MODEL DOCUMENTATION

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Object	**LA18**
Direction	Global Market
Author(s)	IR QR
Reference	Model
Diffusion	Restricted
Version	1.0
Complexity	Medium
Proposed tier	1

Contents

1 General information

Model Development Information	
Model ID	No reference
Model Name	Strike Equivalent
Business unit	Natixis / Global Market / Trading FI
Model purpose	Model
Name of the model development team	IR QR
Model development start date	
Model development completion date	
Date of the most recent revision	
Model Implementation Information	
Implementation plateform	ARM
Integration plateform	Summit
Name of model implementation team	DSI
Date of implementation	
Model Governance Information	
Name of the model owner	Pascal Amiel
Date model deployment approved for initial Business use	
Most recent date model approved for continued Busines use	
Approved uses	LD-M-FIC4-INFLATION
Model restrictions	Payoff dependent
Approved users	
Name of the model validator lead for the most recent validation	MRM
Date of the most recent prior validation/review	
Type of the most recent prior validation/review	Periodical Review
Model Key Documentation	
Model documentation names	New Livret A pricing
Previous validation report names	
Business requirement documentation name	New Livret A pricing
Specification documentation name	
IT implementation sign off documentation name	
User acceptance report name	
Regulatory references	
Calibration files name	
Back testing documentation name	
Stress testing documentation name	

2 Description

2.1 Purpose and use

This document aims at describing the pricing methodology of the LA18 option, an inflation-indexed cap floor option, used as an hedging instrument against the interest rate variations of the Livret A, the most popular French saving account.

Without taking convexity adjustments into account, a LA18 option is sensitive to the following risk factors:

Payoff	Risk factors
LA18 Option	CPI curve Discount rate curve Forecast rate curve Volatility smile of forecast rates Volatility smile of YOY index Forecast rate/YOY correlation and cross correlation

2.2 In House or Vendor

The LA18 model is developed by the IR QR team in the ARM In-house library and integrated by the DSI in Summit trading platform (Vendor platform).

2.3 Outputs

The output obtained by the model is the price of the LA18 option. The Greeks are calculated by Summit, a repricing with a bump on the risk factor is done with the model (finite differences). The difference between the two prices gives the sensitivities.

2.4 Appropriateness of the combination

The LA18 option cash flows can be seen as two-dimensional basket options. To price it through a bivariate formula, a moment matching projection is used.To take into account that the basket assets are smiled, a strike equivalent procedure is done. This combination enables to price the LA18 option quickly.

2.5 Pricer use

The LA18 pricer is used by the following stakeholders :

- Model Owner: Pascal Amiel
- Model Users:
  - FO IRD :
    - Trading : pricing and hedging
    - Structures : pricing and testing
    - Quants : development, testing, support and maintenance
    - Sales : pricing
  - DRM : risk monitoring via VaR, CVA and other risk indicators
  - SDR : P L production and consensus contributions
- Model Developer (Quants): for model development and maintenance
- Model Implementer: DSI

3 Payoff

3.1 Cash flow

Since February 2020, a new formula have been proposed to compute the annual interest rate of the Livret A, R^{LA}. It can be now reviewed by up to two times a year, depending on the level of inflation and monetary conditions. Considering a six month observation period $[T, T_\delta]$, a floor rate R_F of 0.5% and denoting by $\left(Y_{T_y^i}\right)_{i=1}^n$, n successives monthly year-on-year (YOY) and $\left(E_{T_f^j}\right)_{j=1}^m$, m daily successives Eonia rates over that period , R^{LA} is obtained in a simple manner at time T_δ by :

$$R_{T_\delta}^{LA} = \max\left(R_F, \frac{1}{2n}\sum_{i=1}^{n} Y_{T_y^i} + \frac{1}{2m}\sum_{j=1}^{m} E_{T_f^j}\right) \tag{3.1}$$

with $T_y^1 = T_f^1 = T$ and $T_y^n = T_f^m = T_\delta$.

The LA18 option enables investors indebted at the Livret A rate to protect themselves by locking the payment at a maximum rate $K \geq R_F$. Denoting by F an arbitrary market rate (IR) and w its associated weight, the $k-th$ cash flow of the LA18 option payed at date $T_p^k \geq T_\delta^k$ is then obviously given by :

$$C_{w,F}^k = \left(\frac{(1-w)}{n}\sum_{i=1}^{n} Y_{T_y^{i,k}} + \frac{w}{m}\sum_{j=1}^{m} F_{T_f^{j,k}} - K\right)_+ \Bigg|_{w=\frac{1}{2}}^{F=E} \tag{3.2}$$

Remark 1 Introducing the generic notation $C_{w,F}$ enables us to descibes it in most cases and to define other inflation derivatives. For example, setting $w = 0$ leads to the definition of the average inflation option.

3.2 Features

In order to give a global description of that derivative, its main features are listed in the table bellow.

	DATA	NOTATION
SCHEDULE	Start date	T_S
	End date	T_E
	Day count	DC
	Payment gap	P_g
	Payment frequency	P_f
IR	Reset Frequency	R_f^F
	Reset lag	R_l^F
YOY	Reset Frequency	R_f^Y
	Reset lag	R_l^Y
	Month begin	
OTHER	Strike	K
	Notional	N
	Interest rate weight	w^F
	Strike equiv iteration	n_s

LetT1, . . . , TNbe the set of dates at which the values of the basket are contributingto the contract payoff

3.3 Example

4 Diffusion model x numerical method description

4.1 Framework

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a probability space, where $\mathbb{P}$ is the historical probability and $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by a multidimensional brownian motion $\left(W_t^l\right)_{l=1}^d$ with $d \geq n+m$ such that

- $\forall i, j = 1, ..., n$: $\quad d\left\langle W_t^i, W_t^j \right\rangle = \theta_{ij}^Y dt$
- $\forall i, j = n+1, ..., n+m$: $\quad d\left\langle W_t^i, W_t^j \right\rangle = \theta_{ij}^F dt$
- $\forall i = 1, ..., n$ and $\forall j = n+1, ..., n+m$: $\quad d\left\langle W_t^i, W_t^j \right\rangle = \rho_{ij} dt$

We suppose moreover that the market is arbitrage-free and introduce the risk neutral probability measure $\mathbb{Q}$, define by the riskless savings account numeraire β. According to these definitions, the standard valuation formula of the LA18 option at time $t \leq T_S$ is then given by

$$V_t = \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{k=1}^{l} \frac{\beta_t}{\beta_{T_p^k}} C_k\right] = \sum_{k=1}^{l} P(t, T_p^k) \mathbb{E}_t^{\mathbb{Q}_{T_p^k}}\left[C_k\right] \tag{4.1}$$

with $P(t, T_p^k)$, the T_p^k- maturity zero coupon bond value at time t and $\mathbb{Q}_{T_p^k}$ the T_p^k forward mesure. Omitting $k-$ subscript, the real challenge is thus to estimate the following basket option

$$P_{\alpha,\beta}(t, K) = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[\left(\sum_{i=1}^{n} \alpha_i Y_{T_i} + \sum_{j=1}^{m} \beta_j F_{T_j} - K\right)_+\right] \tag{4.2}$$

4.2 Moment matching

We note $W_t^{T_p}$ the brownian motion under Q_{T_p} and assume shifted lognormal dynamics for the pay lagged fowards defined by $\tilde{Y}_t^T = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[Y_T\right]$ and $\tilde{F}_t^T = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[F_T\right]$ such that

$$\begin{aligned} d\tilde{Y}_t^{T_i} &= \mu_i \left(\tilde{Y}_t^{T_i} + s_i\right) dW_t^{T_p,i}, \quad \forall i = 1, ..., n && (4.3) \\ d\tilde{F}_t^{T_j} &= \nu_j \left(\tilde{F}_t^{T_j} + r_j\right) dW_t^{T_p,j}, \quad \forall j = n+1, ..., m && (4.4) \end{aligned}$$

with $(s_i, \mu_i)_{i=1}^n$ and (r_j, ν_j) the shifts and the strike equivalent volatilities of year-over-year and forward rates respectively. As done before, using the last fixing date $T = T_n = T_m$, the basket option value rewrittes as

$$\tilde{P}_{\alpha,\beta}(t, \tilde{K}) = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[\left(\tilde{Y}_T + \tilde{F}_T - \tilde{K}\right)_+\right] \tag{4.5}$$

with $\tilde{Y}_T = \sum_{i=1}^n \alpha_i \tilde{Y}_{T_i}^{T_i}$, $\quad \tilde{F}_T = \sum_{j=1}^m \beta_j \tilde{F}_{T_j}^{T_j}$ and $\tilde{K} = K - \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j$.

In order to price $\tilde{P}_{\alpha,\beta}$ quickly through a bilog formula, a moment matching procedure is used to approximate

$$\tilde{P}_{\alpha,\beta}(t,\tilde{K}) \approx BiLog\left(\bar{Y}_t, \bar{F}_t, \mu_{\bar{Y}}, \nu_{\bar{F}}, \rho\right) \tag{4.6}$$

where $\left(\bar{Y}, \bar{F}\right)$ are two lognormal random variables with volatility $(\mu_{\bar{Y}}, \nu_{\bar{F}})$ and correlation $\rho dt = d\left\langle \ln(\bar{Y}_t), \ln(\bar{F}_t)\right\rangle$ such that $\forall i = \{1,2\}$

$$\left\{\begin{array}{lcl} \mathbb{E}_t\left[\bar{Y}_T^i\right] & = & \mathbb{E}_t\left[\tilde{Y}_T^i\right] \\ \mathbb{E}_t\left[\bar{F}_T^i\right] & = & \mathbb{E}_t\left[\tilde{F}_T^i\right] \\ \mathbb{E}_t\left[\bar{Y}_T\bar{F}_T\right] & = & \mathbb{E}_t\left[\tilde{Y}_T\tilde{F}_T\right] \end{array}\right.$$

By identification we get

$$\begin{aligned} \mu_{\bar{Y}}^2 &= \ln\left(\frac{\sum_{i,j=1}^{n}\alpha_i\alpha_j\tilde{Y}_t^{T_i}\tilde{Y}_t^{T_j}e^{\mu_i\mu_j\theta_{ij}^Y}}{\sum_{i,j=1}^{n}\alpha_i\alpha_j\tilde{Y}_t^{T_i}\tilde{Y}_t^{T_j}}\right) \\ \nu_{\bar{F}}^2 &= \ln\left(\frac{\sum_{i,j=n+1}^{n+m}\beta_i\beta_j\tilde{F}_t^{T_i}\tilde{F}_t^{T_j}e^{\nu_i\nu_j\theta_{ij}^F}}{\sum_{i,j=n+1}^{n+m}\beta_i\beta_j\tilde{F}_t^{T_i}\tilde{F}_t^{T_j}}\right) \\ \mu_{\bar{Y}}\nu_{\bar{F}}\rho &= \ln\left(\frac{\sum_{i=1}^{n}\sum_{j=n+1}^{n+m}\alpha_i\beta_j\tilde{Y}_t^{T_i}\tilde{F}_t^{T_j}e^{\mu_i\nu_j\rho_{ij}}}{\sum_{i=1}^{n}\sum_{j=n+1}^{n+m}\alpha_i\beta_j\tilde{Y}_t^{T_i}\tilde{F}_t^{T_j}}\right) \end{aligned}$$

4.3 Strike equivalent method

In reality, the market option quotes provide volatility smiles for the basket assets $\left(\tilde{Y}_t^{T_i}\right)_{i=1}^{n}$ and $\left(\tilde{F}_t^{T_j}\right)_{j=n+1}^{n+m}$. In consequence, these random variables aren't strictly lognormal. The strike equivalent method split the basket option strike K into a set of asset strike $(p_i)_{i=1}^{n}$ and $(q_j)_{j=1}^{m}$ such that

$$\left\{(p_i)_{i=1}^{n}, (q_j)_{j=1}^{m}\right\} = \arg\left(\min\left(\mathbf{Q}\left((p_i)_{i=1}^{n}, (q_j)_{j=1}^{m}\right)\right)\middle|\,\mathbf{P}\left((p_i)_{i=1}^{n}, (q_j)_{j=1}^{m}\right) = K\right) \tag{4.7}$$

with

$$\begin{aligned}
\mathbf{Q}\left((p_i)_{i=1}^{n}, (q_j)_{j=1}^{m}\right) &= \sum_{i,j=1}^{n} \Omega_{i,j}^{-1} p_i p_j + \sum_{i,j=1}^{m} \Omega_{i+n,j+n}^{-1} q_i q_j + 2\sum_{i=1}^{n}\sum_{j=1}^{m} \Omega_{i,j+n}^{-1} p_i q_j \\
\mathbf{P}\left((p_i)_{i=1}^{n}, (q_j)_{j=1}^{m}\right) &= \frac{1}{2}\sum_{i=1}^{n} \alpha_i X_i^0 e^{-\frac{\mu_i^{0,2}}{2} + p_i \mu_i^0} + \frac{1}{2}\sum_{j=1}^{m} \beta_j Y_j^0 e^{-\frac{\nu_j^{0,2}}{2} + q_j \nu_j^0}
\end{aligned}$$

where the matrix $\Omega \in \mathcal{M}\left((n+m)^2, [-1,1]\right)$ is defined as:

$$\Omega = \begin{pmatrix} \theta^x & \rho \\ \rho & \theta^y \end{pmatrix} \tag{4.8}$$

Introducing the set of variables: $(z_i)_{i=1}^{n+m}$, $(Z_i)_{i=1}^{n+m}$ and $(\sigma_i)_{i=1}^{n+m}$

- $\forall i = 1, \cdots, n$

$$\begin{aligned}
z_i &= p_i \\
Z_i &= \alpha_i X_i^0 e^{-\frac{\mu_i^{0,2}}{2}} \\
\sigma_i &= \mu_i^0
\end{aligned} \tag{4.9}$$

- $\forall j = 1, \cdots, m$

$$\begin{aligned}
z_{n+j} &= q_j \\
Z_{n+j} &= \beta_j Y_j^0 e^{-\frac{\nu_j^{0,2}}{2}} \\
\sigma_{n+j} &= \nu_j^0
\end{aligned} \tag{4.10}$$

we build the Lagrange function H :

$$\lambda \mapsto H(\lambda) = \sum_{i,j=1}^{n+m} \Omega_{i,j}^{-1} z_i z_j + \lambda\left(\sum_{i=1}^{n+m} Z_i e^{\sigma_i z_i} - K\right), \quad \lambda \in \mathbb{R} \tag{4.11}$$

Assuming that the solution is small enough to guarantee $\forall i, \quad \sigma_i z_i(\lambda) \ll 1$ such that $e^{\sigma_i z_i} \simeq 1 + \sigma_i z_i$, we solve the following system

$$\left\{\begin{array}{l}
\sum_{j=1}^{n+m} \Omega_{i,j}^{-1} z_j + \sigma_i^2 Z_i z_i = -\lambda \sigma_i Z_i \\
\sum_{i=1}^{n+m} Z_i e^{\sigma_i z_i(\lambda)} - K = 0
\end{array}\right. \tag{4.12}$$

We define the matrix Λ and the vector Γ such that $\forall i,j=1,\cdots,n+m$

$$\begin{aligned}\Lambda_{ij}^{-1} &= \Omega_{i,j}^{-1}+\sigma_i^2 Z_i\delta_{ij} \\ \Gamma_i &= -\sigma_i Z_i\end{aligned} \tag{4.13}$$

Thus $\partial_{z_i}H(\lambda)=0$

$$z(\lambda)=\lambda\Lambda\Gamma \tag{4.14}$$

We solve $\partial_\lambda H(\lambda)=0$:

$$\lambda\Gamma\Lambda\Gamma=K-\sum_{i=1}^{n+m}Z_i \tag{4.15}$$

and finally the solution is:

$$\begin{aligned}\lambda^* &= \frac{K-\sum_{i=1}^{n+m}Z_i}{\Gamma\Lambda\Gamma} \\ z^* &= \frac{\Lambda\Gamma}{\Gamma\Lambda(\lambda)\Gamma}\left(K-\sum_{i=1}^{n+m}Z_i\right)\end{aligned} \tag{4.16}$$

We sum up the several steps of the computation of the strike equivalent method:

- we start by assigning the Atm volatility to each asset $\left\{\left(\mu_i^0\right)_{i=1}^n,\left(\nu_j^0\right)_{j=1}^m\right\}$
- we determine the induced decomposition of the strike $K=\sum_{i=1}^n\alpha_i p_i^0+\sum_{j=1}^m\beta_j q_j^0$
- we iterate the process by assuming new volatilities $\left(\mu_i^1\right)_{i=1}^n=\left(\mu_i\left(p_i^0\right)\right)_{i=1}^n$ and $\left(\nu_j^1\right)_{j=1}^m=\left(\nu_j\left(q_j^0\right)\right)_{j=1}^m$
- this provides another decomposition $K=\sum_{i=1}^n\alpha_i p_i^1+\sum_{j=1}^m\beta_j q_j^1$ and define a sequence of strike equivalent set. The algorith stop at the n_s-th iteration.

$$\begin{pmatrix}\left(X_i^0\right)_{i=1}^n & \left(\mu_i^0\right)_{i=1}^n \\ \left(Y_j^0\right)_{j=1}^m & , \left(\nu_j^0\right)_{j=1}^m\end{pmatrix}\to\begin{pmatrix}\left(p_i^0\right)_{i=1}^n & \left(\mu_i^1\right)_{i=1}^n \\ \left(q_j^0\right)_{j=1}^m & , \left(\nu_j^1\right)_{j=1}^m\end{pmatrix}\to\cdots\to\begin{pmatrix}\left(p_i^{n_s}\right)_{i=1}^n & \left(\mu_i^{n_s+1}\right)_{i=1}^n \\ \left(q_j^{n_s}\right)_{j=1}^m & , \left(\nu_j^{n_s+1}\right)_{j=1}^m\end{pmatrix}$$

Remark 2 If there is no smile this sequence converge at step $k=0$.

References