Indian Origins of the Pythagorean Theorem

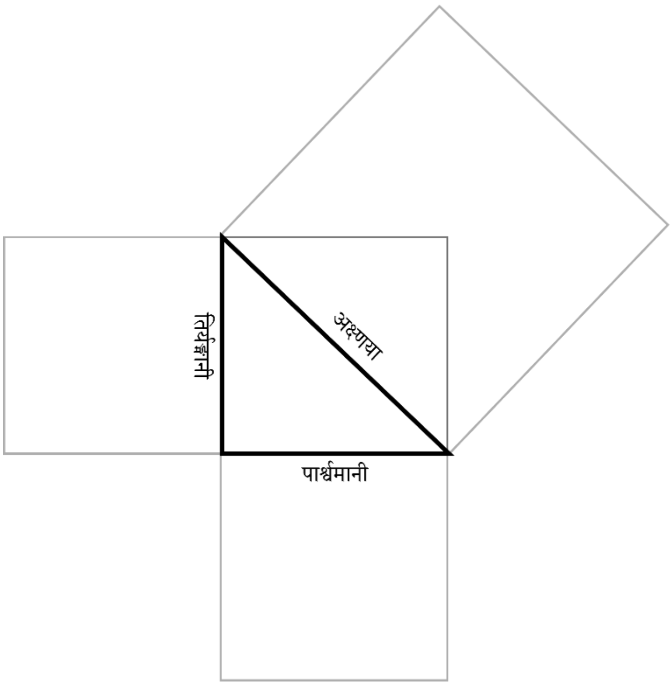
The Pythagorean theorem is arguably the most fundamental idea of Euclidean geometry as we know it. It is, in principle, a relation between the sides of a right-angled triangle. It is associated with the Greek polymath Pythagoras, who belonged to the 1st Century BCE. Although, it has been widely accepted that the said theorem was in use in ancient India, Babylonia, Egypt and China, centuries before the birth of Pythagoras himself. In India, for instance, the earliest explicit references to this theorem go back to the **śulbasūtra-s**, which are dated to the 5th-6th centuries BCE.

The **śulbasūtra-s** are texts that contain aphorisms describing in detail the construction of fire altars to be used in śrauta rituals. The **śulbasūtra-s** are numerous: attributed to different individuals throughout the course of Indian vedic history. The writers of the **śulbasūtra-s** are called **śulbakāra-s**. They constitute a part of a larger set of **sūtra**-s (like the śrautasūtras and grhyasūtras) which describe and govern the nature and procedures of vedic rituals. The most well-known are attributed to **Baudhāyana, Mānava, Āpastamba** and **Kātyāyana** - approximately in chronological order.

In order to construct altars of specific shapes and dimensions, the **śulbasūtra-s** describe many ideas that are associated with geometrical mathematics. They thus contain references to the construction of right-angled triangles as well.

The **Baudhāyana śulbasūtra** states the theorem of the right-angled triangle in aphorism 12 of the first chapter of the text:

> दीर्घचतुरश्रस्य अक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतः तदुभयं करोति । १.१२ ।
>
> dīrghacaturaśrasya akṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutaḥ tadubhayaṃ karoti | 1.12 |
>
> The rope corresponding to the diagonal of an elongated quadrilateral (rectangle) produces what is made by the lateral and vertical sides individually.



In principle, this aphorism refers to the “area produced” by each of the ropes forming the sides and the diagonals, which is mathematically the square of the magnitude of the lengths. We must note, that since the **śulbasūtra-s** are directives on building fire-altars for rituals, the **śulbakāra-s** refer to the usage of ropes and sticks for physical construction.

The **Kātyāyana śulbasūtra** seems to have a very similar statement of the theorem, with the terms **pārshvamānī** and **tiryaṅmānī** interchanged. **refer KSS 2.7.* The **Āpastamba śulbasūtra** too has a very similar statement to that found in the **Baudhāyana śulbasūtra**. **refer ASS 1.4* .

The **Mānava śulbasūtra** has a different approach in the statement of the theorem. It differs both in terminology and principle.

> आयामम् आयामगुणं विस्तारम् विस्तरेणतु । समस्य वर्गमूलं यत् तत् कर्णं तद्विदो विदुः ॥ १०.१० ॥
>
> āyāmam āyāmaguṇaṃ vistāram vistareṇatu | samasya vargamūlaṃ yat tat karṇaṃ tadvido viduḥ || 10.10 ||
>
> The length multiplied by the length, and the breadth multiplied by the breadth. The square-root of the sum (of the aforementioned magnitudes) is the hypotenuse, this is known to scholars.

This statement seems to be most explicit in terms of mathematical description, in comparison to the other **śulbasūtra-s**. Interestingly, the entire 12th chapter of the **Mānava śulbasūtra** deals with the computation of a rectangle's diagonal when the length and breadth are known.

In the succeeding aphorisms of the **Baudhāyana śulbasūtra** , a number of integral examples are enlisted.

> तासां त्रिकचतुष्कयोः द्वादशिकपञ्चिकयोः पञ्चदशिकाष्टिकयोः सप्तिकचतुर्विंशिकयोः द्वादशिकपञ्चत्रिंशिकयोः पञ्चदशिकषट्त्रिंशिकयोः इत्येतासु उपलब्धिः । १.१३ ।
>
> tāsāṃ trikacatuṣkayoḥ dvādaśikapañcikayoḥ pañcadaśikāṣṭikayoḥ saptikacaturviṃśikayoḥ dvādaśikapañcatriṃśikayoḥ pañcadaśikaṣaṭtriṃśikayoḥ ityetāsu upalabdhiḥ | 1.13 |
>
> 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, 15 and 36 – the rule stated above is apparent (in these pairs).

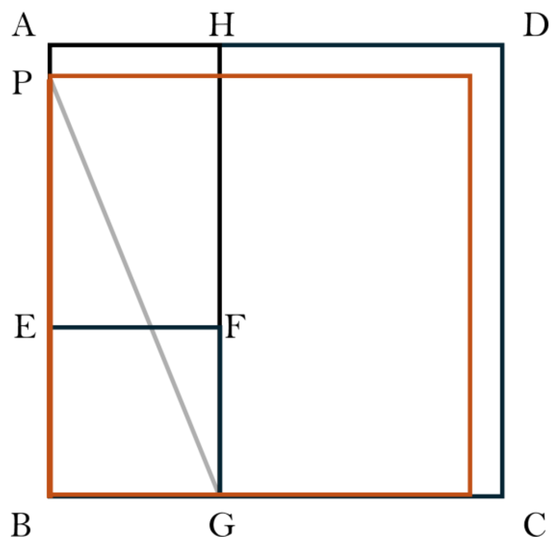
The examples provided here, are pairs that would provide "Pythagorean triplets" as evidently seen. Furthermore, the **Āpastamba Śulbasūtra** seems to generalise the idea that if (a,b,c) satisfy the theorem, (na, nb, nc) are also possible triplets. This provides insights as to how the **śulbakāra-s** may have obtained triplets enlisted as examples. **refer ASS 5.3*

Now, the question that arises is whether the Indian **śulbakāra-s** knew the scope, application and proof of this theorem, apart from an observational statement.

The **sūtra-s** describe various geometrical constructions that would necessarily require deep knowledge of the theorem. These include constructions of squares with areas equal to the sum or difference of areas of two given squares, constructions of isosceles triangles, quadrilaterals, and even perpendicular lines. One such example is the aphorism 2.2 of the **Baudhāyana śulbasūtra** describes the mechanism of constructing a square with the area of the difference of two given squares.

> चतुरश्रात् चतुरश्रं निर्जिहीर्षण्यावन्निर्जिहीर्षेत्तस्य करण्या वर्षीयसो वृध्रमुल्लिखेत् । वृध्रस्य पार्श्वमानीम् अक्ष्णयेतरत्पार्श्वमुपासंहरेत्। सा यत्र निपतेत्तदपच्छिन्द्यात् । चिन्नया निरस्तम् । २.२ ।
>
> caturaśrāt caturaśraṃ nirjihīrṣaṇyāvannirjihīrṣettasya karaṇyā varṣīyaso vṛdhramullikhet | vṛdhrasya pārśvamānīm akṣṇayetaratpārśvamupāsaṃharet| sā yatra nipatettadapacchindyāt | cinnayā nirastam | 2.2 |
>
> If it is desired to remove a square from another, a (rectangular) part is cut off from the larger (square) with the side of the smaller one to be removed; the (longer) side of the cut-

off (rectangular) part is placed across so as to touch the opposite side; by this contact (the side) is cut off. With the cut-off (part) the difference (of the two squares) is obtained.



Let ABCD be the larger square and EBGF be the smaller. The aphorism suggests that the rectangle ABGH is cut off from the square ABCD by BG. The side GH of the cutoff portion is allowed to project onto the side AB, where it intersects at a point P (GH=GP). Hence, BP is the side of the square with the difference of areas between the two given squares.

This interesting application can be proven easily with the theorem stated:

area of constructed square = $BP^2 = GP^2 - BG^2$

$BP^2 = GH^2 - BG^2$

$BP^2 = AB^2 - BG^2$ = *Required difference of areas*

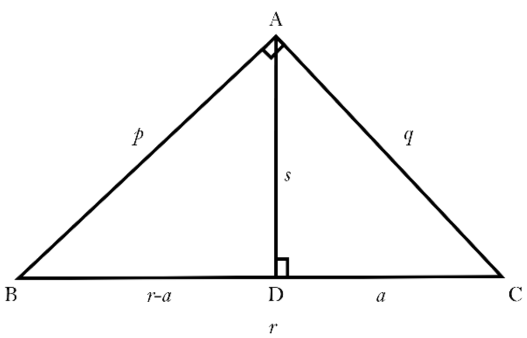
These procedural references in Vedic texts highlight the extent to which such theorems were in vogue and the knowledge of its implications.

Unlike Euclidean mathematicians, Indian mathematical texts seldom provided direct proofs of results mentioned thereof. It was left to commentators and teachers to provide for. This is evident in the case of numerous mathematical advancements recorded by Indian mathematicians like **Brahmagupta, Bhāskara I** and **Bhāskara II**.

For example, consider the works of **Bhāskara II**, who authored the **Līlāvatī** and **Bījagaṇita** – which are textbooks on arithmetic, geometry and algebra – in the 12th Century AD. In his statement of the theorem, he defines the nomenclature of the sides of a right-angled triangle as **bāhu** (base – also **bhuja**), **koṭi** (altitude) and **karṇa** (hypotenuse).

Explanations of the theorem provided herein are enunciated in commentaries "**Buddhivilāsinī"** by **Gaṇeśa** and "**Sūryaprakāśa"** by **Sūryadāsa** on the **Līlāvatī** and **Bījagaṇita** respectively. Interesting interpretations of the theorem are expressed both algebraically and geometrically, by the commentators. **refer Līlāvatī 134 & Vijaganita 146, Colebrooke 1817.*

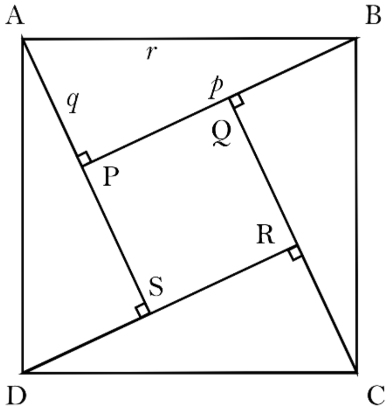
One proof is algebraic in nature, and uses principles of proportionality. The text uses the example of a triangle with base and height with magnitudes 15 and 20 respectively. A rough understanding is as follows:



- The given right-angled triangle is made to rest on its hypotenuse, and an altitude from the hypotenuse to the right-angle is drawn.
- The resulting two triangles are similar to each other and the original given triangle. Thus, since lengths in similar triangles are proportional, the theorem may be derived.
- From triangles BAC and ADC, $q^2=r^2-ar$; and from triangles BAC and ADB, $p^2=ar$.
- Thus, $\mathbf{q^2 = r^2 - p^2 \rightarrow p^2 + q^2 = r^2}$

The second proof uses geometry to reach the result. It considers the area of the triangle and resultant squares.

- Four of the given right-triangles are arranged to form a square ABCD with its side equal in magnitude to the hypotenuse of each triangle.
- Assuming the sides of the triangle to be p, q, and r, where r is the hypotenuse, the side of the square PQRS is evidently p-q, and its area is $(p-q)^2$. p and q are taken to be 20 and 15 respectively in the text.



- The area of the entire square ABCD is r^2, and that of each triangle is $\frac{1}{2}pq$.
- Since area is additive,

$$4(\tfrac{1}{2}pq) + (p-q)^2 = r^2 \quad \text{(total area of all triangles and square PQRS)}$$
$$2pq + p^2 + q^2 - 2pq = r^2$$
$$\mathbf{p^2 + q^2 = r^2}$$

This proof is explained is the **Buddhivilāsinī** of **Gaṇeśa**.

This seminal rule of geometry was independently discovered in India centuries ago. This theorem came to be called the rule of the base (भुजः) , altitude (कोटिः) and hypotenuse (कर्णः), or the **bhuja-koṭi-karṇa-nyaya** (भुज-कोटि-कर्ण-न्यायः).

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