Module 04: Graph Algorithms

Analysis and Design of Algorithms

Ammar Sherif

Nile University

1. Graphs & Graph Algorithms
2. Unweighted Graphs
3. Weighted Graphs
4. Path Construction

What are Graphs?

Graphs

From a mathematical perspective, consist of a nodes/vertices and edges.

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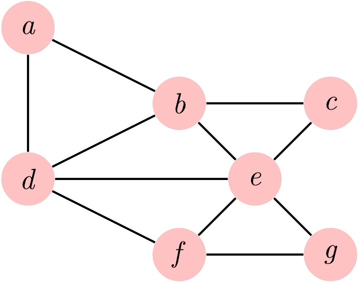
From a mathematical perspective, consist of a nodes/vertices and edges.

- Nodes/Vertices



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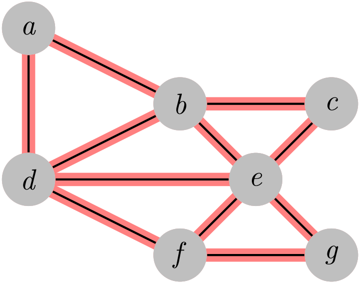
- Edges



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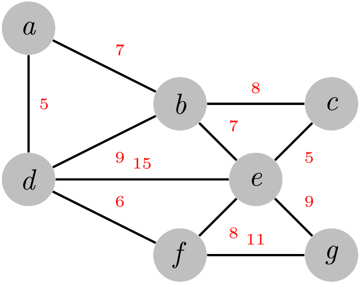

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- Nodes/Vertices
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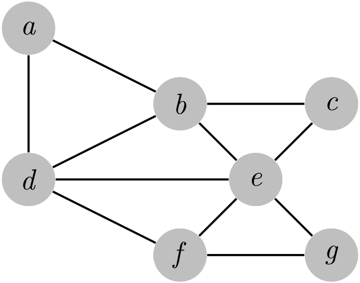
- Weighted Graph
- Unweighted Graph



What are Graphs?

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- Nodes/Vertices
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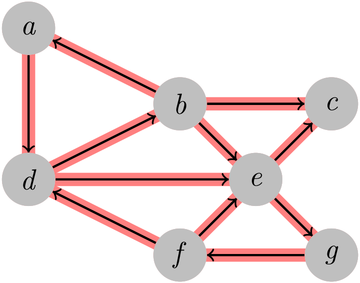
- Weighted Graph
- Unweighted Graph
- Directed Graph



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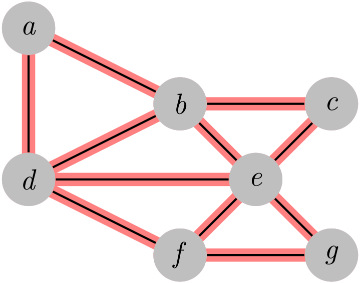

Why Graphs & Graph Algorithms?

- Shortest Path and Route planning

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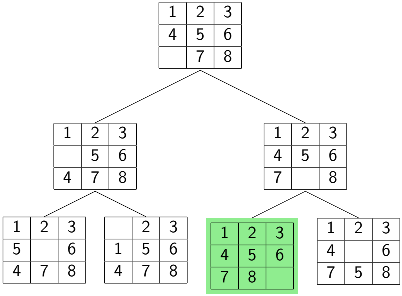

Why Graphs & Graph Algorithms?

- Shortest Path and Route planning
- Robotics
    - Warehouses
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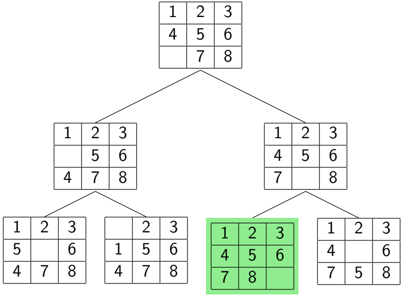
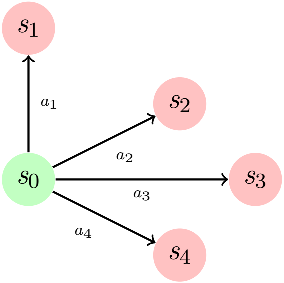
- Games
- Optimization Problems



Why Graphs & Graph Algorithms?

- Shortest Path and Route planning
- Robotics
    - Warehouses
    - Space Robots
    - Rescue Robots
- Games
- Optimization Problems
- Any decision-based problem



Graph Representation

Algorithms are implemented via programming, so how to represent graphs?

What our representation should provide?

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- Weights of links

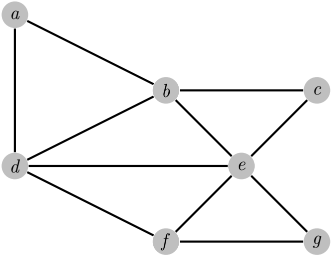
Graph Representation

Algorithms are implemented via programming, so how to represent graphs?

What our representation should provide?

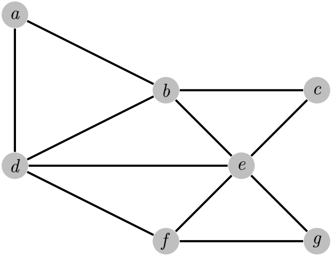
- Know the neighbors
- Weights of links
- list of nodes/edges

Graph Representation



Graph Representation

- Adjacency Matrix



	a	b	c	d	e	f	g
a	–	1		1			
b	1	–	1	1	1		
c		1	–		1		
d	1	1		–	1	1	
e		1	1	1	–	1	1
f				1	1	–	1
g					1	1	–

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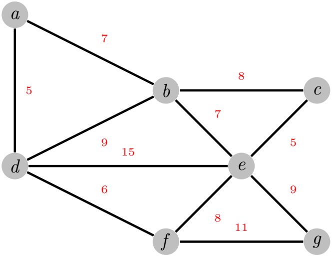
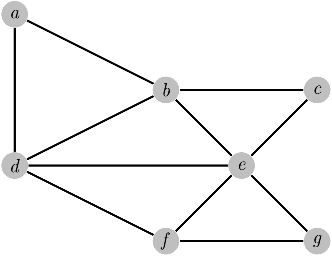


	a	b	c	d	e	f	g
a	–	7		5			
b	7	–	8	9	7		
c		8	–		5		
d	5	9		–	15	8	
e		7	5	15	–	8	9
f				8	8	–	11
g					9	11	–

Graph Representation

- Adjacency Matrix
- Adjacency List



a	b	c	d	e	f	g
↓	↓	↓	↓	↓	↓	↓
b	a	b	a	b	d	e
d	c	e	b	c	e	f
	d		e	d	g	
	e		f	f		
				g		

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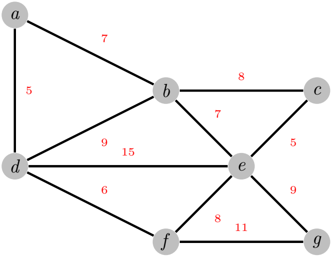


a	b	c	d	e	f	g
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$b:7$	$a:7$	$b:8$	$a:5$	$b:7$	$d:8$	$e:9$
$d:5$	$c:8$	$e:5$	$b:9$	$c:5$	$e:8$	$f:11$
	$d:9$		$e:15$	$d:15$	$g:11$	
	$e:7$		$f:8$	$f:8$		
				$g:9$		

1. Graphs & Graph Algorithms

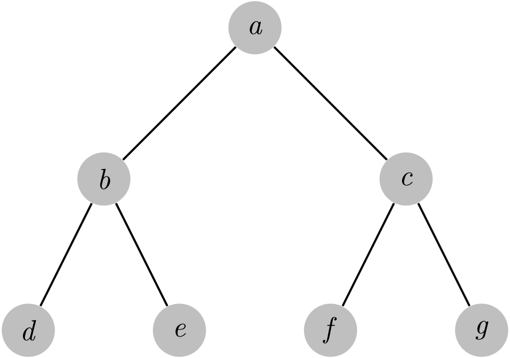
2. Unweighted Graphs
   - Depth First Search (DFS)
   - Breadth First Search (BFS)

3. Weighted Graphs

4. Path Construction

Depth First Search (DFS)

Depth has the max priority



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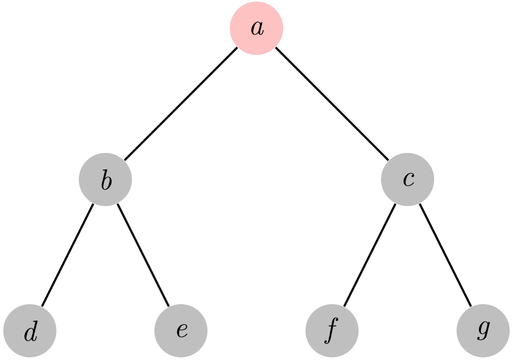

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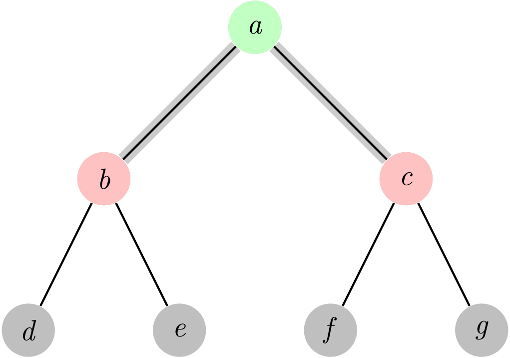
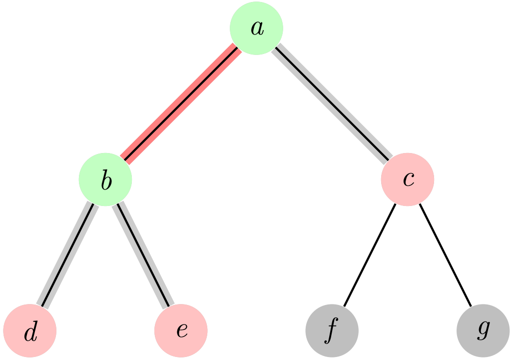


Depth First Search (DFS)

Depth has the max priority

Child $\longleftarrow$ Parent

$b \longleftarrow a$



Depth First Search (DFS)

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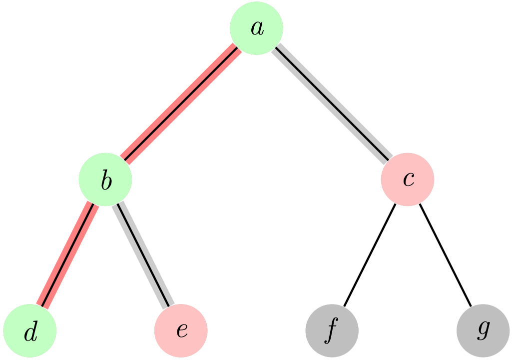
Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$



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$b \longleftarrow a$
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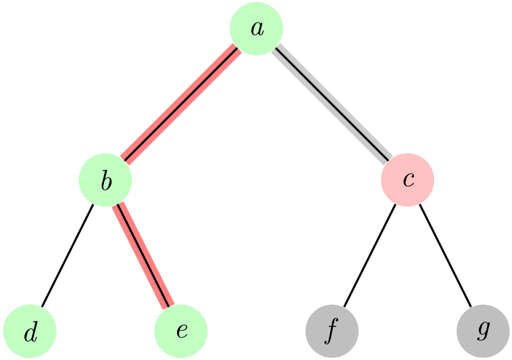

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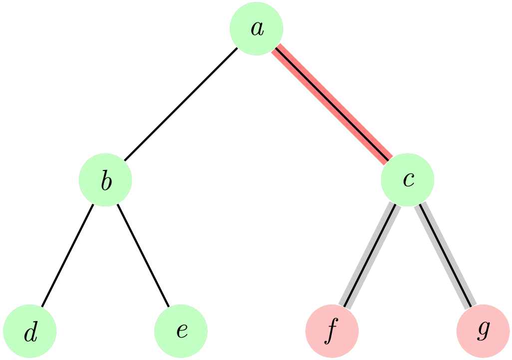

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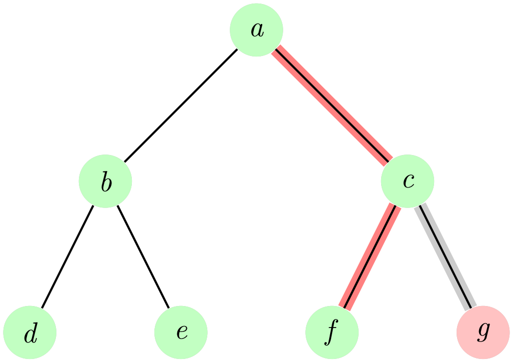
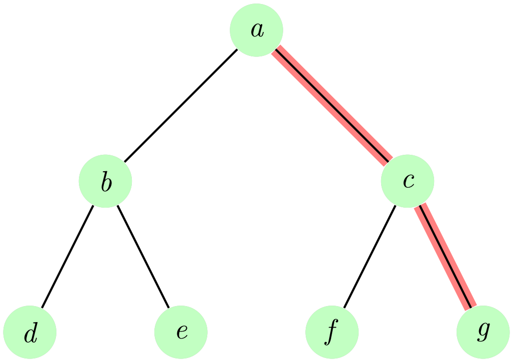


Depth First Search (DFS)

Therefore, the below table summarizes how did we get to any node through our traversal

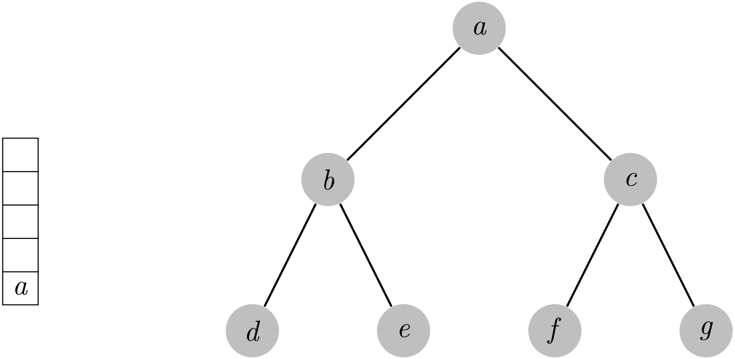
Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$



Depth First Search (DFS)

Let's do it again, to notice the **pattern** of nodes to be visited



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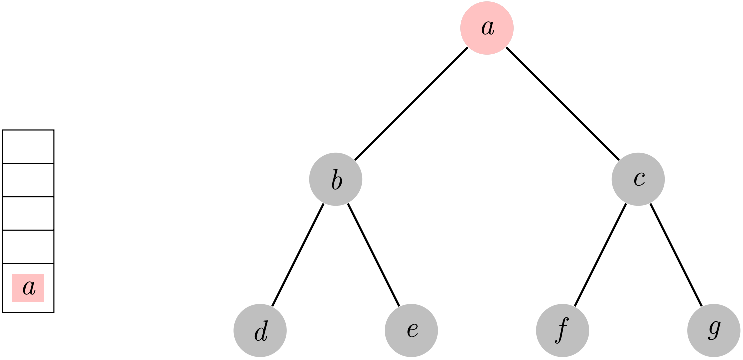

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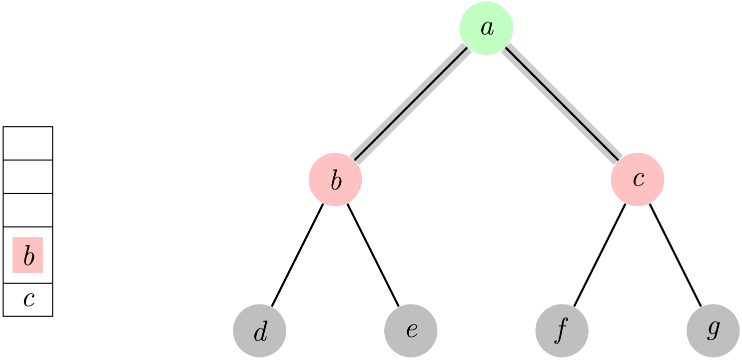

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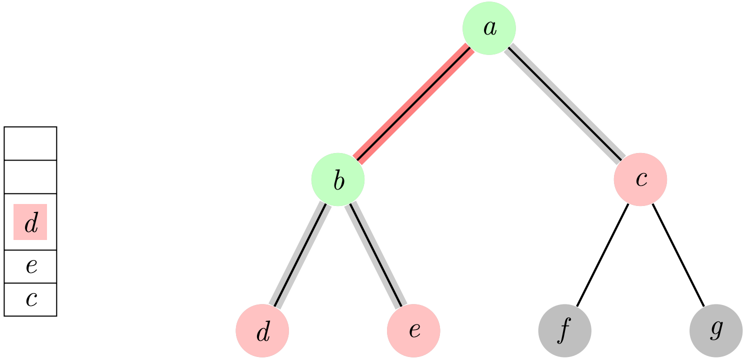

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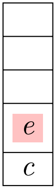
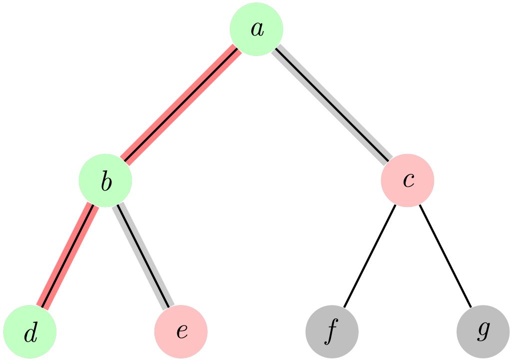
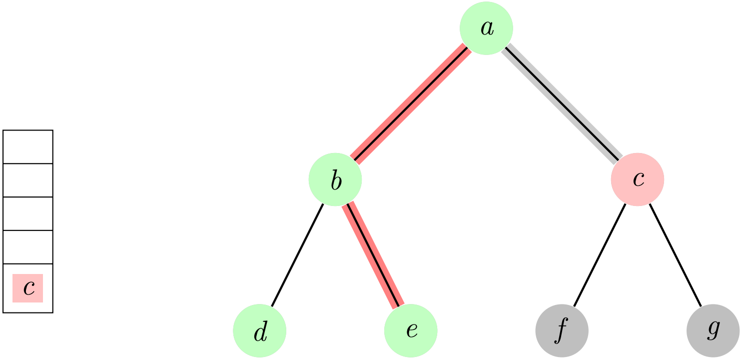




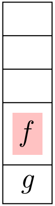
Depth First Search (DFS)

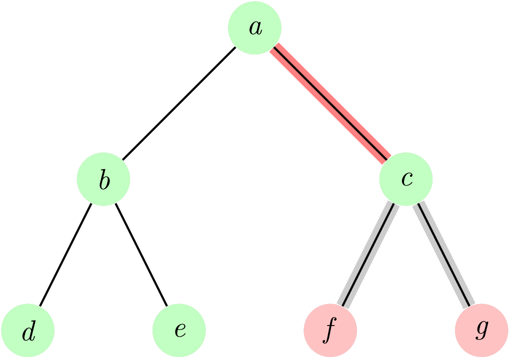
Did you get the pattern of nodes to be visited?



Depth First Search (DFS)

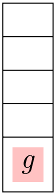
Did you get the pattern of nodes to be visited? **Last** inserted element is **first** to explore.

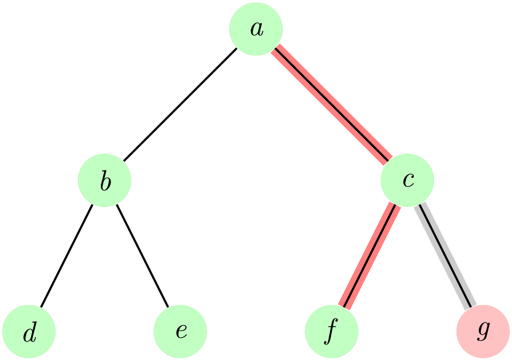




Depth First Search (DFS)

Did you notice what might be this structure? Hint: Last-in First-out

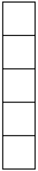


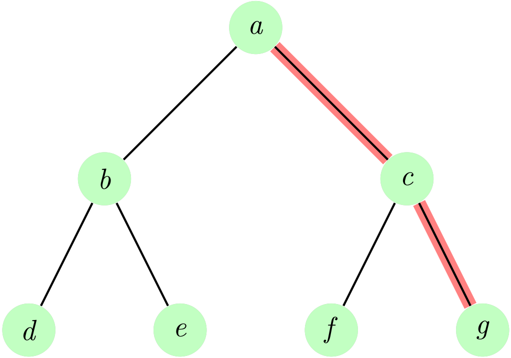


Depth First Search (DFS)

Did you notice what might be this structure? Hint: Last-in First-out

Yes, it is *Stack*





Depth First Search (DFS)

Algorithm 1: DEPTH-FIRST($root$)

```
def S to be Stack;
visited ← {};
S.push(root);
while S ≠ ϕ do
    node ← S.pop();
    if node ∉ visited then
        visited ← visited ∪ {node};
        for n ∈ adjacent(node) do
            S.push(n);
        end
    end
end
```

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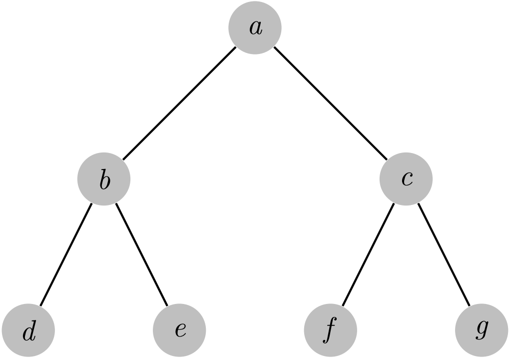
Breadth First Search (BFS)

Breadth has the max priority



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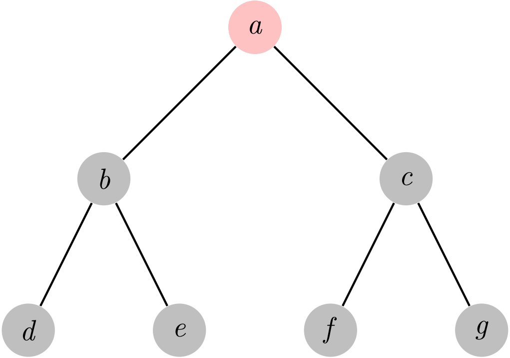

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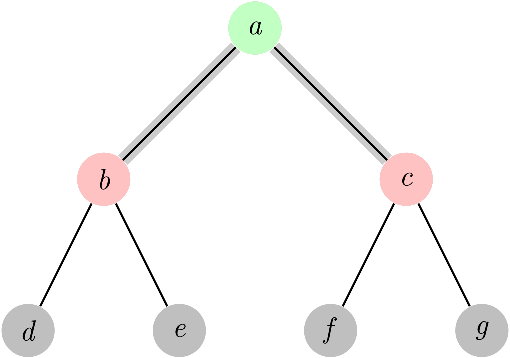

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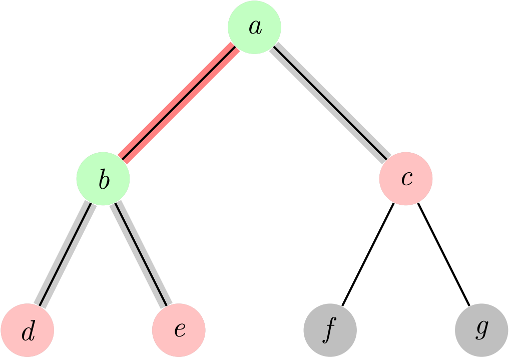

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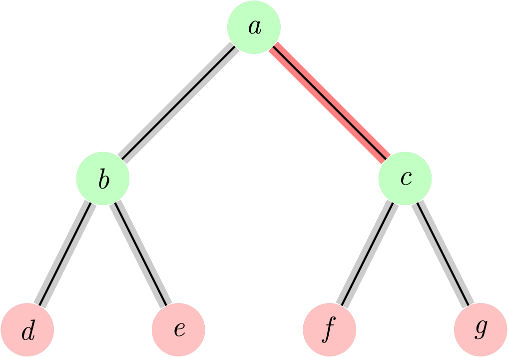

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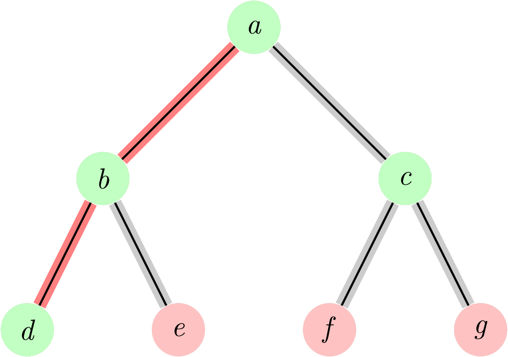

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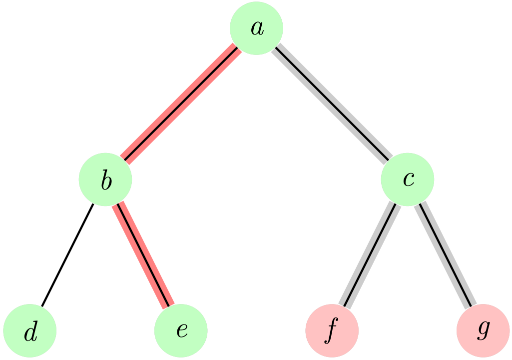

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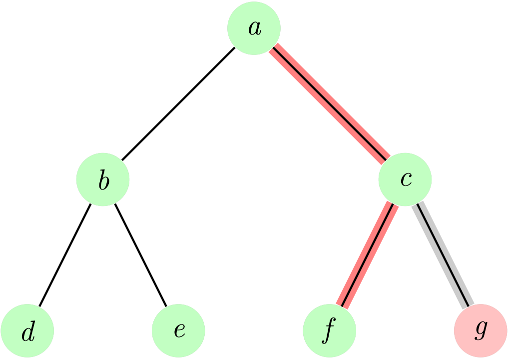

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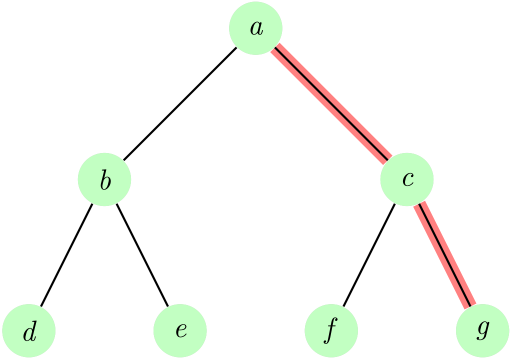
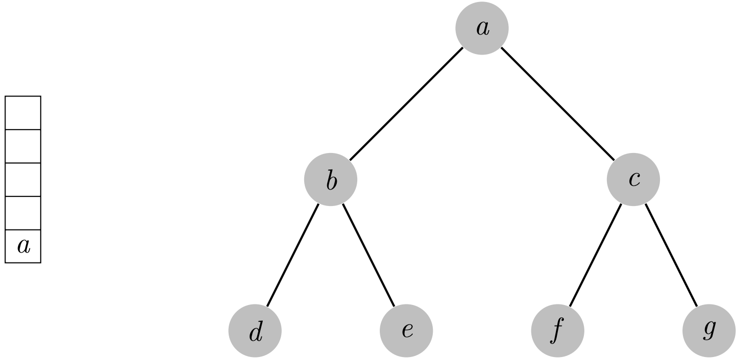


Breadth First Search (BFS)

Let's do it again, to notice the **pattern** of nodes to be visited



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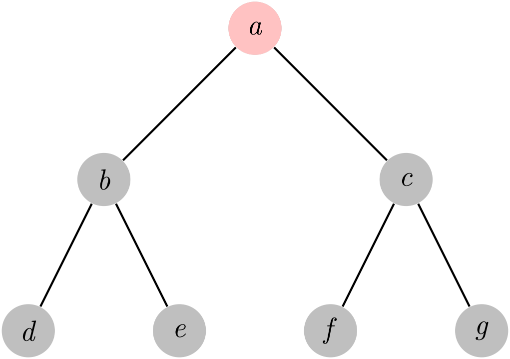

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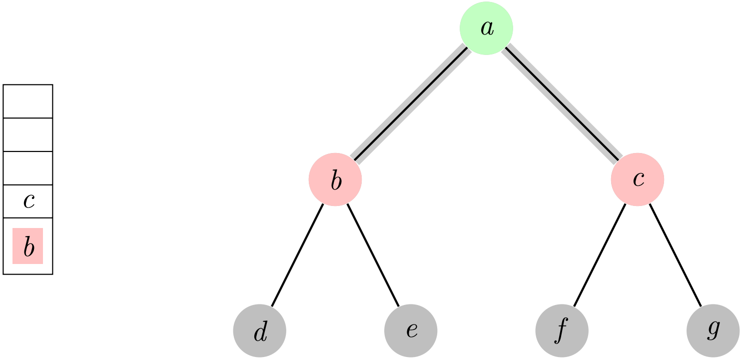

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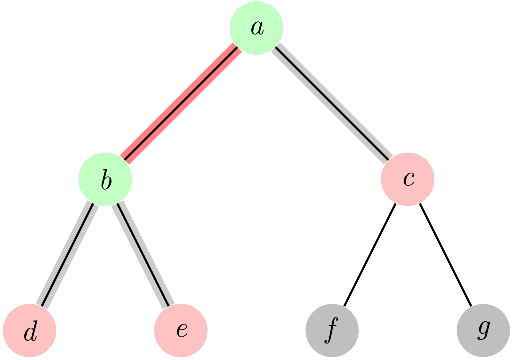
e
d
c



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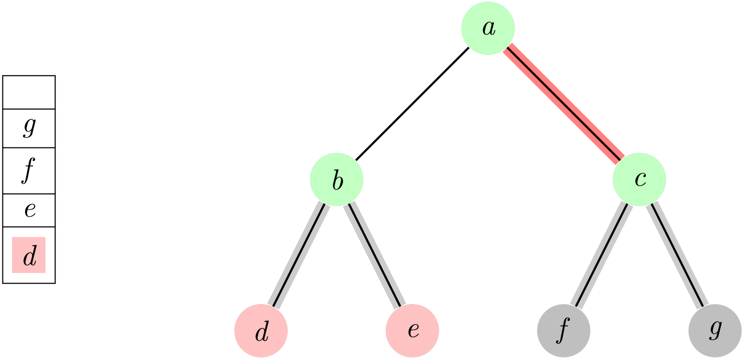

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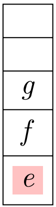
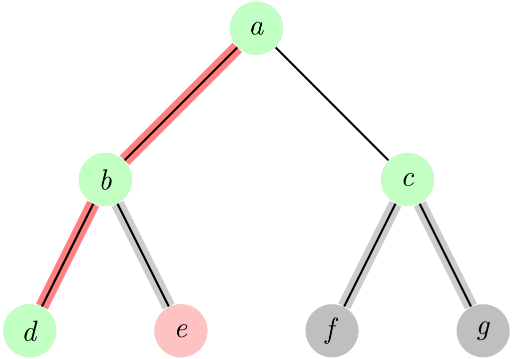
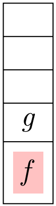
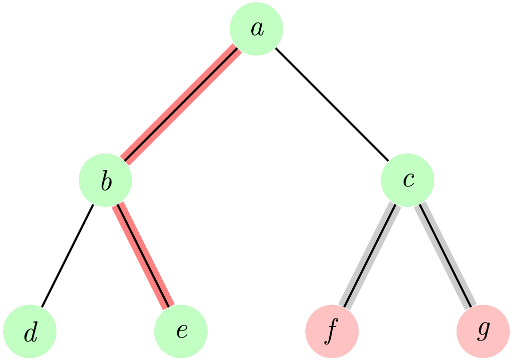




Breadth First Search (BFS)

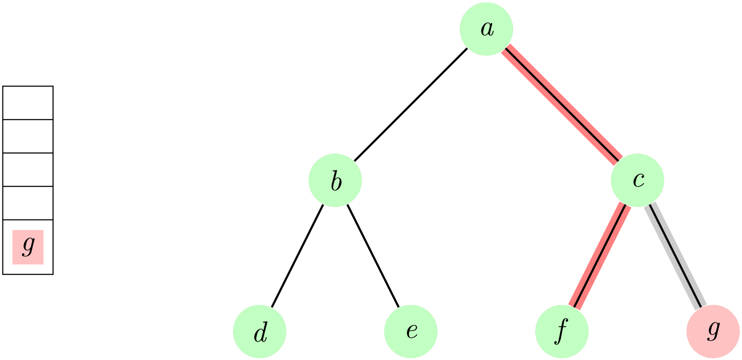
Did you get the pattern of nodes to be visited? **First** inserted element is **first** to explore.





Breadth First Search (BFS)

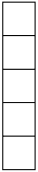
Did you notice what might be this structure? Hint: First-in First-out

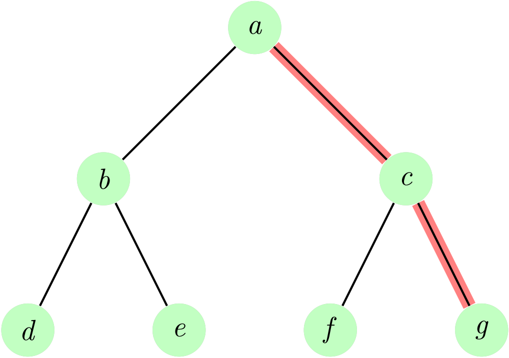


Breadth First Search (BFS)

Did you notice what might be this structure? Hint: First-in First-out

Yes, it is *Queue*





Breadth First Search (BFS)

Algorithm 2: BREADTH-FIRST($root$)

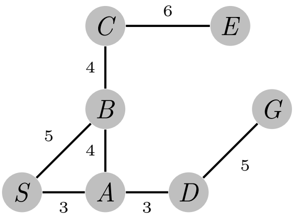
```
def S to be Queue;
visited ← {};
S.enqueue(root);
while S ≠ ϕ do
    node ← S.dequeue();
    if node ∉ visited then
        visited ← visited ∪ {node};
        for n ∈ adjacent(node) do
            S.enqueue(n);
        end
    end
end
```

Branch and Bound

Right now we have weights, now what should we prioritize?

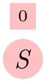
Branch and Bound

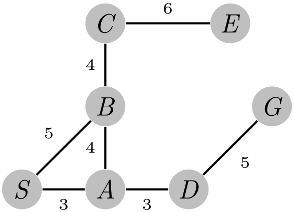
Right now we have weights, now what should we prioritize? Min/Max **weights**



Branch and Bound

Right now we have weights, now what should we prioritize? Min/Max **weights**

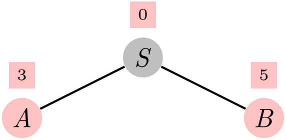


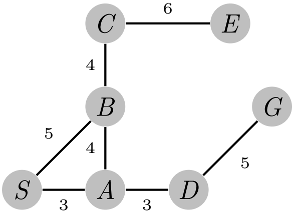


$\boxed{S : 0}$

Branch and Bound

Right now we have weights, now what should we prioritize? Min/Max **weights**



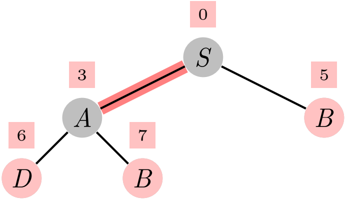


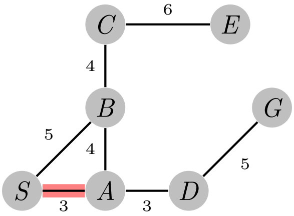
iterations/visits: 1

$A:3$	$B:5$

Branch and Bound

Right now we have weights, now what should we prioritize? Min/Max **weights**



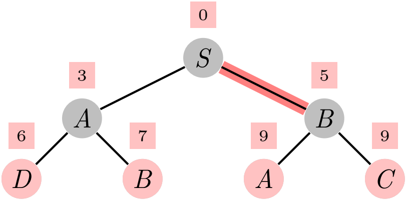


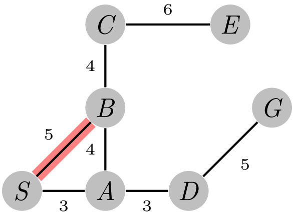
iterations/visits: 2

$B:5$	$D:6$	$B:7$

Branch and Bound

Can you notice the pattern of the structure?



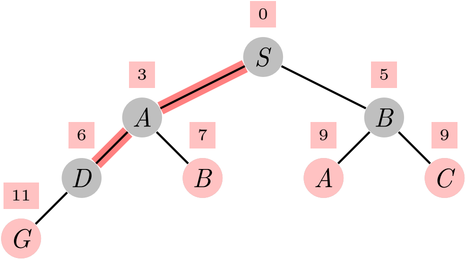


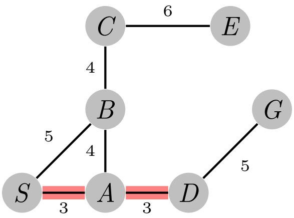
iterations/visits: 3

$D:6$	$B:7$	$A:9$	$C:9$

Branch and Bound

Can you notice the pattern of the structure? Also, we found G, so shall we stop?



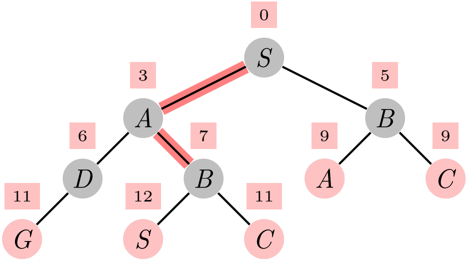


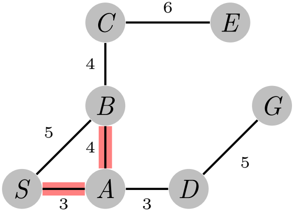
iterations/visits: 4

$B:7$	$A:9$	$C:9$	$G:11$

Branch and Bound

Have you noticed what happened?



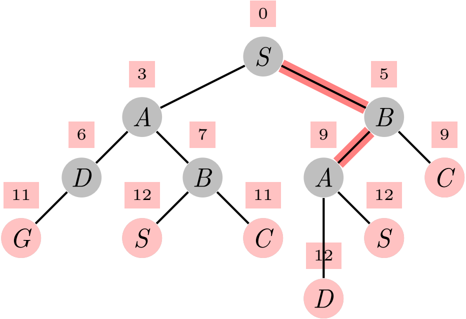


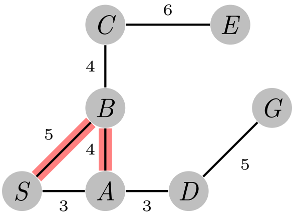
iterations/visits: 5

$A:9$	$C:9$	$C:11$	$G:11$	$S:12$

Branch and Bound

Now, what is such structure?



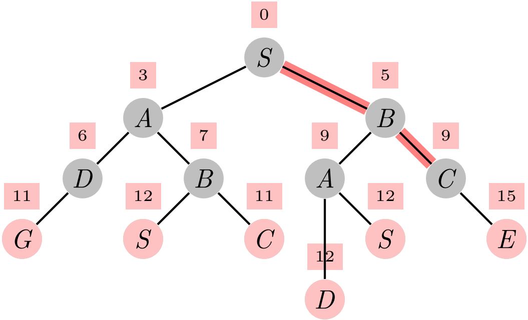


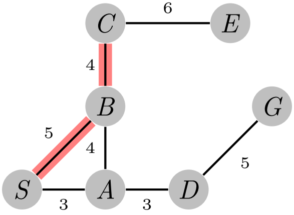
iterations/visits: 6

$C:9$	$C:11$	$G:11$	$S:12$	$D:12$	$S:12$

Branch and Bound

Now, what is such structure?



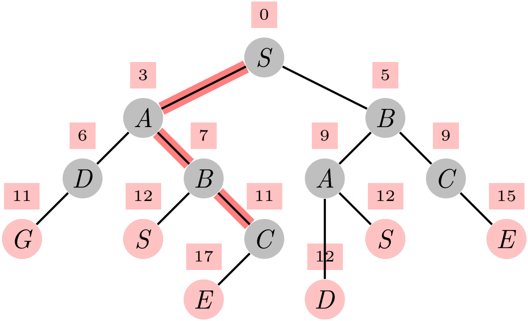


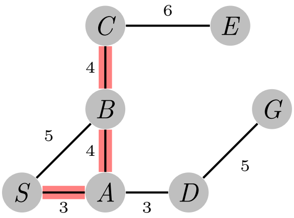
iterations/visits: 7

$C:11$	$G:11$	$S:12$	$D:12$	$S:12$	$E:15$

Branch and Bound

Now, what is such structure?



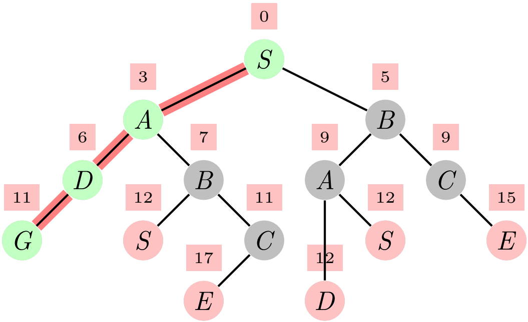


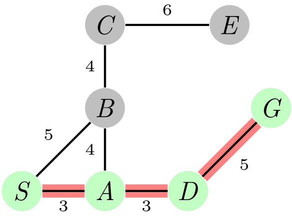
iterations/visits: 8

$G:11$	$S:12$	$D:12$	$S:12$	$E:15$	$E:17$

Branch and Bound

Yes, it is a **Priority Queue**, where the priority is the overall path cost.





iterations/visits: 9

$S:12$	$D:12$	$S:12$	$E:15$	$E:17$

Branch and Bound

Algorithm 3: BRANCH-BOUND($root, goal$)

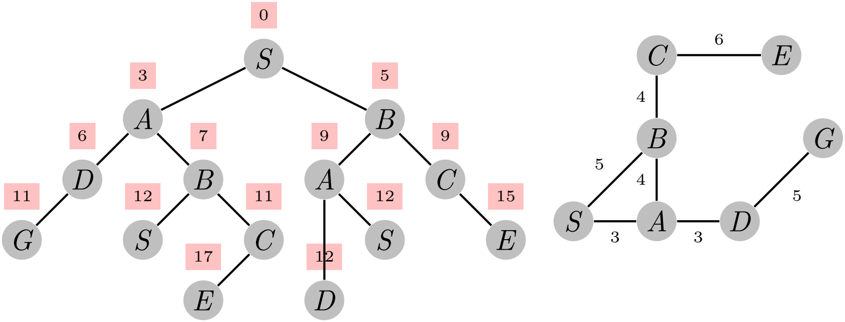
```
def PQ to be Priority Queue
PQ.enqueue(root, 0)
node, path_cost ← root, 0
while PQ ≠ φ ∧ node ≠ goal do
    node, path_cost ← PQ.dequeue()
    for n, c ∈ adjacent(node) do
        // loop over the links and their costs
        PQ.enqueue(n, path_cost + c)
    end
end
```

Search Space Pruning

One question over the previous algorithm is: **cannot we do better?**

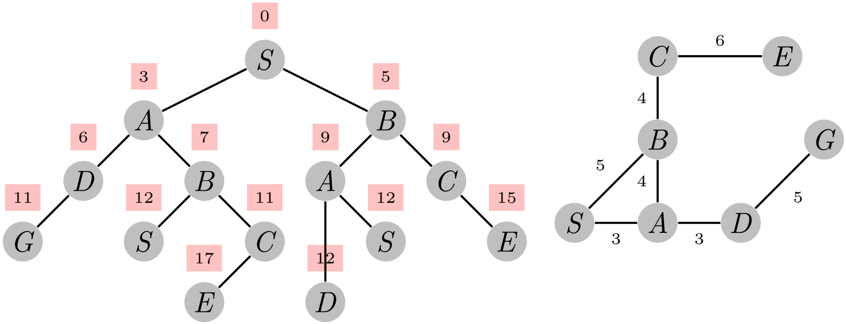
Search Space Pruning

To answer this, we need to check the produced **search space**, on the left.



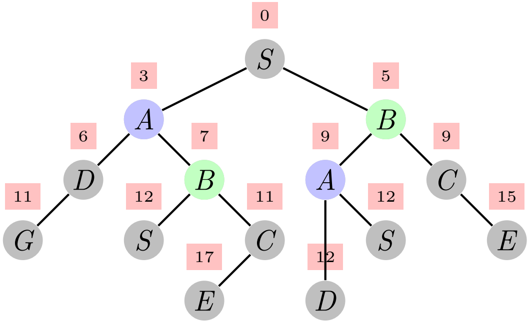
Search Space Pruning

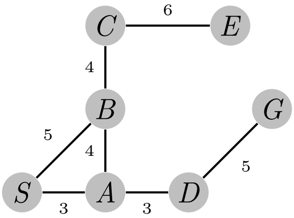
We know the **denser** the search space, more nodes to visit, the more time our algorithms takes. Hence, ***pruning*** it, eliminating some of the nodes, should improve it, how can we do this?



Search Space Pruning

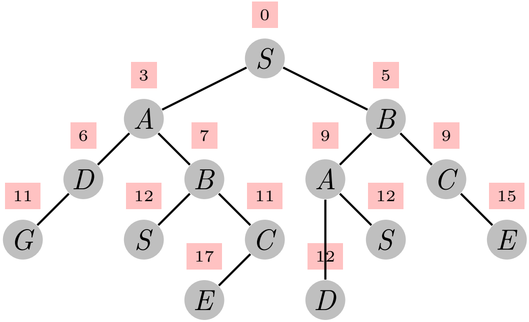
One way is to remove duplicates; do you think we need to check A or B twice? We can do this using a simple **visited list**

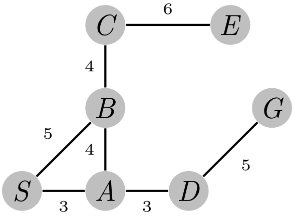




Search Space Pruning

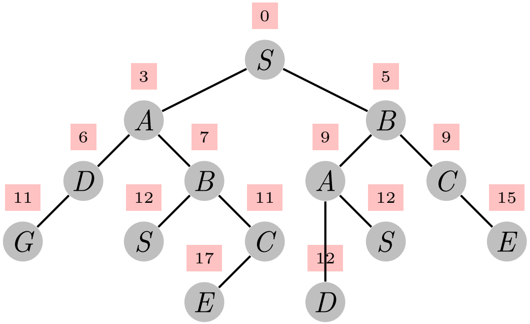
Could you think of other techniques to prune the tree?

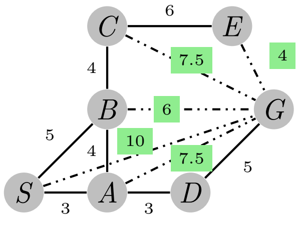




Search Space Pruning

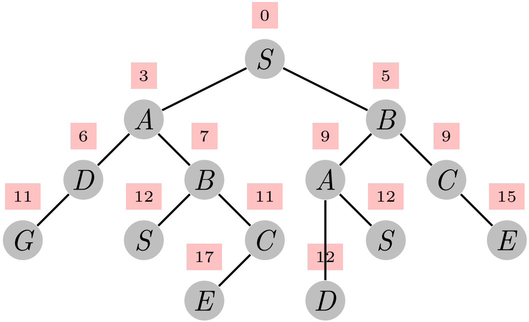
One solution is if we could have an estimate of where the goal is; this may make us more oriented towards searching in particular regions than others; this is briefly, the **heuristics**.

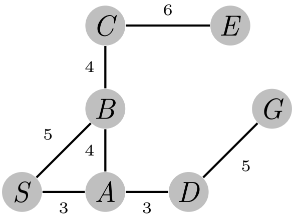




Search Space Pruning

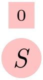
We are covering both the **visited list** and **heuristic** techniques in the next subsections, applying them to branch and bound.

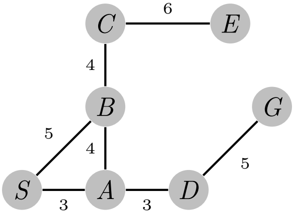




Adding visited list to Branch and Bound

Let's follow on the same graph, but while using a **visited list**.



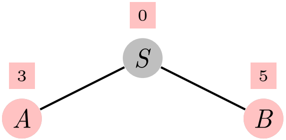


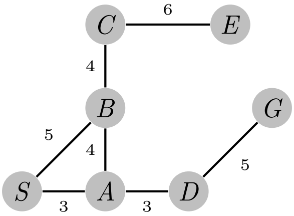
PQ: $\boxed{S : 0}$

$\boldsymbol{visited}$:

Adding visited list to Branch and Bound

Let's follow on the same graph, but while using a **visited list**.





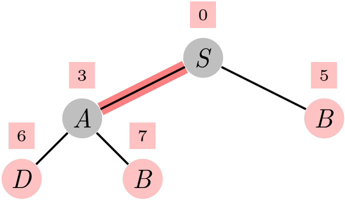
iterations/visits: 1

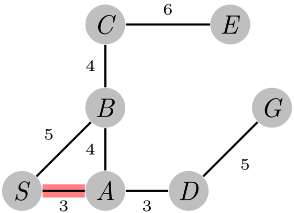
PQ: $\boxed{A:3}\boxed{B:5}$

$\boldsymbol{visited}$: $\boxed{S}$

Adding visited list to Branch and Bound

Let's follow on the same graph, but while using a **visited list**.



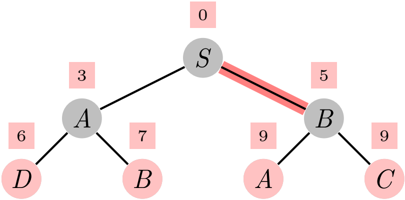


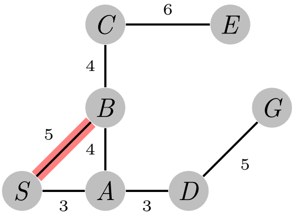
iterations/visits: 2

PQ:	$B:5$	$D:6$	$B:7$
$visited$:	S	A	

Adding visited list to Branch and Bound

Let's follow on the same graph, but while using a **visited list**.





iterations/visits: 3

PQ:

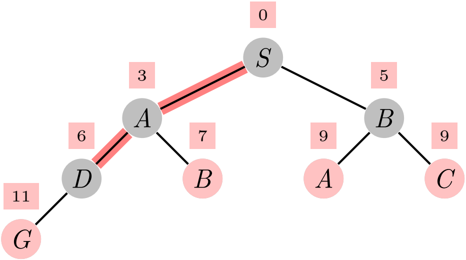
$D:6$	$B:7$	$A:9$	$C:9$

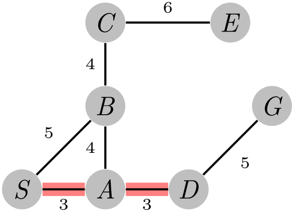
$\boldsymbol{visited}$:

S	A	B

Adding visited list to Branch and Bound

Now, the queue has B, should we visit it next?





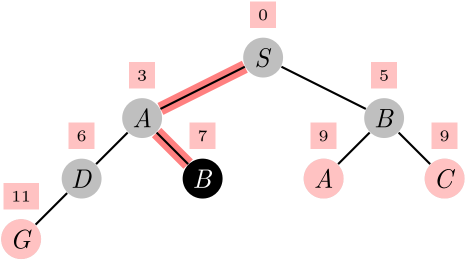
iterations/visits: 4

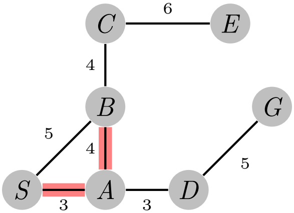
PQ: $\boxed{B:7}\boxed{A:9}\boxed{C:9}\boxed{G:11}$

$\boldsymbol{visited}$: $\boxed{S}\boxed{A}\boxed{B}\boxed{D}$

Adding visited list to Branch and Bound

Well, no: B is already visited, so there is already a shorter path to it.





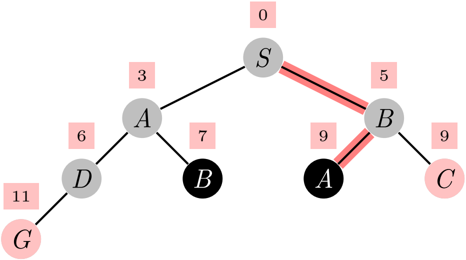
iterations/visits: 5

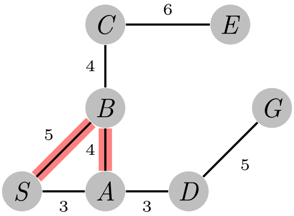
PQ: | $A:9$ | $C:9$ | $G:11$ |

$\boldsymbol{visited}$: | S | A | B | D |

Adding visited list to Branch and Bound

Again, the next was A, so it is blocked as well.





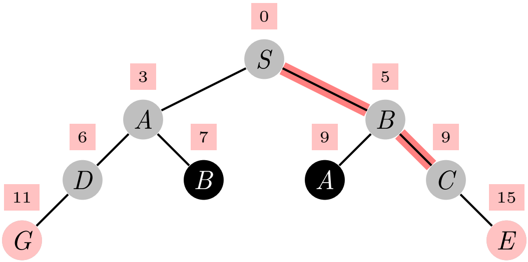
iterations/visits: 6

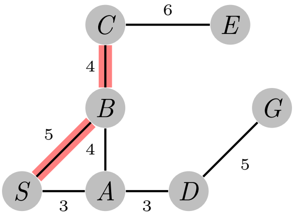
PQ: $C:9$ | $G:11$

$\boldsymbol{visited}$: S | A | B | D

Adding visited list to Branch and Bound

We, then, proceed





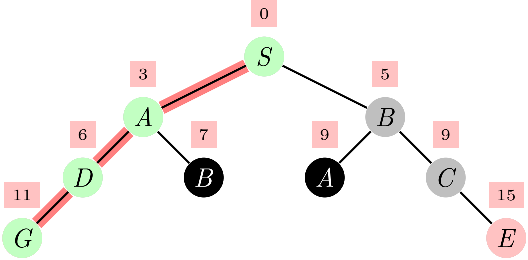
iterations/visits: 7

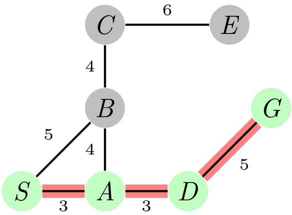
PQ:	$G:11$	$E:15$

$\boldsymbol{visited}$:	S	A	B	D	C

Adding visited list to Branch and Bound

Finally, we got our solution. Could you see the pruning effect?





iterations/visits: 8

PQ: $\boxed{E : 15}$

$\boldsymbol{visited}$: | S | A | B | D | C |

Adding visited list to Branch and Bound

Algorithm 4: BRANCH-BOUND-VISITED($root, goal$)

```
def PQ to be Priority Queue
visited ← {}
PQ.enqueue(root, 0)
node, path_cost ← root, 0
while PQ ≠ ϕ ∧ node ≠ goal do
    node, path_cost ← PQ.dequeue()
    if node ∉ visited then
        for n, c ∈ adjacent(node) do
            // loop over the links and their costs
            PQ.enqueue(n, path_cost + c)
        end
    end
end
```

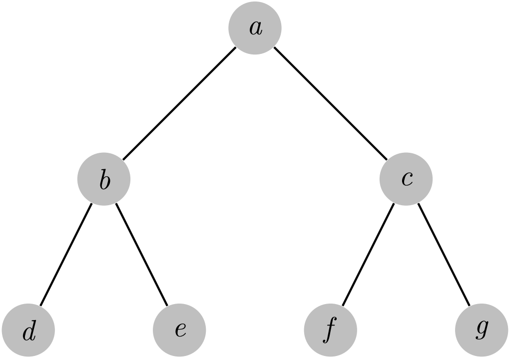
Path Construction from Traversing Algorithm

Cool; now, we have shown how many traversing algorithms, *which visits the nodes*, and checked the pseudo-code, but and important question is

how to construct a path from a traversing algorithm? (DFS for example)

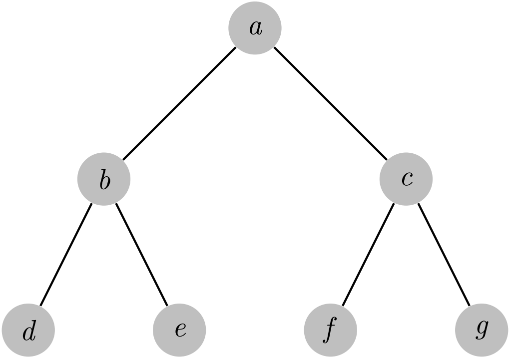
Path Construction from Traversing Algorithm

First, to differentiate between traversing and path, we revisit the same DFS tree as before; the ordering of the visited nodes is a, b, d, e, c, f, g.



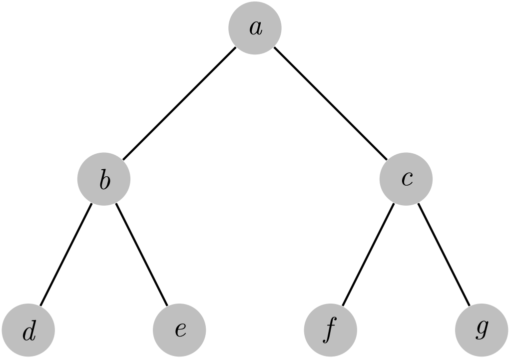
Path Construction from Traversing Algorithm

the ordering of the visited nodes is a, b, d, e, c, f, g. Does this mean that the **path** according to our algorithm from $a \to f$ is a, b, d, e, c, f?



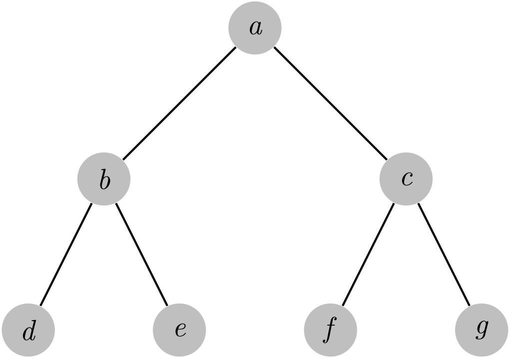
Path Construction from Traversing Algorithm

the ordering of the visited nodes is a, b, d, e, c, f, g. Does this mean that the **path** according to our algorithm from $a \to f$ is a, b, d, e, c, f? well, **No**. This is just a traversing order.



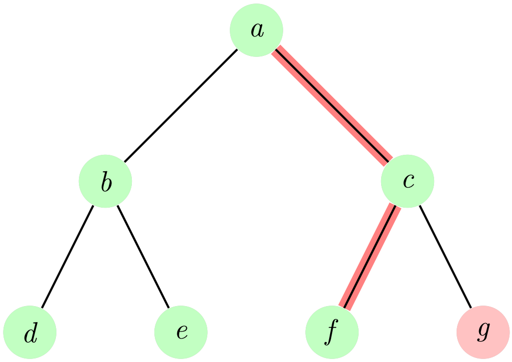
Path Construction from Traversing Algorithm

In that case, what is the path constructed from $a \to f$ according to our algorithm?



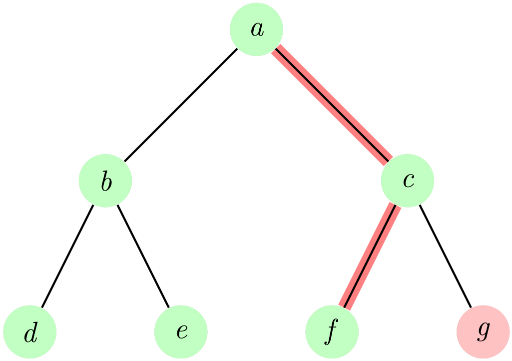
Path Construction from Traversing Algorithm

In that case, what is the path constructed from $a \rightarrow f$ according to our algorithm? It is a, c, f as we see from our previous execution.



Path Construction from Traversing Algorithm

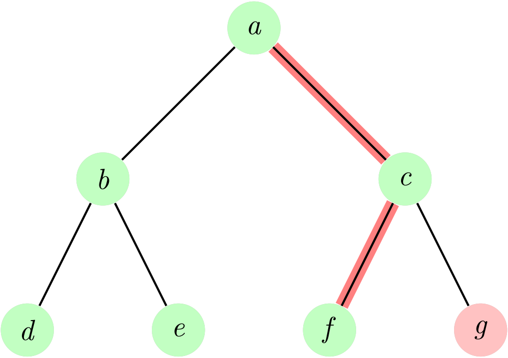
so, how could we construct such path from our algorithm?



Path Construction from Traversing Algorithm

so, how could we construct such path from our algorithm?

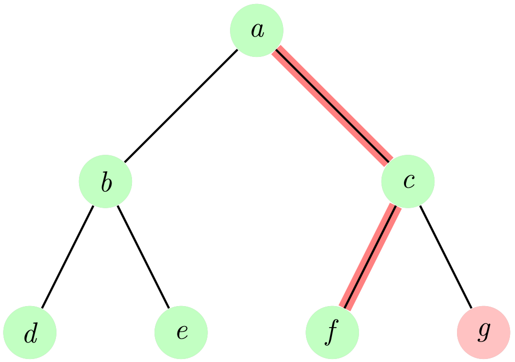
- store the whole paths, instead of just nodes



Path Construction from Traversing Algorithm

so, how could we construct such path from our algorithm?

- store the whole paths, instead of just nodes; Storage hungry



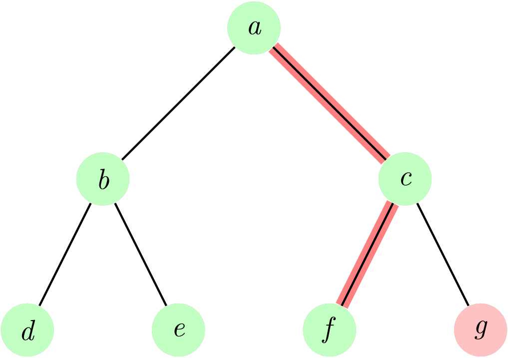
Path Construction from Traversing Algorithm

so, how could we construct such path from our algorithm?

- using **parent-child** structure.

Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$

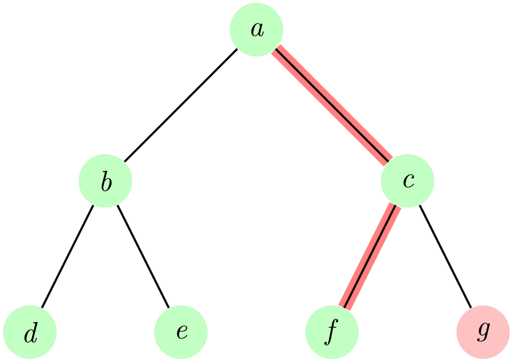


Path Construction from Traversing Algorithm

start from your goal, f, and move backward until getting your start.

Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$

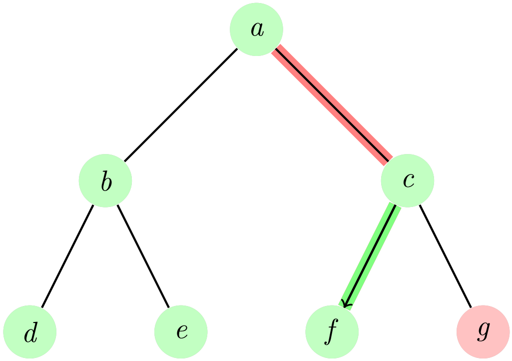


Path Construction from Traversing Algorithm

Path: $c \to f$

Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$

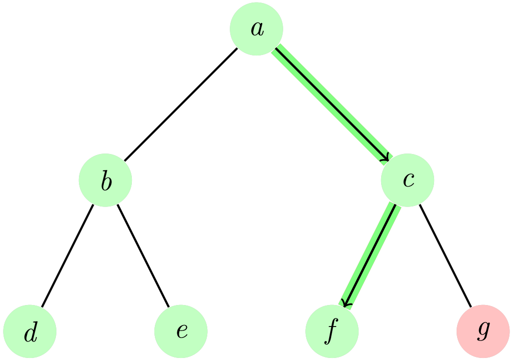


Path Construction from Traversing Algorithm

Path: $\boxed{a \rightarrow c \rightarrow f}$

Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$

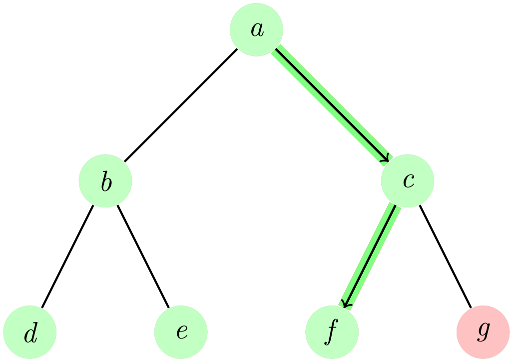


Path Construction from Traversing Algorithm

We can this technique for all the traversing algorithms, mentioned within this module.

Child $\longleftarrow$ Parent

$b \longleftarrow a$
$d \longleftarrow b$
$e \longleftarrow b$
$c \longleftarrow a$
$f \longleftarrow c$
$g \longleftarrow c$



Thanks!