7.1 A

3.

$y' = 12\sin 2x + 16\cos 2x$

$y'' = 24\cos 2x - 32\sin 2x$

可计算 $y''+y'+\frac{5}{2}y = 24\cos 2x - 32\sin 2x + 12\sin 2x + 16\cos 2x + \frac{5}{2}(-6\cos 2x + 8\sin 2x)$

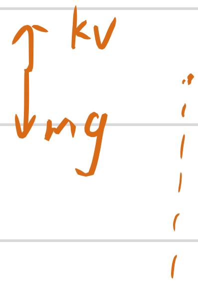
$= 25\cos 2x$

$y(0) = -6 \quad y'(0) = 16$

经验证 $y = -6\cos 2x + 8\sin 2x$ 是此微分方程的解

4.

设人和伞总质量为m，空气阻力为$\vec{f} = -k\vec{v}$，重力加速度是 g



由牛顿第二定律 $\begin{cases} m\dot{v} = mg - kv \\ v|_{t=0} = 0 \end{cases}$

解此方程 $\dot{v} + \frac{k}{m}\left(v - \frac{mg}{k}\right) = 0$

解出 $\dfrac{v - \frac{mg}{k}}{v_0 - \frac{mg}{k}} = e^{-\frac{k}{m}t}$

$v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-\frac{k}{m}t}$

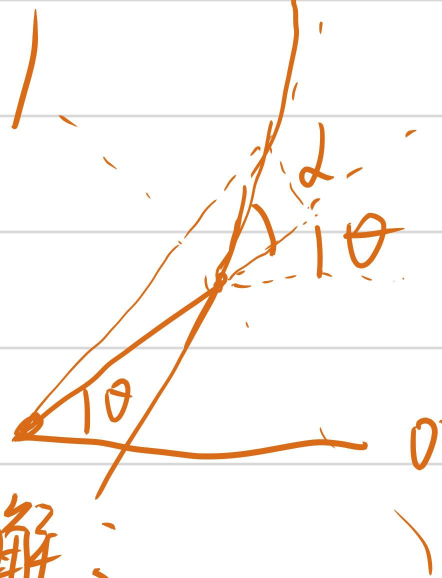
$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

5. (2) 令 $x - C = z$

$z^2 + y^2 = 1 \Rightarrow zdz + ydy = 0 \Rightarrow z + y\frac{dy}{dz} = 0 \Rightarrow x - C + y\frac{dy}{dx} = 0$

(4) $x - C_1 + (y - C_2)\frac{dy}{dx} = 0$

6. (1)



下解：

$$\begin{cases} \tan\theta = \dfrac{y}{x} \\ y' = \tan(\alpha+\theta) \end{cases}$$

or $\tan\alpha = \dfrac{r\,d\theta}{dr} \Rightarrow \boxed{r = Ce^{\frac{\theta}{\tan\alpha}}}$

令 $\dfrac{y}{x} = z$

则 $\dfrac{dy}{dx} = z + x\dfrac{dz}{dx}$

$$z + x\frac{dz}{dx} = \frac{\tan\alpha + z}{1-\tan\alpha\cdot z}$$

$$x\frac{dz}{dx} = \frac{\tan\alpha\,(1+z^2)}{1-\tan\alpha\cdot z}$$

$$\frac{1-\tan\alpha\, z}{\tan\alpha\,(1+z^2)}dz = \frac{1}{x}dx$$

两边积分

$$\frac{1}{\tan\alpha}\int\frac{dz}{1+z^2} - \int\frac{z\,dz}{1+z^2}$$

$$= \ln|x|$$

$$\frac{1}{\tan\alpha}\arctan z - \frac{1}{2}\ln(1+z^2) = \ln|x| + \ln\frac{1}{C}$$

即 $\dfrac{1}{\tan\alpha}\arctan\dfrac{y}{x} - \dfrac{1}{2}\ln\left(1+\dfrac{y^2}{x^2}\right) = \ln|x| + \ln\dfrac{1}{C}$

代入 $\dfrac{y}{x} = \tan\theta \quad x = r\cos\theta$ 得

$$\frac{\theta}{\tan\alpha} - \ln\left|\frac{1}{\cos\theta}\right| = \ln|r\cos\theta| + \ln\frac{1}{C}$$

$$\frac{\theta}{\tan\alpha} = \ln|r| + \ln\frac{1}{C}$$

即 $\boxed{r = Ce^{\frac{\theta}{\tan\alpha}}}$

(3) 

切线方程(在(x_0,y_0)处)

$$y=y'(x-x_0)+y_0$$

$x=0$时 $y=-y'x_0\,y_0$

$y=0$时 $y'(x-x_0)=-y_0$ $x=-\frac{y_0}{y'}+x_0$

三角形面积 $|-y'x_0y_0|\left|-\frac{y_0}{y'}+x_0\right|=a^2$

即 $|x_0y_0(-y_0+x_0y')|=a^2$

Case 1

$x_0y_0(-y_0+x_0y')=a^2$

即 $xy(xy'-y)=a^2$

$y'=\frac{y}{x}+\frac{a^2}{x^2y}$

令 $\frac{y}{x}=z$

有 $z+x\frac{dz}{dx}=z+\frac{a^2}{x^3z}$

$$\frac{dz}{dx}=\frac{a^2}{x^4z}$$

$$\frac{1}{2}z^2=-\frac{a^2}{3}\frac{1}{x^3}+C$$

代入$z=\frac{y}{x}$

$$y=\pm\sqrt{-\frac{2}{3}\frac{a^2}{x}+2Cx^2}$$

Case 2

$x_0y_0(-y_0+x_0y')=-a^2$

即 $xy(xy'-y)=-a^2$

$y'=\frac{y}{x}+\frac{(-a^2)}{x^2y}$

令 $\frac{y}{x}=z$

有 $z+x\frac{dz}{dx}=z+\frac{(-a^2)}{x^3z}$

$$\frac{dz}{dx}=\frac{(-a^2)}{x^4z}$$

$$\frac{1}{2}z^2=+\frac{a^2}{3}\frac{1}{x^3}+C$$

代入$z=\frac{y}{x}$

$$y=\pm\sqrt{+\frac{2}{3}\frac{a^2}{x}+2Cx^2}$$

(7)

$y' = kx$

$\frac{dy}{dx} = kx$

$y = \frac{1}{2}kx^2 + C$

补充：求 $y' = \frac{y(1-x)}{x}$ 的通解

$\frac{dy}{dx} = y\left(\frac{1}{x} - 1\right)$

$\frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx$

$\ln|y| = \ln|x| - x + C$

$|y| = e^{\ln|x| - x + C}$

$y = \pm|x|e^{-x+C}$