Nonlinear Systems II

APM1137 - Numerical Analysis

Department of Mathematics, Institute of Arts and Sciences
Far Eastern University, Sampaloc, Manila

Order of Convergence

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If positive constants λ and α exists with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ **converges to** p **of order** α **with asymptotic error constant** λ**.**

- An iterative technique of the form $p_n = g(p_{n-1})$ is said to be of order α if the sequence $\{p_n\}_{n=0}^{\infty}$ converges to the solution $p = g(p)$ of order α.
- In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.
- The asymptotic constant affects the speed of convergence but not to the extent of the order.
    - If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent.**
    - If $\alpha = 2$, the sequence is **quadratically convergent.**

For large n,

linearly convergent

$$|p_n - p| \approx \lambda |p_{n-1} - p| \approx \lambda^2 |p_{n-2} - p| \approx \cdots \approx \lambda^n |p_0 - p|$$

quadratically convergent

$$|p_n - p| \approx \lambda |p_{n-1} - p|^2 \approx \lambda \left[\lambda |p_{n-2} - p|^2\right]^2 = \lambda^3 |p_{n-1} - p|^4 \approx \cdots \approx \lambda^{2n-1} |p_0 - p|^{2n}.$$

Theorem

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose in addition, that g' is continuous on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a,b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \text{ for } n \geq 1,$$

converges only linearly to the unique fixed point $p \in [a,b]$.

Theorem

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ such that, for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| \leq \frac{M}{2}|p_n - p|^2.$$

Multiple Roots

A solution p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$, we can write

$$f(x) = (x-p)^m q(x), \text{ where } \lim_{x \to p} q(x) \neq 0.$$

Theorem

The function $f \in C^1[a,b]$ has a simple zero at p in (a,b) if and only if $f(p) = 0$, but $f'(p) \neq 0$.

Theorem

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p in (a,b) if and only if

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^(m-1)(p), \text{ but } f^m(p) \neq 0.$$

Example

Let $f(x) = e^x - x - 1$. (a) Show that f has a zero of multiplicity 2 at $x = 0$. (b) Show that Newton's method with $p_0 = 1$ converges to this zero but not quadratically.

Solution (a):

$$f(x) = e^x - x - 1, \quad f'(x) = e^x - 1 \text{ and } f'' = e^x$$

So

$$f(0) = e^0 - 0 - 1 = 0, \quad f'(0) = e^0 - 1 = 0 \text{ and } f''(0) = e^0 = 1.$$

Example

Let $f(x) = e^x - x - 1$. (a) Show that f has a zero of multiplicity 2 at $x = 0$. (b) Show that Newton's method with $p_0 = 1$ converges to this zero but not quadratically.

Solution (b):

n	pn	n	pn
0	1.0	9	$2.7750 \times$ 10-3
1	0.58198	10	$1.3881 \times$ 10-3
2	0.31906	11	$6.9411 \times$ 10-4
3	0.16800	12	$3.4703 \times$ 10-4
4	0.08635	13	$1.7416 \times$ 10-4
5	0.04380	14	$8.8041 \times$ 10-5
6	0.02206	15	$4.2610 \times$ 10-5
7	0.01107	16	$1.9142 \times$ 10-6
8	0.005545		

Aitken's Δ^2 Method

Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p. This method constructs a sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ that converges more rapidly to p than does $\{p_n\}_{n=0}^{\infty}$.

Assume the signs of $p_n - p, p_{n+1} - p$ and $p_{n+2} - p$ agree and n sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

This leads to the definition of the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, where

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Example

The sequence $\{p_n\}_{n=0}^{\infty}$, where $p_n = \cos(1/n)$ converges linearly to $p = 1$.

n	p_n	$\hat{p}_n$
1	0.54030	0.96178
2	0.87758	0.98213
3	0.94496	0.98979
4	0.96891	0.99342
5	0.98007	0.99541
6	0.98614	
7	0.98981	

Definition

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference** Δp_n is defined by

$$\Delta p_n = p_{n+1} - p_n, \text{ for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \text{ for } k \geq 2.$$

This implies that

$$\Delta^2 p_n = \Delta(p_{n+1} - p_n) = \Delta p_{n+1} - \Delta p_n = (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n)$$

Thus we can also write $\hat{p}_n$ as

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \text{ for } n \geq 0.$$

Fundamental Theorem of Algebra)

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root.

Corollary

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, ..., m_k$, such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k} .$$

Corollary

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x.

Horner's Method

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1} x_0 \text{ for } k = n-1, n-2, \ldots, 1, 0.$$

Then $b_0 = P(x_0)$. Moreover, if

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P(x) = (x - x_0) Q(x) + b_0$$

Horner's method is actually synthetic division.

Use Horner's method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

	Coefficient of x^4	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant term
$x_0 = -2$	$a_4 = 2$	$a_3 = 0$	$a_2 = -3$	$a_1 = 3$	$a_0 = -4$
		$b_4x_0 = -4$	$b_3x_0 = 8$	$b_2x_0 = -10$	$b_1x_0 = 14$
	$b_4 = 2$	$b_3 = -4$	$b_2 = 5$	$b_1 = -7$	$b_0 = 10$

So,

$$P(x) = (x+2)(2x^3 - 4x^2 + 5x - 7) + 10.$$

Thus $P(-2) = b_0 = 10$.

Horner's method gives a faster way of computing P and P' in Newton's method since

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1,$$

differentiating with respect to x gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \text{ and } P'(x_0) = Q(x_0).$$

Example

Find an approximate to a zero of $P(x) = 2x^4 - 3x^2 + 3x - 4$ using Newton's method with $p_0 = -2$ and synthetic division to evaluate $P(p_n)$ and $P'(p_n)$

Solution:

$$
\begin{array}{r|rrrrrl}
-2 & 2 & 0 & -3 & 3 & -4 & \\
 & & -4 & 8 & -10 & 14 & \\
\hline
 & 2 & -4 & 5 & -7 & 10 & = P(-2).
\end{array}
$$

$$
\begin{array}{r|rrrrl}
-2 & 2 & -4 & 5 & -7 & \\
 & & -4 & 16 & -42 & \\
\hline
 & 2 & -8 & 21 & -49 & = Q(-2) = P'(-2)
\end{array}
$$

Then $p_1 = p_0 - \dfrac{P(p_0)}{P'(p_0)} = -2 - \dfrac{10}{-49} \approx -1.796$

$$
\begin{array}{r|rrrrrl}
-1.796 & 2 & 0 & -3 & 3 & -4 & \\
 & & -3.592 & 6.451 & -6.197 & 5.742 & \\
\hline
 & 2 & -3.592 & 3.451 & -3.197 & 1.742 & = P(x_1) \\
 & & -3.592 & 12.902 & -29.368 & & \\
\hline
 & 2 & -7.184 & 16.353 & -32.565 & = Q(x_1) & = P'(x_1).
\end{array}
$$

Then use this to solve for p_2.

Horner's Method Algorithm

To evaluate the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at its derivative at x_0:

`INPUT` degree n; coefficients $a_0, a_1, \ldots, a_n; x_0$

`OUTPUT` $y = P(x_0); z = P'(x_0)$.

Horner's Method Algorithm

Step 1 `Set` $y = a_n$; (*Compute* b_n *for* P)
$z = a_n$ (*Compute* b_{n-1} *for* Q).

Step 2 `For` $j = n-1, n-2, \ldots, 1$,
`set` $y = x_0 y + a_j$; (*Compute* b_j *for* P)
$z = x_0 z + y$; (*Compute* b_{j-1} *for* Q)

Step 3 `Set` $y = x_0 y + a_0$. (*Compute* b_0 *for* P.)

Step 4 `OUTPUT` (y, z);
`STOP.`

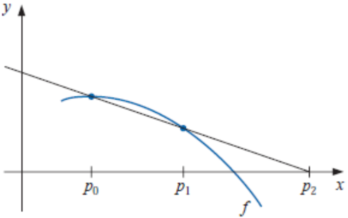
Complex Zero: Müller's Method

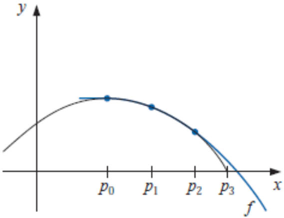
Approximate a complex root begining with a complex initial approximation.

Theorem

If $z = a + bi$ is a complex zero of multiplicity m of the polynomial $P(x)$ with real coefficients, then $z = a - bi$ is also a zero of multiplicity m of the polynomial $P(x)$, and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

Mϋller's method uses three initial approximations, p_0, p_1, and p_2, and determines the next approximation p_3 by considering the intersection of the x-axis with the parabola through
$(p_0, f(p_0)), (p_1, f(p_1)), and (p_2, f(p_2))$.





Consider

$$P(x) = a(x-p_2)^2 + b(x-p_2) + c$$

that passes through $(p_0, f(p_0)), (p_1, f(p_1)), and (p_2, f(p_2))$. The constants a, b, c are determined from

$$\begin{aligned} P(p_0) &= a(p_0-p_2)^2 + b(p_0-p_2) + c \\ P(p_1) &= a(p_1-p_2)^2 + b(p_1-p_2) + c \\ P(p_2) &= a(p_2-p_2)^2 + b(p_2-p_2) + c \end{aligned}$$

To determine p_3, a zero of P, we apply the quadratic formula to $P(x) = 0$. However, because of round-off error problems caused by the subtraction of nearly equal numbers, we apply the rationalized form of the fomula

$$p_3 - p_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

The sign is chosen to agree with the sign of b.

$$p_3 = p_2 - \frac{2c}{b + sgn(b)\sqrt{b^2 - 4ac}}$$

Once p_3 is determined, the procedure is reinitialized using p_1, p_2, and p_3 in place of p_0, p_1, and p_2 to determine the next approximation, p_4.

The method gives approximate complex roots when $b^2 - 4ac < 0$.

Müller's Algorithm

To find a solution to $f(x) = 0$ given three approximations, p_0, p_1, and p_2:

INPUT p_0, p_1, p_2; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = 3$.

Step 2 While $i \le N_0$ do Steps 3–7.

Step 3 $b = \delta_2 + h_2 d$;
$D = (b^2 - 4f(p_2)d)^{1/2}$. *(Note: May require complex arithmetic.)*

Step 4 If $|b - D| < |b + D|$ then set $E = b + D$
else set $E = b - D$.

Step 5 Set $h = -2f(p_2)/E$;
$p = p_2 + h$.

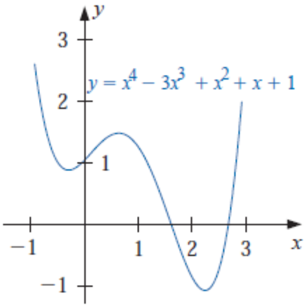
Müller's Algorithm

Step 6 If $|h| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 7 Set $p_0 = p_1$; *(Prepare for next iteration.)*
$p_1 = p_2$;
$p_2 = p$;
$h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = i + 1$.

Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(The procedure was unsuccessful.)
STOP.

find the complex roots of the polynomial



The initial approximations are chosen such that the parabola passing through these points do not intersect the x-axis and thus have complex roots.

$p_0 = 0.5,\ p_1 = -0.5,\ p_2 = 0$		
i	p_i	$f(p_i)$
3	$-0.100000 + 0.888819i$	$-0.01120000 + 3.014875548i$
4	$-0.492146 + 0.447031i$	$-0.1691201 - 0.7367331502i$
5	$-0.352226 + 0.484132i$	$-0.1786004 + 0.0181872213i$
6	$-0.340229 + 0.443036i$	$0.01197670 - 0.0105562185i$
7	$-0.339095 + 0.446656i$	$-0.0010550 + 0.000387261i$
8	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$
9	$-0.339093 + 0.446630i$	$0.000000 + 0.000000i$