Linear Systems

APM1137 - Numerical Analysis

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Linear Systems

Linear systems of equations are associated with many problems in engineering and science, as well as with applications of mathematics to the social sciences and the quantitative study of business and economic problems.

We consider direct methods for solving a linear system of n equations in n variables. Such a system has the form

$$\begin{aligned} E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ &\qquad\qquad \vdots \\ E_1: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_2 \end{aligned}$$

Gaussian Elimination

Gaussian elimination is a method for solving matrix equations of the form

$$\mathbf{Ax=b}.$$

by performing elementary row operations to put the augmented matrix into the upper triangular form.

This process of computing the unknowns from a system that is in upper-triangular form is called **back substitution.**

From the linear system $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Form the augmented matrix

$$\tilde{A} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Perform row operations to get the form

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right]$$

Then backward substitution can be performed

$$x_n = \frac{a_{n,n+1}}{a_{nn}}$$

$$x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$\vdots$$

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

for each $i = n-1, n-2, \ldots, 2, 1$.

Note that when a pivot element is 0, the procedure cannot continue thus a switch in rows must be made. i.e. If $a_{kk} = 0$, interchange the kth row with the pth row where p is the smallest integer greater than k for which $a_{pk} \neq 0$.

If after k row operation, $a_{pk} = 0$ for all p, $k + 1 \leq p \leq n$, then the linear system does not have a unique solution and the procedure stops. If after al n operation, $a_{nn} = 0$, then the linear system does not have a unique solution and the procedure stops.

Gaussian Elimination with Backward Substitution

To solve the $n \times n$ linear system

$$\begin{aligned}
E_1 &: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\
E_2 &: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\
&\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
E_n &: \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1}
\end{aligned}$$

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \leq i \leq n$ and $1 \leq j \leq n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4. (*Elimination process.*)

> ***Step 2*** Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$.
> If no integer p can be found
> then OUTPUT ('no unique solution exists');
> STOP.
>
> ***Step 3*** If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.
>
> ***Step 4*** For $j = i+1, \ldots, n$ do Steps 5 and 6.
>
> > ***Step 5*** Set $m_{ji} = a_{ji}/a_{ii}$.
> >
> > ***Step 6*** Perform $(E_j - m_{ji}E_i) \rightarrow (E_j)$;

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$.

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)
STOP.

Pivoting Strategies

To reduce round-off error, it is often necessary to perform row interchanges even when the pivot elements are not zero.

If the pivot element is small in magnitude compared to the elements in the succeeding row and same collumn, then the multiplier will be much larger than 1 thus increases the round-off error in the computations.

Partial Pivoting

Select an element in the same column as the pivot that is below the diagonal and has the largest absolute value; specifically, we determine the smallest $p \geq k$ such that

$$|a_{pk}| = \max_{k \leq i \leq n} |a_{ik}|$$

and interchange row k and row p.

Gaussian Elimination with Partial Pivoting

To solve the $n \times n$ linear system

$$\begin{aligned} E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\ E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\ &\quad \vdots \qquad\qquad\qquad\quad \vdots \\ E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1} \end{aligned}$$

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$ where $1 \leq i \leq n$ and $1 \leq j \leq n+1$.

OUTPUT solution $x_1, \ldots, x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n$ set $NROW(i) = i$. (*Initialize row pointer.*)

Step 2 For $i = 1, \ldots, n-1$ do Steps 3–6. (*Elimination process.*)

Step 3 Let p be the smallest integer with $i \leq p \leq n$ and
$|a(NROW(p), i)| = \max_{i \leq j \leq n} |a(NROW(j), i)|$.
(*Notation:* $a(NROW(i), j) \equiv a_{NROW_i, j}$.)

Step 4 If $a(NROW(p), i) = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 5 If $NROW(i) \neq NROW(p)$ then set $NCOPY = NROW(i)$;
$NROW(i) = NROW(p)$;
$NROW(p) = NCOPY$.
(*Simulated row interchange.*)

Step 6 For $j = i+1, \ldots, n$ do Steps 7 and 8.

Step 7 Set $m(NROW(j), i) = a(NROW(j), i)/a(NROW(i), i)$.

Step 8 Perform $(E_{NROW(j)} - m(NROW(j), i) \cdot E_{NROW(i)}) \rightarrow (E_{NROW(j)})$.

Step 9 If $a(NROW(n), n) = 0$ then OUTPUT ('no unique solution exists');
STOP.

Step 10 Set $x_n = a(NROW(n), n+1)/a(NROW(n), n)$.
(*Start backward substitution.*)

Step 11 For $i = n-1, \ldots, 1$

$$\text{set } x_i = \frac{a(NROW(i), n+1) - \sum_{j=i+1}^{n} a(NROW(i), j) \cdot x_j}{a(NROW(i), i)}.$$

Step 12 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)
STOP.

Scaled Partial Pivoting

Places the element in the pivot position that is largest relative to the entries in its row.

The first step in this procedure is to define a scale factor s_i for each row as $s_i = \max_{1 \leq j \leq n} |a_{ij}|$. If we have $s_i = 0$ for some i, then the system has no unique solution since all entries in the ith row are 0.

Assuming that this is not the case, the appropriate row interchange is determined by choosing the least integer $p \geq i$ with

$$\frac{|a_{pi}|}{s_p} = \max_{i \leq k \leq n} \frac{a_{ki}}{s_k}$$

and interchange row i and row p.

Gaussian Elimination with Scaled Partial Pivoting

The only steps in this algorithm that differ from those of Algorithm 6.2 are:

Step 1 For $i = 1, \ldots, n$ set $s_i = \max_{1 \le j \le n} |a_{ij}|$;
if $s_i = 0$ then OUTPUT ('no unique solution exists');
STOP.
set $NROW(i) = i$.

Step 2 For $i = 1, \ldots, n-1$ do Steps 3–6. *(Elimination process.)*

Step 3 Let p be the smallest integer with $i \le p \le n$ and

$$\frac{|a(NROW(p), i)|}{s(NROW(p))} = \max_{i \le j \le n} \frac{|a(NROW(j), i)|}{s(NROW(j))}.$$

Complete Pivoting

Complete (or maximal) pivoting at the kth step searches all the entries a_{ij}, for $i = k, k+1, \ldots, n$ and $j = k, k+1, \ldots, n$, to find the entry with the largest magnitude. Both row and column interchanges are performed to bring this entry to the pivot position.

Jacobi and Gauss-Siedel Iterative Techniques

the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late eighteenth century. Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

An iterative technique to solve the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}$ and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converges to $\mathbf{x}$.

Jacobi's Method

The Jacobi iterative method is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \text{ for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij}x_j^{(k-1)} \right) + b_i \right], \text{ for } i = 1, 2, \ldots, n.$$

Example

The linear system $A\mathbf{x} = \mathbf{b}$ is given by

$$\begin{array}{lcrcrcrcrcl}
E_1 & : & 10x_1 & - & x_2 & + & 2x_3 & & & = & 6 \\
E_2 & : & -x_1 & + & 11x_2 & - & x_3 & + & 3x_4 & = & 25 \\
E_3 & : & 2x_1 & - & x_2 & + & 10x_3 & - & x_4 & = & -11 \\
E_4 & : & & & 3x_2 & - & x_3 & + & 8x_4 & = & 15
\end{array}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}$$

Definition

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

Solution We first solve equation E_i for x_i, for each $i = 1, 2, 3, 4$, to obtain

$$\begin{aligned}
x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5},\\
x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11},\\
x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10},\\
x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}.
\end{aligned}$$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$\begin{aligned}
x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000,\\
x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727,\\
x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000,\\
x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750.
\end{aligned}$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after ten iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} |< 10^{-3}.$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$. ■

In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some fixed matrix T and vector $\mathbf{c}$. After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c},$$

for each $k = 1, 2, 3, \ldots$. This should be reminiscent of the fixed-point iteration studied before.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is split into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$= D - L - U.$$

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i, then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \ldots. \tag{7.6}$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j. \tag{7.7}$$

Example

In the previous example,

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}.$$

Jacobi Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While $(k \leq N)$ do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij} XO_j) + b_i \right].$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
(*The procedure was successful.*)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP.

The Gauss-Seidel Method

In Jacobi's method, the components of $\mathbf{x}^{(k-1)}$ are used to compute all the components of $\mathbf{x}^{(k)}$. But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$. It seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values. That is, to use

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i} -1(a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) + b_i \right]$$

for each $i = 1, 2, \ldots, n$. This modification is called the **Gauss-Seidel iterative technique**

Example

Using the same linear system in the previous example,

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5},\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11},\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10},\\
x_4^{(k)} &= -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}.
\end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in Example 1 required twice as many iterations for the same accuracy. ■

with the definitions of D, L, and U given previously, we have the Gauss-Seidel method represented by

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots. \tag{7.9}$$

Letting $T_g = (D-L)^{-1}U$ and $\mathbf{c}_g = (D-L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g. \tag{7.10}$$

For the lower-triangular matrix $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

Gauss-Seidel Iterative

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n; the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A; the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$; the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL; maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While $(k \leq N)$ do Steps 3–6.

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right].$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$;
(The procedure was successful.)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(The procedure was successful.)
STOP.

General Iteration Methods

Theorem

For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \text{ for each } k \geq 1,$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$.

$\rho(T)$ is the spectral radius of the matrix T defined as

$$\rho(T) = \max |\lambda|$$

where λ is an eigenvalue of T. If λ is a complex number equal to $a + bi$, then define $|\lambda| = \sqrt{a^2 + b^2}$

Corrollary

If $\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

- $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$
- $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \dfrac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$

matrix norms

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty, \quad \text{the } l_\infty \text{ norm,}$$

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|A\mathbf{x}\|_2, \quad \text{the } l_2 \text{ norm.}$$

Theorem

If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = b$.

Theorem

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:
(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$; (ii) $1 < \rho(T_j) < \rho(T_g)$;
(iii) $\rho(T_j) = \rho(T_g) = 0$; (iv) $\rho(T_j) = \rho(T_g) = 1$.