Interpolation and Polynomial Approximation

APM1137 - Numerical Analysis

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One of the most useful and well - known classes of functions mapping the set of real numbers into itself is the **algebraic polynomials**,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, \ldots, a_n$ are real constants.

- they uniformly approximate continuous functions
- given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired.
- the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials.
- polynomials are often used for approximating continuous functions.

Weierstrass Approximation Theorem

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial, $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \text{ for all } x \text{ in } [a, b].$$

The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.

A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

Lagrange Interpolating Polynomials

Determining a polynomial of degree one that passes through the distinct points (x_0, y_0) and $(x_1, y1_)$ is the same as approximating a function f for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial *interpolating*, or agreeing with, the values of f at the given points.

Using this polynomial for approximation within the interval given by the endpoints is called **polynomial interpolation**.

Lagrange Interpolating Polynomials

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The linear **Lagrange interpolating polynomial** through (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1, L_0(x_1) = 0, L_1(x_0) = 0, and L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So P is the unique polynomial of degree at most one that passes through (x_0, y_0) and (x_1, y_1).

Example

Determine the linear Lagrange interpolating polynomial that passes through the points (2, 4) and (5, 1).

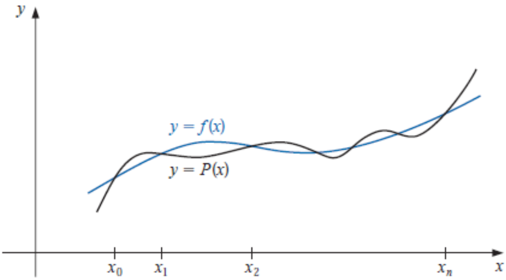
$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \text{ and } L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-3)$$

So

$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -x+6$$

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n+1$ points

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$



For each $k = 0, 1, \ldots, n$, construct a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$.

To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires the numerator of $L_{n,k}(x)$ contain the term

$$(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)$$

To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$.

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x_k - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

The interpolating polynomial defined with this $L_{n,k}$ is called the n**th Lagrange interpolating polynomial**.

Theorem

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \text{ for each } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where, for each $k = 0, 1, \ldots, n$,

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{x_k - x_i}$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree

Example

a) Use the numbers (called *nodes*) $x_0 = 2, x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.
(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Solution (a)

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$
$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2-4)} = -\frac{16}{15}(x-2)(x-4)$$
$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75)$$

also

$$f(x_0) = f(2) = \frac{1}{2}, \quad f(x_i) = f(2.75) = \frac{4}{11}, \text{ and } f(x_2) = f(4) = \frac{1}{4}$$

Thus

$$P(x) = \sum_{k=0}^{2} f(x_k)L_k(x)$$

$$= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75$$

$$= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

Solution (b)

$$f(3) \approx P(3) = \frac{29}{88} \approx 0.32955$$

Remainder Term or Bound for Error

The next step is to calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial.

Theorem

Suppose $x_0, x1_, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x \in [a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)(\xi(x))}}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n),$$

where $P(x)$ is the interpolating polynomial defined in the previous theorem.

Example

Determine the error form for the polynomial in the previous example and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

Solution
since $f(x) = x^{-1}$, then

$$f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3}, \text{ and } f'''(x) = -6x^{-4}.$$

The second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2)$$
$$= -(\xi(x))^{-4}(x - 2)(x - 2.75)(x - 4),$$

for $\xi(x) \in (2, 4)$

The highest value of $(\xi(x))^{-4}$ on $(2,4)$ is $(xi(2))^{-4} = 1/16$

we need to determine the maximum value on $(2,4)$ of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22.$$

$$D_x\left[\frac{35}{4}x^2 + \frac{49}{2}x - 22\right] = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7)$$

The critical points occur at

$$x = \frac{7}{3} \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108} \text{ and } x = \frac{7}{2} \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16}$$

Hence the maximum error is

$$\left|\frac{f'''(\xi(x))}{3!}(x-2)(x-2.75)(x-4)\right| \le \left|-\frac{1}{16}\right|\left|-\frac{9}{16}\right| \approx 0.03516$$

Data Approximation

A frequent use of Lagrange polynomials involves the interpolation of tabulated data.

An explicit representation of the polynomial might not be needed, only the values of the polynomial at specified points.

The function underlying the data might not be known so the explicit form of the error cannot be used.

The table lists values of a function f at various points. The approximations to $f(1.5)$ obtained by various Lagrange polynomials that use this data will be compared to try and determine the accuracy of the approximation.

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

The polynomials can be determined such that the interval bounded by the nodes contains 1.5.

Lagrange Polynomial	**Nodes**
P_1	1.3, 1.6
P_2	1.3, 1,6, 1.9
$\hat{P}_2$	or 1.0, 1.3, 1.6,
P_3	1.3, 1.6, 1.9, 2.2
$\hat{P}_3$	or 1.0, 1.3, 1.6, 1.9
P_4	1.0, 1.3, 1.6, 1.9, 2.2

Lagrange Polynomial	**Approximation Value**
$P_1(1.5)$	0.5102968
$P_2(1.5)$	0.5112857
$\hat{P}_2(1.5)$	0.5124715
$P_3(1.5)$	0.5118302
$\hat{P}_3(1.5)$	0.5118127
$P_4(1.5)$	0.5118200

Because $P_3(1.5), \hat{P}_3(1.5)$, and $P_4(1.5)$ all agree to within 2×10^{-5} units, we expect this degree of accuracy for these approximations. We also expect $P_4(1.5)$ to be the most accurate approximation, since it uses more of the given data.

Neville's Method

Definition

Let f be a function defined at $x_0, x_1, x_2, \ldots, x_n$, and suppose that $m_1, m_2, \ldots, m_k$ are k distinct integers, with $0 \leq m_i \leq n$ for each i. The Lagrange polynomial that agrees with $f(x)$ at the k points $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$ is denoted $P_{m_1,m_2,\ldots,m_k}(x)$.

Theorem

Let f be defined at $x_0, x_1, \ldots, x_k$, and let x_j and x_i be two distinct numbers in this set. Then
$P(x) = \dfrac{(x - x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i - x_j)}$ is
the kth Lagrange polynomial that interpolates f at the $k + 1$ points $x_0, x_1, \ldots, x_k$.

The theorem implies that the interpolating polynomials can be generated recursively

$$P_{0,1} = \frac{1}{x_1 - x_0}[(x - x_0)P_1 - (x - x_1)P_0],$$
$$P_{1,2} = \frac{1}{x_2 - x_1}[(x - x_1)P_2 - (x - x_2)P_1],$$
$$P_{0,1,2} = \frac{1}{x_2 - x_0}[(x - x_0)P_{1,2} - (x - x_2)P_{0,1}]$$

x_0	P_0				
x_1	P_1	$P_{0,1}$			
x_2	P_2	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

To simplify the multiple subscripts form, define Q_{ij} denoting the interpolating polynomial of degree j on the $(j+1)$ numbers

$x_{i-j}, x_{i-j+1}, \ldots, x_{i-1}, x_i$

x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

from the previous example,

1.0	0.7651977				
1.3	0.6200860	0.5233449			
1.6	0.4554022	0.5102968	0.5124715		
1.9	0.2818186	0.5132634	0.5112857	0.5118127	
2.2	0.1103623	0.5104270	0.5137361	0.5118302	0.5118200

Neville's Method Algorithm

To evaluate the interpolating polynomial P on the $n+1$ distinct numbers $x_0, \ldots, x_n$ at the number x for the function f:

INPUT numbers $x, x_0, x_1, \ldots, x_n$; values $f(x_0), f(x_1), \ldots, f(x_n)$ as the first column $Q_{0,0}, Q_{1,0}, \ldots, Q_{n,0}$ of Q.

OUTPUT the table Q with $P(x) = Q_{n,n}$.

Step 1 For $i = 1, 2, \ldots, n$
for $j = 1, 2, \ldots, i$

$$\text{set } Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}.$$

Step 2 OUTPUT (Q);
STOP.

Divided Differences

Iterated interpolationwas used in the previous section to generate successively higher-degree polynomial approximations at a specific point. Divided-difference methods are used to successively generate the polynomials themselves.

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) =$$
$$a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}),$$

for appropriate constants $a_0, a_1, \ldots, a_n$.

Note that

$$a_0 = P_n(x_0) = f(x_0) \text{ and } f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1)$$

so

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

.

Divided Difference Notation

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

So $P_n(x)$ can be rewritten in a form called **Newton's Divided-Difference:**

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

Newton's Divided Difference Algorithm

To obtain the divided-difference coefficients of the interpolatory polynomial P on the $(n+1)$ distinct numbers $x_0, x_1, \ldots, x_n$ for the function f:

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), f(x_1), \ldots, f(x_n)$ as $F_{0,0}, F_{1,0}, \ldots, F_{n,0}$.

OUTPUT the numbers $F_{0,0}, F_{1,1}, \ldots, F_{n,n}$ where

$$P_n(x) = F_{0,0} + \sum_{i=1}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j). \quad (F_{i,i} \text{ is } f[x_0, x_1, \ldots, x_i].)$$

Step 1 For $i = 1, 2, \ldots, n$

For $j = 1, 2, \ldots, i$

set $F_{i,j} = \dfrac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$. $\quad (F_{i,j} = f[x_{i-j}, \ldots, x_i].)$

Step 2 OUTPUT $(F_{0,0}, F_{1,1}, \ldots, F_{n,n})$;

STOP.