PCA
ASSIGNMENT 3

Q1. Eigenvalues and eigenvectors are concepts in linear algebra associated with square matrices. An eigenvector of a square matrix A is a non-zero vector v such that when A is applied to v, the resulting vector is parallel to v. In mathematical terms, it can be represented as $Av = \lambda v$, where λ is a scalar known as the eigenvalue corresponding to that eigenvector.

Eigen-decomposition is an approach to decompose a square matrix into its constituent eigenvalues and eigenvectors. Mathematically, it's represented as $A = Q\Lambda Q^{-1}$, where Q is a matrix whose columns are the eigenvectors of A, and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A. This decomposition is significant because it simplifies many matrix operations and can reveal important properties of the original matrix.

For example, consider the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

To find its eigenvalues and eigenvectors, we solve the equation $Av = \lambda v$. Upon solving, we get the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$, and the corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Q2. Eigen-decomposition is the process of decomposing a square matrix into a set of eigenvectors and eigenvalues. It's significant because it simplifies many matrix operations, reveals important properties of the original matrix, and helps in solving systems of linear equations, analyzing dynamic systems, and understanding geometric transformations.

Q3. For a square matrix to be diagonalizable using the Eigen-Decomposition approach, it must have a full set of linearly independent eigenvectors. In other words, the matrix A must have n linearly independent eigenvectors, where n is the size of the matrix. A brief proof can be provided using the fact that if a matrix has n linearly independent eigenvectors, it can be diagonalized.

Q4. The spectral theorem states that for any symmetric matrix, the eigenvalues are all real, and the eigenvectors corresponding to distinct eigenvalues are orthogonal. In the context of the Eigen-Decomposition approach, the spectral theorem ensures that the decomposition of a symmetric matrix yields real eigenvalues and orthogonal eigenvectors. For example, in Principal Component Analysis (PCA), which relies on Eigen-Decomposition, the spectral theorem ensures that the principal components (eigenvectors) are orthogonal, simplifying the interpretation of the data.

Q5. Eigenvalues of a matrix can be found by solving the characteristic equation $\text{det}(A - \lambda I) = 0$, where A is the matrix, λ is the eigenvalue, and I is the identity matrix. Eigenvalues represent the scaling factor by which the corresponding eigenvectors are scaled when the matrix is applied to them.

Q6. Eigenvectors are vectors that remain in the same direction after the application of a linear transformation represented by a matrix, albeit possibly scaled by a scalar factor (the eigenvalue). They are related to eigenvalues through the equation $Av = \lambda v$, where A is the matrix, v is the eigenvector, and λ is the eigenvalue.

Q7. Geometrically, eigenvectors represent directions in the vector space that remain unchanged (except for scaling) when a linear transformation (represented by the matrix) is applied. Eigenvalues represent the scaling factor by which the eigenvectors are stretched or compressed along those directions.

Q8. Eigen-decomposition has various real-world applications, including Principal Component Analysis (PCA) in data analysis and compression, Markov chains in probability and statistics, vibration analysis in mechanical engineering, and quantum mechanics in physics.

Q9. Yes, a matrix can have multiple sets of eigenvectors and eigenvalues, especially if it's a non-diagonalizable matrix. In such cases, the matrix might have fewer linearly independent eigenvectors than its dimensionality.

Q10. Eigen-Decomposition is extensively used in data analysis and machine learning:

1. Principal Component Analysis (PCA): PCA is a technique used for dimensionality reduction and data visualization. It relies on Eigen-Decomposition to find the principal components, which are the eigenvectors of the covariance matrix of the data.

2. Singular Value Decomposition (SVD): SVD is another method for matrix decomposition, closely related to eigen-decomposition. It's used in various machine learning tasks like collaborative filtering, image compression, and feature extraction.

3. Eigenfaces: In facial recognition systems, Eigenfaces uses Eigen-Decomposition to represent faces as linear combinations of a small number of characteristic faces (eigenvectors), which helps in identifying individuals from images.