewrittes as

$$
2a\left(\alpha^2+\beta^2-2\alpha\beta\rho_{atm}\right)+\alpha\beta b^2 \quad < \quad 0
$$

that is strike independante and equivelent to $\rho_{\max} = \left(\rho_{atm} - \frac{b^2}{4a}\right) < \frac{\alpha^2+\beta^2}{2\alpha\beta}$.

Remarque 1 *As* $|\rho_{atm}| \leq 1$, $\alpha^2+\beta^2-2\alpha\beta\rho_{atm} \geq \alpha^2+\beta^2-2\alpha\beta = (\alpha-\beta)^2 > 0$ *then inevitably* $a < 0$.