Nonlinear Systems I

APM1137 - Numerical Analysis

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A certain population grows continuously with time at a rate proportional to the number present at that time with constant immigration rate. Suppose there are 1,000, 000 individuals initially, and 435,000 individuals immigrate into the community in the first year, and there are 1,564,000 individuals present at the end of one year. Determine the birth rate of the population.

A certain population grows continuously with time at a rate proportional to the number present at that time with constant immigration rate. Suppose there are 1,000, 000 individuals initially, and 435,000 individuals immigrate into the community in the first year, and there are 1,564,0000 individuals present at the end of one year. Determine the birth rate of the population.

Let $N(t)$ be the number of population at time t, and let λ be the birth rate. Then the rate of growth of the population is

$$\frac{dN(t)}{dt} = \lambda N(t) + 435,000 \tag{1}$$

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$$\frac{dN(t)}{dt} = \lambda N(t) + 435,000$$

Using the method of integrating factors, we have

$$\begin{aligned}
\frac{dN(t)}{dt} &= \lambda N(t) + 435,000 \\
\frac{dN(t)}{dt} - \lambda N(t) &= 435,000 \\
e^{-\lambda t}\frac{dN(t)}{dt} - \lambda N(t)e^{-\lambda t} &= 435,000e^{-\lambda t} \\
\left(e^{-\lambda t}N(t)\right)' &= 435,000e^{-\lambda t} \\
\int \left(e^{-\lambda t}N(t)\right)' dt &= \int e^{-\lambda t}435,000dt
\end{aligned}$$

$$\int \left(e^{-\lambda t}N(t)\right)' dt = \int 435,000e^{-\lambda t}dt$$
$$e^{-\lambda t}N(t) = -\frac{1}{\lambda}435,000e^{-\lambda t} + C$$

At $t = 0$, and with the given initial population, $N(0) = 1,000,000$, we have

$$e^{-\lambda(0)}N(0) = -\frac{1}{\lambda}435,000e^{-\lambda(0)} + C$$
$$1,000,000 = -\frac{1}{\lambda}435,000 + C$$
$$C = 1,000,000 + \frac{435,000}{\lambda}$$

Therefore the solution to differential equation (1) is given by obtained by

$$
\begin{aligned}
e^{-\lambda t}N(t) &= -\frac{435,000}{\lambda}e^{-\lambda t} + 1,000,000 + \frac{435,000}{\lambda} \\
N(t) &= -\frac{435,000}{\lambda} + e^{\lambda t}\left(1,000,000 + \frac{435,000}{\lambda}\right) \\
N(t) &= 1,000,000e^{\lambda t} + \frac{435,000}{\lambda}\left(e^{\lambda t} - 1\right)
\end{aligned}
$$

Note that what we need to solve is the value of the birth rate, λ.

Using the given value of $N(1) = 1,564,0000$, we get

$$1,564,0000 = 1,000,000e^{\lambda} + \frac{435,000}{\lambda}\left(e^{\lambda} - 1\right)$$

How do we solve for λ?

root-finding problem

This process involves finding a root, or solution, of an equation of the form $f(x) = 0$, for a given function f.

The first technique, based on the Intermediate Value Theorem, is called the **Bisection**, or **Binary-search, method**.

Bisection Method

Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. The Intermediate Value Theorem implies that a number p exists in (a, b) with $f(p) = 0$.

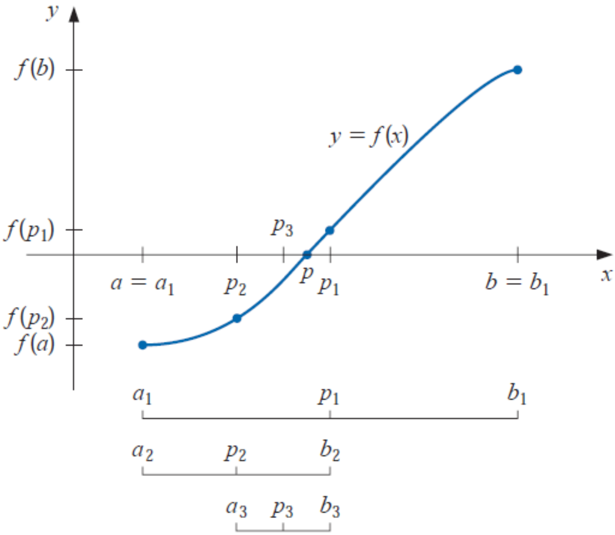
We assume for simplicity that the root in this interval is unique. The method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p.

Bisection Method

Set $a_1 = a$ and $b_1 = b$, and let $p_1 = \dfrac{a_1 + b_1}{2}$

- If $f(p_1) = 0$, then $p = p_1$, and we are done.
- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
  - If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
  - If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$, and so on until we find p such that $f(p) = 0$ or very close to 0.


y
f(b)
y = f(x)
f(p1)
p3
a = a1
p2
p
p1
b = b1
x
f(p2)
f(a)
a1
p1
b1
a2
p2
b2
a3
p3
b3

Bisection Method Algorithm

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

`INPUT` endpoints a, b; tolerance TOL; maximum number of iterations N0.

`OUTPUT` approximate solution p or message of failure.

Bisection Method Algorithm

Step 1 `Set` i = 1;
FA = f (a).

Step 2 `while` i $<=$ N0 `do` Steps 3–6.

- Step 3 `Set` p = a + (b - a)$\backslash$ 2; (*Compute pi.*)
  FP = f (p).
- Step 4 `If` FP = 0 or (b - a)$\backslash$2 $<$ TOL `then`
  `OUTPUT` (p); (*Procedure completed successfully.*)
  `STOP`.
- Step 5 `Set` i = i + 1.
- Step 6 `If` FA $\cdot$ FP $>$ 0 `then` set a = p; (*Compute ai, bi.*)
  FA = FP
  `else set` b = p. (FA is unchanged.)

Step 7 `OUTPUT` ('Method failed after N0 iterations, N0 =', N0); (*The procedure was unsuccessful*.)
`STOP`.

Other stopping methods

select a tolerance $\varepsilon > 0$ and generate $p_1, ..., p_N$ until one of the following conditions is met:

$$|p_N - p_{N-1}| < \varepsilon \tag{2}$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0 \tag{3}$$

$$|f(p_N)| < \varepsilon \tag{4}$$

Remarks

- There are sequences $\{p_n\}_{n=0}^{\infty}$ with the property that the differences $p_n - p_{n-1}$ converge to zero while the sequence itself diverges from p.
- It is also possible for $f(p_n)$ to be close to zero while p_n differs significantly from p.
- Without additional knowledge about f or p, Inequality (3) is the best stopping criterion to apply because it comes closest to testing relative error.
- Choose the interval $[a, b]$ to be as small as possible because it is possible to have more than one root of the function in an interval.

Example

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine the approximation to the root that is accurate to at least 10^{-4}.

Solution

Since f is continuous and $f(1) = -5$ and $f(2) = 14$, then by the Intermediate Value Theorem, f has a root in $[1, 2]$.

$$a_1 = 1, \quad b_1 = 2, \quad p_1 = \frac{1+2}{2} = 1.5$$

Since $f(p_1) = f(1.5) = 2.376, \quad \text{set } a_2 = 1, \quad b_2 = p_1 = 1.5$

Example

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine the approximation to the root that is accurate to at least 10^{-4}.

Solution

$$a_2 = 1, \quad b_2 = 1.5, \quad p_2 = \frac{1 + 1.5}{2} = 1.25$$

Since $f(p_2) = f(1.25) = -1.796875$,

set $a_3 = p_2 = 1.25, \quad b_3 = 1.5$

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

Errors

absolute error

Suppose that p^* is an approximation to p. The absolute error is

$$|p - p^*|$$

relative error

Suppose that p^* is an approximation to p. The relative error is

$$\frac{|p - p^*|}{|p|},$$

provided that $p \neq 0$.

Errors

approximation to significant digits

The number p^* is said to approximate p to t significant digits (or figures) if t is the largest nonnegative integer for which |

$$\frac{|p - p^*|}{|p|} < 5 \times 10^{-t}$$

Remarks

The relative error is generally a better measure of accuracy than the absolute error because it takes into consideration the size of the number being approximated.

We often cannot find an accurate value for the true error in an approximation. Instead we find a bound for the error, which gives us a "worst-case" error.

In the previous example, we want the error to be less than 10^{-4}.

p_{13} approximates p with absolute error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070.$$

and relative error $\dfrac{|p - p_{13}|}{|p|} < \dfrac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9 \times 10^{-5}$

Drawbacks of the Bisection Method

It is relatively slow to converge

A good intermediate approximation might be inadvertently discarded.

However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods.

Theorem

Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1.$$

This theorem gives only a bound for approximation error and that this bound might be quite conservative.

Example

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$

Solution:
Fromt the theorem, we find an integer n that satisfies

$$|p_n - p| \leq \frac{2-1}{2^n} = 2^{-n} < 10^{-3}.$$

$$2^n < 10^{-3} \text{ implies } \log 2^n < \log 10^{-3} = -3$$

thus we have

$$n = \frac{-3}{\log 2} \approx 9.96$$

Hence 10 iterations will ensure an approximation accurate to within 10^{-3}.

Definition

The number p is a **fixed point** for a given function g if $g(p) = p$.

Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as $g(x) = x - f(x)$ or as $g(x) = x + 3f(x)$.

Conversely, if the function g has a fixed point at p, then the function defined by $f(x) = x - g(x)$ has a zero at p.

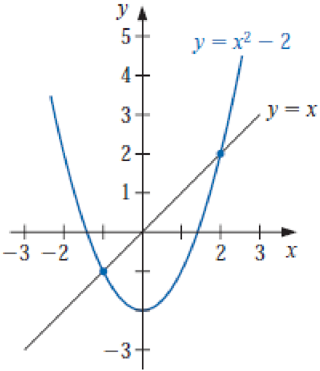
Example

Determine any fixed points of the function $g(x) = x^2 - 2$.

Solution:

$$\begin{gathered} g(p) = p \\ p^2 - 2 = p \\ p^2 - p - 2 = 0 \\ (p-2)(p+1) = 0 \end{gathered}$$

Thus the fixed points of g are 2 and -1


y
5
4
3
2
1
−3
$y = x^2 - 2$
$y = x$
−3 −2
2 3 x

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

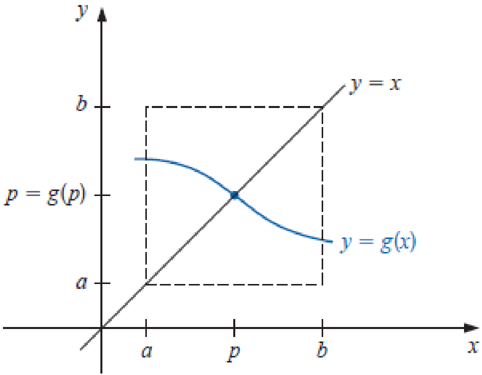
Theorem

(i) If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.

(ii) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b),$$

then there is exactly one fixed point in $[a, b]$.


y
b
p = g(p)
a
y = x
y = g(x)
a
p
b
x

Example

Show that $g(x) = \dfrac{x^2 - 1}{3}$ has a unique fixed point on the interval $[-1, 1]$

Solution:
Clearly, g is continuous. The maximum and minimum values of $g(x)$ for $x \in [-1, 1]$ must occur either at the endpoints or when the derivative is 0.

Since $g'(x) = \dfrac{2x}{3}$, then $g'(x)$ exists on $[-1, 1]$ and $g'(x) = 0$ at $x = 0$. Thus the maximum and minimum values of $g(x)$ occur at $x = -1, 0, 1$.

$$g(-1) = 0, g(1) = 0, \text{ and } g(0) = -1/3$$

So an absolute maximum for $g(x)$ on $[-1, 1]$ occurs at $x = -1$ and $x = 1$, and an absolute minimum at $x = 0$.

This shows that g is continuous on $[-1, 1]$ and $g(x) \in [-1, 1]$ for all $x \in [-1, 1]$. Hence g has at least one fixed point in $[-1, 1]$.

Moreover,

$$|g'(x)| = |\frac{2x}{3}| \leq \frac{2}{3}, \text{ for all } x \in (-1, 1).$$

Therefore g satisfies all the hypothesis of the theorem and has a unique fixed point on $[-1, 1]$.

Remark

Note that this theorem gives only sufficient but not necessary condition for a function to have a unique fixed point at a given interval. The function g in the example has a uniques fixed point on $[3,4]$ but $g(4) = 5$ and $g'(4) > 1$.

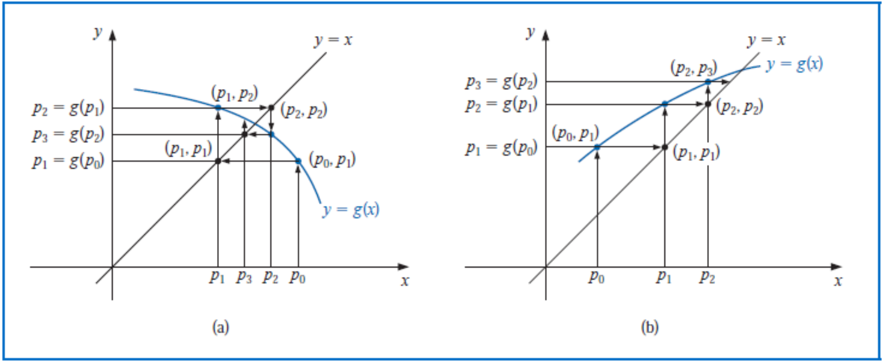
Fixed-Point Iteration

To approximate the fixed point of a function g, we choose an initial approximation p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, for each $n \geq 1$.

If the sequence converges to p and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

and a solution to $x = g(x)$ is obtained.


y
y = x
p_2 = g(p_1)
p_3 = g(p_2)
p_1 = g(p_0)
(p_1, p_2)
(p_2, p_2)
(p_1, p_1)
(p_0, p_1)
y = g(x)
p_1 p_3 p_2 p_0
x
(a)
y
y = x
y = g(x)
(p_2, p_3)
p_3 = g(p_2)
p_2 = g(p_1)
(p_2, p_2)
(p_0, p_1)
p_1 = g(p_0)
(p_1, p_1)
p_0
p_1
p_2
x
(b)

Fixed-Point Iteration Algorithm

To find a solution to $g(p) = p$ given an initial approximation p_0

`INPUT` initial approximation p0; tolerance TOL; maximum iterations N0

`OUTPUT` approximate solution p or message failure

Step 1 `Set` i=1.

Step 2 `While` i<=N0 `do` steps 3-6.

- Step 3 `set` p=g(p0) (*compute pi*)
- Step 4 `If` |p-p0|<TOL `then`
  `OUTPUT` (p;)
  `STOP`.
- Step 5 `Set` i=i+1.
- Step 6 `Set` p0=p (*update p0*)

Step 7 `OUTPUT` ('The method failed after N0 iterations') (*the procedure was unsuccessful.*)
`STOP`.

Example

$x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$.

To convert this to the fixed point form $x = g(x)$, we can use

(a) $x = g(x) = x - x^3 - 4x^2 + 10$

(b) $x = g(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$

(c) $x = g(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g(x) = \left(\dfrac{10}{4 + x}\right)^{1/2}$

(e) $x = g(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

Fixed-Point Theorem

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, g' exists on (a, b) and that a constat $0 < k < 1$ exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), n \geq 1,$$

converges to the unique fixed point $p \in [a, b]$.

Corollary

If g satisfies the hypotheses of the Fixed-Point Theorem, then the bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1.$$

From the different forms of g in the previous example

(a) $x = g(x) = x - x^3 - 4x^2 + 10$
does not map $[1,2]$ onto itself and $|g'(x)| > 1$ for all $x \in [1,2]$
there is no reason to expect convergence

(b) $x = g(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$
does not map $[1,2]$ onto itself and $\{p_n\}_{n=0}^{\infty}$ is not defined when $p_0 = 1.5$. Moreover, there is not interval containing $p \approx 1.365$ such that $|g'(x)| < 1$
there is no reason to expect convergence

From the different forms of g in the previous example

(c) $x = g(x) = \frac{1}{2}(10 - x^3)^{1/2}$
this fails to satisfy the conditions of the theorem at $[1, 2]$ however at $[1, 1.5]$, g maps $[1, 1.5]$ onto itself and $|g'(x)| \leq 0.66$ so the fixed-point theorem confirms the convergence.

(d) $x = g(x) = \left(\frac{10}{4 + x}\right)^{1/2}$
$|g'(x)| < 0.15$ for all $x \in [1, 2]$. the bound and magniture of $g'(x)$ is much smaller than in (c) which explains the more rapid convergence.

(e) $x = g(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$
converges much more rapidly than our other choices.

Newton's Method

Newton's method is one of the most powerful and well-known numerical methods for solving a root-finding problem.

Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small". Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))$$

Since $|p - p_0|$ is assumed to be small, then $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0)$$

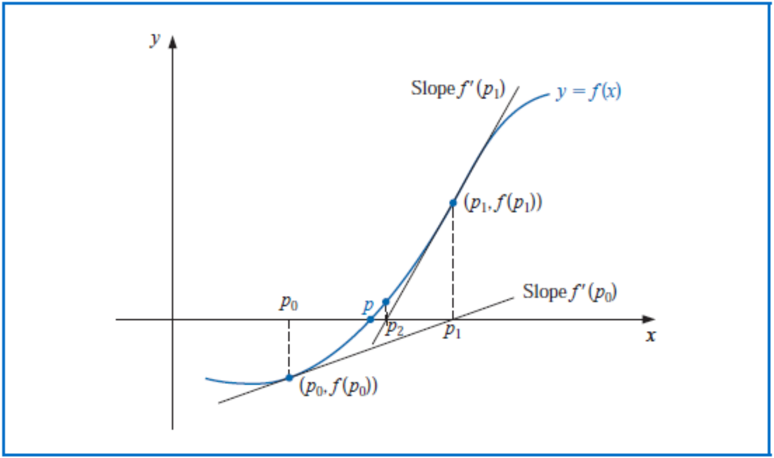
Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1$$

Starting with the initial approximation p_0, the approximation p_1 is the x-intercept of the tangent line to the graph of f at $(p_0, f(p_0))$. The approximation p_2 is the x-intercept of the tangent line to the graph of f at $(p_1, f(p_1))$ and so on.


y
Slope $f'(p_1)$
$y = f(x)$
$(p_1, f(p_1))$
Slope $f'(p_0)$
p_0
p
p_2
p_1
x
$(p_0, f(p_0))$

Newton's Method Algorithm

To find a solution to $f(x) = 0$ given an initial approximation p_0

`INPUT` initial approximation p0; tolerance TOL; maximum iterations N0

`OUTPUT` approximate solution p or message of failure

Step 1 `Set` i=1.

Step 2 `While` i<=N0 `do` steps 3-6.

- Step 3 `set` p=p0-f(p0)/f'(p0) (*compute pi*)
- Step 4 `If` |p-p0|<TOL `then`
  `OUTPUT` (p;)
  `STOP.`
- Step 5 `Set` i=i+1.
- Step 6 `Set` p0=p (*update p0*)

Step 7 `OUTPUT` ('The method failed after N0 iterations') (*the procedure was unsuccessful.*)
`STOP.`

Example

Consider the function $f(x) = \cos x - x = 0$. Approximate a root of f using fixed point method and using Newton's Method

Fixed point method The solution to the root finding problem is the same as teh solution to the fixed point problem $x = \cos x$. The graph of $y = \cos x$ implies that a single fixed point p lies in $[0, \pi/2]$

Newton's method Apply Newton's method starting with $p_0 = \pi/4$. Generate the sequence defined for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}$$

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

	Newton's Method
n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Theorem

Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=0}^{\infty}$ converging to p for initial value approximatin $p_0 \in [p-\delta, p+\delta]$.

The theorem states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen. It also implies that the bound of the derivative of g, indicates the speed of convergence of the method, decreases to 0 as the procedure continues.

Secant Method

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation. Frequently, $f(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

Since

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$$

If p_{n-2} is close to p_{n-1}, then

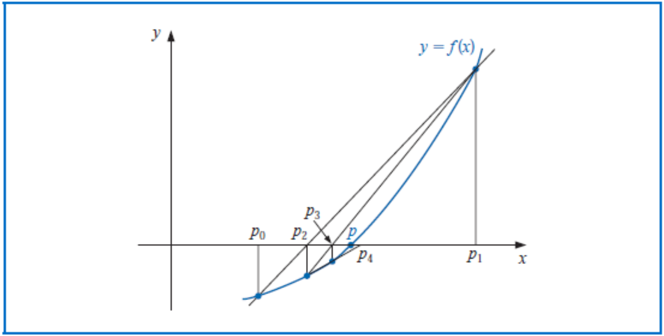
$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}}$$

Using this approximation for f' in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Remarks

Starting with the two initial approximations p_0 and p_1, the approximation p_2 is the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. The approximation p_3 is the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on.



Secant Method Algorithm

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1.

`INPUT` initial approximatins p0,p1; tolerance TOL; maximum iterations

`OUTPUT` appropriate solution p or message failure.

```
Step 1  Set i=2.
        q0=f(p0)
        q1=f(p1)
Step 2  While i<=N0 do steps 3-6.
    Step 3  set p=p1-q1(p1-p0)/q1-q0) (compute pi)
    Step 4  If |p-p0|<TOL then
            OUTPUT (p;)
            STOP.
    Step 5  Set i=i+1.
    Step 6  Set p0=p
            p0=p1
            q0=q1
            p1=p
Step 7  OUTPUT ('The method failed after N0 iterations') (the procedure
        was unsuccessful.)
        STOP.
```

Method of False Position

The method of **False Position** (also called Regula Falsi) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations.

First choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$.

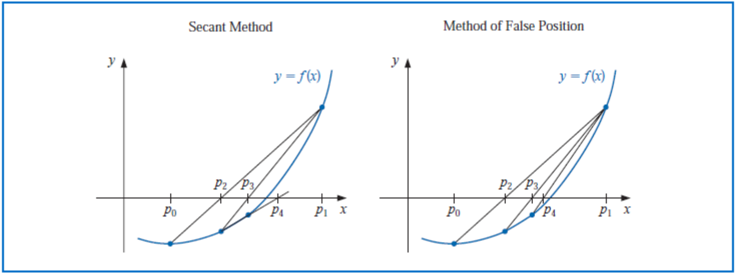
The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$.

To decide which secant line to use to compute p_3, consider $f(p_2) \cdot f(p_1)$.

If $f(p_2) \cdot f(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$.

If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.

In the latter case a relabeling of p2 and p1 is performed. The relabeling ensures that the root is bracketed between successive iterations.


Secant Method
Method of False Position
y
x
y = f(x)
p0
p1
p2
p3
p4

Example

Use the method of False Position to find a solution to $x = cosx$

Start with initial approximations $p_0 = 0.5$ and $p_1 = \pi/4$

	False Position	Secant	Newton
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

Remarks

The added insurance of the method of False Position commonly requires more calculation than the Secant method, just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations.