Jump-Square Root Model for Credit Intensity

$\boxed{\text{Models - Technical Notes}}$

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1 The Jump-Square Root Dynamic

One of the identified drawbacks of the Hull-White model is the non-zero probability of credit intensity.

1.1 Dynamic

Consider a time dependent credit intensity process of the form

$$\begin{aligned} \lambda_t &= i(0,t) + y(t) && (1) \\ dy(t) &= \big(a(t) - \kappa y(t)\big)dt + \sigma(t)\sqrt{i(0,t) + y(t)}dW_t^{\mathbb{Q}} + dJ_t^{\ell,\mu} && (2) \\ y(0) &= 0 \end{aligned}$$

where $(W_t^{\mathbb{Q}})_{t\geq 0}$ is a standard Brownian motion and denotes $J_t^{\ell,\mu}$ any jump that occurs at time t of a pure-jump process $(J_t^{\ell,\mu})_{t\geq 0}$, independent of $(W_t^{\mathbb{Q}})_{t\geq 0}$, whose jump sizes are independent and exponentially distributed with mean μ and whose jump times are those of an independent Poisson process with mean jump arrival rate ℓ. , and $i(0,t)$ is the instantaneous default intensity defined by

$$i(0,t) = -\partial_t \ln(Q_{0,t})$$

1.2 Survival Probability

Starting from the equation (1), we now turn to the search for survival probability reconstitution formula

$$Q(t,T) = \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_t^{\mathbb{Q}} \Big[e^{-\int_t^T \lambda(s)ds} \big| \mathcal{F}_t \Big] = \frac{Q(0,T)}{Q(0,t)} \exp\big(c(t,T) - b(t,T)y(t)\big) \tag{3}$$

where c and b satisfy a system of Riccati ODEs

$$\begin{aligned} -\frac{db}{dt} + \kappa b + \frac{1}{2}\sigma(t)^2 b^2 &= 1, && (4) \\ \frac{dc}{dt} - a(t)b + \frac{1}{2}\sigma(t)^2 i(0,t) b^2 + \ell \frac{\mu b}{1+\mu b} &= 0, && (5) \end{aligned}$$

subject to the terminal conditions $c(T,T) = b(T,T) = 0$

Proof.

$$\begin{aligned} dQ(t,x_t) &= \Big(i(0,t)dt + c'(t,T)dt - b'(t,T)y(t)dt - b(t,T)dy(t) \\ &+ \frac{1}{2} b(t,T)^2 d\langle y \rangle_t \Big) Q(t,x_t) + \big[Q(t, x_{t^-} + Y_t^\mu) - Q(t, x_{t^-}) \big] \\ &= \Big(i(0,t)dt + c'(t,T)dt - b'(t,T)y(t)dt \\ &- b(t,T)\big(\big(a(t) - \kappa y(t)\big)dt + \sigma(t)\sqrt{i(0,t)+y(t)} dW_t^{\mathbb{Q}} + dJ_t^{\ell,\mu} \big) \\ &+ \frac{1}{2} b(t,T)^2 d\langle y \rangle_t \Big) Q(t,x_t) + \big[Q(t, x_{t^-} + Y_t^\mu) - Q(t, x_{t^-}) \big] \\ &= \Big(i(0,t) + (\kappa b(t,T) - b'(t,T) + \frac{1}{2} b(t,T)^2 \sigma(t)^2) y(t) \Big) Q(t,x_t) dt \\ &+ \Big(c'(t,T) - b(t,T)a(t) + \frac{1}{2} b(t,T)^2 i(0,t)^2 \Big) Q(t,x_t) dt \\ &+ \ell \frac{\mu b(t,T)}{1 + \mu B(t,T)} Q(t,x_t) dt \end{aligned}$$

□

1.3 Exact Resolution of b

1.3.1 Constante Volatility

$$b(t,T) = \frac{2(1 - e^{-\delta(T-t)})}{(\kappa+\delta)(1 - e^{-\delta(T-t)}) + 2\delta e^{-\delta(T-t)}} \text{ where } \delta = \sqrt{2\sigma^2 + \kappa^2} \tag{6}$$

1.3.2 Piecewise Constant Volatility

In the case when we are given a time grid $0 = t_0 < t_1 < \cdots < t_n$ on which the volatility $\sigma(t)$ can be assumed piecewise constant. *i.e.* $\forall t \in [t_i, t_{i+1}[$, $\sigma(t) = \sigma_i$.

We introduce the extended transform $\Psi_{t,T}$ by

$$\begin{aligned}
\forall t \leq T,\ \Psi_{t,T}(u,v) &\triangleq \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-u\lambda_T - v\int_t^T \lambda_s ds}\Big|\mathcal{F}_t\Big] \\
&= e^{-u\, i(0,T) - v\int_t^T i(0,s)ds}\ \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-u\, y(T) - v\int_t^T y(s)ds}\Big|\mathcal{F}_t\Big] \\
&= e^{-u\, i(0,T)}\left(\frac{Q_{0,T}}{Q_{0,t}}\right)^v e^{C_{t,T}(u,v) - B_{t,T}(u,v)y(t)}
\end{aligned}$$

Proposition 1. *The functional $B_{t,T}(u,v)$ satisfy the recursive equation*

$$\forall (u,v) \in \mathbb{C}^2,\ \forall i \leq j,\ B_{t_{i-1},t_j}(u,v) \quad = \quad B_{t_{i-1},t_i}\big(B_{t_i,t_j}(u,v), v\big)$$

Proof.

$$\begin{aligned}
\Psi_{t_{i-1},t_j}(u,v) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-u\lambda_{t_j} - v\int_{t_{i-1}}^{t_j} \lambda(s)ds}\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[e^{-u\lambda_{t_j} - v\int_{t_{i-1}}^{t_j} \lambda(s)ds}\big|\mathcal{F}_{t_i}\big]\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-v\int_{t_{i-1}}^{t_i} \lambda(s)ds}\mathbb{E}^{\mathbb{Q}}\big[e^{-u\lambda_{t_j} - v\int_{t_i}^{t_j} \lambda(s)ds}\big|\mathcal{F}_{t_i}\big]\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-v\int_{t_{i-1}}^{t_i} \lambda(s)ds}\Psi_{t_i,t_j}(u,v)\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_i}}\right)^v e^{C_{t_i,t_j}(u,v)}\mathbb{E}^{\mathbb{Q}}\Big[e^{-v\int_{t_{i-1}}^{t_i} \lambda(s)ds} e^{-B_{t_i,t_j}(u,v)y(t_i)}\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_i}}\right)^v e^{C_{t_i,t_j}(u,v)} e^{B_{t_i,t_j}(u,v)i(0,t_i)}\mathbb{E}^{\mathbb{Q}}\Big[e^{-v\int_{t_{i-1}}^{t_i} \lambda(s)ds} e^{-B_{t_i,t_j}(u,v)\lambda_{t_i}}\Big|\mathcal{F}_{t_{i-1}}\Big] \\
&= e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_i}}\right)^v e^{C_{t_i,t_j}(u,v)} e^{B_{t_i,t_j}(u,v)i(0,t_i)}\Psi_{t_{i-1},t_i}\big(B_{t_i,t_j}(u,v), v\big) \\
&= e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_{i-1}}}\right)^v e^{C_{t_i,t_j}(u,v)} e^{C_{t_{i-1},t_i}\left(B_{t_i,t_j}(u,v),v\right) - B_{t_{i-1},t_i}\left(B_{t_i,t_j}(u,v),v\right)y(t)}
\end{aligned}$$

$$\begin{aligned}
e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_{i-1}}}\right)^v e^{C_{t_i,t_j}(u,v)} \qquad & e^{C_{t_{i-1},t_i}\left(B_{t_i,t_j}(u,v),v\right) - B_{t_{i-1},t_i}\left(B_{t_i,t_j}(u,v),v\right)y(t)} \\
&= e^{-u\, i(0,t_j)}\left(\frac{Q_{0,t_j}}{Q_{0,t_{i-1}}}\right)^v e^{C_{t_{i-1},t_j}(u,v) - B_{t_{i-1},t_j}(u,v)y(t)}
\end{aligned}$$

$$\begin{aligned}
B_{t_{i-1},t_j}(u,v) &= B_{t_{i-1},t_i}\big(B_{t_i,t_j}(u,v), v\big) \\
C_{t_{i-1},t_j}(u,v) &= C_{t_i,t_j}(u,v) + C_{t_{i-1},t_i}\big(B_{t_i,t_j}(u,v), v\big)
\end{aligned}$$

$$e^{A_{t_{i-1},t_j}(u,v) - B_{t_{i-1},t_j}(u,v)y_{t_{i-1}}} \quad = \quad e^{A_{t_i,t_j}(u,v) + A_{t_{i-1},t_i}(B_{t_i,t_j}(u,v),v) - B_{t_{i-1},t_i}(B_{t_i,t_j}(u,v),v)y_{t_{i-1}}}$$

$$\forall (u,v) \in \mathbb{C}^2, \forall i < j,\ B_{t_{i-1},t_j}(u,v) \quad = \quad B_{t_{i-1},t_i}(B_{t_i,t_j}(u,v), v)$$

$\square$

One can deduce the functional b as

$$\forall\, 0 \leq t_{i-1} < t_j,\ b(t_{i-1}, t_j) \quad = \quad B_{t_{i-1},t_j}(0,1)$$

As parameters are constant on the time grid, the function $(u,v) \mapsto B_{t_{i-1},t_i}(u,v)$ can be computed in closed form

$$B_{t,T}(u,v) \quad = \quad \frac{(2v - \kappa u)(1 - e^{-\delta(v)(T-t)}) + \delta(v)u\,(1 + e^{-\delta(v)(T-t)})}{(\kappa + \delta(v) + u\sigma^2)(1 - e^{-\delta(v)(T-t)}) + 2\delta(v)e^{-\delta(v)(T-t)}} \text{ where } \delta(v) = \sqrt{2v\sigma^2 + \kappa^2}$$

For all $t \in [t_j, t_{j+1}]$ and $j \in [0, N-1]$, b could be deduced and simplified as below:

$$b(t,T) = \frac{(\frac{\kappa-\delta_k}{\sigma_k^2})(\frac{\kappa+\delta_{N-1}}{\kappa-\delta_{N-1}})e^{\sum_{j=k}^{N-1}\delta_j(t_{j+1}-t_j)} - \frac{\kappa+\delta_k}{\sigma_k^2}}{1 - \frac{\kappa+\delta_{N-1}}{\kappa-\delta_{N-1}}e^{\sum_{j=k}^{N-1}\delta_j(t_{j+1}-t_j)}} \text{ where } \delta_j = \sqrt{2\sigma_j^2 + \kappa^2} \tag{7}$$

1.3.2.1 Numerical implementation

1.4 Algorithm for $a(t)$

The solution of the first Riccati equation (4) can stand as pre-processing step. This does not depend on $a(\cdot)$.
Setting $t = 0$ in equation (3) establishes the fundamental calibration requirement that $c(0,T) = c(0,T;a(\cdot)) = 0$ for all T which, combined with (5), defines a Volterra integral equation of the first kind for $a(\cdot)$.

$$0 = c(t,t) - c(0,t) = \int_0^t a(s)b(s,t)ds - \int_0^t \frac{\sigma(s)^2}{2} i(0,s)b^2(s,t)ds - \int_0^t \frac{\ell\mu b(s,t)}{1+\mu b(s,t)}ds \tag{8}$$

$$f(s) \quad = \quad \alpha s + \beta\int_0^s b(u,t)du + \delta\ln(1 + \mu b(s,t)) \tag{9}$$

$$f'(s) \quad = \quad \alpha + \beta b(s,t) + \delta\frac{\mu b'(s,t)}{1+\mu b(s,t)} \tag{10}$$

$$= \quad \frac{\alpha + \alpha\mu b(s,t) + \beta b(s,y) + \beta\mu b(s,t)^2 + \delta\mu b'(s,t)}{1+\mu b(s,t)} \tag{11}$$

$$= \quad \frac{\alpha + \alpha\mu b(s,t) + \beta b(s,t) + \beta\mu b(s,t)^2 + \delta\mu(\kappa b + \frac{1}{2}\sigma(t)^2 b^2 - 1)}{1+\mu b(s,t)} \tag{12}$$

$$\tag{13}$$

hence

$$\alpha - \mu\delta \quad = \quad 0 \tag{14}$$

$$\mu\alpha + \beta + \mu\kappa\delta \quad = \quad \mu \tag{15}$$

$$\beta + \frac{1}{2}\sigma(t)^2\delta \quad = \quad 0 \tag{16}$$

$$\begin{aligned}
\alpha - \mu\delta &= 0 & (17)\\
\mu^2\delta - \big(\frac{1}{2}\sigma(t)^2 + \mu\kappa\big)\delta &= \mu & (18)\\
\beta + \frac{1}{2}\sigma(t)^2\delta &= 0 & (19)
\end{aligned}$$

$$\begin{aligned}
\alpha &= \frac{\mu^2}{\mu(\mu+\kappa) - \frac{1}{2}\sigma^2(t)} & (20)\\
\delta &= \frac{\mu}{\mu(\mu+\kappa) - \frac{1}{2}\sigma^2(t)} & (21)\\
\beta &= \frac{-\frac{1}{2}\sigma(t)^2\mu}{\mu(\mu+\kappa) - \frac{1}{2}\sigma^2(t)} & (22)\\
& & (23)
\end{aligned}$$

Given a time grid $0 = t_0 < t_1 < \cdots < t_n$ and assuming that $a(\cdot)$ is piecewise constant at a level a_i over the time interval $]t_i, t_{i+1}]$, we can use the following iterative algorithm.

1. For a given i, assume that a_i is known for $j < i$.
2. Compute $\Theta(t_i) = \frac{1}{2}\int_0^{t_{i+1}} \sigma(s)^2 i(0,s) b(s,t_{i+1})^2 ds - \int_0^{t_i} \Phi(s) b(s,t_{i+1}) ds$
3. Compute a_i as the solution to $\Theta(t_i) - \Phi_i \int_{t_i}^{t_{i+1}} b(s,t_{i+1}) ds = 0$
4. Repeat step 1-3 for all $i = 0, 1, \ldots, N-1$

2 Discretization schemes for simulating $y(t)$

$$d\lambda_t = \kappa\big(\Psi(t_i) - \lambda(t_i)\big)dt + \sigma(t)\sqrt{\lambda(t)}\,dZ(t) \qquad (24)$$

where

$$\kappa\Psi(t) = \Phi(t) + i'(t) + \kappa i(t)$$

The easiest choice is given by the Euler scheme. Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a discretization of the interval $[0, T]$.

The Euler scheme reads:

$$\begin{aligned}
\tilde{\lambda}(t_{i+1}) &= \tilde{\lambda}(t_i) + \kappa\big(\Psi(t_i) - \tilde{\lambda}(t_i)\big)(t_{i+1} - t_i)\\
&+ \sigma(t_i)\sqrt{\tilde{\lambda}(t_i)}\,\big(Z(t_{i+1}) - Z(t_i)\big) \qquad (25)
\end{aligned}$$

starting from $\tilde{\lambda}(t_0) = i(0)$. Although the regularity conditions that ensure a better convergence for the Milstein scheme are not satisfied here (the diffusion coefficient is not

Lipschitz), one may try to apply it anyway.
The related equation for $\tilde{y}(t_i)$ is as follows:

$$\begin{aligned} \tilde{\lambda}(t_{i+1}) &= \tilde{\lambda}(t_i) + \kappa\big(\Psi(t_i) - \tilde{\lambda}(t_i)\big)(t_{i+1} - t_i) + \sigma(t_i)\sqrt{\tilde{\lambda}(t_i)}\,\big(Z(t_{i+1}) - Z(t_i)\big) \\ &+ \frac{1}{4}\sigma(t_i)^2\Big(\big(Z(t_{i+1}) - Z(t_i)\big)^2 - (t_{i+1} - t_i)\Big) \end{aligned} \tag{26}$$

The major drawback of the previous explicit schemes is that they do not ensure positivity of $\tilde{\lambda}(t_i)$. Two "modified version" of the Euler scheme have been proposed

1. Deelstra and Delbaen :

$$\tilde{\lambda}(t_{i+1}) = \tilde{\lambda}(t_i) + \kappa\big(\Psi(t_i) - \tilde{\lambda}(t_i)\big)(t_{i+1} - t_i) + \sigma(t_i)\sqrt{\tilde{\lambda}(t_i)\mathbf{1}_{\{\tilde{\lambda}(t_i)>0\}}}\,\big(Z(t_{i+1}) - Z(t_i)\big)$$

2. Diop

$$\tilde{\lambda}(t_{i+1}) = \Big|\tilde{y}(t_i) + \kappa\big(\Phi(t_i) - \tilde{y}(t_i)\big)(t_{i+1} - t_i)\sigma(t_i)\sqrt{\tilde{\lambda}(t_i)}\,\big(Z(t_{i+1}) - Z(t_i)\big)\Big|$$