MODEL DOCUMENTATION

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Object	**LA18**
Direction	Global Market
Author(s)	IR QR
Reference	Model
Diffusion	Restricted
Version	1.0
Complexity	Medium
Proposed tier	1

Contents

1 General information

Model Development Information	
Model ID	No reference
Model Name	Strike Equivalent
Business unit	Natixis / Global Market / Trading FI
Model purpose	Model
Name of the model development team	IR QR
Model development start date	
Model development completion date	
Date of the most recent revision	
Model Implementation Information	
Implementation plateform	ARM
Integration plateform	Summit
Name of model implementation team	DSI
Date of implementation	
Model Governance Information	
Name of the model owner	Pascal Amiel
Date model deployment approved for initial Business use	
Most recent date model approved for continued Busines use	
Approved uses	LD-M-FIC4-INFLATION
Model restrictions	Payoff dependent
Approved users	
Name of the model validator lead for the most recent validation	MRM
Date of the most recent prior validation/review	
Type of the most recent prior validation/review	Periodical Review
Model Key Documentation	
Model documentation names	New Livret A pricing
Previous validation report names	
Business requirement documentation name	New Livret A pricing
Specification documentation name	
IT implementation sign off documentation name	
User acceptance report name	
Regulatory references	
Calibration files name	
Back testing documentation name	
Stress testing documentation name	

2 Description

2.1 Purpose and use

This document aims at describing the pricing methodology of the LA18 option, an inflation-indexed cap floor option, used as a hedging instrument against the interest rate variations of the Livret A, the most popular French saving account.

Without taking convexity adjustments into account, a LA18 option is sensitive to the following risk factors:

Payoff	Risk factors
LA18 Option	CPI curve Discount rate curve Forecast rate curve Forecast rate volatility smile YoY index volatilty smile YoY/YoY correlation YoY/Forecast rate correlation Forecast rate/Forecast rate correlation

2.2 In House or Vendor

The LA18 model is developed by the IR QR team. It is implemented in ARM, the in-house pricing library and used in Summit, the vendor front to back system.

2.3 Outputs

The current model is used to calculate the price of the LA18 option. The resulting price is used by summit to compute the greeks by means of bumping the risk factors, repricing and taking the finite difference.

2.4 Appropriateness of the combination

The LA18 option cash flows can be seen as a two-dimensional basket option. A moment matching projection is performed to allow the pricing with a bivariate formula. To take into account the volatility smile, a strike equivalent routine is applied. This choice is made to reduce the pricing time.

2.5 Pricer use

The LA18 pricer is used by the following stakeholders :

- Model Owner: Pascal Amiel
- Model Users:
  - FO IRD :
    - Trading : pricing and hedging
    - Structuring : pricing and testing
    - Quants : development, testing, support and maintenance
    - Sales : pricing
  - DRM : risk monitoring via VaR, CVA and other risk indicators
  - SDR : P&L production and consensus contributions
- Model Developer (Quants): for model development and maintenance
- Model Implementer: DSI

3 Payoff

Since February 2020, the Livret A rate is fixed by Banque de France twice a year on January 15 and July15. These rates are published as the applicable fixing rates for February 1^{st} and August 1^{st} respectively and are computed as the average of the French inflation and EONIA rates over the same observation window, currently of six months with a 0.5% floor.

3.1 Cash flow

Considering a six month interest period $[T_{\delta^-}, T]$, a floor rate R_F of 0.5% and denoting by $\left(Y_{T_y^i}\right)_{i=1}^{n}$, n successive monthly year-on-year (YoY) and $\left(F_{T_f^j}\right)_{j=1}^{m}$, m daily successive EONIA rates over that period , the Livret A rate is obtained in a simple manner at time T by $R_T^{LA} = \max\left(R_F, \mathrm{LA18}_T\right)$ where :

$$\mathrm{LA18}_T = \frac{1}{2n}\sum_{i=1}^{n} Y_{T_y^i} + \frac{1}{2m}\sum_{j=1}^{m} F_{T_f^j} \tag{3.1}$$

with $T_y^1 = T_e^1 = T_{\delta^-}$ and $T_y^n = T_e^m = T$. The LA18 option enables investors indebted at the Livret A rate to protect themselves by locking the payment at a maximum rate $K \geq R_F$. Denoting by w a weighting coefficient , the $k-th$ cash flow of the LA18 option payed at date $T_p^k \geq T^k$ is then given by :

$$C_w^k = \left(\frac{(1-w)}{n}\sum_{i=1}^{n} Y_{T_y^{i,k}} + \frac{w}{m}\sum_{j=1}^{m} F_{T_f^{j,k}} - K\right)_+ \Bigg|_{w=\frac{1}{2}} \tag{3.2}$$

Remark 1 Introducing the generic notation C_w enables us to describe it in most cases and to define other inflation derivatives. For example, setting $w = 0$ leads to the definition of the average inflation option.

3.2 Features

In order to give a global description of the LA18, its main features are listed in the table bellow.

	NOTATION	DESCRIPTION	USUAL VALUE
SCHEDULE	T_S	start date	
	T_E	end date	
	δ_c	day count	30/360
	p_g	payment gap	
	p_f	payment frequency	half yearly
YOY	r_l^y	reset lag	-8M
	r_f^y	reset frequency	monthly
EONIA	r_l^e	reset lag	-7M
	r_f^e	reset frequency	daily

3.3 Example

4 Diffusion model x numerical method description

4.1 Framework

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a probability space, where $\mathbb{P}$ is the historical probability and $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by a multidimensional brownian motion $\left(W_t^l\right)_{l=1}^d$ with $d \geq n+m$ such that

- $\forall i,j = 1,...,n$: $\qquad d\left\langle W_t^i, W_t^j \right\rangle = \theta_{ij}^Y dt$
- $\forall i,j = 1,...,m$: $\qquad d\left\langle W_t^{n+i}, W_t^{n+j} \right\rangle = \theta_{ij}^F dt$
- $\forall i = 1,...,n$ and $\forall j = 1,...,m$: $\qquad d\left\langle W_t^i, W_t^{n+j} \right\rangle = \rho_{ij} dt$

We suppose moreover that the market is arbitrage-free and introduce the risk neutral probability measure $\mathbb{Q}$, defined by the riskless savings account numeraire β. According to these definitions, the standard valuation formula of the LA18 option at time $t \leq T_S$ is then given by

$$V_t = \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{k=1}^{N} \frac{\beta_t}{\beta_{T_p^k}} C_k\right] = \sum_{k=1}^{N} P(t, T_p^k)\mathbb{E}_t^{\mathbb{Q}_{T_p^k}}\left[C_k\right] \tag{4.1}$$

with $P(t,T_p^k)$, the T_p^k- maturity zero coupon bond value at time t and $\mathbb{Q}_{T_p^k}$ the T_p^k forward mesure. Omitting the $k-$ subscript, the real challenge is to estimate the following basket option

$$P_{\alpha,\beta}(t,K) = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[\left(\sum_{i=1}^{n} \alpha_i Y_{T_i} + \sum_{j=1}^{m} \beta_j F_{T_j} - K\right)_+\right] \tag{4.2}$$

4.2 Moment matching

We note $W_t^{T_p}$ the brownian motion under Q_{T_p} and assume shifted lognormal dynamics for the pay lagged forwards defined by $\tilde{Y}_t^T = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[Y_T\right]$ and $\tilde{F}_t^T = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[F_T\right]$ such that

$$\begin{aligned} d\tilde{Y}_t^{T_i} &= \mu_i\left(\tilde{Y}_t^{T_i} + s_i\right) dW_t^{T_p,i}, \quad \forall i = 1,...,n && (4.3)\\ d\tilde{F}_t^{T_j} &= \nu_j\left(\tilde{F}_t^{T_j} + r_j\right) dW_t^{T_p,n+j}, \quad \forall j = 1,...,m && (4.4)\end{aligned}$$

with $(s_i,\mu_i)_{i=1}^n$ and $(r_j,\nu_j)_{j=1}^m$ the shifts and the strike equivalent volatilities of YoY and EONIA forward rates respectively. As shown before, using the last fixing date $T = T_n = T_m$, the basket option value rewrittes as

$$\tilde{P}_{\alpha,\beta}(t,\tilde{K}) = \mathbb{E}_t^{\mathbb{Q}_{T_p}}\left[\left(\tilde{Y}_T + \tilde{F}_T - \tilde{K}\right)_+\right] \tag{4.5}$$

with $\tilde{Y}_T = \sum_{i=1}^n \alpha_i \tilde{Y}_{T_i}^{T_i}$, $\quad \tilde{F}_T = \sum_{j=1}^m \beta_j \tilde{F}_{T_j}^{T_j}$ and $\tilde{K} = K - \sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j$.

In order to price $\tilde{P}_{\alpha,\beta}$ quickly through a bivariate formula, a moment matching procedure is used to approximate

$$\tilde{P}_{\alpha,\beta}(t,\tilde{K}) \approx BiLog\left(\bar{Y}_t, \bar{F}_t, \mu_{\bar{Y}}, \nu_{\bar{F}}, \rho\right) \tag{4.6}$$

where $\left(\bar{Y},\bar{F}\right)$ are two lognormal random variables with volatility $(\mu_{\bar{Y}},\nu_{\bar{F}})$ and correlation $\rho dt = d\left\langle \ln(\bar{Y}_t), \ln(\bar{F}_t)\right\rangle$ such that $\forall i = \{1,2\}$

$$\left\{\begin{array}{lcl} \mathbb{E}_t\left[\bar{Y}_T^i\right] & = & \mathbb{E}_t\left[\tilde{Y}_T^i\right] \\ \mathbb{E}_t\left[\bar{F}_T^i\right] & = & \mathbb{E}_t\left[\tilde{F}_T^i\right] \\ \mathbb{E}_t\left[\bar{Y}_T\bar{F}_T\right] & = & \mathbb{E}_t\left[\tilde{Y}_T\tilde{F}_T\right] \end{array}\right.$$

By identification we get

$$\begin{aligned} \mu_{\bar{Y}}^2 &= \ln\left(\frac{\sum_{i,j=1}^n \alpha_i\alpha_j \tilde{Y}_t^{T_i}\tilde{Y}_t^{T_j} e^{\mu_i\mu_j\theta_{ij}^Y}}{\sum_{i,j=1}^n \alpha_i\alpha_j \tilde{Y}_t^{T_i}\tilde{Y}_t^{T_j}}\right) \\ \nu_{\bar{F}}^2 &= \ln\left(\frac{\sum_{i,j=1}^m \beta_i\beta_j \tilde{F}_t^{T_i}\tilde{F}_t^{T_j} e^{\nu_i\nu_j\theta_{ij}^F}}{\sum_{i,j=1}^m \beta_i\beta_j \tilde{F}_t^{T_i}\tilde{F}_t^{T_j}}\right) \\ \mu_{\bar{Y}}\nu_{\bar{F}}\rho &= \ln\left(\frac{\sum_{i=1}^n\sum_{j=1}^m \alpha_i\beta_j \tilde{Y}_t^{T_i}\tilde{F}_t^{T_j} e^{\mu_i\nu_j\rho_{ij}}}{\sum_{i=1}^n\sum_{j=1}^m \alpha_i\beta_j \tilde{Y}_t^{T_i}\tilde{F}_t^{T_j}}\right) \end{aligned}$$

4.3 Strike equivalent method

4.3.1 Purpose

In reality, the market option quotes provide volatility smiles $(\mu_i(k), \nu_j(k))$ for the basket assets $\left(\tilde{Y}_t^{T_i}\right)_{i=1}^n$ and $\left(\tilde{F}_t^{T_j}\right)_{j=1}^m$. In consequence, these random variables aren't strictly lognormal. To apply the moment matching method, a strike choice (k_i^*, k_j^*) is needed such that

$$\begin{aligned} \mu_i^* &= \mu_i(k_i^*) \\ \nu_j^* &= \nu_j(k_j^*) \end{aligned}$$

The strike equivalent method relies on the assumption that the price of a basket option at a strike K should be determined from the most likely configuration of its assets conditional on arriving at this level at maturity. Starting from arbitraries standard deviations $(\tilde{\mu_i}, \tilde{\nu_j})$ and denoting by M the set of points on the hypersurface $\Gamma_{\tilde{K}}$ such that

$$\Gamma_{\tilde{K}} = \left\{ (x,y), \quad \sum_{i=1}^n \tilde{\alpha_i} e^{\mu_i x_i} + \sum_{j=1}^m \tilde{\beta_j} e^{\nu_j y_j} - \tilde{K} = 0 \right\} \tag{4.7}$$

with $\tilde{\alpha_i} = \alpha_i \left(\tilde{Y}_t^{T_i} + s_i\right) e^{\frac{-\tilde{\mu}_i^2}{2}}$ and $\tilde{\beta_j} = \beta_j \left(\tilde{F}_t^{T_j} + r_j\right) e^{\frac{-\tilde{\nu}_j^2}{2}}$, the mathematical description of this method is given by

$$\left\{ (p_i)_{i=1}^n, (q_j)_{j=1}^m \right\} = \underset{(x,y)\in M}{\arg\min} \left(\sum_{i,j=1}^n \Omega_{i,j}^{-1} x_i x_j + \sum_{i,j=1}^m \Omega_{i,j}^{-1} y_i y_j + 2 \sum_{i=1}^n \sum_{j=1}^m \Omega_{i,j}^{-1} x_i y_j \right) \tag{4.8}$$

where $\Omega = \begin{pmatrix} \theta^Y & \rho \\ \rho & \theta^F \end{pmatrix} \in \mathcal{M}\left((n+m)^2, [-1,1]\right)$ is the correlation matrix defined in the previous section. At the end of the procedure, the equivalent strikes are then given by

$$\begin{cases} \forall i = (1, \cdots, n), & k_i^* = \frac{\tilde{\alpha}_i}{\alpha_i} e^{\tilde{\mu}_i p_i} \\ \forall j = (1, \cdots, m), & k_j^* = \frac{\tilde{\beta}_j}{\beta_j} e^{\tilde{\nu}_j p_j} \end{cases} \tag{4.9}$$

4.3.2 Resolution

To resolve this minimization problem, we use the method of Lagrange multipliers. Introducing the variable $(z_k)_{k=1}^{n+m}$, $(\gamma_k)_{k=1}^{n+m}$ and $(\sigma_k)_{k=1}^{n+m}$ such that

- $\forall k = 1, \cdots, n$

$$\begin{aligned} z_k &= p_k \\ \gamma_k &= \tilde{\alpha_k} \\ \sigma_k &= \mu_k \end{aligned}$$

- $\forall l = 1, \cdots, m$

$$\begin{aligned} z_{n+l} &= q_l \\ \gamma_{n+l} &= \tilde{\beta_l} \\ \sigma_{n+l} &= \nu_l \end{aligned}$$

the Lagrange function H is define by

$$H_\lambda = \left(\sum_{k,l=1}^{n+m} \Omega_{k,l}^{-1} z_k z_l \right) + \lambda \left(\sum_{k=1}^{n+m} \gamma_k e^{\sigma_k z_k} - \tilde{K} \right), \quad \lambda \in \mathbb{R}$$

An admissible solution is find by resolving the system :

$$\begin{cases} \partial_\lambda H(\lambda) = \sum_{k=1}^{n+m} \gamma_k e^{\sigma_k z_k} - \tilde{K} = 0 \\ \partial_{z_k} H(\lambda) = \sum_{l=1}^{n+m} \Omega_{k,l}^{-1} z_l + \sigma_k \gamma_k \lambda e^{\sigma_k z_k} = 0, \quad \forall k = 1, \cdots, n+m \end{cases}$$

For making things a bit easier, we assume that the solution is small enough to guarantee

$$\forall k, \quad e^{\sigma_k z_k(\lambda)} \simeq 1 + \sigma_k z_k(\lambda)$$

This linearization force us to handle the strike equivalent method as an iterative process but enables to write the solution in a simple manner :

$$z^* = \lambda^* \Lambda \Gamma \qquad \text{with} \qquad \lambda^* = \frac{\tilde{K} - \sum_{k=1}^{n+m} \gamma_k}{\Gamma \Lambda \Gamma} \tag{4.10}$$

where the matrix Λ and the vector Γ are defined $\forall k, l = 1, \cdots, n+m$ by

$$\begin{aligned} \Lambda_{kl}^{-1} &= \Omega_{k,l}^{-1} + \delta_{kl} \gamma_k \sigma_k^2 \\ \Gamma_k &= -\sigma_k \gamma_k \end{aligned} \tag{4.11}$$

We sum up the several steps of the computation of the strike equivalent method:

- we start by assigning the Atm volatility to each asset $\left\{ \left(\mu_i^0\right)_{i=1}^n, \left(\nu_j^0\right)_{j=1}^m \right\}$
- we determine the induced decomposition of the strike $K = \sum_{i=1}^n \alpha_i k_i^0 + \sum_{j=1}^m \beta_j k_j^0$
- we iterate the process by assuming new volatilities $\left(\mu_i^1\right)_{i=1}^n = \left(\mu_i\left(k_i^0\right)\right)_{i=1}^n$ and $\left(\nu_j^1\right)_{j=1}^m = \left(\nu_j\left(k_j^0\right)\right)_{j=1}^m$
- this provides another decomposition $K = \sum_{i=1}^n \alpha_i k_i^1 + \sum_{j=1}^m \beta_j k_j^1$ and define a sequence of strike equivalent set. The algorithm stops at the $n_s - th$ iteration.

$$\begin{pmatrix} \left(\tilde{\alpha}_i^0\right)_{i=1}^n & \left(\mu_i^0\right)_{i=1}^n \\ \left(\tilde{\beta}_j^0\right)_{j=1}^m & , \left(\nu_j^0\right)_{j=1}^m \end{pmatrix} \to \begin{pmatrix} \left(k_i^0\right)_{i=1}^n & \left(\mu_i^1\right)_{i=1}^n \\ \left(k_j^0\right)_{j=1}^m & , \left(\nu_j^1\right)_{j=1}^m \end{pmatrix} \to \cdots \to \begin{pmatrix} \left(k_i^{n_s}\right)_{i=1}^n & \left(\mu_i^{n_s+1}\right)_{i=1}^n \\ \left(k_j^{n_s}\right)_{j=1}^m & , \left(\nu_j^{n_s+1}\right)_{j=1}^m \end{pmatrix}$$

5 Model development tests

5.1 Consistency tests

5.2 Strike equivalent convergence

5.3 Price sensitivity to market data

6 Implementation

6.1 Implementation description

Waiting for DSI.

6.2 Implementation test(s)

Waiting for DSI.

7 Model management

7.1 Maintenance of the model

As model developer and implementer, the quantitative team and the MOA are responsible for the maintenance of the model when it is needed.

7.2 Process for ongoing monitoring

According to the market evolution, the model users are able to raise the alarm if they point out any inconsistence of the model.

7.3 Change control management

The model owner is responsible for any change in the control management.

References