Ordinary Differential Equations

APM1137 - Numerical Analysis

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Initial Value Problem

Approximating the solution $yt(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t,y), \text{ for } a \leq t \leq b,$$

subject to an initial condition $y(a) = \alpha$.

Extended to the approximating the solution to a system of first-order differential equations of the form

$$\begin{aligned}
\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \ldots, y_n), \\
\frac{dy_2}{dt} &= f_2(t, y_1, y_2, \ldots, y_n), \\
&\vdots \\
\frac{dy_n}{dt} &= f_n(t, y_1, y_2, \ldots, y_n),
\end{aligned}$$

for $a \leq t \leq b$, subject to the initial conditions

$$y_1(a) = \alpha_1, y_2(a) = \alpha_2, \ldots, y_n(a) = \alpha_n.$$

Also to examine the relationship of a system of this type to the general nth-order initial value problem of the form

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}),$$

for $a \leq t \leq b$, subject to the initial conditions

$$y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(n-1)}(a) = \alpha_n.$$

Differential equations are used to model problems in science and engineering that involve the change of some variable with respect to another.

In common real-life situations, the differential equation that models the problem is too complicated to solve explicitly, and one of two approaches is taken to approximate the solution.

- Modify the problem by simplifying the differential equation to one that can be solved explicitly and then use the solution of the simplified equation to approximate the solution to the original problem.
- Use methods for approximating the solution of the original problem. Approximation methods give more accurate results and realistic error information.

Definition

A function $f(t,y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|,$$

whenever (t,y_1) and (t,y_2) are in D. The constant L is called a **Lipschitz constant** for f.

Show that $f(t,y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t,y)|1 \leq t \leq 2 \; and \; -3 \leq y \leq 4\}$.

Solution:
For each pair $(t,y_1),(t,y_2) \in D$,

$$|f(t,y_1) - f(t,y_2)| = |t|y_1| - t|y_2|| = |t|||y_1| - |y_2|| \leq 2|y_1 - y_2|$$

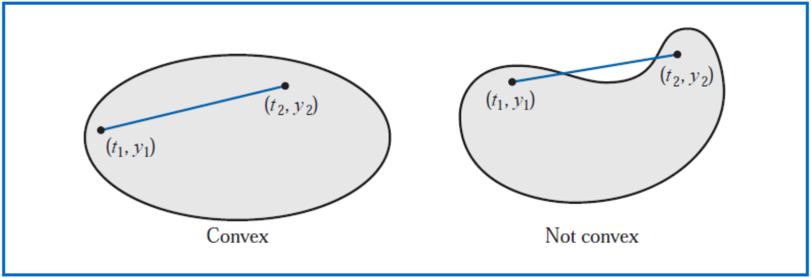
Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2.

Definition

A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t, y_1) and (t, y_2) are in D, then

$$((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$$

also belongs to D for every $\lambda \in [0, 1]$



Theorem

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \text{ for all } (t, y) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem

Suppose that $D = \{(t,y) | a \leq t \leq b \; and \; -\infty \leq y \leq \infty\}$ and that $f(t,y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Example

Show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0.$$

Applying the Mean Value theorem to

$$f(t, y) = 1 + t\sin(ty),$$

we find that when $y_1 < y_2$, a number $\xi \in (y_1, y_2)$ exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t, \xi) = t^2 \cos(t\xi).$$

Thus

$$|f(t,y_2) - f(t,y_1)| = |y_2 - y_1||t^2 \cos(t\xi)| \leq 4|y_2 - y_1|,$$

and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$.

Additionally, $f(t,y)$ is continuous when $0 \leq t \leq 2$ and $-\infty < y < \infty$, so the theorem implies that a unique solution exists to this initial-value problem.

How do we determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution?

Definition

The initial-value problem

$$\frac{dy}{dt} = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

is said to be a **well-posed problem** if :

- A unique solution $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all $t \in [a,b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

  $$\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \leq t \leq b \quad z(a) = \alpha + \delta_0, \quad (**)$$

  has a unique solution $z(t)$ that satisfies

  $$|z(t) - y(t)| < k\varepsilon \text{ for all } t \in [a,b].$$

The problem specified by (**) is called a **perturbed problem** associated with the original problem. It assumes the possibility of an error being introduced in the statement of the differential equation, as well as an error δ_0 being present in the initial condition.

Theorem

Suppose $D = \{(t, y) | a \le t \le b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is well-posed.

Example

Show that

$$\frac{dy}{dt} = y - t^2 + 1, 0 \leq t \leq 2, y(0) = 0.5$$

is well posed on $D = \{(t, y)|0 \leq t \leq 2 \ and \ -\infty < y < \infty\}$.

Since

$$\left|\frac{\partial(y - t^2 + 1)}{\partial y}\right| = |1| = 1,$$

This implies that f satisfies a Lipschitz condition in y on D with Lipschitz constant 1. Since f is continous on D, then the problem is well-posed.

Euler's Method

The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dx} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

- Approximations to y will be generated at various values, called **mesh points**, in the interval $[a, b]$.
- The approximate solution at other points in the interval can be found by interpolation.
- Chose a positive integer N and select the mesh points

  $$t_i = a + ih, \quad \textit{for each } i = 0, 1, 2, \ldots, N.$$

  $h = (b - a)/N$ is called the **step size**.

Use Taylor's Theorem to derive Euler's method. Suppose that $y(t)$, the unique solution to the IVP, has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}).

Since $h = (t_{i+1} - t_i)$, and because y satisfies the IVP, we have

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i),$$

Euler's method constructs $w_i \approx y(t_i)$ for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + hf(t_i, w_i), \text{ for each } i = 1, 2, \ldots, N-1 \end{aligned}$$

This is called the **difference equation** associated with Euler's method.

Example

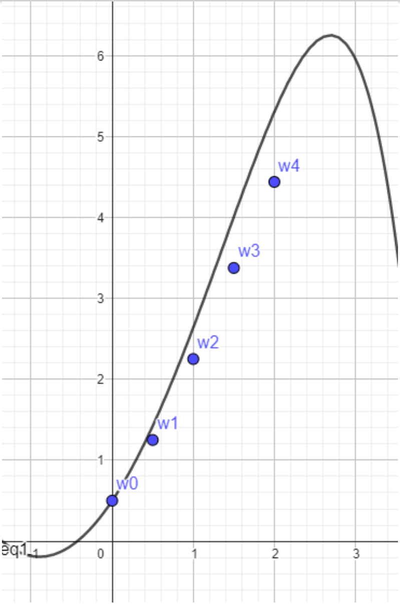
Approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

for $h = 0.5$.

Solution:
$f(t, y) = y - t^2 + 1$

$$\begin{aligned}
w_0 &= y(0) = 0.5 \\
w_1 &= w_0 + 0.5(w_0 - (0.0)^2 + 1) = 0.5 + 0.5(1.5) = 1.25 \\
w_2 &= w_1 + 0.5(w_1 - (0.5)^2 + 1) = 1.25 + 0.5(2.0) = 2.25 \\
w_3 &= w_2 + 0.5(w_2 - (1.0)^2 + 1) = 2.25 + 0.5(2.25) = 3.375 \\
w_4 &= w_3 + 0.5(w_3 - (1.5)^2 + 1) = 3.375 + 0.5(2.125) = 4.4375
\end{aligned}$$


w4
w3
w2
w1
w0
eq1
6
5
4
3
2
1
0
-1
1
2
3

Algorithm

Euler's

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N + 1)$ values of t.

Step 1 Set $h = (b - a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3, 4.

Step 3 Set $w = w + hf(t, w)$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 4 OUTPUT (t, w).

Step 5 STOP.

Figure 5.2

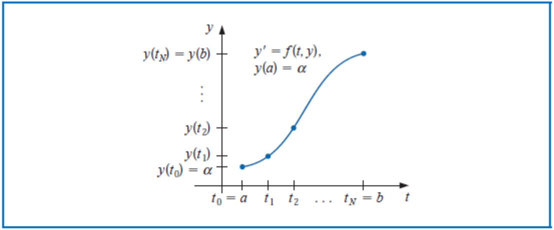


Figure 5.3

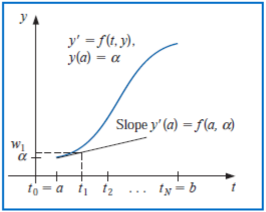
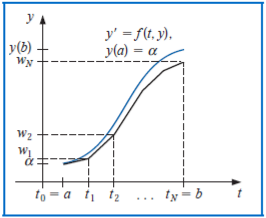


Figure 5.4



Note that the error grows slightly as the value of t increases. This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

Runge-Kutta Methods

Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

The first step in deriving a Runge-Kutta method is to determine values for a_1, α_1, and β_1 with the property that $a_1 f(t + \alpha_1, y + \beta_1)$ approximates

$$f(t, y) + \frac{h}{2} f'(t, y)$$

with error no greater than $O(h^2)$.

Since

$$f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot y'(t)$$

and $y'(t) = f(t,y)$, we have

$$f(t,y) + \frac{h}{2} f'(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y) \cdot f(t,y)$$

Expanding $f(t+\alpha_1, y+\beta_1)$ in its Taylor polynomial of degree one about (t, y) gives

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t, y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t, y)$$
$$+a_1\beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 \cdot R_1(t+\alpha_1, y+\beta_1),$$

where

$$R_1(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2} f(\xi, \mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2} f(\xi, \mu)$$

for some ξ between t and $t+\alpha_1$ and μ between y and $y+\beta_1$.

matching the coefficients of f and its derivatives would give

$$f(t,y) : a_1 = 1 \quad \frac{\partial f}{\partial t}(t,y) : a_1\alpha_1 = \frac{h}{2} \quad \frac{\partial f}{\partial y}(t,y) : a_1\beta_1 = \frac{h}{2}f(t,y)$$

Thus the parameters are

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2}f(t,y)$$

Therefore,

$$f(t,y)+\frac{h}{2}f'(t,y)=f\left(t+\frac{h}{2},y+\frac{h}{2}f(t,y)\right)-R\left(t+\frac{h}{2},y+\frac{h}{2}f(t,y)\right)$$

and

$$R\left(t+\frac{h}{2},y+\frac{h}{2}f(t,y)\right)=$$
$$\frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}f(\xi,\mu)+\frac{h^2}{4}f(t,y)\frac{\partial^2 f}{\partial t\partial y}(\xi,\mu)+\frac{h^2}{8}(f(t,y))^2\frac{\partial^2 f}{\partial y^2}f(\xi,\mu)$$

If all the second-order partial derivatives of f are bounded, then $R\left(t+\frac{h}{2},y+\frac{h}{2}f(t,y)\right)$ is $O(h^2)$.

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf(t_i + (\frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)), \text{ for each } i = 1, 2, \ldots, N-1$$

Modified Euler Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \text{ for each}$$

$$i = 1, 2, \ldots, N-1$$

Example

Use the Midpoint method and the Modified Euler method with $N = 10, h = 0.2, t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

Midpoint method: $w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218$;
Modified Euler method: $w_{i+1} = 1.22wi - 0.0088i^2 - 0.008i + 0.216$, for each $i = 0, 1, \ldots, 9$.

The first two steps of these methods give Midpoint method:
$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828;$
Modified Euler method:
$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826,$

Midpoint method:
$w_2 = 1.22(0.828) - 0.0088(0.2)^2 - 0.008(0.2) + 0.218 = 1.21136;$
Modified Euler method:
$w_2 = 1.22(0.826) - 0.0088(0.2)^2 - 0.008(0.2) + 0.216 = 1.20692$

t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

Runge-Kutta Order Four

$$
\begin{aligned}
w_0 &= \alpha, \\
k_1 &= hf(t_i, w_i), \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right), \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right), \\
k_4 &= hf(t_{i+1}, w_i + k_3), \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}
$$

for each $i = 0, 1, \ldots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives. We introduce the notation k_1, k_2, k_3, k_4 into the method is to eliminate the need for successive nesting in the second variable of $f(t, y)$.

Runge-Kutta (Order Four)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5.

Step 3 Set $K_1 = hf(t, w)$;
$K_2 = hf(t + h/2, w + K_1/2)$;
$K_3 = hf(t + h/2, w + K_2/2)$;
$K_4 = hf(t + h, w + K_3)$.

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 5 OUTPUT (t, w).

Step 6 STOP.

Use the Runge-Kutta method of order four with $h = 0.2$, $N = 10$, and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The approximation to $y(0.2)$ is obtained by

$$w_0 = 0.5$$

$$k_1 = 0.2f(0, 0.5) = 0.2(1.5) = 0.3$$

$$k_2 = 0.2f(0.1, 0.65) = 0.328$$

$$k_3 = 0.2f(0.1, 0.664) = 0.3308$$

$$k_4 = 0.2f(0.2, 0.8308) = 0.35816$$

$$w_1 = 0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816) = 0.8292933.$$

t_i	Exact $y_i = y(t_i)$	Runge-Kutta Order Four w_i	Error $\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324000	3.7323401	0.0000599
1.6	4.2834838	4.2834095	0.0000743
1.8	4.8151763	4.8150857	0.0000906

Systems of Differential Equations

An ***m*th-order system** of first-order initial-value problems has the form

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$
$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$
$$\vdots$$
$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m), \tag{5.45}$$

for $a \leq t \leq b$, with the initial conditions

$$u_1(a) = \alpha_1,\ u_2(a) = \alpha_2,\ \ldots,\ u_m(a) = \alpha_m. \tag{5.46}$$

The object is to find m functions $u_1(t), u_2(t), \ldots, u_m(t)$ that satisfy each of the differential equations together with all the initial conditions.

Definition

The function $f(t, y_1, \ldots, y_m)$, defined on the set

$$D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\}$$

is said to satisfy a **Lipschitz condition** on D in the variables $u_1, u_2, \ldots, u_m$ if a constant $L > 0$ exists with

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|, \tag{5.47}$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D. ■

Theorem

Suppose that

$$D = \{(t, u_1, u_2, \ldots, u_m) \mid a \le t \le b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\},$$

and let $f_i(t, u_1, \ldots, u_m)$, for each $i = 1, 2, \ldots, m$, be continuous and satisfy a Lipschitz condition on D. The system of first-order differential equations (5.45), subject to the initial conditions (5.46), has a unique solution $u_1(t), \ldots, u_m(t)$, for $a \le t \le b$. ■

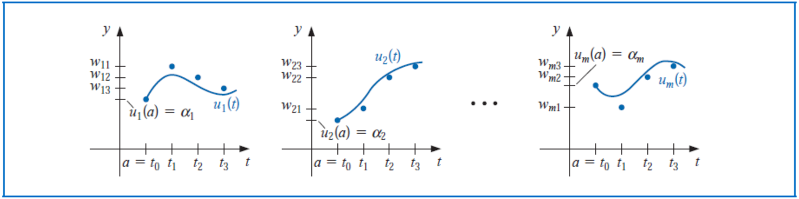
Generalized Runge-Kutta Method of Order 4

Let an integer $N > 0$ be chosen and set $h = (b-a)/N$. Partition the interval $[a,b]$ into N subintervals with the mesh points

$$t_j = a + jh, \quad \text{for each } j = 0, 1, \ldots, N.$$

Use the notation w_{ij}, for each $j = 0, 1, \ldots, N$ and $i = 1, 2, \ldots, m$, to denote an approximation to $u_i(t_j)$. That is, w_{ij} approximates the ith solution $u_i(t)$ of (5.45) at the jth mesh point t_j. For the initial conditions, set (see Figure 5.6)

$$w_{1,0} = \alpha_1, \ w_{2,0} = \alpha_2, \ \ldots, \ w_{m,0} = \alpha_m. \tag{5.48}$$


y
w_{11}
w_{12}
w_{13}
$u_1(a) = \alpha_1$
$u_1(t)$
$a = t_0$ t_1 t_2 t_3 t
y
w_{23}
w_{22}
w_{21}
$u_2(t)$
$u_2(a) = \alpha_2$
$a = t_0$ t_1 t_2 t_3 t
y
$u_m(a) = \alpha_m$
w_{m3}
w_{m2}
w_{m1}
$u_m(t)$
$a = t_0$ t_1 t_2 t_3 t

Suppose that the values $w_{1,j}, w_{2,j}, \ldots, w_{m,j}$ have been computed. We obtain $w_{1,j+1}$, $w_{2,j+1}, \ldots, w_{m,j+1}$ by first calculating

$$k_{1,i} = hf_i(t_j, w_{1,j}, w_{2,j}, \ldots, w_{m,j}), \quad \text{for each } i = 1, 2, \ldots, m; \tag{5.49}$$

$$k_{2,i} = hf_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{1,1}, w_{2,j} + \frac{1}{2}k_{1,2}, \ldots, w_{m,j} + \frac{1}{2}k_{1,m}\right), \tag{5.50}$$

for each $i = 1, 2, \ldots, m$;

$$k_{3,i} = hf_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{2,1}, w_{2,j} + \frac{1}{2}k_{2,2}, \ldots, w_{m,j} + \frac{1}{2}k_{2,m}\right), \tag{5.51}$$

for each $i = 1, 2, \ldots, m$;

$$k_{4,i} = hf_i(t_j + h, w_{1,j} + k_{3,1}, w_{2,j} + k_{3,2}, \ldots, w_{m,j} + k_{3,m}), \tag{5.52}$$

for each $i = 1, 2, \ldots, m$; and then

$$w_{i,j+1} = w_{i,j} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}), \tag{5.53}$$

for each $i = 1, 2, \ldots, m$. Note that all the values $k_{1,1}, k_{1,2}, \ldots, k_{1,m}$ must be computed before any of the terms of the form $k_{2,i}$ can be determined. In general, each $k_{l,1}, k_{l,2}, \ldots, k_{l,m}$ must be computed before any of the expressions $k_{l+1,i}$. Algorithm 5.7 implements the Runge-Kutta fourth-order method for systems of initial-value problems.

Runge-Kutta Method for Systems of Differential Equations

To approximate the solution of the mth-order system of first-order initial-value problems

$$u_j' = f_j(t, u_1, u_2, \ldots, u_m), \quad a \le t \le b, \quad \text{with} \quad u_j(a) = \alpha_j,$$

for $j = 1, 2, \ldots, m$ at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; number of equations m; integer N; initial conditions $\alpha_1, \ldots, \alpha_m$.

OUTPUT approximations w_j to $u_j(t)$ at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t = a$.

Step 2 For $j = 1, 2, \ldots, m$ set $w_j = \alpha_j$.

Step 3 OUTPUT $(t, w_1, w_2, \ldots, w_m)$.

Step 4 For $i = 1, 2, \ldots, N$ do steps 5–11.

Step 5 For $j = 1, 2, \ldots, m$ set
$k_{1,j} = h f_j(t, w_1, w_2, \ldots, w_m)$.

Step 6 For $j = 1, 2, \ldots, m$ set

$k_{2,j} = h f_j\left(t + \frac{h}{2}, w_1 + \frac{1}{2}k_{1,1}, w_2 + \frac{1}{2}k_{1,2}, \ldots, w_m + \frac{1}{2}k_{1,m}\right)$.

Step 7 For $j = 1, 2, \ldots, m$ set

$k_{3,j} = h f_j\left(t + \frac{h}{2}, w_1 + \frac{1}{2}k_{2,1}, w_2 + \frac{1}{2}k_{2,2}, \ldots, w_m + \frac{1}{2}k_{2,m}\right)$.

Step 8 For $j = 1, 2, \ldots, m$ set
$k_{4,j} = h f_j(t + h, w_1 + k_{3,1}, w_2 + k_{3,2}, \ldots, w_m + k_{3,m})$.

Step 9 For $j = 1, 2, \ldots, m$ set
$w_j = w_j + (k_{1,j} + 2k_{2,j} + 2k_{3,j} + k_{4,j})/6$.

Step 10 Set $t = a + ih$.

Step 11 OUTPUT $(t, w_1, w_2, \ldots, w_m)$.

Step 12 STOP.

Higher-Order Differential Equations

A general mth-order initial-value problem

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \le t \le b,$$

with initial conditions $y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(m-1)}(a) = \alpha_m$ can be converted into a system of equations in the form (5.45) and (5.46).

Let $u_1(t) = y(t), u_2(t) = y'(t), \ldots,$ and $u_m(t) = y^{(m-1)}(t)$. This produces the first-order system

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2, \quad \frac{du_2}{dt} = \frac{dy'}{dt} = u_3, \quad \cdots, \quad \frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m,$$

and

$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)}) = f(t, u_1, u_2, \ldots, u_m),$$

with initial conditions

$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2, \quad \ldots, \quad u_m(a) = y^{(m-1)}(a) = \alpha_m.$$

Example

Transform the the second-order initial-value problem

$$y'' - 2y' + 2y = e^{2t} \sin t, \qquad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = -0.4, \ y'(0) = -0.6$$

into a system of first order initial-value problems, and use the Runge-Kutta method with $h = 0.1$ to approximate the solution.

Solution Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$. This transforms the second-order equation into the system

$$u_1'(t) = u_2(t),$$
$$u_2'(t) = e^{2t} \sin t - 2u_1(t) + 2u_2(t),$$

with initial conditions $u_1(0) = -0.4, \ u_2(0) = -0.6$.

The initial conditions give $w_{1,0} = -0.4$ and $w_{2,0} = -0.6$. The Runge-Kutta Eqs. (5.49) through (5.52) on page 330 with $j = 0$ give

$$k_{1,1} = hf_1(t_0, w_{1,0}, w_{2,0}) = hw_{2,0} = -0.06,$$

$$k_{1,2} = hf_2(t_0, w_{1,0}, w_{2,0}) = h\left[e^{2t_0}\sin t_0 - 2w_{1,0} + 2w_{2,0}\right] = -0.04,$$

$$k_{2,1} = hf_1\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) = h\left[w_{2,0} + \frac{1}{2}k_{1,2}\right] = -0.062,$$

$$\begin{aligned} k_{2,2} &= hf_2\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) \\ &= h\left[e^{2(t_0+0.05)}\sin(t_0 + 0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{1,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{1,2}\right)\right] \\ &= -0.03247644757, \end{aligned}$$

$$k_{3,1} = h\left[w_{2,0} + \frac{1}{2}k_{2,2}\right] = -0.06162832238,$$

$$k_{3,2} = h\left[e^{2(t_0+0.05)}\sin(t_0+0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{2,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{2,2}\right)\right]$$

$$= -0.03152409237,$$

$$k_{4,1} = h\left[w_{2,0} + k_{3,2}\right] = -0.06315240924,$$

and

$$k_{4,2} = h\left[e^{2(t_0+0.1)}\sin(t_0+0.1) - 2(w_{1,0} + k_{3,1}) + 2(w_{2,0} + k_{3,2})\right] = -0.02178637298.$$

So

$$w_{1,1} = w_{1,0} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) = -0.4617333423$$

and

$$w_{2,1} = w_{2,0} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}) = -0.6316312421.$$

The value $w_{1,1}$ approximates $u_1(0.1) = y(0.1) = 0.2e^{2(0.1)}(\sin 0.1 - 2\cos 0.1)$, and $w_{2,1}$ approximates $u_2(0.1) = y'(0.1) = 0.2e^{2(0.1)}(4\sin 0.1 - 3\cos 0.1)$.

t_j	$y(t_j) = u_1(t_j)$	$w_{1,j}$	$y'(t_j) = u_2(t_j)$	$w_{2,j}$	$\lvert y(t_j) - w_{1,j}\rvert$	$\lvert y'(t_j) - w_{2,j}\rvert$
0.0	−0.40000000	−0.40000000	−0.6000000	−0.60000000	0	0
0.1	−0.46173297	−0.46173334	−0.6316304	−0.63163124	3.7×10^{-7}	7.75×10^{-7}
0.2	−0.52555905	−0.52555988	−0.6401478	−0.64014895	8.3×10^{-7}	1.01×10^{-6}
0.3	−0.58860005	−0.58860144	−0.6136630	−0.61366381	1.39×10^{-6}	8.34×10^{-7}
0.4	−0.64661028	−0.64661231	−0.5365821	−0.53658203	2.03×10^{-6}	1.79×10^{-7}
0.5	−0.69356395	−0.69356666	−0.3887395	−0.38873810	2.71×10^{-6}	5.96×10^{-7}
0.6	−0.72114849	−0.72115190	−0.1443834	−0.14438087	3.41×10^{-6}	7.75×10^{-7}
0.7	−0.71814890	−0.71815295	0.2289917	0.22899702	4.05×10^{-6}	2.03×10^{-6}
0.8	−0.66970677	−0.66971133	0.7719815	0.77199180	4.56×10^{-6}	5.30×10^{-6}
0.9	−0.55643814	−0.55644290	1.534764	1.5347815	4.76×10^{-6}	9.54×10^{-6}
1.0	−0.35339436	−0.35339886	2.578741	2.5787663	4.50×10^{-6}	1.34×10^{-5}