十一章

11.1A

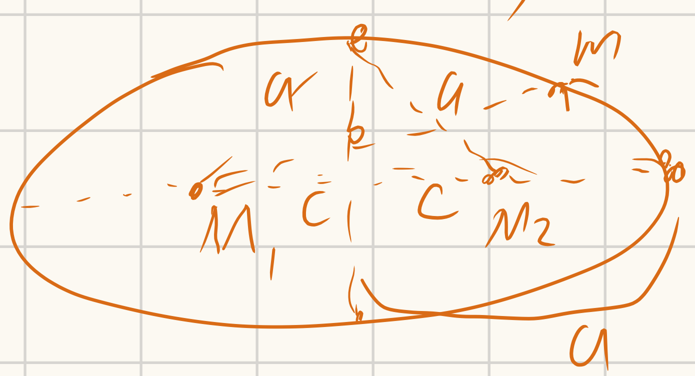
3.

$$W=\int_L \vec{f}\cdot d\vec{r}=\int_L x\,dx+y\,dy+z\,dz$$

$$=\int_0^{2\pi}\left(a\cos t(-a\sin t)+a\sin t\,a\cos t+bt\cdot b\right)dt$$

$$=\int_0^{2\pi}b^2t\,dt=\frac{1}{2}b^2t^2\Big|_0^{2\pi}=\frac{1}{2}b^2 4\pi^2=2\pi^2b^2$$

4、



设 m 到 M 的距离为 r 则 $\vec{F}=\frac{GMm}{r^2}(-\vec{e_r})$

显然这是保守力 $\oint_L \vec{F}\cdot d\vec{l}=\iint_S(\nabla\times\vec{F})\cdot d\vec{s}=0$

引力对 m 所作功为 $W=\int_L \vec{F}\cdot d\vec{l}=GMm\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$

① 质点 M 在左侧

$r_2=a$, $r_1=a+c$ $\quad W=GMm\left(\frac{1}{a}-\frac{1}{a+c}\right)$

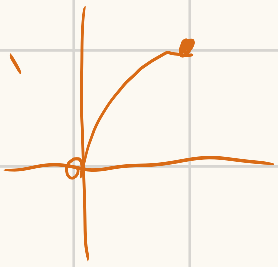
② M在右侧

$r_2 = a \quad r_1 = a - c \quad W = GMm\left(\frac{1}{a} - \frac{1}{a+c}\right)$

$dr \quad r d\theta \quad r\sin\theta\, d\varphi$

$x\sin x\, d$
$(-) x\, d\cos$
$-x\cos x$
$-x\cos$

5、



$\partial_x \; \partial_y$
$P \; Q$
$\partial_x Q - \partial_y P$

$x^2\cos x\, dx$
$x^2\, d\sin x$
$x^2\sin x - \sin x\, 2x\, dx$
$x^2\sin x - 2[-x\cos x + \sin$
$x^2\sin x + 2x\cos x - 2\sin$

$$I = \int_L (x^2 + 2xy)\,dx + (x^2 + y^4)\,dy$$

$$= \int_0^1 \left(x^2 + 2x\sin\frac{\pi}{2}x\right)dx + \int_0^1 \left(x^2 + \sin^4\frac{\pi}{2}x\right)\cos\frac{\pi}{2}x \cdot \frac{\pi}{2}\,dx$$

$$= \int_0^1 \left(x^2 + \frac{\pi}{2}x^2\cos\frac{\pi}{2}x + 2x\sin\frac{\pi}{2}x\right)dx + \sin^4\frac{\pi}{2}x\, d\left(\sin^4\frac{\pi}{2}x\right)$$

$$= \frac{1}{3}x^3 + \int \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2}x\right)^2 \cos\frac{\pi}{2}x\, d\frac{\pi}{2}x + \int 2\left(\frac{2}{\pi}\right)^2 \left(\frac{\pi}{2}x\right)\sin\frac{\pi}{2}x\, d\left(\frac{\pi}{2}x\right) + \frac{1}{5}\sin$$

$$= \frac{1}{3}x^3 + \frac{4}{\pi^2}\left(\left(\frac{\pi}{2}x\right)^2\sin\frac{\pi}{2}x + 2\left(\frac{\pi}{2}x\right)\cos\frac{\pi}{2}x - 2\sin\frac{\pi}{2}x\right)$$

$$+ 2\left(\frac{2}{\pi}\right)^2\left(-\frac{\pi}{2}x\cos\frac{\pi}{2}x + \sin\frac{\pi}{2}x\right)\Bigg|_0^1 + \frac{1}{5}\sin^5$$

$$= \frac{1}{3}(1-0) + \frac{4}{\pi^2}\left(\frac{\pi^2}{4} + 0 - 2\right) + 2\,\frac{4}{\pi^2}(0+1)$$

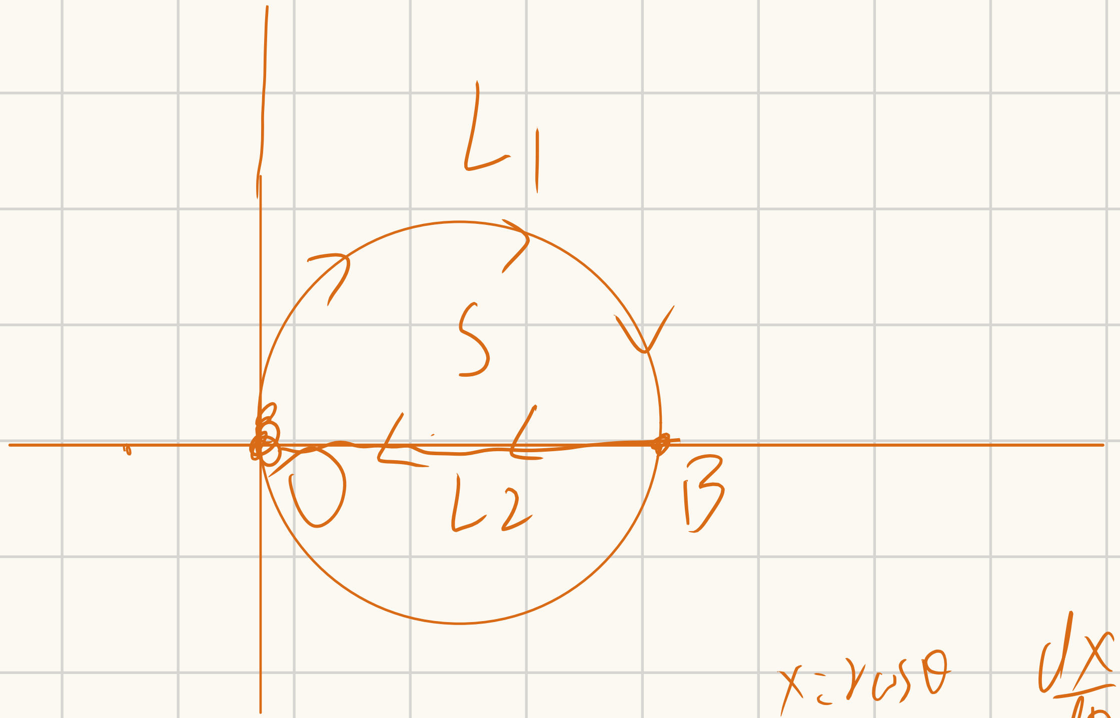
$$= \frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\pi^2}{4} - 2\right) + \frac{8}{\pi^2} + \frac{1}{5}$$

$= \frac{1}{3} + 1 - \frac{8}{\pi^2} + \frac{8}{\pi^2} [illegible] = \frac{1}{3} + \frac{1}{5} = \frac{[illegible]}{15} = \frac{[illegible]}{15}$

$= \frac{23}{15}$

6.

(1)



$L: r = 2\cos\theta \quad \theta\ \frac{\pi}{2} \to 0$

$\frac{dr}{d\theta} = -2\sin\theta$

$x = r\cos\theta \quad \frac{dx}{d\theta} = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right) = (- [illegible]$

$\frac{dy}{d\theta} = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) = ([illegible]$

$L: y^2 = 1 - (x-1)^2 = -x^2 + 2x \quad x\ 0 \to 2 \quad \frac{dy}{dx} = -2x + 2$

$I = \int_0^2 (x^2 - x^2 + 2x)\,dx + (x^2 + x^2 - 2x)(-2x+2)\,dx$

$= \int_0^2 2x + 2(x^2 - x) \cdot 2(-x+1)\ dx$

$= \int 2x + 4(-x^3 + x^2 + x^2 - x)\ dx$

$= \int 2x + (-4x^3 + 4x^2 + x^2 - 4x)\ dx$

$= \int_0^2 -4x^3 + 5x^2 - 2x\ dx$

$= -x^4 + \frac{5}{3}x^3 - x^2 \Big|_0^2$

$10\left(\frac{4}{3} - \frac{6}{3}\right)$

$10\left(-\frac{2}{3}\right)$

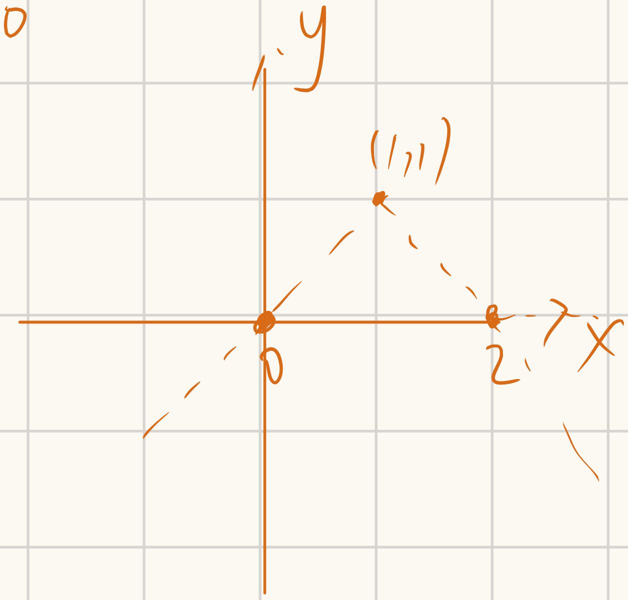
$$= -16 + \frac{5}{3}\times 8 - 4 = \frac{40}{3} - 20 = -\frac{20}{3}$$

(2)

$L = \begin{cases} y=0 \\ x: 0 \to 2 \end{cases}$

$$I = \int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$$

(3)



$L = L_1 \cup L_2$

$L_1: \begin{cases} y=x \\ x: 0 \to 1 \end{cases}$ $\quad L_2: \begin{cases} y = 2-x \\ x: 1 \to 2 \end{cases}$

$(2-x)^2 = x^2$

$$I = \left(\int_{L_1} + \int_{L_2}\right) \cdots$$

$$\int_{L_1} (x^2+x^2)\,dx + (x^2-x^2)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

$$L_2: \int_1^2 (x^2+(2-x)^2)\,dx + (x^2-(2-x)^2)(-dx)$$

$$= \int_1^2 (2x^2-4x+4 + 4-4x)\,dx$$

$= \int_1^2 (2x^2 - 8x + 8)\,dx$

$= \frac{2}{3}x^3 - 4x^2 + 8x \Big|_1^2$

$= \frac{2}{3}(8-1) - 4(4-1) + 8 = \frac{2}{3}\times 7 - 12 + 8$ $\quad dx =$

$= \frac{14}{3} - 4 = \frac{2}{3}$

$x^2+y^2=1 \quad \frac{dx}{y} = -\frac{dy}{x} \quad dy =$

$\int x^3 y\,dx = -\int x^3 y \frac{y}{x}\,dy$

11.2A

(3) $z=0$ 投影为 $D=\{(x,y)\mid x^2+y^2\le 1\}$

$I = \iint_D x^2\sqrt{1-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta\sqrt{1-r^2}\; r\,d\theta\,dr$

$= \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 r^2\sqrt{1-r^2}\cdot r\,dr$

$r^2\sqrt{1-r^2}\,\frac{1}{2}\,dr^2$

$= \frac{1+\cos 2\theta}{2}\Big|_0^{2\pi} \cdot \left(-\frac{1}{3}r^3(1-r^2)^{\frac{3}{2}} - \frac{2}{15}(1-r^2)^{\frac{5}{2}}\right|$

$\left(\frac{2}{15}\right)$

$= \pi \cdot \frac{2}{15} = \frac{2}{15}\pi$

(2) $z=-\sqrt{1-x^2-y^2}$

$I_2=-I_1=-\frac{2}{15}\pi$

(3) $I_3=2I_1=\frac{4}{15}\pi$

6.  $z=\frac{1}{2}(x^2+y^2)$ $0\le z\le 2$ $\iint \vec{E}\cdot d\vec{S}=\iiint(\nabla\cdot\vec{E})\,dV$

$V=\{(x,y,z)\mid x^2+y^2\le 2z,\ 0\le z\le 2\}$

$I_1=\iint_{\Sigma_1}(z^2+x,0,-z)\cdot d\vec{\Sigma}$

$I_2=\iint_{\Sigma_2}(z^2+x,0,-z)\cdot d\vec{\Sigma}$

Σ_2为$\{(x,y,z)\mid x^2+y^2\le 4,\ z=2\}$的上侧

$I_2=\iint(4+x,0,-2)\cdot d\vec{\Sigma}$

$=0+0+\iint_{x^2+y^2\le 4}(-2)\,dxdy=-2\cdot\pi 2^2=-8\pi$

$I_3=\iiint_V \nabla\cdot(z^2+x,0,-z)\,d\tau$

$=\iiint(1+0-1)\,d\tau=0$

$I_3=I_1+I_2$

$I_1=-I_2=8\pi$

Way 2

$z = \frac{1}{2}(x^2+y^2)$ $\qquad z - z(x,y) = 0$

Σ的方向为 $(z'_x, z'_y, -1) = (x, y, -1)$

$$I = \iint_{\Sigma} (z^2+x, 0, -z)\cdot d\vec{S}$$

$$= \iint_{x^2+y^2\le 4} (z^2+x, 0, -z)\cdot(x, y, -1)\,dx\,dy$$

$$= \iint_{x^2+y^2\le 4} (z^2x + x^2 + z)\,dx\,dy$$

$z = \frac{r^2}{2}$

$$= \int_0^{2\pi}\int_0^2 \left(\left(\frac{r^2}{2}\right)^2 r\cos\theta + r^2\cos^2\theta + \frac{r^2}{2}\right) r\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(\frac{1}{4}r^6\cos\theta + r^3\left(\cos^2\theta + \frac{1}{2}\right)\right) dr$$

$$= \int_0^{2\pi} d\theta \left(\frac{1}{4}\times\frac{1}{7}r^7\cos\theta + \frac{1}{4}r^4\left(\cos^2\theta + \frac{1}{2}\right)\right)\Bigg|_0^2$$

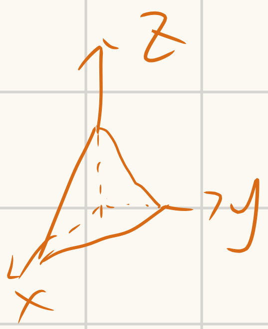
$$\int_0^{2\pi}\left(\frac{32}{7}\cos\theta + 4\cos^2\theta + 2\right) d\theta$$

$$= 4\pi + 4\pi = 8\pi$$

$\frac{2^7}{4} = 2^5$ $\quad 4\cdot4\cdot2 = 16\cdot2 = 32$

$\frac{2^4}{2}\cdot\frac{1}{2} = 2^2 = 4$

7.



$$\oiint \vec{F}\,d\vec{S} = \iiint \nabla\cdot\vec{F}\,dV$$

$$I = \iint_{\Sigma} (xy, yz, xz)\cdot d\vec{S}$$

$$= \iiint_V \nabla\cdot(xy, yz, xz)\,dV$$

$$= \iiint_V (y+z+x)\,dV$$

因对称性 $= 3\iiint_V x\,dV$

$$V = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2-x,\ 0 \le z \le 2-x-y\}$$

$$I = 3\int_0^2 x\,dx \int_0^{2-x} dy \int_0^{2-x-y} dz$$

$$= 3\int_0^2 x\,dx \int_0^{2-x} (2-x-y)\,dy$$

$$= 3\int_0^2 x\,dx \left((2-x)y - \frac{1}{2}y^2\right)\Big|_0^{2-x}$$

$$3\int_0^2 x\left((2-x)^2 - \frac{1}{2}(2-x)^2\right)dx$$

$$3\int_0^2 \frac{1}{2} x\left(4+x^2-4x\right)dx$$

$$= \frac{3}{2}\int_0^2 (4x-4x^2+x^3)\,dx$$

$$= \frac{3}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= \frac{3}{2}\left(2\times 4 - \frac{4}{3}\times 8 + \frac{1}{4}\times 16\right)$$

$$= \frac{3}{2}\left(8 - \frac{32}{3} + 4\right) = \frac{3}{2}\left(12 - \frac{32}{3}\right) = \frac{\cancel{3}}{2}\,\frac{36-32}{\cancel{3}} = \frac{4}{2}$$

$$= 2$$

$$I = 2$$

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

$$\omega = P\,dx + Q\,dy$$

$$d\omega = dP\wedge dx + dQ\wedge dy$$

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$dP\wedge dx = -\frac{\partial P}{\partial y}\,dx\wedge dy$$

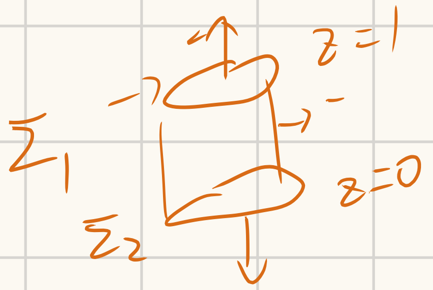
$$dQ\wedge dy = \frac{\partial Q}{\partial x}\,dx\wedge dy$$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy$$

$$\int_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy = \int_{\partial\Omega} P\,dx + Q\,dy$$

11.3B

5.



$$I = \iint_{\Sigma} (xy - xz, 0, x-y) \cdot d\vec{S}$$

$$= \left(\iint_{\Sigma_{all}} - \iint_{\Sigma_1} - \iint_{\Sigma_2}\right) (xy - xz, 0, x-y) \cdot d\vec{S}$$

$$= \iiint_V \nabla \cdot (xy - xz, 0, x-y)\, dV - \iint_{x^2+y^2\le 1} (xy-x, 0, x-y)\cdot(0,0,1)\, dxdy$$

$$\iiint \{x^2+y^2 \le 1,\ 0\le z\le 1\} \qquad - \iint_{x^2+y^2\le 1} (xy, 0, x-y)\cdot(0,0,-1)\, dxdy$$

$$= \iiint_V (y-z)\, dxdydz + 0$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r\sin\theta - z)\cdot r\, d\theta\, dr \cdot dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr \int_0^1 r(r\sin\theta - z)\, dz$$

$$r\left(r\sin\theta\, z - \frac{1}{2}z^2\right)\Big|_0^1$$

$$r\left(r\sin\theta - \frac{1}{2}\right)$$

$$\int_0^1 \left(r^2\sin\theta - \frac{1}{2}r\right) dr$$

$$\left(\frac{1}{3}r^3\sin\theta - \frac{1}{2}\cdot\frac{1}{2}r^2\right)\Big|_0^1$$

$$\int_0^{2\pi} \left(\frac{1}{3}\sin\theta - \frac{1}{4}\right) d\theta$$

$$= -\frac{2\pi}{4} = -\frac{\pi}{2}$$

11.4 A

$1 \quad r \quad r\sin\theta$

$$\iint_S \vec{E}\cdot d\vec{s} = \iiint_V (\nabla\cdot\vec{E})\cdot dV \qquad \nabla\cdot\vec{a} = \frac{1}{H_1H_2H_3}\left(\frac{\partial(a_1H_2H_3)}{\partial q_1}\right.$$

4. 令 $r=\sqrt{x^2+y^2+z^2}$

$$\nabla\cdot\left(\frac{\vec{e_r}}{r^2}\right) = 4\pi\delta(r) \qquad = \frac{1}{r^2\sin\theta}\frac{\partial(r^2}{\partial r} \qquad = \frac{1}{r^2}\frac{\partial(a}{\partial r}$$

$$I = \iint_\Sigma \left(\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\right)\cdot d\vec{s}$$

$$= \iint_\Sigma \frac{1}{r^2}\vec{e_r}\cdot d\vec{s}$$

$$= \iiint_V \nabla\cdot\left(\frac{\vec{e_r}}{r^2}\right) dV$$

$$= \iiint_V 4\pi\delta(r)\, dV = 4\pi$$

6.

$$I = \iint_\Sigma (y-z,\ z-x,\ x-y)\cdot d\vec{s}$$

$$= \left(\iint_{\Sigma_{all}} - \iint_{\Sigma_1}\right)(\quad)\cdot d\vec{s}$$

$$= \iiint_V \nabla\cdot(y-z,\ z-x,\ x-y)\, dV - \iint_{\substack{\Sigma_1 \\ x^2+y^2\le 1,\ z=1}} (y-1,\ 1-x,\ x-y)\cdot d\vec{s}$$

$$= 0 - \iint_{x^2+y^2\le 1} (y-1,\ 1-x,\ x-y)\cdot(0,0,-1)\, dx\, dy$$

$$= \iint_{x^2+y^2\le 1} (x-y)\, dx\, dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr\, (r\cos\theta - r\sin\theta)$$

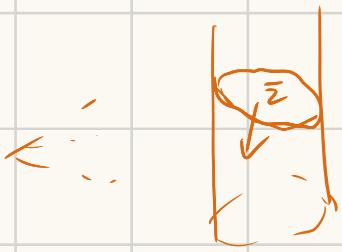
$$= \int_0^{2\pi} d\theta \ \frac{1}{2} r^2 (\cos\theta - \sin\theta) \Big|_0^1$$

$$= \frac{1}{2} \int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta$$

$$= 0$$

8、 $L = \begin{cases} x^2+y^2=1 \\ x-y+z=2 \end{cases}$

取 $z=0$ 投影为 $x^2+y^2=1$



$$\oint \vec{a}\cdot d\vec{l} = \iint (\nabla\times\vec{a})\cdot d\vec{S}$$

$$I = \int_L (z-y,\ x-z,\ x-y)\cdot d\vec{l}$$

$$= \iint_\Sigma \nabla\times(z-y,\ x-z,\ x-y)\cdot d\vec{S}$$

$$= \iint_\Sigma \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial_x & \partial_y & \partial_z \\ z-y & x-z & x-y \end{vmatrix} \cdot d\vec{S}$$

$$= \iint_\Sigma (-1-(-1),\ -(1-1),\ 1-(-1)) \cdot d\vec{S}$$

$$= \iint_\Sigma (0,0,2)\cdot d\vec{S}$$

$$= \iint_{x^2+y^2\le 1} (0,0,2)\cdot(-1,1,-1)\, dxdy$$

$$= -2\pi$$