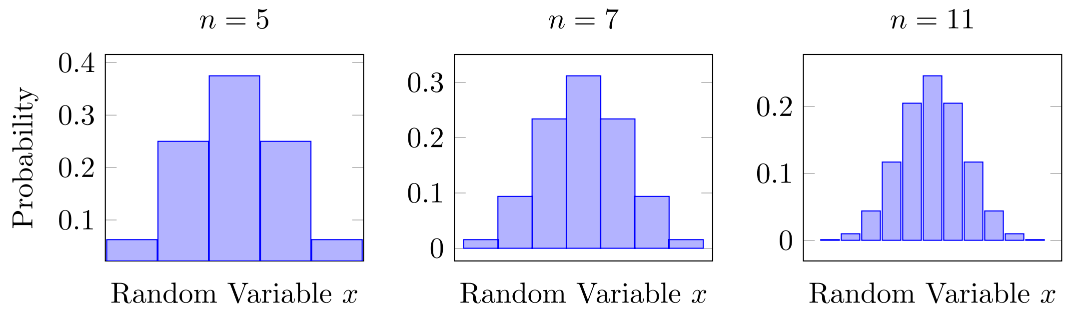
1 Probability Density Function

Consider histograms of a large (almost ∞) population with different class sizes (e.g. students' height distribution), where n is the number of classes:

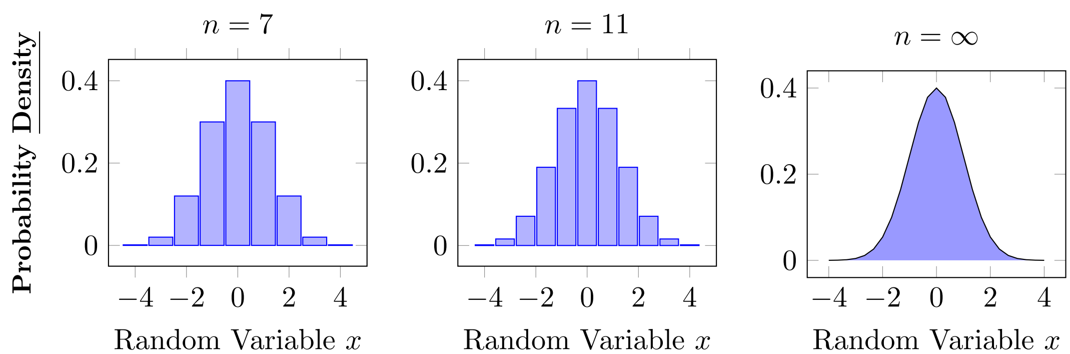


If we take the limit that the number of classes n goes to infinity, the distribution becomes *continuous*, and the probability of each class approaches 0, i.e., $P(x) \to 0$.

Example 1: Temperature distributions can be considered continuous. Compare the probability of the room temperature being exactly 26°C, and that of the temperature being between 25.95°C and 26.05°C.

Note that the **probability density** (probability per unit of the random variable) stays constant as the class size is changed. Thus, we take probability density as the vertical axis for continuous distributions. Such a function is called a **probability density function** (PDF).

$$\text{Probability density} = \frac{\text{Probability}}{\Delta x}$$



$$\text{Probability} = \text{Probability density} \times \Delta x$$

Example 2: Uniform distribution from 0 to 10. PDF: $f(x) = 0.1$

Probability given PDF

For a given probability density function $f(x)$, the probability that $a < x < b$ is

$$\boxed{P(a < x < b) = \int_a^b f(x)dx}$$

In particular, $P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x)dx =$

Statistics given PDF The formulae for discrete distributions can be directly translated to the continuous case. just by replacing sum with integral and $P(x)$ with $f(x)dx$.

- **Mean** $$\mu = \sum_x xP(x) \quad \Rightarrow \quad \mu = \int_{-\infty}^{\infty} xf(x)dx$$
- **Standard deviation** $$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)} \quad \Rightarrow \quad \sigma = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx}$$
- **Expectation values of functions of x**
$$E[g(x)] = \sum_x g(x)P(x) \quad \Rightarrow \quad E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

  E.g., $\sigma = \sqrt{E((x-\mu)^2)} = \sqrt{E(x^2) - \mu^2}$ as in the case of discrete distributions.

How to compute integrals using R

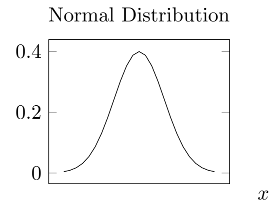
integrand = function(x){ *function to be integrated* }
integrate(integrand, *lower bound*, *upper bound*)

2 Normal Distribution

The normal distribution (or the **Gaussian** distribution) is the most commonly used distribution because

The probability density function of the normal distribution is

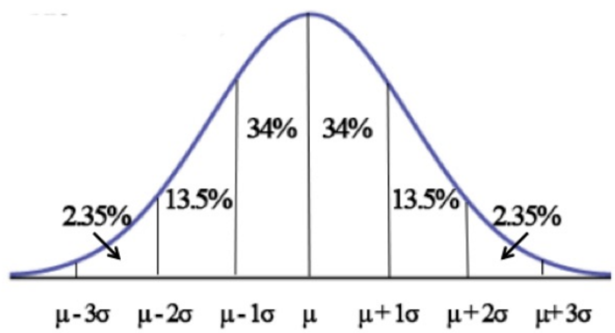
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



where μ is the mean, and σ is the standard deviation.

2.1 Properties of the normal distribution

- Mean = Median = Mode, that is, symmetrical about the mean
- Inflection points at $x = \mu \pm \sigma$
- The **empirical rule** ($68 - 95 - 99.7$ **rule**) is exact.

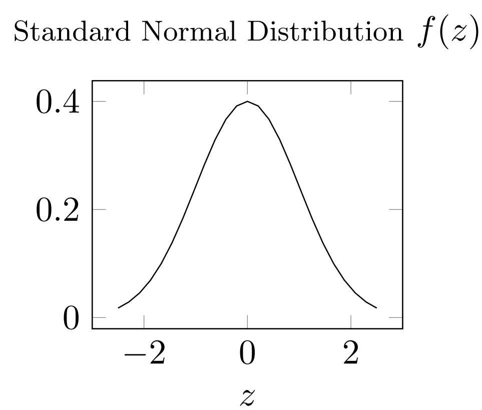


2.2 Standard Normal Distribution

Using the z-score ($z = \dfrac{x-\mu}{\sigma}$) instead of the raw random variable x, the PDF in terms of z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\boxed{}}$$

which has the mean of 0 and the standard deviation of 1.

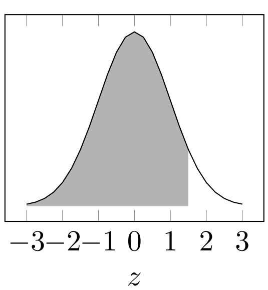


Example 3: The measured values of atomospheric CO_2 concentration is *normally distributed* with the mean of 349 ppm and the standard deviation of 24 ppm. Calculate the probability that a given measurement is within 349 ± 24 ppm.

3 How to compute probabilities with the standard normal distribution

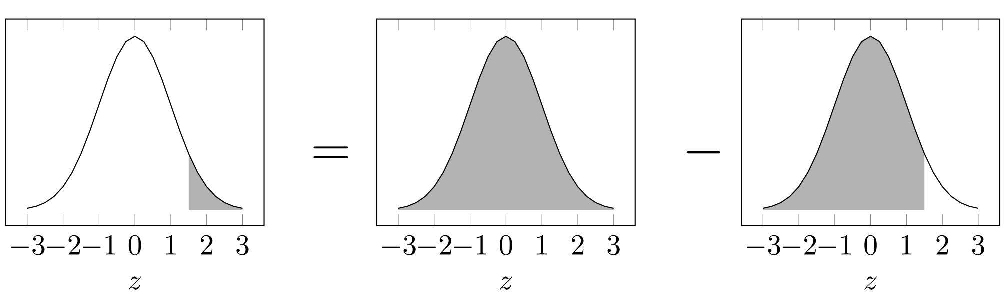
1. $P(z < a) \cdots$ Read the probability directly from the z-table, or use the R command "pnorm(a)", which gives you $P(z < a)$.

   e.g. $P(z < 1.54) =$

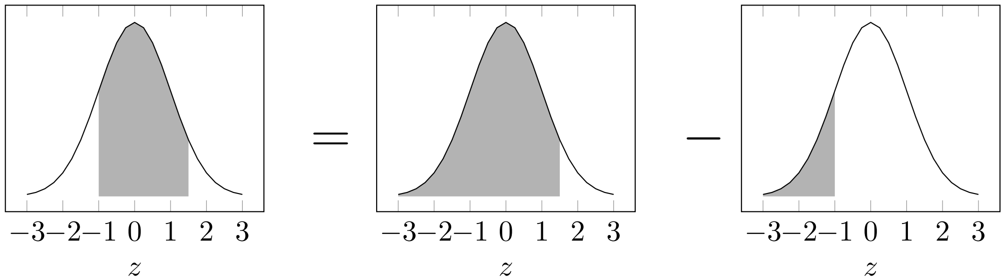


2. $P(z > a) = 1 - P(z < a)$

   e.g. $P(z > 1.54) = 1 - P(z < 1.54) =$



3. $P(a < z < b) = P(z < b) - P(z < a)$
   e.g. $P(-1 < z < 1.54) = P(z < 1.54) - P(z < -1) =$



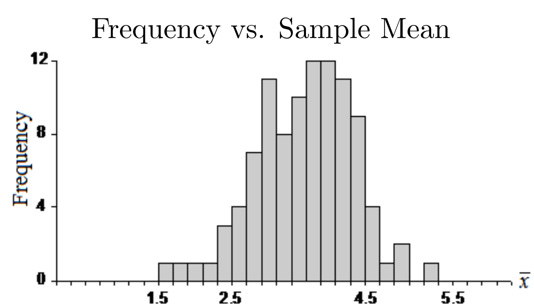
Example 4: Find the probability discussed in the previous example $P(-1 < z < 1)$, using the method described above.

4 Sampling Distributions

Sampling a population is done to infer properties of the *underlying population.* A random sample of size n is a set of n objects drawn from a population in such a way that any set of size n has an equal chance of being drawn.

Example 5: Suppose we have 5 dice and roll them together. This constitutes a single random sample of size $n = 5$ from the sample space of all possible outcomes of the roll of a die. Let x_1 be the number rolled on the first die, x_2 on the second die, etc.

Suppose we roll the 5 dice 100 times. This constitutes 100 samples, each of size $n = 5$. The statistics $\bar{x},\ s, \cdots$ will have different values for different samples. For example we might observe the following **frequency distribution of the sample means**:



Note the resemblance to a normal distribution. This happens even though the probability distribution of x (the number appearing on a single toss of a die) is uniform.

We observe that taking samples of size larger than 1 tends to cluster the sample means $\bar{x}$ about the population mean μ. The larger the sample size the greater the clustering effect.

Example 6: Coin-toss: Let Head$\to$ 1, Tail $\to$ 0.

Sample size: $n = 3$

Sample	$\bar{x}$
{1, 1, 0}	0.67
{0, 0, 0}	0
{1, 1, 1}	1

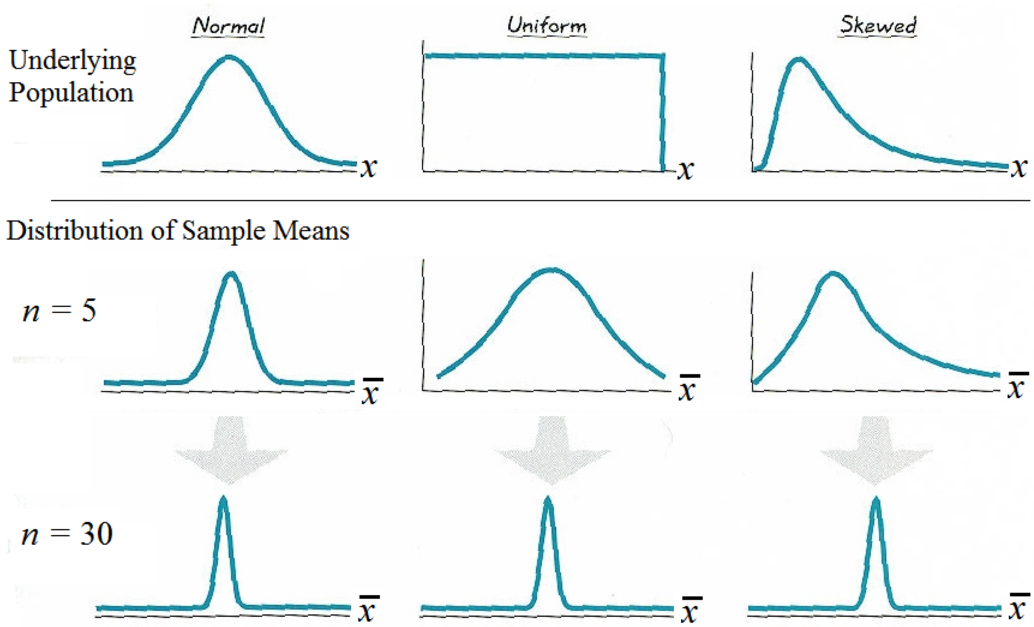
Sample size: $n = 10$

Sample	$\bar{x}$
{1,0,1,1,1,1,1,0,1,0}	0.7
{1,0,1,1,0,1,0,0,1,1}	0.6
{1,0,0,1,1,0,0,1,0,1}	0.5

Sample size: $n = 20$

Sample	$\bar{x}$
{0,1,1,0,1,0,0,0,1,0,0,1,0,1,1,0,1,1,0,0}	0.45
{0,0,1,1,0,0,1,1,0,1,1,0,1,1,0,1,1,0,0,1}	0.55
{1,0,1,1,0,0,1,0,1,1,1,1,0,1,1,0,0,0,1,1}	0.6

The following diagram compares this effect for three different probability distributions. The top row shows the underlying population's probability distribution. The rest of the rows show the **sampling distribution of the sample mean** and how it changes as the sample size increases.



5 The Central Limit Theorem (CLT)

The above example illustrates the Central Limit Theorem. This theorem states that if random samples of size n are drawn from *any* population with mean μ and standard deviation σ, then, when n is large,

- **The distribution of the sample means is approximately a normal distribution**. This approximation becomes better as n gets larger. (Note that if the population is normally distributed to begin with, then so is the sampling distribution, regardless of sample size.)
- The mean of the sampling distribution of the sample means $\mu_{\bar{x}}$ is the same as the mean of the underlying population μ. $\boxed{\mu_{\bar{x}} = \mu}$
- The standard deviation of the sampling distribution of the sample means (**standard error**) is $\boxed{\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}}$, where σ is the standard deviation of the underlying population. (Note that $\sigma_{\bar{x}}$ is inversely proportional to $\sqrt{n}$. This explains why large samples are more accurate than small samples.)

Sampling distribution is approximately normal when

1. the sample size is large enough (typically $n \gtrsim 30$). [CLT]

   > **When calculating probability we need the standard deviation σ of the population. If it is known, say from previous sampling, then we should use it. If it is not known, then for large samples we can simply replace it with s. The CLT still applies and we can use the z table.**

2. the underlying population is normally distributed.

The CLT does not apply and the z table cannot be used when the sample is small and σ is not known. In this case another distribution, the Student t distribution, may be used.

Example 7: A certain population has a mean $\mu = 8.0$ and a standard deviation $\sigma = 0.6$. If a random sample of 36 observations is taken, what is the probability that the sample mean $\bar{x}$ will be within 0.196 of the population mean $\mu = 8.0$?

Example 8: In a protein synthesis plant, the probability of a manufactured protein molecule being left-handed is $p = 0.8$ (thus, 20% right-handed). Estimate the probability that less than 70 out of 100 molecules are left-handed.

0.62%

Example 9: The average wage of chemical engineers is \$52.30 per hour. Suppose we randomly select 50 people. What is the probability that the sample mean for these 50 people is less than \$45.00? Assume σ =\$12.

$P(\bar{x} < 45.00) = 8.93 \times 10^{-6}$

Example 10: The mean voltage of the emergency power supply at some factory is 126 V, and the standard deviation is 15.7 V. If a sample of 25 measurements is made, find the probability that the mean of the sample will be greater than 128.3 V.

$P(\bar{x} > 128.3) = 0.233$

Table of Standard Normal Probabilities: $P(z < a)$

a	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990