1 Definitions

Size of A

Set: A set is a collection of distinct objects (numbers, etc.).

$A=\{a,b,c,d\}$ $c\in A$ (c element in A) $|A|=4$

$B=\{4,2,3,6,17\}$ $6\in B$ $|B|=5$

$C=\{3,4,2\}\subseteq B$ ($C\subseteq B$ C is a subset of B)

Empty set $\{\ \}=\phi$ is a subset of any set

Random experiment: An experiment in which the outcome is not known in advance.

Coin flip, Rolling die

Sample space: The set of all possible outcomes of the random experiment. Often denoted by S.

Event: Any subset of S (i.e. any collection of outcomes).

Simple event: one outcome.

Example 1: A given person's blood type is determined by the presence or lack of A- and B-antigens in red blood cells. Let

A = 1 if A-antigen is present, otherwise 0
B = 1 if B-antigen is present, otherwise 0

The blood type of an individual can be expressed as (A, B). What is the sample space?

$S=\{(0,0)$ "type O", $(1,0)$ "type A", $(0,1)$ "type B", $(1,1)$ "type AB"$\}$

Example 2: Flipping a coin, what's the sample space

$S=\{H,T\}$

Example 3: Flipping two coins, what's the sample space

$S=\{HH,HT,TH,TT\}$

Example 4: Flipping three coins, what's the sample space

$S=\{HHH,HHT,HTH,THH,TTH,THT,HTT,TTT\}$

Example 5: Rolling a fair six sided die once, what's the sample space?

$S=\{1,2,3,4,5,6\}$, $|S|=6$

Example 6: Rolling a fair six sided die twice, what's the sample space?

$$S=\begin{bmatrix}(1,1),(1,2),\ldots,(1,6)\\(2,1),(2,2),\ldots,(2,6)\\\vdots\\(6,1),(6,2),\ldots,(6,6)\end{bmatrix},\quad |S|=36$$

2 The probability of an event A occurring

Denoted $P(A)$. If the outcomes are known to be equally likely then we define:

$$P(A) = \frac{\text{number of outcomes in A}}{\text{number of outcomes in S}}. = \frac{|A|}{|S|}$$

Example 7: Find the probability of a given person having AB blood type in the Example 1, assuming that all simple events in the sample space are equally likely. (Warning: This is highly hypothetical!)

$$P(AB) = \frac{|\{(1,1)\}|}{|\{(0,0),(0,1),(1,0),(1,1)\}|} = \frac{1}{4}$$

Example 8: Rolling a fair six sided die twice, what's the probability that sum is 3?

$$A_3 = \{(x,y) \mid x+y=3\} = \{(1,2),(2,1)\}$$

$$P(A_3) = \frac{|A|}{|S|} = \frac{2}{36} = \frac{1}{18}$$

Example 9: Rolling a fair six sided die twice, what's the probability that sum is 7?

$$A_7 = \{(x,y) \mid x+y=7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$P(A_7) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

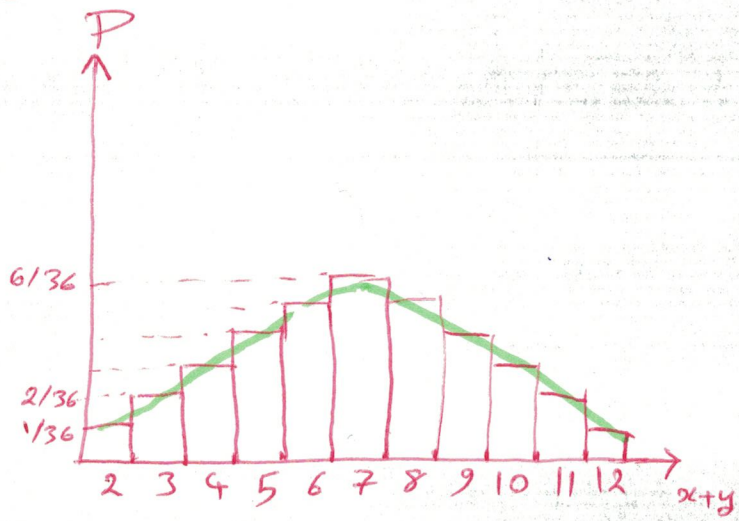
$$P(A_2) = P(A_{12}) = \frac{1}{36}$$

$$P(A_3) = P(A_{11}) = \frac{2}{36}$$

$$P(A_4) = P(A_{10}) = \frac{3}{36}$$

$$P(A_5) = P(A_9) = \frac{4}{36}$$

$$P(A_6) = P(A_8) = \frac{5}{36}$$



3 Law of large numbers

For a large number of trials (experiments)

$$P(A) \approx \frac{\text{number of times that event A occurred}}{\text{total number of trials}}$$

~~**Example 10:** A large number of trials of coin toss.~~

If n is the number of trials then

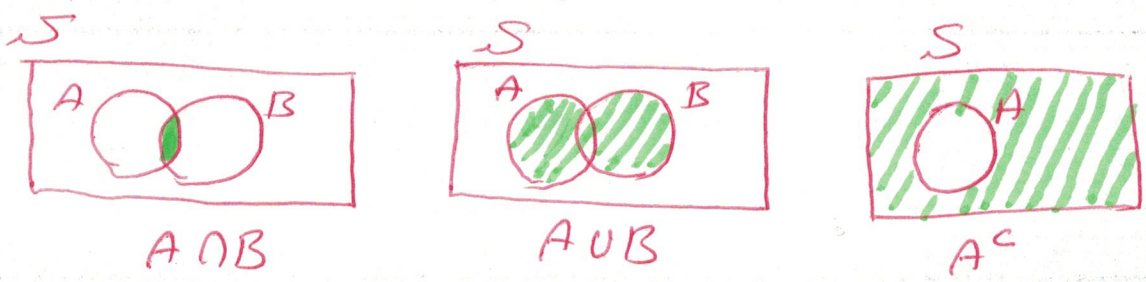
$$P(\text{Heads}) \longrightarrow \frac{1}{2} \quad \text{as} \quad n \longrightarrow \infty$$

4 Relationship Between Events

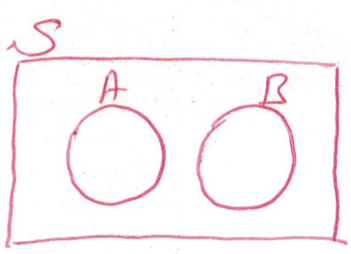
The relationships between any events A and B are described by the logical operations of and, or and not.

- "A and B" is defined as the set of outcomes that belong to both A and B simultaneously. (Intersection, $A \cap B$)
- "A or B" is defined as the set of outcomes that belong to either A or B or both. (Union, $A \cup B$)
- "not A" is defined as the set of outcomes that do not belong to A. (Complement, $\bar{A}$ or A^c)

The relationships can be illustrated with Venn diagrams.



Disjoint or mutually exclusive events: $A \cap B = \emptyset$



$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

$A \cap B = \emptyset$

$A = \{1\}$

$B = \{2, 3\}$

$A \cap B = \emptyset$

Example 11: Blood types: (A: A-antigen present, B: B-antigen present)

$A = \{(1,0),(1,1)\}$ $\quad A \cap B = \{(1,1)\}$ $\quad A \cup B = \{(1,0),(0,1),(1,1)\}$

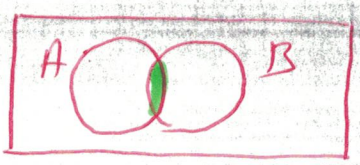
$B = \{(0,1),(1,1)\}$ $\quad (A \cup B)^c = \{(0,0)\}$ $\quad (A \cup B)^c = A^c \cap B^c$

$A^c = \{(0,1),(0,0)\}$ $\quad (A^c \cap B^c) = \{(0,0)\}$

$B^c = \{(1,0),(0,0)\}$

5 The Addition Rule for the Probability of the Union of Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Example 12: $A = \{1,2,3\}$, $B = \{2,4,6\}$ then

Assume $S = \{1,2,3,4,5,6\} \Rightarrow P(A) = \frac{3}{6} = \frac{1}{2} = P(B)$

$P(A \cup B) = P(\{1,2,3,4,6\}) = \frac{5}{6}$ $\quad P(A \cap B) = P(\{2\}) = \frac{1}{6}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$

If A and B are disjoint, i.e., $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) \quad (\overset{0}{\overset{\shortparallel}{-P(A \cap B)}})$$

Example 13: $A = \{1,3\}$, $B = \{2,4,6\}$ then

$P(A) = \frac{2}{6}$ $\quad P(B) = \frac{3}{6}$ $\quad P(A \cup B) = P(\{1,2,3,4,6\}) = \frac{5}{6}$

$P(A) + P(B) = \frac{2}{6} + \frac{3}{6} = \frac{5}{6} = P(A \cup B)$

In particular, since $A \cup A^c = S$

$$P(A \cup A^c) = P(A) + P(A^c) = 1$$

$A \cap A^c = \phi$
$A \cup A^c = S$

Example 14: In the sample space of rolling a die if $A = \{1, 3\}$, then $A^c =$

More generally, if $A_1, A_2, \cdots$ are disjoint and $A_1 \cup A_2 \cup \cdots = S$ then

$$\sum_i P(A_i) = P(\bigcup_i A_i) = P(S) = 1$$

6 The Multiplication Rule for the Probability of the Intersection of Events

Conditional probability: Probability that event B will occur, given that event A has already occurred. Denoted $P(B|A)$. ($B|A$ is read as "B given A")

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Independence of events: Two events A and B are *independent* if the occurrence of one does not affect the probability of the occurrence of the other. In equation form, B is independent of A if $P(B|A) = P(B)$. If two events are not independent, then we say that they are dependent.

Example 15: Genders of randomly chosen two people from a large population.

F: 1st Person is a Female $\quad P(M|F) = P(M) = 0.5$

M: 2nd Person is a Male $\quad$ So M & F are independent

6.1 Multiplication Rule

Using the conditional probability, $P(A \cap B)$ can be written as:

$$P(A \cap B) = P(B|A) \cdot P(A) \qquad \text{(MR1)}$$

or equivalently,

$$P(A \cap B) = P(A|B) \cdot P(B) \qquad \text{(MR2)}$$

This is called the multiplication rule. If A and B are independent events then this simplifies to

$$P(A \cap B) = P(A) \cdot P(B)$$

since $P(B|A) = P(B)$, $P(A|B) = P(A)$.

Example 16: 0.4% of women, 8% of men have congenital colour blindness. What is the probability of a randomly chosen person is colour blind as well as male?

F: Person Female $\quad P(C|F) = 0.004 \quad P(C|M) = 0.08$

M: Person Male $\quad P(M) = P(F) = 0.5$

C: Person Color Blind $\quad P(C \cap M) = P(C|M) \cdot P(M)$

$= 0.08 \times 0.5 = 0.04$

6.2 Bayes' Theorem

Combining Equations (MR1) and (MR2) above, $P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$, and dividing both sides by $P(A)$ we obtain so called *Bayes' theorem*:

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

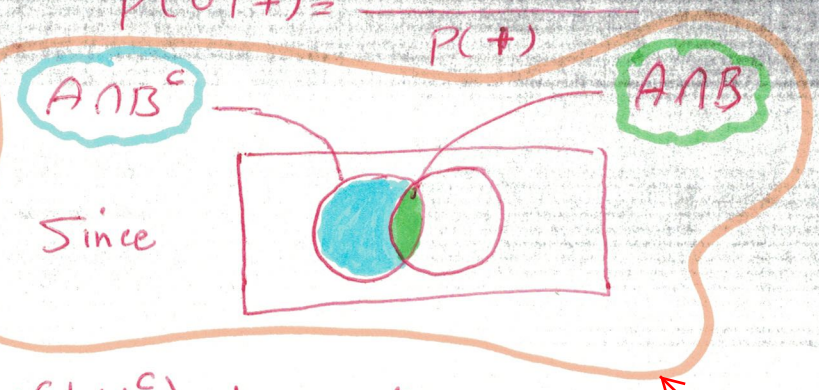
Example 18: Suppose a drug test gives 99% true positive results for drug users and 98% true negative results for non-drug users. Suppose 0.5% of people are users of the drug. If a randomly selected individual tests positive, what is the probability that he is a user?

U : a given person is a drug user, U^c : a given person is a non-drug user

$+$: a given person is tested positive, $-$: a given person is tested negative

$$\begin{cases} P(U \mid +) = ? \\ P(+ \mid U) = 0.99 \\ P(+^c \mid U^c) = 0.98 \\ P(U) = 0.005 \end{cases}$$

$$P(U \mid +) = \frac{P(+ \mid U)\, P(U)}{P(+)}$$



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We have $P(+ \mid U^c) + P(+^c \mid U^c) = 1$ ★

$P(+ \cap U) + P(+ \cap U^c) = P(+)$ ★★

$$\therefore P(+) = P(+ \mid U)\, P(U) + P(+ \mid U^c)\, P(U^c) \quad \text{(expanding ★★)}$$

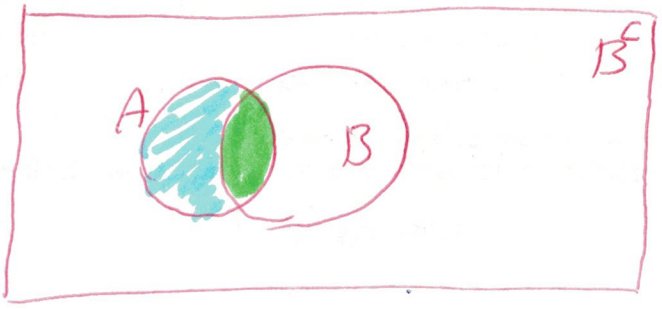
$$= P(+ \mid U)\, P(U) + \underbrace{[1 - P(+^c \mid U^c)]}_{\text{from ★}}[1 - P(U)]$$

$$= (0.99)(0.005) + [1 - 0.98][1 - 0.005]$$

$$= 0.02485$$

$$P(U \mid +) = \frac{(0.99)(0.005)}{0.02485} = 0.199$$

For any A & B



$$P(A \cap B) + P(A \cap B^c) = P(A)$$

$$P(B \cap A) + P(B^c \cap A) = P(A)$$

$$\frac{P(B \cap A)}{P(A)} + \frac{P(B^c \cap A)}{P(A)} = \frac{P(A)}{P(A)}$$

$$P(B \mid A) + P(B^c \mid A) = 1$$

$\Rightarrow$ if $B = +$ and $A = U^c$ we have

$$P(+ \mid U^c) + P(+^c \mid U^c) = 1$$

7 Combinatorics

If you are to pick r objects out of the total of n objects, there are different ways to count the number of possible ways to do this.

- Ordered sampling with replacement n^r

Example 19: If any number can be used for each digit, how many telephone numbers (3+3+4 digits) can you make? $\{0,1,2,\dots,9\}$

10 10 10 | 10 10 10 | 10 ... 10

$= 10 \times 10 \times \cdots \times 10 = 10^{10}$

Example 20: How many 4 digit PIN number can we make?

$\{0,1,2,\dots,9\}$ 10 10 10 10 $10 \times 10 \times 10 \times 10 = 10{,}000$

- **Permutation**: Ordered sampling without replacement

$$n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!} \equiv {}_n\mathrm{P}_r$$

Example 21: How many 3-letter passwords can be made with all three letters being different? (There are 26 letters in the alphabet. Order matters here.)

$\underbrace{\{a,b,\dots,z\}}_{26}$ 26 25 24 $26 \times 25 \times 24 = 15{,}600 = \frac{26!}{(26-3)!} =$

Example 22: How many 4 digit with no digit repeating PIN number can we make?

$\{1,2,\dots,9,0\}$ 10 9 8 7 $P_{10\,4} = \frac{10!}{(10-4)!} = \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!}} = 5040$

- **Combination**: Unordered sampling without replacement

$$\frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} = \frac{n!}{(n-r)!r!} = \frac{{}_nP_r}{r!} \equiv {}_n\mathrm{C}_r$$

Example 23: Let's pick 3 numbers from {1,2,3,4}. There are $4 \times 3 \times 2 = 24$ ways (permutations) to do it. They are

{1,2,3}, {1,3,2}, {2,3,1}, {2,1,3}, {3,1,2}, {3,2,1} $\cdots$ permutations of {1,2,3}
{1,2,4}, {1,4,2}, {2,4,1}, {2,1,4}, {4,1,2}, {4,2,1} $\cdots$ permutations of {1,2,4}
{1,3,4}, {1,4,3}, {3,4,1}, {3,1,4}, {4,1,3}, {4,3,1} $\cdots$ permutations of {1,3,4}
{2,3,4}, {2,4,3}, {3,4,2}, {3,2,4}, {4,2,3}, {4,3,2} $\cdots$ permutations of {2,3,4}

Note that each row consists of triplets of numbers ordered in different ways. There are $3! = 3 \times 2 \times 1 = 6$ ways to arrange 3 numbers. Thus, the numbers of ways you can choose 3 people (unordered) is

$${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{4 \times 3 \times 2}{6} = 4 \quad \text{or} \quad {}_4C_3 = \frac{4!}{(4-3)!3!} = \frac{24}{6} = 4$$

Example 24: How many ways can you pick 5 people out of 20?

$$C_{20\,5} = \frac{20!}{(20-5)!\,5!} = \frac{20!}{15!\,5!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times \cancel{15!}}{\cancel{15!}\,5!}$$

$= 15{,}504$